BLOCK 2

DIFFERENTIAL EQUATIONS

BLOCK 2 INTRODUCTION

The second block is titled **Differential Equations**. In the last Block of the course BECC 102, you had studied difference equations and their applications to economics. Like difference equations, differential equations, too, are useful for studying dynamic processes, that is processes that take place over time. However, unlike difference equations which show discrete dynamic processes, differential equations are utilised to study continuous dynamic processes, and are equations that involve derivatives. This block also has two units. In the block on difference equations in BECC 102, the units were organized as linear and non-linear equations. In the present course, the block on differential equations has two units organized as first-order and second-order differential equations. Unit 3 is titled **First-order Differential Equations**, and unit 4 is titled **Second-order Differential Equations**.

UNIT 3 FIRST-ORDER DIFFERENTIAL EQUATIONS

Structure

3.0 OBJECTIVES

After going through the Unit, you should be able to:

- define a differential equation;
- explain the concept of the order and degree of a differential equation;
- discuss how to set up and solve differential equations of first order and first degree depending on whether they are of the variable separable for, linear, or reducible ; and
- analyse some applications of differential equations to economic problems.

3.1 INTRODUCION

In economics, we often study the interaction between various variables. Of these variables, one is designated as the dependent variables and the others as the independent variables. When the values of the independent variables change, the value of the dependent variable changes accordingly. So, given an independent variable x and a dependent variable y and a relation $y = f(x)$, we can find the rate of change, $\frac{dy}{dx}$, of y with respect to x. This differential coefficient helps us in investigating a number of properties of a given function. This is the subject matter of differential calculus and a number of earlier units have been devoted to it. Now we are faced with the reverse problem. In economics and many other areas like physics, chemistry, biology etc, it sometimes so happens that we are given the rate of change of one (dependent) variable with respect to one or more (independent) variables and we are supposed to find the underlying relationship between the variables. For example, we may be interested in finding a demand function given that the price elasticity for this demand is 1. Now, we know that if $x = x(P)$ is a demand function, where P is the price per unit of the commodity and x is the quantity demanded at this price, then the price elasticity of demand is given by $\eta_d = \frac{pdx}{xdp}$, So, the above problem reduces to " find $x = x(p)$ given that $\frac{pdx}{xdp} = \eta_d$ (1)

An expression like (1), which involves differential coefficients, is called a differential equation and we are supposed to find its solution. In this unit we shall introduce differential equations and apply them to solve some problems in economics. We now formally define what we mean by a differential equation and its solution and give methods for solving various types of differential equations.

3.2 PRELIMINARY CONCEPTS

A differential equation is an equation, which involves the independent and dependent variables, and the derivatives of the dependent variable with respect to the independent variables. Some examples of differential equations are

1) $\frac{dy}{dx} + y = xe^{-x}$ 2) $(x+y)\frac{dy}{dx} = x - y$

3) $\frac{d^2y}{dx^2} + a^2 y = \sin ax$ 4) $\left(\frac{d^3y}{dx^3}\right)^2 + \left(\frac{dy}{dx}\right)^4 = x$

5) $\left(1 + \left(\frac{dy}{dx}\right)^2\right) = \frac{d^2y}{dx^2}$ 6) $\frac{\partial^2 z}{\partial x^2} + a^2 \frac{\partial^2 z}{\partial y^2} = 0$

A differential equation is said to be **ordinary** if it involves only one independent variable. If it involves more than one independent variables it is said to be a **partial** differential equation. Note that in the case of an ordinary differential equation, only ordinary derivatives of the dependent variable are involved whereas partial differential equations contain partial derivatives of the dependent variable with respect to the independent variables. The differential equations in example 1) to 5) above are ordinary, whereas the differential equation in example 6) is a partial differential equation. In this unit, we shall deal with ordinary differential equations only. The **order** of a differential equation is the order of the highest order derivative present in it. Recall that if y is a function of x, then $\frac{dy}{dx}$ is the first order derivative of y, $\frac{d^2y}{dx^2}$ is the second order derivative of y and so on, In general $\frac{d^ny}{dx^n}$ is the nth order derivative of y with respect to x. The **degree** of a differential equation is the degree of the highest order derivative present in it. The differential equations in examples 1) and 2) above are both of order 1 and degree 1. The differential equation in Example 3) has order 2 and degree 1. The differential equation of Example 4) has order 3. Its degree is 2, the degree of the highest order derivative $\frac{d^3y}{dx^3}$, present in the equation. The order of the differential equation in example 5) is 2. To find its degree, we rewrite it as

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = \left(\frac{d^2y}{dx^2}\right)^2$$

and then deduce that its degree is 2. So, before finding the degree of a differential equation, we must free all the derivatives involved from radicals and fractions.

A **Solution** of a differential equation is a relation between the independent variable and the dependent variable, which, along with the derivatives obtained from it, satisfies the given differential equation. Thus, if $f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}\right) = 0$ is a differential equation, then a function $y = f(x)$ is a solution of this differential equation if $F\left(x, f(x), f'(x), \ldots, f^n(x)\right) = 0$ A solution of a differential equation of order n is said to be its general solution if it contains exactly n arbitrary constants. A solution of a differential equation obtained from its general solution by assigning particular values to the constants is called a particular solution.

In this unit, we shall learn to solve the following types of differential equations:

1) First order differential equations in which variables are separable.
2) First order linear differential equations.
3) Second order linear differential equations with constant coefficients.

Let us see what these are.

3.3 DIFFERENTIAL EQUATIONS OF FIRST ORDER AND FIRST DEGREE

A differential equation of first order and first degree is an equation of the type

$$\frac{dy}{dx} = f(x, y) \qquad (1)$$

That is, it is a relation between x, y and $\frac{dy}{dx}$ and the only derivative present in the equation is of first order and its degree is 1. We first learn how to solve a differential equation of order 1 and degree 1 in which variables are separable in the sense that it can be so expressed that the coefficient of dx contains terms containing x only and the coefficient of dy contains terms containing y only.

3.3.1 Separable Variable Equations

A differential equation of first order and first degree, in which the function f can be expressed as $f(x,y)=\frac{g(x)}{h(y)}$ where g (x) is a function of x only and h(y) is a function of y only, is called a separable variable equation. The solution of such a differential equation is easy to obtain as it can be rewritten as

$$h(y)\frac{dy}{dx}=g(x) \qquad (2)$$

Integrating (3) with respect to x, we obtain

$$\int h(y)dy=\int g(x)dx+c$$

which is a general solution of the differential equation (2).

Let us take some examples.

Example 1. Solve the differential equation $\frac{dy}{dx}=\frac{x}{y}$.

Solution : $\frac{dy}{dx}=\frac{x}{y}\Rightarrow ydy=xdx$. Note that the variables have been separated. On integrating, we get

$$\int ydy=\int xdx$$

$$\Rightarrow \frac{y^2}{2}=\frac{x^2}{2}+C \text{ or } y^2+x^2=k$$

Note that in the last equation, which is a solution of the given differential equation, we have replaced the constant $2C$ by k. We could have written the constant as $2C$ or C or we could have denoted it by some other symbol. The only important thing is that is a constant of integration.

Example 2. Solve the differential equation $x\sqrt{1+y^2}dx=y\sqrt{1+x^2}dx$

Solution : The given differential equation can be written as

$$\frac{x}{\sqrt{1+x^2}}dx=\frac{y}{\sqrt{1+y^2}}dy$$

Since the variables have been separated, on integrating we get

$$\int\frac{x}{\sqrt{1+x^2}}dx=\int\frac{y}{\sqrt{1+y^2}}dy$$

$$\Rightarrow\sqrt{1+x^2}=\sqrt{1+y^2}+C$$

which is the general solution of the given differential equation. Note that each of the integrals above can be integrated by substituting the expression inside the square root in the dominator, equal to *t*.

Example 3. Find the demand function $x = x(p)$ given that $\frac{pdx}{xdp} = -1.$

Also find x if it is given that when $P = 1, x$ takes the value 5.

Solution : Note that the variables in this differential equation are easily separable as it can be written as

$$\frac{dx}{x} = -\frac{dp}{p}$$

On integrating we get

$$\log x = \log p + C \qquad (3)$$

as its general solution. This represents the class of demand functions given by the given differential equation as different values of the constant C give rise to different demand functions. A particular solution of the equation can be obtained using the given condition that when $p = 1$, we have $x = 5$. Substituting these values in (3), we get $log\ 5 = log\ 1 + C$ and since $log\ 1 = 0$, we get C = $log\ 5$ and thus a particular solution of the given differential equation is log x $+ log\ p = log\ 5$ or $xp = 5$, applying properties of logarithms. Thus, we get $x = \frac{p}{5}$ as the demand function which the given differential equation represents for the given initial condition.

3.3.2 Linear Differential Equations

A differential equation of the type

$$\frac{dy}{dx} + Py = Q \qquad (4)$$

where P and Q are functions of x only, is called a linear differential equation of order 1.

In a linear differential equation the independent variable and all its derivatives present in the equation are of degree 1 and are never multiplied together. The differential equation (4) is of order 1 because derivatives of order 2 or more are not present. The solution of a linear differential equation is obtained as follows :

Step 1 : Identify the expressions P and Q, which are functions of x only.

Step 2 : Find the Integrating Factor (I.F.), which is given by $e^{\int pdx}$

Step 3 : Write the solution, which is given by

$$y \times \text{I.F.} = \int (Q \times \text{I.F.})\, dx$$

We take some examples to illustrate the procedure.

Example 4 . Solve the differential equation $\frac{dy}{dx}+\frac{y}{x}=\frac{x^2+1}{x}$.

This is a linear differential equation with $P=\frac{1}{x}$ and $Q=\frac{x^2+1}{x}$

The integrating factor is

$$\text{I.F.}=e^{\int pdx}=e^{\int \frac{1}{x}dx}=e^{\log x}=x$$

Therefore, the general solution of the given differential equation is

Example 5 : Solve the following differential equations :

$$\frac{dy}{dx}+xy=x^3$$

Solution : Here $P = x$, $Q = x^3$. Therefore, the integrating factor is

$$\text{I.F.}= e^{\int Pdx}=e^{\int xdx}=e^{\frac{x^2}{2}}$$

$\therefore$ The general solution of the given differential equation is

$$ye^{x^2/2}=\int x^3e^{x^2/2}dx$$

$=2\int te^t dt$ .Integrating by parts, we get

$$=2(t-1)e^{t+c}-e^t+C$$

$$=2\left(\frac{x^2}{2}-1\right)e^{\frac{x^2}{2}}+C$$

$$\left[\begin{array}{l} Put\ \frac{x^2}{2}=t \\ \therefore x^2=2t \\ \Rightarrow xdx=dt \end{array}\right]$$

Check Your Progress 1

1) Solve the following differential equations

(a). $ydx-(x^2-1)dy=0$ (b) $(x+xy^2)dx+y\sqrt{1-x^2}dy=0$

(c). $x^2dy-2xdx=2xy^2dx-dy$ (d) $\frac{dx}{dt}=-8xt^3$

(e). $\sqrt{1+x^2+y^2}+x^2y^2+xy\frac{dy}{dx}=0$

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2) Solve the following differential equations :

a) $x\frac{dy}{dx}+y=x^2+1$

b) $\frac{dy}{dx}+\frac{1}{x\log x}y=\frac{2}{x}$

c) $\frac{dy}{dx}+\frac{y}{1+x^2}=\frac{1}{1+x^2}$

d) $y\log y\frac{dx}{dy}+x-\log y=0$

e) $\frac{dy}{dx}-\frac{y}{x}=1+\frac{2}{x}$

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3.3.3 Equations Reducible to Linear Form

Some times a differential equation is not linear but can be reduced to linear form. A differential equation of the type

$$\frac{dy}{dx}+Py=Qy^{11} \qquad (5)$$

where P and Q are functions of x only, can be reduced to linear form. The procedure is as follows.

Step 1 : Divide the whole equation by y^{11}. The equation (5) becomes

$$\frac{1}{y^{11}}\frac{dy}{dx}+P\frac{1}{y^{m-1}}=Q \qquad (6)$$

Step 2: Put $\frac{1}{y^{n-1}}=z$. On differentiating with respect to x, this gives

$$(1-n)\frac{1}{y^n}\frac{dy}{dx}=\frac{dz}{dx}$$

Making these substitutions in (6), we get

$$\frac{1}{(1-n)}\frac{dz}{dx}+Pz=Q \qquad (8)$$

Or $$\frac{dz}{dx}+P'z=Q'$$

Where $P'=p(n-1) and Q'=Q(n-1)$. This is a linear differential equation with z as the dependent variable and can be easily solved.

Step 3 : Solve (7). This gives us a relation between z and x. Substitutiong $z = \frac{1}{y^{n-1}}$

In this relation, we obtain a solution of the given differential equation.

We take an example to illustrate the procedure.

Example 6 . Solve the differential equation $\frac{dy}{dx}+xy=x^3y^3$

Solution Dividing by y^3, the given differential equation becomes

$$\frac{1}{y^3}\frac{dy}{dx}+x\frac{1}{y^2}=x^3 \qquad (8)$$

We put $\frac{1}{y^2}=z$. Differentiating with respect to x, this gives

$$-\frac{2}{y^3}\frac{dy}{dx}=\frac{dz}{dx}$$

or

$$\frac{1}{y^3}\frac{dy}{dx}=-\frac{1}{2}\frac{dz}{dx}$$

Substituting in (8), we get $-\frac{1}{2}\frac{dz}{dx}+xz=x^3$, which can be rewritten as

$$\frac{dz}{dx}-2xz=-2x^3$$

The differential equation is linear in z with P = -2x and Q = -2x^3. Therefore, the integrating factor is $e^{\int Pdx}=e^{-x^2}$ and the solution of (8) is

$$ze^{-x^2}=\int -2x^3e^{-x^2}dx=-(x^2+1)e^{-x^2}+C$$

Substituting back $z=\frac{1}{y^2}$ in the above equation, we get

$$\frac{1}{y^2}e^{-x^2}=-(x^2+1)e^{-x^2}+C \qquad (9)$$

which can be written as $(x^2+1)y^2+Cy^2=1$ and is the general solution of the given differential equation.

Check Your Progress 2

1) Solve the following differential equations:

a) $2\frac{dy}{dx} = \frac{y}{x} + \frac{y^2}{x^2}$

b) $\frac{dy}{dx} + xy = x^3 y^3$

c) $\frac{dy}{dx} + \frac{x}{1-x^2} y = x\sqrt{y}$

d) $\frac{dy}{dx} + \frac{x}{y} = x^2 y^6$

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2) Solve the following differential equations:

a) $\frac{dy}{dx} + 2xy = y^2 e^{x2} \log x$

b) $ydx = (10y^3 - 2x)dy$

c) $(x^3 y^3 + xy)dx = dy$

d) $x\frac{dy}{dx} + y = y^2 \log x$

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3.4 ECONOMIC APPLICATIONS

As mentioned earlier, a number of situations arise when problems in economics get translated into differential equations and the solutions of these differential equations give the desired relation in the relevant economic variables. One such situation was described in the beginning of the last section and the corresponding differential equation was solved in example 3 to obtain the desired demand function we take some more examples to illustrate our point. So although we give general applications of differential equations in economics, remember that when we want to depict economic dynamics involving continuous time, like capital accumulation and economic growth, we use differential equation involving derivatives with respect to time.

Example. The price elasticity of demand of a commodity is given by $\frac{5p}{(p+3)(p-2)}$. Find demand function given that at p=3, 6 units are demanded

Solution : Here p is the price per unit of the commodity. Let x denote the quantity demanded. Elasticity of demand is given by

$$e_d = -\frac{p}{x}\frac{dx}{dp}$$

$$i.e. -\frac{p}{x}\frac{dx}{dp} = \frac{5p}{(P+3)(p-2)}$$

which on integrating gives

$$\log\frac{p-2}{p+3} = \log\frac{c}{x}$$

$$\Rightarrow x = c\frac{(p+3)}{p-2}$$

Now, we are given that when $p = 3$, $x = 6$. This gives

$6 = 6\ c$ i.e. $c = 1$

So, the required demand function is $x = \frac{p+3}{P-2}$

Example : The marginal propensity to consume is given by

$$\frac{dc}{dI} + \frac{2I}{I^2+1}C = \frac{1}{I^2+1}$$

Where C is the consumption and I denotes income. Find the consumption function given that $C = 100$ when $I = 2$

Solution : The differential equation above is a linear differential equation with C as the dependent and I as the independent variable. Here $P = \frac{2I}{I^2+1}, Q = \frac{1}{I^2+1}$

$\therefore$ The general solution is

$$c\left(I^2+1\right) = \int\left(I^2+1\right)\frac{1}{I^2+1}dI$$

= I + K where K is the constant of integration

Applying the initial condition, C = 100 where I = 2, we get

K= 498

∴ The Consumption function is

$$C = \frac{I + 498}{I^2 + 1}$$

Example : Solow Growth Model.

Consider a production function $Q = f\,(K, L)$ where Q denotes output, K denotes capital and L denotes Labour. Obviously, K >0, L > 0. If the prodcution function is taken to be linear and homgenous, it can be rewritten as

$$Q = L\phi(k)$$

where $k = \frac{K}{L}\, and \phi$ is a function of k. This can be seen easily. For example, if $Q = \frac{L^{5/4} + K^{5/4}}{L^{1/4} + K^{1/4}}$ then Q is a homogeneous function of degree i.e. it is a linear homogeneous function. It can be rewritten as

$$Q = L\left(\frac{1 + \left(\frac{K}{L}\right)^{5/4}}{1 + \left(\frac{K}{L}\right)^{1/4}}\right)$$

$$= L\left(\frac{1 + k^{5/4}}{1 + k^{1/4}}\right)$$

$$= L\phi\left(k\right)$$

In his growth model, professor Solow shows that given a rate of growth of labour λ, the economy would eventually reach in a state of steady growth in which investment will grow at the rate λ. The assumptions made in Solow's model are

$$\frac{dk}{dt} = sQ$$

$$\frac{dL/dt}{L} = \lambda$$

where s represents a constant marginal prospensity to save and λ represent constant rate of growth of labour.

With these assumptions the fundamental equation of the below growth model comes out to be

$$\frac{dk}{dt} = s\phi(k) - \lambda k$$

Let us take a particular production function $\theta = K^{1/3}L^{2/3}$

i.e. $Q = L\left(\frac{K}{L}\right)^{1/3} = Lk^{1/3}$

$\therefore \varphi(k) = k^{1/3}$

With this value of $\phi(k)$, the differential

equation (23) becomes

$$\frac{dk}{dt} = sk^{1/3} - \lambda k$$

$$\frac{d\lambda}{dk} + \lambda k = sk^{1/3}$$

which is a linear differential equation with $P = \lambda$ and $Q = sk^{1/3}$ and can be solved to obtain the relation

$$k^{1/3} = \left[k^{1/3}(0) - \frac{s}{\lambda}\right]e^{-1/3\lambda t} + \frac{s}{\lambda}$$

where $k(0)$ is the initial value of the capital labour ratio k.

This relation determines the time path of k.

Check Your Progress 3

1) Given the demand and supply function

$Q_d = 40 - 2p - 2p^1 - p^{11}, Q_s = -5 + 3p$

with p(0) = 12, P^1 (0) =1, find P (t) on the assuption that the market is always cleared.

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2) Investment function is given by I(t) = 200 $e^{0.4t}$ where I (t) = $\frac{dk}{dt}$ and L denotes the capital at time t and the initial Block is K(0) = 90. Find the time path of capital stock.

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3.5 LET US SUM UP

This unit introduced you to dynamic processes, where time elapses in a continuous fashion. The basic tool to understand such processes is the differential equation, which expresses the change in dependent variable with respect to change in independent variables. Differential equations involve derivatives where the dependent variable is differentiated with respect to the independent variable. It is important to note that in the study of dynamics in economics, since we are aiming to depict the passage of time, the independent variable is time. So, the derivative in differential equations that depict dynamic economic processes is always with respect to time.

The unit began by explaining the concept of order of a differential equation as also its degree. Attention was limited to equations of the first degree. Finally, the unit discussed some economic applications, including an outline of the Solow growth model.

3.6 ANSWERS/HINTS TO CHECK YOUR PROGRESS EXERCISES

Check Your Progress 1

1) See subsections 3.3.1 and 3.3.2 and answer.

2) See subsection 3.3.2 and answer.

Check Your Progress 2

1) See subsection 3.3.3 and answer.

2) See subsection 3.3.3 and answer.

Check Your Progress 3

1) See section 3.4 and answer.

2) See section 3.4 and answer.

UNIT 4 SECOND ORDER DIFFERENTIAL EQUATIONS

Structure

4.0 AIMS AND OBJECTIVES

You studied first-order differential equations in the previous unit. In this unit, you will be presented with a discussion on differential equation of the second degree, that is, differential equations where the highest order of the derivative appearing in the equation is two. After going through the unit, you will be able to:

- Define a second order differential equation;
- Distinguish between linear and non-linear second-order differential equations;
- Specify homogeneous as well as non-homogeneous second-order differential equations;
- Discuss various methods to solve second-order differential equations; and
- Describe some economic applications of second-order differential equations.

4.1 INTRODUCTION

In this block we are discussing dynamic processes in economics. To study these processes in economics, we are equipping ourselves with the mathematical tools needed to study these. Dynamic processes (in economics) mean changes over time. So we are studying mathematical methods which enable us to understand how economic variables change as a result of change in time, that is how these variables change over time.

We did study dynamic processes even in your first course on mathematical methods in economics, that is, course BECC 102. But there we studied discrete dynamic processes. The mathematical tool we used to study discrete dynamic processes was difference equation. In the present course, we are studying continuous dynamic processes, that is, dynamic processes where time changes continuously. The mathematical tool we are studying to help us understand continuous dynamic processes in economics is differential equations.

In the Block on difference equations in course BECC 102, we had divided the study of difference equations into linear and non-linear difference equation. Here we have divided the study of differential equations into first-order and second-order differential equations. In the previous unit, you studied first-order differential equations. In this unit, we study second-order differential equations.

To understand the whole idea of second-order differential equations, let us go over the meaning of differential equations. A differential equation is an equation that involves derivatives. In general differential equations in mathematics, the derivative can be with respect to any variable. In economics, since we use differential equations to study dynamics, and dynamics are processes that take place over time, the derivative is usually with respect to time. Now, coming to the order of the differential equations, the order of a differential equation is the highest order of the derivative that appears in the equation. So in this equation, we study differential equations where there are derivatives no higher than second order. Like we mentioned earlier, we have not divided the study of differential equations into linear and non-linear differential equations, unlike what we did in the case of difference equations. In fact, for simplicity, in this Unit on second-order differential equations, we study only linear differential equations of the second degree.

Please remember a basic feature of differential equations. Unlike simple algebraic equations, which have the value(s) of a variable or variables as the solution, in the case of differential equations the solution is a function, an equation. To solve a differential equation, we look for a function, an equation, which will depict the entire time path of the variable under discussion. Now, let us look closely at the fact that in second-order differential equations we use second-order derivatives. What is the significance of second-order derivatives with respect to time? Let us consider a non-economics context. In basic physics, think of distance (or displacement) as a variable. Differentiating distance with respect to time we get velocity. Differentiating velocity with respect to time, we get acceleration. Thus the second-derivative of displacement with respect to time is acceleration. In economics you can think of a change in general price level with respect to time as inflation. Second-order derivative with respect to time shows the rate of change of inflation, how inflation changes over time. Similarly, differentiating capital with respect to time shows investment. Second-order derivative of capital with respect to time shows the change in investment over time. So solving a second-order differential equation where capital is differentiated twice with respect to time, we will get as solution the equilibrium time path of capital.

Now that you have got an intuitive idea about second-order differential equations, let us sketch out the plan of this unit. In the next section we lay out the structure of second-order differential equations with constant coefficients. We deal throughout with linear equations only. Equations with constant coefficients mean equations where the coefficients attached with the variables of the equations are themselves not functions of time, unlike the variables that are functions of time. In the subsequent section, we look at equations that are homogeneous. Homogeneous equations have no terms where the variable of the equation does not appear, or to put it another way, each term in the equation has the variable of the equation and together the terms add to zero. In section 4.4 we discuss non-homogeneous equations. The following section, section 4.5, discusses some properties of solutions to second-order linear differential equations. After this the unit discusses some applications to economics of second-order differential equations.

Let us introduce a notational convention that would be used throughout the unit. Basically you will come across the terms $\frac{dx}{dt}$ or $\frac{d^2x}{dt^2}$ (first or second-order derivative of time. Usually when a variable x is differentiated the derivative can be written x' and second-order derivative can be written x''. We shall be using these notations sometimes in the unit. Of course, the variable to be differentiated can be written x or y or some other letter. Another notational convention concerns derivative with respect to time. If x is differentiated once with respect to time, we denote it $\dot{x}$. The second-order derivative of x with respect to time is denoted $\ddot{x}$. Keep this notation in mind

4.2 LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

So far we have been dealing with differential equations of first order only. We now learn how to solve differential equations of higher order. We shall restrict ourselves to second order differential equations only. The general form of a second-order differential equation is

$$\ddot{x} = f(t, x, \dot{x})$$

Within second order linear differential equations, we discuss only a special class of differential equations, the linear equations with constant coefficients. A differential equation of order n is said to be linear if it can be written as

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + P_{n-1} \frac{dy}{dx} + P_n y = X$$

where $P_1, P_2, \ldots P_n$ and X are functions of x only. If $P_1, P_2, \ldots, P_n$ are constants, we call it a linear differential equation with constant coefficients. If n is 2, we get a linear differential equation of order 2.

Notice that in the above equation, we have denoted the variable by y, and taken derivatives with respect to x. suppose we take an equation involving x, and take derivatives with respect to t, the general second-order linear differential equation can be depicted

$$\ddot{x} + a(t)\dot{x} + b(t)x = f(t)$$

Here the coefficients a and b are functions of time. The equation is equal to a variable that is a function of time. If coefficients a and b were not functions of time but were constant, we would get a linear second-order differential equation with constant coefficients.

So, a linear differential equation of degree 2 with constant coefficients looks like

$$a_0 \frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = X \qquad (1)$$

When a_0, a_1, a_2 are constants and X is a function of x only. If we write D for the symbol $\frac{d}{dx}$ and D^2 *for* $\frac{d^2}{dx^2}$, then $\frac{dy}{dx}$ can be written as Dy and $\frac{d^2 y}{dx^2}$ *as* $D^2 y$ and the differential equation (1) can be rewritten as

$$(a_0 D^2 + a_1 D + a_2) y = X \qquad (2)$$

or

$$f(D)y = x \text{ where } f(D) = a_0 D^2 + a_1 D + a_2$$

We first learn how to solve differentiation equation (2) when X = 0.

This is called a homogeneous equation. This is what we discuss in the next section.

4.3 SOLVING HOMOGENEOUS EQUATIONS

Solution to the differential equation $f(D)y = 0$

To solve the differential equation

$$f(D)y = 0 \qquad (3)$$

we first solve the equation

$$f(m) = 0 \qquad (4)$$

which is obtained by replacing D by m is the expression $f(D)$. The solution of (3) can then be written directly, depending on the nature of the roots of the equation $f(m) = 0$. We consider various cases by one. The equation $f(m) = 0$ is called the auxiliary equation (A.E.) of the differential equation (3).

Case 1: The roots of the auxiliary equation f (m) = 0 are real and distinct

If $m = m_1, m_2$. Are the roots of (14) and $m_1 \neq m_2$, the solution of (13) is given by

$$y = c_1 e^{m^{1x}} + c_2 e^{m^{2x}} \text{ where } c_{1,} c_2 \text{ are constants.}$$

Example 1 : Solve the differentiation equation $(D^2 - 5D + 6)y = 0$

Here $f(D) = D^2 - 5D + 6$ and therefore $f(m) = 0$ implies

$m^2 - 5m + 6 = 0$

which gives $m = 2$ and 3.

Thus, a solution of the given differential equation is $y = c_1 e^{2x} + c_2 e^{3x}$

Case 2 : The roots of the auxiliary equation are real and repeated.

If $m = X$ is a repeated root of the A.E. of the given differential equation. The solution of the given differential .equation is

$$y = (c_1 + xc_2)e^{\lambda x}$$

Example 2 Solve the differential equation $(D^2 + 4D + 4)y = 0$.

Solution: The A.E. is $\begin{aligned} &m^2 + 4m + 4 = 0 \\ &\Rightarrow m = -2, -2 \end{aligned}$

Therefore, $m = -2$ is a repeated root of the a.e. of the given differential equation. Therefore, the general solution of the given differential equation is

$y = \left(C_1 + xC_2\right)e^{-2x}$

Case 3 : The roots of the A.E. are imaginary (complex number with non-zero imaginary part). First note that if $\alpha + i\beta$ is a root of an equation with real coefficients, then $\alpha - i\beta$ has to be a root of the equation.

So, if L $\alpha \pm i\beta$ are roots of the A.E., then the solution the differential equation is given by

$$y = c^{\alpha x}\left[c_1 \cos \beta x + c_2 \sin \beta x\right] \tag{5}$$

Equation (5) gives us a standard method of writing the solution of a differential equation if its A.E. has roots $\alpha + i\beta$ and $\alpha - i\beta$ This is a convenient formula for writing the solution which, in fact, is obtained as follows:

Since $\alpha + i\beta$ and $\alpha - i\beta$ are distinct (though not real) roots of A.E., the solution can be written as

$$y = Ae^{(\alpha+i\beta)x} + Be^{(\alpha-i\beta)x} \quad \text{where A, B are constants}$$

$$= e^{\alpha x}\left[Ae^{i\beta x} + Be^{-i\beta x}\right]$$

$$= e^{\alpha x}\left[A(\cos\beta x + i\sin\beta x) + B\left(\cos\beta x - i\sin\beta x\right)\right]$$

using the fact that

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

$$= e^{\alpha x}\left[(A+B)\cos\beta x + i(A\_B)\sin\beta x\right]$$

$$= e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x)$$

where $C_1 = A+B, C_2 = i(A-B)$

Example 3: Solve $(D^2 + D + 1)y = 0$

Solution: the A.E. is $m^2 + m + 1 = 0$ and has the roots

$$m - \frac{1 \pm \sqrt{3}i}{2}$$

$$\therefore \alpha = -\frac{1}{2}, \beta = \frac{\sqrt{3}}{2}$$

And the solution of the given differential equation can be written as

$$y = e^{\frac{-1}{2}x}\left[c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x\right]$$

The methods given above are valid even where we are dealing with linear differential equation of higher order. If we are dealing with linear differentiation equation of order n, its A.E. will have n roots. If all the roots $x_{1,}x_{2,\ldots,}x_n$ are distinct, the solution is written as

$$c_1e^{\lambda_1 x} + c_2e^{\lambda_2 x} + \ldots\ldots - + c_ne^{\lambda_n x}$$

If a root λ is repeated *r* times, it contributes

$$(c_1 + xc_2 + \ldots\ldots + x^r c_r)e^{\lambda x}$$

to the solution

Similarly if a complex root, $\lambda + i\beta$,with non-zero imaginary part, is repeated twice it contributes

$$e^{\lambda x}\left[(c_1 + xc_2)\cos\beta x + (c_3 + xc_4)\sin\beta n\right]$$

to the solution. The case when a complex root is repeated a higher number of times can be similarly taken care of. Thus, if a differential equation of order 7 has m =2, 3, 4, 4,4, -1 $\pm 7i$ as the roots of its A.E., its solution would be

$$y = c_1e^{2x} + C_2e^{3x} + (C_3 + xC_4 + x^2C_3)e^{4x}$$

$$+ e^{-x}\left[c_6\cos 7x + c_1\sin 7x\right]$$

Check Your Progress 1

1) Solve the following linear differentiation equation

i) $\left(D^2 - 9D + 18\right)y = 0$

ii) $\left(D^2 - 6D + 9\right)y = 0$

iii) $(D^2 + D + 4)y = 0$

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Solve the following equations

i) $(D^2 + 4D + 4)y = 0$

ii) $6\frac{d^2y}{dx^2} + 13\frac{dy}{dx} + 6y = 0$

iii) $\frac{d^2y}{dx^2} - \frac{dy}{dx} + 1 = 0$

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4.4 NON-HOMOGENEOUS EQUATIONS

So far, we have been dealing with linear differential equations, in which X =0. In this subsection, we consider differential equations in which X is not equal to zero. This becomes an non-homogeneous equation.

Solution of the differential equation $f(D)y = X, X \neq 0$

When X ≠ 0, the solution of the differential equation

$$f(\mathrm{D})y = X, \mathrm{X} \neq 0 \tag{6}$$

consists of two parts : the complementary function (C.F.) and the particular integral (P.I). The Complementary function for the differential equation (5) is the solution of the corresponding differential equation $f(D)\ Y = 0$ given in (3) and is obtained using methods described in unit 3. The particular integral for the differential equation (6) can be any particular solution of (4), without arbitrary constants and depends on the function X. The general solution of (5) is the sun of the C.F. and the P.I. We describe below methods for finding Particular Integrals of (5), given *X*. Again we consider only the second-degree equation (1).

Case 1: When X is a constant.

Since P.I can be any solution of equation (1), with X = a, where a is a constant.,

$$a_0 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = X \qquad (7)$$

We try the solution y = k, k a constant. This implies $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$

and (7) reduces to $a_2 k = a_1$ This gives k = $\frac{a}{a_2}$

$\therefore$ We get the P.I., $y_p = \frac{a}{a_2}$ (8)

provided $a_2 \neq 0$

In case, $a_2 = 0,$ the expression $\frac{a}{a_2}$ is invalid and the P.I. given in equation (8) does not work. Since the constant solution has failed to work, we try the solution $y = kx$.

But $y = kx$ implies

$$\frac{dy}{dx} = k \text{ and } \frac{d^2y}{dx^2} = 0 \qquad (9)$$

Also, when $a_2 = 0,$ the differential equation (7) becomes

$$a_0 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} = a$$

Substituting the values of $\frac{dy}{dx} and \frac{d^2y}{dx_2}$ from (9),

this equation reduces to $a_1 k = a$ or $k = \frac{a}{a_1}$

Thus, we get the particular integral P.I = kx as

$$y_p = \frac{a}{a_1} x$$

Again this requires that $a_1 \neq 0$. If $a_1 = 0$, this solution does not work and we have to pass on to next simplest possible solution, which is $y = kx^2$. This, on substitution in the differential equation, gives $k = \frac{a}{2}$ and so the P.I. is

$$y_p = \frac{a}{2}x^2.$$

Summarising, the above discussion, we see that the particular integral of

$$a_0\frac{d^2y}{dx_2} + a_1\frac{dy}{dx} + a_2 y = a$$

is given by

$$y_p = \begin{pmatrix} \frac{a}{a_2} & \text{if } a_2 \neq 0 \\ \frac{a}{a_1}x & \text{if } a_2 = 0 \text{ but } a_1 \neq 0 & \vdots \\ \frac{a}{a_2}x^2 \text{ if } a_2 = a_1 = 0 & \cdots \end{pmatrix}$$

and using this P.I. , we can write the general solution of the given differential equation as

$$y = C.F. + P.I.$$

We take some examples to illustrate the procedure

Example 4. Solve the differential equation $\frac{d^2y}{dx^2} + 11\frac{dy}{dx} + 28y = 16.$

Solution : In this case, the A.E. is $7m^2+11m+28 = 0$ and has roots $m = -7, -4$. Therefore, its complementary function is C.F. = $C_1 e^{-7x} + C_2 e^{-4x}$

Also, the coefficient of y is non-zero, we can take the P.I. as $\frac{16}{28} = \frac{4}{7}$

$\therefore$ A general solution of the given differential equation is

$$y = \text{C.F.} + \text{P.I.}$$

$$= C_1e^{-7x} + C_2e^{-4x} + \frac{4}{7}$$

Example 5 : Solve the differential equation $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} = 11$

Solution It is easy to see that the C.F. of this differential equation is

$$C.F. = c_1 + c_2e^{-5x}$$

Since coefficient of y is zero but coefficient of $\frac{dy}{dx}$ is 5 (non-zero), therefore, we take

P.I. = $\frac{11x}{5}$

As its particular integral and hence the general solution is given by

$$y = c_1 + c_2 e^{-5x} + \frac{11x}{5}$$

Example 6 : Solve $7\frac{d^2y}{dx^2} = 6.$

Solution : In this case since coefficients of y and $\frac{dy}{dx}$ are both zero, we take

P.I. as $\frac{6}{2}x^2$.   i.e. $y_p = 3x^2$.

You can find the C.F. and write the general solution of this differential equation.

Case 2 : When *X* is not a constant.

When, in a given differential equation with constant coefficient , f (D)y = x, where expression X on the right hand side a function of D, we can find its particular integral using the method of undetermined coefficients.

This method is applicable to differential equations of the type:

$$a_0 y'' + a_1 y' + a_2 y = x$$

Where X is such that the form of a particular solution y_p may be guessed. The process is as follows : we assume for y_p an expression similar to that of X containing unknown coefficient. This expression is a combination of terms involved in X and its possible derivatives. The unknown coefficients are then determined by inserting y_p and its derivatives in the given differential equation. We illustrate this method with the help of some examples. We also help to illustrate the nature of solutions to second-order linear differential equations

4.5 BEHAVIOUR OF SOLUTIONS

Example 7: Solve the differential equation

$$y'' + y' + y = 5e^{4x} \quad (10)$$

Solution We find only the particular integral and leave the C.F part to the reader. Here X = 5 e^{4x}. Since all possible derivatives of

e^{4x} involve e^{4x}, we take as $y_p = ke^{4x}$. Therefore,

$y_P' = 4ke^{4x}, y_P'' = 16ke^{4x}$

Substituting in (8) , we get

$16ke^{4x} + 4ke^{4x} + ke^{4x} = 5e^{4x}$

$\Rightarrow 21ke^{4x} = 5e^{4x}$

$\Rightarrow 21k = 5, \Rightarrow k = \dfrac{5}{21}$

$\therefore y_P = \dfrac{5}{21}e^{4x}$

The general solution of (10) is obtained, as usual, by adding C.F. and P.I.

Example 8 : Solve the differential equation

$$y'' + 2y' + 2y = e^x + \sin x \qquad (11)$$

Solution The C.F. in this case is

C.F. = $e^{-x}(c_1\cos x + C_1 \sin x)$

To find the P.I., we note that in this case $X = e^x + \sin x$. Since derivative of e^x is e^x itself and that of Sin x is Cos x, we include e^x , Sin x and Cos x in the particular integral. The P.I. is taken as $y_p = a\ e^x + b \sin x + c \cos x$. Substituting in (11), we get

$(ae^x - b\sin x - c\cos x) + 2(ae^x + b\cos x - c\sin x + 2(ae^x + b\sin x + c\cos x)$

$= e^x + \sin x$

$\Rightarrow 5ae^x + (b - 2c)\sin x + (c + 2b)\cos x = e^x + \sin x$

$\Rightarrow 5a = 1, b - 2c = 1, c + 2b = 0$

$\Rightarrow a = \dfrac{1}{5}, b = \dfrac{1}{5}, c = \dfrac{-2}{5}$

$\therefore y_P = \dfrac{1}{5}e^x + \dfrac{1}{5}\sin x - \dfrac{2}{5}\cos x$

and hence the general solution of (11) is

y = C.F. + P.I.

$= e^x(c_1 \cos x + c_2 \sin x) + \dfrac{1}{5}e^x + \dfrac{1}{5}\sin x - \dfrac{2}{5}\cos x$

Check Your Progress 2

1) What is the difference between a homogeneous and non-homogeneous second-degree differential equation?

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2) Solve the following differential equations:

   i) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 1 + x^2$

   ii) $y'' + 2y' + y = x - e^x$

   iii) $y'' + y + 1 = e^{2x}$

   iv) $y'' + 5y' + 3y = 6t^2 - t - 1$

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4.6 ECONOMIC APPLICATIONS

In this section we discuss a couple of applications to economics of second-order differential equations. Let us begin with an application to consumer demand.

4.6.1 Consumer Demand

In some markets, the consumers may try to anticipate trends in the price prevailing. They base their demand for the good in question not only on the current market price, but also on how fast the price has been changing. Suppose that the price p is a function of time denoted by t, that is *p(t)*. The first-derivative $p'(t)$ denotes the rate of change of price, and the second derivative $p''(t)$ or $\ddot{p}$ shows the rate at with $p'(t)$ or $\dot{p}$ changes. If $\dot{p} > 0$ but $\ddot{p} < 0$, the buyers will see that although the price is increasing, it is doing so at a diminishing rate, and they may have the foresight that the prices will level off in due course.

Let us bring in both demand and supply factors in the analysis of the market. We consider only price p and do not introduce a separate notation like p^e for expected prices. Suppose we put not only the price of the good, but also first- and second- derivatives of prices with respect to time as arguments in both the demand and supply functions. Let quantity supplied be denoted Q_s and be equal to

$Q_s = f[p(t)p'(t), p''(t)]$ and similarly quantity demanded Q_d be written

$Q_d = g[p(t)p'(t), p''(t)]$

If we take linear forms of supply and demand we have supply and demand functions of the form

$$Q_d = \alpha - \beta p + k\dot{p} + l\ddot{p}$$

$$Q_s = -\gamma + \delta p + u\dot{p} + v\ddot{p}$$

Here we assume $\alpha, \beta, \gamma, \delta > 0$ but place no restrictions on the signs of k,l,u and v in order to take into account the consumer's expectations about price. If k > 0, then a rising price will make Q_d rise.. On the other hand l depicts the consumer's expectations about the trend in rise of prices. Similarly u and v depict expectations about prices and change in prices on the part of sellers. If we assume the market clears at a point in time, as well as over time, we get the equilibrium condition as

$$\ddot{p} + \frac{k}{l}\dot{p} - \frac{\beta + \delta}{l} p = -\frac{\alpha + \gamma}{l}$$

This is a linear second-order differential equation whose solution and the time path of price may be investigated.

4.6.2 Inflation and Unemployment

Initially an inverse relation was posited in a macroeconomic context between the rate of *growth* of money wages and the rate of unemployment. If w is the money wage and the rate of *growth* of money wage is $\frac{\dot{W}}{W} = w$ and the rate of unemployment is U, then there was held to be a negative relation between w and U:

$$w = f(U) \text{ with } f'(U) < 0$$

In course of time there was thought to be a negative relation between the rate of unemployment and the rate of inflation. If P denotes price then the rate of growth of P is denoted by $\frac{\dot{P}}{P} = p$. This p is the rate of inflation. If we suppose there is mark-up pricing in the sense of a rise in wages leading to a rise in prices, then a rise in w will put upward pressure on p. This means a rise in money wages will lead to inflation. On the other hand a rise in labour efficiency (denoted by E) will lead to a downward pressure on p. We can thus write

$$p = w - E$$

if we take a linear version of the equation $w = f(U)$ and substitute *f(U)* in the equation linking p and w we get

$$p = \alpha - \beta U - E \text{ with } \alpha,\ \beta > 0$$

Now if we denote expected inflation by p^e, then the *expectations-augmented* Phillips curve, as suggested by economists like Milton Friedman and Edmund Phelps, should be written (using our notations) as

$$w = f(U) + hp^e \text{ with } 0 < h \le 1$$

The above equation says that if people have inflationary expectations over a period of time, then rate of increase in money wages will be influenced by these expectations. And since we have earlier seen that w influences p (the rate of inflation), we can write the *expectations augmented* Phillips curve as

$$p = \alpha - \beta U - E + hp^{e.}$$

We need to have a theory about how these expectations are formed. Friedman had hypothesized an adaptive expectations formulation of the type

$$\frac{dp^e}{dt} = \phi\left(p - p^e\right) \;\; 0 < \phi \leq 1$$

The above says that if p exceeds p^e, then there will be an upward revision in the expected rate of inflation.

Since p or p^e are first derivatives of time then the *expectations-augmented* Phillips curve equation, coupled with the adaptve expectations equation, will give rise to a linear second-order differential equation, whose solution we can investigate. The solution will tell us something about the time path of the general price level.

Check Your Progress 3

1) Let the demand and supply be

   $Q_d = 9 - \dot{P} + 3\ddot{P}$

   $Q_s = -1 + 4P - \dot{P} + 5\ddot{P}$

   with $P(0) = 4$ and $\dot{P}(0) = 4$

   Find the price path, assuming the market clears at every point of time. Also determine if the time is convergent.

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2) Present a simple model of the interaction of unemployment and inflation, using second-order differential equations.

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4.7 LET US SUM UP

This Unit was the second and the final of our discussion on differential equations. The unit took up the study of differential equations where the earlier unit had stopped. We studied about second-order differential equations. We began the unit by recalling the idea of a differential equation, and the significance of second-order derivatives in differential equations. The unit discussed first the basic formulation of a linear second-order differential equation. Then we went on t see what a linear second-order differential equation looks like if it has constant coefficients. Subsequently the unit talked about homogeneous equations as well as non-homogeneous second-order linear differential equations. Following this the unit discussed the nature of the solution of second-order linear differential equations. Finally the unit discussed two economic applications of linear second-order differential equations, one about expectations in a simple microeconomic demand-supply analysis and the second about the expectations augmented Phillips curve in a macroeconomic context.

4.8 ANSWERS/HINTS TO CHECK YOUR PROGRESS EXERCISES

Check Your Progress 1

1) Read sections 4.2 and 4.3 and answer.

2) Read section 4.3 and answer.

Check Your Progress 2

1) Read section 4.4 and answer.

2) Read sections 4.4 and 4.5 and answer.

Check Your Progress 3

1) See subsection 4.6.1 and answer.

2) See subsection 4.6.2 and answer.