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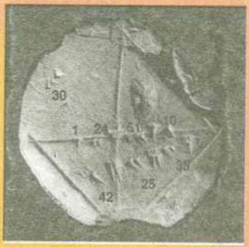
Indira Gandhi National Open University
School of Computer and Information Sciences

BCS-054
Computer Oriented Numerical Techniques



Babylonian clay tablet YBC 7289 for value of $\sqrt{2}$(c.1800-1600 BC)

Aryabhatta (c. 476-550 AD)

Varahmihira (c. 505-587 AD)

Babylonian Mathematical clay tablet Pimpton 322 (c.1800 BC)

Vedic Period: Sulbha/ Shulbha sutras composed by **Baudhayana** (c. 800 BCE), **Manav** (c. 750 BCE), **Apastamba** (c. 600 BCE) and **Katyayana** (3rd Century, BCE) contain, among other mathematical contributions, examples of Pythagorean triples and formula for the value of $\sqrt{2}$ giving accuracy up to 5 places after the decimal.

Brahmagupta (c. 598-670 AD)

Muhammad ibn Musa al-Khwarizmi (c.780-c.850 AD)

A page from Hisab al-jabar w' al Muqabala by al-Khwarizmi (c. 820 AD)

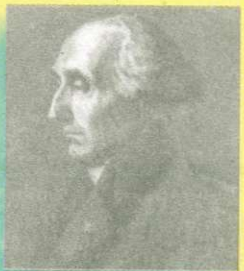
Fragment from Euclid's Elements (c.100 AD)

Isaac Newton (1642-1727 AD)

Leonhard Euler (1707-1783 AD)

A page from Bakhshali Manuscript (Between 2nd Cent. BC, 3rd Cent. AD)

A page from (Chinese): The Nine Chapters on the Mathematical Art (c. 2nd Cent. AD)



Joseph-Louis Lagrange (1736-1813 AD)

Johann Carl Friedrich Gauss (1777-1855 AD)

A page from Ganita-Kaumudi (1356 AD) on Magic Squares

Interpolation

2

"शिक्षा मानव को बन्धनों से मुक्त करती है और आज के युग में तो यह लोकतंत्र की भावना का आधार भी है। जन्म तथा अन्य कारणों से उत्पन्न जाति एवं वर्गगत विषमताओं को दूर करते हुए मनुष्य को इन सबसे ऊपर उठाती है।"

–इन्दिरा गांधी

Symbols	Notations	Symbols	Notations
$\in$	belongs to	$\lim_{x\to a} f(x)$	limit of f(x) as x tends to a
$\supset$	Contains	$P_n(x)$	nth degree polynomial
$< (\leq)$	less than (less than or equal to)	$f'(x)$	derivative of f(x) with respect to x
$> (\leq)$	greater than (greater than or equal to)	$f^{(n)}(x)$	nth derivative of f(x) with respect to x
R	set of real numbers	$\approx$	approximately equal to
C	set of complex numbers	$\propto$	alpha
n!	n.(n – 1). . . 3. 2. 1 (n factorial)	β	beta
[ ]	closed interval	γ	gama
] [	open interval	ε	epsilon
|x|	absolute value of a number x	π	pi
$i.e.$	that is	$\sum$	capital sigma
$\sum_{i=1}^{n} a_i$	$a_1 + a_2 + \cdots + a_n$	ζ	zeta
$x \to a$	x tends to a		

"Education is a liberating force, and in our age it is also a democratising force, cutting across the barriers of caste and class, smoothing out inequalities imposed by birth and other circumstances."

—Indira Gandhi

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BCS-054
COMPUTER ORIENTED NUMERICAL TECHNIQUES

Block

2

INTERPOLATION

PROGRAMME DESIGN COMMITTEE

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SOCIS, IGNOU, New Delhi

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Jamia Millia, New Delhi

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Dept. of CS, CDLU, Sirsa

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Sh. Akshay Kumar, Associate Prof.
SOCIS, IGNOU, New Delhi

Dr. P. K. Mishra, Associate Prof.
Dept. of CS, BHU, Varanasi

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SOCIS, IGNOU, New Delhi

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Dr. Naveen Kumar, Reader
SOCIS, IGNOU, New Delhi

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SOMS, IGNOU, New Delhi

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Associate Professor

Dr. Sudhansh Sharma
Asst. Professor

Dr. P. V. Suresh
Associate Professor

Dr. Naveen Kumar
Reader

PREPARATION TEAM

Prof. Radhey S. Gupta (Content Editor)
Former Professor & Head
Department of Mathematics
I.I.T. Roorkee, Roorkee

Prof. Manohar Lal
SOCIS, IGNOU
New Delhi

Course Coordinator : Prof. Manohar Lal, SOCIS, IGNOU, New Delhi

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BLOCK INTRODUCTION

This block has three units–all about interpolation, i.e., about finding value of a dependent variable corresponding to a given value of the independent variable, where finite number of pairs of values (independent variable, dependent variable) are given.

In Unit 1, first the concept of interpolation is explained. Next, some of the well-known operators useful for interpolation are defined; followed by discussion of inter-relation between these operators. Finally, application of these operators in solving problems is illustrated through examples.

In Unit 2, some of the well-known methods for interpolation are discussed, for the special case when the values of independent variable are equidistant, at which values of the dependent variable, are known. The methods are based on: Newton's Forward Difference Formula, Newton's Backward Difference Formula, Stirling's Formula and Bessel's Formula. Each of the methods is mainly explained through examples of its application. In order to discuss the methods, first, we introduce and illustrate the concepts of Basic Difference Table and Operator Difference tables.

In Unit 3, some of the well-known methods for interpolation are discussed, when the values of independent variable are not necessarily equidistant, at which value of the dependent variable, are known. The methods include Lagrange's Method, Lagrange's Method: Inverse Interpolation, method based on Divided Difference Tables and method based on Newton's Divided Difference. Each of the methods is mainly explained through examples of its application.

UNIT 1 OPERATORS

Structure

1.0 INTRODUCTION

Let us suppose that values of a function y = f (x) are known for some n, at (n + 1) points, say $x = x_i$, i = 0(1) n and further suppose that $x_0 < x_1 < x_2 \ldots < x_n$. The notation *i = 0(1) n* stands for i varies from 0 to n, with i increasing by 1 after each iteration The function f(x) may or may not be known explicitly (i.e., in some form like $f(x) = 5 \sin(x) + 2x^2$, etc.). But y_i, or $f(x_i)$ the value of the function f(x) at x_i is known for all *i = 0(1) n*. Thus, we are given (n + 1) pair of values $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ and the problem is to find the value of y or f(x) at some intermediate point x in the interval $[x_o, x_n]$. The process of computing/determining this value, for an x in the interval $[x_o, x_n]$ is known as 'interpolation'. The x_i's, at which the values of y are known, are called tabular points

Definition : Interpolation is the process of finding the value of a dependent variable, say, y, for an arbitrary given value of the independent variable x in the interval $[x_o, x_n]$, where, for some *n*, the values of the function y = f (x) are known at (n + 1) points of the independent variable. And, for other points of the interval $[x_o, x_n]$, the values may not be known.

There are several methods for interpolation in the special case, when the values x_i, *i = 0(1)* are equally spaced, i.e., when the differences $x_{i+1} - x_i$ of two successive values x_{i+1} and x_i, are all equal to the same quantity. Before discussing these methods, we need to introduce some operators. We will discuss these operators in Section 1.2. These operators will be used in interpolation problems when the given tabular points are equally spaced.

The term *operator* may be new to most of us. In Section 1.2, we will be using expressions like $\Delta f(x) = f(x+h) - f(x)$, which can be understood as 'Δ *acts on functions like f as a sort of function*'. Thus, operator may be understood as a function from a set of functions to a set of functions.

1.1 OBJECTIVES

After going through this unit, you should be able to

- explain the concept of interpolation;
- define the various operators used for interpolation : forward difference operator, backward difference operator, central difference operator, averaging operator, shift operator and differential operator;
- tell the relations between various operators; and
- apply these operators to various functions.

1.2 OPERATORS AND THEIR PROPERTIES

As mentioned in Introduction, for interpolation in the special case, when the tabular values x_i, $i = 0(1)n$ are equally spaced, there are several methods. Before discussing these methods, we need to introduce some operators, which will be used in interpolation problems in the special case. Now, we discuss these operators.

For expressing a fact that a dependent variable, say, y, is functionally dependent on an independent variable, say, x; we may use the notation $y = f(x)$. And, then the value of y at any point x_i, may be denoted either by $f(x_i)$ *or by* y_i, *or even by* $y(x_i)$. Also, when ever required, we may use *either y or f(x)*, to refer to the dependent variable. Generally, in theoretical discussions, as in this section, we use the notations $f(x)$ *and* $f(x_i)$ *etc.*; and in applications, as in most of the rest of the unit, we use the notations y, y_i, *or even*, $y(x_i)$ *etc.*

To mention explicitly, in this section, we use the notations $f(x)$ *and* $f(x_i)$,etc. respectively to denote *the* dependent variable and the value of the dependent variable at a tabular point x_i etc. Also, stating again, the discussion in the section, is relevant only for those interpolation problems, for which tabular points x_i are equally spaced.

Let the differences of two successive tabular values x_{i+1} and x_i be denoted by h. In other words, $\boldsymbol{x_{i+1} - x_i = h}$., *or*, $\boldsymbol{x_{i+1} = x_i + h}$

Next, we introduce the operators. Each of the operators is about either about difference, sum or average of the values of the dependent variable at two tabular points, *h* distance apart. The distinction between the different operators, as will be observed below, lies in the selection of the two tabular points.

In the following, we specify (a) the name of the operator, like *Forward difference* (b) the short hand used, like *FD* (c) the notation, like Δ (d) the name of the notation in English, like *delta* and finally (e) the formula for the operator, like $\Delta f(x) = f(x + h) - f(x)$.

(i) *Forward difference (FD) operator : Δ (Delta)*

$$\Delta f(x) = f(x + h) - f(x)$$

(ii) Backward Difference (BD) operator : ∇ (Inverted Delta)

$$\nabla f(x) = f(x) - f(x - h)$$

(iii) Central Difference (CD) operator : δ (Small Delta)

$$\delta f(x) = f\left(x + \frac{1}{2}h\right) - f\left(x - \frac{1}{2}h\right)$$

(iv) Averaging operator : μ (mew)

$$\mu f(x) = \frac{1}{2}\left\{f\left(x + \frac{1}{2}\right) + f\left(x - \frac{1}{2}h\right)\right\}$$

(v) Shift operator : E

$$E\,f(x) = f(x + h)$$

(vi) Differential Operator : D

$$Df(x) = \frac{d}{dx}f(x)$$

The operators can be applied repeatedly. **For example, in case of FD operator we may have :**

$$\Delta f(x) = f(x + h) - f(x)$$

$$\Delta^2 f(x) = \Delta.\Delta f(x) = \Delta\{f(x + h) - f(x)\}$$

$$= \Delta f(x + h) - \Delta f(x) = f(x + 2h) - 2f(x + h) + f(x), \text{ etc.}$$

The first application of Δ is called First Forward Difference, similarly, two consecutive applications of Δ, denoted by Δ^2, is called Second Forward Difference and so on. Continuing like this, for a natural number k, the k consecutive applications of Δ, denoted by Δ^k, is called ***k^{th} Forward Difference***.

Similarly we can express BD etc. as follows :

$$\nabla^2 f(x) = \nabla.\nabla f(x) = f(x) - 2f(x-h) + f(x-2h) \text{ etc.}$$

And, then on the lines of our discussion above for Δ, we may also define, for any natural number k, k^{th} differences etc. for other operators. Thus, ***the notations*** ∇^k, δ^k, μ^k, E^k ***and*** D^k, denote respectively, k^{th} backward difference, k^{th} central difference, averaging operator, k^{th} shift operator, k^{th} differential operator (or k^{th} derivative).

In case of shift operator we can also have even for a fraction p such that,

$$E^p f(x) = f(x + p \times h).$$

We mentioned earlier that generally, in theoretical discussions, as in this section, we use the notations *f(x) and f(x_i) etc.*; and in applications, as in the most of the rest of the unit, we use the notations *y, y_i, or even, y(x_i) etc.* In view of the fact that most of the time, we will be concerned with applications, we repeat some of the above discussion in terms of the variable y, instead of f(x); and its values in terms of y_i, instead of f (x_i), at tabular points x_i .We will further extend the discussion in terms of y and y_i. We will discuss in detail only for forward difference and backward difference operators.

For defining the various operators, in terms of y's and y_i's, let us repeat that y = f(x) and (n + 1) tabular points x_i, for i = 0 (1) n are given/known to us. And, denoting by y_i the value f(x_i), y_i's are also known. The various forward difference operators are defined as follows :

First forward differences : $\Delta y_i = y_{i+1} - y_i$, for $i = 0(1)n - 1$

Second forward differences : $\Delta^2 y_i = \Delta(\Delta y_i) = \Delta(y_{i+1} - y_i)$

$$= \Delta y_{i+1} - \Delta y_i = (y_{i+2} - y_{i+1}) - (y_{i+1} - y_i) = y_{i+2} - 2y_{i+1} + y_i,$$

for $i = 0(1)n - 1$

In general, for a natural number k,

k^{th} forward difference $\Delta^k y_i = \Delta^{k-1} y_{i+1} - \Delta^{k-1} y_i$ $\quad i = 0\ (1)\ n - k$ . . . (1)

These differences are usually expressed as in the *example* difference table *Table 1* below, the quantities in each column representing the differences between the quantities in the preceding column. The differences are usually placed vertically midway between the quantities being subtracted.

The forward differences are indicated with shaded entries in the table, along the diagonal from first row to some middle row, in *Table 1* below :

Table 1 : Difference Table

I	x_i	y_i	1^{st} diff.	2^{nd} diff.	3^{rd} diff.	4^{th} diff.
0	–1	6.5625				
			7.5000			
1	0	14.0625		–15.0000		
			–7.5000		12.0000	
2	1	6.5625		–3.0000		24.0000
			–10.5000		36.0000	
3	2	–3.9375		33.0000		**24.0000**
			22.5000		**60.0000**	
4	3	18.5625		**93.0000**		
			115.5000			
5	4	**134.0625**				

It should be noted that if the k^{th} differences Δ^k_i are constants, then all the differences of an order higher than k are zero.

It follows, from formula (1), by taking k = 1 and, then k = 2, that

$$y_1 = y_0 + \Delta y_0$$

$$y_2 = y_1 + \Delta y_1 = (y_0 + \Delta y_0) + (\Delta^2 y_0 + \Delta y_0) = y_0 + 2\,\Delta y_0 + \Delta^2\, y_0$$

These results can be written symbolically as :

$$y_1 = (1 + \Delta)\, y_0,$$

$$y_2 = (1 + \Delta)\, y_1 = (1 + \Delta)[(1 + \Delta)\, y_0] = (1 + \Delta)^2\, y_0.$$

By induction,

$$y_k = (1 + \Delta)^k y_0,\ \ k = 1, 2, \ldots\ldots\ldots\ldots, \qquad \ldots (2)$$

or, in expanded form,

$$y_k = y_0 + k\Delta y_0 + \frac{k(k-1)}{2!}\,\Delta^2\, y_0 + \frac{k(k-1)(k-2)}{3!}\,\Delta^3 y_0 + \ldots\ldots + \Delta^k \qquad \ldots (3)$$

Formula (3) enables us to write an expression of every value y_k in terms of y_0 and the operators Δy_0, $\Delta^2 y_{0,}$ ....

The discussion in respect of Backward differences will be exactly on the lines of our discussion above in respect of Forward Difference Operators. Again, we assume that the pairs of points (x_i, y_i), i = 0 (1)n, are given. Then,

First backward differences $\nabla y_i = y_i - y_{i-1}$ $\qquad i = n\,(-1)\,1$

Second backward differences $\nabla^2 y_i = \nabla y_i - \nabla y_{i-1}$ $\qquad i = n\,(-1)\,2$

(Note that the values of i now start with n and decrease by 1 and so on...)

k-th backward differences $\nabla^k y_i = \nabla^{k-1} y_i - \nabla^{k-1} y_{i-1}$ $\qquad i = n\,(-1)\,k \qquad \ldots (4)$

The backward differences ∇y_5 are indicated **in bold**, along the diagonal from last row to some middle row, in *Table 1* above

From formula (4),

$$\nabla y_n = y_n - y_{n-1},$$

$$\nabla^2\, y_n = \nabla y_n - \nabla\, y_{n-1} = y_n - 2y_{n-1} + y_{n-2}$$

These results can be written symbolically as :

$$y_{n-1} = y_n - \nabla y_n = (1 - \nabla)\, y_n$$

$$y_{n-2} = y_n - 2\,\nabla y_n + \nabla^2 y_n = (1 - \nabla)^2\, y_n$$

$$y_{n-k} = (1 - \nabla)^k\, y_n \qquad \ldots (5)$$

in which $(1 - \nabla)^k$ is an operator on y_n with the exponent on the ∇ indicating the order of the differences. ∇, as mentioned earlier, is called the backward differences operator. Formula (5), when expanded, reads

$$y_{n-k} = y_n - k\nabla y_n + \frac{k(k-1)}{2!}\,\nabla^2 y_n - \frac{k(k-1)(k-2)}{3!}\,\nabla^3 y_n + \ldots\ldots\ldots \qquad \ldots (6)$$

Formula (6) enables us to represent every value y_{n-k} in terms of y_n and the backward differences $\nabla^k y_n$.

As can be observed easily, the various operators are related to each other. These relations between operators are discussed in the next section.

1.3 INTERRELATIONS BETWEEN OPERATORS

The operators are interrelated in the sense that one operator can be expressed in terms of the other. Except for relations of D to other operators, we will derive some of the other relations between the operators as given in Table 2 below. Let us start with relations of Δ, ∇ and δ to E.

Relation of the operators to the operator E:

$$\Delta f(x) = f(x+h) - f(x) = (E-1)\, f(x) \text{ or } \Delta \equiv E - 1$$

$$\nabla f(x) = f(x) - f(x-h) = \left(1 - E^{-1}\right) f(x) \text{ or } \nabla \equiv 1 - E^{-1}$$

$$\delta f(x) = f\left(x + \frac{1}{2}h\right) - f\left(x - \frac{1}{2}h\right) = \left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right) f(x) \text{ or } \delta \equiv E^{\frac{1}{2}} - E^{-\frac{1}{2}}$$

Then E in terms of other operators : From $\Delta \equiv E - 1$, we get $E = \Delta + 1$ From $\nabla - 1 \equiv E^{-1}$, we get $E = (\nabla - 1)^{-1}$.

Table 2 : Interrelations between various Operators

	E	**Δ**	**∇**	**δ**	***hD***
E	E	$1 + \Delta$	$(1 - \nabla)^{-1}$	$1 + \frac{1}{2}\delta^2 + \delta\sqrt{1 + \frac{\delta^2}{4}}$	e^{hD}
Δ	$E - 1$	Δ	$\nabla(1 - \nabla)^{-1}$	$\delta\sqrt{1 + \frac{\delta^2}{4}} + \frac{1}{2}\delta^2$	$E^{hD} - 1$
∇	$1 - E^{-1}$	$\Delta(1 + \Delta)^{-1}$	∇	$\delta\sqrt{1 + \frac{\delta^2}{4}} - \frac{1}{2}\delta^2$	$1 - e^{-hD}$
δ	$E^{1/2} - E^{-1/2}$	$\nabla(1 + \nabla)^{-1/2}$	$\nabla(1 + \nabla)^{-1/2}$	δ	$2\sinh\frac{hD}{2}$
μ	$\frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)$	$\left(1 + \frac{\Delta}{2}\right)(1 + \Delta)^{-1/2}$	$\left(1 - \frac{\nabla}{2}\right)(1 + \nabla)^{-1}$	$\sqrt{1 + \frac{\delta^2}{4}}$	$\cosh\frac{hD}{2}$
h D	In E	In $(1 + \Delta)$	$-$ In $(1 - \nabla)$	$2\sinh^{-1}\frac{\delta}{2}$	h D

Ex. 1 : Find E and Δ in terms of δ.

Ex. 2 : Express μ in terms of δ.

1.4 APPLICATIONS OF OPERATORS ON SOME FUNCTIONS

In this section we calculate value obtained on applying the operators to some simple functions. First, we consider, application of Δ on polynomial functions.

(i) When f(x) = c, a constant

$$\Delta f(x) = \Delta \,.\, cx^0 = c\Delta x^0 = c\,\{(x+h)^0 - x^0\} = c\,(1-1) = 0$$

(ii) When f(x) = x, identity function

$$\Delta\, f(x) = \Delta\, x = (x+h) - x = h$$

$$\Delta^2 f(x) = \Delta h = 0$$

(iii) When $f(x) = x^2$, the square function

$$\Delta x^2 = (x+h)^2 - x^2 = 2hx + h^2$$

$$\Delta^2 x^2 = \Delta\,(\Delta x^2) = \Delta\,(2hx + h^2) = 2h\,.\,\Delta x + \Delta h^2 = 2h^2$$

$$\Delta^3 x^2 = \Delta\,(\Delta^2 x^2) = 0.$$

It may be stated (without proof) that Proceeding in this manner it can be shown that if $f(x) = x^n$ then $\Delta^n x^n = n!\,h^n$ and $\Delta^{n+1} x^n = 0$. In general, for any natural number j, if $f(x) = x^j$ then $\Delta^j = j!\,h^j$ and $\Delta^k = 0$, for all $k \geq j$.

Ex. 3 : Find $\Delta^n f(x)$, where f(x) a polynomial of degree n, say

$$f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_n.$$

1.5 SUMMARY

In this unit, we defined a number of important concepts including the following ones :

Definition : Interpolation is the process of finding the value of a dependent variable, say, y, for an arbitrary given value of the independent variable x in the interval $[x_o, x_n]$, where, for some *n*, the values of the function y = f (x) are known at (n + 1) points of the independent variable. And, for other points of the interval $[x_o, x_n]$, the values may not be known.

Definitions of Standard Operators :

(i) *Forward difference (FD) operator : Δ (Delta), with* $\Delta f(x) = f(x+h) - f(x)$

(ii) Backward Difference (BD) operator : ∇ (Inverted Delta) with $\nabla f(x) = f(x) - f(x-h)$

(iii) Central Difference (CD) operator : δ (Small Delta) with

$$\delta f(x) = f\left(x + \frac{1}{2}h\right) - f\left(x - \frac{1}{2}h\right)$$

(iv) Averaging operator : μ (mew) with $\mu f(x) = \frac{1}{2}\left\{f\left(x + \frac{1}{2}\right) + f\left(x - \frac{1}{2}h\right)\right\}$

(v) Shift operator : E with $Ef(x) = f(x+h)$

(vi) Differential Operator : D with $Df(x) = \frac{d}{dx} f(x)$

Higher Order Operators :

Forward difference $\Delta^k y_i = \Delta^{k-1} y_{i+1} - \Delta^{k-1} y_i \qquad i = 0\,(1)\,n-k \qquad \ldots (1)$

From which by taking k =1 and, then k = 2, we gets

$$y_1 = y_0 + \Delta y_0$$

$$y_2 = y_1 + \Delta y_1 = (y_0 + \Delta y_0) + (\Delta^2 y_0 + \Delta y_0) = y_0 + 2\,\Delta y_0 + \Delta^2\, y_0$$

These results can be written symbolically as

$$y_1 = (1 + \Delta)\, y_0,$$

$$y_2 = (1 + \Delta)\, y_1 = (1 + \Delta)[(1 + \Delta)\, y_0] = (1 + \Delta)^2\, y_0.$$

By induction

$$y_k = (1 + \Delta)^k\, y_0, \qquad k = 1, 2, \ldots\ldots\ldots\ldots,$$

or, in expanded form

$$y_k = y_0 + k\Delta y_0 + \frac{k(k-1)}{2!}\,\Delta^2\, y_0 + \frac{k(k-1)(k-2)}{3!}\,\Delta^3 y_0 + \ldots + \Delta^k.$$

Then we obtained relations between these operators and applied these to some functions.

1.6 SOLUTIONS/ANSWERS

Ex. 1 : By squaring both sides of $\delta \equiv E^{\frac{1}{2}} - E^{-\frac{1}{2}}$, we get $\delta^2 \equiv E + E^{-1} - 2$ which implies $E + E^{-1} = \delta^2 + 2$. Multiplying both sides by E, we get $E^2 + 1 = (\delta^2 + 2)\,E$, i.e., $E^2 - (\delta^2 + 2)\,E + 1 = 0$. Solving the quadratic equation, we get

$$E = 1 + \left(\frac{\delta^2}{2}\right) \pm \left\{\left(\frac{\delta}{2}\sqrt{(\delta^2 + 4)}\right)\right\}$$

Δ in terms of δ :

As $$\Delta = E - 1 = \left(\frac{\delta^2}{2}\right) \pm \left(\frac{\delta}{2}\right)\sqrt{(\delta^2 + 4)}$$

In order to make appropriate choice between + and –, let us consider the case of minus first

In this case, we will find that $\Delta = E - 1 = \ldots = -\nabla$ which is not true.

Hence, $$E = 1 + \left(\frac{\delta^2}{2}\right) + \left\{\left(\frac{\delta}{2}\right)\sqrt{(\delta^2 + 4)}\right\}$$ and

Δ in terms of δ : $$\Delta = E - 1 = \left(\frac{\delta^2}{2}\right) + \left\{\left(\frac{\delta}{2}\right)\sqrt{(\delta^2 + 4)}\right\}$$

Ex. 2 : (By definition) $\mu f(x) = \frac{1}{2}\left\{f\left(x + \frac{1}{2}\right) + f\left(x - \frac{1}{2}h\right)\right\} = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)\, f(x)$

Therefore $\mu = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)$. Also, we know $\delta \equiv E^{\frac{1}{2}} - E^{-\frac{1}{2}}$

Hence, $4\,\mu^2 - \delta^2 = 4$, which gives $\mu = \sqrt{1 + \frac{\delta^2}{4}}$

Ex. 3 : $\Delta^n f(x) = a_0 \Delta^n x^n + a_1 . \Delta^n x^{n-1} + a_2 \Delta^n . x^{n-2} + ... + a_n \Delta^n . 1$

First of the difference on R.H.S becomes $a_0 \ (n! h^n)$, and, the rest of differences will be zero.

Hence, the answer is $= a_0 \ (n! h^n)$.

UNIT 2 INTERPOLATION WITH EQUAL INTERVALS

Structure

2.0 INTRODUCTION

From Unit 1, we know that Interpolation is the process of finding the value of a dependent variable, say, y, for an arbitrary given value of the independent variable x in the interval $[x_0, x_n]$, where, for some n, the values of the function y = f (x) are known at (n + 1) points of the independent variable.

In the previous unit, we discussed the tools, namely, the various difference operators including the Forwarding Difference operator : Δ (read as Delta), the Backward Difference operator : ∇ (Inverted Delta) and the Central Difference operator : δ (Small Delta), and also mentioned, and even derived some, relations between them. However, we did not discuss any method/formula, which can be used in solving an interpolation problem. In this unit and the next unit, we mention a number of formulas, which can be used for interpolation. The suitability of different methods for the purpose of interpolation, depends on the provided data set (x_i, y_i), and the relation of this set to the value of x, at which value of y is to be estimated.

In this unit, we derive and discuss those formulas which are used, when the data set (x_i, y_i), i = 0 (1) n is such that x_i's are equally spaced, and then use these formulas in solving interpolation problems. In the next unit, we discuss the general case.

2.1 OBJECTIVES

After going through this unit, you should be able to

- define and construct the (basic) difference table;
- define and construct the three operator difference tables from the (basic) difference table;
- decide which differencing operator, and hence, which operator difference table to use, depending upon the data set and the point of interpolation;
- use the selected operator difference table in defining interpolating polynomial and, then use the polynomial in solving interpolating problems;
- define interpolating polynomials based on each of the methods : Newton's FD, Newton's BD, Stirling's, Bessel's, when the tabular points in the data set, are equidistant; and
- use the polynomials, mentioned in the previous point, in solving interpolating problems.

2.2 BASIC DIFFERENCE TABLE AND OPERATOR DIFFERENCE TABLES

Difference Table : A difference table is central to all interpolation formulas (see Table 2 below). Suppose we are given the data (x_i, y_i), $i = 0, 1, 2 \ldots$ where x_i are equally spaced and $x_{i+1} - x_i = h$ or $x_{i+1} = x_i + h$ and y_i represents value of y corresponding to $x = x_i = x_0 + ih$. We can express Forward Difference (FD) of y_i as follows :

$$\Delta y_i = y_{i+1} - y_i;\ \ \Delta^2 y_i = y_{i+2} - 2y_{i+1} + y_i \quad \text{etc.}$$

It will be convenient if we express Δ in terms of E i.e. $\Delta \equiv E - 1$. In that case,

$$\Delta y_i = (E-1)y_i = Ey_i - y_i = y_{i+1} - y_i$$

$$\Delta^2 y_i = (E-1)^2 y_i = (E^2 - 2E + 1)y_i = y_{i+2} - 2y_i + y_i, \text{ etc}$$

We can obtain similar expressions for Backward Difference (BD) and for Central Difference (CD).

We consider an example of calculating FDs, where, values of x_i with corresponding value of y_i are given. In Table 2 below, the values of x_i's are entered in column 1, corresponding values of y_i's are entered in column 2. The first four corresponding differences in next four columns are tabulated vertically.

Example 1

Table 2 : Difference Table

i	x_i	y_i	1st diff.	2nd diff.	3rd diff.	4th diff.
0	– 1	6.5625				
			7.5000			
1	0	14.0625		– 15.0000		
			– 7.5000		12.0000	
2	1	6.5625		– 3.0000		24.0000
			– 10.5000		36.0000	
3	2	– 3.9375		33.0000		24.0000
			22.5000		60.0000	
4	3	18.5625		93.0000		
			115.5000			
5	4	134.0625				

Note : Above values are taken from $y = x^4 - 8.5x^2 + 14.0625$. Note that fourth differences are constant.

As was mentioned earlier, the goal in interpolation is to estimate value of the dependent variable y at some value of the independent variable x, at which the value of y is not given. In this respect, various interpolation polynomials/formulae will be studied and applied. For each of these formulae, a difference table, derived from the difference table, in which the differences are expressed in terms of some operator (like forward, backward, or central, etc.), is used. We call such a derived table an Operator Difference Table, e.g., Forward Difference Table, Backward Difference Table, etc.

In view of these facts, we construct a general (basic) difference table, namely Table 2, above, and then show how various operator difference tables are obtained

by interpreting the differences in difference table, in terms of respective difference operators. In a particular problem, we will be given the values for the left-most two columns (viz., x_i, y_i), and from these given values, we calculate the values for rest of the difference table as given above in Table 2.

In the next column first differences, Δy_i, $i = 0(1)4$ are computed and placed as shown in the table. Then second differences $\Delta^2 y_i = \Delta(\Delta y_i)$, $i = 0(1)3$, followed by third differences $\Delta^3 y_i = \Delta(\Delta^2 y_i)$, $i = 0\ (1)2$ and finally fourth differences $\Delta^4 y_i = \Delta(\Delta^3 y_i), i = 0,1$ are computed and placed in successive columns.

The above difference table **can be interpreted as Forward Difference Table, according to Table 3 shown below,** if we read the values parallel to the highlighted diagonal in Table 2 above. Thus, the forward differences are

$\Delta y_0 = 7.5000$, $\Delta^2 y_0 = -15.0000$, $\Delta^3 y_0 = 12.0000$, $\Delta^4 y_0 = 24.0000$

$\Delta y_1 = -7.5000$, $\Delta^2 y_1 = -3.0000$, $\Delta^3 y_1 = 36.0000$, $\Delta^4 y_1 = 24.0000$

$\Delta y_2 = -10.500$, $\Delta^2 y_2 = 33.000$, $\Delta^3 y_2 = 60.000$ and so on ...

Table 3 : Differences expressed in terms of forward difference operator

i	x_i	y_i	*1st diff.*	*2nd diff.*	*3rd diff.*	*4th diff.*
0	x_0	$\underline{y_0}$				
			$\underline{\Delta y_0}$			
1	x_1	y_1		$\Delta^2 \underline{y_0}$		
			Δy_1		$\Delta^3 \underline{y_0}$	
2	x_2	y_2		$\Delta^2 y_1$		$\Delta^4 \underline{y_0}$
			Δy_2		$\Delta^3 y_1$	
3	x_3	y_3		$\Delta^2 y_2$		
			Δy_3			
4	x_4	y_4				

Next, we give below Table 4 expressing backward differences. This is done with a view to show how the same difference Table 2 can be used to find backward differences.

Table 4 : Differences expressed in terms of backward difference operator

i	x_i	y_i	*1st diff.*	*2nd diff.*	*3rd diff.*	*4th diff.*
0	x_0	y_0				
			∇y_1			
1	x_1	y_1		$\nabla^2 y_2$		
			∇y_2		$\nabla^3 y_3$	
2	x_2	y_2		$\nabla^2 y_3$		$\nabla^4 \underline{y_4}$
			∇y_3		$\nabla^3 \underline{y_4}$	
3	x_3	y_3		$\nabla^2 \underline{y_4}$		
			$\nabla \underline{y_4}$			
4	x_4	$\underline{y_4}$				

The same difference **Table 2, again shown below, can now be interpreted as backward Difference Table, according to Table 4 above,** if we read the values parallel to the highlighted diagonal. Thus, the backward differences are

$$\nabla y_4 = 115.5000,\ \nabla^2 y_4 = 93.0000,\ \nabla^3 y_4 = 60.0000,\ \nabla^4 y_4 = 24.0000 \text{ and}$$

$$\nabla y_3 = 22.5000,\ \nabla^2 y_3 = 33.0000,\ \nabla^3 y_3 = 36.0000,\ \nabla^4 y_3 = 24.0000 \text{ and so on.}$$

Table 2 : Difference Table

i	x_i	y_i	1st diff.	2nd diff.	3rd diff.	4th diff.
0	– 1	6.5625				
			7.5000			
1	0	14.0625		– 15.0000		
			– 7.5000		12.0000	
2	1	6.5625		– 3.0000		24.0000
			– 10.5000		36.0000	
3	2	– 3.9375		33.0000		**24.0000**
			22.5000		**60.0000**	
4	3	18.5625		**93.0000**		
			115.5000			
5	4	**134.0625**				

It must be observed that in Table 3, y_0 and its differences appear sloping downwards, in Table 4, y_4 and its differences appear in an upward slope while in Table 5 below, y_i and its **even** differences appear in a straight line (see y_2, $\delta^2 y_2$ and $\delta^4 y_2$ below).

Table 5 : Differences expressed in terms of central difference operator

i	x_i	y_i	*1st diff.*	*2nd diff.*	*3rd diff.*	*4th diff.*
0	x_0	y_0				
			$\delta y_{\frac{1}{2}}$			
1	x_1	y_1		$\delta^2 y_1$		
			$\delta y_{\frac{3}{2}}$		$\delta^3 y_{\frac{3}{2}}$	
2	x_2	y_2	————	$\delta^2 y_2$	————	$\delta^4 y_2$
			$\delta y_{\frac{5}{2}}$		$\delta^3 y_{\frac{5}{2}}$	
3	x_3	y_3		$\delta^2 y_3$		
			$\delta y_{\frac{7}{2}}$			
4	x_4	y_4				

In the light of the discussion above, each of the entries in the Difference Table 2 can be given three names in terms of appropriate Δ, ∇, δ and some y's. It is important to note that **the same difference table is always made** in the manner of Table 2, for a given set of pairs of values (x_i, y_i), while the differences may be expressed in terms of FD, BD or CD as required. That is, in the column of first differences we have,

$$\mathbf{7.5000} = y_1 - y_0 = \Delta y_0 = \nabla y_1 = \delta y_{\frac{1}{2}}$$

$$\mathbf{-7.5000} = y_2 - y_1 = \Delta y_1 = \nabla y_2 = \delta y_{\frac{3}{2}} \text{ etc.}$$

In the column of second differences we have,

$$-\mathbf{15.0000} = \Delta y_1 - \Delta y_0 = \Delta(\Delta y_0) = \Delta^2 y_0$$

$$= \nabla y_2 - \nabla y_1 = \nabla(\nabla y_2) = \nabla^2 y_2$$

$$= \delta y_{\frac{3}{2}} - \delta y_{\frac{1}{2}} = \delta(\delta y_1) = \delta^2 y_1 \text{, etc.}$$

Ex. 1 : Construct a difference table for the data :

x	:	1	2	3	4
f(a)	:	7	13	18	25

and highlight the forward differences for f(1) by **BOLD** and backward differences of f(4) by shading.

Ex. 2 : Compute the difference table for the following set of data :

X	1	2	3	4	5	6	7	8
Y	2.105	2.808	3.614	4.604	5.857	7.451	9.467	11.985

and highlight the forward differences for f(1) by **BOLD** and backward differences of f(8) by shading.

2.3 INTERPOLATION FORMULAS

Interpolation Formulas

We shall now discuss some interpolation formulas using Forward Difference **(FD)**, Backward Difference **(BD)** and Central Differences **(CD)**. Let us suppose **we have to compute the value denoted by y_p of y corresponding to $x = x_p$.** We have $x_p = x_0 + ph$ or $p = \frac{x_p - x_0}{h}$ and $y_p = E^p y$. Expressing E in terms of Δ or ∇ we get FD and BD interpolation formulas, respectively. We choose x_0 so as to include more terms in the formula.

Remark : It may be noted that the value y_p of y, corresponding to the value x_p of x, is already expressed in terms of the operator E, as E^p. The four formulas, namely, Newton's FD Formula, Newton's BD Formula, Stirling's Formula and Bessel's Formula, discussed below, just re-express the operator E in terms of respectively Δ, ∇ or δ.

(i) Newton's FD Formula

$$y_p = E^p y_0 = (1+\Delta)^p y_0 = \left(1 + p\Delta + \frac{p(p-1)}{2!}\Delta^2 + \frac{p(p-1)(p-2)}{3!}\Delta^3 + \ldots\right) y_0$$

$$= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots$$

This formula is used to compute the value of y near the upper end of the table such that $0 < p < 1$.

(ii) Newton's BD Formula

$$y_p = E^p y_0 = (1-\nabla)^{-p} y_0 = \left(1 + p\nabla + \frac{p(p+1)}{2!}\nabla^2 + \frac{p(p+1)(p+2)}{3!}\nabla^3 + \ldots\right) y_0$$

$$= y_0 + p\nabla y_0 + \frac{p(p+1)}{2!}\nabla^2 y_0 + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_0 + \ldots$$

This formula is used near the lower end of the table choosing x_0 such that $-1 < p < 0$. The CD formulas are used to find the value in the middle of the table. We can derive several CD formulas with tedius mathematical manipulations.

Next, two popular CD formulas, i.e., in terms of δ, are given below :

(iii) Stirling's Formula

$$y_p = y_0 + p\mu\delta y_0 + \frac{p^2}{2!}\delta^2 y_0 + \binom{p+1}{3}\mu\delta^3 y_0 + \frac{p^2(p^2-1^2)}{4!}\delta^4 y_0 + \ldots$$

$$= y_0 + p\frac{1}{2}\left(\delta y_{\frac{1}{2}} + \delta y_{-\frac{1}{2}}\right) + \frac{p^2}{2!}\delta^2 y_0 + \frac{(p+1)p(p-1)}{3!}\frac{1}{2}\left(\delta^3 y_{\frac{1}{2}} + \delta^3 y_{-\frac{1}{2}}\right)$$

$$+ \frac{p^2(p^2-1^2)}{4!}\delta^4 y_0 + \ldots$$

This formula should be used for $-0.25 \le p \le 0.25$.

(iv) Bessel's Formula

$$y_p = \mu y_{\frac{1}{2}} + \left(p - \frac{1}{2}\right)\delta y_{\frac{1}{2}} + \binom{p}{2}\mu\delta^2 y_{\frac{1}{2}} + \binom{p}{2}\frac{1}{3}\left(p-\frac{1}{2}\right)\delta^3 y_{\frac{1}{2}} + \binom{p+1}{4}\mu\delta^4 y_{\frac{1}{2}} + \ldots$$

$$= \frac{1}{2}(y_0 + y_1) + \left(p-\frac{1}{2}\right)\delta y_{\frac{1}{2}} + \binom{p}{2}\cdot\frac{1}{2}(\delta^2 y_1 + \delta^2 y_0) + \binom{p}{2}\frac{1}{3}\left(p-\frac{1}{2}\right)\delta^3 y_{\frac{1}{2}}$$

$$+ \binom{p+1}{4}\frac{1}{2}(\delta^4 y_1 + \delta^4 y_0) + \ldots$$

This formula is used for $-0.25 \le p \le 0.75$.

2.4 EXAMPLES OF APPLICATION OF NEWTON'S FD FORMULA

We recall Newton's FD Formula :

$$y_p = E^p y_0 = (1+\Delta)^p y_0 = \left(1 + p\Delta + \frac{p(p-1)}{2!}\Delta^2 + \frac{p(p-1)(p-2)}{3!}\Delta^3 + \ldots\right)y_0,$$

where, **the value to computed is denoted by y_p of y corresponding to the given value of the independent variable $x = x_p$**, and p is given by $x_p = x_0 + ph$,

i.e., $$p = \frac{x_p - x_0}{h} \text{ and } y_p = E^p y.$$

Example 2

From Table 2, find the value of y for $x = 0.5$ using appropriate formula.

Solution

Since $x_p = 0.5$ is near upper end of the table, we use FD formula. Choose $x_0 = 0$ so that $p = \frac{0.5-0}{1} = 0.5$.

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0$$

$$= 14.0625 + 0.5\times(-7.5) + \frac{(0.5)(-0.5)}{2}\times(-3) + \frac{(0.5)(-0.5)(-1.5)}{6}\times 36$$

$$+\frac{(0.5)(-0.5)(-1.5)(-2.5)}{24}\times 24$$

$$=14.0625 - 3.750 + 0.3750 + 2.2500 - 0.9375 = 12.0000$$

Example 3

Find the Newton's forward-difference interpolating polynomial which agrees with the table of values given below. Hence obtain the value of f(x) at x = 1.5.

x	1	2	3	4	5	6
f(x)	10	19	40	79	142	235

Solution

Table : Forward Differences

x	f(x)	Δf	$\Delta^2 f$	$\Delta^3 f$		
1	10					
		9				
2	19		12			
		21		6		
3	40		18		0	
		39		6		0
4	79		24		0	
		63		6		
5	142		30			
		93				
6	285					

The Newton's forward-differences interpolation polynomial is

$$f(x) \approx f_0 + (x-1)\,\Delta f_0 + \frac{(x-1)(x-2)}{2!}\,\Delta^2 f_0 + \frac{(x-1)(x-2)(x-3)}{3!}\,\Delta^3 f_0$$

$$= 10 + (x-1)(9) + \frac{(x-1)(x-2)}{2}(12) + \frac{(x-1)(x-2)(x-3)}{6}(6)$$

On simplification we get

$$f(x) \approx x^3 + 2x + 7$$

$$f(1.5) = (1.5)^3 + 2(1.5) + 7 = 13.375$$

Example 4

Given below are the values of $y = e^x$ tabulated for x = 1.00 (0.20) 2.00 :

x :	1.00	1.20	1.40	1.60	1.80	2.00
y :	2.7183	3.3201	4.0552	4.9530	6.049	7.3891

Approximate the value of y at x = 1.12

Solution

x	1.00		1.20		1.40		1.60		1.80		2.00
$y = e^x$	2.7183		3.3201		4.0552		4.9530		6.049		7.3891
Δy		0.6018		0.7351		0.8978		1.0966		1.3395	
$\Delta^2 y$			0.1333		0.1627		0.1988		0.2429		
$\Delta^3 y$				0.0294		0.0361		0.0441			
$\Delta^4 y$					0.0067		0.0080				
$\Delta^5 y$						0.0013					

As 1.12 is near the beginning end of the given table, we will use forward differences for x =1.00.

$$p = (1.12 - 1.00)/0.20 = 0.6;$$

y(1.12) = 2.7183 + (0.6) × 0.6018 + ((0.6) (0.6 – 1)/2) × 0.1333 + [(0.6) (0.6 – 1) (–1.4)/6] × 0.0294 + [(0.6) (0.6 – 1)(–1.4)/24] × 0.0067 + [(0.6) (0.6 – 1) (–1.4) (– 2.4)/120] × 0.0013 = 2.713 + 0.36108 – 0.015996 + 0.016464 – 0.000225 + 0.000030 **= 3.0648.**

Ex. 3 : Construct Newton's forward difference table for the data :

x	1	2	3	4	5
f(x)	1	4	9	16	25

Hence approximate f(1.7) from Newton's forward difference interpolating polynomial.

Ex. 4 : Using forward Differences, show that the following data :

x	– 1	0	1	2	3	4
f(x)	6	1	0	3	10	21

represents a second degree polynomial. Find this polynomial and an approximate value of f(2.5).

Ex. 5 : Using forward differences, show that the following data represents a third degree polynomial.

x	– 3	– 2	– 1	0	1	2	3
f(x)	– 29	– 9	– 1	1	3	11	31

Find the polynomial and obtain the value of f (0.5).

Ex. 6 : Estimate the missing term in the following data valid it is represents a polynomial of degree.

x	:	1	2	3	4	5
f(x)	:	3	7	?	21	31

Ex. 7 : Evaluate the missing term in the following :

x	:	100	101	102	103	104
f(x)=log x	:	2.000	2.0043	?	2.0128	2.0170

Ex. 8 : Compute the backward differences for the following set of data :

x	1	2	3	4	5	6	7	8
y	2.105	2.808	3.614	4.604	5.857	7.451	9.467	11.985

2.5 EXAMPLES OF APPLICATION OF NEWTON'S BD FORMULA

We recall Newton's BD Formula :

$$y_p = E^p y_0 = (1-\nabla)^{-p} y_0 = \left(1 + p\nabla + \frac{p(p+1)}{2!}\nabla^2 + \frac{p(p+1)(p+2)}{3!}\nabla^3 + \ldots\right) y_0$$

$$= y_0 + p\nabla y_0 + \frac{p(p+1)}{2!}\nabla^2 y_0 + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_0 + \ldots$$

where, **the value to computed is denoted by y_p of y corresponding to the given value of the independent variable $x = x_p$**, and p is given by $x_p = x_0 + ph$,

i.e., $$p = \frac{x_p - x_0}{h} \text{ and } y_p = E^p y.$$

Example 5

From Table 2, find the value of y when x = 3.5 using an appropriate formula.

Solution

Since $x_p = 3.5$ is near the lower end of the table we use BD formula choosing

$x_0 = 4$, hence $p = \frac{3.5-4}{1} = -0.5$

$$y_p = y_0 + p\nabla y_0 + \frac{p(p+1)}{2!}\nabla^2 y_0 + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_0 + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_0$$

$$= 134.0625 + (-0.5)(115.5) + \frac{(-0.5)(0.5)}{2}\times 93 + \frac{(-0.5)(0.5)(1.5)}{6}\times 60$$

$$+ \frac{(-0.5)(0.5)(1.5)(2.5)}{24}\times 24$$

$$= 134.0625 - 57.75 - 11.625 - 3.75 - 0.9375 = 60.000$$

Example 6

Find Newton's backward difference form of interpolating polynomial for the data

X	4	6	8	10
f(x)	19	40	79	142

Hence interpolate f (9).

Solution

We have

Table 6 : Backward Difference

x	f(x)	∇f	$\nabla^2 f$	$\nabla^3 f$
4	19			
		21		
6	40		18	
		39		6
8	79		24	
		63		
10	142			

$$h = 2,\ p = \frac{(x-10)}{h},\quad p = \frac{(x-10)}{2}$$

$$P_n(x) = 142 + (x-10)\,\frac{63}{2} + \frac{(x-10)\,(x-8)}{2!}\times\frac{24}{4} + \frac{(x-10)\,(x-8)\,(x-6)}{3!}\times\frac{6}{8}$$

$$f(9) \approx P_n(9) = 142 - \frac{63}{2} - 3 - \frac{3}{8} = 107.125$$

Ex. 9 : Find the Newton's backward differences interpolating polynomial for the data of Example 3.

Ex. 10 : Given below are the values of $y = e^x$ tabulated for $x = 1.00\ (0.20)\ 2.00$:

x :	1.00	1.20	1.40	1.60	1.80	2.00
y :	2.7183	3.3201	4.0552	4.9530	6.049	7.3891

Approximate the value of y at $x = 1.68$.

Ex. 11 : Estimate the value of f (1.45) from the data given below :

x	1.1	1.2	1.3	1.4	1.5
f(x)	1.3357	1.5095	1.6984	1.9043	2.1293

2.6 EXAMPLES OF APPLICATION OF STIRLING'S FORMULA

We recall Stirling's Formula :

$$y_p = y_0 + p\mu\delta y_0 + \frac{p^2}{2!}\delta^2 y_0 + \binom{p+1}{3}\mu\delta^3 y_0 + \frac{p^2\left(p^2-1^2\right)}{4!}\delta^4 y_0 + \ldots$$

$$= y_0 + p\frac{1}{2}\left(\delta y_{\frac{1}{2}} + \delta y_{-\frac{1}{2}}\right) + \frac{p^2}{2!}\delta^2 y_0 + \frac{(p+1)p(p-1)}{3!}\frac{1}{2}\left(\delta^3 y_{\frac{1}{2}} + \delta^3 y_{-\frac{1}{2}}\right)$$

$$+ \frac{p^2(p^2-1^2)}{4!}\delta^4 y_0 + \ldots$$

where, **the value to computed is denoted by y_p of y corresponding to the given value of the independent variable $x = x_p$**, and p is given by $x_p = x_0 + ph$,

i.e., $$p = \frac{x_p - x_0}{h} \text{ and } y_p = E^p y$$

This formula should be used for $-0.25 \le p \le 0.25$.

Example 7

From Table 2, find the value of y, when x = 1.75, using a central difference Stirling's formula.

Solution

Using Stirling's formula, taking $x_0 = 2.0$ so that $p = \dfrac{1.75 - 2.0}{1} = -0.25$

$$y_p = y_0 + p.\frac{1}{2}\left(\delta y_{\frac{1}{2}} + \delta y_{-\frac{1}{2}}\right)\frac{p^2}{2}\delta^2 y_0 + \frac{(p+1)p(p-1)}{3!}.\frac{1}{2}\left(\delta^2 y_{\frac{1}{2}} + \delta^3 y_{-\frac{1}{2}}\right)$$

$$+ \frac{p^2\left(p^2-1^2\right)}{4!}\delta^4 y_0 + \ldots$$

The differences used are as follows :

$$\begin{array}{cccccc}
x_{-1} & y_{-1} & & & & \\
 & & \delta y_{-\frac{1}{2}} & & \delta^3 y_{-\frac{1}{2}} & \\
x_0 & y_0 & & \delta^2 y_0 & & \delta^4 y_0 \\
 & & \delta y_{\frac{1}{2}} & & \delta^3 y_{\frac{1}{2}} & \\
x_1 & y_1 & & & &
\end{array}$$

$$y_p = -3.9375 + (-0.25)\cdot\frac{1}{2}(22.50 - 10.50) + \frac{(-0.25)^2}{2} \times 33.00$$

$$+\frac{(-0.25+1)(-0.25)(-0.25-1)}{6}\times\frac{1}{2}(60.00+36.00)+\frac{(-0.25)^2\left(0.25^2-1\right)}{24}\times 24$$

$$=-3.9375-1.5+1.03125+1.875-0.05859375$$

$$=-2.58984375$$

2.7 EXAMPLES OF APPLICATION OF BESSEL'S FORMULA

We recall Bessel's Formula :

$$y_p=\mu y_{\frac{1}{2}}+\left(p-\frac{1}{2}\right)\delta y_{\frac{1}{2}}+\binom{p}{2}\mu\delta^2 y_{\frac{1}{2}}+\binom{p}{2}\frac{1}{3}\left(p-\frac{1}{2}\right)\delta^3 y_{\frac{1}{2}}+\binom{p+1}{4}\mu\delta^4 y_{\frac{1}{2}}+\ldots$$

$$=\frac{1}{2}(y_0+y_1)+\left(p-\frac{1}{2}\right)\delta y_{\frac{1}{2}}+\binom{p}{2}.\frac{1}{2}\left(\delta^2 y_1+\delta^2 y_0\right)+\binom{p}{2}\frac{1}{3}\left(p-\frac{1}{2}\right)\delta^3 y_{\frac{1}{2}}$$

$$+\binom{p+1}{4}\frac{1}{2}\left(\delta^4 y_1+\delta^4 y_0\right)+\ldots$$

where, **the value to be computed is denoted by y_p of y corresponding to the given value of the independent variable $x = x_p$**, and p is given by $x_p = x_0 + ph$,

i.e., $\quad p=\dfrac{x_p-x_0}{h}$ and $y_p=E^p y$.

This formula is used for $-0.25 \le p \le 0.75$.

Example 8

From Table 2, find the value of y for x = 1.75 using Bessel's formula.

Solution

We choose $x_0=1$ so that $p=\dfrac{1.75-1.0}{1}=0.75$

Bessel's formula is given by,

$$y_p=\frac{1}{2}(y_0+y_1)+\left(p-\frac{1}{2}\right)\delta y_{\frac{1}{2}}+\binom{p}{2}\frac{1}{2}\left(\delta^2 y_0+\delta^2 y_1\right)+\binom{p}{2}\frac{1}{3}\left(p-\frac{1}{2}\right)\delta^3 y_{\frac{1}{2}}$$

$$+\binom{p+1}{4}\frac{1}{2}\left(\delta^4 y_0+\delta^4 y_1\right)+\ldots$$

The differences used in the formula are as follows :

$$\begin{array}{cccccc}
x_0 & y_0 & & \delta^2 y_0 & & \delta^4 y_0 \\
 & & \delta y_{\frac{1}{2}} & & \delta^3 y_{\frac{1}{2}} & \\
x_1 & y_1 & & \delta^2 y_1 & & \delta^4 y_1
\end{array}$$

$$y_p=\frac{1}{2}(6.5625-3.9375)+(0.75-0.5)(-10.5)+\frac{(0.75)(0.75-1)}{2}.\frac{1}{2}(-3.0+33.0)$$

$$+\frac{(0.75)(0.75-1)}{2}.\frac{1}{3}(0.75-0.5)\times 36+\frac{(1.75)(1.75-1)(1.75-2)(1.75-3)}{24}\frac{1}{2}(24+24)$$

$$=\frac{1}{2}(2.625)-(0.25)(10.5)-(0.1875)(7.5)-0.28125+0.41015625$$

$$=-1.3125-2.625-1.40625-0.28125+0.41015625$$

$$=-2.58984375.$$

Note : Usually the computations are required to be done up to second or third differences and at the most fourth differences and an approximate value of y is obtained.

2.8 SUMMARY

In this unit, we

(i) defined the concept of the (basic) difference table, and then illustrated through an example, how to construct a difference table,

(ii) explained how the differences in the difference table, can be interpreted in terms of Forward Difference Operator; in this case, we call the difference table as Forward Difference Table. The same exercise was repeated, in place of Forward Difference Operator, for each of the Backward Difference Operator and Central Difference Operator.

(iii) we stated four well-known formulae for interpolation, for the special case when the tabular points are equally spaced. In this respect, it may be noted that the value y_p of y, corresponding to the tabular value x_p of x, is already expressed in terms of the operator E, as E^p, i.e., $y_p = E^p\, y_0$. The four formulas, stated below, namely, Newton's FD Formula, Newton's BD Formula, Stirling's Formula and Bessel's Formula, discussed below, just re-express the operator E in terms of respectively Δ, ∇ or δ :

(i) Newton's FD Formula

$$y_p = E^p y_0 = (1+\Delta)^p y_0 = \left(1 + p\Delta + \frac{p(p-1)}{2!}\Delta^2 + \frac{p(p-1)(p-2)}{3!}\Delta^3 + \ldots\right) y_0$$

$$= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots$$

This formula is used to compute the value of y near the upper end of the table such that $0 < p < 1$.

(ii) Newton's BD Formula

$$y_p = E^p y_0 = (1-\nabla)^{-p} y_0 = \left(1 + p\nabla + \frac{p(p+1)}{2!}\nabla^2 + \frac{p(p+1)(p+2)}{3!}\nabla^3 + \ldots\right) y_0$$

$$= y_0 + p\nabla y_0 + \frac{p(p+1)}{2!}\nabla^2 y_0 + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_0 + \ldots$$

This formula is used near the lower end of the table choosing x_0 such that $-1 < p < 0$.

Next, two popular CD formulas, i.e., in terms of δ, are given below.

The CD formulas are used to find the value in the middle of the table. We can derive several CD formulas with tedius mathematical manipulations.

(iii) Stirling's Formula

$$y_p = y_0 + p\mu\delta y_0 + \frac{p^2}{2!}\delta^2 y_0 + \binom{p+1}{3}\mu\delta^3 y_0 + \frac{p^2(p^2-1^2)}{4!}\delta^4 y_0 + \ldots$$

$$= y_0 + p\frac{1}{2}\left(\delta y_{\frac{1}{2}} + \delta y_{-\frac{1}{2}}\right) + \frac{p^2}{2!}\delta^2 y_0 + \frac{(p+1)p(p-1)}{3!}\frac{1}{2}\left(\delta^3 y_{\frac{1}{2}} + \delta^3 y_{-\frac{1}{2}}\right)$$

$$+ \frac{p^2(p^2-1^2)}{4!}\delta^4 y_0 + \ldots$$

This formula should be used for $-0.25 \le p \le 0.25$.

(iv) Bessel's Formula

$$y_p = \mu y_{\frac{1}{2}} + \left(p-\frac{1}{2}\right)\delta y_{\frac{1}{2}} + \binom{p}{2}\mu\delta^2 y_{\frac{1}{2}} + \binom{p}{2}\frac{1}{3}\left(p-\frac{1}{2}\right)\delta^3 y_{\frac{1}{2}} + \binom{p+1}{4}\mu\delta^4 y_{\frac{1}{2}} + \ldots$$

$$= \frac{1}{2}(y_0 + y_1) + \left(p-\frac{1}{2}\right)\delta y_{\frac{1}{2}} + \binom{p}{2}\cdot\frac{1}{2}\left(\delta^2 y_1 + \delta^2 y_0\right) + \binom{p}{2}\frac{1}{3}\left(p-\frac{1}{2}\right)\delta^3 y_{\frac{1}{2}}$$

$$+ \binom{p+1}{4}\frac{1}{2}\left(\delta^4 y_1 + \delta^4 y_0\right) + \ldots$$

This formula is used for $-0.25 \le p \le 0.75$.

and, finally,

(iv) explained how the various operator difference tables can be used to find the respective interpolating polynomials, and further, explained how these polynomials, in turn, can be used to estimate value of dependent variable for some value of the independent variable when the tabular points are necessarily equidistant.

2.9 SOLUTIONS/ANSWERS

Ex. 1 :

x	f(x)	Δf	Δ^2	Δ^3
1	**7**			
		6		
2	13		**– 1**	
		5		**3**
3	18		2	
		7		
4	25			

Since four data values are given, a polynomial of degree 3 can be passed through them. Hence, fourth differences are 0 and third differences have the same value.

Forward differences of y_1 are bold and backward differences of y_8 are shaded.

Ex. 2 :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	**2.105**				
		0.703			
2	2.808		**0.103**		
		0.806		**0.081**	
3	3.614		0.184		**– 0.002**
		0.990		0.079	
4	4.604		0.263		– 0.001
		1.253		0.078	
5	5.857		0.341		+ 0.003
		1.594		0.081	
6	7.451		0.422		– 0.001
		2.016		0.080	
7	9.467		0.502		
		2.518			
8	11.985				

Thus, first, second, third and fourth forward differences for y_1 are respectively

$$\Delta y_1 = 0.703,\quad \Delta^2 y_1 = \mathbf{0.103},\quad \Delta^3 y_1 = \mathbf{0.081},\quad \Delta^4 y_1 = -\mathbf{0.002}$$

Similarly, first, second, third and fourth forward differences for y_2 are respectively

$$\Delta y_2 = 0.806,\quad \Delta^2 y_2 = \mathbf{0.184},\quad \Delta^3 y_3 = \mathbf{0.079},\quad \Delta^4 y_2 = -\mathbf{0.001}$$

and so on.

Forward differences of y_1 are bold and backward differences of y_8 are shaded.

Ex. 3 : Forward Difference Table :

x	f(x)	Δf	$\Delta^2 f$	$\Delta^3 f$	
1	**1**				**We may notice h = 1**
		3			
2	4		**2**		
		5		**0**	
3	9		2		
		7		0	
4	16		2		
		9			
5	25				

$f(x) = f(x_0) + ((x - x_0)/h)\ \Delta f(x_0) + ((x - x_0)(x - x_1)/h^2\ 2!)\ \Delta^2 f(x_0) + ((x - x_0)(x - x_1)(x - x_2)/h^3\ 3!)\ \Delta^3 f(x_0) + \ldots = \mathbf{1} + (\mathbf{x} - 1)\ 3 + (\mathbf{x} - 1)(\mathbf{x} - 2)\ 2/2! + \mathbf{0}$
$= \mathbf{1} + \mathbf{3x} - 3 + (\mathbf{x} - 1)(\mathbf{x} - 2) = x^2$

$f(1.7) = (1.7)^2 = 2.89$.

Ex. 4 : Forward Difference Table

x	f(x)	Δf	$\Delta^2 f$ ...	
–1	**6**			
		–5		
0	1		**4**	
		–1		**0**
1	0		4	
		3		0
2	3		4	
		7		0
3	10		4	
		11		
4	21			

Since third and higher order differences are zeros, f(x) represents a second order polynomial

$$f(x) \approx P_2(x) = f_0 + \frac{(x - x_0)}{h} \Delta f_0 + \frac{(x - x_0)(x - x_1)}{2!h^2} \Delta^2 f_0$$

$$= 6 + (x + 1)(-5) + \frac{(x + x_0)(x - 0)}{2}(4) = 2x^2 - 3x + 1$$

$f(2.5) \approx 6$

Ex. 5 : Forward Difference Table

x	f(x)	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
– 3	**– 29**				
		20			
– 2	– 9		**– 12**		
		8		**6**	
– 1	– 1		– 6		**0**
		2		6	
0	1		0		0
		2		6	
1	3		6		0
		8		6	
2	11		12		
		20			
3	31				

$$f(x) \approx P_3(x) = f(x_0) + \frac{(x - x_0)}{h} \Delta f_0 + \frac{(x - x_0)(x - x_1)}{2!h^2} \Delta^2 f_0$$

$$+ \frac{(x - x_0)(x - x_1)(x - x_2)}{3!h^3} \Delta^3 f_0$$

$$= (-29) + (x + 3)\,20 - 6(x + 3)(x + 2) + (x + 3)(x + 2)(x + 1)$$

$$= (-29) + (x + 3)[\,20 + (x + 2)\{-6 + (x + 1)\}\,]$$

$$= x^3 + x + 1, \text{ hence, } \; f(0.5) \approx P_3(0.5) = 1.625$$

Ex. 6 : Let the missing value be a.

x	f(x)	Δf	Δ^2	Δ^3
1	3			
2	7	4	a – 11	39 – 3a
3	A	a – 7	28 – 2a	3a – 39
4	21	21 – a	a – 11	
5	31	10		

Since, third degree differences are to be same 39 – 3a = 3a – 39, or 6a = 78 and a f(3) = 13.

Ex. 7 : The value for $x = 102$. As four values of $f(x)$ are known, for $x = 1$, we get, $\Delta^4 f(x) = 0$.

x :	100	101	102	103	104
f(x)=log x :	2.000	2.0043	a	2.0128	2.0170

$\Delta f(x)$: 0.0043 $a - 2.0043$ $2.0128 - a$ 0.0042

$\Delta^2 f(x)$: $a - 2.0086$ $4.0171 - 2a$ $-2.0086 + a$

$\Delta^3 f(x)$: $(4.0171 - 2a) - (a - 2.0086)$;; $(-2.0086 + a) - (4.0171 - 2a)$

$= 6.0257 - 3a$;;; $= -6.0257 + 3a$

As 4 values are given, Third differences must be equal

So $a = 2.0086$ i.e. $\log 102 = 2.0086$.

Ex. 8 : The same table as in **Ex. 2** will be formed. But order and names of differences will be different.

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1	2.105				
		0.703			
2	2.808		0.103		
		0.806		0.081	
3	3.614		0.184		– 0.002
		0.990		0.079	
4	4.604		0.263		– 0.001
		1.253		0.078	
5	5.857		0.341		+ 0.003
		1.594		0.081	
6	7.451		0.422		**– 0.001**
		2.016		**0.080**	
7	9.467		**0.502**		
		2.518			
8	11.985				

Thus, first, second, third and fourth backward differences of y_8 **are respectively**

$$\nabla y_8 = 2.518, \quad \nabla^2 y_8 = 0.502, \quad \nabla^3 y_8 = 0.080, \quad \nabla^4 y_8 = -0.001$$

Also, first, second, third and fourth backward differences of y_7 **are respectively**

$$\nabla y_7 = 2.016, \quad \nabla^2 y_7 = 0.422, \quad \nabla^3 y_7 = 0.081, \quad \nabla^4 y_7 = 0.003$$

and so on.

Ex. 9 : Backward Difference Table

	x	f(x)	∇f	$\nabla^2 f$	$\nabla^3 f$	
x_0	1	10				
			9			
x_1	2	19		12		
			21		6	
x_2	3	40		18		0
			39		6	… 0
x_3	4	79		24		0
			63		6	
x_4	5	142		30		
			93			
x_5	6	235				

$$P(x) = f_5 + (x - x_5)\nabla f_5 + \frac{(x - x_5)(x - x_4)}{2!}\nabla^2 f_5$$

$$+ \frac{(x - x_5)(x - x_4)(x - x_3)}{3!}\nabla^3 f_5$$

$$= 235 + 93(x - 6) + 15(x - 6)(x - 5) + (x - 4)(x - 5)(x - 6)$$

$$= x^3 + 2x + 7$$

Ex. 10 : Given below are the values of $y = e^x$ tabulated for $x = 1.00\ (0.20)\ 2.00$:

x:	1.00	1.20	1.40	1.60	1.80	2.00
$y = e^x$:	2.7183	3.3201	4.0552	4.9530	**6.049**	7.3891
Δy :	0.6018	0.7351	0.8978	**1.0966**	1.3395	
$\Delta^2 y$:	0.1333	0.1627	**0.1988**	0.2429		
$\Delta^3 y$:	0.0294	**0.0361**	0.0441			
$\Delta^4 y$:	**0.0067**	0.0080				
$\Delta^5 y$:	**0.0013**					

As 1.68 is closest to 1.80 among the values of x at which y is known, we will use backward *(because, 1.80 is near the back end 2.00)* differences for x =1.80.

p = (1.68 – 1.80)/0.20 = – 0.6 and

$y(1.68) = 6.0496 + (-0.6)\ 1.0966 + [(-0.6)\ ((-0.6)+1)/2\ ] \times 0.1988 + [(-0.6)\ (-0.6+1)$
$(-0.6+2)/6] \times 0.0361 + [(-0.6)\ (0.4)\ (1.4)\ (2.4)/24] \times 0.0067$

$= 6.049 - 0.65796 - 0.023856 - 0.002022 - 0.000225$

$= 5.36554 \approx 5.3655.$

Ex. 11 : As 1.45 is nearest to 1.50, which is last value of x for which of y = f(x) is given. Hence, Backward Difference Table

x	f(x)	∇f	$\nabla^2 f$	$\nabla^3 f$	$\nabla^3 f$
1.1	1.3357				
		0.1738			
1.2	1.5095		0.0151		
		0.1889		0.0019	
1.3	1.6984		0.0170		0.0002
		0.2059		0.0021	
1.4	1.9043		0.0910		
		0.2050			
1.5	2.1293				

$x_n = 1.5,\ x = 1.45,\ h = 0.1$

$$\therefore \quad p = s = \frac{(x - x_n)}{h} = \frac{1.45 - 1.5}{0.1} = -0.5$$

$$f(x) = f_n + s\nabla f_n + \frac{s(s+1)}{2!}\nabla^2 f_n + \frac{s(s+1)(s+2)}{3!}\nabla^3 f_n$$

$$+ \frac{s(s+1)(s+2)(s+3)\nabla^4 f_n}{4!}$$

$$= 2.1293 - 0.1125 - 0.00239 - 0.00013 - 0.0000075$$

$$= 2.01427 \approx 2.0143.$$

UNIT 3 INTERPOLATION WITH UNEQUAL INTERVALS

Structure

3.0 INTRODUCTION

From Unit 1, we know that Interpolation is the process of finding the value of a dependent variable, say, y, for an arbitrary given value of the independent variable x in the interval $[x_0, x_n]$, where, for some *n*, the values of the function y = f (x) are known at (n + 1) points $x_0, x_1, \ldots, x_n$ of the independent variable.

In Unit 1, we discussed the tools, useful, when tabular points are equidistant, namely, the various difference operators including the Forward Difference operator : Δ (read as Delta), the Backward Difference operator : ∇ (Inverted Delta) and the Central Difference operator : δ (Small Delta), and also mentioned, and even derived some, relations between them.

In Unit 2, we derived and discussed those interpolating formulas which are used, when the data set (x_i, y_i), i = 0 (1) n is such that x_i's are equally spaced, and then used these formulas in solving interpolation problems.

In this unit, we discuss the general case, i.e., derive and discus those interpolating formulas which are used, when the data set (x_i, y_i), i = 0 (1) n is such that the x_i's are not necessarily equally spaced, and then use these formulas in solving interpolation problems.

3.1 OBJECTIVES

After going through this unit, you should be able to

- define and construct the Divided difference table;
- define interpolating polynomials based on each of the methods : Lagrange's, Lagrange's Inverse Interpolation and Newton's Divided Difference Method; and
- use the polynomials, mentioned in the previous point, in solving interpolating problems.

3.2 LAGRANGE'S METHOD

Let us suppose we have (n + 1) pair of values (x_i, y_i), i = 0 (1) n where y_i denotes a value, of dependent variable y = f (x) corresponding to the value x_i of independent variable x. We assume that $x_0 < x_1 < x_2 < \ldots < x_n$ and that the interval between two successive values of x is not necessarily same i.e, $x_i - x_{i-1}$, i = 1 (1) n may or may not be same all over. We shall now discuss interpolation methods with unequal intervals.

Lagrange's Method

In Lagrange's method a polynomial is fitted passing through all the given points. Obviously, if (n + 1) points (x_i, y_i), i = 0 (1) n are given, a polynomial of highest degree n can be fitted which is represented as,

$$y(x) = L_0(x)y_0 + L_1(x)y_1 + \ldots + L_i(x)y_i + \ldots + L_n(x)y_n$$

where $$L_i(x) = \frac{(x-x_0)(x-x_1)\ldots(x-x_{i-1})(x-x_{i+1})\ldots(x-x_n)}{(x_i-x_0)(x_i-x_1)\ldots(x_i-x_{i-1})(x_i-x_{i+1})\ldots(x_i-x_n)};$$

$L_i(x), i = 0(1)n$ are known as Lagrange's coefficients and each of them is a polynomial of degree n. It should be noted that they satisfy the property :

$$L_i(x_j) = 1,\ i = j$$
$$= 0,\ i \neq j$$

The Lagrange's method can be used for inverse interpolation i.e. we can express x as function of y : $x(y) = \sum_{i=0}^{n} L_i(y)\, x_i$.

where $$L_i(y) = \frac{(y-y_0)(y-y_1)\cdots(y-y_{i-1})(y-y_{i+1})\cdots(y-y_n)}{(y_i-y_0)(y_i-y_1)\cdots(y_i-y_{i-1})(y_i-y_{i+1})\cdots(y_i-y_n)}.$$

Linear Interpolation : Let just two values of a function f(x) be given at two points, say, x_0 and x_1, i.e. the data consists of $(x_0, f(x_0))$ and $(x_1, f(x_1))$. Then, linear interpolation gives the value of f (x) at $x = x_a$, $x_0 < x_a < x_1$ as

$$f(x_a) = \frac{(x_a - x_1)}{x_0 - x_1} f(x_0) + \frac{(x_a - x_0)}{x_1 - x_0} f(x_1).$$

This process of finding $f(x_a)$ is called linear interpolation.

Example 1

Using Lagrange's method to find the value of y when $x = 2.5$ from the following data :

i	0	1	2	3
x_i	0	0.5	1.5	3.0
y_i	− 6	− 1.875	0.375	0

Compute up to four places of decimal only.

Solution

From Lagrange's method

$$y(x) = L_0(x)y_0 + L_1(x)y_1 + L_2(x)y_2 + L_3(x)y_3$$

$$L_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}$$

$$= \frac{(2.5-0.5)(2.5-1.5)(2.5-3.0)}{(0-0.5)(0-1.5)(0-3.0)}$$

$$= \frac{2.0\times1\times(-0.5)}{(-0.5)\times(-1.5)\times(-3.0)} = 0.4444$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}$$

$$= \frac{(2.5-0)(2.5-1.5)(2.5-3.0)}{(0.5-0)(0.5-1.5)(0.5-3.0)} = \frac{2.5\times1\times(-0.5)}{0.5\times(-1.0)\times(-2.5)} = -1.0$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}$$

$$= \frac{(2.5-0)(2.5-0.5)(2.5-3.0)}{(1.5-0)(1.5-0.5)(1.5-3.0)} = \frac{2.5\times2\times(-0.5)}{1.5\times1\times(-1.5)} = 1.1111$$

$$L_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

$$= \frac{(2.5-0)(2.5-0.5)(2.5-1.5)}{(3.0-0)(3.0-0.5)(3.0-1.5)} = \frac{2.5\times2.0\times1}{3.0\times2.5\times1.5} = 0.4444$$

$$y(2.5) = 0.4444\times(-6) + (-1.0)\times(-1.875) + (1.1111)(0.375) + (0.4444)\times0$$

$$\approx -2.666 + 1.875 + 0.4167 + 0 \approx -0.3743$$

Example 2

Use linear interpolation to find f(0.3) for $f(x) = 5^x$.

Solution

Take the data at the points $x_0 = 0$, $x_1 = 1$,

Now, $f(x) = 5^x$. Hence, the data is (0, 1), (1, 5).

By the Linear Interpolation formula, we get,

$$f(0.3) = \frac{0.3-1}{0-1}(1) + \frac{0.3-0}{1-0}(5) = 2.2$$

Example 3

Find the Lagrange's interpolating polynomial for the following data :

X	$\frac{1}{4}$	$\frac{1}{3}$	1
f(x)	–1	2	7

Solution

$$L_0(x) = \frac{\left(x-\frac{1}{3}\right)(x-1)}{\left(\frac{1}{4}-\frac{1}{3}\right)\left(\frac{1}{4}-1\right)} = 16\left(x-\frac{1}{3}\right)(x-1)$$

$$L_1(x) = \frac{\left(x-\frac{1}{4}\right)(x-1)}{\left(\frac{1}{3}-\frac{1}{4}\right)\left(\frac{1}{3}-1\right)} = -18\left(x-\frac{1}{4}\right)(x-1)$$

$$L_2(x) = \frac{\left(x-\frac{1}{3}\right)\left(x-\frac{1}{4}\right)}{\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)} = 2\left(x-\frac{1}{3}\right)\left(x-\frac{1}{4}\right)$$

Hence $P_2(x) = L_0(x)(-1) + L_1(x)(2) + L_2(x)(7)$

$$P_2(x) = -16\left(x - \frac{1}{3}\right)(x-1) - 36\left(x - \frac{1}{4}\right)(x-1) + 14\left(x - \frac{1}{3}\right)\left(x - \frac{1}{4}\right)$$

Example 4

Given f(x) = sin x, f(0.1) = 0.09983, f(0.2) = 0.19867, use the method of linear interpolation to find f(0.16).

Solution

Using the Lagrange interpolation formula, we obtain

$$f(0.16) = \frac{0.16-0.2}{0.1-0.2}(0.09983) + \frac{0.16-0.1}{0.2-0.1}(0.19867)$$

$$= 0.039932 + 0.119202 = 0.159134.$$

Example 5

If f(1) = − 3, f(3) = 9, f(4) = 30 and f(6) = 132, find the Lagrange's interpolation polynomial of f(x). Also find the value of f when x = 5.

Solution

We have $x_0 = 1, x_1 = 3, x_2 = 4, x_3 = 6$

and $f_0 = -3, f_1 = 9, f_2 = 30, f_3 = 132.$

The Lagrange's interpolating polynomial $P_3(x)$ is given by

$$P_3(x) = L_0(x)\, f_0 + L_1(x)\, f_1 + L_2(x)\, f_2 + L_3(x)\, f_3 \qquad \ldots (3.1)$$

where $L_0(x) = \dfrac{(x-3)(x-4)(x-6)}{(1-3)(1-4)(1-6)} = -\dfrac{1}{30}(x^3 - 13x^2 + 54x - 72)$

$$L_1(x) = \frac{(x-1)(x-4)(x-6)}{(3-1)(3-4)(3-6)} = \frac{1}{6}(x^3 - 11x^2 + 34x - 24)$$

$$L_2(x) = \frac{(x-1)(x-3)(x-6)}{(4-1)(4-3)(4-6)} = -\frac{1}{6}(x^3 - 10x^2 + 27x - 18)$$

$$L_3(x) = \frac{(x-1)(x-3)(x-4)}{(6-1)(6-3)(6-4)} = \frac{1}{30}(x^3 - 8x^2 + 19x - 12)$$

Substituting these in Eq. (3.1) we have :

$$P_3(x) = -\frac{1}{30}(x^3 - 13x^2 + 54x - 72)(-3) + \frac{1}{6}(x^3 - 11x^2 + 34x - 24)$$

$$\times (9) - \frac{1}{6}(x^3 - 10x^2 + 27x - 18)(30)$$

$$+ \frac{1}{30}(x^3 - 8x^2 + 19x - 12)(132)$$

which gives on simplification

$$P_3(x) = x^3 - 3x^2 + 5x - 6.$$

$$f(5) \approx P_3(5) = (5)^3 - 3(5)^2 + 5 \times 5 - 6 = 125 - 75 + 25 - 6 = 69$$

Example 6

Find a polynomial of degree ≤ 2 with the properties p(1) = 5, p(1.5) = – 3, p(3) = 0.

Solution

The data given is (1, 5), (1.5, – 3), (3, 0). Using the Lagrange interpolation formula, we obtain the second degree polynomial as :

$$p(x) = \frac{(x-1.5)\,(x-3)}{(1-1.5)\,(1-3)}(5) + \frac{(x-1)\,(x-3)}{(1.5-1)\,(1.5-3)}(-3) + \frac{(x-1.5)\,(x-1.5)}{(3-1)\,(3-1.5)}(0)$$

$$= 5(x-1.5)\,(x-3) + 4(x-1)\,(x-3) = 9x^2 - 38.5x + 34.5$$

Example 7

Find the interpolating polynomial that fits the data :

x_k	0	1	2	5
f_k	2	3	12	147

using the Lagrange interpolation formula.

Solution

We have

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}f(x_1) +$$

$$\frac{(x-x_0)\,(x-x_1)\,(x-x_3)}{(x_2-x_0)\,(x_2-x_1)\,(x_2-x_3)}f(x_2) + \frac{(x-x_0)\,(x-x_1)\,(x-x_2)}{(x_3-x_0)\,(x_3-x_1)\,(x_3-x_2)}f(x_3)$$

$$= \frac{(x-1)\,(x-2)\,(x-5)}{(0-1)\,(0-2)\,(0-5)} \times 2 + \frac{(x-0)\,(x-2)\,(x-5)}{(1-0)\,(1-2)\,(1-5)} \times 3 +$$

$$\frac{(x-0)\,(x-1)\,(x-5)}{(2-0)\,(2-1)\,(2-5)} \times 12 + \frac{(x-0)\,(x-1)\,(x-2)}{(5-0)\,(5-1)\,(5-2)} \times 147$$

$$= -\frac{(x-1)(x-2)(x-5)}{5} + \frac{3}{4}x\,(x-2)\,(x-5) - 2x\,(x-1)\,(x-5)$$

$$+ \frac{49}{20}x\,(x-1)\,(x-2)$$

Example 8

Compute f ′ (2.0) from the following data of values :

x	1.8	1.9	2.0	2.1
f (x)	6.05	6.69	7.39	8.17

Solution

Since, four data points are given, the data represents a cubic polynomial, P(x). We can construct the polynomial P ′ (x) and find P ′ (2.0). Alternatively, we can construct a quadratic polynomial P(x) using the data values (1.9, 6.69), (2.0, 7.39), (2.1, 8.17) and find P ′ (2.0). The cubic polynomial is given by :

$$P(x) = \frac{(x-1.9)(x-2.0)(x-2.1)}{(-0.1)\,(-0.2)\,(-0.3)}(6.05) + \frac{(x-1.8)(x-2.0)(x-2.1)}{(+0.1)\,(-0.1)\,(-0.2)}(6.69)$$

$$+ \frac{(x-1.8)(x-1.9)(x-2.1)}{(+0.2)\,(0.1)\,(-0.1)}(7.39) + \frac{(x-1.8)(x-1.9)(x-2.0)}{(0.3)\,(0.2)\,(0.1)}(8.17)$$

Differentiating P (x), we have,

$$P'(x) = -\left(\frac{6.05}{0.006}\right)(3x^2 - 12x + 11.99) + \left(\frac{6.69}{0.002}\right)(3x^2 - 11.8x + 11.58)$$

$$-\left(\frac{7.39}{0.002}\right)(3x^2 - 11.6x + 11.19) + \left(\frac{8.17}{0.006}\right)(3x^2 - 11.4x + 10.82)$$

Hence, $P'(2.0) = 10.08 - 66.9 + 39.95 + 27.33 = 7.46$

Alternatively, the quadratic polynomial is given by

$$P(x) = \frac{(x-2)(x-2.1)}{(-0.1)(-0.2)}(6.69) + \frac{(x-1.9)(x-2.1)}{(0.1)(-0.1)}(7.39)$$

$$+ \frac{(x-1.9)(x-2.0)}{(0.2)(0.1)}(8.17)$$

$$P'(x) = +334.5(2x - 4.1) - 739(2x - 4) + 408.5(2x - 3.9)$$

Hence, $P'(2.0) = -33.45 + 40.85 = 7.40.$

Ex .1 : Find Lagrange's interpolating polynomial for the following data. Hence obtain f(2).

x	0	1	4	5
f(x)	8	11	68	123

Ex. 2 : Using the Lagrange's interpolation formula, find the value of y when x = 10.

X	5	6	9	11
F(x)	12	13	14	16

Ex. 3 : Using Lagranges interpolation formula, find the value of f(4) from the following data :

x	1	3	7	13
f(x)	2	5	12	20

Ex. 4 : In the following table h is the height above the sea level and p the barometric pressure. Calculate p when h = 5280.

h	:	0	4763	6942	10594
p	:	27	25	23	20

Ex. 5 : The following table is given :

x	:	0	1	2	5
f(x)	:	2	3	12	147.

Find the interpolating polynomial that fits this data.

Ex. 6 : The following values of the function f(x) for values of x are given :

$f(1) = 4,\ f(2) = 5,\ f(7) = 5,\ f(8) = 4.$

Find the value of f(60).

Ex. 7 : By means of Lagrange's formula, prove that

$$y_1 = y_3 - 0.3(y_5 - y_{-3}) + 0.2(y_{-3} - y_{-5})$$

where $y_i = y(i)$, the value of the dependent variable y at x = i.

3.3 LAGRANGE'S METHOD : INVERSE INTERPOLATION

One of the advantages of methods of interpolation for not necessarily equidistant tabular points x_i over methods for necessarily equidistant tabular points, is that even the value of independent variable x may be interpolated from the values of those of the dependent variable y.

This is in view of the fact that, even when the values of the independent variable x may be equidistant, the values of the dependent variable y may not be equidistant. Therefore the earlier methods of interpolation with equidistant tabular points cannot be applied for inverse interpolation.

In inverse interpolation, in a table of values of x find y = f(x), one is given a number $\overline{y}$ and wishes to find the point $\overline{x}$ so that $f(\overline{x}) = \overline{y}$, where f(x) is the tabulated function. For this, we naturally assume that the function y = f(x) has a unique inverse x = g(y) in the range of the table. The problem of inverse interpolation simply reduces to interpolation with x-row and y-row in the given table interchanged so that the interpolating points now become $y_0, y_1, \ldots, y_n$ (same as $f_0, f_1, \ldots, f_n$ i.e. $f(x_i) = f_i = y_i$) and the corresponding function values are $x_0, x_1, \ldots, x_n$ where the function is x = g(y). Since the points $y_0, y_1, \ldots, y_n$ are not necessarily equally spaced, this interpolation can be done by Lagrange's form of interpolation (also by Newton's divided difference form discussed later). By Lagrange's formula

$$P_n(y) = \sum_{i=0}^{n} x_i \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{(y - y_j)}{(y_i - y_j)}$$

and $$\overline{x} \approx P_n(\overline{y}).$$

This process is called inverse interpolation.

Example 9

Find the value of x when y = 3 from the following table of values :

x	4	7	10	12
y	–1	1	2	4

Solution

The Lagrange's interpolation polynomial of x is given by

$$x = P(y) = \frac{(y-1)(y-2)(y-4)}{(-1-1)(-1-2)(-1-4)}(4) + \frac{(y+1)(y-2)(y-4)}{(1-(-1))(1-2)(1-4)}(7)$$

$$+ \frac{(y+1)(y-1)(y-4)}{(3)(1)(-2)}(10) + \frac{(y+1)(y-1)(y-2)}{(5)(3)(2)}(12)$$

So $$P(3) = \frac{(2)(1)(-1)}{-(2)(3)(5)} \times (4) + \frac{(4)(1)(-1)}{(2)(-3)} \times (7)$$

$$+ \frac{(4)(2)(-1)}{-(3)(2)}(10) + \frac{(4)(2)(1)}{(5)(2)(3)}(12)$$

$$= \frac{4}{15} - \frac{14}{3} + \frac{40}{3} + \frac{48}{15} = \frac{182}{15} = 12.1333$$

$\therefore$ $$x(3) \approx P(3) = 12.1333.$$

Ex. 8 : From the following table, find the Lagrange's interpolating polynomial, which agrees with the values of x at the given value of y. Hence find the value of x when y = 2.

X	1	19	49	101
Y	1	3	4	5

Ex. 9 : Using the inverse Lagrange interpolation, find the value of x when y = 3 from the following table of values :

X	36	54	72	144
Y	– 2	1	2	4

3.4 DIVIDED DIFFERENCE (DD) TABLE

The **Divided difference Table** is a **generalization of Forward Divided Difference Table,** which we studied in Section 2.2 of Unit 2. The generalization is in the sense that now the tabular points x_i are not necessarily equidistant.

Let us suppose that values of a function f(x) are provided for $x = x_0, x_1, x_2, \ldots$, etc. which are not necessarily equi-distant. We define the divided difference (DD) as follows :

First DD :

$$f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}, \frac{f(x_2) - f(x_1)}{x_2 - x_1}, \ldots, \frac{f(x_{k+1}) - f(x_k)}{x_{k+1} - x_k} \quad \text{etc.}$$

Second DD :

$$f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}, f(x_k, x_{k+1}, x_{k+2}) = \frac{f(x_{k+1}, x_{k+2}) - f(x_{k+1}, x_k)}{x_{k+2} - x_k}$$

Third DD :

$$f(x_0, x_1, x_2, x_3) = \frac{f(x_1, x_2, x_3) - f(x_0, x_1, x_2)}{x_3 - x_0} \quad \text{and so on.}$$

We form a DD Table as follows :

Table 6 : Divided Difference (DD) Table

X	f(x)	1st DD	2nd DD	3rd DD	4th DD
x_0	$f(x_0)$				
		$f(x_0, x_1)$			
x_1	$f(x_1)$		$F(x_0, x_1, x_2)$		
		$f(x_1, x_2)$		$f(x_0, x_1, x_2, x_3)$	
x_2	$f(x_2)$		$F(x_1, x_2, x_3)$		$f(x_0, x_1, x_2, x_3, x_4)$
		$f(x_2, x_3)$		$f(x_1, x_2, x_3, x_4)$	
x_3	$f(x_3)$		$F(x_2, x_3, x_4)$		
		$f(x_3, x_4)$			
x_4	$f(x_4)$				

The remaining divided differences of higher orders are defined inductively as follows : The kth divided differences relative to $x_i, x_{i+1}, \ldots, x_{i+k}$ is defined as follows :

$$f(x_i, x_{i+1}, \ldots, x_{i+k}) = (f(x_{i+1}, \ldots, x_{i+k}) - f(x_i, x_{i+1}, \ldots, x_{i+k-1}))/(x_{i+k} - x_i),$$

where the (k –1)st divided differences $f(x_i, \ldots, x_{i+k-1})$ and $f(x_{i+1}, \ldots, x_{i+k})$ have already been determined. From the above, we can see that, by definition, the kth divided difference is obtained by dividing the (k –1)st divided differences by $(x_{i+k} - x_i)$.

Remark : Some authors use rectangular brackets (namely, '['and']') instead of the round parentheses (namely, '('and')'). Thus k^th^ divided difference is denoted as :

$$f[x_i, x_{i+1}, \ldots, x_{i+k}] = \frac{f[x_{i+1}, \ldots, x_{i+k}] - f[x_i, \ldots, x_{i+k-1}]}{x_{i+k} - x_i}$$

Example 10

If $f(x) = x^3$, find the value of f (a ,b, c).

Solution

$$f(a, b) = \frac{f(b) - f(a)}{b - a} = \frac{b^3 - a^3}{b - a} = b^2 + ba + a^2$$

Similarly $f(b, c) = b^2 + bc + c^2$

Hence $$f(a, b, c) = \frac{f[b, c] - f[a, b]}{c - a}$$

$$= \frac{b^2 + bc + c^2 - b^2 - ba - a^2}{c - a}$$

$$= \frac{(c - a)(c + a + b)}{(c - a)}$$

$$= a + b + c$$

Example 11

Find the Newton's divided difference Table for the following data :

X	– 4	–1	0	2	5
f(x)	1245	33	5	9	1335

Solution

Divided Difference Table

x	f(.)	f(., .)	f(., ., .)	f)., ., ., .)	f)., ., ., .,.)
– 4	1245				
		– 404			
– 1	33		94		
		– 28		– 14	
0	5		10		3
		2		13	
2	9		88		
		442			
5	1355				

Ex. 10 : If $f(x) = \frac{1}{x}$, show that $f(a, b, c, d) = -\frac{1}{abcd}$.

Ex. 11 : Form the divided difference table for the following data :

X	1	2	3	5	6
f(x)	1	3	7	21	31

Ex. 12 : From the table of values given below, obtain the divided difference interpolation table :

X	0	2	3	4
F(x)	– 4	6	26	64

3.5 NEWTON'S DIVIDED (DD) FORMULA

The Newton's DD formula is :

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2)$$
$$+ (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3) + \ldots$$

Remark : The Newton's DD formula is a sort of generalization of the Newton's FD Formula *(where, FD is meant for only in cases of equidistant tabular points)* **:**

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots$$

where, $p = \frac{x_p - x_0}{h}$ and $y_p = E^p y$ and, $\Delta y_0 = (y_1 - y_0)$

Example 12

Form the Newton's divided difference Table of Example 11 above, for the following data, form interpolating polynomial for

X	– 4	–1	0	2	5
f(x)	1245	33	5	9	1335

Also approximate f(1) from the polynomial.

Solution

The Divided Difference Table of Example 11 is :

x	f(.)	f(., .)	f(., ., .)	f)., ., ., .)	f)., ., ., .,.)
– 4	1245				
		– 404			
– 1	33		94		
		– 28		– 14	
0	5		10		3
		2		13	
2	9		88		
		442			
5	1355				

Hence, $P_4(x) = 1245 - 404(x + 4) + 94(x + 4)(x + 1) - 14(x + 4)(x + 1)x +$

$3(x + 4)(x + 1)x(x - 2)$

$f(1) \approx P_4(1) = -5$

Example 13

Prepare a DD table for the following data :

x :	– 03	0	1	3	4
f(x)	– 28	2	4	32	70

Find f(2) using Newton's DD formula.

Solution

I	x_i	$f(x_1)$	1^{st} DD	2^{nd} DD	3^{rd} DD	4^{th} DD
0	– 3	– 28				
1	0	2	$\frac{2+28}{0+3}=10$			
				$\frac{2-10}{1+3}=-2$		
2	1	4	$\frac{4-2}{1}=2$		$\frac{4+2}{3+3}=1$	0
				$\frac{14-2}{3-0}=4$		
3	3	32	$\frac{32-4}{2}=14$		$\frac{8-4}{4-0}=1$	
				$\frac{38-14}{4-1}=8$		
4	4	70	$\frac{70-32}{1}=38$			

Newton's DD formula is given by

$$y(x)=f(x_0)+(x-x_0)f(x_0,x_1)+(x-x_0)(x-x_1)f(x_0,x_1,x_2)$$
$$+(x-x_0)(x-x_1)(x-x_2)f(x_0,x_1,x_2,x_3)$$

Hence, $y(2) = -28 + (2 + 3) \times 10 + (2 + 3) \times (2 - 0)(-2) + (2 + 3)(2 - 0)(2 - 1) \times 1$

$= -28 + 50 - 20 + 10 = 12$

Example 14

From the following table of values, find the Newton's form of interpolating polynomial approximating f(x).

X	– 1	0	3	6	7
f(x)	3	– 6	39	822	1611

Also find the approximate value of the function f(x) at x = 2.

Solution

We note that the values of x, that is x_i, are not equally spaced. To find the desired polynomial, we form the table of divided differences of f(x).

Table 2

x		f(.)	f(., .)	f(., ., .)	f)., ., ., .)	f)., ., ., .,.)
x_0	−1	3				
			−9			
x_1	0	−6		6		
			15		5	
x_2	3	39		41		1
			261		13	
x_3	6	822		132		
			789			
x_4	7	1611				

Newton's interpolating polynomial $P_4(x)$ is given by

$$P_4(x) = f(x_0) + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x_2]$$

$$+ (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3)$$

$$+ (x - x_0)(x - x_1)(x - x_2)(x - x_3) f(x_0, x_1, x_2, x_3) \qquad \ldots (3.2)$$

The divided differences $f(x_0)$, $f(x_0, x_1)$, $f(x_1, x_1, x_2)$, $f(x_0, x_1, x_2, x_3)$, $f(x_0, x_1, x_2, x_3)$ are those which lie along the diagonal at $f(x_0)$ as shown by the dotted line. Substituting the values of x_i and the values of the divided differences in Eq. (3.2), we get

$$P_4(x) = 3 + (x + 1)(-9) + (x + 1)(x - 0)(6) + (x + 1)(x - 0)(x - 3)(5)$$

$$+ (x + 1)(x - 0)(x - 3)(x - 6)(1)$$

This on simplification gives

$$P_4(x) = x^4 - 3x^3 + 5x^2 - 6$$

$$\therefore \quad f(x) \approx P_4(x)$$

$$f(2) \approx P_4(2) = 16 - 24 + 20 - 6 = 6.$$

Ex. 13 : From the divided differences table of Ex. 11, show that the following data :

X	1	2	3	5	6
f(x)	1	3	7	21	31

represents a second degree polynomial. Also, obtain this polynomial. Hence, find the approximate value of f(4).

Ex. 14 : From the table of values given below, obtain the value of y when x = 1 using

(a) divided difference interpolation formula, and

(b) Lagrange's interpolation formula.

X	0	2	3	4
F(x)	−4	6	26	64

In case of (a), you may use table of **Ex. 12.**

3.6 SUMMARY

In this unit, we

- defined the concept of the divided difference table, and then illustrated through examples, how to construct a divided difference table;
- stated two well-known formulae for interpolation, for the general case when the tabular points are not necessarily equally spaced : (a) Lagrange's Method, and its use in Inverse Interpolation, and (b) Newton's Divided Difference Method;
- explained how, using each of the methods, to derive interpolating polynomials; and
- explained how these polynomials, mentioned under the previous point, can be used to estimate value of dependent (and in the case of Lagrange's Method, even for independent) variable, for some value of the independent/dependent variable, when the tabular points are not necessarily equidistant.

3.7 SOLUTIONS/ANSWERS

Ex. 1 : $P(x) = x^3 - x^2 + 3x + 8$, $f(2) = 18$

Ex. 2 : 14.6667

Ex. 3 : 6.6875

Ex. 4 : Use Lagrange Interpolation p (5280) = 24.8.

Ex. 5 : By Lagrange's formula, we get, $f(x) = x^3 + x^2 - x + 2$.

Ex. 6 : Applying Lagrange's formula for the above values, we get

$$f(x) = \frac{[-x^2 + 9x + 16]}{6}$$

Hence, $f(6) \approx 5.667$.

Ex. 7 : We are required to obtain y_1, the value of y at x = 1 while y_5, y_{-5}, y_{-3}, y_3, are given. By Lagranges formula, we have

$$y_1 = \frac{(1+3)(1-3)(1-5)}{(-5+3)(-5-3)(-5-5)} y_{-5} + \frac{(1+5)(1-3)(1-5)}{(-3+5)(-3-3)(-3-5)} y_{-3}$$

$$+ \frac{(1+5)(1+3)(1-5)}{(3+5)(3+3)(3-5)} y_3 + \frac{(1+5)(1+3)(1-3)}{(5+5)(5+3)(5-3)} y_5$$

$$= y_3 - 0.3\,(y_5 - y_{-3}) + 0.2\,(y_{-3} - y_{-5}).$$

Ex. 8 : Let x = g(y). The Lagrange's interpolating polynomial P(y) of g(y) is given by

$$P(y) = -\frac{1}{24}(y^3 - 12y^2 + 47y - 60) + \frac{19}{4}(y^3 - 10y^2 + 29y - 20)$$

$$-\frac{49}{3}(y^3 - 9y^2 + 23y - 15) + \frac{101}{8}(y^3 - 8y^2 + 19y - 12)$$

which, on simplification gives

$$P(y) = y^3 - y^2 + 1$$

when $y = 2,\ x \approx P(2) = 5$.

Ex. 9 : $x = g(y)$

$$P(y) = \frac{(y-1)(y-2)(y-4)}{(-3)(-4)(-6)}.(36) + \frac{(y+2)(y-2)(y-4)}{(3)(-1)(-3)}.(54)$$

$$+ \frac{(y+2)(y-1)(y-4)}{(4)(1)(-2)}.(72) + \frac{(y+2)(y-1)(y-2)}{(6)(3)(2)}.(144)$$

$$x \approx P(3) = \frac{(2)(1)(-1)}{-72} \times (36) + \frac{(5)(1)(-1)}{9} \times (54)$$

$$+ \frac{(5)(2)(1)}{8} \times (72) + \frac{(5)(2)(1)}{36} \times (144)$$

$$= 1 - 30 + 90 + 40 = 101$$

Ex. 10 : $f[a, b] = \dfrac{\frac{1}{b} - \frac{1}{a}}{b - a} = -\dfrac{1}{ab}$

$$f[b, c] = -\frac{1}{bc}$$

and $f[c, d] = -\dfrac{1}{cd}$

$$f[a, b, c] = \frac{1}{abc},\ f[b, c, d] = \frac{1}{bcd}$$

$$f[a, b, c, d] = \frac{f[b, c, d,] - f[a, b, c]}{d - a} = -\frac{1}{abcd}$$

Ex. 11 : The required Divided Difference Table is :

x	f(x)	f(., .)	f(., ., .)		
1	1				
		2			
2	3		1		
		4		0	
3	7		1		0
		7		0	
5	21		1		
		10			
6	31				

Ex. 12 : The required Newton's Divided difference table is :

x	f(x)	f(., .)	f(., ., .)	f(., ., ., .)
0	− 4			
		5		
2	6		5	
		20		1
3	26		9	
		38		
4	64			

Ex. 13 : From the difference table of **Ex. 11**, we observe that since third and higher order divided differences are zeros, given data represents a second degree polynomial

$$P_2(x) = f[x_0] + (x - x_0]\, f[x_0, x_1] + (x - x_0)(x - x_1)\, f[x_0, x_1, x_2]$$

$$= 1 + (x - 1) \,.\, 2 + (x - 1)(x - 2) \,.\, 1 = x^2 - x + 1\; f(4) \approx P_2(4) = 13$$

$$f(4) \approx P_2(4) = 13$$

Ex. 14 : Lagrange's interpolation polynomial

$$P(x) = \frac{1}{6}(x^3 - 9x^2 + 26x - 24) + \frac{3}{2}(x^3 - 7x^2 + 12x)$$

$$- \frac{26}{3}(x^3 - 6x^2 + 8x) + 8(x^3 - 5x^2 + 6x)$$

$$= x^3 + x - 4 \text{ on simplification.}$$

From the Newton's Divided Difference table of **Ex. 12**, we get

$$P(x) = -4 + (x - 0)\,5 + x(x - 2)(5) + x(x - 2)(x - 3)(1)$$

$$= x^3 + x - 4$$

$$f(1) \approx p(1) = -2.$$

NOTES

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