BLOCK 4

MULTIVARIATE OPTIMISATION

BLOCK 4 INTRODUCTION

The final Block of the course is titled **Multivariate Optimisation.** This Block has three units. Unit 8, titled unconstrained optimisation, discusses optimisation of multivariate functions that are not subject to any constraints. The next unit, unit 9 discusses constrained optimisation, and has this as the title. The final unit of this Block, and of the course, discusses certain specialised topics related to optimisation, particularly multivariate optimisation, both constrained and unconstrained.

UNIT 8 UNCONSTRAINED OPTIMISATION

Structure

8.0 OBJECTIVES

After going through this Unit, you will be able to:

- Explain the concept of a constraint in optimisation exercises;
- State the meaning of total differential;
- Describe the first-order and second-order conditions of optimisation subject to constraints;
- Explain the meaning of quadratic forms; and
- Discuss some economic applications of optimisation subject to constraints.

8.1 INTRODUCTION

In the previous unit, we have concentrated on objective functions with only one choice variable. However, this is a very restrictive assumption. Frequently we come across situations where more than one variable is involved. For example, a multi-product firm has to choose the optimal product mix that will enable the firm to maximise its overall profit. Even if we consider a firm producing a single output – usually the revenue earned by the firm depends not only on the quantity produced, but also on some other factors such as advertising expenditure, price charged by a competing firm, etc. It is more realistic to assume that the revenue (R) accruing to the firm is a function of its quantity (Q), advertising expenditure (A) and price charged by a competing firm (P). Hence it can be expressed as:

$$R = R(Q, A, P)$$

Now the question is: how does one of these variables influence the total revenue? To understand this, we have to take recourse to 'partial differentiation'. The latter enables us to capture the effect of each of these variables on R keeping others as constant.

Before deriving the first and second order conditions of optimisation of objective functions with more than one variable, let us consider the differential version of optimisation condition in one choice variable case. Recall the optimisation conditions laid down in course BECC-102 were in terms of "derivatives" as against "differentials". To prepare the background for solving optimisation problems with multivariate objective functions, it is important to see how these conditions can also be expressed in terms of differentials.

8.2 THE DIFFERENTIAL VERSION OF OPTIMISATION CONDITIONS

8.2.1 First-Order Condition

Consider the following function:

$$z = f(x) \tag{1}$$

At the maximum as well as the minimum points of the function, value of z is stationary. In other words, it is necessary for an extremum of z that $dz = 0$ as x varies. This constitutes the first-order condition for an extremum in the differential form. To verify that this condition is equivalent to the first derivative condition of zero slope, let us differentiate equation (1) totally. Total differentiation of equation (1) yields the following:

$$dz = f'(x)dx \tag{2}$$

Note if $dx = 0$, dz is automatically equal to zero. However, this is not what the first-order condition is all about. The first-order condition requires that at the extremum points, even with infinitesimal changes in *x, dz* should be equal to zero. Now with $dx \neq 0$, *dz* can be zero only if $f'(x) = 0$. Hence, the first-derivative condition $f'(x) = 0$ and the first differential condition "dz = 0 for arbitrary nonzero values of dx", are equivalent.

8.2.2 Second-Order Condition

The sufficient condition for a stationary point to be also a relative maximum is that $dz < 0$ in the immediate neighbourhood of that point. In other words, *z* is decreasing as we move away from this point either to the left or to the right. The fact that $dz = 0$ at the maximum point but $dz < 0$ on the two sides of the point means that *dz* is decreasing as we move away from the former in either direction. Consequently, the sufficient of a stationary value *z* to be a relative maximum is that $d(dz) < 0$ i.e. $d^2z < 0$ for arbitrary non-zero value of *dx*. This constitutes the second-order condition of maximisation in differential form. Again to verify that this is equivalent to the second-order derivative conditions, we totally differentiate equation (2). Total differentiation of equation (2) gives us:

$$d^2 z = d(f'(x)dx)$$

Now $dx = constant$ (arbitrary non-zero value), therefore,

$$\begin{aligned} d^2 z &= [df'(x)]dx \\ &= [f''(x)dx]dx \\ &= f''(x)(dx)^2 \\ d^2 z &= f''(x)(dx)^2 \end{aligned}$$

Note, $(dx)^2 = (constant)^2 > 0$

Therefore, for $d^2 z < 0; f''(x) < 0$

This again shows that the second-order differential condition is equivalent to the second-order derivative condition of maximisation. Analogously, for a stationary value of *z* to be a relative minimum, it is sufficient that $d(dz) > 0$ i.e. $d^2z > 0$. This is the sufficient condition of minimisation in differential form.

8.3 EXTREMUM VALUES OF A FUNCTION OF TWO VARIABLES

8.3.1 First-Order Condition for Objective Function with Two Variables

Assume that

$$z = f(x, y) \tag{3}$$

The first-order necessary condition for an extremum (either maximum or minimum) again involves dz = 0. However, now since there are two choice variables, the first-order condition is modified as follows:

dz* = *0* for arbitrary non-zero values of *dx* and *dy

The rationale behind this is similar to the explanation of the condition dz = 0 for the one variable case: an extremum point must necessarily be a stationary point; at a stationary point *dz = 0* ever for infinitesimal change in the two variables *x* and *y*. Totally differentiating equation (3), we get

$$dz = f_x dx + f_y dy$$

where $f_x = {df}/{dx}$ = partial derivative with respect to *x* and $f_y = {df}/{dy}$ = partial derivative with respect to *y*.

Now $dx \neq 0; dy \neq 0$ (A)

And at the stationary point $dz = 0$ (B)

(A) and (B) can hold simultaneously only if $f_x = f_y = 0$.

Hence the first-order condition for optimisation (location of extremum points) for an objective function with two variables is:

$$f_x = f_y = 0 \text{ or } {\partial z}/{\partial x} = {\partial z}/{\partial y} = 0$$

As in the earlier discussion, the first-order condition is necessary but not sufficient. To develop the sufficiency condition, we must look into the second-order to the differential, which is related to second-order partial derivatives.

Example*:*

Assuming that the second-order condition is satisfied, find out the profit-maximising values of quantity (*Q*) and advertising expenditure (*A*) for a producer with the following profit function (*Π*):

$$\Pi = 400 - 3Q^2 - 4Q + 2QA - 5A^2 + 48A \tag{4}$$

The first-order condition for profit-maximisation requires that the partial derivative

$${\partial \Pi}/{\partial Q} = {\partial \Pi}/{\partial A} = 0.$$

Partially differentiating equation (4) with respect to *A* keeping *Q* as constant gives us:

$${\partial \Pi}/{\partial A} = 2Q - 10A + 48$$

Partially differentiating equation (4) with respect to *Q* keeping *A* as constant gives us:

$\partial\Pi/\partial Q = -6Q - 4 + 2A$

Setting $\partial\Pi/\partial Q = \partial\Pi/\partial A = 0$ as the first-order condition we get:

$$2Q - 10A + 48 = 0 \quad (5)$$

and

$$-6Q - 4 + 2A = 0 \quad (6)$$

Equations (5) and (6) can be written as:

$$Q - 5A = -24 \quad (7)$$

$$-3Q + A = 2 \quad (8)$$

From equation (8), we get $A = 3Q + 2$. Substituting for A in equation (7) and we then solve for Q as follows:

$$Q - 5(2 + 3Q) = -24$$
$$Q - 10 - 15Q = -24$$
$$-14Q = 14$$
$$Q = 1$$

Hence $A = 2 + 3*1 = 5$.

So, we obtain that Q =1 and A = 5 are the quantity and advertising expenditure level which maximises profit for the firm using the first-order necessary condition.

8.3.2 Second-Order Partial Derivatives and Total Differentials

Note the partial derivatives are themselves function of the independent variables. Hence, they are capable of generating 'partial derivatives of higher order' through repeated differentiation. In the subsequent section, we shall show how the second-order partial derivatives are being generated. The former plays an important role in formulating second-order condition or the sufficiency condition for optimisation of objective functions with two independent variables.

Second-Order Partial Derivatives:

Once again let us start with the following objective function:

$z = f(x,y)$

The two first-order partial derivatives are: $f_x = df/dx$ = partial derivative with respect to x and $f_y = df/dy$ = partial derivative with respect to y.

Now the functions $f_x(x,y)$ and $f_y(x,y)$ are generated by partially differentiating the objective function are themselves functions of 'x' and 'y'. Consequently, we can measure the rate of change in f_x with respect to x keeping y constant. This generates second-order partial derivatives, symbolically represented as f_{xx}.

$$f_{xx} = \partial/\partial x (f_x) = \partial/\partial x (\partial z/\partial x) = \partial^2 z/\partial^2 x$$

The notation f_{xx} has a double subscript signifying that the primitive objective function *f(x,y)* has been partially differentiated with respect to x twice. Analogously we can find the second-order partial derivatives with respect to y. Since f_y is a function of y (and also x), we can measure the rate of change f_y with respect to y, keeping x as constant.

$$f_{yy} = \partial/\partial y (f_y) = \partial/\partial y (\partial z/\partial y) = \partial^2 z/\partial^2 y$$

The notation f_{yy} has a double subscript indicating that the primitive objective function *f(x,y)* has been partially differentiated with respect to y twice. This generates two other partial derivatives defined as follows:

$$f_{yx} = \partial/\partial y (f_x) = \partial/\partial y (\partial z/\partial x) = \partial^2 z/\partial y \partial x$$

$$f_{xy} = \partial/\partial x (f_y) = \partial/\partial x (\partial z/\partial y) = \partial^2 z/\partial x \partial y$$

These are also known as "**cross**" (or "**mixed**") partial derivatives.

Two important aspects have to be borne in mind.

- Firstly, even though f_{xy} and f_{yx} has been defined separately, they are identical to each other as long as the two cross partial derivatives are continuous functions (**Young's Theorem**). Hereafter, we shall assume that the cross derivatives are identical unless stated otherwise, i.e. f_{xy} = f_{yx}
- Secondly, each of the second order partial derivatives, i.e., f_{xx}, f_{yy} and f_{xy}, like the first partial derivative f_x and f_y, may also be functions of x and y.

Example:

Determine f_8, f_{22} and f_{12} for the following function:

$$f(x_1, x_2) = x_1^2 x_2^2 - x_1 x_2 + 3x_1 - 2x_2$$

$$f_1 = \frac{\partial f}{\partial x_1} = 2x_1 x_2^2 - x_2 + 3 \qquad (9)$$

$$f_2 = \frac{\partial f}{\partial x_2} = 2x_1^2 x_2 - x_1 - 2 \qquad (10)$$

f_8 is obtained by partially differentiating equation (9) with respect to x_1. Hence,

$$f_{11} = \frac{\partial}{\partial x_1}(f_1) = 2x_2^2$$

f_{22} is obtained by partially differentiating equation (10) with respect to x_2. Hence,

$$f_{22} = \frac{\partial}{\partial x_2}(f_1) = 2x_1^2$$

The cross derivative f_{12} is obtained by differentiating equation (10) with respect to x_1. Note that in this case following Young's theorem, same result would have been obtained by differentiating equation (9) with respect to x_2. Hence,

$$f_{12} = \frac{\partial}{\partial x_1}(f_2) = \frac{\partial^2 f}{\partial x_1 x_2} = 4x_1x_2 - 1$$

Therefore the solution:

$$f_{11} = 2x_2^2$$
$$f_{22} = 2x_1^2$$
$$f_{12} = 4x_1x_2 - 1$$

Second-Order Total Differential:

The concept of partial derivatives enables us to write the total differential of a function. Recall for a function $z = f(x,y)$, the total differential can be expressed as:

$$dz = f_x dx + f_y dy \tag{11}$$

where dx and dy are non-zero arbitrary infinitesimal change in x and y respectively and to be treated as constants. Consequently, dz depends only on f_x and f_y and since f_x and f_y are themselves functions of x and y, dz like z is also a function of the two choice variables x and y.

The second-order total differential, $d^2z \equiv d(dz)$ is a measure of the change of dz itself and is expressed in terms of the second-order partial derivatives defined earlier. To obtain d^2z we need to once again totally differentiated dz, using equation (8).

$$\begin{aligned}
d^2z &\equiv d(dz) \\
&= \frac{\partial(dz)}{\partial x}dx + \frac{\partial(dz)}{\partial y}dy \\
&= \frac{\partial}{\partial x}(f_x dx + f_y dy)dx + \frac{\partial}{\partial y}(f_x dx + f_y dy)dy \\
&= (f_{xx}dx + f_{xy}dy)dx + (f_{yx}dx + f_{yy}dy)dy \\
&= f_{xx}(dx)^2 + f_{xy}dydx + f_{yx}dxdy + f_{yy}(dy)^2 \\
&= f_{xx}(dx)^2 + 2f_{xy}dxdy + f_{yy}(dy)^2,\ as\, f_{yx} = f_{xy}
\end{aligned}$$

In other words, the second-order total differential depends on second-order partial derivatives.

$$d^2z = f_{xx}(dx)^2 + 2f_{xy}dxdy + f_{yy}(dy)^2 \qquad \textbf{(A)}$$

Recall, dz measures the rate of change in z, while d^2z measures the rate of change of dz. If $d^2z > 0$ then this implies that dz is increasing and if $d^2z < 0$ then this implies that dz is decreasing.

Example:

Given $z = 2x^3 + 4xy - y^2$, find dz and d^2z.

Step 1: To find dz, we need to obtain the first-order partial derivatives f_x and f_y respectively.

Partially differentiating the given equation with respect to x, we get

$$f_x = \frac{\partial z}{\partial x} = 6x^2 + 4y = 2(3x^2 + 2y) \qquad (12)$$

Partially differentiating the given equation with respect to y we get

$$f_y = \frac{\partial z}{\partial y} = 4x - 2y = 2(2x - y) \qquad (13)$$

Now

$$dz = f_x dx + f_y dy$$
$$dz = 2(3x^2 + 2y)dx + 2(2x - y)dy$$

Step 2: To obtain d^2z we need to find out the second-order partial derivatives, f_{xx}, f_{xy} and f_{yy}. To find f_{xx}, we partially differentiate equation (12) with respect to x to get the following:

$$f_{xx} = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}[2(3x^2 + 2y)] = 2 * 6x = 12x$$

To find f_{yy}, we partially differentiate equation (13) with respect to y to get the following:

$$f_{yy} = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}[2(2x - y)] = -2$$

To find f_{xy}, we partially differentiate equation (13) with respect to x to get the following:

$$f_{xy} = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}[2(2x - y)] = 4$$

Now

$$d^2z = f_{xx}(dx)^2 + 2f_{xy}dxdy + f_{yy}(dy)^2$$

Substituting in the above equation for f_{xx}, f_{xy} and f_{yy}, we get

$$d^2z = 12x(dx)^2 + 2*4dxdy - 2(dy)^2$$
$$d^2z = 12x(dx)^2 + 8dxdy - 2(dy)^2$$

Solution:

$$dz = 2(3x^2 + 2y)dx + 2(2x - y)dy$$

$$d^2z = 12x(dx)^2 + 8dxdy - 2(dy)^2$$

8.3.3 Second-Order Condition for Objective Function with Two Variables

Let us now examine the sufficiency condition for optimisation where the objective function has two decision variables. Using the concept of d^2z, we can state the second-order sufficient condition for a maximum of $z = f(x,y)$ as follows:

***$d^2z < 0$* for arbitary non-zero values of *dx* and *dy*.**

The rationale behind this is similar to that of the d^2z condition explained in the case where the objective function has one variable. Analogously, the second-order sufficiency condition for a minimum of $z = f(x,y)$ is the following:

***$d^2z > 0$* for arbitary non-zero values of *dx* and *dy*.**

Note that d^2z is a function of the second order partial derivatives f_{xx}, f_{xy} and f_{yy}. Intuitively, it is clear that the second-order sufficiency condition can be translated in terms of these derivatives. However, the actual translation would require knowledge of quadratic form – the discussion of which is given in section 8.5. Hence, we will <u>state</u> the main results here:

For any values of *dx* and *dy*, not both zero:

$$d^2z < 0 \text{ iff } f_{xx} < 0; f_{yy} < 0 \text{ and } f_{xx}f_{yy} > f_{xy}^2$$

and

$$d^2z > 0 \text{ iff } f_{xx} > 0; f_{yy} > 0 \text{ and } f_{xx}f_{yy} > f_{xy}^2$$

To sum up, the first-order and the second-order condition for optimisation in case of an objective function $z = f(x,y)$ is depicted in the following Table:

Table 8.1: First-Order and Second-Order Condition for Optimisation

	Maximum	**Minimum**
First-Order Necessary Condition	$f_x = f_y = 0$	$f_x = f_y = 0$
Second-Order Sufficiency Condition	$f_{xx}, f_{yy} < 0$ and $f_{xx}f_{yy} > f^2{}_{xy}$	$f_{xx}, f_{yy} > 0$ and $f_{xx}f_{yy} > f^2{}_{xy}$

The second order condition is applicable only after the first-order condition has been fulfilled.

In example 1, we had solved for the optimal product (Q) and the advertising expenditure (A), assuming that second-order condition is satisfied at the optimal point. Now let us examine whether it is actually satisfied or not.

Recall that

$$\Pi = 400 - 3Q^2 - 4Q + 2QA - 5A^2 + 48A$$

$$\Pi_A = {\partial\Pi}/{\partial A} = 2Q - 10A + 48 \tag{14}$$

$$\Pi_Q = {\partial\Pi}/{\partial Q} = -6Q - 4 + 2A \tag{15}$$

Setting them equal to zero and solving for Q and A yields $Q^* = 1$ and $A^* = 5$, where * denotes the optimal level.

For the second-order condition we need to derive the following partial derivatives:

$$\frac{\partial\Pi^2}{\partial A^2}, \frac{\partial\Pi^2}{\partial Q^2} \text{ and } \frac{\partial\Pi^2}{\partial A\partial Q}$$

Partial differentiation of equation (14) with respect to A gives us:

$$\Pi_{AA} = \frac{\partial\Pi^2}{\partial A^2} = -10 < 0 \tag{16}$$

Partial differentiation of equation (15) with respect to Q gives us:

$$\Pi_{QQ} = \frac{\partial\Pi^2}{\partial Q^2} = -6 < 0 \tag{17}$$

Partial differentiation of equation (15) with respect to A gives us:

$$\Pi_{AQ} = \frac{\partial\Pi^2}{\partial A\partial Q} = 2$$

Now

$\Pi_{AA} * \Pi_{QQ}$ = -10 * -6 = 60

and $(\Pi_{AQ})^2 = (2)^2 = 4$

Hence $\Pi_{AA} * \Pi_{QQ} > (\Pi_{AQ})^2$ (18)

From (16), (17) and (18), it follows that the second-order condition is satisfied for the output level (Q) equal to 1 and the advertising expenditure (A) equal to 5.

Example:

Find the extreme values of the following function and that determine whether it is a maxima or a minima.

$$z = -x^2 + xy - y^2 + 2x + y \tag{19}$$

For a point (x_0, y_0) to be an extrema, it is necessary that the first-order condition is satisfied at that point.

First-order condition: $f_x = f_y = 0$ or $\partial z/\partial x = \partial z/\partial y = 0$

Partially differentiating equation (19) with respect to x we get,

$$f_x = \frac{\partial z}{\partial x} = -2x + y + 2 \tag{20}$$

Partially differentiating equation (19) with respect to y we get,

$$f_y = \frac{\partial z}{\partial y} = x - 2y + 1 \tag{21}$$

Using the first-order necessary condition, set equation (20) and (21) to zero and solve for x and y to locate the extrema. Hence,

$$-2x + y + 2 = 0 \tag{22}$$

$$x - 2y + 1 = 0 \tag{23}$$

Multiplying equation (23) with 2 we get:

$$2x - 4y + 2 = 0 \tag{24}$$

Adding equation (22) and equation (24) we get:

$$-2x + y + 2 + 2x - 4y + 2 = 0$$
$$-3y + 4 = 0$$
$$y = 4/3$$

Substituting $y = 4/3$ in equation (23) we can obtain the value for x:

$$x - 8/3 + 1 = 0$$
$$x = 11/3$$

Hence $x = 8/3$ and $y = 4/3$ satisfies the first-order necessary condition for an extrema.

To verify whether the second order condition is satisfied at this point and examine whether it is a maximum or a minimum, we need to determine the following second order partial derivatives: f_{xx}, f_{xy} and f_{yy}.

Partially differentiating equation (20) with respect to x we get,

$$f_{xx} = \frac{\partial^2 z}{\partial x^2} = -2 < 0 \tag{25}$$

Partially differentiating equation (21) with respect to y we get,

$$f_{yy} = \frac{\partial^2 z}{\partial y^2} = -2 < 0 \tag{26}$$

Partially differentiating equation (21) with respect to *x* we get,

$$f_{xy} = \frac{\partial^2 z}{\partial x \partial y} = 1$$

Now using these values, we see that

$$f_{xx} f_{yy} = -2 * -2 = 4; \; f_{xy}^2 = 1 * 1 = 1; \text{this implies } f_{xx} f_{yy} > f_{xy}^2 \tag{27}$$

From, (25), (26) and (27), it follows that the second-order condition for a maximum is satisfied at *x =8/3* and *y = 4/3*. The corresponding stationary values of z (which is also a relative maxima of the function!) is thus solved by substituting the optimal values of *x* and *y* in the objective function given by equation (19). We get *z = -17/9*.

Solution: *x = 8/3*, *y = 4/3* and *z = -17/9* is relative maximum.

Check Your Progress 1

1) Consider the following utility function of a consumer:

   a) $U = q_1^2 + q_2^2$

   b) $U = q_1 + q_2 + 2q_1 q_2 - 0.01(q_1^2 + q_2^2)$

   c) $U = Aq_1^{\alpha} q_2$ where $A > 0; \alpha > 0$

   Find the four second-order partial derivatives, determine their signs and interpret the results economically.

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2) Find the first-order (*dU*) and the second-order total differentials (d^2U) of the three functions given in Problem 1.

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8.4 QUADRATIC FORMS

The expression for d^2z in equation (A) some pages earlier exemplifies what is known as 'quadratic forms'. A polynomial expression in which each of the components is of second degree (sum of exponents in each term equals 2) constitutes a quadratic form. In this section, we will go on to express quadratic equation in matrix form. Thereafter we shall state the conditions of "positive definiteness" and "negative definiteness". The latter plays an important role in locating extreme values of objective functions with multiple numbers of decision variables.

The quadratic equation in general form with 'n' number of variables can be expressed as:

$$\begin{aligned} Q =& a_{11}x_1^2 + a_{12}x_1x_2 + a_{13}x_1x_3 + \ldots\ldots + a_{1n}x_1x_n \\ &+ a_{21}x_1x_2 + a_{22}x_2^2 + a_{23}x_2x_3 + \ldots\ldots + a_{2n}x_2x_n \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &+ a_{n1}x_1x_n + a_{n2}x_2x_n + a_{n3}x_3x_n + \ldots\ldots + a_{nn}x_n^2 \end{aligned}$$

Assuming that $a_{ij} = a_{ji}$, we get

$$\begin{aligned} Q =& a_{11}x_1^2 + a_{12}x_1x_2 + a_{13}x_1x_3 + \ldots\ldots + a_{1n}x_1x_n \\ &+ a_{12}x_1x_2 + a_{22}x_2^2 + a_{23}x_2x_3 + \ldots\ldots + a_{2n}x_2x_n \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &+ a_{1n}x_1x_n + a_{2n}x_2x_n + a_{3n}x_3x_n + \ldots\ldots + a_{nn}x_n^2 \end{aligned}$$

Suppose X is a (n x 1) column vector comprising of the n variables and A is (n x n) is a square and symmetric matrix comprising of the coefficients a_{ij}'s, i.e.

$$X = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix}_{n\times 1} \quad \text{and } A = \begin{bmatrix} a_{11} & a_{12} & . & a_{1n} \\ a_{12} & a_{22} & . & a_{2n} \\ . & . & . & . \\ a_{1n} & a_{2n} & . & a_{nn} \end{bmatrix}_{n\times n}$$

Then Q can be expressed as a product of these matrices, i.e.

$Q = X'AX$ where X' indicates transpose of X.

Example:

Consider the quadratic equation $Q = x_1^2 + x_2^2 - 2x_1x_2$. Express it in matrix form.

$$Q = x_1^2 + x_2^2 - 2x_1x_2 \tag{28}$$

Rearranging the individual components in Q, equation (28) can be written as:

$$\begin{aligned} Q =& x_1^2 - x_1x_2 \\ &- x_1x_2 + x_2^2 \end{aligned}$$

In his case $a_8 = 1$, $a_{12} = a_{21} = -1$ and $a_{22} = 1$

Hence $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}_{2\times 2}$ and $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{2\times 1}$

Consequently $Q = X'\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}X$

Once we have defined a quadratic form, we can go on to lay down the conditions that need to be satisfied for positive definiteness and negative definiteness.

Fact 1 : The quadratic form $Q = X'AX$ in n variables is positive definite when it takes a positive value for any values of the variables (not all zero).

Fact 2 : The quadratic form $Q = X'AX$ in n variables is negative definite when it takes a negative value for any values of the variables (not all zero).

Fact 3 : The quadratic form $Q = X'AX$ in n variables is positive definite if and only if the principal minors of determinant of A are all positive. Note A by definition is a square, symmetric and non-singular matrix. The principal minor is obtained by deleting the last *(n-i)* rows and *(n-i)* columns of A.

Fact 4 : The quadratic form $Q = X'AX$ is negative definite if and only if the principal minors of the determinant of A is alternate in signs, with the first principal minor being negative.

Example: Determine $Q = x_1^2 + x_2^2$ is positive or negative definite.

$$Q = x_1^2 + 0 \times x_1 x_2 + 0 \times x_1 x_2 + x_2^2$$

Hence $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$ and $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{2\times 1}$

Therefore, $|A_{11}| = 1 > 0$ and $|A_{22}| = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1 > 0$

This implies that Q is positive definite.

Example: Determine whether $Q = x_1^2 + 6x_2^2 + 3x_3^2 - 2x_1x_2 - 4x_2x_3$ is positive or negative definite

The equation can be rewritten as:

$$Q = x_1^2 - x_1x_2 + 0 \times x_1x_3$$
$$- x_1x_2 + 6x_2^2 - 2x_2x_3$$
$$+ 0 \times x_1x_3 - 2x_2x_3 + 3x_3^2$$

In this case, $a_8 = 1$, $a_{22} = 6$, $a_{33} = 3$, $a_{13} = a_{31} = 0$, $a_{12} = a_{21} = -1$, $a_{23} = a_{32} = -2$

So,

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 6 & -2 \\ 0 & -2 & 3 \end{bmatrix}_{3\times 3}$$

The principal minors of $|A|$ are as follows:

$$|A_{11}| = 1 > 0$$

$$|A_{22}| = \begin{vmatrix} 1 & -1 \\ -1 & 6 \end{vmatrix} = 6 - 1 = 5 > 0$$

$$|A_{33}| = \begin{vmatrix} 1 & -1 & 0 \\ -1 & 6 & -2 \\ 0 & -2 & 3 \end{vmatrix} = 1(18-4) + 1(-3) + 0(2-0) = 11 > 0$$

Therefore, the quadratic form is positive definite.

8.5 SECOND-ORDER TOTAL DIFFERENTIAL AS A QUADRATIC FORM

Recall the second-order condition in differential form for an objective function with two variables, i.e. $z = f(x,y)$ is

$$d^2z = f_{xx}(dx)^2 + 2f_{xy}dxdy + f_{yy}(dy)^2$$

The above can be expressed as:

$$d^2z = f_{xx}(dx)^2 + f_{xy}dxdy$$
$$+ f_{xy}dxdy + f_{yy}(dy)^2$$

Expressing d^2z in matrix form, we get

$$d^2z = X'AX \text{ where } X = \begin{bmatrix} dx \\ dy \end{bmatrix}_{2\times 1} \text{ and } A = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix}_{2\times 2}$$

The second-order sufficiency condition for extremum requires d^2z to be positive definite (for a minimum) and negative definite (for a maximum) regardless what values dx and dy take (as long as they both are not equal to zero).

Hence for a minimum,

$d^2z > 0 \Leftrightarrow$ principal minors of $A = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix}_{2\times 2}$ are all positive. In other words, $|A_{11}| : f_{xx}; f_{yy} > 0$ and $|\mathrm{A}_{22}|$, i.e. $f_{xx}f_{yy} - (f_{xy})^2 > 0 \Rightarrow f_{xx}f_{yy} > (f_{xy})^2$ (C)

And for a maximum

$d^2z < 0 \Leftrightarrow$ principal minors of $A = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix}_{2\times 2}$ are alternate in sign. In other words, $|A_{11}| : f_{xx}; f_{yy} < 0$ and $|\mathrm{A}_{22}|$, i.e. $f_{xx}f_{yy} - (f_{xy})^2 > 0 \Rightarrow f_{xx}f_{yy} > (f_{xy})^2$ (D)

We have already depicted conditions (C) and (D) in Table 1.

8.6 OBJECTIVE FUNCTIONS WITH MORE THAN TWO VARIABLES

8.6.1 First-Order Condition for Extremum if Objective Function has More than Two Variables

Let us consider a function with three variables

$$z = f(x_1, x_2, x_3) \tag{29}$$

with first-order partial derivatives f_1, f_2 and f_3 and second-order partial derivatives $f_{ij}\,(\equiv \frac{\partial^2 z}{\partial x_i \partial x_j})$ with $j = 1, 2, 3$. On the basis of Young's theorem we get $f_{ij} = f_{ji}$ for all $i \neq j$.

As mentioned earlier, an extremum point (maximum or minimum) corresponds to a stationary value of 'z'. In other words, to have an extremum of z, it is necessary that

$dz = 0$ for arbitrary values of dx_1, dx_2 and dx_3, not all zero. Totally differentiating equation (29) we get:

$$dz = f_1 dx_1 + f_2 dx_2 + f_3 dx_3 \tag{30}$$

where $f_1 = \frac{\partial f}{\partial x_1}, f_2 = \frac{\partial f}{\partial x_2}$ and $f_3 = \frac{\partial f}{\partial x_3}$. Since, dx_1, dx_2 and dx_3 are arbitrary (infinitesimal) changes in the independent variables, not all zero, the only way to ensure a zero dz is to have $f_1 = f_2 = f_3 = 0$. Once again we see that the necessary condition for an extremum is that all the first-order partial derivatives are equal to zero.

8.6.2 Second-Order Condition for Extremum if Objective Function has More than Two Variables:

If the first-order condition for an extremum is fulfilled then the sufficiency condition that needs to be satisfied is as follows: at a stationary value of z if we find that d^2z is positive definite then this will suffice to establish z as a minimum. Analogously, at a stationary value of z if we find that d^2z is negative definite then this is sufficient to establish z as a maximum.

As before, the expression for d^2z can be obtained by totally differentiating equation (30). Recall $f_i = f_i(x_1, x_2, x_3)$; and dx_i measures arbitrary non-zero constant change for all

$i = 1, 2, 3$. Hence, $dz = \Phi(x_1, x_2, x_3)$.

To get the exact form, total differentiation of equation (30) gives us the following:

$$\begin{aligned}
d^2z &= d(dz)\\
&= \frac{\partial(dz)}{\partial x_1}dx_1 + \frac{\partial(dz)}{\partial x_2}dx_2 + \frac{\partial(dz)}{\partial x_3}dx_3\\
&= \frac{\partial}{\partial x_1}(f_1dx_1 + f_2dx_2 + f_3dx_3)dx_1\\
&\quad + \frac{\partial}{\partial x_2}(f_1dx_1 + f_2dx_2 + f_3dx_3)dx_2\\
&\quad + \frac{\partial}{\partial x_3}(f_1dx_1 + f_2dx_2 + f_3dx_3)dx_3\\
&= (f_{11}dx_1 + f_{12}dx_2 + f_{13}dx_3)dx_1\\
&\quad + (f_{12}dx_1 + f_{22}dx_2 + f_{23}dx_3)dx_2\\
&\quad + (f_{13}dx_1 + f_{23}dx_2 + f_{33}dx_3)dx_3
\end{aligned}$$

Assuming $f_{ij} = f_{ji}$

Or

$$\begin{aligned}
d^2z = f_{11}dx_1^2 &+ f_{12}dx_1dx_2 + f_{13}dx_1dx_3\\
+f_{12}dx_1dx_2 &+ f_{22}dx_2^2 + f_{23}dx_2dx_3\\
+f_{13}dx_1dx_3 &+ f_{23}dx_2dx_3 + f_{33}dx_3^2
\end{aligned} \tag{31}$$

Note that this is in the form of a quadratic equation with three variables dx_1, dx_2 and dx_3 and the coefficients expressed in terms of the second-order partial derivatives. We can thus express equation (31) in matrix form as shown below:

Let $X = \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}_{3\times 1}$ and $A = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{12} & f_{22} & f_{23} \\ f_{13} & f_{23} & f_{33} \end{bmatrix}_{3\times 3}$

Here the matrix consisting of second-order partial derivatives as elements is called a Hessian matrix.

Then

$$d^2z = X'AX \tag{32}$$

where A is by definition a square symmetric matrix. As stated earlier the second-order sufficiency condition requires that d^2z is positive definite if z is a minimum. Using Fact 3, recall that d^2z is positive definite if and only if the principal minor determinants of the Hessian A are all positive, i.e. $f_{11}, f_{22} > 0, \begin{vmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{vmatrix} > 0$ and $|A| > 0$

Using Fact 4, the second-order condition for maximization of an objective function with three decision variables reduces to d^2z is a maximum (negative definite) if and only if the principal minor determinants of the Bordered-Hessian A are alternate in sign with the first principal minor being negative, i.e. $|A_{11}| < 0, |A_{22}| > 0, |A_{33}| < 0 \Rightarrow f_{11} < 0, \begin{vmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{vmatrix} > 0$ and $|A| < 0$

The requisite necessary and sufficient condition for optimisation of an objective function with three decision variables is presented in tabular form in Table 2.

Table 8.2: Condition for Extremum: *$z = f(x_1, x_2, x_3)$*

Condition	Maximum	Minimum
First-Order	$f_1 = f_2 = f_3 = 0$	$f_1 = f_2 = f_3 = 0$
Second-Order	$f_{11} < 0$ and $f_{11}f_{22} > (f_{12})^2$ and $\begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{12} & f_{22} & f_{23} \\ f_{13} & f_{23} & f_{33} \end{vmatrix} < 0$	$f_{11} > 0$ and $f_{11}f_{22} > (f_{12})^2$ and $\begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{12} & f_{22} & f_{23} \\ f_{13} & f_{23} & f_{33} \end{vmatrix} > 0$

Example :

Find the extreme values of $z = -x_1^3 + 3x_1x_3 + 2x_2 - x_2^2 - 3x_3^2$ (33)

For the first-order condition, we need to partially differentiate equation (33) with respect to x_1, x_2 and x_3 and set it to be equal to zero.

$$\frac{\partial z}{\partial x_1} = -3x_1^2 + 3x_3 = 0$$

$$\frac{\partial z}{\partial x_2} = 2 - 2x_2 = 0$$

$$\frac{\partial z}{\partial x_3} = 3x_1 - 6x_3 = 0$$

Solving these three equations yields us the following solution:

$$(x_1^*, x_2^*, x_3^*) = \begin{cases} (0,1,0) \text{ implying } z^* = 1 \\ (\tfrac{1}{2}, 1, \tfrac{1}{4} \text{ implying } z^* = \tfrac{17}{16} \end{cases}$$

The second-order partial derivatives can be rearranged to get the following determinant:

$$|A| = \begin{vmatrix} -6x_1 & 0 & 3 \\ 0 & -2 & 0 \\ 3 & 0 & -6 \end{vmatrix}$$

The principal minors of $|A|$ are:

$$|A_{11}| = |-6x_1|; |A_{22}| = \begin{vmatrix} -2 & 0 \\ 0 & -6 \end{vmatrix}; \text{and } |A_{33}| = |A|$$

At $x_1 = 0$, $|A_{11}| = 0$ which does not satisfy the second-order condition as stated earlier. Hence (0,1,0) is ruled out as a possible extremum.

At $x_1 = 1/2$, $|A_{11}| = -3; |A_{22}| = 12$ and $|A_{33}| = -18$. This duly alternates in sign. Consequently, $z^* = \tfrac{17}{16}$ is a maximum.

Example:

Find the extreme value of the following function:

$$z = 29 - (x_1^2 + x_2^2 + x_3^2)$$

The first-order condition for extremum requires that the simultaneous satisfaction of the following three equations based on the first-order partial derivatives:

$$f_1 = \frac{\partial z}{\partial x_1} = -2x_1 = 0$$

$$f_2 = \frac{\partial z}{\partial x_2} = -2x_2 = 0$$

$$f_3 = \frac{\partial z}{\partial x_3} = -2x_3 = 0$$

There exists only a unique solution $x_1^* = x_2^* = x_3^* = 0$. This means that there is only one stationary value of z i.e. $z^* = 29$.

To evaluate whether this is a relative extremum the second-order condition must be fulfilled. The Hessian determinant (defined earlier) of this function is:

$$|A| = \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{12} & f_{22} & f_{23} \\ f_{13} & f_{23} & f_{33} \end{vmatrix} = \begin{vmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{vmatrix}$$

To verify the second-order condition, we need to see the sign of the principal minors of the Hessian.

Note that

$$|H_{11}| = -2 < 0, |H_{22}| = \begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} = 4 > 0 \text{ and } |H_{33}| = \begin{vmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{vmatrix} = -8 < 0 .$$

This duly alternates in sign with the first principal minor being negative. Thus we conclude that $z^* = 29$ is a maximum.

Check Your Progress 2

1) Express each quadratic form given below as a matrix products involving a square symmetric coefficient matrix:

   a) $z = 3x_1^2 - 4x_1x_2 + 7x_2^2$

   b) $z = 6q_1q_2 - 2q_1^2 - 5q_2^2$

   c) $z = x_1^2 + x_2^2 + x_3^2 - x_1x_3 - x_2x_3$

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2) Ascertain whether the quadratic forms given in Problem 1 are positive definite or negative definite.

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3) Obtain the extreme values, if any of the following functions (indicate whether it is maximum or minimum)

   a) $z = 5 - (x_1^2 + x_2^2 + x_3^2)$

   b) $z = e^{2x} - e^{y} + e^{w} - 2(x + e^{w}) + y$

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8.7 FUNCTIONS WITH n VARIABLES

We have already defined the first-order and second-order conditions for optimisation of functions with three variables. The concept can easily be extended to generate the necessary and sufficient conditions of optimisation of functions with *n – variables* defined below:

$$z = f(x_1, x_2, ..., x_n) \qquad (34)$$

First-Order Condition:

The first-order (necessary) condition for extremum is that z is stationary at that point, i.e., $dz = 0$ for arbitrary constant change in $x_1, x_2, ..., x_n$, not all equal to zero.

Totally differentiating equation (34) we get,

$$dz = f_1 dx_1 + f_2 dx_2 + ... + f_n dx_n$$

For first-order condition to hold, we need $f_1 = f_2 = ... = f_n = 0$

Second-Order Condition:

The second-order differential, as seen earlier, can be expressed in Quadratic form. The relevant Hessian determinant is:

$$|A| = \begin{vmatrix} f_{11} & f_{12} & f_{13} & . & f_{1n} \\ f_{12} & f_{22} & f_{23} & . & f_{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ f_{1n} & f_{2n} & f_{3n} & . & f_{nn} \end{vmatrix}_{n \times n}$$

where f_{ij}'s are the second-order partial derivatives and following Young's theorem $f_{ij} = f_{ji}$ for all $i \neq j$. The n-principal minors $|A_{11}|, |A_{22}|, ..., |A_{nn}|$, as defined before, are formed by deleting the last (n-i) rows and (n-i) columns.

8.8 ECONOMIC APPLICATION OF OPTIMISATION PROBLEM

8.8.1 Multi-plant Monopolist

In this case, we shall examine the case of a monopolist who produces a homogenous product in two different plants. The analysis has been restricted to two plants for simplicity but can easily be generalized to 'n' plants.

Suppose the aggregate market demand (Q) faced by the monopolist as a function of price (P) alone as follows:

$$Q = 100 - 2P$$
$$P = 100 - \tfrac{1}{2}Q \qquad (35)$$

Assume that the cost of producing the output in two different plants is the following:

$$C_1 = 10Q_1 \text{ and } C_2 = \tfrac{1}{4}Q_2^2 \tag{36}$$

where C_1 and C_2 is the total cost of producing the product in plant 1 and in plant 2 respectively. Q_1 and Q_2 is the amount of the homogenous product produced in plant 1 and in plant 2 respectively. Obviously,

$$Q = Q_1 + Q_2 \tag{37}$$

The total profit function faced by the monopolist is:

$$\Pi = R - C_1 - C_2$$

where R is the total revenue.

Now,

$$\begin{aligned} R &= PQ \\ &= (100 - \tfrac{1}{2}Q)Q \\ &= (100 - \tfrac{1}{2}(Q_1 + Q_2))(Q_1 + Q_2) \\ &= 100(Q_1 + Q_2) - \tfrac{1}{2}(Q_1 + Q_2)^2 \\ &= 100Q_1 + 100Q_2 - \tfrac{1}{2}(Q_1^2 + Q_2^2 + 2Q_1Q_2) \end{aligned}$$

Hence,

$$R = -\tfrac{1}{2}Q_1^2 - \tfrac{1}{2}Q_2^2 + 100Q_1 + 100Q_2 - Q_1Q_2 \tag{38}$$

Substituting the value of R from equation (38) into the profit function, we get

$$\Pi = -\tfrac{1}{2}Q_1^2 - \tfrac{1}{2}Q_2^2 + 100Q_1 + 100Q_2 - Q_1Q_2 - C_1 - C_2$$

Substituting for C_1 and C_2 on the basis of equation (36), we get

$$\Pi = -\tfrac{1}{2}Q_1^2 - \tfrac{1}{2}Q_2^2 + 100Q_1 + 100Q_2 - Q_1Q_2 - 10Q_1 - \tfrac{1}{4}Q_2^2$$

$$\Pi = -\tfrac{1}{2}Q_1^2 - \tfrac{3}{4}Q_2^2 + 90Q_1 + 100Q_2 - Q_1Q_2$$

To solve for Q_1 and Q_2 we set the first-order partial derivatives

$\dfrac{\partial \Pi}{\partial Q_1} = \dfrac{\partial \Pi}{\partial Q_2} = 0$ and we get

$$\begin{aligned} \Pi_1 &= \frac{\partial \Pi}{\partial Q_1} = -Q_1 + 90 - Q_2 = 0 \\ Q_1 + Q_2 &= 90 \end{aligned} \tag{39}$$

$$\begin{aligned} \Pi_2 &= \frac{\partial \Pi}{\partial Q_2} = -\frac{3}{2}Q_2 + 100 - Q_1 = 0 \\ 2Q_1 + 3Q_2 &= 200 \end{aligned} \tag{40}$$

Solving equations (39) and (40) simultaneously yields the following output combinations:

$Q_1^* = 70$ and $Q_2^* = 20$

Our next step is to verify that the second-order condition is satisfied at this point. The relevant Hessian determinant is:

$$|A| = \begin{vmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12} & \Pi_{22} \end{vmatrix} = \begin{vmatrix} -1 & -1 \\ -1 & -3/2 \end{vmatrix}$$

where Π_{ij}'s are the second-order partial derivatives. The sufficiency condition for maximization requires that the principal minors of the Hessian alternates in sign with the first principal minor being negative, i.e.

$|A_{11}| = \Pi_{11} < 0$ and

$|\mathrm{A}_{22}| = |A| > 0$

Note that

$|\Pi_{11}| = -1 < 0$ and

$$|A| = \begin{vmatrix} -1 & -1 \\ -1 & -3/2 \end{vmatrix} = 1/2 > 0$$

Thus the necessary and sufficient conditions for profit maximization are both satisfied for $Q_1^* = 70$ and $Q_2^* = 20$.

The monopolist's profit at this output combination i.e. $\Pi^* = 3525$ is the maximum.

Solution: Output produced in first plant = 70 units

Output produced in second plant = 20 units

Maximum Profit = 3525 units

8.8.2 Price Discriminating Monopolist

A monopolist need not sell his entire output in one market. In some situations he is able to sell his output in two or more different market, at different price and thereby increase his aggregate profit. A fundamental requirement for price discrimination is that the buyers cannot buy the product from one market and resell it into another. Price discrimination is often possible in markets that are regionally separated for instance 'home' and 'abroad'. It is also often encountered for products like 'electricity' where resale of such commodities is not feasible.

Assume, for simplicity that a monopolist, sells his product in two markets whose demand functions are as follows:

In Market 1: $p_1 = 80 - 5q_1$ where p_1, q_1 are price charged and quantity sold in the first market.

In Market 2: $p_2 = 180 - 20q_2$ where p_2, q_2 are price charged and quantity sold in the second market.

The aggregate cost function of the monopolist is: $C = 50 + 20\ (q_1 + q_2)$

The revenue earned by the monopolist from Market 1 is the following:

$$R_1 = p_1 q_1 = (80 - 5q_1)q_1 = 80q_1 - 5q_1^2$$

The revenue earned by the monopolist from Market 2 is the following:

$$R_2 = p_2 q_2 = (180 - 20q_2)q_2 = 180q_2 - 20q_2^2$$

The aggregate profit accruing to the monopolist is

$\prod = R_1 + R_2 - C$

Substituting for R_1, R_2 and C we get:

$$\Pi = 80q_1 - 5q_1^2 + 180q_2 - 20q_2^2 - 50 - 20q_1 - 20q_2 \qquad \text{(E)}$$

The First Order Condition for maximisation requires the simulataneous solution of the following equations:

$$\Pi_1 = \frac{\partial \Pi}{\partial q_1} = 80 - 10q_1 - 20 = 0$$

$$or \quad q_1^* = 6$$

$$\Pi_2 = \frac{\partial \Pi}{\partial q_2} = 180 - 40q_2 - 20 = 0$$

$$or \quad \mathrm{q}_2^* = 4$$

The next step is to see whether the Second Order Condition of maximisation is fulfilled at (q_1^*, q_2^*).

For this we require the principal minors of the requisite Hessian Determinant to alternate in signs, with the first being negative.

The Hessian in this case is the following:

$$|A| = \begin{vmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12} & \Pi_{22} \end{vmatrix} = \begin{vmatrix} -10 & 0 \\ 0 & -40 \end{vmatrix}$$

Note $/A_8/ = -10 < 0$ and $/A_{22}/ = /A/ = 400 > 0$

This implies that the second order condition is satisfied for this output combination.

The maximum profit earned by the price discriminating monopolist can be calculated by substituting for $q_1^* = 6$ *and* $q_2^* = 4$ in equation (E).

Hence

II = 80(6) – 5(36) + 180(4) – 20(16) – 50 – 20(6) – 20(4) = 450

Solution: Output sold in Market 1 = 6 units

Output sold in Market 2 = 4 units

Maximum Profit = 450 units

Check Your Progress 3

1) A monopolist uses one input, *X*, which he purchases at the fixed price = *5* to produce his output, *q*. The demand and production functions are $p = 85 - 2q$ and $q = 2\sqrt{x}$ respectively. Determine the values of *p*, *q*, and *x* at which the monopolist maximises his profits.

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2) Let the demand and cost functions of a monopolist be $p = 100 - 3q + 4\sqrt{A}$ and $C = 4q^2 + 10q + A$, where p denotes the price, q the quantity and A the level of advertising expenditure. Find the values of A, q and p that maximises his profit.

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8.9 LET US SUM UP

In this unit, we extended the discussion that was carried out the in relevant unit of the previous course which discussed unconstrained optimisation in the case of only one dependent variable. The analytical process of locating 'extremal points' or in other words the 'optimisation process' was explored in more general terms in this unit. The entire analysis was based on unconstrained optimisation of functions. The restrictive assumption of objective function with only one decision variable was relaxed in this unit and the former was allowed to be a multivariate function. The first and second order conditions that need to be satisfied to classify a point either as a relative 'maximum' or as a 'minimum' for an objective function with multiple decision variables was presented. Finally, we applied these mathematical tools in the context of economics, with examples from the case of a multi-plant monopolist and discriminating monopolist.

8.10 ANSWERS/HINTS TO CHECK YOUR PROGRESS EXERCISES

Check Your Progress 1

1) Read sections 8.2 and 8.3

2) Read section 8.3

Check Your Progress 2

1) Read section 8.4

2) Read section 8.6

3) Read subsection 8.6.2

Check Your Progress 3

1) Read section 8.8

2) Read section 8.8

UNIT 9 CONSTRAINED OPTIMISATION WITH EQUALITY CONSTRAINS

Structure

9.0 OBJECTIVES

After reading the Unit you should be able to:

- Explain the effect of a constraint on an optimisation problem;
- Describe the process of obtaining the stationary values in a constrained optimisation problem;
- Describe the second-order conditions in constrained optimisation problems; and
- Discuss some economic applications of constrained optimization problems.

9.1 INTRODUCTION

In the previous unit, you studied the topic of optimisation. This finds numerous applications in economics. You were made familiar with the case where the dependent variable is a function of several independent variables. However, the optimisation was unconstrained. In economic applications, unconstrained optimisation is a relatively rare case. More often, we find instances of constrained optimisation, that is optimisation subject to a constraint. What this means is that there is a side condition on the optimisation exercise. The domain of the function that is sought to be optimised is restricted by one or more side relations. Consider the case of utility maximization by a consumer. An individual as consumer has unlimited wants. But she is constrained by her budget set. So she maximizes her utility subject to her budget constraint. Similarly a producer would want to minimize her cost subject to a given level of output.

This unit is concerned with an exposition of constrained optimisation. The unit begins with a discussion of how to find stationary values of the objective function. It is shown that in simple cases we can use the method of substitution. But in more complex cases, different methods have to be employed. We will see the importance of the Lagrangian multiplier and how it is used in static constrained optimization. After this the unit discusses second-order conditions for constrained optimization. After discussing these conditions, the unit will go on to discuss some economic applications.

9.2 FINDING THE STATIONARY VALUES

9.2.1 The Method of Substitution

With the help of an example from co-ordinate geometry, we shall learn to maximize/minimize a function subject to a constraint which restricts the domain of the function.

Consider a simple example. Find the smallest circle centred on (0,0) which has a point common with the straight line x+y = 10.

The equation for the circle is $x^2+y^2 = r^2$. The smallest circle will be one with the smallest radius. The restriction is that the circle must have a point in common with a given straight line. Without this restriction, the smallest circle can easily be seen to be a circle with radius zero, i.e. a point.

Thus our problem is the following:

Minimise $x^2 + y^2$ subject to x + y = 10

Here the unconstrained solution x=0, y=0 will not be available. The constraint prohibits this solution. What we have to do is to consider as the domain only those values of x and y for which x + y = 10. So we see that the constraint has diminished the domain. How to find the solution? From the constraint we find $y = 10 - x$. Now if we substitute this into the minimand $x^2 + y^2$ we get $x^2 + (10 - x)^2$, which incorporates the constraint. Let us now minimize this expression

$$\frac{d}{dx}\left[x^2 + (10-x)^2\right] = 2x + 2(10-x)(-1) = 4x - 20$$

For a stationary value $4x - 20 = 0$ or $x = 5$

To check whether the stationary value is truly a minimum we differentiate the function once again with respect to x. Thus,

$$\frac{d^2}{dx^2}\left[x^2 + (10-x)^2\right]$$

$$= \frac{d}{dx}(4x - 20)$$

$$= 4 > 0$$

Which proves that we have minimized the expression $x^2 + y^2$

We know that the constraint makes $y = 10 - x = 5$, thus giving us a solution $x = 5, y = 5$ for which $x^2 + y^2 = 50$.

Let us check on the method used for the problem above, and see whether we can formulate a general method for all such problems. We have a function, in general $f(x, y)$ which we shall call the objective function. This is to be maximised or minimised (as specified), subject to the constraint equation which has the general form $g(x, y) = 0$. (The actual constraint above should be written as $x + y - 10 = 0$ to conform to the general form).

In solving the problem our method involved several steps:

Step 1 : Solve the constraint equation to get one variable in terms of the other. Above we found y in terms of x. Thus, $y = 10 - x$

Step 2 : *Substitute* this solution in the objective function to ensure that the domain of the function is a set of pairs of values of x and y which satisfy the constraint. Note that the objective function thus modified is a function of x only.

Step 3 : Differentiate this modified function to get the first derivative. Find the value of x for which this derivative equals zero.

Step 4 : Put this value of x in the constraint to find the value of y.

Step 5 : Calculate the value of the objective function for this pair of values of x and y.

Step 6 : Check the second order condition to find whether the stationary point is an extremum.

In general it may not be easy or indeed possible to solve the constraint equation. In such cases it would appear that we are stuck at step (1) of the procedure described above. For instance, we may have a constraint equation like $x^3 + 2x^2y + 9y^3 - 2y - 117 = 0$. To find y in terms of x, or x in terms of y from this complicated equation is extremely difficult. Suppose now that the problem has been referred to a mathematician and while we wait for the solution we prepare the ground for our computation.

When the mathematician finds the solution it will be of the form y = h(x). Let us proceed to the next step. In step(2) the objective function f(x,y) is transformed into f(x,h(x)). Step(3) requires to differentiate this function and what we shall get is $f_x + f_y \frac{dh(x)}{dx}$. If we know $\frac{dh(x)}{dx}$ then we set the expression equal to zero and solve for x, completing step (3). The rest of the steps present no problem.

Note that the exact form of the function h(x) is not necessary in the solution of our problem. What we need is the derivative of this problem in order to follow our method of steps (1) to (6). But how could we find the derivative of a function without knowing the function itself?

Strangely enough there is a theorem which tells us how and the conditions when this is possible. It is the implicit function theorem.

The Implicit Function Theorem

If F(x,y) is continuous with continuous derivatives F_x and F_y and if $F(x_0,y_0) = 0$ but $F_y(x_{0,}y_0)\neq 0$, then

i) there exists a rectangle $x_1 \leq x \leq x_2$, and $y_1 \leq y \leq y_2$ such that for every $x \in [x_1, x_2]$ the equation F(x,y) = 0 determines exactly one value y = m (x), $y \in [y_1, y_2]$ i.e., within this rectangle, y can be expressed as a function of x, y = m(x).

ii) This function satisfies $y_0 = m(x_0)$, and for every $x \in [x_1, x_2]$, F(x, m(x)) = 0.

iii) This function m(x) is continuous and differentiable, and its derivative

$$\frac{dy}{dx} = \frac{dm(x)}{dx} = -\frac{F_x}{F_y}$$

(This is because taking total differential of the function F(x,y) = 0, $F_x dx + F_y dy=0$, i.e., $\frac{dy}{dx} = -\frac{F_x}{F_y}$)

We shall not prove this theorem here, but let us translate the theorem into informal language. Let us go back to our problem of finding y in terms of x from our constraint equation whose general form is g(x,y) = 0. Thus our problem is to maximize (or minimize) F(x,y), $F_y\neq 0$ subject to g(x,y)=0. The theorem says that if

i) the function g is continuous with continuous derivatives and

ii) $g_y(x,y) \neq 0$ at a point (x_0, y_0) which satisfied the equation, then we can use the steps outlines earlier even without finding the function y =h(x). In then we can use the steps outlined earlier even without finding the function y=h(x). In the third step we will need h′(x) and the theorem states that $h'(x) = -\frac{g_x}{g_y}$.

Let us apply this method to the following problem.

Find the maximum value of $f(x,y)=x^2 + y^2$ subject to $x^2 + y^2-4x-2y+4=0$.

$g_y = \frac{\delta}{\delta y}(x^2 + y^2-4x-2y+4) = 2y-2$ (for $y\neq 1$ this is non-zero).

By the implicit function theorem whenever $g_y \neq 0$, we can express y as a function of x, namely y=h(x) (even though we do not know the exact form). Substituting into f(x,y) we get

$f(x,y) = f(x,h(x)) = x^2 + (h(x))^2$

A stationary value of z would make $\frac{dz}{dx} = 0$.

$\frac{dz}{dx}$=2x +2h(x)(h'(x))=2x+2y(h'(x))=2x+2y($-\frac{g_t}{g_y}$)

$=2x + 2y\left(-\frac{2x-4}{2y-2}\right) = -x + 2y$

that is, we must have x=2y at the stationary value of z.

Applying step (4) as used above gives us $5y^2-10y+4=0$ from which $y=1\pm\frac{1}{\sqrt{5}}$

and

thus $x=2(1\pm\frac{1}{\sqrt{5}})$. The objective function (step 5) now becomes

$x^2 + y^2 = (1\pm\frac{1}{\sqrt{5}})^2 = 2(3\pm\sqrt{5})$

The last step would involve a somewhat complicated procedure involving second order conditions, but we postpone this till later. In this particular case $2(3+\sqrt{5})$ is a maximum and $2(3-\sqrt{5})$ is a minimum.

We now discuss a method which uses a round-about mode and derives the answer by initially complicating the problem further by introducing another variable. This is known as the Lagrange multiplier method.

9.2.2 The Lagrange Multiplier Method

Consider once more the problem:

Maximise f(x, y) subject to g(x, y) = 0

If $g_y(x, y) \neq 0$, then y = h(x), so that the problem is transformed into:

Maximise f(x,h(x))

1st order condition gives us $f_x + f_y h'(x) = 0$ ……..(i)

g(x,h(x)) = 0 is an identity so that we have

$g_x + g_y h'(x) = 0$ ……..(ii)

Now define $\lambda = \frac{f_y}{g_y}$. Multiplying (ii) by λ, we get

$\lambda g_x + f_y h'(x) = 0$ …….(iii)

From (i) and (iii), $f_x - \lambda g_x = 0$ …….(1)

From $\lambda = \frac{f_y}{g_y}, f_y - \lambda g_y = 0$ …….(2)

Also the constraint is g(x,y) = 0 …….(3)

(1), (2) and (3) are three equations in three variables, x, y and λ. When we solve these three simultaneous equations we get the value of x and y which solve our problem; we also get the value of λ which is Lagrangian multiplier, some thing which does not seem to have any relevance to our problem. We shall see presently that it is full of importance and meaning.

We started this exercise with the assumption g_y (x, y) ≠ 0. You should work out the consequences of this proceeding in the same way as above, except that now one should define $\lambda = \frac{f_x}{g_x}$. The result is that we land with the same set of equation as (1), (2) and (3) above. In a problem a large number of variables, say x_1, x_2, ………….., x_{17}, as long as one partial derivative of constraint function with respect to some variable is non-zero we shall get the same set of equations which are symmetrical with respect to all variables.

Let us apply the lagrangian multiplier method to the following problem:

Example

Minimise $f(x,y) = (x-1)^2 + y^2$ subject to $g(x,y) = y^2 - 4x = 0$.

Define a new function $L(x,y,\lambda) = f(x,y) - \lambda g(x,y)$

$$= (x-1)^2 + y^2 - \lambda (y^2 - 4x)$$

Find the stationary point of L by considering the first order equations.

$L_x = f_x - \lambda g_x = 2(x-1) + 4\lambda = 0$

$L_y = f_y - \lambda g_y = 2y - 2\lambda y = 0$

$L\lambda = -g(x,y) = -(y^2-4x) = 0$

From (ii) $y(1-\lambda)$ 0. For y =0, we get from the other equations x=0, $\lambda = \frac{1}{2}$.

The other alternative λ=1 gives us from (i) x=-1 which conflicts with (iii). The stationary point therefore is at x=0, y = 0 which gives the constrained minimum as $(-1)^2 + (0)^2 = 1$. We shall here ignore the by product $\lambda = \frac{1}{2}$.

Note carefully that we have not suggested that the stationary values of L (x, y, λ) is either a maximum or minimum. Indeed where a constrained extremum can be found, the associated lagrangean function does not have either a maximum or minimum point. For the function L (x, y, λ) the stationary point is **saddle-point**, a minimum in one direction and a maximum from another direction. A saddle point is a higher dimensional analogue of an inflection point.

Also note that function L has been artfully devised so that its first order conditions coincide with the conditions necessary for the solutions of the constrained extremum. It offers the desirable characteristic of symmetry with respect to all variables .

We see that the stationary value, $\frac{f_x}{f_y} = \frac{g_x}{g_y}$. This really means for x, y which solve the problem we have tangency between the constraint and a level curve of the function f(x, y) obtained from the equation (x, y) = c. For various values of c we get various level curves of f and the constrained extremum occurs at appoint where a level curve of f just touches the constraint curve.

Now let us see how this method is useful in Economics. Constrained optimization problems abound in Economics. Frequently some function is maximized subject to some constraint. For example, in consumer equilibrium, we maximise satisfaction subject to income or budget constraint similarly, for producer equilibrium, we obtain a given output at minimum factor cost subject to resource constraint. For constrained optimisation we use langragean multiplier. We explain below the various steps needed for this.

1) State clearly the function to be optimized. This is called the objective function (OF)
2) Identify the constraint function (CF) and reduce it to $C - ax - by = 0$ form (i.e. implicit function)
3) Create another function (z or v) such that $v = OF + \lambda CF$ where λ is some proportion (λ is read as lamda)
4) Find $V_x = 0, V_y = 0$ and solve for x and y. If clear values of x and y are not obtainable, then find $V_\lambda = 0$ (This will be the constraint function). With the help of $V_x = 0$, $u_y = 0$ and $V_\lambda = 0$, the required solution (i.e. values of x, y) can be obtained.

Let us consider an example:

Maximise $5x^2 + 6y^2 - xy$, subject to constraint $x + 2y = 24$

Solution: Given OF : $5x^2 + 6y^2 - xy$

CF : $x + 2y = 24 \Rightarrow 24 - x - 2y = 0$

Let $v = OF + \lambda CF = 5x^2 + 6y^2 - xy + \lambda(24 - x - 2y = 0)$

or $v = 5x^2 + 6y^2 - xy + 24\lambda - \lambda x - 2\lambda y$

$$v_x = 10x + 0 - y + 0 - \lambda - 0 = 0 \quad \text{or} \quad 10x - y = \lambda \qquad (1)$$

$$v_y = 0 + 12y - x + 0 - 0 - 2\lambda = 0 \quad \text{or} \quad 12y - x = 2\lambda \qquad (2)$$

From (13) and (14) we get $20x-2y=12y-x \; x=\frac{2}{3}y$ (3)

Since we do not get clear values of x and y,
therefore we find $v_\lambda = 2u - x - 2y = 0$ (4)

From (15) and (16), we get

$$24-\frac{2}{3}y-2y=0 \quad \text{or} \quad \left(\frac{2}{3}+2\right)y=24 \quad \text{or} \quad y=9$$

From (15) $x=\frac{2}{3}\times 9=6$

Hence constrained maximization takes place when $x=6$ and $y=9$

A consumers utility function is given as $u=(y+1)(x+2)$. If his budget constraint is $2x+5y=51$, how of x and y he should consumer to maximise his satisfaction.

Sol. Given OF : $u=(y+1)(x+2)=xy+x+2y+2$

CF : $51-2x-5y=0$

Let $v=OF+\lambda CF$ where λ is langragean multiplier.

or
$$v=xy+x+2y+2+\lambda(51-2x-5y)$$
$$=xy+x+2y+2+51\lambda-2\lambda x-5\lambda y$$

$$v_x=y+1-2\lambda-0=0 \quad \text{or} \quad \lambda=\frac{y}{2}+\frac{1}{2} \tag{5}$$

$$v_y=x+0+2+0+0-0-5\lambda=0 \quad \text{or} \quad \lambda=\frac{x}{5}+\frac{2}{5} \tag{6}$$

From (17) and (18), we get $\frac{y}{2}+\frac{1}{2}=\frac{x}{5}+\frac{2}{5}$ or $\frac{y+1}{2}=\frac{x+2}{5}$

$$5y+5=2x+4 \quad \text{or} \quad x=\frac{5y+1}{2} \tag{7}$$

Since no clear solution emerges, therefore, we find v_λ.

$$v_\lambda=51-2x-5y=0 \tag{8}$$

From (19) and (20), we get

$$\frac{51-x(5y+1)}{2}-5y=0 \quad \text{or} \quad 51-5y-1-5y=0 \quad \text{or}$$

$$10y=50, y=5$$

$$x=\frac{5y+1}{2}=\frac{5\times5+1}{2}=\frac{26}{2}=13$$

Therefore, the solution is $x = 13, y = 5$. That is the consumer will consumer 13 units x and 5 units of y.

Another example:

Use the method of langragean multiplier to find equilibrium consumption of two goods x and y on the basis of the following information.

Utility function of the consumer is $u = xy + 2x$

Price of x =Rs.4/-, Price of y =Rs.2/- and consumer money income = Rs.60/-

Solution We first obtain constraint function. It is given by

$xp_x + yp_y = M$ or $4x + 2y = 60$ or $60 - 4x - 2y = 0$

We are also given objective function: $4 = xy + 2x$

Applying the langragean multiplier, we get $v = OF + \lambda CF$ as

$$v = xy + 2x + \lambda(60 - 4x - 2y) = xy + 2x + 60\lambda - 4\lambda x - 2\lambda y$$

$$v_x = y + 2 + 0 - 4\lambda - 0 = 0 \quad \text{or} \quad 2\lambda = x \quad \text{or} \quad \lambda = \frac{1}{2}x \qquad (9)$$

$$v_y = x + 0 + 0 - 0 - 2\lambda = 0 \quad \text{or} \quad 2\lambda = x \quad \text{or} \quad \lambda = \frac{1}{2}x \qquad (10)$$

From (9) and (10), we get $\frac{1}{2}x = \frac{y}{4} + \frac{1}{2}$ or $x = \frac{y}{2} + 1$ (11)

Since no clear solution emerges, therefore, we find v_λ as.

$$v_\lambda = 60 - 4x - 2y = 0 \qquad (12)$$

From (11) and (12), we get, $60 - 4\left(\frac{y}{2} + 1\right) - 2y = 0$

or $y = \frac{56}{4} = 14$

$x = \frac{y}{2} + 1 = \frac{14}{2} + 1 = 8$. Thus solution is $x = 8, y = 14$. Consumer is in equilibrium (or gets maximum satisfaction) when he purchases 8 units of good x and 14 units of y.

Check Your Progress 1

1) Explain the concept of : (a) objective function (b) constraint.

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2) Outline the method of substitution in obtaining a solution to a constrained optimisation problem.

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3) Describe the method of Lagrange multiplier to obtain a solution to a constrained optimisation problem, outlining the various steps.

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9.3 SECOND ORDER CONDITIONS

As explained earlier, optimisation consists of both maximisation and minimisation. We want to maximise one which is useful to us and minimise what is costly to us. For example, we want to maximise bus service in a city and minimise pollution . A student would like to maximise marks with minimum efforts.

We would like to obtain conditions of sufficiency for optimisation subject to constraints. We saw that both for maximisation and for minimisation, the first order conditions are the same. since we were searching for stationary values, we formed a Lagrangian function and set the first-order partial derivatives to zero. The second-order conditions help us determine conditions for maximization or minimization. Le us study the second-order conditions now. For that we will need to build up some tools to which we now turn. The first of these is is the concept of total differential. You have already studied this.

Total Differential

The concept of total differential is very useful in constrained optimisation. In discussing the concept of total differential, remember that if we have a function $f(x, y)$ then the symbol f_x stands for $\frac{\partial f}{\partial x}$

If $z = f(x, y)$ is a homogeneous function of first degree, then the total differential dz can be expressed as:

$dz = f_x d_x + f_y d_y = \frac{\partial z}{\partial x} \cdot d_x + \frac{\partial z}{\partial y} d_y$ approximately

It may be noted that this formula holds good whether x and y are dependent or independent variables. The expression dz shows the increment in the function $z = f(x, y)$ when there is an infinitesimal increments in x and well as y. For example

if $z = x^3 + y^3$, then total differential can be expressed as $dz = f_x d_x + f_y d_y = 3x^2 d_x + 3y^2 d_y$

The following rules on total differential will be found useful. Let z and w represent two functions of x and y, then

1) $d(w \pm z) = dw \pm dz$
$= (f_x d_x + f_y d_y) \pm (g_x d_x + g_y d_y)$

2) $d(wz) = w.dz + z\,dw$
$= w(g_x d_x + g_y d_y) + z(f_x d_x + f_y d_y)$ (Product rule)

3) $d\left(\frac{w}{z}\right) = \frac{z.dw - wdz}{z^2}$
$= \frac{z(f_x d_x + f_y d_y) - w(g_x d_x + g_x(d_y))}{z^2}$ (Quotient Rule)

4) $z - f(u)$ and $u - g(x, y)$, then
$dz = f'(u) \cdot du$, where du is differential of u which in turn is a function of x and y.
Example. $z = u^n$, where $u = f(x, y)$, then
$dz = \frac{d}{dx}(u^n) \cdot du = nu^{n-1} \cdot du$

Let us now solve some problems on total differentials.

1) Find du when $u = 3x^3 + 2y^2 + y^3$

Sol. Total differential du is given by:
$du = f_x d_x + f_y d_y = 9x^2 d_x + (uy + 3y^2) d_y$
$= 9x^2 d_x + y(u + 3y) d_y$

2) Find total differentials y the following functions

a) $u = \frac{x^2 - y^2}{x^2 + y^2}$ b) $w = e^{x^2 - y^2}$ c) $u = \log(x^2 + y^2)$

Sol. a) $u = \frac{x^2 - y^2}{x^2 + y^2}$, apply quotient rule

$$\frac{z\left(f_x d_x + f_y d_z\right) - w\left(g_u d_u + f_y d_y\right)}{z^2}$$

$$= \frac{\left(x^2 + y^2\right) d\left(x^2 - y^2\right) - \left(x^2 - y^2\right) d\left(x^2 + y^2\right)}{\left(x^2 + y\right)^2}$$

$$= \frac{\left(x^2 + y^2\right)\left(2xdx - 2ydy\right) - \left(x^2 - y^2\right)\left(2xdx + 2ydy\right)}{\left(x^2 + y^2\right)^2}$$

$$= \frac{4xy^2 dx - 4x^2 ydy}{\left(x^2 + y^2\right)^2}$$

b) $w = e^{x^2 - y^2}$

Put $u = x^2 - y^2$ so that $w = e^u$ and $dw = e^u du$ (13)

Also $du = d\left(x^2\right) - d\left(y^2\right) = 2xdx - 2ydy$ (14)

From (11) and (9), we get

$$dw = e^{x^2 - y^2} \cdot \left(2xdx = 2ydy\right) = 2xe^{x^2 - y^2} dx - 2ye^{x^2 - y^2} dy$$

c) $u = \log\left(x^2 + y^2\right)$

Let us try it by using the formula:

$$du = f_x d_x + f_y d_y = \frac{1 \times xx}{\left(x^2 + y^2\right)} dx + \frac{1 \times 2y}{\left(x^2 + y^2\right)} dy$$

$$= \frac{2x}{x^2 + y^2} dx + \frac{2y}{\left(x^2 + y^2\right)} dy = \frac{2xdx + 2ydy}{\left(x^2 + y^2\right)} = \frac{2\left(xdu + ydy\right)}{\left(x^2 + y^2\right)}$$

But for studying second-order conditions in the case of constrained optimization, we need to be able to compute second order differentials.

We may state here a basic result. Given the equation $z = f\left(x, y\right)$, its second-order total differential is:

$$d^2 z = f_{xx} dx^2 + 2f_{xy} dxdy + f_{yy} dy^2 + f_y d^2 y$$

Now let us consider the second-order sufficiency conditions in the case of constrained optimum. Before proceeding, it is requested to go through the previous unit again carefully, as there the concepts of total differentials and quadratic forms were discussed, and these come in handy to study constrained optimisation.

To proceed, consider the objective function $dg = 0$

$$z = f\left(x, y\right)$$

subject to the constraint

$$g(x,y) = c$$

Here c is a constant.

Proceeding systematically, we write the Lagrangian function as

$$L = f(x,y) + \lambda\left[c - g(x,y)\right]$$

For stationary values of L, regarded as a function of the three variables λ, x, y, the necessary condition is

$$L_x = f_x - \lambda g_x = 0$$

$$L_y = f_y - \lambda g_y = 0$$

$$L_\lambda = c - g(x,y) = 0$$

For the second-order necessary-and-sufficient conditions still are related to the algebraic sign of the second-order total differential d^2z evaluated at a stationary point, as in the case of unconstrained optimization that you studied in unit 8 in block 4, but there is one difference in the case of constrained optimization. In constrained optimization, we are concerned with the sign definiteness or semi-definiteness of d^2z, not for all possible values of dx and dy, but only for those dx and dy values satisfying the linear constraint $g_x dx + g_y dy = 0$. Thus the second-order *necessary* conditions are:

For maximum of z, d^2z negative semidefinite, subject to $dg = 0$

For minimum of z, d^2z positive semidefinite, subject to $dg = 0$.

The second-order *sufficient* conditions are:

For maximum of z, d^2z negative definite, subject to $dg = 0$

For minimum of z, d^2z positive definite, subject to $dg = 0$.

Let us concentrate on the second-order sufficient conditions. In unit 8, we saw that the second-order sufficient could be expressed using the Hessian determinant. However, in the case of constrained extremum, we come across what is called a *bordered* Hessian. This determinant used in this case in nothing but the original determinant, with a border (row) placed on top and a similar border (column) on the left. Moreover, this border is merely composed of the coefficients from the constraint, with a zero in the principal diagonal. We had expressed the second-order differential as: given the equation $z = f(x,y)$, its second-order total differential is:

$$d^2z = f_{xx}dx^2 + 2f_{xy}dxdy + f_{yy}dy^2 + f_y d^2y\,.$$

Now recall the first-order conditions that we mentioned a little while ago:

$$L_x = f_x - \lambda g_x = 0$$

$$L_y = f_y - \lambda g_y = 0$$

$$L_\lambda = c - g(x, y) = 0$$

We can partially differentiate the derivatives again to get:

$$L_{xx} = f_{xx} - \lambda g_{xx}$$

$$L_{yy} = f_{yy} - \lambda g_{yy}$$

$$L_{xy} = f_{xy} - \lambda g_{xy} = L_{yx}$$

Making use of the Lagrangian, we can express d^2z as:

$$d^2z = L_{xx}dx^2 + L_{xy}dxdy + L_{yx}dydx + L_{yy}dy^2$$

When we apply the bordered Hessian to the above case, we get the conditions

d^2z is positive definite subject to $dg = 0$ if $\begin{pmatrix} 0 & g_x & g_y \\ g_x & L_{xx} & L_{xy} \\ g_y & L_{yx} & L_{yy} \end{pmatrix}$ (determinant) < 0

The condition for negative definiteness is similar with the sign of bordered Hessian reversed (> 0)

For the n-variable, general case, the condition for relative constrained extremum for

$$z = f(x_1, ..., x_n)$$

subject to $g(x_1, ..., x_n) = c$

with $L = f(x_1, ..., x_n) + \lambda\left[c - g(x_1, ..., x_n)\right]$

The first-order necessary condition for a maximum is

$$L_\lambda = L_1 = ... = L_n = 0$$

Here the subscript denotes partial derivative of L with respect to that variable.

The first order necessary condition for a minimum is the same.

The second-order sufficient condition for a maximum is

$$\left|\bar{H}_2\right| > 0; \left|\bar{H}_3\right| < 0; \left|\bar{H}_4\right| > 0 .....(-1)^n \left|\bar{H}_n\right| > 0$$

For a minimum, the second order condition is that

$$\left|\bar{H}_1\right|, \left|\bar{H}_2\right|, ..., \left|\bar{H}_n\right| < 0$$

Here the H with the bar on top stands for the bordered Hessian, and the subscripts stand for various order of the determinant.

Check Your Progress 2

1) What is a bordered Hessian?

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2) State the second-order sufficient condition for a minimum in the case of constrained minimum.

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9.4 ECONOMIC APPLICATIONS

9.4.1 Consumer's Equilibrium

We first consider a simple case: maximisation of a utility function subject to a budget constraint. Let the utility function be u(x, y) and the budget constraint $p_x x + p_y y = M$. Using the Lagrangian multiplier method we find that the necessary first order conditions require that $\frac{u_x}{u_y} = \frac{p_x}{p_y}$ and that the values of x and y are such that the budget constraint is satisfied. The second order condition would involve second order partial derivatives of function u, but these partial derivatives have to be defined on the set of x, y which satisfies the constraint. The constraint equation $p_x x + p_y y = M$ yields $p_x dx + p_y dy = 0$. We can solve for $dx = -\frac{p_y}{p_x} dy$. Now let us go back to the concept of quadratic form we discussed earlier. For maximum utility we must have stationary point at which the quadratic form $d^2u = u_{xx} + 2u_{xy}dxdy + u_{yy}dy^2$ is negative definite. In the constrained maximisation at hand we do not examine all dx and dy; instead we limit ourselves to dx, dy which fit the constraint, namely where $dx = -\frac{p_y}{p_x}$ dy is satisfied. To ensure this we substitute and get

$$u_{xx}\left(\frac{p_y}{p_x}dy\right)^2 + 2u_{xy}\left(\frac{p_y}{p_x}dy\right)dy + u_{yy}dy^2$$

$$= \left[\frac{p^2{}_y}{p^2{}_x}u_{xx} - 2\frac{p_y}{p_x}u_{yy}\right]dy^2$$

$$= (u_{xx}p_y^2 - 2u_{xy}p_x p_y + u_{xy}p_y^2$$

$$= -\begin{vmatrix} 0 & p_x & p_y \\ p_x & u_{xx} & u_{xy} \\ p_y & u_{yx} & u_{yy} \end{vmatrix} \frac{dy^2}{p_x^2}$$

The determinant is known as the bordered Hessian determinant for obvious reasons.

For a maximum the quadratic form defined by d^2u has to be negative; when we constraint the function the quadratic form is transformed, but this form also must be negative. As d^2y and p^2x are both positive and the entire expression starts with a negative sign, the bordered Hessian determinant must be **positive** for a maximum. For a minimum by a similar set of arguments the determinant must be **negative**.

The utility maximisation problem discussed above is simple on two counts: (i) the constraint is linear; and (ii) only two variables are considered. A problem involving a non-linear constraint is to minimise the cost of attaining a given level of utility . Formally this problem can be stated as:

Minimise $p_x x + p_y y$ subject to $u(x,y) = \bar{u}$.

Here the constraint is given indifference curve which is curve convex to origin. Again any problem involving n variables, n > 2 cannot be handled by the technique discussed above. The mathematics for both types of complexity is too involved to present here but we shall state the general result without proof.

Suppose that the problem is:

Maximise $f(x_1,x_2, \ldots\ldots x_n)$ subject to $g\,(x_1,x_2, \ldots\ldots x_n) \neq 0$ and that g is a non-linear function. Then the second order condition is the boarded Hessin determent(D)

$$\begin{vmatrix} 0 & g_1 & g_2 & g_n \\ g_1 & f_{11} - \lambda g_{11} & f_{12} - \lambda g_{12} & f_{1n} - \lambda g_{1n} \\ g_2 & f_{21} - \lambda g_{21} & f_{22} - \lambda g_{22} & f_{nn} - \lambda g_{2n} \\ g_n & f_{n1} - \lambda g_{1n} & f_{n2} - \lambda g_{2n} & f_n - \lambda_n g_{2n} \end{vmatrix}$$

has the sign $(-1)^n$.

and the principal minors have the sign $(-1)^t$ where t is the order of the principal minor $t \geq 2$.

For a minimum D and its principal minors must all be negative. Let us try this on problem of minimising cost to attain a given level of utility.

9.4.2 Cost and Supply

Assume a smooth production function with variable inputs, say *L* and *K*, standing for labour and capital. Let the production function be $Q = Q(L, K)$ where $Q_L, Q_K > 0$. If w and r are the prices of labour and capital respectively,the problem here is one of minimizing the cost

$$C = wL + rK$$

subject to the output constraint $Q(L, K) = Q_o$

Hence the Lagrangian function is

$$Z = wL + rK + \lambda\left[Q_o - Q(L, K)\right].$$

The first-order condition are

$$Z_\lambda = Q_o - Q(L, K) = 0$$

$$Z_L = w - \lambda \frac{\partial Q}{\partial L} = 0$$

$$Z_K = r - \lambda \frac{\partial Q}{\partial K} = 0$$

The last two conditions imply the condition

$$\frac{w}{\partial Q / \partial L} = \frac{r}{\partial Q / \partial K} = \lambda$$

The denominators above denote the marginal products of the inputs.

Thus at the point of optimal input combination, the price-marginal product ratio must be the same for each input. Since this ratio measures the amount of outlay per marginal product of each input, the Lagrangian multiplier can be interpreted as the marginal cost of production in the optimum state.

The above equation can also be written as

$$\frac{w}{r} = \frac{MP_L}{MP_K}$$

This says that the ratio of marginal products, which is the marginal rate of technical substitution is equal to the input prices.

Check Your Progress 3

1) Briefly set out how the consumer's equilibrium can be found using the Lagrange multiplier method.

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2) How would you use the concept of constrained optimization to explain least-cost combination of input use by a firm/

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9.5 LET US SUM UP

This unit extended the discussion started in the previous about optimization. The previous unit had discussed unconstrained optimization. This unit provided a discussion about optimization in the presence of constraints. Constraints, as we saw, introduce restrictions so that the constrained optimum in general differs from free optimum. The constraint, or the side relation essentially narrows down the domain, and therefore the range of the objective function.

The unit then went on to a discussion of how to find stationary values. It mentioned that for very simple problems, a simple way could be substituting the conditions of the constraint into the objective function, but pointed out that this method may not work in more complicated cases. Then the crucial, and one of the key tools for a student of economics, the Lagrange multiplier method was discussed. The way the Lagrangian function is set up was explained, and the idea of the multiplier was discussed.

Following this, the unit went on to a discussion of second-order conditions for constrained optimization. The unit elaborated on the concept of bordered Hessian and its use in deriving second-order condition for constrained optimization. Finally the unit went on to discuss applications of constrained optimization in economics. Instances of constrained optimization in economics is ubiquitous, and constrained optimization forms a central tool and technique in economics.

9.6 ANSWERS/HINTS TO CHECK YOUR PROGRESS EXERCISES

Check Your Progress 1

1) See subsection 9.2.1

2) See subsection 9.2.1

3) See subsection 9.2.2

Check Your Progress 2

1) See section 9.3

2) See section 9.3

Check Your Progress 3

1) See subsection 9.4.1

2) See subsection 9.4.2

UNIT 10 DUALITY

Structure

10.0 OBJECTIVES

After studying the Unit, you should be able to:

- Explain the idea of comparative statics and distinguish it from statics;
- Distinguish between comparartive statics in the context of optimisation from that in non-optimisation situations;
- Define the maximum value function;
- State the Envelope theorem;
- Explain the notion of duality in optimisation analysis; and
- Describe some economic applications of maximum value function, envelope theorem and duality.

10.1 INTRODUCTION

In unit 8, you were acquainted with the techniques to optimise a function, where the optimisation is without constraints. Also, in the preceding unit, you studied constrained optimisation. This unit discusses some topics related to optimisation. But for that you have to keep in mind that an optimum point is a special type of equilibrium situation.

We know that an equilibrium is a point of rest at which no agent has any incentive to change her behaviour. But it is of interest to us to see how equilibrium values change as parameters change. For example think of utility maximisation subject to budget constraint. Carrying out this exercise gives us the demand function. The demand for a good is a function of the prices and the consumer's income. However, in the utility maximisation exercise, the prices and income are parameters. We can see how utility would change if prices and/or income change. Comparative statics studies how equilibrium values change as parameters change. It could be in an optimisation or non-optimisation context.

In the next section, we present a discussion about the concept of comparative statics, both in the optimisation context as well as non-optimisation context. Section 10.3 deals with a very important concept used in comparative static analysis in the optimisation context, namely, the maximum value function. This basically looks that how the value of the optimised objective function changes as the parameters in the optimisation exercise change. The section, further, discusses the crucial envelope theorem, which provides surprising insight into the optimisation exercise. The envelope theorem is discussed for both constrained as well as unconstrained optimisation exercise. This section also provides an interpretation of the Lagrangean multiplier in the constrained optimisation exercise. The subsequent section discusses some economic applications of the maximum value function and the envelope theorem, and as such, discusses the indirect utility function, Roy's identity and Hotelling's Lemma.

Section 10.5 takes up a discussion of an important idea in the context of constrained optimisation, namely, duality. Roughly speaking it involves converting a given constrained optimisation problem into its 'dual', in order to obtain some powerful insights and results. Converting an original problem into its dual means a new optimisation exercise where the original maximisation (minimisation) problem is converted into a minimisation (maximisation) problem, and where the constraint of the original problem appears as the objective function of the new problem; moreover, the parameters in the original problem become the arguments in the new objective function. The section explains the concept of duality through an example of utility maximisation. It notes for example, that the dual of the utility maximization problem is the expenditure minimization problem.

Section 10.6 applies duality theory to some economic settings, and explains the compensated demand function and Shephard's lemma.

10.2 UNDERSTANDING COMPARATIVE STATICS

You are now acquainted with the notions of optimisation, both with and without constraints. You are also familiar with the idea of equilibrium, which essentially means a state of rest. In the earlier blocks you were introduced to the idea of statics, which means that the variables are not dated, time does not enter explicitly into the picture. In this section, we propose to deal with the concept of comparative statics, which, as the name suggests, means a comparison to two static situations. The idea is that given a situation of statics and where equilibrium prevails, that is, we have come to know the values of the endogenous variables in equilibrium, and where the situation is of rest, what would happen, if the values of the parameter were to change. For example, in a demand-supply scenario, suppose we know the equilibrium values of the quantity of the goods being exchanged as well as the equilibrium price that prevails, what would happen if the parameters, like income, or the prices of other goods, were to change? Similarly, given a scenario of optimisation, say utility maximisation by the consumer, suppose we have found out the optimal quantities of the goods for which the utility is maximised subject to the budget constraint, what will happen if the parameter values that appear in the constraint (in this case the budget constraint) were to change, like prices and income in the budget constraint. In the following sub-section we will discuss comparative statics in a context where, optimisation, though perhaps implicitly present, is not explicitly taken into consideration. In the subsection after that, we consider comparative statics in an optimisation context.

10.2.1 Comparative Statics in non-Optimisation Context

Comparative statics, as we have seen, is concerned with the comparison of different equilibrium states that are associated with different values of parameters and exogenous variables. We start by assuming a given equilibrium state and see what happens when there is a disequilibriating change. The initial equilibrium position will be disturbed, and consequently, exogenous variables would undergo certain adjustments. If we assume that a new equilibrium position will be attained as a result of change in the values of the parameters and exogenous variables, then comparative static analysis inquires how the new equilibrium position compares with the old. If we are interested only in the direction of change, then the comparative static exercise is qualitative; if we look at the direction as well as the magnitude of the change, then it is quantitative comparative statics.

A pair of demand and supply functions determines the equilibrium price and the output when we solve the two equations simultaneously. Graphically this means that the point of inter section of the two curves, demand and supply; give us the equilibrium price and the quantity. What happens when one of the curves say the demand curve is shifted? We get another intersection which is the equilibrium price and quantity pair for the shifted demand curve and the supply curve.

This is an example of comparative statics, where we compare two equilibrium configurations. It is called statics because time does not play any part in this analysis. We can think of our problem as a problem involving two separate positions of demand curve, one for a specific value of parameter, say income, and the other for another value of income. What we want to determine is the equilibrium price and quantity change in response to variations in the value of the parameter.

Let us consider a linear demand and supply model. Let us write the equations for the model:

Example 1

q–D(p,y) = 0, the demand equation (1)

q–S(p) =0, the supply equation (2)

If we solve the two equations simultaneously we should get the values of equilibrium p and q corresponding to any given values of y. If the two equations are linear then this is very easy.

Let the two equations be

q = a + bp + α y, the demand equation (3)

q = c + dp, the supply equation (4)

Now, q –bp = a + α y

q –dp = c

$$\therefore q = \frac{\begin{vmatrix} a+\alpha y & -b \\ c & -d \end{vmatrix}}{\begin{vmatrix} 1 & -b \\ 1 & -d \end{vmatrix}} = \frac{ad - \alpha yd + bc}{b-d}$$

$$= \frac{bc-ad}{b-d} - \frac{\alpha d}{b-d} y$$

$$= \frac{ad-bc}{d-b} + \frac{\alpha d}{d-b} y$$

$$\text{Also, } p = \frac{\begin{vmatrix} 1 & a+\alpha y \\ 1 & c \end{vmatrix}}{\begin{vmatrix} 1 & -b \\ 1 & -d \end{vmatrix}}$$

$$= \frac{c-a-\alpha y}{b-d} = \frac{c-a}{b-d} - \frac{\alpha}{b-d} y$$

$$= \frac{a-c}{d-b} + \frac{\alpha}{d-b} y$$

The solution is, $p=\frac{a-c}{d-b}+\frac{\alpha}{d-b}y; q=\frac{ad-bc}{d-b}+\frac{\alpha d}{d-b}y.$

Now $\frac{dp}{dy}=\frac{\alpha}{d-b}$: d is the slope of supply curve, normally positive; b is the slope of the demand curve and therefore negative, so that d–b > 0. When α is positive (i.e., income elasticity of demand positive) we see that the higher equilibrium price corresponds to a higher value of income. We can do a similar exercise for q, the equilibrium quantity.

Now let us consider a non-linear model.

When we do not have linear equations as in (3) and (4) solutions may not be easy. In any case for comparative statics the exact solution is not necessary; what is necessary is to find the derivatives of the equilibrium price and quantity with the respect to the parameter income. Can we bypass the question of the exact solution and yet find the derivative of this solution with respect to the parameter?

The answer is yes, if we take help of the **implicit function theorem**.

If $f^1(x_1, x_2, \ldots\ldots, x_n; a_1, a_2, \ldots\ldots, a_t)$

$f^2(x_1, x_2, \ldots\ldots, x_n; a_1, a_2, \ldots\ldots, a_t)$

$f^n(x_1, x_2, \ldots\ldots, x_n; a_1, a_2, \ldots\ldots, a_t)$

are continuously differentiable at

$\left(x_1^0, x_2^0, \ldots\ldots, x_n^0; a_1^0 a_2^0, \ldots\ldots, a_t^0\right)$ and if $f^1\left(x_1^0, x_2^0, \ldots\ldots, x_n^0; a_1^0 a_2^0, \ldots\ldots, a_t^0\right)=0$

if i = 1, 2, ……, n and if

$$J=\begin{vmatrix} f_1^1 & f_2^1 & \ldots\ldots & f_n^1 \\ f_1^2 & f_2^2 & \ldots\ldots & f_n^2 \\ . & . & \ldots\ldots & \\ f_1^n & f_2^n & \ldots\ldots & f_n^n \end{vmatrix} \neq 0 \text{ where } f=\frac{\partial f}{\partial x_3}$$

then there exists a neighbourhood R of $\left(a_1^0 a_2^0, \ldots\ldots, a_t^0\right)$ and a set of functions (unique) $h^1(a_1, a_2, \ldots\ldots, a_t)$, I = 1, 2, ……,n letting $x_1=h^1$, such that

i) $x_1^0=h\left(a_1^0 a_2^0, \ldots\ldots, a_t^0\right)$, i = 1, 2, …….n.

ii) $f^1\left[h^1, h^2, \ldots\ldots, h^n; a_1, a_2, \ldots\ldots a_t\right]=0$ for all $a_1 \in R$

iii) h^1 are continuously differentiable on R.

This complicated statement really amounts to saying that a solution of the set of equations written as implicit function exists; this solution makes every variable a function of the parameter ; and that when these solutions are substituted in the equations , they are transformed into identities so that it is now possible to differentiate both sides of these identities and set them equal

– something which is not valid for a mere equation (Differentiate the equation $x^2 - 4x + 3=0$ and check that for the solution x = 1, or x = 3 the derivative of equation $x^2 - 4x + 3$ is not equal to zero).

The condition which is necessary for these results is that the Jacobian determinant should be non-zero, which guarantees that the equations are independent and that a solution exists. We do not want the explicit solution here. The determining J in the previous page is a Jacobian determinant.

Let us use this theorem on equations (1) and (2) above.

The general form is written first so tat it is easy to see the relevance of the theorem:

$f^1(q, p, y) = q - D(p,y) = 0$ …… (1)

$f^2(q, p, y) = q - S(p) = 0$ …… (2)

By the implicit function theorem there exist functions q(y) and p(y) which substituted into (1) and (2) give us

$q(y) - D(p,(y), y) = 0$

$q(y) - S(p,(y)) = 0$

Note that these are identities.

On differentiation we can write

$$\frac{dq}{dy} - \frac{dD}{dp}\frac{dp}{dy} = \frac{dD}{dy} \qquad (3)$$

$$\frac{dq}{dy} - \frac{dS}{dp}\frac{dp}{dy} = 0 \qquad (4)$$

These are two linear equations in the two variables $\frac{dq}{dy}$ and $\frac{dp}{dy}$. Solving by Cramer's rule,

$$\frac{dq}{dy} = \frac{\begin{vmatrix} \frac{dD}{dy} & -\frac{dD}{dp} \\ 0 & -\frac{dS}{dp} \end{vmatrix}}{\begin{vmatrix} 1 & -\frac{dD}{dp} \\ 1 & -\frac{dS}{dp} \end{vmatrix}} = \frac{\frac{dD}{dy}\frac{dS}{dp}}{\frac{dS}{dp} - \frac{dD}{dp}}$$

and $$\frac{dp}{dy} = \frac{\begin{vmatrix} 1 & \frac{dD}{dy} \\ 1 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & -\frac{dD}{dp} \\ 1 & -\frac{dS}{dp} \end{vmatrix}} = \frac{\frac{dD}{dy}}{\frac{dS}{dp} - \frac{dD}{dp}}$$

In this exercise we have used the determinant of co-efficient to divide. This determinant should be non-zero to validate our procedure. Why? Because the Jacobian determinant

$$J \equiv \begin{vmatrix} f_1^1 & f_2^1 \\ f_1^2 & f_2^2 \end{vmatrix} = \begin{vmatrix} \frac{\partial f^1}{\partial q} & \frac{\partial f^1}{\partial p} \\ \frac{\partial f^2}{\partial q} & \frac{\partial f^2}{\partial p} \end{vmatrix} = \begin{vmatrix} \frac{\partial q}{\partial q} & -\frac{\partial D}{\partial p} \\ \frac{\partial q}{\partial q} & -\frac{\partial S}{\partial p} \end{vmatrix}$$

Indeed this determinant is the Jacobian determinant necessary for the application of the method of the implicit function theorem.

You should now check the results for the linearised versions in equations (3) and (4) to confirm that they are special cases of the more general result now obtained for (1) and (2).

10.2.2 Optimisation and Comparative Statics

In optimisation problems, there is not merely the idea of equilibrium. The primary exercise is optimising an objective either without constraints, or subject to constraints. The objective function shows a dependent variable as a function of one or more independent variables. Moreover, there could be parameters that enter the objective function or the constraint, or both. Now, using calculus techniques, suppose we have found the optimum value of the independent variable(s) for which the objective function is maximised. Now corresponding to optimal values of the independent variable(s) there would be an optimal value of the dependent variable. So we find the optimal value of this dependent variable.

We can sense that as the values of the parameters change, so will the *optimal* value of the dependent variable. This relation between the optimal value of the dependent variable as a function of the parameters is the subject matter of comparative statics in optimisation context. Let us make it clear by an example. Suppose we maximise utility function:

Maximise $U = U\left(x_1, x_2, ..., x_n\right)$

subject to $p_1x_1 + ... + p_nx_n = m$

Here x_1 to x_n are n goods, and p_1 to p_n their prices. Income is denoted by m. Here the utility is the dependent variable, the goods are the independent variable, the prices and income are parameters. Now suppose we have found the optimal values of the goods, say $\left(x_1^*,...,x_n^*\right)$. This enables us to find the optimal value of U which we denote as U^*. What comparative statics deals with are the changes that will be brought about in the values of U^* as the prices and incomes change. We shall deal with this is greater detail in the next section.

10.3 MAXIMUM VALUE FUNCTIONS AND THE ENVELOPE THEOREM

10.3.1 Maximum Value Function

Suppose we have an objective function $u = f(x, y, \alpha)$. Here α is a parameter. After solving, let the values of x and y that solve for u be x^*, y^*. Then, given the values x^*, y^*, and by varying α, we will get a function $u^* = f\left(x^*(\alpha), y^*(\alpha), \alpha\right) = V(\alpha)$. Here, V(.) is called a maximum value function. What the function means is that the maximum value of the objective will be influenced by the values of the independent variables that maximise the objective function. Thus the maximum value of u will be influenced by the values of x and y that maximise u. This might seem obvious and a tautology. But the maximum value function further shows that the values of x and y at which u is maximised are themselves functions of the parameter α. Moreover the parameter α itself independently influences the maximum value of u. Thus the maximum value of the objective function is a function of the values of x and y that maximise u (and these values are functions of the parameter), as well as of the parameter. Thus indirectly, the maximum value of the objective function is a function of the parameter.

10.3.2 The Envelope Theorem for Unconstrained Optimisation

We have understood the concept of the maximum value function. Now we discuss a very important theorem with regard to the maximum value function, namely the envelope theorem. In this subsection we explain the envelope theorem with regard to the maximum value function in the case of unconstrained optimisation, while in the next subsection we consider the envelope theorem pertaining to the case of constrained optimisation.

Recall the chain rule that you studied in the unit on partial differentiation. Let us use that to differentiate the maximum value function with respect to the parameter. In other words, let us differentiate $V = f\left(x^*(\alpha), y^*(\alpha), \alpha\right)$ with respect to α. Doing so, we get

$$\frac{dV}{d\alpha} = \frac{\partial f}{\partial x}\frac{\partial x^*}{\partial \alpha} + \frac{\partial f}{\partial y}\frac{\partial y^*}{\partial \alpha} + \frac{\partial V}{\partial \alpha}$$

However, from the first order condition of optimisation, we know that

$\frac{\partial f}{\partial x}=0, \frac{\partial f}{\partial y}=0$ and hence the first two terms in the equation above will vanish, and we are left with

$$\frac{dV}{d\alpha}=\frac{\partial V}{\partial \alpha}.$$

The envelope theorem says that, at the optimum, as α varies, with x^* and y^* allowed to adjust, the derivative $\frac{dV}{d\alpha}$ gives the same result we would have got had x^* and y^* been treated as constants. We can see that α enters the maximum value function directly as well as indirectly through x^* and y^*. So, in effect, α enters the maximum value function at three places. The basic idea behind the envelope theorem is that at the maximum, only the direct effect of a change in the parameter needs to be considered. Only the direct effect of the parameter matters, even though the effects of the parameter enters indirectly as well, influencing the endogenous choice variables. Of course, in all this, remember that there can be more than two exogenous variables or more than one parameter.

10.3.3 The Envelope Theorem for Constrained Optimisation

Again, suppose we have an objective function $u=f(x,y,\alpha)$. But now suppose we maximise this function subject to a constraint. Let the constraint be a function

$$h(x,y,\alpha)=0$$

The Lagrangean for this optimisation exercise is

$$L=f(x,y,\alpha)+\lambda\left[0-h(x,y,\alpha)\right]$$

The first-order conditions will be

$$\frac{\partial L}{\partial x}=\frac{\partial f}{\partial x}-\lambda\frac{\partial h}{\partial x}=0$$

$$\frac{\partial L}{\partial y}=\frac{\partial f}{\partial y}-\lambda\frac{\partial h}{\partial y}=0$$

$$\frac{\partial L}{\partial \lambda}=-h(x,y,\alpha)=0$$

Solving this system gives us $x=x^*(\alpha); y=y^*(\alpha);$ and $\lambda=\lambda^*(\alpha)$

Substituting the solution into the objective function, we get

$$u^*=f\left(x^*(\alpha),y^*(\alpha),\lambda^*(\alpha)\right)=V(\alpha)$$

To see how $V(\alpha)$ changes with a change in α, we differentiate V with respect to α and get:

$$\frac{dV}{d\alpha}=\frac{\partial f}{\partial x}\frac{\partial x^*}{\partial \alpha}+\frac{\partial f}{\partial y}\frac{\partial y^*}{\partial \alpha}+\frac{\partial f}{\partial \alpha}$$

In the case of constrained optimisation, unlike in the case of unconstrained optimisation, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ may not be equal to zero (as we had seen in the previous unit), and hence we cannot directly get $\frac{dV}{d\alpha}=\frac{\partial V}{\partial \alpha}$. It is important to remember this. If we substitute the solutions to x and y into the constraint (producing an identity), we get:

$$h\left(x^*(\alpha), y^*(\alpha), \alpha\right)\equiv 0$$

If we differentiate this with respect to α, we get:

$$\frac{\partial h}{\partial x}\frac{\partial x^*}{\partial \alpha}+\frac{\partial h}{\partial y}\frac{\partial y^*}{\partial \alpha}+\frac{\partial h}{\partial \alpha}=0$$

Multiplying throughout by λ, using the expression for the Lagrangean, and putting the resulting expression in the expression for $\frac{dV}{d\alpha}$, we get:

$$\frac{dV}{d\alpha}=\left(\frac{\partial f}{\partial x}-\lambda\frac{\partial h}{\partial x}\right)\left[\frac{\partial x^*}{\partial \alpha}\right]+\left(\frac{\partial f}{\partial y}-\lambda\frac{\partial h}{\partial y}\right)\left[\frac{\partial y^*}{\partial \alpha}\right]+\frac{\partial h}{\partial \alpha}-\lambda\frac{\partial h}{\partial \alpha}=\frac{\partial L}{\partial \alpha}$$

By using the first-order conditions, we get

$$\frac{dV}{d\alpha}=\frac{\partial L}{\partial \alpha}$$

This is the envelope theorem in the context of constrained optimisation. It is possible to observe how the envelope theorem differs under constrained optimisation from the envelope theorem in the context of unconstrained optimisation. In the unconstrained optimisation case, $\frac{dV}{d\alpha}$ was equal to the partial derivative of the *objective function*

with respect to the parameter, while in the constrained optimisation case, $\frac{dV}{d\alpha}$ equals the partial derivative of the Lagrangean with respect to the parameter.

The significance of the envelope theorem in the context of constrained optimisation is that if we want to obtain the change in the maximum value function as a result of the change in the parameter(s), we can as well find the partial derivative of the Lagrangean with respect to the parameter(s).

10.3.4 Interpreting the Lagrangean Multiplier

Consider the problem of maximising the function $u = f(x, y)$ subject to $h(x, y) = c$.

Here we are considering a general constraint function, and not specifically a linear one. The Lagrangean for this problem is

$$L = f(x, y) + \lambda\left[c - h(x, y)\right]$$

The first-order conditions are:

$$\frac{\partial L}{\partial x} = \frac{\partial f}{\partial x} - \lambda\frac{\partial h}{\partial x} = 0$$

$$\frac{\partial L}{\partial y} = \frac{\partial f}{\partial y} - \lambda\frac{\partial h}{\partial y} = 0$$

$$\frac{\partial L}{\partial \lambda} = c - h(x, y) = 0$$

From the first two equations, we have

$$\lambda = \frac{\partial f/\partial x}{\partial h/\partial x} = \frac{\partial f/\partial y}{\partial h/\partial y}$$

The first-order conditions implicitly define the solution

$$x^* = x^*(c),\ y^* = y^*(c),\ \lambda^* = \lambda^*(c)$$

Substituting the solution back into the Lagrangean gives us the maximum value function

$$V(c) = L^*(c) = f\left(x^*(c), y^*(c)\right) + \lambda^*(c)\left[c - h\left(x^*(c), y^*(c)\right)\right]$$

Differentiating with respect to *c*, we get

$$\frac{dV}{dc} = \frac{dL}{dc} = \frac{\partial f}{\partial x}\frac{\partial x^*}{\partial c} + \frac{\partial f}{\partial y}\frac{\partial y^*}{\partial c} + \left[c - h\left(x^*(c), y^*(c)\right)\right]\frac{\partial \lambda^*}{\partial c} - \lambda^*$$

$$(c)\frac{\partial h}{\partial x}\frac{\partial x^*}{\partial c} - \lambda^*(c)\frac{\partial h}{\partial y}\frac{\partial y^*}{\partial c} + \lambda^*(c)\frac{dc}{dc}$$

Rearranging the above equation, we get

$$\frac{dL^*}{dC} = \left[\frac{\partial f}{\partial x} - \lambda^*\frac{\partial h}{\partial x}\right]\frac{\partial x^*}{\partial c} + \left[\frac{\partial f}{\partial y} - \lambda^*\frac{\partial h}{\partial y}\right]\frac{\partial y^*}{\partial c} + \left[c - h\left(x^*, y^*\right)\right]\frac{\partial \lambda^*}{\partial c} + \lambda^*$$

The first three terms on the right-hand side are zero. Thus we are left with

$$\frac{dL^*}{dc} = \lambda^* = \frac{dV}{dc}$$

We can see the interpretation of the Lagrange multiplier in the context of consumer's demand. Let us do this for a three - good case.

$$u^* \equiv u(x_1(p_1, p_2, p_3, M), x_2(p_1, p_2, p_3, M), x_3(p_1, p_2, p_3, M))$$

Differentiating with respect to M

$$\frac{\partial u^*}{\partial M} = u_1 \frac{\partial x_1}{\partial M} + u_2 \frac{\partial x_2}{\partial M} + u_3 \frac{\partial x_3}{\partial M}, \text{ where } u_i = \frac{\partial u}{\partial x_i}, i = 1, 2, 3.$$

From the first order conditions u = λ p_i, i=1, 2, 3.

Then,

$$\frac{\partial u^*}{\partial M} = \lambda \left[p_i \frac{\partial x_1}{\partial M} + p_2 \frac{\partial x_2}{\partial M} + p_3 \frac{\partial x_3}{\partial M} \right]$$

On the other hand

$$p_1 x_1(p_1, p_2, p_3, M) + p_2 x_2(p_1, p_2, p_3, M) + p_3 x_3(p_1, p_2, p_3, M) \equiv M$$

Differentiating with respect to M,

$$p_i \frac{\partial x_1}{\partial M} + p_2 \frac{\partial x_2}{\partial M} + p_3 \frac{\partial x_3}{\partial M} = 1$$

Using this result we get $\frac{\partial \mathrm{u}^*}{\partial \mathrm{M}} = \lambda.$

Thus, the Lagrangian multiplier introduced in a constrained extremum exercise as a mathematical convenience turns out to have a simple meaning, and relevant. It measures the rate at which the value of the objective function changes for a small shift in the constraint. In the utility maximization problem, it stands for the marginal utility of money income.

Check Your Progress 1

1) Explain the concept of comparative statics, and briefly explain how it differs in a non-optimisation context from an optimization context.

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2) What do you understand by the maximum value function.

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3) Explain how the Lagrange multiplier can be considered as a shadow price.

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10.4 SOME ECONOMIC APPLICATIONS OF MAXIMUM VALUE FUNCTIONS AND THE ENVELOPE THEOREM

In this section we take up some applications of the maximum value function and the envelope theorem to Economics. We consider the indirect utility function (as a type of maximum value function). We then demonstrate an application of the envelope theorem to the indirect utility function to obtain a result known as Roy's identity, as well as an application of the envelope theorem to the profit function of a firm to get a result known as Hotelling's lemma.

10.4.1 Indirect Utility Function

$u(x\ [p_x, p_y, I], y[p_x, p_y, I])$ says that utility is a function of the two goods $x$and y each of which in turn is a function of the prices of both goods and income. Here, income is denoted by I. We may thus write utility indirectly as a function of the prices of the goods and income: $v = v\left(p_x, p_y, I\right)$ where v stands for utility. We denote utility by v here because here it is called indirect utility function. The term 'indirect' denotes that utility is a function of the parameters, as the goods' prices, and income are parameters.

We have seen that a (maximum) value function gives the value of the objective function at the optimum. The value function gives the optimal value of the variable to be optimised (the endogenous variable) as a function of the parameters. We have seen in unit 1 what a parameter means. For example, in the utility function, the consumer maximises utility as a function of the goods consumed. Suppose there are two goods: x_1 and x_2. So the utility function is $u = f\left(x_1, x_2\right)$. The consumer maximises this function subject to the constraint $p_1x_1 + p_2x_2 = m$, where p_1 and p_2 are the prices of the two goods, and m is the consumer's income. Let the optimal value of utility maximised be denoted u^* and let the optimal amount of the two goods consumed be x_1^* and x_2^*. Now, we have considered the prices of the two goods and the consumer's income as held constant. Suppose they were to change. In other words, p_1, p_2, ad m are supposed to be constants, which, since specific numbers have not been assigned to them, can be treated as variables. Hence they are constants, which act as variables! These are called parameters.

10.4.2 Roy's Identity

This identity asserts that the individual consumer's ordinary demand function is equal to the negative of the ratio of the partial derivative of the maximum-value functions with respect to the price of the good to the partial derivative of the maximum-value function with respect to the income To derive this, we proceed as follows:

We take the indirect utility function which is the maximum value function for the consumer's direct utility maximisation exercise.

Consider the standard consumer's problem

Max $u(x, y)$

subject to $p_x x + p_y y = m$

From the first-order conditions, we obtain the demand functions

$x = x(p_x, p_y, m)$ and $y = y(p_x, p_y, m)$. Substituting these into the utility function gives

$$u = u\left[x(p_x, p_y, m), y(p_x, p_y, m)\right]$$

$$= V(p_x, p_y, m)$$

V is the indirect utility function. It is the maximum value function.

The consumer's utility maximization problem that we had considered gives the Lagrangian function

$$L = u(x, y) + \lambda\left[m - p_x x - p_y y\right]$$

Now we come to the crucial part, of linking the Lagrangean and the indirect utility function by invoking the Envelope Theorem. This, as you know, says that the partial derivative of the maximum value function with respect to a parameter is the same as the partial derivative of the Lagrangean function with respect to the parameter. Applying the Envelope Theorem here we get

$$\frac{\partial V}{\partial m} = \frac{\partial L}{\partial m} = \lambda$$

Applying the envelope theorem again to partial derivates with respect to the other parameters, the prices, we obtain:

$$\frac{\partial V}{\partial p_x} = \frac{\partial L}{\partial p_x} = -\lambda x(p_x, p_y, m)$$

Similarly, $\dfrac{\partial V}{\partial p_y} = \dfrac{\partial L}{\partial p_y} = -\lambda y\left(p_x, p_y, m\right)$

Hence we obtain

$$\frac{\partial V/\partial p_x}{\partial V/\partial m} = -x$$

$$\frac{\partial V/\partial p_y}{\partial V/\partial m} = -y$$

These last two results are known as Roy's identity

10.4.3 Hotelling's Lemma

Suppose a firm uses two inputs: labour L and capital K. The production function is :

$$y = f\left(K, L\right)$$

Let price of the product be P, the wage rate, w, and the rental of capital r. Then total revenue is $Pf(K, L)$ and total cost is $wL + rK$.

The profit, which the firm wants to maximise is

$$\pi = pf(k, L) - wL - rK$$

The Lagrange function for profit maximization subject to a output constraint is

$$L = (py - wL - rK) + \lambda\left[f\left(L, K\right) - y\right]$$

The first-order conditions are:

$$p - \lambda = 0$$

$$\lambda \frac{\partial f}{\partial L} - w = 0$$

$$\lambda \frac{\partial f}{\partial K} - r = 0$$

$$-y + f\left(L, K\right) = 0$$

These equations give solutions for the input demand functions $L\left(p, w, r\right)$ and $K\left(p, w, r\right)$, and the output supply function $y\left(p, w, r\right)$. Inserting these functions into the profit equation gives the profit function

$$\pi = py\left(p, w, r\right) - wL\left(p, w, r\right) - rK\left(p, w, r\right) = V\left(p, w, r\right)$$

This is the value function. Hence, applying the envelope theorem, we get

$$\frac{\partial V}{\partial p} = \frac{\partial L}{\partial p} = y(p, w, r)$$

$$\frac{\partial V}{\partial w} = \frac{\partial L}{\partial w} = -L(p, w, r)$$

$$\frac{\partial V}{\partial r} = \frac{\partial L}{\partial r} = -K(p, w, r)$$

This means that differentiating the profit function with respect to the prices gives the out-supply function, and the input –demand functions for the firm. This is called Hotelling's lemma.

Check Your Progress 2

1) Explain in what way is an indirect utility function a maximum value function.

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2) Show how Roy's identity can be derived using the envelope theorem.

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3) In the context of a firm, what is Hotelling's lemma?

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10.5 DUALITY AND OPTIMISATION

You now have a good idea of maximum value functions and of the envelope theorem. The time has now come for you to be acquainted with a very important concept related to optimisation—that of duality. Duality is the relationship between two constrained optimisation problems. Roughly speaking duality involves converting a given constrained optimisation problem into its 'dual', in order to obtain some powerful insights and results. Converting an original problem into its dual means a new optimisation exercise where the original maximisation (minimisation) problem is converted into a minimisation (maximisation) problem, and where the constraint of the original problem appears as the objective function of the new problem; moreover, the parameters in the original problem become the arguments in the new objective function. Thus, there are two optimisation problems. One of the problems would be a maximisation problem, while the other would be a minimisation problem. The structure and solution of any one of the problems can provide information about the solution as well as the structure of the other problem.

We try to explain the relationship between primal and dual problem by taking an example from the theory of the consumer. We look at the objective function of the consumer and show how maximising utility subject to the budget constraint has a counterpart dual problem

10.5.1 The Primal Problem

Let us consider the maximisation of utility by a consumer, subject to a budget constraint. Suppose there are two goods x and y. Suppose the consumer had a budget, or income, m. Let the prices of the two goods be p_x and p_y. The primal problem is the usual utility maximisation subject to the budget constraint:

$$\text{Max } U = U(x, y)$$

$$\text{subject to } p_x x + p_y y = m$$

For this problem, there is the usual Lagrangian

$$L = U(x, y) - \lambda\left(m - p_x x + p_y y\right)$$

We obtain the first order conditions, and solving these we can get the demand functions for the two goods.

10.5.2 The Dual Problem

In the dual problem, maximization is changed to minimisation; and the constraint in the primal problem appears as the objective function of the dual. We have considered earlier the case where the consumer maximised her utility function under the condition that her expenditure on the goods she consumed (for simplicity assume she consumes two goods) cannot exceed her income. Now consider the dual problem to the primal problem of utility maximisation. The dual problem consists of *minimising* the expenditure on the two goods subject to the constraint that the utility is maintained at a fixed level $\overline{u}$ .We can set up the dual problem for the consumer as

Minimise $E = p_x x + p_y y$

subject to $u = u(x, y) = \overline{u}$

The objective function in the above problem is the expenditure function.

10.6 SOME ECONOMIC APPLICATIONS OF DUALITY

10.6.1 The Compensated Demand Function

We just saw that the expenditure minimisation problem is the dual to the utility maximization problem. Consider the dual problem once more;

Minimise $E = p_x x + p_y y$

subject to $u = u(x, y) = \overline{u}$

Let us set up the Lagrangian. It is

$$L^d = p_x x + p_y y + \mu\left[\overline{u} - u((x, y))\right]$$

L^d is the Lagrangean in the dual sense, or rather, Lagrangean for the dual optimization exercise (minimization). The quantity μ is the Lagrange multiplier for this dual optimization exercise.

The first-order conditions are as follows:

$$\frac{\partial L^d}{\partial x} = p_x - \mu\frac{\partial u}{\partial x} = 0$$

$$\frac{\partial L^d}{\partial y} = p_y - \mu\frac{\partial u}{\partial y} = 0$$

$$\frac{\partial L^d}{\partial \lambda} = \overline{u} - u(x, y)$$

This system of equations yields a set of solution values that we denote by x^h, y^h, λ^h (the superscript 'h' stands for 'Hicksian', after sir John Hicks, one of the foremost economists). The functions are:

$$x^h = x^h\left(p_x, p_y, \overline{u}\right)$$

$$y^h = y^h\left(p_x, p_y, \overline{u}\right)$$

These are known as Hicksian or compensated demand functions. Of course, there is one equation for μ but we have not shown it here.

10.6.2 Shephard's Lemma

We just acquainted ourselves with the expenditure function. We saw that the dual to the utility maximisation problem is the expenditure minimisation problem:

$$E = p_x x + p_y y$$

subject to $u(x, y) = \overline{u}$

Forming the Lagrangian we get

$$L^d = p_x x + p_y y + \mu\left[\overline{u} - u\left((x, y)\right)\right]$$

Here μ is a Lagrange multiplier. Solving this exercise yields two demand functions for x and y as functions of prices and utility. This is called compensated or Hicksian demand functions.:

$x^h\left(p_x, p_y, u\right)$ and $y^h\left(p_x, p_y, u\right)$

Substituting these into the objective function of the expenditure minimization problem we get:

$$E = p_x x\left(p_x, p_y, \overline{u}\right) + p_y y\left(p_x, p_y, \overline{u}\right) = e\left(p_x, p_y, \overline{u}\right)$$

Here e is the value function (in this case it is the minimum value function).

If we apply the envelope theorem to the expenditure minimization problem, we find

$$\frac{\partial e}{\partial u} = \frac{\partial L}{\partial u} = \overline{u}$$

Here e is the expenditure function and L is the Lagrangian

Furthermore, the envelope theorem gives us

$$\frac{\partial e}{\partial p_x} = \frac{\partial L}{\partial p_x} = x\left(p_x, p_y u\right)$$

and $$\frac{\partial e}{\partial p_y} = \frac{\partial L}{\partial p_y} = y\left(p_x, p_y, u\right)$$

Notice that in the optimization problem,

p_x and p_y appear only in the objective condition.

The situation described by the latter two equations, namely

$$\frac{\partial e}{\partial p_x} = \frac{\partial L}{\partial p_x} = x\left(p_x, p_y u\right)$$

and $\frac{\partial e}{\partial p_y} = \frac{\partial L}{\partial p_y} = y\left(p_x, p_y, u\right)$ are known as Shephard's lemma

Check Your Progress 3

1) What do you understand by duality?

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2) What is a compensated demand function. Explain.

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3) What is Shephard's lemma?

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10.7 LET US SUM UP

This Unit was concerned with a discussion about the impact on equilibrium values of a change in parameter values. Given an equilibrium position, what would happen if the parameters change? How would equilibrium change if the parameters change? The unit began by explaining the concept of comparative statics and distinguished it from statics. The distinction in the use of comparative statics in non-optimisation context and optimisation context was made clear.

The unit then went on to discuss the envelope theorem in the context of optimisation. The maximum value function was explained, and then the envelope theorem was elaborated upon. The envelope theorem was elaborated upon both in the context of constrained as well as unconstrained optimisation. Further, the envelope theorem was applied to give an interpretation of the Lagrangian as a shadow price. Further the unit discussed applications of the envelope theorem to economic problems, to the indirect utility function, Roy's identity and Hotelling's lemma. Following this the unit discussed the crucial concept of duality, and took a primal problem and explained how its dual is to be obtained. The unit took the indirect utility function and its dual, the expenditure function to explain duality. Finally, the unit discussed applications of duality theory to compensated demand functions and to Shephard's lemma.

10.8 ANSWERS/HINTS TO CHECK YOUR PROGRESS EXERCISES

Check Your Progress 1

1) See subsections 10.2.1 and 10.2.2
2) See subsection 10.3.3
3) See subsection 10.3.4

Check Your Progress 2

1) See subsection 10.4.1
2) See subsection 10.4.2
3) See subsection 10.4.3

Check Your Progress 3

1) See section 10.5
2) See subsection 10.6.1
3) See subsection 10.6.2

GLOSSARY

Adjoint of a matrix: The adjoint of a square matrix A is defined as the transpose of the matrix of the cofactors of the determinant of A.

Cofactor: The cofactor of any element a_{ij} of the determinant of A denoted by C_{ij}, is in fact a signed (+ or -) minor. The sign of the minor is determined by the rule: $C_{ij} = (-1)i + jM_{ij}$.

Constraint: : A restriction that reduces the domain of the choice set and does not allow free optimum. It is a side-condition that has to be fulfilled in some optimisation exercises. It changes the value of the global optimum that would have prevailed in the absence of such a constraint.

Determinant: The determinant is a number (scalar) defined for a square matrix only. It is conventionally represented by enclosing the elements of the corresponding matrix within two vertical straight lines.

Diagonal matrix: A diagonal matrix is a square matrix, where there is at least one non-zero element on the principal diagonal and all the off-diagonal elements are zeros.

Dimension or order of a matrix: The number of rows and columns of a matrix constitutes its dimension or order.

Duality: Duality is the relationship between two constrained optimisation problems. Roughly speaking duality involves converting a given constrained optimisation problem into its 'dual', in order to obtain some powerful insights and results. Converting an original problem into its dual means a new optimisation exercise where the original maximisation (minimisation) problem is converted into a minimisation (maximisation) problem, and where the constraint of the original problem appears as the objective function of the new problem; moreover, the parameters in the original problem become the arguments in the new objective function.

Element of a matrix: Each member of a matrix is called an element of the matrix.

Expenditure Function: minimising the expenditure function is the dual of the utility maximization.

Envelope Theorem: the envelope theorem in the context of constrained optimisation is that if we want to obtain the change in the maximum value function as a result of the change in the parameter(s), we can as well find the partial derivative of the Lagrangean with respect to the parameter(s). of course, the envelope theorem is applicable for the unconstrained optimisation case as well.

Hawkins-Simon Condition : This is the condition for positive levels of output, when a_{ij}'s are expressed in physical units. The condition requires that all principal minors of the technology matrix must be positive.

Hotelling's Lemma: differentiating the profit function with respect to the prices gives the output-supply function, and the input –demand functions for the firm. This is called Hotelling's lemma.

Identity Matrix: It is a square matrix with 1s in the principal diagonal and zeros in the off-diagonal places.

Indirect Utility Function: this says that utility is a function of all the goods each of which in turn is a function of the prices of all goods and income. Hence, utility is indirectly a function of prices and income.

Inner-product: It is the product of two vectors.

Input coefficient matrix: It is a matrix of various secondary inputs required by different producing sectors per unit of their output.

Input-output model or input-output transaction matrix : It is formulation that focuses on the interdependence of the producing sector and, the interaction between the producing sectors one hand and household sector of an economy on the other hand.

Inverse of a matrix: Given a square matrix A, if there exists another square matrix B, such that, AB = BA = I, where I is an identity matrix, then B is said to be the inverse of the matrix A. For a matrix A, its inverse is generally denoted by A-1.

Lagrangean Function: A function constructed to solve constrained optimization problems by combining the objective function and the constraint

Lagrange multiplier : A quantity in constrained optimization used in the Lagrangian function; it measure the shadow price of the variable in the objective function.

Matrix: It is a rectangular array of numbers.

Matrix operations: The basic operations of addition and multiplication on matrices are called matrix operations.

Minor: The minor of any element a_{ij} of the determinant of some matrix A, denoted by M_{ij}, is the sub-determinant obtained by deleting ith row and jth column of the determinant of A.

Non-singular matrix: A square matrix A is said to be non-singular, if its determinant is not zero.

Norm of a Vector: it is the length of a vector. It is computed as the square root of the sum of squares of the components of a vector. The norm can also be seen as the square root of the inner product of a vector with itself.

Null matrix: It is a matrix whose elements are all zero. It need not be a square matrix.

Orthogonal vectors: Two vectors, whose inner product yields a zero-valued scalar, are said to be orthogonal (perpendicular) to each other.

Quadratic Form: a quadratic form is a polynomial expression in which each component term has a uniform degree. In a quadratic form, each term is of the second degree.

Roy's Identity: this says that the individual consumer's ordinary demand function is equal to the negative of the ratio of the partial derivative of the maximum-value functions with respect to the price of the good to the partial derivative of the maximum-value function with respect to the income

Shephard's Lemma: this says that differentiating the expenditure function with respect to prices gives us the compensated demand function

Stationary Value: Point at which optimum is found

Scalar: An ordinary number is called a scalar. It is a matrix of d

Singular matrix: A square matrix A is said to be singular, if its determinant is zero.

Square matrix: A matrix, in which the number of rows and columns are equal, is called a square matrix.

Symmetric matrix: It is a square matrix whose transpose is the matrix itself.

Technology matrix: This is a matrix, that is obtained by subtracting a given input coefficient matrix from an identity matrix of appropriate dimension. This matrix is supposed to reflect the technology.

Transpose of a matrix: The transpose of a matrix, say A, is defined as a matrix that is obtained by interchanging the rows and columns of A.

Vector: A vector is an n-tuple of numbers. A vector is thus also a matrix consisting of only one row or one column.

Vector Space: The totality of two-element vectors generated by the various linear combinations of two independent vectors is the two-dimensional vector space. Similarly, the totality of n-dimensional vectors is the n-dimensional vector space.

SUGGESTED READINGS

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