BLOCK 3

LINEAR ALGEBRA

BLOCK 3 INTRODUCTION

The title of Block 3 in this course is **Linear Algebra**. As the name suggests this deals with topics having to do with linear equations and equation systems, and their manipulations. This block has three units. Unit 5 titled **Vectors and Vector Spaces**, discusses vectors, which are ordered n-tuples of numbers. The unit discusses geometric and algebraic properties of vectors. The next unit, unit 6 is titled **Matrices and Determinants**. Matrices are the basis of linear algebra. Determinants are formed from a particular type of matrix called square matrices, and are single numbers. The final unit in Block 3, unit 7, titled **Linear Economic Models**, brings together the concepts and techniques discussed in units 5 and 6, and gives a variety of economic applications of linear algebra in economics.

UNIT 5 VECTORS AND VECTOR SPACES

Structure

5.0 AIMS AND OBJECTIVES

This Unit is the first of three units in Block 3; this Block is concerned with a branch of mathematics called linear algebra. The present unit lays the foundation for the study of linear algebra, and deals with mathematical objects called vectors. After going through the unit, you will be able to:

- Define a vector and distinguish a vector and a scalar;
- Show how vectors can be represented geometrically;
- Carry out algebraic operations on vectors like addition, subtraction, and multiplication of a vector by a scalar;
- Describe the inner product of two vectors;
- Explain the ideas of linear combination and linear dependence of vectors;
- Describe norm and orthogonality of vectors; and
- Discuss the concept of vector spaces.

5.1 INTRODUCTION

This unit begins the study of linear algebra, that is, the study of linear equations, particularly simultaneous linear equations. Linear algebra is the topic of study for you in this block. To study linear algebra we make use of certain mathematical concepts like matrices. Matrices you will study in detail in the next unit. At this stage let us just say that a matrix (the plural of matrix is matrices) is a rectangular array, usually of real numbers. These numbers are arranged in rows (horizontally) and columns (vertically). If we take a single row or a single column of numbers, it is called a vector. That is, a matrix consisting of only a single row or a single column is called a vector.

Let us leave aside matrices for a moment (you just learnt the relationship of vectors and matrices) and study vectors independently. What is a vector? What algebraic operations can be performed on vectors? How do we depict a vector diametrically? What do we understand by a 'vector space'? Furthermore, how are vectors useful in the study of economics? What applications do vectors find in economics? All these we study in this unit.

In the next section, we begin our study of vectors by discussing the definition and meaning of vectors. We mention some specific types of vectors and also see how a vector can be represented diagrammatically. In section 5.3, we study algebraic operations on vectors like addition and subtraction of vectors. We also see what happens when a vector is multiplied by a scalar. In the subsequent section, section 5.4, the unit discusses the idea of linear combination; the idea of linear combination is used to study the linear dependence and independence of vectors. Section 5.5 discusses the product of vectors. This is different from the multiplication of a vector by a scalar, which you study in section 5.3. Section 5.5 discusses the multiplication of a vector by another vector. This is called the inner product of vectors. The next section discusses the concept of a vector 'space'. You will study what is meant by a 'space' in this context, and what a space of vectors means. The final section discusses the idea of the 'length' of a vector (called the norm of a vector) and the situation when two vectors (when depicted geometrically) are perpendicular to each other (called orthogonal vectors). The unit discusses throughout applications of vectors in economics, and relates vectors to concepts that you come across in your study of other courses in economics, like microeconomics and macroeconomics

5.2 VECTORS

Let us go directly to the definition of a vector. A vector is an ordered set of numbers. A vector is characterized not only by the elements contained in it but also by the order in which the elements appear.

5.2.1 Meaning of Vectors

Suppose a consumer is consuming two goods: apple and banana. Suppose we depict the various combination of the consumption of two goods by a pair of numbers, with the first number showing the number of apples and the second number the number of bananas. Suppose the consumer consumes 5 apples and 3 bananas, we can show this as (5,3). So we have shown this by a pair of numbers separated by a comma within parentheses. If we see a pair of numbers like (3,7) in this context, then that would tell us that the consumer has a combination of 3 apples and 7 bananas. A vector is defined as an ordered set of numbers, parameters, or variables. For our purposes, we will consider vectors as ordered set of real numbers. Vectors are usually put in parentheses or square brackets, and we shall use square brackets [ ] to denote vectors. The individual real numbers in vectors are called the components of the vector. Within the brackets, the components are not usually separated by commas, unlike when we depict an ordered pair or ordered n-tuple.

We can use a simple example to elucidate this definition of a vector. Assume

that a consumer buys x_1 and y_1 units of two goods x and y, respectively. We can write these

units either in a column as

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

or in a row as

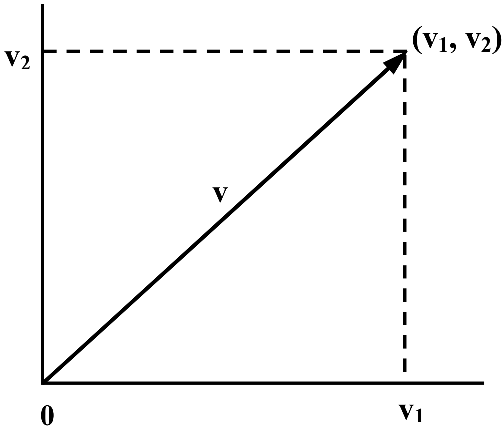
$$[x_1, y_1]$$

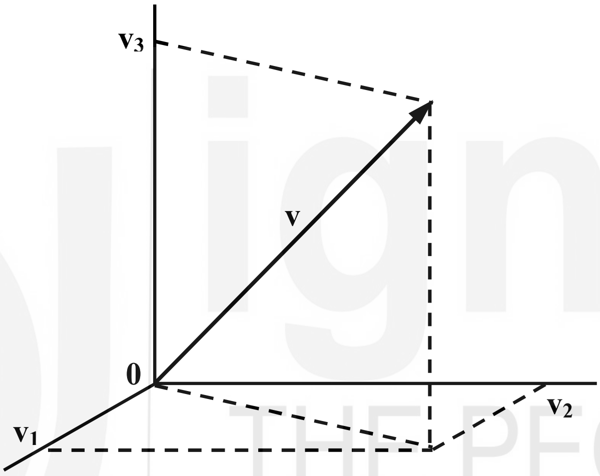
. This column or row of numbers is called a vector. If the vector appears in the form of a column, it is called a *column vector*; if it appears in the form of a row, it is called a *row vector*. Each entry inside the brackets is called a *component* of the vector. Column vectors are generally denoted by bold, lowercase letters such as **u**, **v**, etc. And row vectors are denoted by notations **u'**, **v'**, etc. Please note that the symbol $^{/}$ is also used to denote derivatives in the context of calculus. Please do not confuse between the two**.** Both of the above vectors have two components each, and we say that each is a *2-vector* or each has *dimension* 2. If a vector has three components it would be called a 3-vector, and if a vector has n components, then we can say that it is an *n-vector* or it has dimension *n*. Suppose there are n goods that the consumer is consuming, denoted $(x_1, x_2, ..., x_n)$. This is an n-vector. We can denote it as $\mathbf{x}^{/}$. Here $\mathbf{x}^{/} = (x_1, x_2, ..., x_n)$. It can also be denoted, as a column vector as $\mathbf{x} =$

$$\begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix}$$

Similarly, if the prices of the n goods are $[p_1, p_2, ..., p_n]$ where p_i is the price of the i^{th} good (where i = 1,2,3,…,n), then $[p_1, p_2, ..., p_n]$ can be represented as a (row) vector $\mathbf{p}^{/} = [p_1, p_2, ..., p_n]$. It can also be represented as a column vector.

Those of you who were acquainted with science in school may recall the definition that vectors are objects that have magnitude as well as direction, while scalars have only magnitudes. We will discuss vectors on these lines when we come to the geometric representation of vectors.





5.2.2 Types of Vectors

There are various types of vectors. One type is what is called a *unit* vector. A unit 2-vector is of the form $\mathbf{u}' = [1\ 0]$ or $\mathbf{v}' = [0\ 1]$. Geometrically they coincide with the x-axis and y-axis of the Cartesian plane. We will discuss this further in the next sub-section.

Another type of vectors is called *null* vector ,or *zero* vector. This has only zeroes as components for a 2-vector, a null vector would be

$\mathbf{u}' = [0, 0]$

A third type of vectors is called *equal* vectors. Two vectors are said to be equal vectors only if they have the same dimension and if their corresponding components are equal. For example,

the two row 2-vectors $\mathbf{u'} = [u_1\ u_2]$ and $\mathbf{v'} = [v_1\ v_2]$ are said to be equal vectors only if

$u_1 = v_1$ and $u_2 = v_2$.

The fourth type consists of vectors called *like* vectors and *unlike* vectors. Vectors having the same direction are called like vectors, and vectors with opposite directions are called unlike vectors. Another type of vector is *collinear* vectors. If two vectors lie on the same line or on parallel lines, they are called collinear vectors. Finally, *coplanar* vectors are vectors which lie on the same or parallel planes. All these types of vectors you can appreciate when you do the geometric representation of vectors

5.2.3 Geometric Representation of Vectors

In this subsection we look at how vectors can be represented geometrically. We have to limit ourselves to 2-vectors or 3-vectors since we can draw diagrams of these. In fact, for vectors with four or more components, it is not possible to draw diagrams, and you have to 'visualise' the geometric representation of these in an abstract manner.

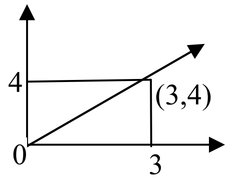
Let us begin with 2-vectors. Suppose we have vectors $\mathbf{u}' = [x_1\ y_1]$ and $\mathbf{v}' = [x_2\ y_2]$. Let $x_1 = 2$,

$y_1 = 0$, $x_2 = 0$ and $y_2 = 0$. Then we can write the vectors as $\mathbf{u}' = [2\ \ 0]$ and $\mathbf{v}' = [0\ \ \ 2]$. Then the two vectors can be depicted graphically as

$\mathbf{u}'$

$\mathbf{v}'$

Suppose we have a vector w = [3 4]. This is a 2-vector so we can make use of the x axis and y-axis to depict the vector. So start at the origin of the Cartesian plane (point (0,0)). Count off three units to the right to reach point 3 on the x-axis. From this point, count off 4 units in the vertical direction (parallel to the y-axis). You will reach the point (3,4). This point on the Cartesian plane can be considered the vector. Alternatively, an arrow originating at (0,0) and going till (3,4) can also be considered the vector. In the above, to reach the point (3,4), you could have gone first in an upward direction for four units on the y-axis, and then moved rightward for three units parallel to the x-axis. You would still have reached the point (3, 4)



So any 2-vector can also be seen as a point in the Cartesian plane (an ordered pair). Similarly, a 3-vector can be viewed as an ordered triple. Extending the analogy, an n-vector can be viewed as an n-tuple.

5.3 ALGEBRAIC OPERATIONS ON VECTORS

In the previous section, we understood the concept of vectors. We also looked at the types of vectors and saw how vectors can be represented geometrically. In this section, we take a look how we can undertake algebraic operations on vectors like addition, subtraction, multiplication, and so on.

5.3.1 Addition of vectors

We shall addition of vectors here. here. Suppose that we have two row 2-vectors: **u'** = [1 2] and **v' =** [2 1]. Then the sum **u'** + **v'**, called the *addition of vectors*, is obtained by adding each

component of **u'** to the corresponding component of **v'.** We have **u'** + **v'** = [1+2 2+1] = [3 3]. Please bear in mind that for addition of two vectors, the number of components in the given vectors (or the dimension of the vectors) has to be the same. We cannot add, say, a 3-vector with a 2-vector.

There are certain properties of vector addition

Property I. Vector addition is commutative. For any two vectors **u** and **v**, **u** + **v** =**v** + **u**.

Property II. Vector addition is associative. For any three vectors **u**, **v**, and **z**, (**u** + **v**) + **z** = **u** + (**v** + **z**).

Property III. Existence of *additive identity*. For any vector **u**, there exists a zero vector **0**, called the additive identity, such that **u** + **0** = **u**.

Property IV. Existence of *additive inverse*. For any vector **u**, there exists a vector called −**u**, such that **u** + (−**u**) = **0**.

5.3.2 Subtraction of Vectors

Assume that we have two vectors: **u'** = [1 4] and **v'** =[3 2]. The difference of these two vectors is called the subtraction of vectors. This is obtained by subtracting each component of **v'** from the corresponding component of **u'**. One thing you have to remember is that subtracting one vector from another vector is the same as adding the negative of the second vector to the first vector. Thus

u' - v' = **u'** +(- **v'**)

In the above example, $\mathbf{u}' - \mathbf{v}' = \mathbf{u}' + (-\mathbf{v}')$

[1 4] – [3 2] = [1 4] + [-3 -2] = [1-3 2- 4] = [-2 2]

Note that $\mathbf{u}' + (-\mathbf{u}')$ or $\mathbf{v}' + (-\mathbf{v}')$ will give rise to a zero vector

5.3.3 Multiplication of a Vector by a Scalar

The last algebraic operation we shall deal with here is the multiplication of a vector by a scalar. A scalar, as we have seen, is a single real number. When we multiply a vector by the scalar, we multiply each component of the vector by the scalar. Suppose r is a scalar and $\mathbf{u}'$ is a vector. Multiplying this vector by r, we get a new vector $r\mathbf{u}'$ which is r times the old vector. Suppose we have a vector [3 2] and we have a scalar 3, the multiplication of the vector with this scalar gives rise to a new vector [9 6]. Note that the direction of the newly generated vector will be reversed if the scalar r is negative. Moreover, scalar multiplication can be performed in combination with vector addition and vector subtraction. This means that given two vectors of the same dimension $\mathbf{u}'$and $\mathbf{v}'$ and the scalar r, $r(\mathbf{u}' \pm \mathbf{v}') = r\,\mathbf{u}' \pm r\mathbf{v}'$.

We now state some properties of scalar multiplication (multiplication of vectors with scalars).

Property 1. Scalar multiplication is associative. If there are scalar $r1$ and $r2$ and if $\mathbf{u}$ is a vector, then $(r1 \times r2)\,\mathbf{u} = r1 \times (r2 \times \mathbf{u})$.

Property 2. Scalar multiplication is distributive. If $r1$ and $r2$ are two scalars and $\mathbf{u}$ and $\mathbf{v}$ are two vectors of the same direction, then $(r1 + r2)\,\mathbf{u} = r1 \times \mathbf{u} + r2 \times \mathbf{u}$, and $r1 \times (\mathbf{u} + \mathbf{v}) = r1 \times \mathbf{u} + r1 \times \mathbf{v}$.

Property 3. Existence of *multiplicative identity*. For any vector $\mathbf{u}$, $1 \times \mathbf{u} = \mathbf{u}$.

Check Your Progress 1

1) What is the difference between a scalar and a vector? Can you think of the relationship between a set (which you studied in BECC 102) and a vector?

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2) If $u' = [3,8,7]$, $v' = [1, 6, 5]$, compute:

   i) $u' + 3v$

   ii) $5u - 2v$

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5.4 LINEAR COMBINATION AND LINEAR DEPENDENCE OF VECTORS

In this section we discuss an important concept derived from the multiplication of vectors by scalars. This concept is called linear combination of vectors. We use this concept to talk about linear dependence of vectors.

Suppose that we have n n-vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n$; and n scalars $k_1, k_2, k_3, \ldots, k_n$. Then we can generate a new n-vector by multiplying the components of each original vector by the corresponding scalars. The new vector generated in this way is called a *linear combination* of the original vectors and is given by $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + \cdots + k_n\mathbf{v}_n$.

Let us consider an example. Suppose we have two 2-vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and scalars k_1 and k_2

Then the linear combination of the given vectors with respect to the given scalars can be written as $k_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + k_2\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} k_1 \times 1 + k_2 \times 2 \\ k_1 \times 2 + k_2 \times 1 \end{bmatrix}$

Building on the concept of linear combination, the next concept we discuss here is the linear dependence of vectors or the linear independence of vectors. Suppose that we have n n-vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n$; and n scalars $k_1, k_2, k_3, \ldots, k_n$, not all zeros. Given these vectors and scalars, if we can write $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + \cdots + k_n\mathbf{v}_n = \mathbf{0}$, then the vectors are said to be linearly dependent. Here $\mathbf{0}$ is a null vector. On the other hand, if there do not exist scalars $k_1, k_2, k_3, \ldots, k_n$, again not all zeros, such that $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + \cdots + k_n\mathbf{v}_n = \mathbf{0}$, then the vectors are said to be linearly independent.

5.5 INNER PRODUCT OF VECTORS

In this section we look at the multiplication of vectors. When one vector is multiplied by another, the result is called inner product, or dot product, or scalar product. Scalar product is different from the multiplication of a vector by a scalar. In scalar product, a vector is multiplied by another vector and the result is a scalar. Here we multiply two vectors but the result is a scalar. That is the reason it is called a scalar product.

Suppose that we have three goods. Suppose that we consider two 3-vectors **q'** = [q_1 q_2 q_3] and **p'** = [p_1 p_2 p_3], where q_i denotes the quantity of the ith good purchased by a consumer and p_i represents the price of the ith good, and i = 1,2, 3. When we multiply the quantity of the ith good in **q'** by the price of the ith good in **p**, we obtain the expenditure on the ith good. Then, when we sum the amounts spent on the three goods, we obtain the total expenditure of the consumer on these three goods. That is, the total expenditure will be equal to $\sum_{i=1}^{3} q_i p_i$

This is also the result of multiplying the vector **q' by** the vector **p**. This product is called the inner product, or scalar product, or dot product, or simply the product of vectors **q'** and **p**. Let us now generalize the last result. Suppose that we have two n-vectors **u'** = [u_1 u_2 . . . u_n] and **v'** = [v_1 v_2 . . . v_n]. Then the scalar product of **u'** and **v**, denoted by **u'.v**, is defined as:

$$\mathbf{u'}.\mathbf{v} = [u_1.v_1 + u_2.v_2 + u_3.v_3 + \cdots + u_n.v_n]$$

$$= \sum_{i=1}^{n} u_i v_i$$

Notice that the product in the above equation is not a vector but a real number. This is the reason why it is called a scalar product. Also notice that in order for the scalar product to exist, **u'** and **v'** must be of the same dimension. We list below three important properties of the scalar product of any three n-vectors **u**, **v**, and **z**, and the scalar k_1:

Property a. Scalar product is commutative: **u'.v** = **v'.u**.

Property b. Scalar product is distributive: **u'**.(**v** + **z**) = **u'.v** + **u'.z**.

Property c. Scalar product is associative: (k_1.**u'**).**v** = **u'**.(k_1.**v**) = k_1.(**u'**.**v**).

Check Your Progress 2

1) If

$u' = [5,1,3], v' = [3,1,-1], w' = [7,5,8], x' = [x_1, x_2, x_3]$, compute:

(i) $u'v$

$(ii) w'x$

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2) Supposing we index an input used by a firm for production by i, i = 1,2,3,…,n. Let the price of input is w_i. How can you depict the total input cost as a product of vectors? Is the product of vectors a scalar or a vector?

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3) When are two vectors linearly independent?

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5.6 VECTOR SPACES

We have looked at the concept of vectors, and how they are different from scalars. You have also studied about sets of real numbers and the notion of Cartesian product, as well as coordinate geometry in course BECC 102. We can define an *n-dimensional vector space* (*n-space* or $\Re^n$) as the set of all the *n*-vectors generated through linear combinations of *n* independent *n*-vectors, although it is not possible to show geometrically. Notice that, since each point in an *n*-space is an ordered *n-tuple*, each *n*-vector represents a point in the *n*-space. Since the *n*-space or $\Re^n$ contains all the real numbers, it is also known as *Euclidian n-space*.

Vector spaces can be thought of as sets which have vectors as elements. Vector spaces are real Euclidean spaces with an additional structure. This structure is derived from the various properties of vectors that you have learnt in this unit. These properties are mentioned below. Let us denote $\Re^n$, by V with V standing for 'vector space. For all vectors u,v,w in V, and all scalars k, m in R (the set of real numbers)

1) **u** + **v** is in V, whenever **u** and **v** are in V

2) **u**+ **v** = **v** + **u**

3) **u** + (**v** + **w**) = (**u** + **v**) + **w**

4) There exists an element **0** in V such that, for all **v** in V, **v** + **0** = **v**

5) For every **v** in V, there is an element **w** (where w = - **v**) such that **v + w = 0**

6) k.**v** is in V, whenever **v** is in V

7) k. (**u** + **v**) = k.**u** + k.**v**

8) (k + m). **u** =k.**u** +m . **u**

9) k . (m . **u**) = m . (k . **u**)

10) 1.**u** = **u**

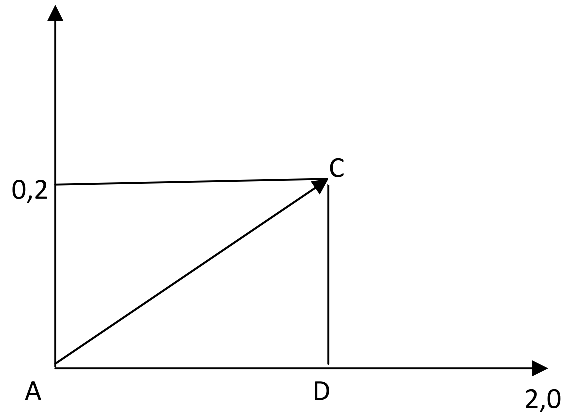
5.7 NORM AND ORTHOGONALITY OF VECTORS

Now we turn to a basic aspect of vectors. In the initial sections of this unit, you learnt that vectors have magnitude and direction. Well, if they have direction, we need to know how long a vector is. We also are interested in finding out the distance between two vectors. Finally, since vectors have length, two vectors may lie at an angle to each other. So having encountered the definition of vectors, and learnt how to carry out algebraic operations on vectors, having learnt about linear combinations and linear dependence, and about vector spaces, you will now be presented a discussion on the length of vectors and the distance and angle between them.

Let us begin with the concept of the length of a vector. To find the length, we look at the end-point of the vector, and the beginning point of the vector, and calculate the difference. Let us consider a simple example. Let the origin of a vector be [0 0]. We have, in this unit proceeded to discuss vectors usually keeping the origin of the vector(s) at [0,0] But of course this need not be the case always. In this example also, let us, as we said, keep the origin of the vector at (0,0). Let this vector be along the y axis (say for 3 units). So the end-point of the vector will be [3,0]. The length of the vector can be found by subtracting its end point from its initial point. In this example the length will be 3 – 0 = 3. This was how the length of a vector is obtained for a 1-space.

Let us now consider the length of a vector in a 2-space or the length of a row 2-vector.

For this we can use the geometric form of the row 2-vector **u'** = [2 2]. We can also write it as the ordered pair (2,2). This is shown in the diagram below



To find out the length of the vector (2,2). We find the origin is at point A(0,0) and the terminal point is at point C (2,2). Notice that the vector makes a right-angled *triangle* CAD. We want to find the length of the vector **u'** = [2 2], that is, the length of the line segment AC or the *distance* from A to C. We can apply *Pythagoras' theorem* to find this distance.

We apply this theorem in the above diagram to find $(AC)^2 = (AD)^2 + (DC)^2$

So the length of AC (or the distance from A to C), denoted $\|AC\|$, is,

$$\|AC\| = \sqrt{(AD^2) + (DC)^2}$$

On similar lines , we can find the lengths of vectors in 3- space (or row 3-vectors) and in *n*-space (or row *n*-vectors). Suppose we have a row 3-vector and a row *n*-vectors **u'** = [x_1 y_1 z_1] and **v'** = [x_1 x_2 . . . x_n], respectively. Then the lengths of these vectors can be given as

$\|u'\| = \sqrt{x_1^2 + y_1^2 + z_1^2}$ and

$\|v'\| = \sqrt{x_1^2 + x_2^2 + x_3^2 + \ldots x_n^2}$, respectively. Take this last vector. Now recall the definition of inner product from the last section. Let us find the inner product of **v'** with itself. We want to find **v'. v**. we know this will be $x_1^2 + x_2^2 + x_3^2 + \ldots x_n^2$.

Looking carefully, we find that $\|v'\| = \sqrt{\mathrm{v'.v}}$

If we have two vectors **u** and **v**, the distance between them is given by $\|u - v\| = \sqrt{(u - v) \bullet (u - v)}$

Any two n-component vectors in $\Re^n$ determine a plane. We can measure the angle θ between **u** and **v**. there is a link between the inner product of **u** and **v,** the length between the two and the angle between them. the result is as follows:

Let **u** and **v** be two vectors in $\Re^n$. Let θ be the angle between them. Then

$$\mathbf{u.v} = \|u\|\|v\|\cos\theta$$

$\cos\theta > 0$ if θ is acute, < 0 if θ is obtuse, and = 0 if θ is a right angle. Two vectors which are at right-angles to each other, that is for which θ =0, are called orthogonal vectors.

Check Your Progress 3

1) What do you understand by:

   i) Orthogonality of two vectors

   ii) Norm of a vector?

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2) Find the length of

   i) $v' = [1, 2]$ (ii) $w' = [1, 2, -2]$

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3) A consumer consumes only apple and oranges. Can you think how you would use the concept of vectors and vector space to depict the consumption of this consumer?

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5.8 LET US SUM UP

This unit set the ball rolling on the discussion on linear algebra. You came to know about vectors which can be seen both as an ordered collection of real numbers, and as objects having both magnitude and direction. Vectors are also a special type of matrices, about which you will study in the next unit. Since vectors and matrices are useful in depiction of linear equations and functions, the study of vectors and matrices is called linear algebra.

The unit began with the definition of vectors. We saw that vectors are ordered sets of real numbers. The individual real numbers are called components of the vector. You learnt how to represent vectors through diagrams. Next, the unit discussed some algebraic operations on vectors, namely addition of vectors, subtraction of vectors and multiplication of a vector by a scalar. Some properties of vector addition as well as of multiplication of a vector by a scalar were discussed.

Following this the unit went on to discuss a construct arising out of multiplication of a vector by a scalar, namely, linear combination. We saw that linear combinations of a given set of vectors is obtained by multiplying the component of each vector by the corresponding scalar, and then by adding the different products. This idea of linear combination was used to discuss the idea of linear dependence, which basically means that for a given set of vectors, the linear combination equals zero.

After this the unit discussed one more operation on vectors, an operation that had not been discussed in the section on algebraic operations on vectors. This is the multiplication of a vector by another vector, called the inner product, or dot product, or scalar product. Following this

The unit discussed the idea of a vector space. In this section you were familiarized with the relationship between real numbers, sets of points, and spaces of vectors. You were able to relate the material in this section with the material on sets, Cartesian products, real space as well as the unit on coordinate geometry that you studied in course BECC 102.

Finally, the unit discussed how we can give a magnitude of the length of a vector. This section also dealt with the distance between two vectors. We saw how this concept is similar to the concept of distance that you learnt about when studying the unit on coordinate geometry in course BECC 102. Finally the unit talked about the angle between two vectors, and the special situation where two vectors are perpendicular to one another; that is, they are orthogonal to each other.

5.9 ANSWERS/HINTS TO CHECK YOUR PROGRESS EXERCISES

Check Your Progress 1

1) See subsection 5.2.1 and answer.
2) See section 5.3 and answer.

Check Your Progress 2

1) See section 5.5 and answer.
2) See section 5.5 and answer.
3) See section 5.4 and answer.

Check Your Progress 3

1) See section 5.7 and answer.
2) See section 5.7 and answer.
3) See section 5.6 and answer.

UNIT 6 MATRICES AND DETERMINANTS

Structure

6.0 OBJECTIVES

After reading this unit you will be able to:

- Explain the concept of a matrix;
- Perform operations of matrix addition, subtraction and multiplication;
- Describe the properties of some very useful special matrices;
- Define the concept of a determinants;
- Introduce the notions of minor and cofactor and, evaluate a higher (more than second) order determinant; and
- Compute the inverse of a matrix.

6.1 INTRODUCTION

In mathematics, we are often concerned with operations involving blocks of numbers and variables. Matrix Algebra is a very powerful but convenient way to handle such situations. As the term matrix suggests, it refers to an array (combination of rows and columns).The building blocks of matrix algebra are arrays of numbers and variables. Let us now be familiar with the concepts and notations of matrix algebra by considering the matrix representation of the following simultaneous equation system

$$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1$$
$$a_{12}x_2 + a_{22}x_2 + ... + a_{2n}x_n = b_2$$
$$...$$
$$a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m$$

In the above equation system, there are m equations involving n variables represented by x's, m x n coefficients represented by a's and n constants represented by b's. The coefficients and the constants together form the parameters of the equation system. Let us now see how these parameters and variables can be represented by different matrices or arrays. If **A** is the array for coefficients, **x** is the array for variables and **b** is the array for constants, then

$$\mathbf{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

In **A,** there are m rows and n columns. The first row consists of the coefficients attached to the variables in the first equation of the given equation system. The second row consists of the coefficients figuring in the second equation and so on. Finally, the last row consists of the coefficients attached to the variables in the last equation of the system.

The array **x** has n rows and one column. In the first row, we have the variable x_1, in the second row we have x_2 and similarly in the last row, we have the variable x_n. The array **b** also consists of m rows and one column. In the first row, we have the constant b_1 of the first equation, the second row has the constant b_2 of the second equation and in the same fashion, the last row has the constant b_m of the last equation. Thus, the three matrices **A**, **x** and **b** contain all the ingredients of the simultaneous equation system under consideration. The number of rows and columns of a particular matrix constitutes its dimension or order. We have seen that the matrix **A** consists of m rows and n columns. Accordingly, the dimension of **A** is said to be (m x n) (pronounced as m-by-n) or, we say that A is an (m x n) matrix. Similarly, **x** is (n x 1) and **b** is (m x 1) matrix. A matrix, in which the number of rows and columns are equal, is called a square matrix.

6.2 MATRIX OPERATIONS

6.2.1 Addition and Subtraction of Matrices

Suppose we have two matrices A = $[a_{ij}]$ and B = $[b_{ij}]$ where a_{ij} and b_{ij} are the typical elements of A and B respectively. The matrix A + B is defined as A + B = $[a_{ij} + b_{ij}]$. What this means is that in order to get the matrix A + B we take the elements in the first row and first column of A and B and add them together to get the element in the first row and first column of the matrix A + B. To get the element in the second row and first column of A + B, we add a_{21} and b_{21} to get $a_{21}+b_{21}$. Now notice that in each case we are adding two numbers. And we know how to do that! In general, to get the element in the ith row and the jth column of A + B, we add a_{ij} and b_{ij} to get $a_{ij}+b_{ij}$. The only problem here is that unless the matrices A and Bare compatible in a particular sense, they cannot be added together to get A+B. Why? Suppose:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \\ -5 & 7 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 2 & 3 \\ 5 & 9 & 2 \end{bmatrix}$$

We want A + B. If we apply our rule we get:

$$\mathrm{A + B} = \begin{bmatrix} 1 & + & 0 & 2 & + & 2 & * & + & 3 \\ 3 & + & 5 & 0 & + & 9 & * & + & 2 \\ -5 & + & * & 7 & + & * & * & + & * \end{bmatrix}$$

Where the (*) symbol represents a missing element. Consider the element in the first row and third column of A+B. To get this we have to add a_{13} and b_{13}. We know b_{13}=3. But A is a (3×2) matrix. So it does not have an element a_{13}! We find that the matrix A+B in this case has several (*) elements for similar reasons. And hence, A+B is meaningless in this case. The point then is that for A+B to be defined, we require that A and B have the same dimension, ie., the same number of rows and the same number of columns.

In a similar manner one can define what one means by subtraction. Given any two matrixes A and B, the matrix A-B = $[a_{ij}-b_{ij}]$. Check that for this definition to be meaningful, A and B must have the same dimensions.

6.2.2 Multiplication of Matrices

This is a bit more complicated. So let us begin with a series of examples.

Example 1: Let A = $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and B = $\begin{bmatrix} 5 \\ 6 \end{bmatrix}$. Now if you asked a mathematician to give you the product matrix AB, he would tell you immediately: AB = $\begin{bmatrix} 17 \\ 39 \end{bmatrix}$. What strikes one immediately is that multiplying a (2×2) matrix by a (2×1) matrix yields a (2×1) matrix. Suppose we want to complicate things for him. Let us give him two matrices:

Example 2: A = $\begin{bmatrix} -5 & 1 & 0 \\ 0 & 2 & -7 \\ 3 & 0 & 5 \end{bmatrix}_{(3\times 3)}$ and B = $\begin{bmatrix} 1 & 0 \\ 0 & 7 \\ 5 & 1 \end{bmatrix}_{(3\times 2)}$

and ask for the product matrix AB. He may take a little more time but in the end he is going to claim:

$$AB = \begin{bmatrix} -5 & 7 \\ -35 & 7 \\ 28 & 5 \end{bmatrix}_{(3\times 2)}$$

But suppose we ask him to give us the product matrix BA. He is going to look at it for a second and will claim that this does not exist! All this is quite troublesome and there has to be some method in his madness. If we ask him to explain he is going to say:

$\mathbf{a_i b_j} = [a_{i1}\, b_{1j} + a_{i2}\, b_{2j} + \ldots\ldots.+ a_{in}\, b_{nj}]$

Let us check this. We turn to Example 1.

Look at A. It has 2 rows, so I = 1, 2; and two columns so j=1, 2. The matrix B has two rows, so i = 1, 2 and one column, so j = 1.

Let us try to compute the element in the first row and first column position of AB. If we follow his rule, then, we get:

$a_{11}\, b_{11} + a_{12}\, b_{21} = 1\times 5+2\times 6 = 17$

Then we want the element in the second row, first column position. This, according to his rule should be:

$a_{21}\, b_{11} + a_{22}\, b_{21} = 3\times 5+4\times 6 = 39$

Suppose we wanted to carry on and get the term in the first row second column position. Then we would have to compute the sum: $a_{11}\, b_{12} + a_{12}\, b_{22}$. But b_{12} and b_{22} do not exist. So we cannot compute this sum. Now we try to compute the element in the second row and second column position. Once again you end up with the same problem. The essential point is that if you have two matrices A and B, where A is of dimension $(m\times n)$ and B is of dimension $(p\times q)$ then AB is defined if and only if n = p and AB is of dimension $(m\times q)$.

Let us return to the introduction of this unit to check why.

$$\begin{bmatrix} 3 & 1 \\ 7 & -2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 20 \end{bmatrix} \qquad (1)$$

is equivalent to:

$3x_1 + x_2 = 0$ (i)

$7x_1 - 2x_2 = 5$ (ii)

$2x_1 + 5x_2 = 20$ (iii)

$$\begin{bmatrix} 3x_1 & + & x_2 \\ 7x_1 & - & 2x_2 \\ 2x_1 & + & 5x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 20 \end{bmatrix}$$

But two vectors are the same if they are equal element by element. So,

$3x_1 + x_2 = 0$ (i)

$7x_1 - 2x_2 = 5$ (ii)

$2x_1 + 5x_2 = 20$ (iii)

Check Your Progress 1

1) Let A = $\begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$ and B = $\begin{bmatrix} 1 & 3 \\ 5 & 8 \end{bmatrix}$

Find i) A+ B

ii) A – B

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2) Let A = $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and B = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$

Can you find A + B? Justify your answer.

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3) Let A = $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ and B = $\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$

Find AB and BA. Is AB = BA?

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4) Given two matrices:

A = $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and B = $\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$

Show that AB $\neq$ BA.

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6.3 SOME SPECIAL MATRICES

Now that we know what we mean by matrix multiplication, we can ask the following question. Given a matrix A, does there exist a matrix, let us call it I, such that:

IA = AI = A …..(4)

If such an I exists, then it would form the counterpart of the number 1 in arithmetic. This is so, because, we know that for any number x

1. x = x. 1 = x.

Now from the last section, we know that if A is of dimension $(m\times n)$ then for the product IA to exist, I must have m columns. Similarly, for AI to exist, I must have n rows. Now look at the product AI. This is of dimension $(m\times n)$. But we require AI = A. So it must be true that m = n. This leads us to conclude that, for the equality (4) to hold A must be a matrix of dimension $(m\times n)$, i.e., the number of rows of A must be the same as the number of columns of A. Such a matrix is called a **Square Matrix.** It follows that I must also be a square matrix of the same dimension as A. The question is: what does I look like? The matrix I, called the **Identity Matrix** has a special form: All the diagonal terms are **1,** and the off-diagonal terms are all zero:

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 & \ldots\ldots & 0 \\ 0 & 1 & 0 & 0 & \ldots\ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots\ldots & 0 \\ 0 & 0 & 0 & 1 & \ldots\ldots & 0 \\ : & : & : & : & :::::::: & : \\ : & : & : & : & :::::::: & : \\ 0 & 0 & 0 & 0 & \ldots\ldots & 1 \end{bmatrix}_{(m\times n)}$$

Another type of matrix that we encounter quite frequently, is a matrix whose elements are all zero. Such a matrix is called a **Null Matrix.** Unlike an Identity Matrix, a Null Matrix does not have to be square. Another important type of matrix is a symmetric matrix. Square matrices of this type, i.e., for which A = A' are called **Symmetric** matrices.

6.4 DETERMINANTS

6.4.1 Concept of a Determinant.

Suppose we have to solve the equations

$a_{11}x_{11} + a_{12}x_2 = b_1$

$a_{21}x_1 + a_{22}x_2 = b_2$

By the method of cross-multiplication we get

$$x_1 = \frac{a_{22}b_1 - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}}; \text{and } x_2 = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}}$$

The denominator $a_{11}a_{22} - a_{12}a_{21}$ is written as $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ and is called a **determinant** of the second order. It is a scalar and in the two by two cases we get the scalar by cross multiplication. A determinant has as many rows as columns. This particular determinant is of the second order because there are two rows and two columns. Two things should be kept in mind about determinants. First, a determinant is only of a square matrix. A non-square matrix cannot have a determinant. Secondly, a determinant is a scalar. It is a single number.

6.4.2 Minors and Co-factors

Consider the following square matrix:

$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 5 & 6 \\ 2 & 2 & 5 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Now consider the element a_{11}=1. Its location in the matrix is at the intersection of the first row and first column of A. If we wipe out the first row and the first column of A we are left with the matrix $M_{11} = \begin{bmatrix} 5 & 6 \\ 2 & 5 \end{bmatrix}$. The determinant of this matrix is called the **minor** of the element a_{11} and is written as $| M_{11} |$. Here $| M_{11}| = 13$. In general, the minor of the element a_{ij} of A is obtained by deleting the ith row and the jth column of A and then computing the determinant of the resulting matrix. So for computing the minor of the element a_{23} of A, we compute wiping out the second row and the third column of A.

$$| M_{23} | = \begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{bmatrix} = \begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix}. \text{ Then } | M_{23} | = 2-4 = -2.$$

Definition: (*Cofactor of a_{ij}*): Let A = $[a_{ij}]$ be any square matrix. If i + j is even then the **cofactor** of the element $a_{ij} = | M_{ij} |$. If I + j is odd then the cofactor of the element a_{ij} is

$- | M_{ij} |$. The cofactor of a_{ij} is written as $| c_{ij} |$.

Now in our example, $| M_{11}| = 13$. Notice that for the element a_{11}, i = 1 and j = 1, so i + j = 2. Hence, from our definition, $| c_{11}| = | M_{11}| = 13$. For the element a_{23}, i = 2. and j = 3, so i + j = 5, is odd. Then $| c_{23} | = -| M_{23} | = -(-2) = 2$.

Definition: (*Cofactor Matrix of A*): Let A = $[a_{ij}]$ be any square matrix. If we replace each element a_{ij} by its cofactor, the resultant matrix, designated by C = $[| c_{ij} |]$, is called the **cofactor matrix** of A.

In our example,

$| c_{11} | = 13 \quad | c_{12} | = -3 \quad | c_{13} | = -4$

$| c_{21} | = -2 \quad | c_{22} | = -3 \quad | c_{23} | = 2$

$| c_{31} | = -8 \quad | c_{32} | = 6 \quad | c_{33} | = -1$

$$\text{so that } C = \begin{bmatrix} 13 & -3 & -4 \\ -2 & -3 & 2 \\ -8 & 6 & -1 \end{bmatrix}$$

6.4.3 Computation of a Third-order Determinant

A determinant of the third order is

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

The rule used above is known as expansion by co-factors.

6.4. 4 Properties of Determinants

There are several rules for manipulating determinants. These should be learned from mathematical textbooks some of which are mentioned in the reading list. We mention here two properties which help us to establish a method of solution of simultaneous equations.

1) $$\begin{vmatrix} ka_{11} & a_{12} & \ldots & a_{1n} \\ ka_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ ka_{n1} & a_{n2} & \ldots. & a_{nn} \end{vmatrix} = k\begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ a_{n1} & a_{n2} & \ldots. & a_{nn} \end{vmatrix}$$

   i.e., multiplying a row or a column of a determinant by any number multiplies the value of determinant by the same number.

2) Adding k times a row or column of a determinant to another row or column respectively of the determinant yields the original determinant, that is, the value of the determinant is unchanged.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21}+ka_{11} & a_{22}+ka_{12} & a_{23}+ka_{13} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

   We may show this in a simpler way with a 2 x 2 determinant.

   Suppose the determinant is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The value is $ad - bc$

   If we add k times the top row to the bottom row, we get

   $\begin{pmatrix} a & b \\ c+ka & d+kb \end{pmatrix} = a(d + kb) - b(c + ka) = ad - bc$, which is the original determinant i.e., the value of determinant is unchanged when a multiple of the element of a column (or row) is added to corresponding element of another column (or row).

Check Your Progress 2

1) i) Is an identity matrix always a square matrix?

ii) Can a square matrix be of dimension $m \times n$, $m \neq n$?

iii) Can a null matrix have one negative element?

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2) Are the following identity matrices?

i) $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ ii) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

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3) What is a determinant? Explain the concept of minor and cofactor.

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6.5 INVERSE OF A MATRIX

In arithmetic we know that for any number, say 7, its inverse is $\frac{1}{7}$. What does this mean? It means that $\frac{1}{7} \times 7 = 7 \times \frac{1}{7} = 1$, i.e., if we multiply $\frac{1}{7}$ by 7 we get 1, and if we multiply 7 by $\frac{1}{7}$ we get 1. Is there a counterpart for matrices? More formally, give a matrix A, does there exist a matrix, called the inverse of A and written as A^{-1}, such that $A^{-1} A = A A^{-1} = 1$.

Notice that for A^{-1} to exist, A must be a square matrix and further, A^{-1} must also be a square matrix with the same dimension as A. As an exercise, check this assertion. Further, if A^{-1} exists, it is unique.

Definition (non-singularity of a matrix): A square matrix is called a non-singular matrix if its inverse exists. If A does not have an inverse, it is called a singular matrix.

The next question is, given a square matrix A, how do we know that it has an inverse? The condition for A to have an inverse is that its determinant does not vanish, i.e. determinant of A, written as | A |, is not equal to zero.

Now that we know what we mean by the inverse of a matrix, we would like to know why acquiring this bit of knowledge is of use to us. It turns out that there is an immediate application. We began this unit by looking at systems of linear equations. Equation (2) summarises such a system in matrix form:

$$\mathbf{Ax} = \mathbf{b} \qquad \text{….. (2)}$$

Suppose we want to find a unique solution to this equation. When does such a solution exist? We shall find that the answer to this question is that a unique solution exists if A^{-1} exists, i.e., if $|A| \neq 0$. Let us look at three examples.

System I : $2x_1 + 3x_2 = 10$

$6x_1 + 9x_2 = 20$

or $\begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$

The equations in System I represent two parallel straight lines. Since straight lines do not intersect. System I does not have a solution. Now look at the A matrix for the system, i.e.,

$\begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix}$ Its determinant

$$|A| = \begin{vmatrix} 2 & 3 \\ 6 & 9 \end{vmatrix} = 2 \times 9 - 3 \times 6 = 0$$

i.e., $|A| = 0$

System II: $3x_1 + x_2 = 10$

$x_1 - x_2 = 0$

or, $\begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$

The system has a unique solution at the point where the 45^0 line in the (x_1, x_2) plane intersects the straight line whose equation is $x_2 = 10 - 3x_1$. The solution is $\left(2\frac{1}{2}, 2\frac{1}{2}\right)$: Now check the A matrix, A = $\begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix}$. Its determinant $|A| = -3-1 = -4 \neq 0$

System III: $x_1 + x_2 = 10$

$x_1 - x_2 = 0$

$x_1 - 5x_2 = -20$

i.e.,

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ -20 \end{bmatrix}$$

The system has a solution (5, 5). Now look at the A matrix,

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -5 \end{bmatrix}$$

Since A is not a square matrix one cannot talk of | A | being zero or non-zero. This allows us to conjecture that $|A| \neq 0$ is at most a sufficient condition for A**x** = **b** to have a solution.

Suppose A is a square matrix and $|A| \neq 0$ in

A**x** = **b** ....(2)

What does the solution look like? Since $|A| \neq 0$, A^{-1} exists, which implies:

$A^{-1}A\mathbf{x} = A^{-1}b$

But $A^{-1}Ax = I$ and $\mathbf{Ix} = x$ so

$\mathbf{x} = A^{-1}b$

is the solution vector. This means that if we can compute A^{-1}, we can directly find the solution vector, $\bar{x}$ by computing $A^{-1}b$. The problem then is to find a way of computing A^{-1}.

Computation of A^{-1}

Before we can describe the procedure for computing A^{-1}, we must define three related concepts. We have come across some of these these earlier.

Definition (*Transpose of a Matrix)*: Let A = $\left[a_{ij}\right]$ be any matrix. The transpose of A, written as A', is obtained by interchanging the columns and rows of A.

Examples: (a) $A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$

To get A′ and construct a matrix whose first column is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Now take the second row of A and complete this matrix by inserting the vector as $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ as the second column. This process yields:

$$A' = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$$

(b) $A = \begin{bmatrix} 1 & 7 \\ 2 & 9 \\ 3 & 8 \end{bmatrix}$

Interchanging columns and rows as in example (a) yields:

$$A' = \begin{bmatrix} 1 & 2 & 3 \\ 7 & 9 & 8 \end{bmatrix}$$

Note, that A is a (3×2) matrix and A′ is a (2×3) matrix in this case.

(c) $A = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 3 & 7 \\ 4 & 7 & 2 \end{bmatrix}$

Check that $A' = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 3 & 7 \\ 4 & 7 & 2 \end{bmatrix}$

This is a curious case. Notice that A′ = A. Square matrices of this type, i.e., for which A = A′ are called **Symmetric** matrices.

Another concept we need is of the cofactor of a matrix. We studied this in the previous section.

Definition: (*Adjugate of Matrix A*): Let A be any square matrix. Then the adjugate of A, written as adj A, is the transpose of the cofactor matrix of A, i.e., adj A = C′ written out more explicitly:

$$\text{adj } A = C' = \begin{bmatrix} c_{11} & c_{21} & \cdots\cdots & c_{n1} \\ c_{12} & c_{22} & \cdots\cdots & c_{n2} \\ : & : & :::: & : \\ : & : & :::: & : \\ c_{1n} & c_{2n} & \cdots\cdots & c_{nn} \end{bmatrix} \text{ where } C = \begin{bmatrix} c_{11} & c_{21} & \cdots\cdots & c_{1n} \\ c_{21} & c_{22} & \cdots\cdots & c_{2n} \\ : & : & :::: & : \\ : & : & :::: & : \\ c_{n1} & c_{n2} & \cdots\cdots & c_{nn} \end{bmatrix}$$

In our example:

$$\text{adj A} = C' = \begin{bmatrix} 13 & -2 & -8 \\ -3 & -3 & 6 \\ -4 & 2 & -1 \end{bmatrix}$$

Now we have all the machinery required to define the inverse of A.

Definition: (*Inverse of A*): Let A be any square matrix such that $|A| \neq 0$. Then, the inverse of A, written as

$$A^{-1} = \frac{C'}{|A|} = \text{adj A} \frac{1}{|A|}$$

Notice that if $|A| = 0$ then A^{-1} is undefined, i.e., A singular. In our example,

$|A| = a_{11} |M_{11}| - a_{12} |M_{12}| + a_{13} |M_{13}|$

$= a_{11} |c_{11}| + a_{12} |c_{12}| + a_{13} |c_{13}|$

$= 1(13) + 2(-3) + 4(-4)$

$= 13 - 6 - 16 = -9 \neq 0$

Hence,

$$A^{-1} = \text{adj A} \frac{1}{|A|} = -\frac{1}{9}\begin{bmatrix} 13 & -2 & -8 \\ -3 & -3 & 6 \\ -4 & 2 & -1 \end{bmatrix}$$

We end this section with an example to show that if A is non-singular then $\bar{x} = A^{-1}b$.

Consider the system of linear equations:-

$3x_1 + x_2 = 20$

$x_1 - x_2 = 0$

i.e., $\begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$

Using the usual method of solving simultaneous equations yields the solution $\bar{x} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$.

What does our method yield?

Step 1: Find $|A|$. $|A| = 3(-1) - -1(1) = -4 \neq 0$

So A is non-singular.

Step 2: Compute C, $C = \begin{bmatrix} -1 & -1 \\ -1 & 3 \end{bmatrix}$

Step 3: Compute Adj C. C' = $\begin{bmatrix} -1 & -1 \\ -1 & 3 \end{bmatrix}$ = Adj C

Notice, C is symmetric.

Step 4: Compute A^{-1}. $A^{-1} = \dfrac{C'}{A} = \dfrac{\begin{bmatrix} -1 & -1 \\ -1 & 3 \end{bmatrix}}{-4} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{3}{4} \end{bmatrix}$

Step 5: Compute $\bar{x}$. $\bar{x} = A^{-1}b = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{3}{4} \end{bmatrix} = \begin{bmatrix} 20 \\ 0 \end{bmatrix}$

$$= \begin{bmatrix} \frac{1}{4}.20 + & \frac{1}{4}.0 \\ \frac{1}{4}.20 + & \left(-\frac{3}{4}\right).0 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

which is exactly what we had found. The only difficulty is that the computation may seem quite tedious compared to the standard method taught in school level algebra. It has to be remembered that for large systems that standard method turns out to be terribly difficult and time-consuming. In such cases the method used here is much easier.

Check Your Progress 3

1) Does the matrix inverse exist for the following system of equations?

   $4x_1 + 6x_2 = 5$

   $12x_1 - 18x_2 = 10$

   Do these straight lines intersect or are they parallel?

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2) Is $|A| \neq 0$ a(i) necessary or (ii) sufficient or a (iii) necessary and sufficient condition for A**x** = **b** to have a solution?

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3) Let A = $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ Find its transpose.

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4) Let A = $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

When does A become symmetric?

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5) Let (i) A = $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

(ii) A = $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

a) Find their adjugate.

b) Hence, find their inverse.

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6.6 LET US SUM UP

In this unit, you have been introduced to matrix algebra. You have learnt the concept of a matrix and how to perform various operations like addition, subtraction and multiplication on matrices. In this connection, some special matrices have also been presented. The concept of a determinant has been introduced and its various properties have been discussed. The ideas of minor and cofactor have been considered and the evaluation of a higher (more than second) order determinant has been explained. Next, the concept of the adjoint of a matrix has been introduced and the procedure for the evaluation of the inverse of a matrix has been discussed. Finally, the inverse matrix method and Cramer's rule for the solution of a simultaneous equation system have been presented.

6.7 ANSWERS OR HINTS TO CHECK YOUR PROGRESS EXERCISES

Check Your Progress 1

1) See subsection 6.2.1 and answer.
2) See subsection 6.2.1 and answer.
3) See subsection 6.2.2 and answer.
4) See subsection 6.2.2 and answer.

Check Your Progress 2

1) i) No, because all off-diagonal elements are not zero.
   ii) No, because all off-diagonal elements are not zero.
   iii) Yes, because all off-diagonal elements are zero and atleast one diagonal element is not zero.
   iv) No, because the given matrix is not square.
2) i) No, because all diagonal elements are not 1.
   ii) No, because diagonal elements are not 1.
3) See section 6. 4 and answer.

Check Your Progress 3

1) See section 6.5 and answer.
2) See section 6.5 and answer.
3) See section 6.5 and answer.
4) See section 6.5 and answer.
5) See section 6.5 and answer.

UNIT 7 LINEAR ECONOMIC MODELS

Structure

7.0 OBJECTIVES

After going through this unit, you will be able to:

- Apply the theory of matrices and linear equation in a market demand and supply Model;
- National Income Model;
- Input-Output Analysis; and
- Markov Models.

7.1 INTRODUCTION

In Unit 4, we have learnt to solve linear simultaneous equations with the help of matrix algebra. Certain problems in business and economics can be meaningfully formulated in terms of linear simultaneous equation systems. Often such systems require solving a large number of equations. Matrix algebra, very conveniently and efficiently, provides a solution to such kind of system. As a result, it finds applications in various fields of business and economics. In fact, matrix algebra is now widely used in areas like input-output models, linear programming and game theory. In this unit, we shall consider some of the simple applications of matrix algebra. We should note that the real power of matrix algebra can be realized in the case of a large number of equations. But here, for the purpose of simplification, we shall confine ourselves to a small number of equations without any loss of essential elements.

7.2 MARKET MODEL

In simple market models, the demand for a commodity and its supply are expressed as functions of its own price only. Here, the equilibrium condition is given by a single equation which is obtained by equating the demand equation with the supply equation. This single equation can then be solved for the equilibrium price. The equilibrium quantity can be obtained by putting the value of price in either the demand equation or the supply equation. However, this simple market formulation presupposes that the demand for and the supply of a commodity are not influenced by other prices. In real world situations, for any commodity, there can be many substitutes and complementary goods. The prices of these goods can also be influenced by the prices obtained for other commodities. Therefore, a better description of demand and supply functions should take into account the effect of the prices of other commodities also. Suppose, there are n interrelated markets for *n* different but related commodities. In this market model, the demand for each commodity will be a function *n* price and similarly, the demand equations and *n* supply equations. For equilibrium, each of these demand equations will have to be equated with the corresponding supply equation. The resultant *n* equations can then be simultaneously solved for obtaining *n* equilibrium prices for *n* commodities under consideration. Finally, *n* equilibrium quantities can be obtained by putting the values of *n* prices in each of either the set of *n* demand equations or the set of *n* supply equations. Thus for obtaining 2*n* pieces of information (*n* prices and n quantities), we have to solve 2*n* equations (representing *n* equilibrium conditions and either *n* demand functions or *n* supply functions). The task may seem to be quite daunting for a large value of *n*. Matrix algebra, however, considerably simplifies the matter. Let us consider a two-commodity market model and see how the solution for the model can be obtained with the help of matrices.

Example

The demand and supply equations for a two-commodity (*x* and *y*) market model are given by

$D_x = 18 - 3P_x + P_y$ $\qquad$ $D_y = 12 + P_x - 2P_y$

$S_x = 2 + 4P_x$ $\qquad$ $S_y = 2 + 3P_y$

where, D_x is the quantity demanded for *x, D_y* is the quantity demanded for *y, S_x* is the quantity supplied for *x*, S_y is the quantity supplied for *y, P_x* is the price of *x* and P_y is the price of *y.*

Find the equilibrium prices and quantities.

Equilibrium condition for *x*

$D_x = S_x = Q_x$

or $18 - 3P_x + P_y = -2 + 4P_x$ $\qquad$ or $\qquad$ $3P_x - 4P_x + P_y = -2 - 18$

$$\text{or } 7P_x + P_y = -20 \qquad (1)$$

Equilibrium condition for *y*

$$D_y = S_y = Q_y$$

$$\text{or} \quad 12 + P_x - 2P_y = -2 + 3P_y \quad \text{or} \quad P_x - 2P_y - 3P_y = -2 - 12$$

$$\text{or} \quad P_x - 7P_y = -14 \qquad (2)$$

Writing equations (1) and (2) together in the matrix form

$$\begin{bmatrix} -7 & 1 \\ 1 & -7 \end{bmatrix} \cdot \begin{bmatrix} P_x \\ P_y \end{bmatrix} = \begin{bmatrix} -20 \\ -14 \end{bmatrix}$$

We shall solve the above equation system for the two prices by the matrix inverse method

Hence, $\begin{bmatrix} P_x \\ P_y \end{bmatrix} = \begin{bmatrix} -7 & 1 \\ 1 & -7 \end{bmatrix}^{-1} \begin{bmatrix} -20 \\ -14 \end{bmatrix}$

Now,

$$\begin{bmatrix} -7 & 1 \\ 1 & -7 \end{bmatrix}^{-1} = \frac{\text{Adj} \begin{bmatrix} -7 & 1 \\ 1 & -7 \end{bmatrix}}{\begin{vmatrix} -7 & 1 \\ 1 & -7 \end{vmatrix}}$$

$$\text{or} \begin{bmatrix} -7 & 1 \\ 1 & -7 \end{bmatrix}^{-1} = \frac{1}{34} \begin{bmatrix} -7 & -1 \\ -1 & -7 \end{bmatrix}$$

$$\therefore \begin{bmatrix} P_x \\ P_y \end{bmatrix} = \frac{1}{34} \begin{bmatrix} -7 & -1 \\ -1 & -7 \end{bmatrix} \cdot \begin{bmatrix} -20 \\ -14 \end{bmatrix} = \frac{1}{34} \begin{bmatrix} 114 \\ 118 \end{bmatrix} = \begin{bmatrix} \frac{77}{17} \\ \frac{79}{17} \end{bmatrix}$$

Thus,

$$P_x = \frac{77}{17} \text{ and } P_y = \frac{79}{17}$$

Equilibrium quantities for the two commodities can be solved for by putting the values of the two prices in either the demand equations or the supply equations. Let us put the values of the two prices in the demand equations.

Thus,

$$Q_x = 18 - 3.\frac{77}{17} + \frac{79}{17} = \frac{194}{17}$$

and

$$Q_y = 12 + \frac{77}{17} - 2.\frac{79}{17} = \frac{143}{17}$$

Check Your Progress 1

1) The market demand equations for wheat and rice are

$Dx = 4 - 10P_x + 7P_y$ and $Dy = 3 + 7P_x - 7P_y$

The supply equations for wheat and rice are

$S_x = 7 + P_x - P_y$ and $S_y = -27 - P_x + 2P_y$

Where, P_x is the price of wheat and P_y is the price of rice.

Find equilibrium prices and quantities by using Cramer's rule.

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2) The demand and supply equations for a two-commodity (*x* and *y*) market model are given by

$D_x = 47 - 2P_x + 2P_y$ $\qquad$ $D_y = 16 + 2P_x - P_y$

$S_x = 7 + 2P_x$ $\qquad$ $S_y = 4 + 2P_y$

where

D_x is the quantity demanded for *x*, D_y is the quantity demanded for *y*, S_x is the quantity supplied for *x*, S_y is the quantity supplied for *y*, P_x is the price of *x* and P_y is the price of *y*.

Use matrix inverse method to find equilibrium prices and quantities.

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7.3 NATIONAL INCOME MODEL

Another useful application of matrix algebra can be in the national income model. Let us consider a simple two-equation national income model for an economy that does not have government and trade relationship with other countries. For such an economy, the national income consists of the following two equations

$$Y = C + I_0 \quad (1)$$

$$C = a + bY \quad (2)$$

Where, Y is national income, C is consumption, I_0 is autonomous investment, a and b are constants. We should note that in national income model, total production and total income generated n the process of production are considered to be equivalent. Thus, Y can be taken as either the total value of production or total income (national income). In this model, the total expenditure that can be incurred on the goods and services produced is the summation of the consumption expenditure and the investment expenditure, that is, $C + I_0$ Equation (1) gives the condition for the equilibrium level of income. It states that national income will be in equilibrium, when, the planned production (Y) is equal to the planned expenditure ($C + I_0$). In other words, for equilibrium, the plan of the producers (suppliers) should match that of the buyers (demanders). Equation (2) presents consumption as a liner function of income with some restrictions on the constants (parameters) a and b. The restrictions are $a > 0$ and $0< b < 1$. It is clear from equations (1) and (2) that Y and C are to be determined from the model whereas I_0, the autonomous investment, is given from outside the model. The variables which are determined from within a given model are called endogenous variables. Thus, Y and C are endogenous variables. The variables whose values are given from outside, are called exogenous variables. Here in this model, investment (I_0) is an exogenous variable. Thus, investment virtually acts as a constant here. Let us try to solve this model for the two endogenous variables Y and C.

Rearranging (1) and (2), we get

$$Y - C = I_0 \quad (3)$$

$$-bY + C = a \quad (4)$$

We should note that in the rearranged equations, the endogenous variables figure on the left hand side of the sign of equality. Writing equations (3) and (4) together in the matrix form

$$\begin{bmatrix} 1 & -1 \\ -b & 1 \end{bmatrix} \cdot \begin{bmatrix} Y \\ C \end{bmatrix} = \begin{bmatrix} I_0 \\ a \end{bmatrix} \quad (7)$$

We can obtain IY I and IC from (7) either by matrix inverse method or by Creamer's rule. We can now consider a numerical example.

Example

Given the following national income model

$$Y = C + I$$

$$C = 20 + \frac{3}{4}Y$$

$$I = 20$$

Find Y and C.

Putting the value of I from the third equation into the first equation

$$Y = C + 20$$

Rearranging this new equation and the second equation

$$Y = C + I \qquad (1)$$

$$-\frac{3}{4}Y + C + 20 \qquad (2)$$

Writing equations (1) and (2) together in the matrix equation form $Ax = b$

$$\begin{bmatrix} 1 & -1 \\ -\frac{3}{4} & 1 \end{bmatrix} \cdot \begin{bmatrix} Y \\ C \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \end{bmatrix} \qquad (7)$$

Where

$$A = \begin{bmatrix} 1 & -1 \\ -\frac{3}{4} & 1 \end{bmatrix}, x = \begin{bmatrix} Y \\ C \end{bmatrix} \text{ and } b = \begin{bmatrix} 20 \\ 20 \end{bmatrix}$$

We shall use Cramer's rule to obtain Y and C From (3).

$$Y = \frac{|A_1|}{|A|} = \frac{\begin{vmatrix} 20 & -1 \\ 20 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ -\frac{3}{4} & 1 \end{vmatrix}} = \frac{40}{\frac{1}{4}} = 160$$

$$C = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} 1 & 20 \\ -\frac{3}{4} & 20 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ -\frac{3}{4} & 1 \end{vmatrix}} = \frac{37}{\frac{1}{4}} = 140$$

Check Your Progress 2

1) Given the national income model

$Y = C + I + G$ (G: Government expenditure)

$C = 20 + \frac{3}{4}Y$

$I = 20$

$G = 10$

Using matrix inverse method, find Y and C.

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2) Let the national income model be

$Y = C + I + G$ (*G:* Government expenditure)

$C = 20 + \frac{3}{4}Y(Y - T)$ (*T*: Lump-sum tax)

$I = 10$

$G = 20$

$T = 20$

Using Cramer's rule, find *Y* and *C*.

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7.4 INPUT-OUTPUT ANALYSIS

7.4.1 Structure

Matrix algebra is very fruitfully used in the area of input-output analysis. The credit for the development of the input-output analysis goes to the noted economist Wassily Leontief. It is a technique that focuses on the interdependence of various producing sectors of an economy. For the purpose of simplification, let us consider an economy with only two producing sectors: agriculture and industry. If we concentrate on agriculture, it will produce a given output of goods in a given time period, say, one year. These goods will be used in various ways. In other words, agricultural products will have several destinations. These destinations are broadly classified into two groups: (a) output going as intermediate input to the agricultural sector itself and to the industrial sector and (b) output going to the final demand sector for final consumption. For example, wheat is an output of agricultural sector and it may be demanded (i) as input in the agricultural sector itself (in the form of seed) for further production of wheat; and in industry for the manufacture of bread and (ii) for final consumption by the final demand sector. In the same manner, the output of the industry may be demanded by agricultural or industrial sectors as intermediates input and by final demand sector for final consumption.

In addition to the intermediate (also known as secondary) inputs, each sector requires primary inputs. This primary input is in the form of services of factors like land, labour, capital and entrepreneurship and is supplied by the final demand sector, also known as household sector.

Let us consider an input-output table involving agriculture and industry. Such a table is often called an input-output transactions matrix.

Table 7.1: Input-Output Transactions Matrix

Producing Sector	**Purchasing Sector**		**Final Demand (Consumption)**	**Total Output**
	1 (Agriculture)	2 (Industry)		
1 (Agriculture) 2 (Industry)	X_{11} X_{21}	X_{12} X_{22}	d_1 d_2	X_1 X_2
Primary Input (labour)	L_1	L_2		

In the above input-output transaction matrix, X_{ij} (i = 1, 2; j = 1, 2) denote the output of i th producing sector that is being used as intermediate input in j th producing sector. In the above table, agricultural sector is denoted by 1 and industrial sector by 2. Thus, we can interpret X_{11} as the output of agricultural sector that is being used as an intermediate input in agriculture. Similarly, X_{21} can be interpreted as the output of industrial sector that is being used as input in agriculture etc. In addition, the output for each producing sector goes for the final use also. Thus, d_1 is the output of agriculture and d_2 is the output of industry going for the final use (consumption). Further, X_1 and X_{12} denote the total output of agriculture and industry respectively. In view of the above introduction, we can write

$$X_{11} + X_{12} + d_1 = X_1 \qquad (1)$$

$$X_{21} + X_{22} + d_2 = X_2 \qquad (2)$$

Finally, the elements in the last row of the above table i.e. , L_1 and L_2 denote the requirement of primary input (say, labour) by the respective sector. These primary inputs, as mentioned earlier, are presumed t be supplied by the household sector. It is clear that our input-output table is an oversimplified representation. However, the table can readily be made more realistic by introducing more producing sectors, additional kinds of final uses (say, investment, government expenditure and net exports) and other categories of primary inputs (that is, land, capital and entrepreneurship).

To proceed further, let us make some important assumptions. These assumptions are (1) each sector produces one homogeneous commodity, (2) the input requirement of each sector per unit of its output remains fixed, which means the level of output in each sector determines the quantity of each kind of input that it uses and (3) the production in each sector is subject to constant returns to scale, that is, a k-times change in each input results in a k-times change in output.

From the assumption of fixed input requirement, we can see that to produce one unit of some commodity *j,* the input of some commodity *I* must be a fixed amount.

Let us denote this amount by a_{ij}, where $a_{ij} = \frac{X_{ij}}{X_j}$. If X_{ij} and X_j denote value of outputs, then a_{ij} can be interpreted in value terms. Thus a_{ij}, = 0.04 can mean that forty paise worth of the *i*th commodity is required to produce one rupee worth of the *j*th commodity. This a_{ij}, is called an *input coefficient.*

We can arrange the input-coefficients of an economy with a given number of producing sectors in the form of a matrix. This matrix is called an *input coefficient matrix.* Thus, for our two-producing sector economy, the input coefficient matrix can be presented as

Table 7.2: Input Coefficient Matrix

Input	**Output**	
	1 (Agriculture)	**2 (Industry)**
1 (Agriculture)	a_{11}	a_{12}
2 (Industry)	a_{21}	a_{22}

We should note that the sum of elements in each column gives the requirement of secondary input to produce a rupee worth of output in that producing sector. As a result, the sum of the elements in each column of the input coefficient matrix should be less than 1, since; it does not include the cost of the primary inputs per rupee worth of the output. Assuming that there is pure competition with free entry, the primary input cost per one rupee worth output for a producing sector should be one minus the relevant column sum of the elements of the input coefficient matrix. For the economy considered above, if the cost of primary inputs per rupee worth of output for the two producing sectors are l_1 and l_2 respectively; then

$$l_1 = 1 - (a_{11} + a_{21})$$

and

$$l_2 = 1 - (a_{12} + a_{22})$$

The total output for each producing sector can be expressed in terms of the input coefficients, by replacing $X_{ij} = a_{ij}X_j$ in equations (1) and (2). Thus, for out two producing sector economy

$$X_1 = a_{11}X_1 + a_{12}X_2 + d_1$$

or $$X_1 - a_{11}X_1 - a_{12}X_2 = d_1$$

or $$(1 - a_{11})X_1 - a_{12}X_2 = d_1 \qquad (3)$$

and similarly

$$X_2 = a_{21}X_1 + a_{22}X_2 + d_2$$

or $\quad -a_{21}X_1 + X_2 - a_{22}X_2 = d_2$

or $\quad -a_{21}X_1 + (1 - a_{22})X_2 - a_{12}X_2 = d_2 \quad (4)$

Writing equations (3) and (4) in the matrix form

$$\begin{bmatrix} 1 - a_{11} & -a_{12} \\ -a_{21} & 1 - a_{22} \end{bmatrix} \cdot \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \quad (7)$$

Suppose

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{21} \end{bmatrix} X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \text{and } d = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

$$(I - A)\, X = d \quad (6)$$

Where I is a (2×2) identity matrix. The matrix (I-A) is called the *technology matrix.*

If (I -A) is non-singular (that is, $|I - A| \neq 0$, equation (6) can be solved for X. Thus

$$X = (I - A)^{-1}\ d \quad (7)$$

It is important to note that for a given technology, as embodied in the technology matrix, equation (7) can be used to determine the total output that is needed to be produced by different producing sectors to satisfy a given final demand for the commodities. (d).

7.4.2 Hawkins-Simon Condition

We know that negative output for any system is meaningless. So, how can we ensure that the solution for equation (7), wherea_{ij}'s are expressed in physical units, will result in positive outputs for the producing sectors? For that, we have to ensure that the *Hawkins-simon Conditions* are satisfied. Let us consider the technology matrix (I−A). According to Hawkins-Simon Condition for positive output, all principal minors of (I−A) should be positive. Thus for our technology matrix

$$(I - A) = \begin{bmatrix} 1 - a_{11} & -a_{12} \\ -a_{21} & 1 - a_{22} \end{bmatrix}$$

The condition for positive output requires that

$1 - a_{11} > 0$ and $1 - a_{22} > 0$ and

$$\begin{vmatrix} 1 - a_{11} & -a_{12} \\ -a_{21} & 1 - a_{22} \end{vmatrix} > 0$$

It is clear that the condition of the diagonal elements being positive means both a_{11} and a_{22} for the above technology matrix should be less than one. We shall now consider a numerical example.

Example

The following is the input coefficient matrix for an economy consisting of an agricultural sector and an industrial sector.

$$\begin{bmatrix} 0.20 & 0.20 \\ 0.70 & 0.20 \end{bmatrix}$$

The final consumption demand for the products of the two sectors have been estimated to be Rs. 400 crore and Rs. 4,670 crore respectively. Calculate the total output of the two sectors. Also estimate their input requirements.

Technology matrix

$$(I-A)=\begin{bmatrix} 1-0.20 & -0.20 \\ -0.70 & 1-0.20 \end{bmatrix} = \begin{bmatrix} 0.80 & -0.20 \\ -0.70 & 0.80 \end{bmatrix}$$

Determinant of the technology matrix

$$|I-A|= \begin{vmatrix} 0.80 & -0.20 \\ -0.70 & 0.80 \end{vmatrix} = 0.64 - 0.14 = 0.70$$

The determinant and the diagonal elements of the technology matrix are positive. Hence, the Hawkins-Simon Condition is satisfied and the system has a positive solution. The final consumption demand vector.

$$d = \begin{bmatrix} 0.80 & -0.20 \\ -0.70 & 0.80 \end{bmatrix}$$

If the total output of the two sectors are X_1 and X_2 respectively, the output vector

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

We know

$$X = (I - A)^{-1}d$$

or $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0.80 & -0.20 \\ -0.70 & 0.80 \end{bmatrix}^{-1} \begin{bmatrix} 400 \\ 4670 \end{bmatrix}$

Now

$$\begin{bmatrix} 0.80 & -0.20 \\ -0.70 & 0.80 \end{bmatrix}^{-1} == \frac{\text{Adj} \begin{bmatrix} -0.80 & -0.20 \\ -0.70 & 0.80 \end{bmatrix}}{\begin{vmatrix} 0.80 & -0.20 \\ -0.70 & 0.80 \end{vmatrix}} = \frac{1}{0.70} \begin{bmatrix} 0.80 & 0.20 \\ 0.70 & 0.80 \end{bmatrix}$$

Hence

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \frac{1}{0.70} \begin{bmatrix} 0.80 & 0.20 \\ 0.70 & 0.80 \end{bmatrix} . \begin{bmatrix} 400 \\ 4670 \end{bmatrix} = \frac{1}{0.70} \begin{bmatrix} 1270 \\ 4000 \end{bmatrix} = \begin{bmatrix} 2700 \\ 8000 \end{bmatrix}$$

Therefore, the total output of the agricultural sector and that of the industrial sector are Rs. 2700 crore and Rs. 8000 crore respectively.

Secondary input requirements:

Suppose

X_{11}: Input of the agricultural sector from the agricultural sector

X_{21}: Input of the agricultural sector from the industrial sector

X_{22}: Input of the industrial sector from the industrial sector

X_{12}: Input of the industrial sector from the agricultural sector

From the definition of input coefficients (which are given) we have

$X_{11} = a_{11}X_1 = 0.20 \cdot 2700 = 700$

$X_{21} = a_{21}X_1 = 0.70 \cdot 2700 = 1770$

$X_{22} = a_{22}X_2 = 0.20 \cdot 8000 = 1600$

$X_{12} = a_{12}X_2 = 0.20 \cdot 8000 = 1600$

Thus

Input of the agricultural sector from the agricultural sector is Rs. 700 crore.

Input of the agricultural sector from the industrial sector is Rs. 1770 crore.

Input of the industrial sector from the industrial sector is Rs. 1600 crore.

Input of the industrial sector from the agricultural sector is Rs. 1600 crore.

Primary input requirements:

Let the primary input requirement of the agricultural sector and that of the industrial sector be L_1 and L_2 respectively. Hence,

$L_1 = l_1 X_1$ and $L_2 = l_2 X_2$ wehre, l_1 and l_2 are the primary input requirements per rupee worth of the output of the agricultural sector and that of the industrial sector respectively. We know

$$l_1 = 1(a_{11} + a_{21}) = 1(0.20 + 0.70) = 0.10$$

$$l_2 = 1(a_{12} + a_{22}) = 1(0.20 + 0.20) = 0.60$$

So

$$L_1 = l_1X_1 = 0.10\ 2700 \quad = 270$$

$$L_2 = l_2X_2 = 0.60\ 8000 \quad = 4800$$

Thus

The primary input requirement of the agricultural sector is Rs. 270 crore.

The primary input requirement of the industrial sector is Rs. 4800 crore.

7.4.3 Open Model and Closed Model

The input-output transactions matrix that we have considered earlier is an example of an *open input-output model.* In such a model, the producing sectors interact with household or final demand sector of an economy through their purchase of primary inputs and sales of final products. We have seen that solution of an open input-output model results in unique levels of output for the producing sectors.

A *Closed input-output model,* on the other hand, is one in which the final demand sector is treated as one producing sector. Thus, the entire output of each producing sector is absorbed by other producing sectors as secondary inputs or each producing products. Here, no portion of the output is sold in the market as final product. Again in this model, producing sectors do not use the primary inputs or the services of factors of production. The model is closed in the sense that all economic activities take place within the production boundary and nothing crosses the production boundary. The solution of a closed input-putout model is not unique where as the solution of an open input-output model is unique. Thus, closed input-output model is more flexible in the sense that it provides us an infinite range of production alternatives.

7.5 MARKOV MODELS

Uncertain events are analyzed using the concept of probability, which is a measure of the likelihood of an outcome occurring of a chance event or experiment. An outcome which is absolutely certain to occur is given a value of 1, while an outcome that cannot occur is given a value of zero. Thus probability of an outcome lies in the closed interval [0,1]. The probability of an event is the number of likely outcomes divided by the number of all possible cases. Thus when we roll a die, has a total of six faces and six numbers from 1 to 6, and has three even numbers: 2, 4, 6. Similarly, drawing an ace from a deck of cards is 4/72, that is, 1/13. This is because there are 72 cards in a deck of cards and there are four aces. The sum of the probabilities of all possible events is equal to 1.

If we consider sequences of events, or observations, or experiments occurring, they are often taken to be independent. A simple way to study sequence of change events is to impose the condition that the probability of the outcome of any observation or experiment depends on the outcome of the immediately preceding observation or experiment, but not on outcomes of other prior observations or experiments. If x_t is the value of x at time *t,* then the probability of x_t depends only on x_{t-1} and not on $x_{t-2}, x_{t-3}.$ A process or sequence of the this type is said to be a first order Markov chain process, first order Markov process, or first order Markov chain.

Suppose that each observation in a sequence of observation has one of finite number of possible outcomes $x_1, x_2, \ldots\ldots x_t.$ The probability of outcome x_j for any given observation depends on at most the outcome of the immediately preceding observation. These probabilities are denoted by p_{ij}, $i = 1,2,\ldots,T$ and $j = 1,2\ldots,T$. The outcomes $x_1, x_2, \ldots x_t$ are called states, and the p_{ij}, are known as transition probabilities of a first-order Markov process. If it is assumed that the process starts at some particular state, we can calculate the probabilities of the various sequences occurring. Thus, we can specify a first-order Markov chain by defining its possible states, specifying the initial probability distribution for these states, and specifying the transition matrix.

The transition probabilities can be summarized in a square matrix. For a Markov chain with states $x_1, x_2, \ldots x_T$ the matrix of transition probabilities is

$$\mathbf{P} = \{p_{ij}\} = \begin{pmatrix} p_{11} & p_{12}\cdots & p_{1T} \\ \vdots & \ddots & \vdots \\ p_{T1} & p_{T2}\cdots & p_{TT} \end{pmatrix}$$

Keep in mind that the sum of elements in each row of the matrix **P** is 1, as the elements of the *i*th row represent the probabilities for all possible transactions when the process is the state x_i. In other words,

$$\sum_{j=1}^{T} p_{ij} = \text{for } i = 1,2,\ldots,T$$

Thus, if the probability distribution of the states on the nth trial is $[p_1, p_2, \ldots\ldots, p_r]$, the probability distribution of the states on trial n +1 is

$$[p_1, p_2, \ldots\ldots, p_r]\begin{pmatrix} p_{11} & \cdots & p_{1r} \\ \vdots & \ddots & \vdots \\ p_{r1} & \cdots & p_{rr} \end{pmatrix} = \left[\sum_{i=1}^{r} p_i p_{il,} \sum_{i=1}^{r} p_i p_{12,} \ldots \sum_{i=1}^{r} p_i p_{ir}\right]$$

The probability distribution of the outcome of the outcomes of the *n*th observation of a first-order Markov process is found by multiplying the initial probability vector with the *n*th power of the transition matrix. If the initial probability vector is denoted by $\mathrm{p}_0$and the vector of probabilities at step n is denoted by P_n, then $\mathrm{p}_1 = \mathrm{p}_0\mathrm{p},\ \mathrm{p}_2 = \mathrm{p}_0\mathrm{p}^2, = \mathrm{p}_2\mathrm{p} = \mathrm{p}_0\mathrm{p}^3, \ldots\ldots, \mathrm{p}_\mathrm{n} = \mathrm{p}_0\mathrm{p}^\mathrm{n}$.

Check Your Progress 3

1) The following is the input coefficient matrix for an economy consisting of an agricultural sector and an industrial sector.

$$A = \begin{bmatrix} 0.20 & 0.40 \\ 0.10 & 0.70 \end{bmatrix}$$

The final consumption demand for the products of the two sectors have been estimated to be Rs. 60 crore and Rs. 40 crore respectively. Calculate the total output of the two sectors. Also estimate their input requirements.

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2) The following is a input-output transactions matrix for a two-producing sector economy for the year: 2019-2020

Table 7.3: Input-Output Transactions Matrix: 2019-2020

Producing Sector	**Purchasing Sector**		**Final Demand Consumption**	**Total Output**
	Agriculture	**Industry**		
Agriculture	17.70	3.27	20	38.77
Industry	7.77	9.77	17	32.70
Primary Input	17.70	19.70		
Total Input	38.77	32.70		

The Planning Commission of the economy feels that the final consumption demand for the products of the two sectors will rise by Rs. 10 Crore and Rs. 7 Crore respectively in the year: 2020-2021. Assuming unchanged technology and fixed prices, what will be the changes in the output of the two sectors? Also present the estimated input-output transactions matrix for the year 2020-2021.

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3) Explain the concept of a transition matrix.

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7.6 LET US SUM UP

In this Unit you have been introduced to some of the simple applications of matrix algebra in business and economics. You have seen, how an analysis of market models for related commodities can be facilitated by matrix algebra. Such a model may often be characterized by a large number of equations simultaneously. This unit has explained how matrix methods for solving an equation system can quite efficiently provide solution to such market models.

National income models for an economy provide insight into the macroeconomic working of an economy. Such models require the solutions for endogenous variables like income, consumption, savings, investment etc. A complex national income model may contain a large number of equations, identities, exogenous variables and endogenous variables. Matrix algebra very elegantly handles such models. In this unit the technique for solving a simple national income model by matrix methods has been explained. The basic technique is equally valid for more complex models.

The producing sectors of an economy are interdependent in the sense that each sector normally buys secondary inputs or intermediate products from other sectors for the purpose of its own production an in turn sells its own product to other sectors for the purpose of their own production. In addition, each sector also sells its product to satisfy the final demands of other sectors of the economy. These producing sectors also buy primary inputs or factors of production from the household sector. Thus production at the disaggregated level entails an interdependence among the producing sectors and, an interaction (again an interdependence) between the producing sectors on one hand and household sector of the economy on the other hand. Thus interdependence is very conveniently described by an input-output system. This is an area that presents an ideal application of matrix algebra. Here, matrix algebra is used both for formulating a theoretical model in terms of an equation system and for deriving some policy implications from it.

Uncertain events can occur in such a way that the probability of occurrence of an event in one state can depend on the occurrence of events in the preceding stage but not on events prior to that. These transition probabilities can be analyzed by using an application of matrices to probability theory, namely Markov chains. Matrices of transition probabilities are the basic tools in this theory.

7.7 ANSWERS/HINTS TO CHECK YOUR PROGRESS EXERCISES

Check Your Progress 1

1) See section 7.2 and answer.

2) See section 7.2 and answer

Check Your Progress 2

1) See section 7.3 and answer.

2) See section 7.3 and answer.

Check Your Progress 3

1) See section 7.4 and answer.

2) See section 7.4 and answer.

3) See section 7.5 and answer.