# UNIT 7 MODELS OF OPTIMAL ECONOMIC GROWTH

**Structure**



## 7.0. OBJECTIVES

After going through the unit, you should be in a position to:

- Describe the concept of inter-temporal consumption;
- Explain the technique of inter-temporal optimisation;
- Discuss the Ramsey model of optimal growth;
- Analyse the implication of the 'Golden Rule of Accumulation;
- Explain the meaning of discounting and time-preference; and
- Discuss the Cass-Koopmans model of optimal growth as an extension of the Ramsey model of growth.

## 7.1 INTRODUCTION

In Units 3 to 6, we have studied the meaning of economic growth and development, several models of growth – and distribution—and have examined, too, the concepts of technical progress as well as factor productivity. A common feature running through all these is that economic agents, the people in the economy, are nowhere to be seen! Sure, we have talked of labour, of capital, that is, the factors of production; we have even discussed various theories of distribution, and the distribution presumably takes place among the factors of production, which is supplied by people. But have we seen people explicitly taking actions, making choices? Recall what you are studying in your microeconomics and macroeconomics courses, particularly the former. There are households, firms, consumers, savers, and investors. All of them make choices. They optimise some objective function (usually subject to constraints). The consumer wants to maximise utility. The owners of firms want to minimise cost, or maximise profits. And so on. Do not economic agents make choices in the context of economic growth? Of course they do. Recall that in Kaldor's theory of distribution, we saw who did the saving and who did the investing. But it was all done slightly mechanically, as though the proportion of income saved is given! The time has come to put some decision-making and exercising of choices into the picture.

This unit attempts to study the growth process of an economy – you may think of it as a neo-classical type economy—with economic agents making decisions. The decisions that we focus on are decisions regarding consumption and saving. Saving now may be considered as deferred consumption, and we look consumption and saving over a period of time—a stream of consumption and saving. Since agents supposedly want to optimise the mix of consumption and saving, hence these models are called ***optimal*** growth models. But please bear in mind that the optimisation that we will look at (about consumption and saving) is taking place over time: it is ***inter-temporal*** optimisation. The agent doing the optimisation can be thought of as % "representative agent" of the economy or it can be the state taking decisions about society, a kind of "social planner". The implications for this unit are that it involves use of some new techniques of optirnisation, which you have not studied either in your course on quantitative techniques or in the course on microeconomics. Now, do not feel apprehensive: we are going to explain whatever techniques are needed right here in this unit.

The next section discusses inter-temporal optimisation techniques. The subsequent section sets out and explains the Ramsey growth model. The section after that deals with optimal saving and what is called the 'golden rule of accumulating', which is a way of bringing in ideas about economic and social justice in the study of economic growth. The following section analyses rhe extension to the Ramsey model by the models of David Cass and Tjalling Koopmans. Recall that both in the Harrod-Domar and Solow models, the saving is taken to be exogenously determined. The main extension that the models that we study in this unit make is that they make the saving rate an endogenously determined variable. It is the optimisation decision of the economic agent that determines the saving rate.

## 7.2 OPTIMISATION OVER TIME

In this section we introduce you to some tools of optimisation over time. In your quantitative techniques course, you are being acquainted with static optimisation, that is, optimisation at a point of time. You are studying in that course how to optimise an objective function, with or without constraints. You are also reading about dynamic processes, which means processes that take place over time. Dynamic processes are being explained in that unit with difference and differential equations. But difference and differential equations are descriptions of the inter-temporal processes and do not bring in optimisation.

In this unit we introduce you to situations where optimisation is carried out in an environment that extends over several time periods. There are two ways of looking at it. Suppose there ate T time periods t =1, 2, 3...T either the decision-maker makes a decisicn at t = 1 for period 1 and all subsequent time periods t = 2,3,4,...,T. Alternatively we can conceptualise that the decision maker optimises, that is, makes a decision so as to optimise, sequentially in each time period. Thus each period is separable and separate decisions are made in each of the periods.

We now present a simple discussion of optimisation over time. Think of a consumer who chooses in only two periods, 1 and 2, where you may think of '1' as 'today' and '2' as 'tomorrow'. Let his consumption be $c_1$ in period 1 and $c_2$ in period 2. You can think of $c_1$ and $c_2$ being plotted on the horizontal axis and vertical axis respectively. Let us first think of the consumer's budget constraint (we are simply extending the discussion of the consumer from your microeconomics course). Let the money with the consumer in the two periods be $m_1$ and $m_2$ respectively. If the consumer can neither borrow nor lend, then she has $m_1$ to spend in period 1 and $m_2$ to spend in period 2.

Suppose the consumer is free to borrow and lend. Let the interest rate be r. Let us derive the budget constraint. Suppose the consumer saves a part of her income in period 1 and lends it out at an interest rate r. in this case her period 1 consumption $c_1$

is less than her period 1 income $m_1$. in the second year she will get back what she had lent in period 1 plus the interest on it. So the budget constraint for this consumer is:

$$c_2 = m_2 + (m_1 - c_1) + r(m_1 - c_1).$$

The first term on the right hand side is her second period income, the second term is the amount she had lent out in period I and the third term is the interest on the loan amount.

Had the consumer borrowed in period 1, then $c_1 \succ m_1$. So the consumer borrows the shortfall $(c_1 - m_1)$ and pays back this amount plus the interest on the loan. In this case her budget constraint is:

$$c_2 = m_2 - r(c_1 - m_1) - (c_1 - m_1)$$

$$= m_2 + (1+r)(m_1 - c_1)$$

If $(m_1 - c_1) \succ 0$ then the consumer earns interest; if it is negative she pays interest.

We can write the budget constraint, after some manipulation, as:

$$c_1 + \frac{c_2}{(1+r)} = m_1 + \frac{m_2}{(1+r)}$$

This expresses the budget in present value terms.

Consider several time periods. If we invest P rupees at interest rate r compounded annually, then after n years the value will become $P(1+r)^n$. Similarly, to get back P rupees n years from now at rate of interest r compounded annually, we need to invest $\frac{P}{(1+r)^n}$ rupees today. Now think of a stream of payments of $c_1$ in year 1, $c_2$ in year 2, ....$c_n$ in year n. Then the present value of the stream of payments is:

$$\frac{c_1}{(1+r)} + \frac{c_2}{(1+r)^2} + \ldots + \frac{c_n}{(1+r)^n}$$

$$= \sum_{i=1}^{n} \left( \frac{c_i}{(1+r)^i} \right)$$

Lei (I+r) be denoted $\beta$.

Then the present value of the stream of payments can be written

$\sum_{i=1}^{n} \beta^i c_i$. We will be using this kind of formulation in the next couple of sections.

Instead of returns directly, we can use inter-temporal utility function.

After having seen how to obtain present value of a stream of payments or a stream of payments, let us see how to optimise such a utility function. For this we have to first get acquainted with certain concepts, such as a functional, We also see how optimisation over time is different from static optimisation, which you have learnt in your courses on quantitative techniques.

When we optimise over a period of time we do not choose a single optimal value of a variable quantity; we choose a variable *function*. A quantity, which is dependent on a variable function, is called a functional. Let us elaborate on the notion of a functional in general terms, because understanding the notion of a functional is crucial to understanding how to solve dynamic optimisation problems, although we do not directly solve such optimisation problems.

In static optimisation, a single optimal magnitude for every choice variable is sought to be obtained. In dynamic optimisation problems, by contrast, the question is, what is the optimal magnitude of a choice variable in each period of time in the given duration or time horizon of the optimisation problem, say, the interval , say [0, T].

Whether we consider discrete time periods, or continuous time period , any dynamic optimisation problem would contain the following ingredients: (a) a given initial point ad a given terminal point, which may extend to infinity (b) a set of admissible *paths* from the initial point to the terminal point (c) a set of **path *values*** and (d) a specified objective to maximise or minimise the path value by choosing the optimal path.

This relation between paths and path value is very important in trying to understand the method of solving for dynamic optimisation problems. Their relation is shown by a special type of mapping – not a mapping from real numbers to real numbers as is the usual case but from entire paths (curves) to real numbers. Usual mappings in economics are from a set containing real numbers to a set whose eiements are functions, to a set of real numbers. Such a mapping is called a functional. Suppose we have a a variable y which deeds on time t. There can be many different types of functions. Let three such functions $y = f(t), y = g(t)$, and $y = h(t)$. Let the value associated with function $f$ be $v_1$, the value with function $g$ be $v_2$ and the value associated with function h be $v_3$. This relationship between the various types of functions, which can be generically denoted by y(t), and the v values is called a functional. We can denote it by

$v = [y(t)]$. Notice that in standard differential calculus, if you have $x = f(g(y))$ it means some variable is a function of y and x in turn is a function of this variable, and hence x is indirectly function of y. this is called a composite functions. There are three methods to solve dynamic optimisation problems. The first is calculus of variations; the second is optimal control, and finally the third is dynamic programming. Calculus of variations and Optimal Control are presented in the unit on Ramsey Growth model in the macroeconomics course.

## 7.3 THE RAMSEY MODEL OF GROWTH

In the late 19th century the well-known economist Eugen Bohm-Bawerk conjectured that people tend to have myopic views and focus too much on the present and ton little on future utility. From this Pigou in 1920, in his book ***The Economics of Welfare*** suggested that if it was indeed the case that people tend to underestimate future utility, then they will not save enough for the future, for themselves and succeeding generations. The implication of this is that actual social saving will be less than the optimal one. Pigou suggested that this is some sort of market failure. If actual saving in a market economy is less than what the optimal saving should be, the question arises: what is the optimal level of saving, and how can we find it?

Frank Ramsey was a young (he died at the age of 26) philosopher and mathematician at Cambridge, who worked on Pigou's question, He used the standard utilitarian philosophy of Jeremy Bentharn, (by which much of microeconomic theory, too, is influenced) to brilliantly derive a theory about inter-temporal consumption and optimal saving decision. From the point of view of history of economic thought, there are several interesting things. We mention these here briefly before going on to an exposition of the Ramsey model. First although we are saying that the Ramsey model is an extension of the Solow model, in terms of bringing in optimisation into the growth process, chronologically, the Ramsey model was presented in a paper in 1926, that is, it preceded the Solow model by decades. Secondly, the Ramsey model was presented even before the publication of Irving Fisher's book The ***Theory of*** Interest, which was published in 1930, and which presented the standard model of two-period optimisation. We present a version of Ramsey's model.

The basic point about Rarnsey's model is that it is a prescriptive theory and not merely predictive. Society has to choose optimal economic growth. You can think of society as being represented by a 'representative agent' or a central planning agency. This representative agent or central planner maximises an objective function. You can think of it as a social welfare function or a utility function. Let us suppose the utility function is as follows. Utility depends an consumption:

$$U = f(C)$$

Utility at each time period t depends on consumption at that time

$$U = U(C_t)$$

Let capital be denoted by K and labour by L. we could have taken several alternative types of utility functions :

$$U = U(C, K)$$

$$U = U\left(\frac{C}{L}\right)$$

$$U = LU\left(\frac{C}{L}\right)$$

However in the literature on optimal growth of the Ramsey type, , the type of utility function most often used is $U = U\left(\frac{C}{L}\right)$ or $U = U(c_t)$ where c = $C/L$. this is a simplified picture of reality and takes only consumption as a basic proxy for all welfare inducing activity. Keynes held that "consumption, to repeal the obvious, is the sole end and object of all economic activity" (Keynes, in *A General* Theory of *Employment* Interest *and* Money, 1936)

Let the time periods be 0,1,2,...,T. (Now we are denoting the starting lime period as 0)

There is thus a stream or consumption over the time periods 0,...,T, namely

$\{C_0, C_1, \ldots C_T\}$ This is mathematically a sequence of consumption. A sequence is nothing but a list of quantities arranged in order. It has a first term, a second term, and so on. A sequence is thus something for which you can say 'first', 'second', 'third', and so on. The sequence of consumption can be denoted $\{C_t\}_{t=0}^{T}$. Thus the optimisation problem is to choose that consumption sequence that maximises utility. This is different from the usual optimisation that you study

Let U(0), U(1), ...,U(T) be the value of the utility function at times 0,1,....t. Thus the value of the objective function at time t is U(t) The idea of optimisation here is to maximise the value of the objective function over the entire path, which is

$$\sum_{t=0}^{T} U(t)$$

since social welfare function, or utility depends on the consumption per person at each time period, the optimsation exercise is to maximise the value of the objective function (here a utility function) over the entire consumption path or sequence. Thus the maximisation problem is max

$$J = \sum_{t=0}^{T} U[c(t)]$$

Strictly speaking the utility is $U_t = U(c_t)$, that is, utility at each point in time, t, is a function of the consumption at *thut* point of time, $c_t$. there is thus a stream of instantaneous utility $\{U_t\} = \{U_0, U_1, \ldots, U_T\}$

Several points are to be noted about the objective function

I) Here J is the value of the sum of the utility function, since the particular utility function here is additive separable, which means that the utility in each period is independent of utility in other periods, and moreover, the total utility is the sum ofutility in all periods combined.

2) Time appears as an argument of the consumption function. This is merely to highlight that consumption depends on time. Putting time as a subscript indicates consumption in a certain time period. The idea is similar.

3) Since we have taken time as discrete variable we use summation. Had we considered continuous time the objective function would have been written

$$J = \int_0 U\left[c(t)\right]dt$$

4) The term J is a **functional.** This means that although utility depends on c and c in turn depends on t, it does not mean there is a simple function- of- function situation so that U becomes an indirect function oft. U is dependent on the ***whole path***, or *the whole function* c. If the *function* c changes, *U* will change. Similarly with J. Thus in the optimisation problem what is being maximised is a functional. Reread the previous section to get an idea of a functional.

The basic idea behind the Ramsey model has been presented above. Ramsey used the mathematical technique called calculus of variations to solve the optimisation problem above, but since that technique is beyond the scope of our discussion we shall not elaborate on the problem. When Ramsey published his paper, calculus of variations was not something with which a lot of economists were familiar.

The way Ramsey actually formulated the problem was as follows. He thought of an ideal level of utility, called bliss point and saw how much actual utility fell short of it.

Let B denote ideal utility, the bliss point, which is same in all time periods. Let $U(c_t)$ be the actual utility from per worker consumption level c. Then, define a function

$B - U(c_t)$ for net utility in period t. Thus for the total path or the whole path, the functional to be optimised becomes

$J = \sum_{t=0}^{T}[B - U(c_t)]$. Here $B-U(c_t)$ represents the net utility obtained from consuming $c_t$ at time t. Of course, Ramsey used this optimisation subject to the constraint of growth of capital stock in the economy, The main issue is whether the time horizon is to be taken to be up to some finite time period like T or should it extend to infinity. We come to that now. Now we discuss certain issues that present themselves in the Ramsey model. We discuss the issue of time horizon, and then we discuss a very important issue of discounting.

If we take time to be finite and we find that some capital stock is left over at the terminal time period T, then optimum has not been reached and $\sum_{t=0}^{T} U\left[c(t)\right]$ would not have been optimised. The solution to this, and to make the problem mathematically tractable is to specify a terminal condition that some capital is retained at time T, and to build this condition into the optimisation problem itself. To get around this problem and to make the solution mathematically less difficult the time horizon is often taken to be till infinity. This can present theoretical problems its own, but we do not go into these now.

The discounting problem can be posed as follows. Given that people do not take the future as seriously as the present, in the sense that from the point of view of the present, individuals tend to underestimate the future problems, should future values be discounted? Ramsey held this to be morally and ethically indefensible. He insisted that future consumption and welfare ought not to, be given less weight than current consumption and welfare. Harrod, too, was against discounting the future. Some have argued that since social welfare is taken to be proxied by the utility of the representative agent, and individuals are taken to be myopic and do discount the future, social welfare ought to be discounted. However, it can be counter-argued that the future need not involve the same representative agent, but actually concerns future generations, who are not present at the time when the current generation undertakes its inter-temporal optimisation exercise. The idea is that social welfare should not depend only on the current members of society, but of all members of society who will live in this society in future. This again has been contested on the ground that all democratic societies make policies for and consider votes of, current members of the society.

**Check Your Progress 1**

1) What are the basic elements or ingredients of a dynamic optimisation problem?

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**2)** Explain the concept of a functional.

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**3)** Describe the basic structure of the optimisalion in the Ramsey growth model.

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# 7.4 THE GOLDEN RULE OF ACCUMULATION

We saw while studying the Solow model in unit three that changes in saving rates raises the levels of capital per labour, output per labour, and so on, but does not change their ***rates*** of growth. In the context of optimal growth, we can ask the question: what is the saving rate that gives rise to the steady path with the highest levels of c (which is C/L, consumption per labour)? The answer turns out that the highest steady state path for c, $\{c_t\}^*$, is the one that corresponds to the highest steady-state value of c. thus we have to find the rate of saving that generates the highest steady state value of c, given the values of n and $\delta$, where n is the exogenously given rate of growth of population or labour, and $\delta$ is the rate of depreciation.

We can consider this further by looking at some basic equations. We know that from a national accounting sense

$$Y = C + I$$

Here C is consumption and I is investment. This implies

$$C = Y - I$$

Writing this in per capita terms and denoting per capita values by lower-case letters we have

$$c = y - i$$

Using what we have learnt in unit three, we write this as

$$c_t = f(k_t) - sf(k_t)$$

Here we are saying that investment equals saving and saving $S = sY$. The lowercase s stands for saving rate. Since there are assumed to be constant returns to scale, we write

$y = f(k)$ the consumption equation is the one that needs to be maximised. But there are constraints. The basic steady-state constraint is that rate of growth of capital per labour should be zero. Thus the basic objective is to maximise

$$c_t = y_t - i_t = f(k_t) - sf(k_t)$$

subject to

$$\dot{k} = 0$$

where the dot on top indicates derivative with respect to time. The restriction $\dot{k}$ can be written $sf(k_t) = (n+6)k_t$ and thus the optimisation problem becomes

$\max\ c_t = f(k_t) - sf(k_t)$

subject to $sf(k_t) = (n+6)k_t$

substituting the constraint into the objective function, we get

$\max\ c_t = f(k_t) - (n+\delta)k_t$.

The first-order condition for the solution to this problem becomes

$\frac{dc}{dk_t} = f'(k_t) - (n+\delta) = 0$ which implies:

$f'(k_t) = (n+\delta)$

This equation is known as the "Golden Rule of Capital Accumulation" and put forward by Edmund Phelps in 1961. the solution to the equation gives the steady state value of k whose corresponding steady path for $c_t$, $\{c_t\}^*$ is the highest path that can be attained given the values of n and 6. By suitable substitution in the constraint equation, we can obtain the value for the saving rate that corresponds and which is the required saving rate. Let us denote this $s^*$. To understand this diagrammatically, recall or refer back to the second diagram in unit 3. We want to maximise the vertical distance between the $sf(k_t)$ and $f(k_t)$ curves and find the value of k on the horizontal axis at which this is attained.

The phrase "Golden Rule" comes from an analogy to the Biblical "Golden Rule of Conduct" which says, "do unto others as you would have them do unto you". Here, in the context of optimal growth, 'us' means our generation, the current generation, and 'them' means other generations. The idea is that we would like that the earlier had saved at a saving rate $s^*$ so that we would have received an economy on its optimal path that gave us optimal consumption streams. Hence, and similarly, we should follow the same principle*for our succeeding generations and should on our part strive to get a saving rate s so that futme generations can enjoy optimal consumption levels.

**Check Your Progress 2**

1) What do you understand by the "Golden Rule of Accumulation?

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2) Suppose there is technical progress in the neo-classical model? How would then the Golden Rule need to be modified?

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## 7.5 THE CASS KOOPMANS MODEL OF GROWTH

The Cass-Koopmans model is an extension of the Ramsey model. David Cass and Tjalling Koopmans both provided extensions to the Ramsey model. They used discounting, unlike in the Ramsey mode. The underlying reasoning for imposing time preference is more a mathematical one than a logical one - necessary for solving an inter-temporal optimising programme.

They also extended the Phelps Golden Rule formulation by arguing that what the objective function should be maximising is utility and not consumption stream directly.

Let us now set out the Cass-Koopmans formulation. Think of a social utility function $U(c_t)$. Suppose we think of a stream of utility accruing to society $\{U_0, U_1, ...\}$ What is the total utility ? Is it $U_0 + U_1 + U_2 ...$? This is what would be acceptable in the Ramsey model, but the Cass-Koopinans model does not accept this simple formulation. The argument here is that first, the society is not indifferent to the timing of utility receipts, and values and attaches greater value to utility today than to utility say, twenty years from now. Hence society displays time preference. Therefore total utility ia computed as the present discounted value of the utility stream, that is,

$$U = U_0 + \frac{U_1}{1+\rho} + \frac{U_2}{(1+\rho)^2} + .....$$

Here $\rho$ is the rate of time preference or the discount rate. We can denote 1/( 1+ρ) as βcan call β the discount *factor* and denote total utility as

$$U = U_0 + \beta U_1 + \beta^2 U_2 + \beta^3 U_3 + ....$$

or $$U = \sum_i \beta^i U_i$$

Look up section 7.2 to see again the concepts of discounting and time preference. The question is, what justifications can be given to put discounting and time preference into the picture. The first argument can be of course that it need not be justified at all and should be considered simply as taste or preference of the optimising agent or social planner. It is an old dictum in economics that 'the tastes are not to be disputed'. The other argument is to suggest that we no longer think of future generations but 'dynasties'. Each person is concerned about his or her own children, they about their children, and so on. Moreover, each person's utility in the current period is a function also of the utility of his or her children, and their children, and so on. Some economists have proposed that we take a dynastic utility function that stays the same for *that* dynasty for all time. All this necessitates some modification in the objective function to be maximised. The final argument to justify discounting future consumption is provided by economists like Robert Becker. This argument says that the inter-temporal optimisation should not be seen as an exercise in welfare economics or normative economics, that is, it does not involve making ethical statements. Supposing we look at the optimisation exercise as an analysis in positive economics that just *describes* the maximising action of a typical individual who maximises over time over an infinite horizon, who has perfect foresight and is supposed to be myopic. Then discounting necessarily enters the picture.

Now the optimisation exercise can be proceeded with. The Cass Koopmans model begins be stating a simple representative consumer who maximises an inter-temporal utility function:

$$\sum_{t=0}^{\infty} \beta^t U(c_t)$$

with the discount factor β, 0< β< 1. We can write $\beta$ as $\frac{1}{1+\rho}$, where $\rho$ is the discount rate, or rate of time preference.

A characteristic of the above utility function is that it is additively separable over time.

While discussing money in the neo-classical model, we had mentioned that the Sidrauski model would be discussed in unit 7 when we come to it. Now that we have discussed the Ramsey-Cass-Koopmans type of optimal growth model, we describe briefly the Sidrauski model that puts money into the growth model.

The basic idea in the Sidrauski model is to introduce real money balances into the utility function, so that the representative agent derives utility not merely from consuming goods but also from holding real money balances, that is money balances adjusted by the aggregate price level. Thus the utility function now looks as

$$\sum \beta^t U(c_t, m_t)$$ where

$m_t = \frac{M_t}{P_t}$ where $M_t$ is nominal money balances and $P_t$ is the price level at time t

It is assumed that the transactions services provided by money depends on the rate at which money and goods exchange for each other. If the nominal holdings doubles but the price level also doubles, then the transactions services provided by money remains unchanged,

In the Sidrauski model, the basic constraint is the same as in the Cass-Koopmans model:

$c_t + k_t = f(k_t) + (1-6)k_t$. Now in the Sidrauski model, there is an additional constraint because of money holdings

New money is received as cash transfers of real value:

$$v_t = \frac{\tilde{M}_{t+1} - \tilde{M}_t}{P_t}$$

where $\tilde{A}4$ denotes money supplied from outside by the government. It is different from $M_t$, which the money balance chosen by the individual.

The individual budget constraint now becomes

$$P_t(c_t + k_t) + M_{t+1} = P_t\left[f(k) + (1-\delta)k_t\right] + M_t + P_t v_t$$

Although in equilibrium, supply of, and demand for money are equal, we cannot use this fact in the above equation.

If we define inflation rate $\pi$ as $\frac{(P_{t+1} - P_t)}{P_t}$, we get:

$$\frac{M_{t+1}}{P_t} = \left(\frac{M_{t+1}}{P_{t+1}}\right)\left(\frac{P_{t+1}}{P_t}\right) = m_t + (1 + \pi_t)$$

We can then solve for the optimisation exercises using optimal control theory or dynamic programming and investigate the behaviour of the different variables, in stationary state and out of it.

**Check Your Progress 3**

1) What are the reasons for the use of discounting in the Cass-Koopmans model?

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2) Bring out the similarities and dissimilarities of the Cass-Koopmans model with the Ramsey model

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## 7.6 LET US SUM UP

This unit provided a study of models that extend the Marrod-Domar and Solow models to bring in explicitly the idea that saving is endogenous to the models. In order to do so we have to look at the decision-making process of individuals as they make optimal decisions over time. We looked at some techniques of inter-temporal optimisation. We discussed the important concept of a functional.

We discussed in detail the Ramsey model of optimal growth. Rarnsey did not believe in using discounted values of future consumption. The Keynes Ramsey conditions were discussed. Following this, the unit went on to present an analysis of Phelps's Golden Rule of Accumulation. Finally the unit described and analysed in detail the Cass-Koopmans extension to the Ramsey model, and also its comment on the Phelps's Golden .Rule of Accumulation. The Cass-Koopmans model explicitly brings in discounting of the consumption.

## 7.7 KEY WORDS

Additively separable utility function: A type of utility function which is actually a collection of sub-functions each of which is a separate function. For the overall function value, these separable functions can be added.

Discounting: The process of attaching weights to future returns in order to make them as valuable as present returns

Dynamic or Inter-temporal **Optimisation:** Dynamic optimisation is optimisation when the decision-making is done over time. The solution to a dynamic optimisation problem is a function.

Time Preference: The basic idea is that decision-makers are myopic, and prefer the present over the future.

## 7.8 SOME USEFUL BOOKS

Barro, Robert, and Sala- I-Martin. (2002), ***Economic Growth*** (2nd edition) McGraw-Hill Singapore

Hahn,F.H. and Solow, R.M. (1995) A ***Critical.Essay on Modern Macroeconomic Theory.*** M.I.T. Press, Cambridge, Massachusetts.

Jones, Charles I. (2002), ***Introduction to Economic Growth*** (2nd edition). W.W.Norton, New York.

Romer, David. (2001), ***Advanced Macroeconomics*** (2nd edition) McGraw-Hill, Singapore.

Solow, Robert M. (2000), ***Growth*** Theory: An ***Exposition (2nd edition).*** Oxford University Press, Oxford and New York.

## 7.9 ANSWERS/HINTS TO CHECK YOUR PROGRESS EXERCISES

**Check** Your Progress **1**

1) See section 7.2 and answer.
2) See section 7.2 and answer.
3) See section 7.3 and answer.

**Check** Your Progress **2**

1) See section 7.4 and answer.
2) In the equation for change in per-worker output, change in $f(k)$, **this** has to equal $(n+\delta+g)$ where g is the rate of technical change.

**Check** Your Progress **3**

1) See section 7.5 and answer.
2) See section 7.5 and answer.

# UNIT 8 MULTI-SECTOR MODELS OF GROWTH

**Structure**



## 8.0 OBJECTIVES

After going through the unit, you should be able to:

- distinguish between a capital good and a consumption good;
- discuss the Uzawa model of two-sector growth;
- describe the Feldman two-sector growth model;
- differentiate between two-sector and multi-sector growth models; and
- analyse the dynamic Leontief model and the von Neumann multi-sector models of economic growth.

## 8.1 INTRODUCTION

One-sector growth models have the unrealistic assumption that there is a homogeneous good that is used both consumer good or can be instantaneously, costlessly transformed into a capital good. The tenn 'multi-sector' in the title of this unit stands for a class of growth models that has two-sector models as a special case. However, usually, the structures of two-sector and multi-sector models are somewhat different from each other. Two-sector models can be seen as a natural extension and development of these simpler one-sector growth models you have studied earlier. These two-sector models can have fixed-coefficient production function, or smooth production function of the neoclassical type. In a sense, it can be argued that these models—whatever type of production function they may employ—intellectually derive from Marx's schema of two-sector production and growth.

The basic point about two-sector models is they explicitly recognise that consumption and capital goods are inherently different commodities produced in different sectors of the economy. Each sector has its own production function, which is obtained by aggregating-the production functions of the individual firms in that sector.

Multi-sector models are another type of extension of one-sector models where several durable goods are produced. Within this broad characteristic feature, there are models

## 7.7 KEY WORDS

**Additively separable utility function:** A type of utility function which is actually a collection of sub-functions each of which is a separate function. For the overall function value, these separable functions can be added.

**Discounting:** The process of attaching weights to future returns in order to make them as valuable as present returns

**Dynamic or Inter-temporal Optimisation:** Dynamic optimisation is optimisation when the decision-making is done over time. The solution to a dynamic optimisation problem is a function.

**Time Preference:** The basic idea is that decision-makers are myopic, and prefer the present over the future.

## 7.8 SOME USEFUL BOOKS

Barro, Robert, and Sala- I-Martin. (2002), ***Economic Growth*** (2nd edition) McGraw-Hill Singapore

Hahn,F.H. and Solow, R.M. (1995) ***A Critical Essay on Modern Macroeconomic Theory. M.I.T.*** Press, Cambridge, Massachusetts.

Jones, Charles I. (2002), ***Introduction to Economic Growth*** (2nd edition). W.W.Norton, New York.

Romer, David. (2001), ***Advanced Macroeconomics*** (2nd edition) McGraw-Hill, Singapore.

Solow, Robert M. (2000), ***Growth Theory: An Exposition (2nd edition).*** Oxford University Press, Oxford and New York.

## 7.9 ANSWERS/HINTS TO CHECK YOUR PROGRESS EXERCISES

**Check Your Progress 1**

1) See section 7.2 and answer.

2) See section 7.2 and answer.

3) See section 7.3 and answer.

**Check Your Progress 2**

1) See section 7.4 and answer.

2) In the equation for change in per-worker output, change in $f(k)$, this has to equal $(n+\delta+g)$ where g is the rate of technical change.

**Check Your Progress 3**

1) See section 7.5 and answer.

2) See section 7.5 and answer.

with varying parameter characteristics. The models can be fixed-coefficient or of smooth, neoclassical variety, non-produced factor of production like labour may or may not be present, and whether each good is produced only a single activity or there is a whole spectrum of activities. Moreover, there may or may not be joint production.

Quite popular in the 1950s, 1960s and 1970s, two-sector and multi-sector models lost favour by the 1980s when endogenous growth theory burst upon the theory scene with great force. We begin our study with an exposition of some two-sector models in the next section. In the subsequent section, we discuss some multi-sector models.

## 8.2 TWO-SECTOR GROWTH MODELS

### 8.2.1 Introduction and Assumptions

Two-sector models, whether of fixed-coefficient variety or of neo-classical type, all share a common structure. A single homogeneous good is produced using homogeneous capital and labour, which is also considered to be homogeneous. This homogeneous capital good itself is produced in a sector using homogeneous capital and labour. For simplicity you may consider the consumer good as 'corn' and the capital good as 'tractor'. In one sector of the economy, corn is produced using tractors and labour, in a separate sector tractors are produced using tractors and labour. Thus you may think there are two sectors in the economy: one, a farm sector where corn is produced, and a factory sector where tractors are produced. When we think of one-sector models there is supposed to be just one good in the economy, **either** corn or tractor. Either corn is produced with the aid of labour and corn, or tractors are produced using labour and tractors. Thus, in the one-sector models there is only one good, which serves both as consumption as well as capital good.

All two-sector models share a common set of assumptions:

1) There is a single malleable capital good used as an input in both sectors
2) Labour is homogeneous and grows at an exogenously given exponential rate $g_L = n$.
3) Both labour and capital can be shifted without cost and instantaneously from one sector to another.
4) (This assumption is true of neoclassical type two-sector models): the conditions of competitive equilibrium obtain.
5) Capital depreciates at a constant exponential rate $\delta$.

### 8.2.2 The Uzawa Model

We now discuss a very important two-sector growth model put forward by Hirofumi Uzawa in a couple of articles in 1961 and 1963. Meade in 1961 and Kurz in 1963 put similar models forward. This line of research was quite popular in the 1960s, but by the 1970s was on the wane. Uzawa's model is a two-sector extension of the Solow-Swan type neoclassical growth model.

Let there'be two goods, corn, or consumer good, denoted by c, and a capital good, 'tractor', or machine, denoted by m. the assumptions of the model are as follows:

The outputs of the machine and corn are produced subject to well-behaved, constant returns to scale neo-classical production functions:

$$Y_m = f_m\left(K_m, L_m\right)$$

$$Y_c = f_c\left(K_c, L_c\right)$$

where $K_m$ and $L_m$ are the amounts of capital and labour used in the machine sector, and $K_c$ and $L_c$ are the amounts of capital and labour used in the consumer sector. We denote the total amounts of capital and labour by K and L respectively:

$$K = K_c + K_m$$
$$L = L_c + L_m$$

Both capital and labour are shiftable, as we have mentioned as a general assumption, and also, capital is malleable. The production function of the two sectors can be written in per-worker form as:

$$y_m = f(k_m)$$
$$y_c = f(k_c)$$

where $y_m = \frac{Y_m}{L_m}, k_m = \frac{K_m}{L_m}, y_c = \frac{Y_c}{L_c}, k_c = \frac{K_c}{L_c}$

We restate the assumption about labour growth that we had mentioned earlier: the labour force grows at a constant propoitionate rate n, that is $\frac{\dot{L}}{L} = n = g_L$, or $L_t = L_0 e^{nt}$

The next assumption of the Uzawa model is that capital depreciates at a constant rate 6, which is independent of the sector in which the capital is being used.

Since a neo-classical type growth model is being considered, it is assumed that both the sectors are assumed to be maximising profits. This entails the paying of labour and capital the value of their marginal products. Moreover, thewage rate and rental for capital (profit) is the same in both the sectors. Thus

$$P_m \frac{\partial Y_m}{\partial K_m} = r, P_c \frac{\partial Y_c}{\partial K_c} = r$$
$$P_m \frac{\partial Y_m}{\partial L_m} = w, P_c \frac{\partial Y}{\partial L_c} = w$$

where $P_m$ is the price of the machine and $P_c$ is the price of the consumer good. The respective marginal products are easily interpreted. The ratio of wage rate to the rental of capital in each sector is given by

w / r

It can be shown that $r = P_m f'(k_m)$
$r = P_c f'(k_c)$

Where the $f'$ are the physical marginal products of capital in the machine and consumer sector respectively.

Since constant returns to scale are assumed to prevail, Euler's theorem is assumed to hold in each sector, we have

Wages per worker = value of output per worker minus the profits per worker

Hence in the capital goods sector (machine sector) the wage rate can be written as

$$w = P_m y_m - k_m P_m f'(k_m)$$

Similarly for the consumer goods sector, by changing the subscript m to c. Now it can be shown that

$$\frac{w}{r} = \frac{f(k_m)}{f'(k_m)} - k_m$$

$$\frac{w}{r} = \frac{f(k_c)}{f'(k_c)} - k_c$$

In Uzawa's two-sector model, the saving done necessarily equals investment, indeed is investment since saving necessarily implies a demand in the only asset present – capital good. An extreme assumption in two-sector models can be made about saving is that all profits are saved and all wages are consumed. This is the approach adopted by Kaldor, as you have studied in block **2.**

In equilibrium the following conditions are satisfied:

$$Y_m = f_m(K_m, L_m), Y_c = f_c(K_c, L_c)$$

$$P_m \frac{\partial Y_m}{\partial K_m} = r, P_m \frac{\partial Y_m}{\partial L_m} = w$$

$$P_c \frac{\partial Y_c}{\partial K_c} = r, P_c \frac{\partial Y_c}{\partial L_c} = w$$

$$K_c + K_m = K; L_c + L_m = L$$

$$P_c Y_c = wL; P_m Y_m = rK$$

The last is a saving-investment equation.

We can derive the basic equation showing the 'law of motion' of this two-sector model. The machine sector's output has two uses: to produce net increases in the stock of machines (capital good) and partly to replace the depreciating machines.

Thus the rate of change of capital stock is given by:

$$\dot{K} = Y_m - \delta K$$

Using the saving investment equation, we know $Y_m = \frac{rK}{P_m}$. Substituting this value of $Y_m$ in the equation above, we get

$$\dot{K} = \frac{rK}{P_m} - \delta K$$

We also know that $r = P_m f'(k_m)$. Using this, and dividing both sides by K we get

$\frac{\dot{K}}{K} = f'(k_m) - \delta$. We also know that $\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{L}}{L}$. In this equation, we can replace $\frac{\dot{K}}{K}$ by the equation we just obtained, and write n for $\frac{\dot{L}}{L}$ we get

$$\frac{\dot{k}}{k} = f'(k_m) - (n + \delta)$$

Here k is overall capital-labour ratio. This is a weighted average of the capital labour ratios in the two sectors. To make statements about growth in the balanced growth path, we have to remember that a balanced growth path is such that $\dot{k}=0$. So, in the equation above, the left-hand side equals zero and thus

$$f'(k_m)=(n+\delta)$$

So, in the Uzawa model, if we make the Kaldor-like assumption about saving and investment, we can say that balanced growth path implies that the marginal product of capital in the capital goods industry must equal the sum of the growth in the labour force and constant rate of depreciation. Now, the question is, are there circumstances in which the economy eventually can move to the balanced growth path, whatever the initial capital-labour ratio? It turns out that if the capital-labour ratio in the in the consumption goods sector is always grater than, or equal to, the capital-labour ratio in the capital goods sector, then the economy will tend to the balanced growth path, whatever the value of the ratio of the wage-rate to the profit-rate. We do not prove this assertion here as this will take us beyond the scope of the present unit.

### 8.2.3 The Uzawa-Srinivasan Extension

We may briefly mention here that the Uzawa model was extended by Uzawa and T.N. Srinivasan in a later paper to construct a model of optimal growth in the two-sector model. They postulated a decision-maker who wishes to maximise the sum (or integral, as they assumed continuous time) of per capita consumption subject to the different production conditions and conditions about the use of the capital good in both the consumption good and capital goods industry. Their objective function looks like the following:

Max $\int_0^{\infty} y_c e^{\beta t}\,dt$. Here β is the discount factor. To understand why the exponential e appears in the integral, look up unit 3, where the presence of e in continuous compounding and discounting is explained.

Subject to the condition that production functions are of the type we have encountered earlier, as well as the conditions of the stock of capital that we saw in the Uzawa model. The assumptions of two-sector growth models in general, and the Uzawa model in particular, hold as well. Consumption per capita is nothing but output of the consumption goods sector. The Uzawa-Srinivasan model investigates situations where capital-labour ratio in the consumer-goods sector is greater than, or equal to the capital-labour ratio in the capital goods sector, as well as situations where the capital-labour ratio in the consumption-goods sector is less than the capital-labour ratio in the capital goods sector. One of the important results of theirs is that unlike in the Uzawa model, it is not a condition for balanced growth that the capital-labour ratio in the consumer-goods industry has to be greater than, or equal to, the capital-labour ratio of the capital-goods industry. The main thing is the optimisation exercise and thus the optimality of the growth path. One important fact has to be observed about the objective functional (integral in this case) in the Uzawa-Srinivasan model. It is that what is being maximised here is consumption per capita, and not utility. Thus, the usual assumptions of diminishing marginal utility do not hold. However, we may think of the optimisation being carried out by the government or some public sector agency.

### 8.2.4 The, Feldman Model

Now let us discuss a two-sector model, which is quite different from the Uzawa two-sector model in that the production is carried out with fixed-coefficient production functions. This is the two-sector model of Feldman. G.A. Feldman was a Soviet economist who worked on problems of planning in the newly formed Soviet State Planning Commission. In 1927, when the erstwhile Soviet Union embarked on the path of planned development through its Five Year Plans, Feldman constructed a long-term theory of growth.

Feldman's model was heavily influenced by Marx's scheme of extended commodity reproduction, with its two 'departments'. Feldman's model is important because although it came much before their models, it generates results similar to the Harrod-Domar models. In Feldman's model, output growth is only constrained by the availability of capital, and thus, it is also similar to the Lewis model of development for developing nations, about which you will study in block 5.

The assumptions of the Feldman model are as follows. The economy is divided into two sectors or categories, I and 2. Sector 1 produces capital goods and these capital goods can be used either in sector 1 or **2, but *once installed, they cannot be shifted from one sector to another.*** Once a capital good is used in sector 2, it can never be used in sector 1. thus, capital is not shiftable. A portion of currently produced is allocated to sector 1 and the rest to sector ***2*.**

The second assumption of the Feldman model is that production is carried out in both sectors with fixed coefficients technology.

$$Y_1 = \min\left[\frac{K_1}{w_1}, \frac{L_1}{z_1}\right]$$

$$Y_2 = \min\left[\frac{K_2}{w_2}, \frac{L_2}{z_2}\right]$$

Where $Y_1$ is output of sector 1, $Y_2$ is output of sector 2, K and L with relevant subscripts show their use in the two sectors and $w_1$, $w_2$, $z_1$, $z_2$ are the fixed coefficients. It is only capital, however, which is the limiting factor in the growth process. We can describe the technology by $Y_1 = \frac{K_1}{w_1}, Y_2 = \frac{K_2}{z_2}$ and not even bring labour into the picture. The next assumption is that capital does not depreciate, so that change in capital stock equals investment. Another assumption is that the economy is a closed economy, which implies that capital goods cannot be imported. The final assumption is that production of goods in sector 1 is independent of production in the other sector.

We can look at the workings of the Feldman model. Total amount of investment good I, which can be installed in either sector, is nothing but the output of sector 1:

$$I = Y_1 = \frac{K_1}{w_1}$$

Thus, the rate of change in investment $I$ is

$$\dot{I} = \dot{Y}_1 = \frac{1}{w_1}\dot{K}_1$$

Now the rate of change in $K_1$ depends on the proportion of the total output of the investment good sector that is allocated to sector 1. then, by definition,

$$K = I_1 = \alpha I$$

Substituting this into the equation above, we get

$$\dot{I} = \frac{1}{w_1} \times \alpha I$$

$$\Rightarrow \frac{\dot{I}}{I} = \frac{\alpha}{w_1}$$

Thus we see in the Feldman model that proportionate rate of growth of investment depends on a and $w_1$. If the former increases, or tlie latter decreases, the proportionate rate of growth of investment will rise. Let us now look at the consumption good sector. Consumption is nothing but tlie output of sector 2, $Y_2$. What we have is

$C = Y_2 = \frac{K_2}{w_2}$ and the rate of change of consumption goods is

$$\dot{C} = \dot{Y} = \frac{K_2}{w_2}$$

By definition, $K_2 = I, = (1-\alpha) I$. Substituting for $\dot{K}$, we get

$$\frac{\dot{C}}{C} = \frac{(1-\alpha)}{w_2} \times \frac{I}{C}$$

We know that total investment I grows at the rate $\frac{\alpha}{w_1}$ and thus rate of growth of C depends on that of I. Two implications of the Feldman model follow:

1) In general, the rate of consumption and that of investment will not be equal to each other. Over time, the rate of growth of consumption increases until it reaches the long run rate of growth of the economy which is given by tlie rate of growth of investment, $\frac{a}{w_1}$.

2) In the Feldman model, in general the rate of growth of the national income will not be equal to the rate of growth of total investment, but will tend towards that rate, $\frac{a}{w_1}$ in tlie long run. Thus in the Feldman model, one policy prescription is to increase the proportion of current capital goods engaged in producing more capital goods as this will, in the long run, increase the rates of growth of consumption, investment and output over what they otherwise would have been.

The Feldman model is of sorne importance in the study of Indian economic development since it influenced tlie model of growth constructed by P.C. Mahalanobis (some claim that Mahalanobis presented his model independently of Feldman). Mahalanobis was a great Indian statistician and economist, and architect of the Second Five Year Plan in India which espoused a plan model based on heavy industries and import substitution. He had, in a couple of theoretical papers set out the structure of his model. In his models, he had divided the capital goods sector into further two types of capital goods: C-type capital goods that produce consumer goods, and K-type capital goods that produce other capital goods, or 'machines that produce other machines'. The Mahalanobis model is thus an elaboration of Feldman's model applied to capital goods of more than one-type.

**Cheek Your Progress 1**

1) What are the basic, assumptions that are common to all types of two-sector growth models?

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2) Describe the basic structure of the Uzawa model.

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3) In what basic way is the Feldman model different from the Uzawa model?

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## 8.3 MULTI-SECTOR GROWTH MODELS

Perhaps the most straightforward approach to the study of multi-sector growth models would be to extend the one-sector models and two-sector models to add more sectors, each denoting an additional durable good produced by a neo-classical production function of say, the Solow type. Thus we could have several durable goods and a single, non-reproducible factor, labour.

Suppose there are n distinct capital goods, each produced in a separate sector of the economy. Let us suppose labour is homogeneous and that there is a consumption sector which produces a single consumption good. So we have n+1 production sectors, and $K_1, K_2 \ldots, K_n$ denoting stocks of capital goods. We denote the sector producing consumer good by subscript 0. Let $K_{ij}$ denote the quantity of $i^{th}$ capital good employed in the $j^{th}$ sector, and $L_j$ denote the quantity of labour in the jth sector,( i = 1,2,...,n; j = 1,2.....n)

In each sector production takes place with a neo-classical production function:

$$Y_j = f^j\left(L_j, K_{1j}, \ldots, K_{nj}\right)$$

Since these production functions show constant returns to scale, we can write, after multiplying each term by $\frac{1}{Y_j}$

$$1 \equiv f^j\left(a_{0j}, a_{1j}, \ldots, a_{nj}\right)$$

where we define

$$a_{0j} = \frac{L_j}{Y_j}, (j = 0, 1, \ldots, n)$$

and $a_{1j} = \frac{K_{ij}}{Y_j}, j = 0, 1, \ldots, n$

We have the full employment conditions

$$\sum_{j=0}^{n} a_{0j} Y_j$$

and $\sum_{j=0}^{n} a_{ij} Y_j = K_i$

This is the general multi-sector model where several durable capital-type goods are produced. Let us now discuss two multi-sector models which are more restricted in the sense that in these models the production function is of a restricted, linear variety. We study the Leontief and the von Neumann models. You would have studied about the usual static Leontief model in your course on quantitative techniques; here we look at the Leontief model where production takes place over time.

The dynamic Leontief system is an extension of the static input-output system of inter-industry analysis to the dynamic or intertemporal case. Like the static model, the dynamic Leontief model has found wide application in empirical analysis.

The dynamic Leotief system is characterised by having a single production process available for the production of each good. This is the fixed coefficientsassumption.

The dynamic Leontief model is one where the production function is of fixed coefficients form of the type:

$$Y_j = \min\left[\frac{L_j}{a_{0j}}, \frac{K_{1j}}{a_{1j}}, \ldots, \frac{K_{nj}}{a_{nj}}\right], j = 0, 1, \ldots, n$$

The basic idea in dynamising the static Leontief model seems to have come from the Harrod-Domar model. Therefore, the assumption of fixed capital coefficients is essential in the model.

Let us briefly set up the physical structure of the dynamic Leontief system. Let $\mathbf{x}(t)$ be an n-vector whose $i^{th}$ element is $x_i(t)$, the output of the $i^{th}$ good in period t. Let $A = [a_{ij}]$ be an $n \times n$ matrix, where $a_{ij}$ denotes the current input of the $i^{th}$ good used per unit of the $j^{th}$ good. Let $B = [b_{ij}]$ be an $n \times n$ matrix, where $b_{ij}$ denotes the quantity of the $i^{th}$ good invested the $j^{th}$ industry in order to increase the output of that industry by one unit. Let $c_i(t)$ be the final demand, or consumption demand of the $i^{th}$ good in period t. Then the total demand for the $i^{th}$ good in period t is

$$\sum_{j=1}^{n} a_{ij}x_j(t) + \sum_{j=1}^{n} b_{ij}\left[x_j(t+1) - x_j(t)\right] + c_i(t)$$

Let $K_i(t)$ be the total stock of the $i^{th}$ good required as a capital good in the economy in period t, that is, $K_i(t) \equiv \sum_{j=1}^{n} K_{ij}(t)$. In the equation above, we may think of the coefficient $b_{ij}$ as the amount of stock of the ith good necessary as capital good in the production of one unit of the jth good. Then we have

$b_{ij}x_j(t) = K_{ij}(t)$. We can combine the abovementioned two relationships to derive

$K_i(t) = \sum_{j=1}^{n} b_{ij}x_j(t)$. If we assume that capital is freely transferable from one industry to another, and also that capital is fully employed, we have

$$\begin{aligned} I_i(t) &= \Delta K_i(t) \\ &= K_i(t+1) - K_i(t) \\ &= \sum_{j=1}^{n} b_{ij}[x_j(t+1) - x_j(t)] \end{aligned}$$

The basic equation of the dynamic Leontief system can be written as

$$x_i(t)=\sum_{j=1}^{n} a_{ij}x_j(t)+\sum_{j=1}^{n} b_{ij}\left[x_j(t+1)-x_j(t)\right]+c(t)$$

We can write the above in matrix form as

$$x(t)=A\cdot x(t)+B\cdot\left[x(t+1)-x(t)\right]+c(t)$$

The above equation is the basic output equation of the dynamic Leontief system.

Let us now discuss von Neumann's growth model. von Neumann's model deals with production in a linear programming way, with alternative processes and intermediate goods. The model basically has only intermediate goods used in production. There are no primary factors of production; there is not even a final consumer demand sector. Thus the model is a 'closed system' type with only production. There is no autonomous demand and no naturally existing factor like labour.

von Neumann's model is a generalisation of Leontief-type system to permit joint production. It also suggests the relaxing of the condition of a fixed ratio of inputs to outputs, though even here in the basic set-up goods are produced from other goods by processes or activities of the fixed coefficients type. But unlike the Leontief system there are several activities permissible and available. The von Neumann model gives rise to balanced growth at a maximal rate.

We can present the von Neumann model in the following manner.

Suppose the economy produces n commodities indexed by $i, i=1,\dots,n$. Suppose the technologically given production possibilities can be described by the transformation set T such that T has as elements pairs (x,y) of n-vectors such that the production of output $y=(y_1,\dots,y_n)$ is possible technologically from the input $x=(x_1,\dots,x_n)$ only if (x, y) is a member of the set T. One of the assumptions about T is that if

Let there be a finite number, m, of ***production activities*** indexed by ***j*** that is, $j=1,2,\dots m$. Suppose that the jth activity's operation at unit level requires an input $(a_{1j},\dots,a_{nj})$ of the n commodities and produces an output $(b_{1j},\dots,b_{nj})$. We assume that $a_{ij}\geq 0, b_{ij}\geq 0$.

We can define a 'transformation set' that depicts the relationship between the activities and commodities as

$$T=\{(x,y): x\geq Az, y\leq Bz\} \text{ for some } z\geq 0, z=(z_1,\dots,z_m)$$

where $A_{ij}$ and $B_{ij}$ are the matrices of input and output coefficients, respectively.

It is also assumed in the von Neumann model that for any j there is at least one $i$ such that $a_{ij}>0$, that is, every activity uses some commodity as input.

Related to this is the assumption that for every $i$, there is at least one j such that $b_{ij}>0$ that is, every commodity can be produced by some output.

Under these assumptions, the von Neumann model claims that there exists an activity level $z^*$ and a price level $p^*$ such that

$$z^*\geq 0, p^*\geq 0, Bz^*\geq \lambda^* Az^*,$$
$$p^*B\leq \lambda^* A.$$

Here $h$ is an eigenvector. The von Neumann growth rate $h^*$ is the growth rate of the slowest growing sector.

The von Neurnann model has been extended by Michio Morishima, in a series of papers and books in the 1960s, by incorporating a consumption-good sector and in some other ways as well. Labour need not be absent or be produced within the system. Labour can grow at some finite rate. Secondly, Morishima suggests that workers' demand for consumer goods depends not only on wages but also on prices Morishima further suggests that capitalists consume a constant proportion of their income and their demand for consumer goods depends on prices.

Thus, by putting in the possibility of labour and by introducing a consumption sector, Morishima is able to generalize and extend the von Neumann model.

**Check Your Progress 2**

1) Describe the basic structure of the general multi-sector production process.

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2) Bring out the basic dissimilarities between the dynamic Leontief model and von Neumann's model.

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## 8.4 LET US SUM UP

This unit presented extensions to the one-sector growth models in one particular direction. The basic extension had to do with introducing two or more sectors in the economy. Broadly two types of models were discussed in the unit: two-sector models and multi-sector models.

The basic difference between the two types of models is as follows. In two-sector models there are two types of goods: a single homogeneous consumption good produced by homogeneous labour and homogeneous capital, and this capital good is itself produced with the help of homogeneous labour and homogeneous capital good. In multi-sector models, by contrast, there are several durable goods that are produced.

The unit went on to discuss the basic two-sector 'neo-classical-type' growth model constructed by Uzawa. The assumptions of the model were carefully stated, the structure of the model was described, and the implications of steady state and balanced growth were discussed. Uzawa-Srinivasan's extension to the Uzawa model, in terms of making it a optimal growth model were discussed. Another two-sector model, that of Feldman, was discussed, as it is of a fixed coefficient production function type. Finally, some multi-sector growth models were discussed. These 'include the dynamic Leontief model, the von Neumann model, and some extensions by Morishima to this model by considering consumption.

## 8.5 KEY WORDS

Capital ***good:*** a produced good which is used to produce other goods, a produced means of production.

**Closed Models of Production:** inter-industry models of production that do not have a final demand or consumption sector

**Euler's Theorem:** a theorem whose application in economics in the factor markets says that if there are constant returns to scale and if factors are paid the value of the marginal product, the product is exhausted in the payment to factors and there is no surplus left over.

**Fixed coefficient production function:** A production function where the inputs are used in fixed proportions

## 8.6 SOME USEFUL BOOKS

Chakravarty, Sukhamoy (1969) *Capital* and Development Planning MIT Press, Cambridge, Massachusetts

Dorfinan, R., Samuelson, P.A., and Solow, R.M.(1958) Linear Programming and Economic Analysis , McGraw-Hill, New York.

Leontief, W.W. (1966) Input Output Analysis, Oxford University Press, New York.

Morishima, M. (1969). Theory *of* Economic Growth, Oxford University Press, Oxford.

Radner, R. (1963) Notes on the Theory of Planning, Centre of Economic Research, Athens

## 8.7 ANSWERS OR HINTS TO CHECK YOUR PROGRESS EXERCISES

**Check Your Progress 1**

1) See sub-section 8.2.1
2) See sub-section 8.2.2
3) See sub-section 8.2.4

**Check Your Progress 2**

1) See section 8.3
2) See section 8.3

# UNIT 9 ENDOGENOUS GROWTH MODELS

**Structure**



## 9.0 OBJECTIVES

After going through the unit, you should be able to:

- distinguish between exogenous growth models and endogenous growtli models;
- describe the impact of human capital on economic growtli;
- discuss the nature of ideas as a spur to human capital formation arid as an engine of growth;
- analyse how technology can be modelled as an endogenous variable; and
- discuss what impact increasing returns and monopolistic competition have on the growtli process.

## 9.1 INTRODUCTION

Endogenous growth encompasses a wide body of theoretical and empirical work on growtli theory tliat emerged in the 1980s. The work differs from the standard neo-classical model in emphasising that economic growth is caused by forces endogenous to the economic system ratlier than being influenced by factors outside the system. These tnodels do not rely on exogenous technical change to explain the rise in per capita income, nor do these models set great store by the growtli accounting methods and the measurement of the growth accounting 'residual'. Endogenous growth theory began with the works of Paul Romer in 1986 and Robert Lucas in 1988.

Models of endogenous growtli had their origins in two sources: one was to give a coherent explanation of the convergence controversy, and the other, to go beyond a simple world of perfect competition and constant returns to scale in growtli theory. You are studying in your microeconomics course about imperfections in the market, but the assumption in macroeconomics and growth tlieory seems to be that there is perfect competition everywhere in the economy. Although economists had given up this simplifying assumption in many areas of economic theory, it took a long time in growth theory to construct models that bring in imperfect competition. Another standard assumption that was challenged was that (eventual) diminishing returns to individual factors of production would set in, and that the aggregate production function would exhibit constant returns to scale.

The neo-classical Solow model exhibited diminishing returns to capital and labour separately and constant returns to scale. Thus the idea was, like in traditional economic theory, diminishing returns to factors would eventually set in. The new endogenous growth theories extend the Solow model in this respect: they explore the idea that it is possible to stave off diminishing returns, and there can be increasing returns, because of specialisation and investment in knowledge capital. The basic neoclassical model suggests that countries with a higher return to capital (relatively lower stock of capital) would catch up with richer countries. Failure to show empirical convergence led to plans to drop neoclassical assumptions of (1) exogenous technological change (2) equal technological opportunities across different countries

Endogenous growth theory does not require abandoning the neoclassical framework, only extending it. Also, we should not look at growth theory only to investigate the issue of convergence. A different perspective on models of growth is needed. The basic idea in endogenous growth theory is that growth is caused by factors like technology, and technology is not something given exogenously but is determined within the system.

## 9.2 THE NEOCLASSICAL GROWTH MODEL WITH HUMAN CAPITAL

One of the assumptions of standard growth models of the Solow type was that if initial conditions were similar in different countries then eventually the levels of incomes of these countries would converge. Thus it seemed that what was being suggested was that growth rates of poor countries would tend to be faster than those of the developed nations. The growth rates of the poor countries would get faster and faster, while the growth rates of the richer countries would slow down. But looking at the experiences of economic growth of countries in the world, two facts stand out. First, output growth has consistently been much faster than population growth for most of the world. Secondly, countries have remained on disparate growth paths for a very long time and there does not seem to be much convergence. It was also observed that there was a correlation of economic growth with several economic, social and political factors, many of which were influenced by government policies.

In response to these observations there have been attempts to build models where rise in per capita incomes goes on indefinitely. One strand of this research effort involves models that continue to see capital accumulation as the driving force behind economic growth. But capital accumulation is broadened to include human capital. In some of these works, firms keep on adding to their stocks of capital in a perfectly competitive environment with constant returns to scale.

We first look at a body of theory that takes the basic structure of the Solow growth model, and explores modifications that result when human capital is used along with physical capital. Another way is to see human capital as an extension of the factor of production, labour. We have so far considered labour as a single homogeneous factor of production. A class of endogenous growth theory tries to go beyond this simple formulation.

Mankiw, Romer and Weil in a 1992 paper gave a marginal extension to the neoclassical model – include human capital (H) as a distinct factor of production. K and H are allowed to vary together across countries and arrive at decent results for the example discussed earlier. The Mankiw Romer Weil formulation maintains the Solow assumption of decreasing returns to the factor capital. Their model is thus within the tradition

The motivation and structure of the Mankiw, Romer and Weil (MRW) model can be presented as follows. Suppose we consider an economy which produces an output Y using physical capital in combination with human capital, according to a constant returns to scale Cobb-Douglas production function

$Y = K^{\alpha}\left(AH\right)^{1-\alpha}$, where A represents labour augmenting technology that grows at an exogenous rate g. How people accumulate human capital is modelled differently by different writers. According to Mankiw, Romer and Weil, people accumulate human capital the way they accumulate physical capital—by foregoing current consumption. Lucas on the other hand posits that individuals spend by in accumulating skills. The Lucas type of presentation is made here. This idea was first put forward by Kenneth Arrow in a 1962 article on 'The Economic Implications of Learning by Doing'. In our economy, individuals accumulate capital by building and learning new skills and foregoing working for that period. Let q denote the proportion of an individual's time spent learning new skills, and let L be the total amount of raw labour used in production in the economy. Let us assume that unskilled labour learning skills for time q generates skilled labour H according to the function

$$H = e^{\phi q} L$$

where $\phi$ is a constant.. If q = 0, the L = H, and all labour is unskilled. By raising q, a unit of unskilled labour increases the effective units of skilled labour H.

$$\frac{d\ln H}{dq} = \phi \Rightarrow \frac{dH}{dq} = \phi H$$

In the economy, physical capital is accumulated by investing some output instead of consuming it, as in tlie Solow model:

$$\dot{K} = s_K Y - \delta K$$

Let us denote by lower-case letters variables divided by the stock of unskilled labour L. We can rewrite tlie production function in terms of output per worker:

$$y = k^{\alpha}\left(Ah\right)^{1-\alpha}$$

Here $h = e^{\phi q}$. The model assumes that q is constant and given exogenously. Since h too is a constant, this model implies that along a balanced growth path, y and k will grow at the constant rate g, the rate of technical progress. Let us divide tlie per-worker production function by dividing by Ah. We obtain

$$\tilde{y} = \tilde{k}^{\alpha}$$

We can write the capital accumulation equation in terms of state variables as

$$\dot{\tilde{k}} = s_K \tilde{y} - \left(n + g + \delta\right) \tilde{k}$$

the steady-state values of $\tilde{k}$ and $\tilde{y}$ arc obtained by setting $\dot{\tilde{k}} = 0$, which gives us

$$\frac{\tilde{k}}{\tilde{y}} = \frac{s_K}{n + g + \delta}$$ .substituting this into the production function in terms of the equation involving $\tilde{y}$, we can find the steady-state value of tlie output technology ratio $\tilde{y}$:

$$\tilde{y}^{*} = \left(\frac{s_K}{n + g + \delta}\right)^{\frac{a}{1-a}}$$ . If we rewrite this in terms of output per worker, we get

$$y^*(t)=\left(\frac{s_K}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}} hA(t)$$

Here t is introduced to stress that some variables are growing over time.

This gives us some explanation into why some countries are rich and some are poor. Those countries that have a high investment rates in physical capital (That is, those that save more), have high levels of technology, and spend a high propol-tion of time accumulating skills are comparatively richer.

'Knowledge capital' or 'ideas' is slightly different from human capital, and facilitates in the accumulation of the latter. In the new growth theories endogenise technological progress and human capital formation. New knowledge is produced by investment in the research sector. Once knowledge is treated as a public good, there are spillover effects that accrue to other firms. This is what we will discuss in the next section.

Cheek **Your** Progress **1**

1) In what way arc the models that incorporate human capital different From the standard neoclassical model?

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2) Using the human-capital augmented model, can you provide an explanation why in spite of diminishing returns to physical capital, similar savings and similar levels of technology, there may not be convergence in the growth rates of countries?

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## 9.3 ENDOGENOUS TECHNOLOGY, INCREASING RETURNS AND MONOPOLISTIC COMPETITION

In the previous section we looked at the implications for economic growth, of expanding the notion of capital 'to include human capital. We tried to see what answers have been provided for the question 'what is the effect of human capital on economic growth'? In this section we get into a detailed discussion about the nature of human capital itself, and how it is created. The intuitive answer to the rhetorical question above seems to be obvious: human capital will aid growth. But then the question is, how exactly does it affect growth and what is its nature? This is what this section would explore. This section also explores the issue of diminishing returns to capital that was a standard conclusion of the Solow model. We look at attempts to come up with models where diminishing returns to capital do not set in.

Given that we see evidence of cross-country variation in per capita incomes and national growth rates, how far can the neoclassical growth model, with decreasing returns to capital, perfect competition, and exogenous technology, fully explain it?

Let us take a standard Cobb-Douglas production function in the standard (human capital augmented) neoclassical model: $Y = Ae^{\eta t}K^{\alpha}L^{1-\alpha}$. This is constant-returns to scale production function. Here Y is gross domestic output, K is the stock of physical (and human) capital, L is unskilled labour, A is a constant reflecting the technological position of society, and $e^{\eta}$ is the exogenous rate at which the technology evolves. In the above formulation, a indicates the percentage increase in Y from a 1 per cent increase in capital. Here $\alpha$ is obtained from the share of capital in the national income, assuming capital is paid its marginal product. As long as $\alpha$ is less than 1, there are diminishing returns to capital and labour.

In this type of model, saving will create growth for a while. But as the ratio of capital to labour rises, the marginal capital of capital will fall, and the economy will revert back to the steady state and capital, labour and output will all increase at the same rate. Growth in income per worker will continue and will equal $\eta$, the annual rate of productivity measurement. We mat think of $\eta$ as improvement in knowledge. Traditional neoclassical theory says nothing about how $\eta$ is determined. This is where endogenous growth theory steps in and tries to rectify this shortcoming.

The AK model of endogenous growth is the basic type of formulation that captures the essence of endogenous growth. Sergio Rebelo in an article titled "Long Run Policy Analysis and Long Run Growth" published in 1991, introduced the AK model, building upon earlier work by Romer and Lucas. The basic equation of the AK model is $Y = AK$. Here A is to be understood as factors that affect technology, and K includes physical as well as human capital. This model has no diminishing returns. One way to achieve this is to invoke some kind of externalities or spillovers. Another way to stave off diminishing returns is to postulate that an increasing variety or quality of intermediate inputs.

Let us look at the AK model in some detail. We have the basic equation. Suppose capital is accumulated as some of the aggregate output is saved by individuals.

$K = sY - \delta K$, where the dot on top indicates time derivative, s is the rate of saving and $\delta$ denotes depreciation. Here whatever the initial quantity of capital, it will continue to grow since investment in this model is greater than depreciation since sY and $\delta$K are both linear upward sloping lines, since saving is a constant function of Y, and Y is linear in K. We can show it as follows:

$$K = sY - \delta K$$

Dividing both sides by K, we have

$$\frac{K}{K} = s\frac{Y}{K} - \delta$$

On the right-hand side, substitute AK for Y and we get

$$\frac{K}{K} = sA - \delta$$

Let us go back to the growth equation

Y=AK

Taking natural logarithms we get

$$\ln Y = \ln A - \ln K$$

Diffetentiating with respect to time we get

$$\frac{\dot{Y}}{Y} = \frac{\dot{K}}{K}$$

Using this in the equation $\frac{K}{K} = sA - \delta$ we obtain

$$\frac{Y}{Y} = sA - \delta K$$

Thus we see that growth rate of the output is an increasing function of the rate of investment.

The general progression in that area has been to attribute a smaller fraction of observed growth to the residual and a higher fraction to the accumulation of inputs. The way that literature started out was a statement that technology is extremely important because it explains the bulk of growth. Endogenous growth theory admits that technology does not explain, by itself, the majority of growth. Initially, these theorists overstated its importance. But some people say there is no need to understand technology, because it is such a small part of the contribution to growth. They argue that we can just ignore it. It does not follow logically. We know from the Solow model, and this observation has been borne out empirically, that even if investment in capital contributes directly to growth, it is technology that causes the investment in the capital and indirectly causes all the growth. Without technological change, growth would stop. If we look at AF (K, L) we find that there are increasing returns {to factor) in all the relevant inputs A, K and L We need to think of technology as a key input and one that is fundamentally different From traditional inputs. Technology makes increasing returns possible.

Let us look a little closely now at the notion of technology and how it is produced. The neoclassical growth model considers technology as exogenous **and** leaves it unmodeled, although technology is a key ingredient of that theory. The neoclassical model also does not explain differences in technology across economies. What is technology? In development economics the term technology denotes the way inputs to the production process are transformed into output. If we take a normal production function $Y = f(K, L)$, the function $f(.)$ represents technology because it explains how the inputs K and L are transformed into output. In the Cobb-Douglas production function .

$Y = K^a (AL)^{1-\alpha}$, **A** is an index of technology.

It is the generation of new ideas, and the ideas themselves, that improve the technology of production. New ideas allow a given bundle of inputs to produce more output or better output. Paul Romer, in a paper in 1986 and. another in 1990, presented a way of modelling ideas as an engine of growth. Romer's basic argument was that ideas are non-rivalrous in nature as a good. Romer suggested that the nature of ideas as a non-rivalrous good implies the presence of increasing returns to scale. If increasing returns are to be present in a competitive environment with explicit involvement in research, we need to invoke imperfect competition.

Most economic good are rivalrous. This means that the use of a good, say, a shirt, excludes the use of that same shirt by anyone else. By contrast, ideas are non-rivalrous goods. Once an idea has been created, anyone with knowledge of the idea can take advantage of it. However, ideas share a characteristic with other goods: ideas are excludable, at least partially, The degree of excludability is the degree to which the owner of a good or service can charge a price for its use. In the case of idem, patents and copyrights allow the creators of ideas to charge a price or fee. Goods that are both non-rivalrous as well as very low or no excludability are called pure 'public, goods, like national defence. There are goods that are rivalrous but have low excludability like common property resources, like a fishing lake or grazing land. Ideas are non-rivalrous goods but they vary in their degree of excludability. Goods that are excludable allow their producers to capture the benefits they produce. Goods that are non-excludable lead to externalities, which are spillovers of benefits not captured by their producers.

Goods that are rivalrous, as most goods are in economics, need to be produced every time they are sold. For example, each shirt has to be separately produced. Non-rival goods have to be produced only once. Think of a park or a beach. Once it is created, everybody can use these, and they do not have to be created again and again. Ideas are such non-rivalrous goods. Think of the idea of some operating system software for a computer. Every time the physical product embodied in that software has to be produced but not the intellectual property that created it. The implication of the fact that non-rivalrous goods need to be produced only once is that such goods have a fixed cost of production and zero marginal cost. It takes a lot of effort to bring out the first unit of a software, but subsequent copies are produced merely by copying from that first unit.

The nature of costs involving non-rivalrous goods suggests the presence of increasing returns to scale and imperfect competition. Think again of the software development example. Here the idea or knowledge capital required to come up with a new software involves a one-time research costs and effort. However, subsequent units are produced under constant returns to scale. The production of subsequent 100 copies merely requires raising the production of the various units by 100. This is production with fixed costs and zero marginal cost. Since the units after the first one are produced under constant returns to scale, the entire production takes place under increasing returns to scale. The average cost curve will be a downward sloping curve.

The cost structure has an implication for pricing: should the price charged be equal to marginal costs? In perfect competition, firms charge **a** price that is equal to the marginal cost. But if firms that produce ideas were to charge a price equal to marginal costs, they would make losses. Since there are increasing returns to scale, average cost will always be greater than marginal costs, and charging a price (price is average revenue) equal to marginal costs will result in average revenue being less than average cost and thus negative profits. In the production of ideas, producers are given patents and copyrights when they apply for these, and these patents and copyrights are legal mechanisms that grant these producers monopoly powers for some time and allow them to reap profits from their ideas and inventions. This was the case with, say, the'invention of the light bulb or the phonograph.

Romer's contribution was that he combined monopolistic competition and increasing returns to technological advances. His concept of positive spillovers to R&D led to the creation of models where growth could be sustained in a framework of endogenously determined variables that impinge on the growth process. Endogenous growth models introduce monopolistic competition in order to show increasing returns that would sustain time-dependent production functions for technological change. Perfect competition would not allow this since firms would operate at **a** minimal efficient level and increasing firm size would have no effect on output.

Let us now look closely some more at Romer's model of endogenous technical change. Romer begins with a standard production function of the type

$Y = K^{\alpha}(AL_Y)^{1-\alpha}$ where $L_Y$ denotes labour, **A** is taken as the stock of ideas and a lies between 0 and 1. This production function exhibits constant returns to scale in K and L: if we double **the** amount of capital and labour, output will double. But once **A** enters the picture, there are increasing returns to scale. If K, and $L_Y$ are doubled, and the stock of ideas is doubled as well, output will more than double.

The accumulation equation in Romer is similar to that in Solow. Capital accumulates as people save at a constant rate, and there is also depreciation of capital.

$$K = s_K Y - \delta K$$

Labour grows at a constant rate n:

$$\frac{L}{L} = n$$

In Romer's model, what is new is an equation describing the accumulation of ideas, or technical progress. In the neoclassical model, **A** is a productivity term and grows exogenously at a constant rate. The crucial point in Romer's model is that th growth of A is endogenised. The theoretical devise to endogenise the growth of **A** is to conceive of a production function for new ideas. According to the Romer model, there is a stock of knowledge or ideas that has been accumulated till time. t. Then $\dot{A} = \frac{dA}{dt}$ is the number of new ideas produced at any given time. We can think of the number of people engaged in producing new ideas $L_A$ and the rate $\gamma^*$ at which they produce new ideas. Thus

$A = \gamma^* L_A$. We can think of $\gamma^*$ as an increasing function of A, $\gamma^* = f(A)$

We could think of the new ideas appearing as

$y' = \gamma A^{\theta}$ where $\gamma$ and $\theta$ are constants.

In the Romer model, what is the growth rate along a balanced growth path? If a constant fraction of the population is employed in producing ideas, then the model predicts, just like the neoclassical model, that all per capita growth is due to technical progress. Rate of growth of output will be equal to the rate of growth of capital and these will be equal to the rate of growth of technical progress. From this follows the question as to what is the rate of growth of technical progress along the balanced growth path. Along the balanced growth path the rate of growth of ideas, A, is constant.

Romer in his model develops three sectors to depict the generation of ideas: there is a final goods sector, an intermediate-goods sector and an R and D sector. Basically the research sector creates ideas which take the form of new types of capital goods. The research sector sells the rights to the intermediate goods sector to produce a specific type of capital good. Each firm in the intermediate-goods sector is a monopolist. Each monopolist in the intermediate-goods sector manufactures a specific type of capital goods and sells it to the final goods sector which produces the output. Thus the imperfection in the model appears in the intermediate-gods sector.

The Neoclassical model of growth ignores the time-dependent production function for technological advances as well as the fact that firms own these innovations and charge monopoly rents for them. Introduce monopolistic competition in order to show increasing returns that would sustain time-dependent production functions for technological change. Perfect competition would not allow this since firms would operate at a minimal efficient level and increasing firm size would have no effect on output.

**Check Your Progress 2**

1) Explain why the production of ideas involves imperfect competition.

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2) Why do diminishing returns to capital not set in the AK model of production?

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## 9.4 SOME ISSUES IN ENDOGENOUS GROWTH THEORY

What are the implications of the new endogenous growth theory for developing nations? It had long been supposed that developing nations need to push up their rates of saving. This came through in the writings of economists like Arthur Lewis and W.W. Rostow, who insisted that for the economy to get transformed or to "take off" (Rostow's term), it is necessary to substantially raise the savings rate. The first lesson of endogenous growth theory is to place greater emphasis on accumulation of human capital – even more than physical capital, Knowledge capital has to be stressed. The second implication for developing nations is that, since knowledge capital can be acquired by transfer of technology, developing nations would do well to open up their economies and integrate with the world economy. In that case, international transfer of technology-can take place.

Endogenous growth theory suggests that for developing nations that are interested in growth should pay close attention to policy and governance. Particular attention has to be paid to policies For the creation of knowledge capital and suitable investment to this end. The research sector has to be promoted.

We have seen in the earlier part of this unit that endogenous growth theory explains why convergence in growth rates does not always take place as suggested by neo-classical growth theory. Hence developing nations would do well to devise policies that create the situation for higher growth rates. The neoclassical model, augmented by human capital can still predict conditional convergence

How does endogenous growth theory relate to exogenous growth theory? Sometimes endogenous growth theory, because of the chronological time when it appeared, is called 'new' growth theory, to contrast it with neoclassical growth theory of the Solow type, in which technical change is exogenous, and which is called 'old' growth theory. The implication seems to be that the 'old' growth theory is outdated while the new growth theory is 'modern' and also superior because it explains empirical facts better. However, we must realise that technology being important and endogenous is not a new idea and had appeared in the writings of economists like Marx and Schumpeter. Moreover, in the 1950s, there was some work on the endogeneity of technology, though not directly within a model of growth, by Nelson. Similarly, Arrow's seminal work on learning-by-doing appeared as early as 1962, and this is the basic work from which much of the endogenous growth theory has drawn inspiration.

It appeared initially that the endogenous growth theory did a better job of explaining the lack of convergence –or divergence — in per capita growth rates of countries that was empirically observed. However, if we consider the work of Mankiw, Romer and Weil, which augmented the Solow model by expanding the notion of capital to include human capital, and which we might take to be in the tradition of exogenous growth theory, we find it does a reasonably good job of explaining inter-country divergence. But then, Mankiw, Romer and Weil's paper appeared in 1992, and this was after the seminal papers of Romer and Lucas, which were published in the mid to late 1980s, as well as Romer' s 1990 paper on endogenous technical change.

Where the endogenous growth theory appears to have extended the exogenous growth theory seems to be in the way the creation of technology conies about in economies. It brings in explicitly the R and D process and imperfect competition.

**Check Your Progress 3**

1) What are the implications of endogenous growth models for the developing nations?

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2) Briefly compare the exogenous neoclassical model and the AK model of production.

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## *9.5* LET US SUM UP

In this unit, we took a close look at some new growth theories that suggest that the engine of growth is technology, and that this teclinology is not given exogenously, but are endogenously determined. We began by observing the basic implications of the standard neo-classical model like constant returns to scale in all factors and diminishing returns to each factor; the fact that increasing saving does nothing to push up the per capita income growth ***rates;*** only the levels. Initially the growth rates will rise but eventually it will reach a balanced growth state and stay there.

The unit then went on to look at the human- capital-augmented neoclassical mode. The neoclassical model had predicted that there would be convergence in growth rates, but reality was coming up with a different picture. The Mankiw Romer and Weil approach was to put human capital into the neoclassical growth model. The unit discussed this approach in detail.

Following this, the unit took up the presentation of endogenous technical change and its impact on economic growth. Basically the ideas of Romer and others were discussed regarding the creation of technical change as a result of augmenting of knowledge capital. How ideas create human capital was discussed. We saw ideas were non-rival goods which had externalities and spillovers. We saw how R and D created knowledge capital, and how monopolistic competition was required for such creation were discussed.

The unit discussed a class of models and theories that attempt to present a scenario theoretically where diminishing returns to capital do not set in. these are several variants of the so-called AK model where K is physical cum human capital. Why diminishing returns do not appear was explored. Finally the unit discussed some issues related to endogenous growth theory, namely, its implications for developing nations, and its relation with exogenous growth theory.

## 9.6 KEY WORDS

**Endogenous variable: A** variable that is explained by a particular model; a variable whose value is determined by the model's solution.

**Exogenous variable: A** variable that the model takes as given; a variable whose value is taken as given from outside the model, whose value does not depend on the model's solution.

**Non-excludable goods:** goods for which it is difficult to charge a price to the user.

**Rival Goods:** goods, which, when consumed by one person, reduces the amount left over for consumption by other people.

## 9.7 SOME USEFUL BOOKS

Barro, Robert, and Sala- I-Martin. (2002), *Economic Growth* (2nd edition) McGraw-Hill Singapore

Romer, David. (2001), Advanced Macroeconomics (2nd editionj McGraw-Hill, Singapore.

Solow, Robert M. (2000), *Growth Theory: An Exposition (2'''' edition). Oxford* University Press, Oxford and New York.

## 9.8 ANSWERS OR HINTS TO CHECK YOUR PROGRESS EXERCISES

**Check Your Progress 1**

1) See section 9.2

2) See section 9.2

**Check Your Progress 2**

1) See section 9.3

2) See section 9.3

**Check Your Progress 3**

1) See section 9.4

2) See section 9.4

# UNIT 10 STOCHASTIC GROWTH MODELS

**Structure**



## 10.0 OBJECTIVES

After going through this unit, you should be able to:

- explain the concept of a stochastic process;
- discuss the impact of uncertainty on economic growth;
- describe the Brock-Mirman growth model; and
- analyse and evaluate the Real Business Cycle model of stochastic growth.

## 10.1 INTRODUCTION

The word "stochastic" is derived from the Greek word "Stochastikos" which means skilled at aiming and hitting targets, with "stochos" being a target. It may also have been derived from " Stokazesthas" which means the art of guessing. Stochastic' models deal with randomness in economic phenomena. You have come across uncertainty in decision-making in your microeconomics course. We have also acquainted you with dynamic optimisation in unit 7. This unit combines the two types of techniques: making decisions over time, and making them in an environment where uncertainty is present.

We have seen how the Ramsey and Cass-Koopmans models explicitly bring in decision-making into the growth process and allow us to make statements about welfare in a dynamic setting. This unit familiarises you with some models that deal with situations where the decision maker has to make guesses about the future values of variables, and where the economy is hit randomly with external shocks. These shocks can generate 'cycles' and fluctuations in the growth process. The unit looks at the structure and implications of such models. We begin with the Brock-Mirman model which considers a model of the Ramsey-Cass-Koopmans type but where uncertainty' is present. Before introducing the model formally, certain concepts regarding how to deal with uncertainty in economic theory, that is, how decision-making under uncertainty is brought in the standard analysis of decision-making, is presented. The subsequent section looks at a class of theories that emerged in the 1980s, and were about fluctuations in mature industrial economies. These models generally built upon the Brock-Mirman model and presented a hypothesis about why an economy, in the course of its development, will experience fluctuations.

## 10.2 GROWTH UNDER UNCERTAINTY

In this section we extend the simple optimal growth models to see the impact of uncertainty on these models and how the models need to be extended or modified.

What we propose to do is introduce some notion of uncertainty in general, and then see how the standard neoclassical growth model and the optimal growth models need to be extended and what difference in the results is obtained.

Let us begin with an economy with one good and where there is no randomness. This single good can be used either for consumption or for investment. In any period t, the output $y_t$ is divided into consumption for the next period $c_{t+1}$ and capital for the next period, $k_{t+1}$. The relationship between output and capital is given by

$$y_t = f(k_t)$$

The function f is the production function converting capital into output. For ease of exposition, and for the purpose at end we do not show labour as entering the production function. In this one-good economy, given the production function, the basic equation of motion is given by:

$$k_{t+1} = f(k_t) - c_{t+1}$$

We get the above equation by substituting the production function into the condition that $y_t = c_{t+1} + k_{t+1}$

If consumption at time t+1 is assumed to be a function of output in the previous period

$c_{t+1} = c(y_t)$ then indirectly, via the production function, consumption at t+1 is a function of capital at time t

$c_{t+1} = c(f(k_t))$. More generally, it may be written as $c_{t+1} = c(k_t)$. Then the basic equation of motion we had can be written as $k_{t+1} = f(k_t) - c(k_t) \approx h(k_t)$ where the sign $\approx$ denotes 'approximately' or 'can be written as'. This is a difference equation of the type that you are studying in your course on quantitative methods. In the case where there is no uncertainty, the capital stock at any period is determined by the above equation and the initial capital stock $k_0$. To study the model fully, certain assumptions and restrictions are made on the function $f(.)$. It is assumed that the function is continuous and twice differentiable such that

$$\begin{cases} f > o, k > o, f(0) = 0 \\ f' > o, f'' < 0 \\ f'(0) = \infty, f'(\infty) = 0 \end{cases}$$

the economy is said to be in equilibrium (or steady stale) if

$k_{t+1} = k_t$, for all t. There may be a unique equilibrium or multiple equilibria.

Now we introduce a criterion for deciding on an optimal consumption path. However, there is no uncertainty yet. We will introduce uncertainty into the picture in the next step. Consider the consumption path $c_0, c_1, \ldots, c_t, \ldots$ and let the utility of this path be given by

$U(c_0, c_1, \ldots c_t, \ldots) \approx U(C)$ where $C = \{c_t\}$ is the feasible consumption path. For the consumption path to be feasible, there has to be present a path of capital stock

$K = \{k_t\}$ such that $c_t + k_t = f(k_{t-1})$ from the initial point $k_0$

In order to get an optimal consumption path, the following objective function is maximised:

$U(c_0, c_1, \ldots c_t, \ldots) = \sum_{t=0}^{\infty} \beta^t u(c_t)$ where β is the discount factor. This formulation is exactly what you have studied in your unit 7 where we discussed the Cass-Koopmans model. The optimisation exercise is solved using a technique called dynamic programming. This is a method for solving optimisation where several time periods are involved, and t can even go up to infinity. It involves breaking down the n-period problem into a sequence of 2-period problems and working backwards. Dynamic programming was invented in the 1950s by Richard Bellman.

Let us now consider economic growth under uncertainty. We first take up the case where optimisation *is* not explicitly brought into the picture. We consider output given by

$y_t = \mathbf{f}(k_t, \varepsilon_t)$ where ε is a random variable. This has been brought in to capture tho presence of uncertainty in the situation. What the production function is saying is that output depends not only on the stock of capital but also on an unpredictable, random element. There may be natural disasters, or war which may adversely affect production. Rains may fail which might affect agricultural output. You can look up your course on quantitative techniques for the concept of random variables and probability distributions. We assume the ε's are identically and independently distributed. In the case of uncertainty the equation describing the motion of the system is given as

$k_{t+1} = \mathbf{f}(k_t, \varepsilon_t) - c(k_t, \varepsilon_t)$. Here consumption depends on inputs as well as random occurrences. Let $\mathbf{f}(k_t, \varepsilon_t) - c(k_t, \varepsilon_t) = h(k_t, \varepsilon_t)$. Then the previous equation becomes

$k_{t+1} = h(k_{t+1}, \varepsilon_t)$. This is an example of a stochastic process called a Markov process. This shows that the value of a variable in period t+l depends on the value of the variable only at period t (earlier periods do not matter), and the value of a random variable at time period t.

Finally we come to discussing economic growth under uncertainty with optirnisation introduced in the growth process. This is the basic Brock-Mirman model. The Brock Mirman model is an extension of the Cass-Koopmans model that we studied in the unit on optimal economic growth. Brock and Mirman presented two papers in the early 1970s. In the first paper, they considered the case of stochastic optimisation with discounting. In the second paper, they looked at a case where discounting is not present, that is you may think of it as a stochastic version of the Ramsey model. In this unit we discuss the case with discounting as this is the more general case.

Let us understand clearly what is going on. We have a growth process where decisions are being taken explicitly by individuals (for simplification, the representative consutner, or the planner). This is like the Cass-Koopmans model. Now the economy is hit with random shocks and uncertainty. Now look up your course on microeconomics. There you are studying about decision-making under uncertainty. You know that decision-makers want to maximise the expected utility, that is, act as though they are taking the mathematical expectation of the utilities arising in various states. In the dynamic case as well, that is when decisions are taken over time, the decision-makers maximise the expected utility arising out of a stream of consumption. The optimisation exercise looks like

**Max** $E\sum_{t=0}^{t} \beta^t u(c_t)$ subject to

$k_t + c_t = \mathbf{f}(k_{t-1}, \varepsilon_{t-1}), t = 1, 2, \ldots, T$. Of course, T can be infinity as well. It is just that the solution process will undergo some change. In the above optimisation exercise, there is another constraint about the initial values of k and c, which we can write as $k_0$ and $c_0$ respectively. Here the solution procedure involves the use of dynamic programming. To fix ideas, we can present some more examples of decision-making under uncertainty.

We take a simple example of a consumer who has an endowment of wealth. We consider first a simple two-period situation. In the first period the consumer can invest his wealth in assets. Again for simplicity we assume that there are only two assets A1 and A2. One asset gives a certain (sure) return of $\pi_0$ and the other (risky) asset gives a random return of $\overset{*}{\pi}_1$. Suppose the initial endowment of wealth with the consumer in period 1 is $w_1$. Suppose the consumption of the consumer in the two periods are $c_1$ and $c_2$ respectively. Now, suppose the consumer invests a fraction x of his endowment of wealth in the certain, non-risky asset and the rest of this wealth .(fraction 1-x) in the risky asset. His second period wealth will equal his second-period consumption. So the consumer's budget constraint is

$$w_2 = c_2 = (w_1 - c_1)\left[\overset{*}{\pi}_1 x + \pi_0(1-x)\right] = (w_1 - c_1)\overset{*}{\pi}$$

what the right-hand side of the equation is saying is as follows. We know that the consumer has $(w_1 - c_1)x$ earning a return $\overset{*}{\pi}_1$ and $(w_1 - c_1)(1-x)$ earning a return $\pi_0$

Here $\overset{*}{\pi} = \overset{*}{\pi}_1 + \pi_0(1-x)$

The crucial thing here is that since the consumer's return on wealth is uncertain, his second period consumption is uncertain. Let us denote it by c. We assume that the consumer has a utility function of the form

$U\left(c_1, \tilde{c}_2\right) = u(c_1) + \beta u\left(\tilde{c}_2\right)$ where $\beta < 1$ is the discount factor. Let $V_1(w_1)$ be the maximum utility that the consumer can achieve if he has wealth w1 in period 1. it is essentially an indirect utility function, giving utility as a function of wealth. We have

$$V_1(w_1) = \max u(c_1) + \beta E u\left[(w_1 - c_1)\overset{*}{\pi}\right]$$

Differentiating the above equation with respect to time we get the first-order conditions

$$u'(c_1) = \beta E u'(c_2)\tilde{\pi}$$

$$E u'(c_2)\left(\tilde{\pi}_1 - \pi_0\right) = 0$$

The first equation that the marginal utility of consumption in period 1 should be equal to discounted expected marginal utility in period 2.

Now let us consider a T-period problem. Suppose $(c_1, c_2, ..., c_T)$ is a stream of consumption that could be random. Let us suppose the consumer maximises the expected utility form this consumption stream:

$$U(c_1, ...c_T) = \sum_{t=0}^{T} \beta^t Eu(c_t)$$

Suppose at time t the consumer has wealth $w_t$ and invests a fraction $x_t$ in the risky asset, his wealth in period t+1 is given by

$$w_{t+1} = [w_t - c_t]\pi$$

where $r = x_t \pi_1 + (1 - x_t)\pi_0$ is the random portfolio return in period t and t+1

This optimisation problem can be solved using dynamic programming, breaking it down into a sequence of 2-period problems

Consider period T-1. if the consumer has wealth $w_{t-1}$ at this point, the utility that he can get is

$$V_{T-1}(w_{T-1}) = \max u(c_{T-1}) + \beta Eu\left[(w_{T-1} - c_{T-1})\tilde{\pi}\right]$$

this is just the equation that we had got earlier with T-1 replacing 1. the first order conditions will be

$$u'(c_{T-1}) = \beta Eu'(c_T)\tilde{\pi}$$
$$Eu'(\tilde{c}_T)(\tilde{\pi}_1 - \pi) = 0$$

Now we go back to period T-2. If the consumer chooses $(c_{T-2}, x_{T-2})$ then in the next period T-1 he would have a wealth of

$$w_{T-1} = [w_{T-2} - c_{T-2}]\tilde{\pi}$$

From this wealth he will achieve expected utility of $V_{T-1}(w_{T-1})$. The consumer's maximisation problem at period T-2 can be written as

$$V_{T-2}(w_{T-2}) = \max u(c_{T-2}) + \beta EV_{T-1}\left[(w_{T-2} - c_{T-2})\tilde{\pi}\right]$$

This is just like the optimisation exercise for period T – 1 except here the utility for period T –2 is solved using the indirect utility function $V_{T-1}$ rather than the direct utility function u. Indirect utility function V shows utility as a function of wealth w, while direct utility function u shows utility as a function of consumption c.

We can get the first-order condition for period T-2 as

$$u'(c_{T-2}) - \beta EV'(w_{T-1})\tilde{\pi}$$
$$EV'(w_{T-1})(\tilde{\pi}_1 - \pi_0) = 0$$

We can use these to solve for $V_{T-2}(w_{T-2})$ and keep working backwards to solve for T-3,
T-4 and so on.

**Check Your Progress 1**

1) How does the presence of uncertainty affect the basic one-good growth model?

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2) Describe the basic structure of the Brock-Mirman model.

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## 10.3 THE REAL BUSINESS CYCLE MODEL

In the previous section, we looked in detail at the Brock-Mirman model. This section discusses a theory or model of economic fluctuations called the Real Business Cycle model or Real Business Cycle theory. 'Real Business Cycle' is sometimes abbreviated as RBC, and we shall henceforth use this. RRC theory is technically based on the Brock Mirman model. Theoretically, it is a model that looks at the reason for macroeconomic fluctuations, that is, fluctuations in aggregate economic activity.

It is important to grasp the essence of the RBC theory from the perspective of macroeconomics. Basically, it makes two types of propositions. First, it says that although we usually look at long term growth and short term fluctuations in economic activity separately, the same reasons that give rise to long run growth, also cause fluctuations in economic activity, or what traditionally have been called 'business cycles'. The RBC theorists prefer the use of tlie word 'fluctuations' to 'cycles', since the latter conveys a sense of regular recurrence, which might not actually be the case. The

'real' part of the RBC theory suggests that as a theory that attempts to explain business fluctuations, these models play down the role of monetary forces and believe, like the classical economists, that money is a veil. They insist that it is the real forces like production, supply shocks etc that actually lead to fluctuations. Moreover, the RBC theorists play down the demand side of the economy and maintain that the Keynesian (and even the monetarist) approach lays too much emphasis on the aggregate demand. Because RBC theory emphasises on relative prices rather than the absolute price level, and believes that money is neutral, and also because it lays emphasis on supply side forces, this theory displays similarity with the earlier classical theorists of growth and development, about some of whose theories you will study in block 5. It is for this reason that RBC theory is sometimes put together with some other theories and dubbed 'New Classical'.

There can be several kinds of shocks in the economy. These shocks can originate on the demand side as well as on the supply side. There may even be shocks to the economy emanating from monetary and fiscal policies. The RBC theorists focus on productivity shocks. Productivity shocks can be of several types. There can be development of new techniques of production, new management practices, crop failure, and supply shocks coming from outside the economy, and so on. Basically a business cycle theory has to describe the nature of shocks to the system, and has to explain how these shocks affect the key macroeconoinic variables like output, employment, investment. etc. RBC models not only look at productivity shocks, but also focus more on the propagation of shocks to the rest of the economy than other theories.

Now we develop the basic structure of a prototype RBC model. The RBC model builds upon the Brock-Mirman model of the case where discounting is present, but extends that model. Basically the RBG explicitly bring into the utility function a decision regarding labour and leisure, where it is assumed that leisure provides utility and leisure is the time left over from total time after labour has been provided.

The structure is a dynamic optimisation-under-uncertainty case. We can present it as follows.

The idea is to think of maximising the following type of utility function

$E\left[\sum_{j=0}^{\infty}\beta^{j}u\left(c_{t+j},l_{t+j}\right)\right]$. Here c and l denote the representative household's consumption and leisure activities. β is a discount factor that lies between 0 and 1. Leisure is time not devoted to labour. E is the operator showing mathematical expectation, conditional upon information at time t. in the above formulation t denotes the time at which the optimisation exercise is carried out. In our earlier examples we had taken t = 0, and what we are denoting here by j, we had denoted by t

Each 'houschold' (in this formulation, the households are also the firms, the production unit) has access to a production technology of the type $y_t = z_t f\left(n_t^d, k_t^d\right)$.

Here z is the realisation of a random variable depicting technology. The other variables n and k denote labour and capital supplied during time t. the amount of labour supplied is $1-l$ where $l$ is leisure. The budget constraint of the household likes the following:

$$c_t + k_{t+1} = z_t f\left(n_t^d, k_t^d\right) + (1-\delta)k_t - w\left(n_t^d - n_t\right) - r\left(k_t^d - k_t\right)$$

Here the superscript d denotes the amount demanded of the variable, w is the wage rate, and r is the rental for capital. In the last section, we had used w to denote wealth. Do not get confused, and remember the context. What the budget constraint says is that the current consumption and the capital stock for the next period (determined by investment in the current period) has a random component and is used to pay wages and rental for capital.

The RBC model then goes on to carry out, using dynamic programming, a constrained optimisation exercise and further to make statements about shocks to the economy and the generation of fluctuations.

**Check Your Progress 2**

1) What does RBC theory have to say about the cause of economic fluctuations?

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2) Why is RBC theory called 'New Classical'?

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3) In what way does the RBC theory build upon the Brock-Mirman model?

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## 10.4 LET US SUM UP

This unit took up for discussion certain growth models which study growth in the presence of uncertainty. What these models do is to take up the simple Cass-Koopmans type of fluctuations and bring in uncertainty and stochastic processes into the picture. We looked in detail at the Brock-Mirman formulation which was a novel way in that it sought to extend the optimal growth models of the Cass-Koopmans type and explicitly brought in uncertainty. The unit first presented a simple extension of the basic one-sector growth model to the case of uncertainty. Following this, the optimal growth model was briefly discussed. Subsequently, the unit presented the Brock-Mirman model in the case of discounting. Elementary notion of the technique of dynamic programming was presented. This was used to explain the Brock-Mirman model.

The unit also presented the theory of Real Business Cycle that aims at providing an explanation for the recurrent fluctuations observed in mature industrial economies. These theories focus on shocks to the economy that originate on the supply side', and particularly, on productivity shocks. The unit looked at the structure of the RBC model and explained how it builds upon the Brock Mirman model.

## 10.5 KEY WORDS

**Endogenous variable:** A variable that is explained by a particular model; a variable whose value is determined by the model's solution.

**Exogenous variable:** A variable that the model takes as given; a variable whose value is taken as given from outside the model, whose value does not depend on the model's solution.

**Inter-temporal Substitution of Labour: A** key proposition of Real Business Cycle model, which suggests that workers allocate and reallocate labour supply over time depending on the economic incentives they face.

**Random Variable:** A variable whose value is deterniined by chance, that is probability. The different values that a random variable can take, along with the probability of each value, is called a probability distribution.

**Real business cycle theory:** a theory of macroeconomics that stresses that shifts in potential GDP are a primary cause of fluctuations in real GDP; the shifts in potential GDP are usually assumed to be caused by changes in technology.

**Shock:** An, exogenous change in an economic relationship, such as a sudden fall in the output of some commodity, with a sudden spurt in its price.

## 10.6 SOME USEFUL BOOKS

Dixit, Avinash K., and Pindyck, Robert S. (1994) ***Investment Under Uncertainty,*** Princeton University Press, Princeton

Malliaris, A.G., and Brock, W.A. (1982) ***Stochastic Methods in Economics and Finance,*** North- Holland, Amsterdam.

Sargent, Thomas J. (1987). *Dynamic Macroeconomic Theory,* Harvard University Press, Harvard.

Romer, David. (2000 ). *Advanced Macroeconomics,* McGraw Hill, Singapore.

## 10.7 ANSWERS OR HINTS TO CHECK YOUR PROGRESS EXERCISES

**Check Your Progress 1**

1) See section 10.2

2) See section 10.2

**Check Your Progress 2**

1) See section 10.3

2) See section 10.3

3) See section 10.3