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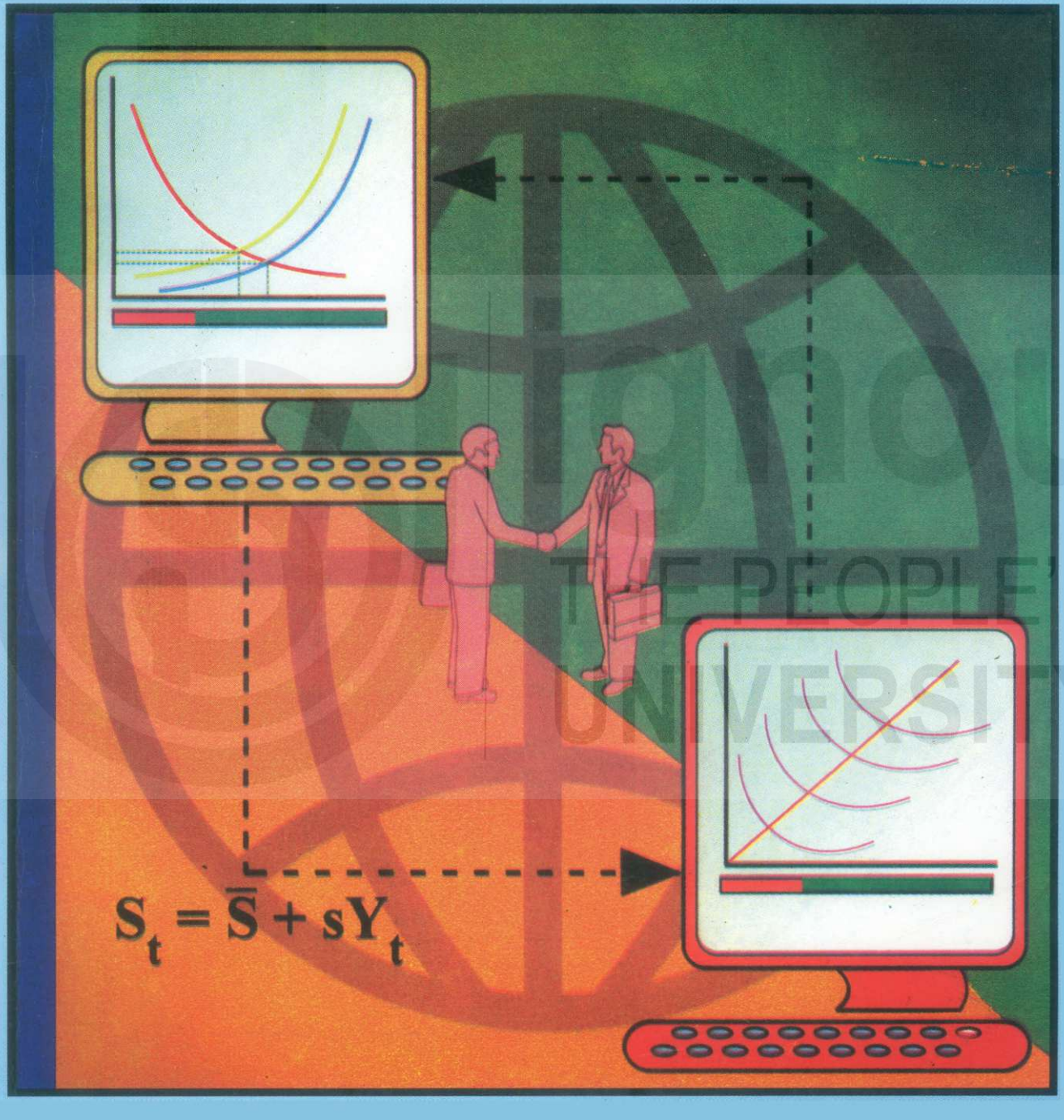
# MEC - 101
# MICROECONOMIC ANALYSIS




## Volume 2

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# MEC-101
# Microeconomic Analysis

Block

# 7
# NON-COOPERATIVE GAME THEORY-I

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# BLOCK 7 NON-COOPERATIVE GAME THEORY-I

## Introduction

This block deals with the non-cooperative game theoretic approach often used in seeking solutions to economic problems in general and microeconomic problems in particular. **Unit 20** introduces the static games with complete information along with the basic concepts that helps understand the method of solution. The concept of Nash equilibrium is introduced by considering pure and mixed strategies. The unit also contains the description of the concept of equilibrium and representation of games through normal form. Some of the applications of static games with complete information are discussed in **Unit 21**. Insights thrown by game theory to problems of tragedy of commons and duopoly markets are important themes covered in the unit. The last unit of the block (**Unit 22**) contains solution to problems when we have situation of dynamic games with complete information. It introduces the concept of backward induction as a refinement of Nash equilibrium and presents the extensive format of a game.

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# UNIT 20 STATIC GAMES OF COMPLETE INFORMATION

**Structure**


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20.2 Some Examples of Strategic Games

20.3 Classifications of Games

- 20.3.1 Cooperative and Non-Cooperative Game Theory
- 20.3.2 Games of Complete Information and Incomplete Information
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20.13 Exercises


## 20.0 OBJECTIVES

After going through this unit, you will be able to understand:

- various classifications of games like cooperative and non-cooperative games along with that of simultaneous move games;

- rules of strategic game or normal form of game;
- basics of the normal form or the strategic form of game;
- solution concept under Dominant Strategy;
- solution concept under Nash equilibrium in pure strategy; and
- solution concept under Mixed Strategy Nash equilibrium.

## 20.1 INTRODUCTION

Game theory studies the *general principles* that explain how people and organisations act in strategic situations in a multiple agent framework. A *strategic situation* involves when two or more interacting players, who make decisions by trying to anticipate the actions and reactions of others. Thus, Game Theory can be described as a strategy theory, or theory of interactive decision making. It is rather a formal study of interdependence, conflict and cooperation. Game theory can be used to explain phenomena in the areas of economics, political theory, evolutionary science and military strategy, to name a few.

Game theory was established as a field in its own right after the 1944 publication of the monumental volume 'Theory of Games and Economic Behavior' by von Neumann and the economist Oskar Morgenstern. It provided much of the basic terminology and problem setup that is still in use today. In 1950, John Nash demonstrated that finite games have always have an equilibrium point, at which all players choose actions which are best for them, given their opponents' choices. This central concept of non-cooperative game theory has been a focal point of analysis since then. Since the 1970s, it has driven a revolution in economic theory. Additionally, it has found applications in sociology and psychology and established links with evolution and biology.Game theory received special attention in 1994 with the awarding of the Nobel Prize in economics to Nash, John Harsanyi and Reinhard Selten.

Game theory studies strategy mainly through the analysis of different "games". A "game" in game theory is a fully explicit structure which characterises players, their set of actions, payoffs and possible outcomes under given rules of playing. Thus a game is described by the following ingredients:

Game Theory is a formal way to consider each of the following items:

a) **Group:** In any game there is more than one decision –maker, each decision maker is referred to as 'player'.

b) **Interaction:** What any one individual player does directly affects at least other player in the group.

c) **Strategy:** An individual player accounts for her interdependence, each player chooses her best action or the strategy.

d) **Rational:** A central assumption in many variants of game theory is that the players are *rational*. A rational player is one who always chooses an action which gives the outcome she most prefers, given what she expects her opponents to do. The goal of game-theoretic analysis then is to predict how the game will be played by rational players, or, to give advice on how best to play the game against opponents who are rational.

Let me now illustrate these four concepts – *group, interaction, strategic* and *rational* – by discussing in detail some of the examples given below.

## 20.2 SOME EXAMPLES OF STRATEGIC GAME

a) *Working on a group project:A case study for the game theory class.*

The group comprises the students jointly working on the case. The interaction arises from the fact that the certain amount of work needs to be done in order to write a paper. Hence if one students slacks off, somebody else has to put in extra hours the night before the paper is due. Strategic play involves estimating the likelihood of freeloaders in the group and rational play requires a careful comparison of the benefits to a better grade against the costs of extra work.

b) *Random drug testing (at Olympics)*

The group is made up of competitive athletes and the International Olympic Committee (IOC). The interaction is both between the athletes – who make decision on training regimens as well as whether or not to use drugs – and with the IOC, which needs to preserve the reputation of the sport. Rational strategic play requires the athletes to make decisions based on their chances of winning and if they dope, their chances of getting caught. Similarly, it requires the IOC to determine drug testing procedures and punishments on the basis of testing procedures and punishments on the basis of testing costs and the value of a clean whistle reputation.

c) *R and D efforts by pharmaceutical companies*

Some estimates suggest that research and development (R and D) expenditures constitutes as 20% of annual sales of US pharmaceutical companies and that on an average the development costs of a new drug is about $350 million dollars. Companies are therefore very concerned about ploughing back their investment in R and D and mitigation of risk in their investment. If we take the set of big drug companies, the strategic interaction arises because the first developer of a drug makes the most profits (thanks to the associated patents).

## 20.3 CLASSIFICATION OF GAMES

In this section we will discuss the various classifications of games.

### 20.3.1 Cooperative and Non-Cooperative Game Theory

Game theory comprises two branches: **Cooperative game theory and non-cooperative game theory**. Note that 'cooperative' and 'non-cooperative' are technical terms and are not an assessment of the degree of cooperation among agents in the model. A cooperative game can as much model extreme competition as a non-cooperative game can model cooperation.

1) **Cooperative Game:** A game in which players/ agents negotiate and enforce their contracts (contract is binding through third parties) so that they can make joint strategies. Firms buying raw material or buying a service can negotiate for price settlement. Cooperative game theory models analyses situations where agents can cooperate to create value by joining coalitions, but also where agents compete to capture value, in unstructured interactions. It can be thus called coalitional as it abstracts from the detailed procedures and focuses on how the value creation

abilities of each coalition of agents can bear on the agents' ability to capture value.

Its formalism comprises a set of agents and a function that returns the value each subset of agent (i.e., each coalition) can create on its own (the characteristic function). The characteristic function is the input to a solution concept, which returns the value captured by each agent (their imputation). Because no structure is put on the bargaining procedure, the solution concept models the outcome of the negotiations occurring among the agents, accounting for the value each coalition can create. While the solution concepts are the **Core** and the **Shapley value**, but a number of others have been developed (for further details see Osborne and Rubinstein (1994) and Owen (1995)).

The **core** is the set of imputations such that each coalition receives at least as much as it can create on its own. This concept formalizes a notion of individual and coalitional self-interest. The core is thus appropriate to model the competitive nature of free-form competitive interactions. However, the core may not exist and is not necessarily unique. Non-existence of the core signals the intrinsic instability of a particular interaction. Non-uniqueness of the core implies that competitive forces alone are not enough to determine each agent's value capture and that negotiating abilities also come into play. An agent's added value (or marginal contribution) is the value lost to the grand coalition (the coalition of all agents) if the agent is not included. The added-value principle, implied by the core, states that under competitive free-form interaction an agent can capture no more than her added value. Otherwise, the other agents would be better off to exclude her and trade only among themselves.

The **Shapley value** is the average added value of an agent to all possible coalitions. It always exists and is unique. It emphasizes the fairness of the division of value among agents. It has been used to model the allocation of costs among users of a common facility and to measure political power.

2) **Non-Cooperative Game:** In game theory, a non-cooperative *game theory* is concerned with the analysis of strategic choices where players (or agents) make decisions independently, out of their own interest. Non-Cooperative game theory models the actions of agents, maximizing their utility in a defined procedure, relying on a detailed description of the moves and information available to each agent. Thus, in the paradigm of non-cooperative game theory, the details of the ordering and timing of players' choices are crucial to determine the outcome of a game. In the non-cooperative game theory players cannot write any binding contract. Cooperation can, and often does, arise in non-cooperative models of games, when players find it in their own best interests i.e., that strategy if played in equilibrium must be self-enforcing.

Non-cooperative game is the natural way to start a lesson on Game Theory.

### 20.3.2 Games of Complete Information and Incomplete Information

Non-Cooperative games can be classified into (i) **Games of Complete Information** and (ii) **Games of Incomplete information.**

Under non-cooperative games, a category of game in which knowledge about other players, utility functions, payoffs, strategies and "types" of players are common knowledge i.e., those information are available to all participants. Then such games are called **Games of Complete Information.**

On the other hand, in a **game of incomplete information**, players may not possess full information about their opponents. Some players may possess private information that the others should take into account when forming expectations about how a player would behave. A typical example is an auction: each player knows her own utility function (i.e., the valuation for the item), but does not know the utility function of the other players. It is often assumed that the players have some statistical information about the other players. For example, in an auction, each player knows that the valuations of the other players are drawn from some probability distribution. In this case, the game is called a **Bayesian game**.

### 20.3.3 Strategic (or Normal Form) and Extensive Form Games

Games of Complete Information again can be classified as Strategic and Extensive form games.

The *strategic form* (also called *normal form*) is the basic type of game studied in non-cooperative game theory, where players take their decision simultaneously. Thus in a strategic game any player does not know the action or the strategy of the rival players before taking her own move.

The rules of the game are defined by the *players, actions,* possible *strategies* and the *payoffs* of a game.

*Players* are the agents who are involved in the game of decision making.

A *strategy* is a complete contingent plan, or decision rule, that specifies how the player will act in every possible distinguishable circumstance in which she might be called upon to move. All the players have preferences and goals. That is to say, like all normal people, these agents prefer some options over another. The players will choose actions or strategies that increase the chances of getting their preferred outcomes.

*Outcomes/payoffs:* The key to game theory is that my payoff is contingent on the action/strategy of the other players playing the game. Thus, a game has different outcomes that result from each possible combination of choices of the players. An outcome is represented by a separate *payoff* for each player, which is a number (also called *utility*) that measures how much the player likes the outcome.

The *extensive form* (also called the *sequential form)*, also called a *game tree*, is more detailed than the strategic form of a game. It is a complete description of how the game is played over time. This includes the order in which players take actions, the information that players have at the time they must take those actions and the times at which any uncertainty in the situation is resolved. A game in extensive form may be analysed directly, or can be converted into an equivalent strategic form.

## Check Your Progress 1

1) What is a cooperative game?

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2) State the meaning of a sequential game.

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3) What do you mean by common knowledge in a game with complete information?

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4) Define a strategy used in game theory.

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5) State the meaning of Nash equilibrium.

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6) What is a non-cooperative game?

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7) State the meaning of payoff in a game.

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8) Define a pure strategy.

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9) Give the meaning of weakly dominant strategy.

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10) There are two players, 1 (kicker) and 2 (goalie). Each has two actions, $a_i \in \{L, R\}$ to denote left or right. The kicker wins when they choose opposite directions while the goalie wins if they choose the same direction. Using 1 to denote a win and $-1$ to denote a loss, write the game in a matrix form.

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11) Provide an example of a game in which there is more than one weakly dominant strategy equilibrium.

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12) Give an example of a game in which there is no weakly dominant strategy equilibrium.

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13) Provide an example of a game in which there is more than one weakly dominant strategy equilibrium.

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## 20.4 RULES OF THE GAME

Every game is played by a set of rules which have to specify four issues.

1) Who is playing the group of players that strategically interacts.
2) What they are playing with – the alternative actions of choices, the strategies, that each player has available.
3) When each players get to play (in what order)
4) How much they stand to gain (or lose) from the choices made in the game.
5) How much are the players in a game supposed to know about the rules? In game theory it is standard to assume **common knowledge** about the rules. Everybody has a knowledge about the rules means that if you asked any two players in the game, a question about who, what, when or how much, they would give you the same answer.

The two principal representations of (the rules of) a game are called, respectively, the *Normal (or strategic) form* of a game and the *Extensive form* of a game.

## 20.5 NORMAL FORM OF GAME UNDER COMPLETE INFORMATION

A normal form is a model of interacting decision makers (we call them *players* or *agents*). Each player has a set of possible actions. The model captures the interaction between the players by allowing each player to be affected by the actions of all players, not only her own action. Each player simultaneously chooses a *strategy* and the combination of strategies chosen by the players, determine a *payoff* of each player. Each player has ordinal preferences about the action profile – payoffs received by the player for each combination of strategies that could be chosen by the players.

The Normal form of game thus specifies:

1) The players in the game
2) The strategies available to each player
3) The payoff received by each player, for each combination of strategies that could be chosen by the players.

Suppose the game consisting of n players and let the representative player is denoted by player I ( i = 1, 2, ..... n). Let $S_i$ denotes the set of strategies available to each player $i$ and let $s_i$ denotes the an arbitrary specific strategy ($s_i \in S_i$), to indicate the strategy $s_i$ is the member of the set of strategy $S_i$. Let $(s_1, s_2, s_3, \ldots \ldots s_n)$ denotes a combination of strategies one for each player, let $u_i$ denote the player i's payoff $u_i(s_1, s_2, \ldots \ldots s_n)$, where players choses the strategies $(s_1, s_2, \ldots \ldots s_n)$. Thus the normal form of $n$-players game is specified as $G = \{S_1, S_2, \ldots . S_i, u_1, u_2, \ldots . u_i\} \ldots \ldots \ldots \ldots (20.1)$.

In the normal form of game player's choose their strategies simultaneously. It implies that each player chooses her or her action without the knowledge of the other's choices.

### 20.5.1 Some Well-known Strategic Games

#### 20.5.1.1 Matching of Pennies

a) The game involves two players, child 1 and 2, simultaneously placing a penny on the table with Head or tail at the top. The payoff depends on whether the pennies match or not. If both pennies are heads or tails i.e., (h, h) or (t, t), Child 1 wins and keeps Child 2's penny; if they do not match i.e., (h, t) or (t, h), Child 2 wins and keeps Child 1's penny. This is a *zero-sum game* because one player's gain is the other's loss. The payoffs for Child 1 and 2 are shown in the table below (see Table 20.1), with the first numeral in the four payoff cells representing Child1's payoff and the second numeral Child 2's playoff. As can be seen, the combined playoff for Child1 and 2 in all the four cells is zero.

**Table 20.1: Example of *Zero-Sum Game***

| | | Child 2 | |
|---|---|---|---|
| Child 1 | | Head (h) | Tail (t) |
| | Head (h) | (h, h ) = (1, -1) | (h, t) = (-1, 1) |
| | Tail (t) | (t, h) = (-1, 1) | (t, t) = (1, -1) |

### 20.5.1.2 Battle of Sexes

A husband and wife are in a dilemma of going to movie or to a football match. They each respectively prefer the football game and movie. Again at the same time they prefer to go with the spouse than going alone. See Table 20.1.

**Table 20.2: Battle of Sexes**

| | Wife | | |
|---|---|---|---|
| **Husband** | | **Football (F)** | **Movie(M)** |
| | Football (F) | (f, f ) = (3, 1) | (h, t) = (0, 0) |
| | Movie (M) | (t, h) = (0, 0) | (m, m) = (1, 3) |

### 20.5.1.3 Prisoners' Dilemma (PD) Game

The prisoners' dilemma is a popular example widely used in game theory that shows why two completely rational individuals might not cooperate, even if it appears that it is in their best interests to do so. The game is presented as follows:

Two members ((1 and 2) of a criminal gang are arrested and imprisoned. Each prisoner is in solitary confinement with no means of communicating with the other. Prosecutors lack sufficient evidence to convict the pair on the principal charge. They are being interrogated separately about their crime. Each prisoner is given the opportunity either to: betray the other by testifying that the crime is committed, or to cooperate with the other by remaining silent. The convicts have the options either to confess or not to confess their crime. Based on their joint decision penalty will be decided. If both of them confess the crime will be established and each will get 6 years of imprisonment. If none of them confesses, the crime is not established. So, they will be nominally punished for 1 year. However in case one confesses and the other does not, the crime is established, so the convict who has confessed will be set free for her honesty and the other one will be put under the bar for 9 years. Table 20.3.

**Table 20.3: Prisoners' Dilemma Game**

| | Convict 2 | | |
|---|---|---|---|
| **Convict 1** | | **Confess (c)** | **Defect (d)** |
| | Confess (c) | (c, c )=(-6,-6) | (c, d) =(0, -9) |
| | Defect (d) | (d, c) =(-9,0) | (d, d)=(-1, -1) |

**PD Game in a Generalized Form**

The structure of the traditional Prisoner's Dilemma can be generalized from its original prisoner setting. If both players cooperate, they both receive the reward R for cooperating. If both players defect, they both receive the punishment payoff P. If Player 1 defects while Player 2 cooperates, then Player 1 receives the temptation payoff T, while Player 2 receives the "sucker's" payoff, S. Similarly, if Player 1 cooperates while Player 2 defects, then Player 1 receives the sucker's payoff S, while Player 2 receives the temptation payoff T. This can be expressed in the following normal form (see Table 20.4).

**Table 20.4: Prisoner's Dilemma in Generalized Form**

| | Player 2 | | |
|---|---|---|---|
| **Player 1** | | **Cooperate (c)** | **Defect (d)** |
| | Cooperate (c) | (R, R ) | (S, T) |
| | Defect (d) | (T, S) | (P, P) |

The condition that must hold for a prisoner's dilemma game, in the strong sense, is that if the payoffs follows the inequality: $T > R > P > S$. The payoff relationship R > P implies that mutual cooperation is superior to mutual defection, while the payoff relationships T > R and P > S imply that defection is the dominant strategy for both agents. That is, mutual defection is the unique Nash equilibrium in the game (i.e., the only outcome from which each player could only do worse by unilaterally changing strategy). The dilemma then is that mutual cooperation yields a better outcome than mutual defection but it is not the rational outcome because from a self-interested perspective, the choice to cooperate, at the individual level, is irrational. This sort of outcome is described by Mancur Olson (1965) as – *individual rationality produces collective irrationality.*

The prisoner's dilemma game is widely used in the study of game theory.

1) Payoff structure associated with PD game arises in many strategic situations in economic, social, political, biological competitions. This game has a wide range of applicability from the strategic stand point.

2) Curious nature of equilibrium outcome achieved in such game. Both players follow conventional wisdom in choosing their dominant strategy, but the resulting equilibrium outcome yields them a payoff that are lower than that could have achieved if they have chosen their dominated strategy. In casual usage, it may be applied to situations in which two entities could gain important benefits from cooperating or suffer from the failure to do so, but find it merely difficult or expensive, to coordinate their activities to achieve cooperation.

### 20.5.1.4 Hawk-Dove (or, Chicken) Game

The chicken game, also known as the hawk-dove game, is a model of conflict for two players in game theory. The principle of the game is that while it is for both players beneficial if the other player yields, their own optimal choice depends on what their opponent is doing: if their opponent yields, the player should not, but if the opponent fails to yield, the player should.

The name "chicken" has its origins in a game in which two drivers drive towards each other on a collision course: one must swerve, or both may die in the crash, but if one driver swerves and the other does not, the one who swerved will be called a "chicken", meaning a coward; this terminology is most prevalent in political science and economics.

The name "Hawk-Dove" refers to a situation in which there is a competition for a shared resource and the contestants can choose either conciliation or conflict; this terminology is most commonly used in biology and evolutionary game theory. From a game-theoretic point of view, "chicken" and "hawk-dove" are identical; the different names stem from parallel development of the basic principles in different research areas. See **Table 20.5 and Table 20.6.**

**Table 20.5: Chicken Game**

| | **Player 2** | | |
|---|---|---|---|
| **Player 1** | | **Swerve (Sw)** | **Straight (St)** |
| | Swerve (Sw) | Tie, Tie | Lose, Win |
| | Straight St) | Win, Lose | Crash, Crash |

**Table 20.6: Chicken Game with Numerical Payoffs**

| | **Player 2** | | |
|---|---|---|---|
| **Player 1** | | **Swerve (Sw)** | **Straight (St)** |
| | Swerve (Sw) | 0, 0 | -1, +1**(NE)** |
| | Straight (St) | +1, -1 **(NE)** | -10, -10 |

Two versions of the payoff matrix for the *Chicken game* are presented here (Table-20.5 and Table 20.6). Two (young) players are engaged in a conflict situation. They may be racing their car towards each other on Main Street, while being egged on by their many friends. If player 1 hangs tough by racing straight and stays in the Centre of the road while the other player swerves by moving out of the way, then all glory is her and the other player eats humble pie (St, Sw). If they both hang tough by racing straight (St, St) they end up with broken bones, while if they both swerve they have their bodies– but not their pride – intact.

In such game the outcomes are such that, where each player would prefer to win over tying, prefer to tie over losing and prefer to lose over crashing. Here, the benefit of winning is 1, the cost of losing is -1 and the cost of crashing is -10. It is mutually beneficial for the players to play different strategies.

### Hawk-Dove Game

In the biological literature, this game is referred to as *Hawk-Dove*. The earliest presentation of a form of the Hawk-Dove game was by John Maynard Smith and George Price in their paper, "The logic of animal conflict". Both in Chicken and Hawk-Dove, it is mutually beneficial for the players to play different strategies. It can be thought of as the opposite of a coordination game (see below for Coordination game), where playing the same strategy Pareto dominates playing different strategies. The underlying concept is that players use a shared resource. In coordination games, sharing the resource creates a benefit for all: the resource is non-rivalrous and the shared usage creates positive externalities. In anti-coordination games the resource is rivalrous but non-excludable and sharing comes at a cost (or negative externality).

The traditional payoff matrix for the Hawk-Dove game is given in **Table 20.7**, where V is the value of the contested resource and C is the cost of an escalated fight. It is (almost always) assumed that the value of the resource is less than the cost of a fight, i.e., $C > V > 0$. If $C \leq V$, the resulting game is not a game of Chicken but is instead a Prisoner's Dilemma. The exact value of the Dove vs. Dove payoff varies between model formulations. Sometimes the players are assumed to split the payoff equally (V/2 each), other times the payoff is assumed to be zero.

While the Hawk-Dove game is typically taught and discussed with the payoffs in terms of V and C, the solutions hold true for any matrix with the payoffs in **Table 20.8**, where W > T > L > X.

**Table 20.7: Hawk-Dove game**

| | | Player 2 | |
|---|---|---|---|
| | | **Hawk (H)** | **Dove (D)** |
| **Player 1** | Hawk (H) | **(V–C)/2, (V–C)/2** | **V, 0** |
| | Dove (D) | **0, V** | **V/2, V/2** |

**Table 20.8: General Form of a Hawk-Dove game**

| | | Player 2 | |
|---|---|---|---|
| | | **Hawk (H)** | **Dove (D)** |
| **Player 1** | Hawk (H) | **X, X** | **W, L(NE)** |
| | Dove (D) | **L, W(NE)** | **T, T** |

### 20.5.1.5 Coordination Game

Coordination games are a formalization of the idea of a **coordination problem**, which is widespread in the social sciences, including economics, meaning situations in which all parties can realize mutual gains, but only by making mutually consistent decisions. A common application is the choice of technological standards.

One way to model a *minimum-effort coordination game* is to let players choose effort levels, 1 or 2, where each unit of effort results in a cost of c < 1. The output per person is determined by the minimum of the effort levels chosen. For example, if both choose an effort level of 1, then for a given per-unit effort cost (c), each player receives a payoff of 1–c, as shown in the upper-left box of **Table 20.9.** Similarly, when both choose a high effort level of 2, they each obtain 2-2c. But if they fail to coordinate with effort choices of 1 and 2, then the minimum is 1 and payoffs are 1-c for the low-effort individual and 1- 2c for the high-effort individual.

**Table 20.9:-Minimum-effort Coordination Game**

| | | Player 2's Effort | |
|---|---|---|---|
| | | **1** | **2** |
| **Player1's Effort** | 1 | (1 –c), (1 – c)**(NE)** | (1 – c), (1 –2c) |
| | 2 | (1 –2c), (1 – c) | (2 –2c), (2 – 2c)**(NE)** |

## 20.6 SOLUTION CONCEPT UNDER DOMINANT STRATEGY

In this section we will discuss the first solution concept of the Static Game under Complete Information.

### 20.6.1 Dominance

Consider a player have two strategies *A* and *B* so that, given any combination of strategies of the other players, the outcome resulting from *A* is better than the outcome resulting from *B*. Then strategy *A* is said to *dominate* strategy *B*. Since all players are assumed to be rational, they make choices which result in the outcome they prefer most, given what their opponents do. If out of N strategies A, B, C, …, N, the outcome from strategy A is better than any other (N-1) strategies, irrespective of the strategies of the other players then there is a compelling reason to think that the rational player will choose it. If a strategy is bad for an agent no matter what the others might be doing there is equally compelling reason to think that a rational player would avoid it. A rational player will never choose to play a dominated strategy.

### 20.6.2 Dominant Strategy Equilibrium

In a normal form of n players game specified as$G = \{S_1, S_2, \ldots\ldots S_i, u_1, u_2, \ldots\ldots u_i\}$. Let $s_i' and s_i''$ be feasible the strategies for player i (i.e., $s_i' and s_i''$ are the members of $S_i$. Strategy $\boldsymbol{s_i'}$ is **strictly dominated** by strategy $\boldsymbol{s_i''}$, if for each feasible combination of the other player's strategies, player i's payoff from playing $s_i'$ is strictly less than i's payoff from playing$s_i''$. i.e., $u_i(s_1, s_2, s_3, \ldots\ldots s_i, \boldsymbol{s_i'}, s_{i+1}, s_n) < u_i(s_1, s_2, s_3, \ldots\ldots s_i, \boldsymbol{s_i''}, s_{i+1}, s_n)$, ..... (20.2) for each combination of strategy $(s_1, s_2, s_3, \ldots\ldots s_{i-1}, s_{i+1}, s_n)$ that can be constructed from the other player's strategy spaces

$(S_1, S_2, \ldots\ldots S_{i-1}, S_{i+1}, \ldots. S_n)$.

If a strategy is not dominated by any other, it is called **undominated strategy**. It is useful to think of a dominated strategy as a bad strategy and an undominated strategy as a good one. Thus a rational player will not play a strictly dominated strategy. Consider a game of two players with strategies High- Low and North-South respectively (**Table 20.10**).

**Table 20.10: Example of Undominated Strategy**

| | | Player 2 | |
|---|---|---|---|
| | | **North (N)** | **South (S)** |
| **Player 1** | High (H) | $\pi_1(H,N), \pi_2(H,N)$ | $\pi_1(H,S), \pi_2(H,S)$ |
| | Low(L) | $\pi_1(L,N), \pi_2(L,N)$ | $\pi_1(L,S), \pi_2(L,S)$ |

In this above example of strategic game the Player's 1 strategy Low (L) is dominated by High (H) if:

a) $\pi_1(H,N) \geq \pi_1(L,N)$ ..... (20.3a)

b) $\pi_1(H,S) \geq \pi_1(L,S)$, ..... (20.3b)

The first inequality says that H yields a higher payoff than L to the player 1, if the player 2 plays her first strategy i.e., North (N). The second inequality says the same is true even if player 2 plays her second strategy.

With at least one of those inequalities being strict, Low is not dominated by High (and vice versa).If it is better against, say, South but does worse against North i.e.,

a) $\pi_1(H,N) > \pi_1(L,N)$, ..... (20.4a)

But b) $\pi_1(H,S) < \pi_1(L,S)$, .....(20.4b).

Consider the *Prisoners Dilemma* game of **Table 20.3**.

1) Each agent has 2 strategies to cooperate (not confess) with one's rival and the other to defect from cooperation (by confessing).
2) Each player has a dominant strategy (to confess) or to defect from the cooperation strategy.
3) Dominant strategy equilibrium is worse for both the agent than the non equilibrium situation or the dominated strategy to cooperate with the rival (not confess, not confess).

The strategy confess is dominating the strategy defect, as it gives the Convict 1 a higher payoff than Defect irrespective of the strategy played by the Convict 2 i.e., whether she confesses or defects $\Rightarrow$

$(i)\pi_1(c,c) = (-6) > \pi_1(d,c) = (-9)$

$(ii)\pi_1(c,d) = (0) > \pi_1(d,d) = (-1)$

Similarly, for the Convict 2 also, confession is dominating the strategy defect, as it gives the Convict 2 a higher payoff than Defect, irrespective of the strategy played by the Convict 1 whether she confesses or defects $\Rightarrow$

$(iii)\pi_2(c,c) = (-6) > \pi_2(d,c) = (-9)$

$(iv)\ \pi_2(c,d) = (0) > \pi_2(d,d) = (-1)$.

So, in this Prisoner's Dilemma game, confession is the Dominant strategy for both the convict. Thus in this game, it is reasonable to predict that both the players will end up confessing in the equilibrium. Hence {Confess, Confess} is the **Dominant Strategy Equilibrium**.

### 20.6.3 Iterated Elimination of Dominant Strategies Equilibrium

The problem with the dominant strategy equilibrium is that in many games a player need not have a dominant or best strategy. Thus the modest objective is instead of searching for the best strategy we will end up in successively eliminating the dominated strategies (See Table- 20.11, Table -20.11a, Table - 20.11b and Table-20.11c for the iterative elimination).

Consider the following game given in **Table 20.11**.

| | | **Player 2** | |
|---|---|---|---|
| | | **Left** | **Right** |
| **Player 1** | Up | (1,1) | (0,1) |
| | Middle | (0,2) | (1,0) |
| | Down | (0,-1) | (0,0) |

For the Player 1, the strategies Up and Middle dominate the strategy Down. Hence Down is the dominated strategy for the player 1. So, for a rational player, it will be irrational to play a *dominated strategy* Down. The reason is that by playing any strategy that dominates (this dominated strategy), the player can guarantee a payoff which is at least as high, irrespective of the moves of the rival players. Hence the player 1 will ever play Down but will rather play either Up or Middle.

The logic could then set in motion a chain reaction. In a game once it is known that the player 1will not play the *dominated strategy* Down, other player might find that that certain of their strategies are in fact dominated.

This is because Player 2 for instance, no longer has to worry how her strategies would perform against player 1's dominated strategies. So, some of the strategies of player 2's, which are only good against player 1's dominated strategies, might turn out to be bad strategies. In that case player 2 will not play those strategies. This might lead to a third round of discovery of bad strategy by some of the other players and so on.

**Table 20.11a**

| | Player 2 | | |
|---|---|---|---|
| **Player 1** | | **Left** | **Right** |
| | Up | (1,1) | (0,1) |
| | Middle | (0,2) | (1,0) |

Thus if it is known to Player 2 that Player 1 will never play Down, the Right looks dominated to her. Therefore the rational Player 2 would never play Right. But then the Player 1 should not worry about Player 2 playing Right. Hence she should choose the strategy Up.

**Table 20.11b**

| | Player 2 | |
|---|---|---|
| | | **Left** |
| **Player 1** | Up | **(1,1)** |
| | Middle | (0,2) |

The strategy (Up, Left) is said to be reached by *iterated elimination of dominated strategies* (IEDS) and the game is said to be *dominance solvable*. If we are able to reach a unique strategy by following this procedure, we call the outcome the solution to IEDS and call the game dominance solvable.

**Table 20.11c**

| | Player 2 | |
|---|---|---|
| | | **Left** |
| **Player 1** | Up | **(1,1)** |

However the process of *iterated elimination of dominated strategies* (IEDS) has two drawbacks. First, it is based on the assumption of *common knowledge* that the players are rational. That is, we need to assume not only that all the players are rational, but also that all the players know that all the players are rational, and that all the players know that all the players know that all the players are rational and so on (the formal definition of common knowledge). Second drawback of *iterated elimination of dominated strategies* (IEDS) is that the process often produces a very imprecise prediction about the play of the game. Consider the following game:

**Table 20.12: Game Without Dominated Strategy**

| | | Player 2 | | |
|---|---|---|---|---|
| | | **Left (L)** | **Centre (C)** | **Right (R)** |
| **Player 1** | Top (T) | (0,4) | (4,0) | (5,3) |
| | Middle (M) | (4, 0) | (0, 4) | (5,3) |
| | Bottom (B) | (3, 5) | (3, 5) | (6, 6) |

In this above game (**Table20.12**) there are no strictly dominated strategies to be eliminated. Since all the strategies in the game survive IEDS, the process produces no prediction whatsoever about the play of the game.

**Check Your Progress 2**

1) What is a strictly dominated strategy?

.....

.....

.....

.....

.....

2) When would a player finds a dominant strategy equilibrium ?

.....

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.....

3) Solve the following game using iterated elimination of strictly dominated strategies.

| | | Player 2 | | |
|---|---|---|---|---|
| | | **left** | **middle** | **right** |
| **Player 1** | up | 1,0 | 1,2 | 0,1 |
| | down | 0,3 | 0,1 | 2,0 |

.....

.....

.....

.....

.....

## 20.7 SOLUTION CONCEPT UNDER THE NASH EQUILIBRIUM IN PURE STRATEGY

Many simultaneous games have no dominant strategy or dominated strategy (like Battle of Sexes game). We take up as a next step to find a solution of the game in which every player or agent does the best she can, given the strategies of the other players.

A pair of strategies satisfies *Nash equilibrium* condition if players' strategies are mutually the best response strategy. The outcome is strategically stable or self enforcing because no single player has an incentive to deviate from her strategy.

In a normal form of n players game specified as$G = \{S_1, S_2, \ldots\ldots S_i, u_1, u_2, \ldots\ldots u_i\}$, the strategies $(s_1^*, s_2^*, \ldots\ldots, s_n^*)$ are a Nash Equilibrium if, for each player i, $s_i^*$ is player I's best response to the strategies specified for the (n-1) other players, $(s_1^*, s_2^*, \ldots s_{i-1}^*, s_{i+1}^*, \ldots\ldots\ldots, s_n^*)$:

$$u_i(s_1^*, s_2^*, \quad \ldots s_{i-1}^*, \boldsymbol{s_i^*}, s_{i+1}^* \ldots\ldots\ldots, s_n^*) \geq u_i(s_1^*, s_2^*, \quad \ldots s_{i-1}^*, \boldsymbol{s_i}, s_{i+1}^* \ldots\ldots\ldots, s_n^*) \ldots\ldots\ldots\ldots (20.5)$$

The above equation (20.3) says that each player i, in playing $\boldsymbol{s_i^*}$,is playing a best response to the other players' strategy choice. This condition includes two requirements of Nash equilibrium:

1) Each player must be playing a best response against a conjecture
2) The conjecture must be correct.

Thus Nash equilibrium includes the first requirement because $\boldsymbol{s_i^*}$ is the ith player's best response against the conjecture $\boldsymbol{s_{-i}^*}$(optimal strategy choice of all the other (n-1) players (except the ith player)). It includes the second requirement because no player has an incentive to change her strategy (from $\boldsymbol{s_i^*}$). Hence $\boldsymbol{s_i^*}$ is stable and each player's conjecture is correct.

For every feasible strategy $\boldsymbol{s_i}$ in $S_i$, that is $\boldsymbol{s_i^*}$ solves

$$Max u_i(s_1^*, s_2^*, \ldots s_{i-1}^*, \boldsymbol{s_i}, s_{i+1}^* \ldots\ldots\ldots, s_n^*) \ldots\ldots\ldots (20.6)$$

Suppose game theory offers the strategies $(s_1', s_2', \ldots\ldots, s_n')$ as a solution to the normal form of game $G = \{S_1, S_2, \ldots\ldots S_i, u_1, u_2, \ldots\ldots u_i\}$. Saying that $(s_1', s_2', \ldots\ldots, s_n')$ is not a Nash equilibrium of G is equivalent of saying that there exists some player i such that $\boldsymbol{s_i'}$ is not a best response to $(s_1', s_2', \ldots s_{i-1}', s_{i+1}', \ldots, s_n')$. That is there exists some $\boldsymbol{s_i''}$in$S_i$, such that

$$u_i(s_1', s_2', \ldots s_{i-1}', \boldsymbol{s_i'}, s_{i+1}', \ldots, s_n') < u_i(s_1', s_2', \ldots s_{i-1}', \boldsymbol{s_i''}, s_{i+1}', \ldots, s_n') \ldots\ldots (20.7)$$

Thus if the theory offers the strategy $(s_1', s_2', \ldots\ldots, s_n')$, as solution but these strategies are not a Nash equilibrium, then atleast one player will have an incentive to deviate from the theory's prediction, so the theory will be falsified by the actual play of the game.

Consider the case of two player 1 and 2, each with two strategies $(a_1, a_2)$ for the player 1 and $(b_1, b_2)$ for the player 2. For example, $(\boldsymbol{a_2, b_1})$ is a *Nash equilibrium* if and only if :

$$\text{(a). } \pi_1(\boldsymbol{a_2, b_1}) \geq \pi_1(\boldsymbol{a_1, b_1}) \ldots\ldots\ldots\ldots (20.8a)$$

$$\text{(b). } \pi_2(\boldsymbol{a_2, b_1}) \geq \pi_2(\boldsymbol{a_2, b_2}), \ldots\ldots\ldots\ldots (20.8b)$$

### 20.7.1 Minimax Method for Zero Sum Game

Zero-sum is a situation in game theory in which one person's gain is equivalent to another's loss, so the net change in benefit is zero. A zero-sum game may have as few as two players, or millions of participants. Zero-sum games are found in game theory, but are less common than non-zero sum games. Poker and gambling are popular examples of zero-sum games since the sum of the amounts won by some players equals the combined losses of the others. Games like chess and tennis, where there is one winner and one loser, are also zero-sum games. In the financial markets, options and futures are examples of zero-sum games, excluding transaction costs. For every person who gains on a contract, there is a counter-party who loses. The game

of *matching pennies* (See Figure 20.1) is as an example of a zero-sum game. Zero-sum games are most often solved with the minimax theorem which is closely related to linear programming duality or with Nash equilibrium. Another variant of zero sum game is the constant sum games where the sum of the payoffs of the players from each outcome is always a constant.

In contrast, situations where participants can all gain or suffer loss together are referred to as non-zero-sum. Thus in non-zero-sum games are games in which the sum of gains and losses by the players are sometimes more or less than what they began with.

In the equilibrium of the zero sum game, each player is choosing her or her best response, given her beliefs about what the others will do. In anticipating such best responses, each player will expect to receive the worst payoff associated with each action and will choose her own action accordingly. She is thus choosing her best payoff from among the set of worst pay offs.

In some game most notably in sports contexts, the interests of the two sides are exactly opposite to each other. Then, for each combination of players' choices, the payoff of one can be obtained by reversing the sign of the payoff of the other. So, the payoff matrix under a zero sum game can be simplistically represented by showing the payoff of one player. Consider the following example of zero sum game where in a soccer game the two players (Offense and Defense) are represented row wise and column wise with the strategies respectively (Run, Short Pass, Medium Pass and Long Pass) and (Run, Pass, Blitz). Suppose the payoff table shows the row player's (Offense) payoff and therefore it is implicit that the Column player's (Defense) payoff will be the negative of the same. The Offense wants the outcome to be a cell with as high a number as possible in each column. The Defense (column) wants the outcome of the cell with as low a number as possible. See **Table 20-20.13**.

Using this logic the Offense figures that for each of her row, the defense or the Column will choose the column with the lowest number in that row. Therefore the Row should choose the row that gives her the highest among the lowest numbers i.e., maximum among the minima – the **maxi-min strategy**. Similarly, the Columns reckon that, for each of her columns, Row will choose the row with the largest number in that column. The Column should choose the column with the smallest number among the largest one or the minimum among the maxima – called the **mini-max strategy.** If the Offense (row's) **maximin** values and the defense's **minimax** values are in the same cell of the payoff matrix, then the outcome is the **Nash equilibrium** of the zero sum game – a concept which we will elaborate in details in the next section. The method of finding equilibrium in the zero sum game is called **maximin-minimax method.** The equilibrium will lead to *Nash equilibrium* in pure strategies, if it exists.

We begin by finding the offense's worst payoff from each her strategy (which is lowest number in each row) and defense's worst payoff in each strategy (which should be the highest number from each column). Thus the offense's worst payoff from Run is 2, worst payoff from Short Pass is 5.6, worst payoff from Medium Pass is 1 and worst payoff from Long Pass is -2. We write the minimum of each row at the last column. The Defense's worst payoff from Run is 10, worst payoff from Pass is 5.6 and worst payoff from Blitz is 13. We write the maximum of each column at the bottom of the column.

The next step is to find out the best of each player's worst possible outcomes, i.e., the largest row minimum and the smallest column maximum. The largest row minima is 5.6, so the offense can ensure itself a gain of 5.6 yards by playing the Short Pass, this is her *maximin*. The lowest of the column maxima is 5.6, so the defense can be sure of holding the offense down to a gain of 5.6 yards by deploying its Pass Defense. This is the defense's *minimax*.

Now from this two given strategy choices, we see the *maximin* and the *minimax* values are belonging to the same cell of the game table. Thus offense *maximin*is the best response to the defense's *minimax* strategy and vice versa, which is basically the *Nash equilibrium* of the game. The equilibrium entails the offense attempting a Short Pass while the defense defends against a Pass. A total 5.6 yards will be gained by the offense (and given up by the defense).

**Table 20.13: Minimax Method**

| | | Defense | | | |
|---|---|---|---|---|---|
| | | **Run** | **Pass** | **Blitz** | **Row Min** |
| Offence | Run | 2 | 5 | 13 | Min = 2 |
| | Short Pass | 6 | 5.6 | 10.5 | Min=5.6 |
| | Medium Pass | 6 | 4.5 | 1 | Min = 1 |
| | Long Pass | 10 | 3 | -2 | Min=-2 |
| | Column Max | Max=10 | Max=3 | Max= 13 | |

If the *minimax method* fails to find an equilibrium in some *zero sum game*, then we conclude that the game has no *Nash equilibrium* is pure strategies (we will address the concept of mixed strategy equilibrium in the next section). Moreover this *minimax method* cannot be applied to the non zero sum games. As in those games the opponent's best is not necessarily the defense's worst.

## 20.7.2 Nash Parables

There are various other ways in which Nash equilibrium concept has been motivated within game theory. These motivations are parables in the sense that we will only offer a verbal description of each one. Some of these motivations have been precisely worked out in mathematical models, some are accepted as simple intuitive proposition. In either case these parables are worth telling because Nash equilibrium is the most widely used solution concept in game theory. The parables will reassure you the reasonableness of the solution concept.

a) **Play Prescription**

One can think of a Nash equilibrium$s^*$as a prescription for play. If the strategy vector is proposed to the players, then it is stable prescription in the sense that no one has an incentive to play otherwise. By playing an alternative strategy, a player would simply lower her payoffs, if she thinks that others are going to follow their part of the prescription.

b) **Pre-play Communication**

How would the players in a game find their way to a Nash equilibrium? One answer that has been proposed is that they could coordinate on a Nash equilibrium by way of *pre-play communication*, i.e., they could coordinate by meeting before (the game is actually played) and discussed their options. It is not credible for the players to agree on anything that is

not a Nash equilibrium because atleast one player would be worse off against such agreement.

c) **Rational Introspection**

A related motivation is rational introspection: each player could ask himself what she expects will be the outcome to a game. Some candidate outcome will appear unreasonable in that there are players who could do better than they are doing, i.e., there will be players not playing a best response. The only time no players appears to be making mistake is when each is playing a best response, i.e., when we are at a Nash equilibrium.

d) **Focal Point**

Another motivation is the idea that a Nash equilibrium forms a focal point for the players in a game, The intuitive idea of a focal point was first advanced by Thomas Schelling in 1960 in his book: *The Strategy of Conflict.* It refers to a strategy vector that stands out from the other strategy vectors because of some distinguishing characteristics. Consider the two Coordination game: (i) two players would have to write down either heads or tails. They will be paid if their choices match. (ii) To players have to meet in Canaught Place, New Delhi and have to choose a time for their meeting. Again they are paid only when their chosen time coincide. Schelling found that in a disproportionate number of cases, students choose *head* in the case (i) and 12:00 pm in the case of (ii). A Nash equilibrium strategy vector is a *focal point* because it has the distinguishing characteristics that each player plays a best response under that strategy vector.

e) **Trial and Error**

If player started by playing a strategy vector that is not a Nash equilibrium, somebody would discover that she could do better. If she changes her strategy choice and we are still not in a Nash equilibrium, somebody else might want to change her strategy. This process of trial and error will go on till we reach Nash equilibrium and then nobody has an incentive to deviate from that strategy choice.

## 20.7.3 Some Common Examples of Nash Equilibrium

Let us now examine the Nash equilibrium concept in the well known strategic games (Figure 20.2 - 20.9), discussed in the Section 20.5.

1) **Battle of Sexes (See Table-20.2)**

Here the two players (Husband and Wife) have two strategy choices Football (F)and Movie (M). This game has two Nash equilibrium (H, H) and (M, M), i.e., either the couple go for Football together or Movie.

The best response to the player 1 i.e., Husband, is to play (F) against his wife's choice of (F) and earns a payoff of 3. If the Husband deviates and play (M) against his Wife choice of (F) he is worse off as he gets 0 (because $3 > 0$). Similarly, playing F is the optimal strategy for Wife (with a payoff 1) against her Husband's strategy of (F). If she deviates and play (M) against her husband's choice of (F), she is worse off as compared to the strategy choices (F, F), because $1>0$.

Similarly, the best response to the Husband, is to play (M) against his wife's choice of (M) and earns a payoff of 1. If the Husband deviates and

play (F) against his Wife choice of (M) he is worse off as he gets 0 (because 1 > 0). Similarly, playing M is the optimal strategy for Wife (with a payoff 3) against her Husband's strategy of (M). If she deviates and play (F) against her husband's choice of (M), she is worse off as compared to the strategy choices (M, M), because 3>0.

So, (H, H) and (M, M) are the two best response strategies of the couple and are therefore Nash equilibrium.

2) **Prisoner's Dilemma (See Table - 20.3)**

Here (Confess, Confess) is the mutually best response strategy of the 2 convicts and therefore the unique Nash equilibrium of the Prisoners' Dilemma game where each convict will be imprisoned for 6 years ($\pi_1(\mathbf{C}, \boldsymbol{C}) = -6$, $similarly \pi_2(\boldsymbol{C}, \boldsymbol{C}) = -6$). Given that one of them stick to the strategy Confess, none of the convicts has an incentive to deviate from the strategy Confess. Because if any of them deviates she will be worse off (e.g., ($\pi_1(\mathbf{NC}, \boldsymbol{C}) = -9$, $similarly \pi_2(\boldsymbol{C}, \boldsymbol{NC}) = -9$).

If both the convicts play Non-Confession (NC, NC) the suspected crime cannot be established because of the absence of eye witness and therefore they would get a nominal punishment of 1 year imprisonment ($\pi_1(\mathbf{NC}, \boldsymbol{NC}) = 0, \pi_2(\boldsymbol{NC}, \boldsymbol{NC}) = 0$). But if any of the players defects from (NC, NC) and confesses his payoff would increased as he would be set free for his honesty ($as \pi_1(\mathbf{C}, \boldsymbol{NC}) = 0$ $and \pi_2(\boldsymbol{NC}, \boldsymbol{C}) = 0$). So for each of the player, there is an incentive to deviate from the strategy Not Confess (NC). Moreover, if any of the player ditches from NC (given the other stick to NC) it becomes really costly for the player who has not deviated. Thus if the Convict 1 deviates from the (NC) and plays C, whereas the Convict 2 plays NC, it is Convict 2 who will be punished for 9 years of imprisonment and the Convict 1 will be set free (i.e., $\pi_2(\boldsymbol{C}, \boldsymbol{NC}) = -9$, $\pi_1(\mathbf{NC}, \boldsymbol{NC}) = 0,$). Similarly, you may check the case when the Convict 2 deviates from (NC) and plays Confess (C), where as the Convict 1 plays (NC). In that case the Convicts 1's payoff will be 9 years of imprisonment and the Convict 2 will be set free ($\pi_1(\mathbf{C}, \boldsymbol{NC}) = -9$), $\pi_2(\boldsymbol{NC}, \boldsymbol{C}) = 0$)).

Hence the strategy (NC, NC) cannot be sustained as an equilibrium strategy. Both the convicts face a credible threat that given he participates (i.e., plays NC), his rivals may defects and push him to 9 long years of imprisonment. This mutual threat pushes them to play safe by choosing (Confess, Confess)i.e., (C, C). This strategy (C, C) is the mutually best response strategy or *Nash equilibrium*with a payoff = (-6, -6), although it is not *socially Pareto optimal*. For, both of them would have been better off by playing (NC, NC).

3) **Hog and Dove or Chicken Game (See Table-20.6 and Table-20.8)**

So far the *Chicken game* is concerned we have mentioned in the earlier section that it is mutually beneficial for the players to play different strategies. Because the loss of swerving is so trivial compared to the crash that occurs if nobody swerves, the reasonable strategy would seem to be to swerve before a crash is likely. Yet, knowing this, if one believes one's opponent to be reasonable, one may well decide not to swerve at all, in the belief that he will be reasonable and decide to swerve, leaving the other player the winner. This unstable situation can be formalized by saying there is more than one Nash equilibrium, which is a pair of

strategies for which neither player gains by changing his own strategy while the other stays the same. In this case, the pure strategy equilibria are the two situations wherein one player swerves while the other does not i.e., (St, Sw) and (Sw, St). See Table - 20.6.

For any matrix with the payoffs in Figure 20.8, where W > T > L > X, the Hog Dove game has multiple Nash equilibrium i.e., (H, D) and (D, H). Consider the strategy (H, D). Given that Player 1 plays H, it is optimal for the Player 2 to play D because his payoff from playing D (i.e., L) exceeds his payoff from playing H (L > X). Similarly, given Player 2 plays D, it is optimal for the Player 1 to play H because his payoff from playing H (i.e., W) exceeds his payoff from playing H (which is X) since (W > X). Therefore the strategy (H, D) is mutually the best response strategy or the Nash equilibrium. You may also check the strategy (D, H), by the similar reasoning it is the mutually best response strategy or the Nash equilibrium.

You may also check why the strategies like (H,H) or (D, D) are not the Nash equilibrium. In such cases, given the other player's strategy, each of the players has an incentive to deviate.

4) **Coordination Game**

In game theory, coordination games are a class of games with multiple pure strategy Nash equilibria in which players choose the same or corresponding strategies. Consider the Minimum-Effort Coordination game given in Figure 20.9. In this game it is optimal for both the players' to play the same strategy i.e., putting low effort (1,1) or high effort (2, 2) as the best response strategy at the equilibrium. So, (1,1) or (2, 2) are the Nash equilibrium strategy.

Notice that the low effort outcome (1, 1) is an equilibrium, since a costly increase in effort by only one person will not raise the amount produced. Given Player 1 plays to put low effort(1), it is optimal for the player 2, to put low effort, because (1- c) > (1- 2c), with 0 < c <1. Similarly, given Player 2 plays to put low effort, it is optimal for the player 1 to put low effort because (1- c) > (1- 2c). As long as c is less than 1, payoffs are maximized when all players choose the highest possible effort. So, high effort outcome (2, 2) is also mutually best response strategy and therefore a Nash equilibrium. In this case you may check that a reduction in effort by only one person will lower the minimum by more than the cost savings c.

Note that any common effort level constitutes a Nash equilibrium, since a costly unilateral increase in effort will not raise the minimum and a unilateral decrease will reduce the minimum by more than the cost when c < 1. However, this argument does not depend on the number of players, so noncritical changes in c and n will not alter the set of Nash equilibria, despite the reasonable expectation that efforts should be high for sufficiently low effort costs and low numbers of participants. See Table-20.9.

## 20.7.4 Relation between the Solution Concepts

The general relation between IEDS and Nash equilibrium can be summarized as follows:

For any game in which there is an outcome to IEDS. It must be the case that the outcome is a Nash equilibrium. The iterated elimination of strictly dominated strategies eliminates all but the one strategy which is the Nash equilibrium of the game. In a n player Normal form of a game, if the strategy is a Nash equilibrium then they survived the iterated elimination of strictly dominated strategies. However not every Nash equilibrium can be obtained as the outcome to IEDS.

Consider a 2 player 3 strategy game. Suppose the strategy $(a^2, b^3)$ is the IEDS solution. In order to establish that this pair is also a Nash equilibrium, we need to show that $a^2$ has a higher pay off than $a^1$ or $a^3$ when played against $b^3$. Likewise $b^3$ has a higher payoff than $b^2$ or $b^1$ against player 1's strategy $a^2$. Suppose $a^1$ is the strategy of the player 1 that is eliminated in the last round of elimination and likewise $b^1$ is the last round of elimination. We can now proceed by the following three step proof:

**Step 1:** Regardless of whether player 2 plays $b^3$or $b^1$, $a^2$ has a payoff as high as $a^1$. In particular against $b^3$, $a^2$ is at least as good a response as $a^1$.

**Step 2:** Similarly,$b^3$ is as good a response for player 2 as $b^1$ against $a^2$.

How is $a^2's$ payoff compare with those of $a^3$? Since $^3$ got eliminated in the very first round it must have been dominated by either$^{21}$. If it is dominated by $^2$ we are done. (Why?) If it is dominated by $^1$, then we know that it does worse than $^1$, against player 2's strategy of $^3$. But in turn $^1$ has no higher a payoff than $^2$ against that same strategy $^3$. We have therefore concluded:

**Step 3:** For player $1^2$ is the best response against$^3$. Likewise$^3$ is the best response against $^2$.

**Table 20.14**

| | | Player 2 | | |
|---|---|---|---|---|
| | | 1 | 2 | 3 |
| Player 1 | 1 | | | |
| | 2 | | | |
| | 3 | | | |

The reverse implication is not necessarily true; there may be Nash equilibria that are not the solution to IEDS. This may be easy to see in any game, such as Battle of Sexes, in which there is a Nash equilibrium but the game is not dominance solvable. Another situation where Nash does not imply IEDS can arise when there are multiple Nash equilibria; some of these Nash equilibria would be eliminated by iterated elimination of dominant strategies.

As regards the relation between dominant strategy solution and Nash equilibrium, exactly the same results hold. First note that every dominant strategy solution is an IEDS outcome, if there is dominant strategy every other strategy is immediately eliminated. But we just learnt that every IEDS outcome is a Nash equilibrium. It follows therefore that every dominant strategy solution is a Nash equilibrium. The converse need not holds. The coordination game has a dominant strategy solution however is also a Nash equilibrium.

**Check Your Progress 3**

1) What is minimax strategy?

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2) Find solution to the following game for pure strategy.

| | L | R |
|---|---|---|
| T | (3,1) | (2,-20) |
| M | (5,0) | (-10,1) |
| B | (-10,0) | (24,2) |

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3) How do you find a relation between *elimination of dominated strategies* (IEDS) and Nash equilibrium.

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## 20.8 MIXED STRATEGY NASH EQUILIBRIUM

This section will present more general class of strategy called mixed strategies that players can profitably employ. Players or agents could follow a strategy of randomizing over the set of available actions according to some probability distribution. Such a strategy is called *Mixed strategy*.

### 20.8.1 Definition and Example

Consider the example of Battle of Sexes game (given in Figure 20.5.2). You may recall that you have two pure strategy Nash equilibrium (f, f) and (m, m). Actually those are not the only choices. For instance the players can toss a coin. If it is head, they can go to the Football match and if it is tail they can go to the Movie instead. Hence there are actually three strategies to choose from:

i) (F, F)

ii) (M, M) and

iii) A coin toss i.e., a strategy with equal likelihood to going to Football or Movie.

Note that the coin toss has clearly enlarged the set of initial choices as strategy (iii). is evidently distinct from (i) and (ii). Strategies (i) and (ii) as well as every other strategy discussed so far are called *Pure Strategies*. A strategy like (iii) is called the *Mixed Strategy*. In fact strategy (iii) is just one of the many mixed strategies.

**Definition:** Suppose a player has M pure strategies$s^1, s^2, \ldots\ldots\ldots, s^M$. A mixed strategy for the player is the probability distribution over his pure strategies; i.e., it is a probability vector $(p^1, p^2, \ldots\ldots\ldots, p^M)$ with $p^k \geq 0, k = 1, 2, \ldots\ldots, M$ $and \sum_{k=1}^{M} p^k$ =1. Every pure strategy is in a trivial sense also a mixed strategy in which $p^1 = 1$ and $p^k = 0$, for all other pure strategies.

The obvious question now is how a mixed strategy payoff will be evaluated? It follows that the right way to evaluate the uncertainty inherent in a mixed strategy is to take its expected payoff. Expected payoffs are computed in two steps:

**Step 1:** Weight the payoff to each pure strategy by the probability with which that strategy is played.

**Step 2:** Add up the weighted payoffs.

Let us recall the Battle Sexes game as shown in **Table-20.14**. Let $\alpha$and $\beta$ denote the profile of mixed strategies of the Husband and Wife. Which mean to say that, Husband is playing the F and M with probabilities $\alpha\ and (1-\alpha)$ and the Wife is playing the F and M with probabilities$\beta\ and\ (1-\beta)$.Let $\pi_H(.,.)$ and $\pi_W(.,.)$ be the payoff functions of the Husband and Wife, respectively. Thus, $\pi_H(F,F)$ denotes the payoff of Husband when both the Husband and Wife are going for Football. Similarly, $\pi_H(M,F)$ denotes the payoff of Husband when Husband going for Football and Wife going for Movie. Similarly, you can interpret $\pi_W(F,M)\ and\ \pi_W(F,F)$.

The Battle of Sexes game has 4 possible outcomes (F, F), (F, M), (M, F), (M, M).

The outcome (F, F) occurs with a probability $(\alpha, \beta)$

The outcome (F, M) occurs with a probability $\{(\alpha.(1-\beta)\}$

The outcome (M, F) occurs with a probability$\{(1-\alpha).\beta)\}$

The outcome (M, M) occurs with a probability $\{(1-\alpha).(1-\beta)\}$

We will now calculate the expected payoffs of Husband$\varepsilon_H(.,\beta)$, for playing the pure strategies (F) and (M) under the assumption that the Wife is playing the Mixed Strategy. Thus, $\varepsilon_H(F,\beta)$ = Expected payoff of Husband for playing the pure strategies (F), under the assumption that the Wife is playing the Mixed Strategy $\beta, (1-\beta)$. Thus,

$\varepsilon_H(F,\beta) = \beta\ \pi_H(F,F) + (1-\beta)\ \pi_H(F,M) = \beta.3 + (1-\beta).0 = 3\beta$.. (20.9)

$\varepsilon_H(M,\beta)$=Expected payoff of Husband for playing the pure strategies (M), under the assumption that the Wife is playing the Mixed Strategy $\beta, (1-\beta)$. Thus,

$\varepsilon_H(M,\beta) = \beta\ \pi_H(M,F) + (1-\beta)\ \pi_H(M,M) = \beta.0 + (1-\beta).1 = (1-\beta)$ . (20.10)

Similarly, we will now calculate the expected payoffs of the Wife $\varepsilon_W(F,\alpha)$, for playing the pure strategies (F) and (M) under the assumption that the Husband is playing the Mixed Strategy with probabilities $\alpha, (1-\alpha)$.

Thus, $\varepsilon_W(F,\alpha)$ = Expected payoff of Wife for playing the pure strategies (F), under the assumption that the Husband is playing the Mixed Strategy $\alpha,(1-\alpha)$.

That is, $\varepsilon_W(\alpha,F) = \alpha\,\pi_W(F,F) + (1-\alpha)\,\pi_H(M,F) = \alpha.1 + +(1-\alpha).0 = \alpha$ ..... (20.11)

$\varepsilon_W(M,\alpha)$ = Expected payoff of Wife for playing the pure strategies (M), under the assumption that the Husband is playing the Mixed Strategy $\alpha,(1-\alpha)$. So,

$\varepsilon_W(\alpha,M) = \alpha\,\pi_W(F,M) + (1-\alpha)\,\pi_H(M,M) = \alpha.0 + +(1-\alpha).3 = 3(1-\alpha)$ ..... (20.12)

**Table 20.14: Battle of Sexes under Mixed Strategy**

| | | Wife | |
|---|---|---|---|
| | | **Football (F)** $\boldsymbol{\beta}$ | **Movie(M)** $\boldsymbol{(1-\beta)}$ |
| Husband | Football (F) $(\alpha)$ | $\pi_H(F,F), \pi_W(F,F)$ = (3, 1) | $\pi_H(F,M), \pi_W(F,M)$ = (0, 0) |
| | Movie (M) $(1-\alpha)$ | $\pi_H(M,F), \pi_W(M,F)$ = (0, 0) | $\pi_H(M,M), \pi_W(M,M)$ = (1, 3) |

Under the Mixed Strategy the expected payoffs of Husband from F and M should be equal (if the expected payoffs are not equal then there can't be mixed strategy equilibrium).

Therefore equating equation (i) and (ii) for Husband. Thus,

$$\varepsilon_H(F,\beta) = \varepsilon_H(M,\beta)$$

$$\Rightarrow\ 3\beta = (1-\beta) \Rightarrow \beta = \frac{1}{4} \Rightarrow\ (1-\beta) = \frac{3}{4} \text{ ..... (20.13)}$$

Repeating the same for the Wife, by equating (iii) and (iv) we get :

$$\varepsilon_W(\alpha,F) = \varepsilon_W(\alpha,M)$$

$$\Rightarrow\ \alpha = 3(1-\alpha) \Rightarrow \alpha = \frac{3}{4} \Rightarrow\ (1-\alpha) = \frac{1}{4} \text{ ..... (20.14)}$$

Therefore, the **Mixed Strategy Nash equilibrium** is:

$$\{\alpha,(1-\alpha),\beta,(1-\beta)\} = \{\left(\tfrac{3}{4},\tfrac{1}{4}\right),\left(\tfrac{1}{4},\tfrac{3}{4}\right)\}$$

Thus by extending the definition of Nash equilibrium to allow for mixed strategy, we simply require that each player's mixed strategy be a best response to other player's mixed strategy. Thus $\alpha,(1-\alpha)$ is the best response of the Husband against the Wife's mixed strategy $\beta,(1-\beta)$ and vice versa. Since any pure strategy can be represented as the mixed strategy that puts zero probability on all of the player's pure strategies, this extended definition subsumes the earlier one.

In a two player normal form of game $G = \{S_1, S_2, u_1, u_2\}$, the mixed strategies $(\alpha^*,\beta^*)$ are a Nash equilibrium if each player's mixed strategy is a best response to the other player's mixed strategy, so that, $\alpha^*$ must satisfy: $\pi_1(\alpha^*,\beta^*) \geq \pi_1(\alpha,\beta^*)$ for every probability distribution over $S_1$ and $\beta^*$ must satisfy : $\pi_2(\alpha^*,\beta^*) \geq \pi_2(\alpha^*,\beta)$, for every probability distribution over $S_2$.

Plugging the values of $\alpha, (1-\alpha)$, *and* $\beta, (1-\beta)$, we can derive the numerical values of the Expected payoffs of the Husband and Wife from equations (20.9), (20.10), (20.11) and (20.12).

From (i) expected payoff of the Husband from the pure strategy F:$\varepsilon_H(F,\beta) = 3\beta = 3.\frac{1}{4} = \frac{3}{4}$

From (ii) expected payoff of the Husband from the pure strategy M:$\varepsilon_H(M,\beta) = (1-\beta) = \frac{3}{4}$

From (iii) expected payoff of the Wife from the pure strategy F:$\varepsilon_W(\alpha,F) = \alpha = \frac{3}{4}$

From (iv) expected payoff of the Wife from the pure strategy M:$\varepsilon_W(\alpha,M) = 3(1-\alpha) = \frac{3}{4}$

The above values of expected payoffs of the Husband and Wife are derived for their pure strategies of F and M when the opponent is playing the mixed strategies.

Now let us consider that both Husband and Wife are playing the mixed strategies. The expected payoff of the Husband from the mixed strategies is

$$\varepsilon_H(\alpha,\beta) = \alpha.\varepsilon_H(F,\beta) + (1-\alpha)\varepsilon_H(M,\beta)$$

$$= \alpha\{\beta\,\pi_H(F,F) + (1-\beta)\pi_H(F,M)\} + (1-\alpha)\{\beta\,\pi_H(M,F) + (1-\beta)\pi_H(M,M)\}\ldots\ldots(20.15)$$

$$= \frac{3}{4}.\frac{3}{4} + \frac{1}{4}.\frac{3}{4} = \frac{3}{4}\left(\frac{3}{4} + \frac{1}{4}\right) = \frac{3}{4}$$

Expected payoff of the Wife from the mixed strategies is:

$$\varepsilon_W(\alpha,\beta) = \beta\varepsilon_W(\alpha,F) + (1-\beta)\varepsilon_W(\alpha,M)$$

$$= \beta\{\alpha\,\pi_W(F,F) + (1-\alpha)\,\pi_H(M,F)\} + (1-\beta)\{\alpha\,\pi_W(F,M) + (1-\alpha)\,\pi_H(M,M)\}\ldots(20.16)$$

$$= \frac{1}{4}\frac{3}{4} + \frac{3}{4}\frac{3}{4} = \frac{3}{4}$$

Thus in the equilibrium the expected payoffs of both the Husband and Wife when both of them are playinga mixed strategy are equalized. Moreover this value is equal to the payoff of the Husband or Wife under pure strategy when the opponent is playing mixed strategy at the equilibrium ( $\alpha^*, \beta^*$).$\Longrightarrow$

$$\varepsilon_H(\alpha,\beta) = \varepsilon_W(\alpha,\beta) = \varepsilon_H(F,\beta) = \varepsilon_H(M,\beta) = \varepsilon_W(\alpha,M) = \varepsilon_W(\alpha,F)$$

*It should be noted in the finite player and finite strategies there are games with no Nash equilibrium in pure strategies but there will always be such an equilibrium in mixed strategies.*

There are 3 possibilities, viz.,

1) If $\varepsilon_H(F,\beta) > \varepsilon_H(M,\beta) \Longrightarrow$ Husband's unique best response is to play pure strategy F.

2)I f $\varepsilon_H(M,\beta) > \varepsilon_H(F,\beta) \Longrightarrow$ Husband's unique best response is to play pure strategy M.

3) If $\varepsilon_H(F,\beta) = \varepsilon_H(M,\beta) \Longrightarrow$ All mixed strategies of the Husband yields the same expected payoff and hence all are the best responses. Same way you can do it for the Wife.

For the best Response function of the Husband:

1) For $\beta = \frac{1}{4} \Rightarrow \varepsilon_H(F,\beta) = \varepsilon_H(M,\beta) \Rightarrow$ Exp. Payoff of Husband from all mixed strategy givesher the same payoff. Thus every mixed strategy is his best response.
2) If $\beta < \frac{1}{4} \Rightarrow \varepsilon_H(F,\beta) < \varepsilon_H(M,\beta) \Rightarrow$Exp. Payoff of Husband from M exceeds his expected payoff from all mixed strategy that assign positive probability to F.$\Rightarrow$Unique best response for Husband is M.
3) If $\beta > \frac{1}{4} \Rightarrow \varepsilon_H(F,\beta) > \varepsilon_H(M,\beta) \Rightarrow$ Expected payoff of Husband from F exceeds his expected payoff from all mixed strategy that assign positive probability to M. That impliesunique best response for Husband is F.

So, the Husband's best response function is:

$$B_H(\beta) = \begin{cases} 0, & and\ \beta < \frac{1}{4} \\ \alpha: 0 \le \alpha \le 1, and \beta = \frac{1}{4} & \\ 1, & \beta > \frac{1}{4} \end{cases}$$

Similarly, Wife's best response is:

$$B_W(\alpha) = \begin{cases} 0, & and\ \alpha < \frac{3}{4} \\ \beta: 0 \le \beta \le 1, & and\ \alpha = \frac{3}{4} \\ 1, & \alpha > \frac{3}{4} \end{cases}$$

This game has three mixed strategy Nash equilibrium (as evident from three points of intersections of the Player 1 and Player 2's Best Response functions): $(\alpha, \beta) = (0, 0), (\frac{3}{4}, \frac{1}{4}), (1, 1)$. **See Figure 20.1.**

Proposition: *Every strategic game in which each player has finitely many strategies has a mixed strategy Nash equilibrium.*

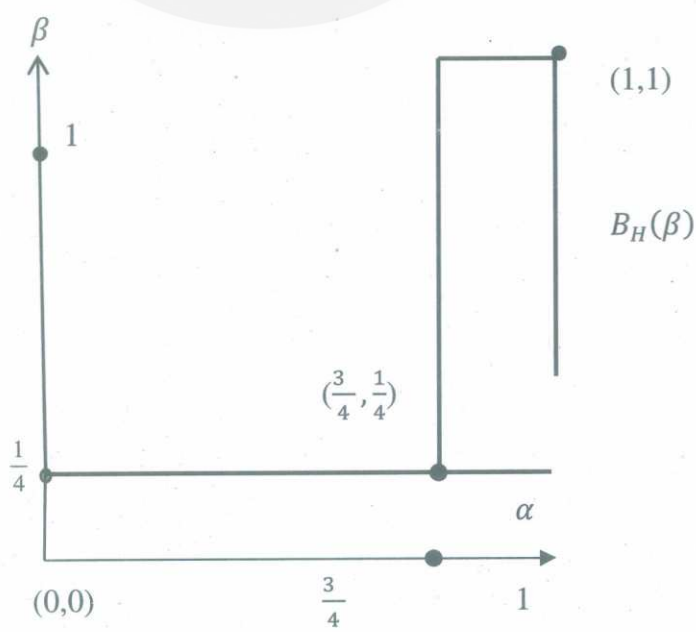


**Fig. 20.1: Mixed Strategy of Battle of Sexes Game**

Recall in the mixed strategy equilibrium we have assumed $\varepsilon_H(F,\beta) = \varepsilon_H(M,\beta)$and $\varepsilon_W(\alpha,M) = \varepsilon_W(\alpha,F)$in the equilibrium.

**Proof:** Let us assume statement 1 and 2 hold.

1) The Husband is playing the mixed strategy with probabilities $\alpha$ *and* $(1-\alpha)$for F and M respectively $(0<\alpha<1)$.
2) But the expected payoff of the husband from F and M (when Wife is playing the mixed strategy) are not equal i.e., $\varepsilon_H(F,\beta) \neq \varepsilon_H(M,\beta)$.

Let us assume $\varepsilon_H(F,\beta) > \varepsilon_H(M,\beta)$

Now from 20.15, we know, the expected payoff of the Husband when both of them are playing the mixed strategies is $\varepsilon_H(\alpha,\beta) = \alpha.\varepsilon_H(F,\beta) + (1-\alpha)\varepsilon_H(M,\beta)$

Differentiating the above equation with respect to $\alpha$ (the probability with which husband plays the F):$\frac{\partial \varepsilon_H(\alpha,\beta)}{\partial \alpha} = 1.\varepsilon_H(F,\beta)(-1).\varepsilon_H(M,\beta) = \varepsilon_H(F,\beta) - \varepsilon_H(M,\beta) > 0$ (because of the assumption)$\Longrightarrow \frac{\partial \varepsilon_H(\alpha,\beta)}{\partial \alpha} > 0 \Longrightarrow$ Expected payoff of the Husband from the mixed strategy is increasing in α. As increases the expected payoff of the Husband from the mixed strategy i.e., $\varepsilon_H(\alpha,\beta)$ increases. So, it is optimal for the husband to put $\alpha = 1$.

Therefore, the husband can maximize his expected payoff $\varepsilon_H(\alpha,\beta)$by playing F with probability 1 and Movie with probability 0.

Hence the contradiction to the statement 1 and in equilibrium $\varepsilon_H(F,\beta) = \varepsilon_H(M,\beta)$.

Similarly, we can show,
if $\varepsilon_H(F,\beta) < \varepsilon_H(M,\beta) \Longrightarrow \frac{\partial \varepsilon_H(\alpha,\beta)}{\partial \alpha} =$
$1.\ \varepsilon_H(F,\beta)(-1).\varepsilon_H(M,\beta) = \varepsilon_H(F,\beta) - \varepsilon_H(M,\beta) < 0$ (because of the assumption)$\Longrightarrow \frac{\partial \varepsilon_H(\alpha,\beta)}{\partial \alpha} < 0 \Longrightarrow$
Expected payoff of the Husband from the mixed strategy is decreasing in α.
$\Longrightarrow$ *As* $\alpha$ increases$\varepsilon_H(\alpha,\beta)$ falls. So it is optimal for the husband to put $\alpha = 0$. Therefore the husband can maximize his expected payoff $\varepsilon_H(\alpha,\beta)$by playing F with probability 0 and M with probability 1.

Hence the contradiction. So in equilibrium,$\varepsilon_H(F,\beta) = \varepsilon_H(M,\beta)$(proved)

**Example:** The following matrix is a partially specified payoff matrix of the two player strategic game is given below. Both the players are playing mixed strategy. Figures in the bracket against each strategy are showing the probabilities of their mixed strategy. This example is illustrating a method of checking whether a mixed strategy profile is a mixed strategy equilibrium.

| | | **Player 2** | | |
|---|---|---|---|---|
| | $(\alpha,\beta)$ | L$(\beta=0)$ | C $(\frac{1}{3})$ | R $(\frac{2}{3})$ |
| Player 1 | T$(\alpha=\frac{3}{4})$ | (., 2) | (3, 3) | (1, 1) |
| | M (0) | (., .) | (0, .) | (2, .) |
| | B $(\frac{1}{4})$ | (., 4) | (5, 1) | (0, 7) |

So we have to check the indicated pair of mixed strategy $\{(\frac{3}{4}, 0, \frac{1}{4}), (0, \frac{1}{3}, \frac{2}{3})\}$ is a Mixed Strategy Nash equilibrium. This also suffices by the properties of Mixed Strategy equilibrium. Note that Husband is playing T and B with positive weights and Wife is playing C and R with positive weights. Consider the player 1's expected payoff from 3 pure strategies (T, M, B) provided player 2 plays mixed strategy:

1) $\varepsilon_1(T, \beta) = 0.(.) + \frac{1}{3}3 + 1.\frac{2}{3} = \frac{5}{3}$

2) $\varepsilon_1(M, \beta) = 0.(.) + \frac{1}{3}.0 + \frac{2}{3}.2 = \frac{4}{3}$

3) $\varepsilon_1(B, \beta) = 0.(.) + \frac{1}{3}.5 + \frac{2}{3}.0 = \frac{5}{3}$

Expected payoff of the Husband when both Husband and Wife are playing mixed strategy:

$\varepsilon_1(\alpha, \beta) = \frac{3}{4}\varepsilon_1(T, \beta) + 0.\varepsilon_1(M, \beta) + \frac{1}{4}\varepsilon_1(B, \beta) = (\frac{3}{4}.\frac{5}{3}) + (0.\frac{4}{3}) + \frac{1}{4}.\frac{5}{3} = \frac{5}{3}$

Thus, in Mixed Strategy Nash equilibrium: $\varepsilon_1(\alpha^*, \beta^*) = \varepsilon_1(T, \beta^*) = \varepsilon_1(B, \beta^*) = \frac{5}{3}$

Similarly, $\varepsilon_1(\alpha^*, \beta^*) > \varepsilon_1(M, \beta)$ →because $\frac{5}{3} > \frac{4}{3}$. This is because player 1's mixed strategy assigns positive probabilities to T and B and zero probability to M.

You may repeat the same exercise for the Player 2.

**Check Your Progress 4**

1) What is a mixed strategy?

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2) Consider the example of Matching of Pennies game (in Table- 20.1).

   i) Show that there is no pure strategy Nash equilibrium.

   ii) Show that if the Player 1 plays Head with probability p and the Player 2 plays Head for sure, then player 1's expected payoff is (2p -1).

   iii) What is the Player 2's expected payoff?

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3) Consider a Hawk-Dove (or Chicken) Game (t = Tough, c= Concede)

| | | Player 2 | |
|---|---|---|---|
| | | Tough (t) | Concede (c) |
| Player 1 | Tough (t) | (1, -1) | (10, 0) |
| | Concede (c) | (0, 10) | (5, 5) |

Consider the Mixed Strategy in which Player 1 plays t with probability $\frac{9}{11}$, while Player 2 plays t for sure. Then what is the player 1's expected payoff?

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4) Show that there is no Mixed Strategy Nash equilibrium in the following game:

| | | Player 2 | | |
|---|---|---|---|---|
| Player 1 | | Left (L) | Middle (M) | Right (R) |
| | Up (U) | (1, 0) | (1, 2) | (0, 1) |
| | Down (D) | (0, 3) | (0, 1) | (2, 0) |

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5) Find the mixed strategy Nash equilibrium of the following game:

| | Player 2 | | |
|---|---|---|---|
| Player1 | | L | R |
| | T | (2, 1) | (0, 2) |
| | B | (1, 2) | (3, 0) |

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## 20.9 LET US SUM UP

A strategic form of game is described by the list of players the strategies available to each player and the payoffs to any strategy combination, one strategy for each player. A dominant strategy gives higher payoffs than every other strategy regardless of what the other player's do. A dominant strategy solution to a game exists when every player has a dominant strategy.

No rational player will play a dominated strategy but would rather play one of his undominated strategies. A rational player would not expect his opponent to play a dominated strategy either. Elimination of dominated strategies can lead to a chain of reaction that successively narrows down how a group of rational

players will act. If there is eventually a unique prediction, it is called the IEDS solution.

Nash equilibrium is the most popular solution concept with wide range of applications. A strategy combination is a Nash equilibrium if each player's strategy choice is a best response against her opponent's choices in that combination. Any IEDS solution has to be a Nash equilibrium however there may be Nash equilibrium that are not IEDS solutions. Same thing is true of a dominant strategy solution, it must be a Nash equilibrium but the reverse is not true.

A mixed strategy is a probability distribution over a player's pure strategies; not every pure strategy needs to be included in every mixed strategy. The payoff to a mixed strategy is computed as expected payoff to its component of pure strategies. A mixed strategy can dominate a pure strategy even if the latter is undominated by every other pure strategy. There are games with no Nash equilibrium in pure strategies but there will always be such an equilibrium in mixed strategies.

## 20.10 KEY WORDS

**Common Knowledge:** A fact is common knowledge if all players know it, and know that they all know it, and so on. The structure of the game is often assumed to be common knowledge among the players.

**Dominating Strategy:** A strategy dominates another strategy of a player if it always gives a better payoff to that player, regardless of what the other players are doing. It weakly dominates the other strategy if it is always at least as good.

**Extensive Game:** An extensive game (or extensive form game) describes with a tree how a game is played. It depicts the order in which players make moves and the information each player has at each decision point.

**Game Theory:** Game theory is the formal study of decision-making where several players must make choices that potentially affect the interests of the other players.

**Game:** A game is a formal description of a strategic situation.

**Mixed Strategy:** A mixed strategy of a player in a strategic game is a probability distribution over the player's action. As a special case, a mixed strategy can be the deterministic choice of one of the given pure strategies.

**Nash Equilibrium:** A Nash equilibrium, also called strategic equilibrium, is a list of strategies, one for each player, which has the property that no player can unilaterally change strategy and get a better payoff.

**Payoff:** A payoff is a number, also called utility, that reflects the desirability of an outcome to a player, for whatever reason. When the outcome is random, payoffs are usually weighted with their probabilities. The expected payoff incorporates the player's attitude towards risk.

**Player:** A player is an agent who makes decisions in a game.

**Rationality:** A player is said to be rational if she seeks to play in a manner which maximizes her own payoff. It is often assumed that the rationality of all players is common knowledge.

**Shapley Value:** States in which costs and rewards are distributed among a number of participants in order of the importance of that each plays in the cooperative activity.

**Core:** Set of feasible allocations that cannot be improved upon by a subset (a coalition) of the economy's consumers

**Strategic Form:** A game in strategic form, also called normal form, is a compact representation of a game in which players simultaneously choose their strategies. The resulting payoffs are presented in a table with a cell for each strategy combination.

**Strategy:** In a game in strategic form, a strategy is one of the given possible actions of a player. In an extensive game, a strategy is a complete plan of choices, one for each decision point of the player.

**Zero-Sum Game:** A game is said to be zero-sum if for any outcome, the sum of the payoffs to all players is zero. In a two-player zero-sum game, one player's gain is the other player's loss, so their interests are diametrically opposed.

## 20.11 SOME USEFUL BOOKS

1) Avinash K. Dixit(2015), Susan Skeath andDavid H. Reiley, *Games of Strategy*, W. W. Norton and Company; 4 edition.

2) Drew Fudenberg and Jean Tirole(1991), *Game Theory*, MIT Press.

3) Drew Fudenberg and David Levine (1998), *The Theory of Learning in Games*, MIT Press,.

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## 20.12 CHECK YOUR PROGRESS ANSWERS/HINTS

**Check Your Progress 1**

1) A game in which players are able to make enforceable contracts.

2) Games in which players make moves at different times or in turn.

3) If all of the players know it (it is mutual knowledge) and all of the players know that all other players know it and all other players know that all other players know that all other players know it, and so on.

4) A strategy consisting of possible moves and a probability distribution (collection of weights) which corresponds to how frequently each move is to be played.

5) A set of strategies, one for each player, such that no player has incentive to unilaterally change her action.

6) In a game where players are unable to make enforceable contracts outside of those specifically modeled in the game.

7) Payoffs are numbers which represent the motivations of players. These may represent profit, quantity, "utility", or other continuous measures (cardinal payoffs), or may simply rank the desirability of outcomes (ordinal payoffs).

8) A specific move or action that a player will follow in every possible attainable situation in a game.

9) A strategy is weakly dominant if, regardless of what any other players do, the strategy earns a player a payoff at least as high as any other strategy, and, the strategy earns a strictly higher payoff for some profile of other players' strategies.

10)

| | Player 2 | | |
|---|---|---|---|
| Player1 | | L | R |
| | L | –1, 1 | 1, –1 |
| | R | 1, –1 | –1, 1 |

11)

| | Player 2 | | |
|---|---|---|---|
| Player1 | | H | T |
| | H | 1, 1 | 1, 1 |
| | T | 1, 1 | 1, 1 |

12)

| | Player 2 | | |
|---|---|---|---|
| Player1 | | H | T |
| | H | 1, –1 | –1, 1 |
| | T | –1, 1 | 1, –1 |

13)

| | Player 2 | | |
|---|---|---|---|
| Player1 | | H | T |
| | H | 1, 1 | 1, 1 |
| | T | 1, 1 | 1, 1 |

**Check Your Progress 2**

1) A strategy is strictly dominated if there is some other strategy which offers a higher utility, regardless of the strategies of the other players.

2) A dominant strategy equilibrium is reached when each player chooses their own dominant strategy.

3) Player 1 has two strategies and player 2 has three. S1={up, down} and S2={left, middle, right}. For player 1, neither up nor down is strictly dominated. Up is better than down if 2 plays left (since 1>0), but down is better than up if 2 plays right (since 2>0). For player 2, however, right is strictly dominated by middle (since 2>1 and 1>0), so player 2 being rational will not play right.

Thus if player 1 knows that player 2 is rational then player 1 can eliminate right from player 2's strategy space. So, if player 1 knows that player 2 is rational then player 1 can play the game as if it was the game depicted below.

| | | Player 2 | |
|---|---|---|---|
| | | left | middle |
| Player 1 | up | 1,0 | 1,2 |
| | down | 0,3 | 0,1 |

In the figure above, down is strictly dominated by up for player 1 and so if player 1 is rational (and player 1 knows that player 2 is rational, so that the second game applies) then player 1 will not play down. Consequently, if player 2 knows that player 1 is rational and player 2 knows that player 1 knows that player 2 is rational (so that player 2 knows that the second game applies) then player 2 can eliminate down from player 1's strategy space, leaving the game looking like below.

| | | Player 2 | |
|---|---|---|---|
| | | left | middle |
| player 1 | Up | 1,0 | 1,2 |

And now left is strictly dominated by middle for player 2, leaving (up, middle) as the outcome of the game. This is process is called the iterated elimination of strictly dominated strategies.

**Check Your Progress 3**

1) Minimizing the possible loss for a worst case (*max*imum loss) scenario.
2) The row player can get a value of 4 (if the other player plays R) or 5 (if the other player plays L), so: $\overline{v_{raw}} = 4$. The column player can get 1 (if the other player plays T), 1 (if M) or 4 (if B). Hence: $v_{col} = 1$.
3) The iterated elimination of strictly dominated strategies eliminates all but the one strategy which is the Nash equilibrium of the game.

**Check Your Progress 4**

1) A mixed strategy is one in which a player plays his available pure strategies with certain probabilities
2) Solve yourself after reading carefully solution concepts through Nash equilibrium.
3) Hint: When player 2 plays t for sure, player 1's expected payoff: $\frac{9}{11}(-1) + (1-\frac{9}{11}).0 = -\frac{9}{11}$ When player 2 plays c for sure, then player 1's expected payoff: $\frac{9}{11}(10) + (1-\frac{9}{11}).5 = \frac{100}{11}$
4) Do yourself after reading Section 20.8
5) Do yourself reading after reading Section 20.8

## 20.13 EXERCISES

1) An employee (player 1) who works for a boss (player 2) can either work (Ω) or shirk(Σ), while his boss can either monitor the employee (M) or ignore him (I). Like most employee-boss relationships, if the employee is working then the boss prefers not to monitor, but if the boss is not monitoring then the employee prefers to shirk. The game is represented in the following matrix:

| | Player 2 | | |
|---|---|---|---|
| Player1 | | M | I |
| | W | 1, 1 | 1, 2 |
| | S | 0, 2 | 2, 1 |

a) Draw the best response functions of each player.

b) Find the Nash equilibrium of this game.

**Answer:** (a)

Let p be the probability that player 1 chooses W and q the probability that player 2 chooses M. It follows that $v_1$ (W, q) > $v_1$ ($S_1$ q) if and only if 1 > 2(1 – q), or $q > \frac{1}{2}$, and $v_2(p, M) > v_2(p, I)$ if and only if p + 2(1 – p) > 2p + (1 – p), or $p < \frac{1}{2}$. It follows that for player 1,

$$BR_1(q) = \begin{cases} p = 0 & \text{if } q < \frac{1}{2} \\ p \in [0,1] & \text{if } q = \frac{1}{2} \\ p = 1 & \text{if } q > \frac{1}{2} \end{cases}$$

And for player 2

$$BR_2(p) = \begin{cases} q = 1 & \text{if } p < \frac{1}{2} \\ q \in [0,1] & \text{if } p = \frac{1}{2} \\ q = 0 & \text{if } p > \frac{1}{2} \end{cases}$$

**Answer:** (b)

Best response corresponds to Nash equilibrium $(p, q) = \left(\frac{1}{2}, \frac{1}{2}\right)$

2) Use the minimax method to get the mixed strategy of the game.

Solve the following game:

$$\begin{array}{c} \\ A \end{array} \begin{array}{c} B \\ \begin{bmatrix} 5 & 1 \\ 3 & 4 \end{bmatrix} \end{array}$$

**Solution**

It is a 2 x 2 matrix and no saddle point exists. So the solution is

$$\begin{array}{c} \\ A \\ \\ 3 \end{array} \begin{array}{c} B \\ \begin{bmatrix} 5 & 1 \\ 3 & 4 \end{bmatrix} \\ 4 \quad\quad 2 \end{array} \begin{array}{c} 1 \\ 4 \end{array}$$

$$V = \frac{a_{11}a_{22} - a_{21}a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{20 - 3}{9 - 4}$$

V = 17 / 5

SA = (x1, x2) = (1/5, 4 /5)

SB = (y1, y2) = (3/5, 2 /5)

3) Consider the following game.

| | a | b | c | d |
|---|---|---|---|---|
| *w* | 2, 0 | 0, 5 | 1, 0 | 0, 4 |
| *x* | 4, 1 | 2, 1 | 0, 2 | 1, 0 |
| *y* | 2, 1 | 5, 0 | 0, 0 | 0, 3 |
| *z* | 0, 0 | 1, 0 | 4, 1 | 0, 0 |

a) Compute the set of mixed strategies.

b) Compute the set of all Nash equilibria.

**Ans.** (a)

*w* is dominated by a mixed strategy putting 2/3 on x and 1/3 on z.

*a* is dominated by mixed strategy putting 3/5 on c and 2/5 on d.

*b* is dominated by mixed strategy putting 3/5 on c and 2/5 on d.

*y* is dominated by a mixed strategy putting ½ on x and ½ on z

*d* is dominated by c.

*x* is dominated by z.

The set of rationalizable strategies is {z} × {c}

b) The only Nash equilbrium is (z,c) because there is only 1 mixed strategy for each player

4) Let $p_1$ and $p_2$ be the prices of a homogeneous product produced by firms 1 and 2 respectively. Each firm chooses their price $p_i \in [0, \infty)$. There is no restriction on price and the firms choose their price simultaneously. $D_i(p_i, p_j)$ is the demand function of the $i^{th}$ firm.

$$D_i(p_i, p_j) = \begin{cases} F(p_i) & \cdots\cdots \text{if } p_i < p_j \\ \frac{1}{2}F(p_i) & \cdots\cdots \text{if } p_i = p_j \\ 0 & \cdots\cdots \text{if } p_i > p_j \end{cases}$$

Both the firms have a constant marginal cost, c. The payoff function of the $i^{th}$ firm is given by $\pi_i(p_i, p_j) = (p_i - c)\, D_i(p_i, p_j)$.

Show that the Nash equilibrium of the game is ($p_1^*=c$, $p_2^*=c$).

Ans. Let ($p_i^*$, $p_j^*$)be another Nash equilibrium of the game. Now there could be following mutually exhaustive possibilities:

i) **Case I** $p_i^* > p_j^* > c$ ➔ In this case the entire market is served by the $j^{th}$ firm. If the $i^{th}$ firm reduces its price slightly such that its price is below the $j^{th}$ firm but above the marginal cost, then it will get the entire market. Therefore, $p_i^*$ is not the best response of the $i^{th}$ firm and hence such a price configuration cannot be the Nash equilibrium.

ii) **Case II** $p_j^* > p_i^* > c$ ➔ Here the $j^{th}$ firm is not behaving optimally (the argument is the same as case I).

iii) **Case III** $p_j^* = p_i^* > c$ ➔ Given the equilibrium price of one firm the other is not behaving optimally. Reducing price by $\xi$ (where $\xi$ is negligibly small positive quantity) one firm can increase its profit given the other firm is charging its equilibrium price.

iv) **Case IV** $p_j^* > p_i^* = c$ ➔ Firm i is not doing its best. It can increase its profit by charging a price, which is greater than c but slightly, lower than the price $p_j^*$ .

v) **Case V** $p_i^* > p_j^* = c$ ➔ Here firm j is not behaving optimally (the argument is same as case

Thus, we see there is no other combination of price in which both the firms are doing their best. Therefore, no other price configuration other than ($p_1^*=c$, $p_2^*=c$) which is a Nash equilibrium.

5) Solve the following normal form game using the method of iterated elimination of strictly dominated strategies.

| | | Player 2 | | |
|---|---|---|---|---|
| | | Left | Middle | Right |
| Player 1 | Up | (1, 0) | (1, 2) | (0, 1) |
| | Down | (0, 3) | (0, 1) | (2, 0) |

Ans. The strategy "Right" of player 2 is strictly dominated by the strategy Middle". Therefore, we can eliminate it from the table. The reduced table is given below:

| | | Player 2 | |
|---|---|---|---|
| | | Left | Middle |
| Player 1 | Up | (1, 0) | (1, 2) |
| | Down | (0, 3) | (0, 1) |

Notice that in the reduced table for player 1, the strategy "Down" is strictly dominated by the strategy "Up". Therefore, we can remove it from the table. The further reduced table is as the following:

| | | Player 2 | |
|---|---|---|---|
| | | Left | Middle |
| Player 1 | Up | (1, 0) | (1, 2) |

Now, there is a single strategy left for player1 and for player 2 the strategy "Left" is strictly dominated by "Middle". Therefore, the equilibrium is (Up, Middle)

6) Can we solve the following normal form game using the mini-max method? If possible then solve it.

| | | Player 2 | | |
|---|---|---|---|---|
| | | Left | Middle | Right |
| Player 1 | Up | 5, 1 | 3, 3 | 1, 5 |
| | Straight | 6, 0 | 2, 4 | 1, 5 |
| | Down | 1, 5 | 0, 6 | 0, 6 |

Ans. It is possible to solve the game with minimax method because it is a constant sum game (the constant being 6)

7) Consider the following game of 2 players. Suppose Player 1 has 3 strategies U, M, D, which are played with probabilities (0.2, 0.3 and 0.5) respectively. Player 2 has 4 strategies L, $M_1$, $M_2$ and R. Player 2 plays L and $M_2$ with probabilities 0.4 and 0.6. Show that player 1's expected payoff is 2.32. What about Player 2's expected payoff?

| | | **Player 2** | | | |
|---|---|---|---|---|---|
| | | L | $M_1$ | $M_2$ | R |
| Player 1 | U | 1, 0 | 4, 2 | 2, 4 | 3, 1 |
| | M | 2, 4 | 2, 0 | 2, 2 | 2, 1 |
| | D | 4, 2 | 1, 4 | 2, 0 | 3, 1 |

8) Show that there is no Mixed Strategy Nash equilibrium in the Prisoner's Dilemma Game.

9) Show that apart from the two pure strategy-equilibrium, there is also a mixed strategy equilibrium in a chicken game (as given in Table-20.6).

10) Find out the unique Nash equilibrium in pure strategy in the following game. Show that there is no Mixed Strategy Nash equilibrium in the following game:

| | | **Player 2** | | |
|---|---|---|---|---|
| | | Left (L) | Centre (C) | Right (R) |
| Player 1 | Top (T) | (0,4) | (4,0) | (5,3) |
| | Middle (M) | (4, 0) | (0, 4) | (5,3) |
| | Bottom (B) | (3, 5) | (3, 5) | (6, 6) |

# UNIT 21 STATIC GAMES WITH COMPLETE INFORMATION: APPLICATIONS

**Structure**


21.0 Objectives

21.1 Introduction

21.2 Game Theoretic Applications in Common Property Resources

- 21.2.1 Prisoners' Dilemma Resource Conservation Game
- 21.2.2 Chicken Game
  - 21.2.2.1 Problem of the CPR Provision – Two Person Chicken Game
  - 21.2.2.2 Chicken Game: Heterogeneous Group and Collective Action
- 21.2.3 Coordination Game in Fishing

21.3 Best Response Function

- 21.3.1 Synergistic Relationship

21.4 Quantity Competition and Price Competition

- 21.4.1 Cournot Competition
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21.5 War of Attrition

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21.7 Let Us Sum Up

21.8 Key Words

21.9 Some Useful Books

21.10 Check Your Progress Answers/Hints

21.11 Exercises


## 21.0 OBJECTIVES

After going through this unit, you will be able to:

- understand the nature of strategic interactions among the agents in using Common Property Resources;
- appreciate application of three different categories of normal form games viz., Prisoners' Dilemma Game, Chicken Game and Coordination Game (already introduced in the Unit 20) in common property resource (CPR) Management;
- derive best response functions for computation of Nash equilibrium in the case when payoff functions are continuous; and
- learn derivation of Nash equilibrium in the case of Cournot Duopoly (quantity competition), Bertrand Duopoly (Price competition), War of Attrition Game and Hotelling's Linear City Model.

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## 21.1 INTRODUCTION

Game theory is a multi-person decision theory that analyses situations in with payoffs to agents. It involves the analysis of conflict, cooperation, and (tacit) communication and extends applications into several fields, such as economics, politics, law, biology, and computer science. Quite a number of examples under discrete payoff structure and continuous payoff structure are presented here to analyse Nash equilibrium in pure strategy. These applications help learn theoretical insights, to understand how to formalize game theoretic problem from an informal description of multi-person decision making.

## 21.2 GAME THEORETIC APPLICATIONS IN COMMON PROPERTY RESOURCES

Common property resources are often explained in terms of the physical attributes of the goods or resources. They share two characteristics: (a) the difficulty of excluding individuals from benefiting from the resource and (b) the subtractability of the benefits consumed by one individual from those available to others. Such characteristics give rise to problems of (i) appropriation and (ii) provision. While the first relates to extracting more than the socially optimal level, the second problem refers to the difficulty of investing for the "adequate" provision of the CPR or in activities related to its improvement or maintenance. Both these problems can be considered to generate negative consequences for users.

**Commonly used games for depicting CPR problems**

Three games, viz., (a) the prisoner's dilemma game (b) the chicken game and (c) the assurance game are used for handling CPR problems. The prisoner's dilemma game has been widely used to represent the appropriation problems encountered in CPRs.

When limits for resource use cannot be established, all users of the CPR will want to use as much as possible so as to maximize their own profit, resulting in over-appropriation of the resource. In the case of natural resources such over-appropriation can lead to overexploitation.

In the case of resource provision, the problem arises with free riders benefiting from the resource (as a public good) provided without having contributed to its provision. Take forexample, investment for the maintenance or improvement of the CPR. This problem has been modeled, depending on the provision technology, with the prisoner's dilemma or with the assurance game. The former having no-one contributing to the resource provision and the latter having players contributing if, and only if, the others are to contribute.

Lastly, the chicken game has been used to illustrate assignment problems when there are different resource locations with different richness.

### 21.2.1 Prisoners' Dilemma Resource Conservation Game

Consider a village pasture which is common property. Let two local villagers graze their cattle there. Note that villagers can graze more cattle with the same effort cost.

**Table 21.1: Prisoners' Dilemma Game**

| | | Player 2 | |
|---|---|---|---|
| | | **Participate (p)** | **Defect (d)** |
| Player 1 | Participate (p) | (s, s ) | (v, r) |
| | Defect (d) | (r, v) | (t, t) |

But beyond a certain level grazing, the pasture can no more be sustainable. If the villagers desire to maintain its sustainability, the number of cattle grazing there must be restraint so that the extraction of the resources does not exceed the regeneration capacity of the pasture. When such a participation occurs in the conservation of the commons, it results in payoff $(s, s)$given in Table 21.1.

However, if any of the participants them defects from participation (given the other player participates) then defector's payoff increases and the one who is participating gets nothing. That is,$(p, d) = (v, r)$ or,$(d, p) = (r, v)$. So there is always an incentive among the villagers to free-ride or overgraze (defect).

Thus, mutual defection or free riding $(d, d)$ becomes the *dominant strategy equilibrium.* The strategy, defection, for each of the villager dominates the strategy participation, irrespective of the other players' choice of action (or participation is dominated by the strategy defection for each). This sort of outcome is described by Mancur Olson (1965) as – *individual rationality produces collective irrationality.*

Here the cooperation is not sustainable as each of the villagers faces a credible threat that given he participates, his rivals may defect which will push him to zero payoff. Such a mutual threat pushes them to play safe by choosing defect $(d, d)$.The outcome is unique pure strategy Nash equilibrium $(d, d)$, with payoff $=(t, t)$. This strategy $(d, d)$ is the mutually best response strategy or *Nash equilibrium* although it is not *social / Pareto optimal* and both the villagers over exploit the pasture.

Basically, lack of institutional control may lead the individuals to over exploit the natural resource base. So long as there are multiple users competing for the resource, and pay-off structure follows inequality, $r > s > t > v$, extraction by one creates negative externalities on others causing Hardin's *tragedy of the commons.*

## 21.2.2 Chicken Game (or, Hawk-Dove Game)

A chicken game refers to a set up that typically describes two players heading toward each other. If the players continue on the same path, they bump into each other; if one swerves out of the way and other doesn't, the swerver "loses" and is labeled the chicken, while the second, implicitly braver player wins. Hawk-Dove game also is based on the description of the same scenario in which players compete for resources by being either aggressive or passive.

Both Chicken and Hawk-Dove are anti-coordination games, in the sense that these can be thought of as the opposite of a coordination game. The underlying idea here is that players use a shared resource. In coordination games, sharing the resource creates a benefit for all as the resource is non-rivalrous, and the shared usage creates positive externalities. In anti-coordination games the resource is rivalrous but non-excludable and sharing comes at a cost (or negative externality).

### 21.2.2.1 Problem of the CPR Provision – Two Person Chicken Game

Consider a situation when everybody agrees something has to be done, but the problem is who will actually do it?

There may be many situations where collective problem is solved through unilateral actions. The situation arises when collective goods can be obtained at sufficiently low costs in relation to its benefits. This may create an incentive to the bigger stake holders to undertake the responsibility of procuring it. Given the total gains associated with the production of collective goods or avoidance of collective bad is pretty large in relation to individual's costs share in avoiding it, unilateral contribution may be the Nash equilibrium.

Consider two small scale marine fishermen A and B, who stands threatened by the invasion of a fleet of foreign trawlers in their traditional fishing zone. Their income is presently 10 units. But to maintain the level of present catches they need to have inshore water legally protected against the encroachments of foreign boats. They can secure the legal protection from the government by making a lobbying effort (which is a public good), which costs altogether, say Rs. 4.

**Table 21.2: Fishermen Game**

| | | **Fisherman 2** | |
|---|---|---|---|
| | | **Cooperation (C)** | **Defection (D)** |
| Fisherman 1 | Cooperation (C) | (x, x ) = (8, 8) | (a, b) = (6, 10) **(NE)** |
| | Defection (D) | (b, a) = (10, 6) **(NE)** | (y, y) = (2, 2) |

If they cooperate by sharing the costs, their payoffs are: $(C, C) = (x = 10 - 2 = 8, x = 10 - 2 = 8)$. But if only one cooperates and the other defects, then the payoffs are: $(C, D) = (b = 10 - 4, a = 10) = (6, 10)$ or, $(D, C) = (a = 10, b = 6)$.

The lobbying is so rewarding that it pays a single agent to bear the entire costs. The agents without lobbying effort $(D, D)$ will face a disastrous situation in their payoff ( $y = 2, y = 2$). So there are two-pure strategy Nash equilibrium $(C, D)$or $(D, C)$. Politically active agent (or sucker) will unilaterally bear the entire costs of procuring the collective goods.

In this kind of game the players have no dominant strategy (unlike the PD game). As we have seen in PD game, each player prefers others to take the responsibility of procuring all goods, while he refrains from moving. However contrary to the PD games, in this kind of game, each player is willing to take full responsibility for that action if the other refrains to do anything. The consequence of nobody doing the action is so disastrous that either of the players would undertake it if the other refuses to do it.

Other Applications of Chicken Game are:

- Maintenance and surveillance of irrigation system
- Maintenance and surveillance of pastures
- Maintenance and surveillance of grazing fields
- Maintenance and surveillance of hunting grounds

In some fishing communities the problem of conflict of appropriation of public goods is solved by reorganizing the claim of the first entrants into the fishing grounds. There may be some conflicting situations when in case of catching of lobsters or bottom dwelling species, the number of fishermen exceeds the number of locations, where enough lobsters can be caught to justify the effort involved.

### 21.2.2.2 Chicken Game: Heterogeneous Group and Collective Action

Olson (1965) predicts that richer party will necessarily contribute because he has greater stake. To appreciate such a result, consider a case of two herders. The rich one has a herd of prized goats, whose furs are used for making upscale Kashmeri sweaters to be exported and sold in some expensive salons of Paris. The other poor herder has a herd of humble goat used for making pet meat. The payoff matrix of the game is shown in Table 21.3. This game has two pure strategy equilibrium $(C, D)$ or $(D, C)$ (leading to an asymmetric payoffs), and one mixed strategy equilibrium, where all outcomes have positive probability of occurrence. $(C, D)$is the focal point equilibrium, where it is likely that the rich herder will play $C$ in equilibrium.

**Table 21.3: Game with Asymmetric Payoffs**

| | | **Poor Herder** | |
|---|---|---|---|
| | | **Cooperation (C)** | **Defection (D)** |
| Poor Herder | Cooperation (C) | (x, x ) = (15, 3) | (13, 5) **(NE)** |
| | Defection (D) | (17, 0) **(NE)** | (2, 0) |

If we assume that the rich is more powerful in terms of economic and political power, it is more likely that class has abetter ability to take prior moves than the poorer or weaker party. If this is true, then the appropriate tool for analysis of decision problem involving rich and poor is not the normal form or strategic of game but the extensive or sequential form of chicken game (which will be introduced in the Unit 22). The solution concept in that case is subgame perfect Nash equilibrium.

## 21.2.3 Coordination Game in Fishing

In a coordination game, players benefit from working together. So the solutions to some of problems arise only on cooperation. In this class of games, there is no incentive for either party to cheat since it will result in a worse outcome than that from cooperating.

Assurance Game also provides insights to solution of problems through cooperation. It goes by a generic name "Stag Hunt" game due to its following formulation: Two hunters can either jointly hunt a stag (an adult deer) or individually hunt a rabbit. Hunting a stag is challenging and requires mutual cooperation. If either of the hunters hunts a stag alone, the chance of success is minimal. Thus, hunting a stag is most beneficial although requires a lot of trust among the participants.

In tragedy of the commons, therefore, we need to look for solutions through coordination game instead of Prisoners' Dilemma type of game. For, the incentive to conserve depends on whether others are willing to conserve.

To see that suppose that there are two fishermen (A and B), independently deciding whether to put one or two boat at the sea for catching fish. Given the resource constraint, each of them taking two trawler boats are not sustainable

and will thus result in low payoff (2, 2) as given in Table 21.4. So a cooperative behaviour is reflected by the strategy of taking (one Boat), whereas free riding is reflected when taken (two Boats).

There are two pure strategies Nash equilibrium: good equilibrium - (1 Boat, 1 Boat); bad equilibrium - (2 Boat, 2 Boat). If both A and B can convey properly their intention regarding cooperation there is a chance of having good equilibrium - (1 Boat, 1 Boat), where both will have fairly good payoff of (8, 8). However there is a chance of having a bad equilibrium as well. Note that in this game a genuine trust problem persists regarding other player's intended action. A may fear that B will abstain from cooperation not because B prefers free-riding but because B's expectation about A;sbehaviour is pessimistic and vice versa for B vis-à-vis on A.

**Table 21.4: Example of Coordination Game**

| | | **Fisherman B** | |
|---|---|---|---|
| | | **1 Boat** | **2 Boat** |
| Fisherman A | 1 Boat | (8, 8) **(NE)** | (1, 6) |
| | 2 Boat | (6, 1) | (2, 2) **(NE)** |

**Check Your Progress 1**

1) What is a 'Common Resource'?

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2) Write a short note on 'Common Property Resources'.

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3) Why the players choose strategies that are not Pareto optimal in common property of grazing field.

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4) What is focal point equilibrium?

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5) Why Chicken game is called anti-coordination game?

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## 21.3 BEST RESPONSE FUNCTION

If each player has a fewer actions, we can find the Nash equilibrium by examining each action profile and see if it satisfies the condition of Nash equilibrium. In more complicated games where the payoff functions are *continuous*, we obtain the Nash equilibrium through the *best response function.* In a two player game, the action profile $s_i^*$ $(i = 1, 2)$ is a Nash equilibrium of a strategic game if and only if each player's action is a best response to the other player's action. So, $s_1^* = b_1(s_2^*)$ and$s_2^* = b_2(s_1^*)$.

Thus each player has a unique best response to every action of the other player. $(s_1^*, s_2^*)$is the Nash equilibrium if player 1's action $s_1^*$ is her best response to player 2's action $s_2^*$ and player 2's action $s_2^*$ is the best response to player 1's action $s_1^*$.

### 21.3.1 Synergistic Relationship

Note that in a synergistic relationship if two individuals devote more effort, both are better off. To see that, consider two individuals $(i = 1, 2)$ involved in synergistic relationship of marriage.When they devote more effort they are better off. So their set of actions is their effort levels given that of the other. As each individual i has infinitely many actions,the best response functions of each individual have to be constructed. For that we take their preferences as reflected by their payoff functions.

Payoff function of the ith individual is:

$\pi_i = a_i(c + a_j - a_i)$, $a_i$ = Non negative number = Effort level of the individual i, c > 0, constant.

$\pi_j = a_j(c + a_i - a_j)$, $a_j$ = Non negative number = Effort level of the individual$i$, c > 0.

First order condition (FOC) :$\frac{\partial \pi_i}{\partial a_i} = (c + a_j - a_i) - a_i = 0$ ..... (21.1) and

$\frac{\partial \pi_j}{\partial a_j} = (c - a_j + a_i) - a_j = 0$ ..... (21.2)

Solving for $a_i$ from the equation 21.1 we get$a_i = \frac{c+a_j}{2} = b_i(a_j) i.e.$,best response function of the ith individual. Note that it is a linear function with positive slope $(\frac{1}{2})$ in the $a_j a_i$ plane. Similarly, the best response function of the jth player (from 21.2) is$a_j = \frac{c+a_i}{2} = b_j(a_i)$. Again, it a linear function with positive slope (2) in the $a_j a_i$ plane (see, Figure 21.1). It may be seen that the best response function $b_j(a_i)$ is steeper than $b_i(a_j)$, and both are positive.

The second order condition (SOC): $\frac{\partial^2 \pi_i}{\partial a_i^2} = -2\, a_i < 0$ and $\frac{\partial^2 \pi_j}{\partial a_j^2} = -2\, a_j < 0$ and $\frac{\partial^2 \pi_i}{\partial a_i \partial a_j} = 1 > 0$. So the effort levels $a_i$ and $a_j$ are *strategic complements*.

Solving the two best response functions (equation 21.1 and 21.2) we get:

$b_i(a_j) = a_i = \frac{c+a_j}{2}$ and $b_j(a_i) = a_j = \frac{c+a_i}{2}$.

The solution or the Nash equilibrium of this example has obtained the best response functions such that, $a_i^* = a_j^* = c$.

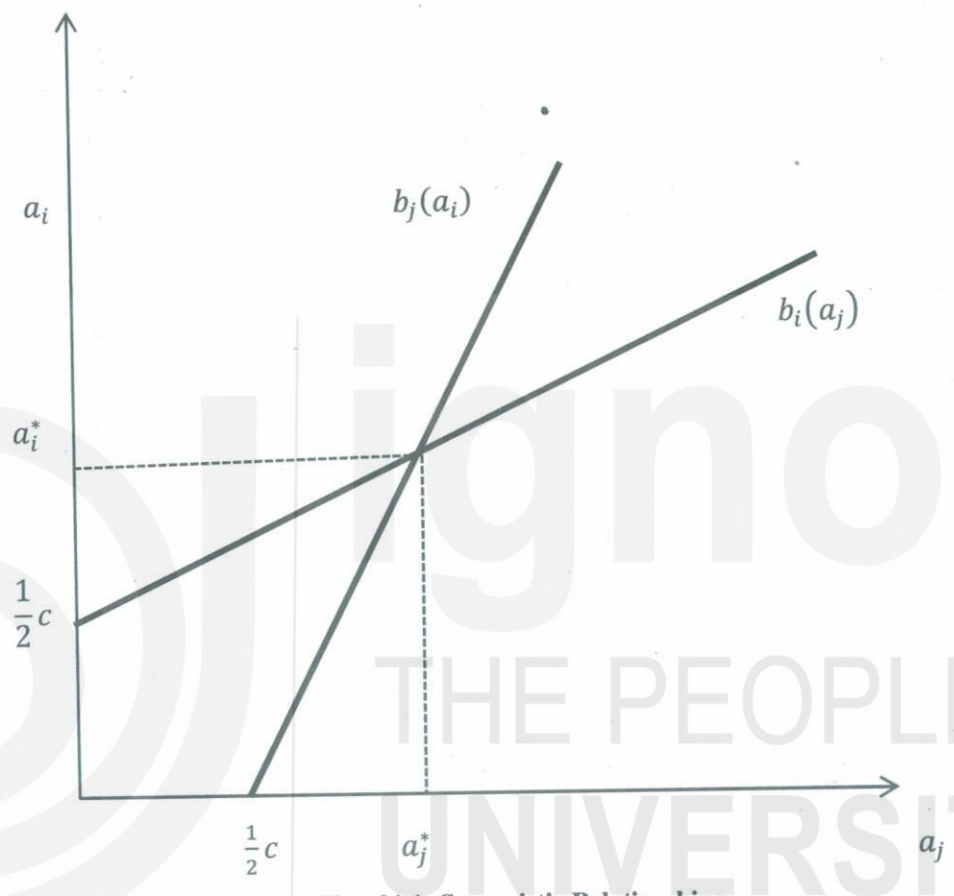


**Fig.: 21.1: Synergistic Relationships**

## Check Your Progress 2

1) Define Best Response function.

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2) What is strategic complement and strategic substitute?

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3) Discuss the meaning of a synergistic relationship.

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## 21.4 QUANTITY COMPETITION AND PRICE COMPETITION

In this section we will be presenting the case of duopoly firms simultaneously competing with each other through quantity or price. If the strategic interaction is through quantity the firms are under Cournot competition. If it is through price, then the firms are under Bertrand competition.

### 21.4.1 Cournot Competition

Firms compete with each other by simultaneously choosing the level of output of a homogenous product believing that the output levels of their rival firms remain unchanged. Thus, output is the strategic variable.

With entry and exit of the firms are restricted, let us consider a duopoly market structure with two symmetric firms, summarized by their demand function and identical cost function.

Market demand function: $p(Q) = a - bQ = a - b(q_1 + q_2)$ , .... (21.3)

where, $Q = q_1 + q_2$, a, b > 0.

Precisely, $p(Q) = \begin{cases} a - bQ, & for Q < a/b \\ 0, \text{for } Q > a/b, \end{cases}$ and negative output is ruled out.

Cost function: $C_i(q_i) = c_i q_i$, i = 1, 2. ..... (21.4)

Each firm's strategy space can be defined as: $s_i = [0, \infty)$, output is continuously divisible and negative output is not feasible.

Thus the payoff function of the 2 firms: $\pi_1(q_1, q_2), \pi_2(q_1, q_2)$

There is interdependence in each firm's Payoff functions. Payoff functions are simply the profit functions and are realized by a combination of strategies chosen by competing firms. If $q_i$ increases then $\pi_i(.)$ also increases till a certain point. However, if $q_i$ increases, then the profit of the rival firm $\pi_j(.)$ falls. So if both firms increase their output, aggregate profit may as well fall.

Equilibrium in the Cournot duopoly is defined by Nash equilibrium. The nature of interdependence is a sort of Prisoner's Dilemma game where $(q_1^*, q_2^*)$ is said to be Cournot Nash equilibrium if and only if

$\pi_1(q_1^*, q_2^*) \geq \pi_1(q_1, q_2^*)$ ..... (21.5a)

$\pi_2(q_1^*, q_2^*) \geq \pi_2(q_1^*, , q_2)$ ..... (21.5b)

In order to derive the Nash equilibrium output $(q_1^*, q_2^*)$ we have to derive the best response functions (reaction function) of both the firms.

Consider the profit function of the firm 1: $\pi_1(q_1, q_2) = p(Q).q_1 - c_1 q_1$

Thus, $\pi_1(q_1, q_2) = q_1(a - bq_1 - bq_2) - c_1q_1$.

Maximizing the profit function of the firm 1 with respect to.$q_1$, we get the

FOC: $\frac{\partial \pi_1(q_1,q_2)}{\partial q_1}$ $=(a - bq_1 - bq_2) - bq_1 - c_1 = 0$ , or,$(a - 2bq_1 - bq_2) - c_1 = 0$

Therefore, $q_1 = \frac{(a-c_1)}{2b} - \frac{q_2}{2} = R_1(q_2)$ .....(21.6)

$R_1(q_2) \Longrightarrow$*Reaction function of the firm 1* or, the *best responsefunction* of the firm 1. $R_1(q_2)$is a linear function decreasing in $(q_2)$, slope = $\frac{\partial q_1}{\partial q_2} = -\frac{1}{2} < 0$ and the intercept = $\frac{(a-c_1)}{2b} > 0$ .

Similarly, we have to derive the reaction function or the best response function of the firm 2.

Consider the profit function of the firm 2: $\pi_2(q_1, q_2) = p(Q).q_2 - c_2q_2$

Then, $\pi_2(q_1, q_2) = q_2(a - bq_1 - bq_2) - c_2q_2$.

Maximizing the profit function of the firm 2 with respect to$q_2$we get

FOC: $\frac{\partial \pi_2(q_1,q_2)}{\partial q_2}$= $(a - bq_1 - bq_2) - bq_2 - c_2 = 0$ , or,$(a - bq_1 - 2bq_2) - c_2 = 0$

Therefore, $q_2 = \frac{(a-c_2)}{2b} - \frac{q_1}{2} = R_2(q_1)$ .....(21.7)

$R_2(q_1)$ *is* the *reaction function of the firm 2* or, the *best response function* of the firm 2. $R_2(q_1)$is a linear function, decreasing in $(q_1)$, with slope $\frac{\partial q_2}{\partial q_1} = -\frac{1}{2} < 0$ and the intercept = $\frac{(a-c_2)}{2b} > 0$ . (Given that $\frac{\partial q_2}{\partial q_1} = -\frac{1}{2} or, \frac{\partial q_1}{\partial q_2} = -2$).

Thus, both the reaction functions are downward sloping straight lines. If both are plotted on the $(q_2, q_1)$ plane, $R_2(q_1)$ is steeper than $R_1(q_2) \Longrightarrow \frac{\partial q_1}{\partial q_2}\Big|R_1(q_2) = -\frac{1}{2}$and$\frac{\partial q_1}{\partial q_2}\Big|R_2(q_1) = -2$. If the rival's output increases, it is optimal for the incumbent firm to reduce the output. The intuition is quite simple, i.e., if the ith firm increases its output (for a given level of output of the rival), price will fall. In order to maintain the high price the rival firm will have to drop its output. Taking the second order cross partial derivative of the two firms' profit function we get $\frac{\partial^2 \pi_1(q_1,q_2)}{\partial q_2 \partial q_1} = -b < 0$. Thus, $q_1 and q_2$ are *strategic substitutes* of each other. As the firm 2 expands its output$q_2$, marginal profit $\frac{\partial \pi_1(q_1,q_2)}{\partial q_1}$ of the firm 1 falls. Same is true for firm 2. In such case the reaction functions are always negatively sloped. However, if the second order cross partial derivative is positive, then $q_1 and q_2$ are *strategic complements* of each other. In that case reaction functions are positively sloped.

Given the reaction functions of firms 1 and 2, we have a unique intersection which defines the *Cournot Nash equilibrium*. Solving the reaction functions (equation 21.6 and 21.7) corresponds to the intersection of the functions. Thus,

$q_1^* = \frac{a-2c_1+c_2}{3b}$, $q_2^* = \frac{a+c_1-c_2}{3b}$ ..... (21.8)

$\Longrightarrow Nash\ Equilibrium \Longrightarrow$ Equilibrium output of firm 1 and 2. See Figure (21.2).

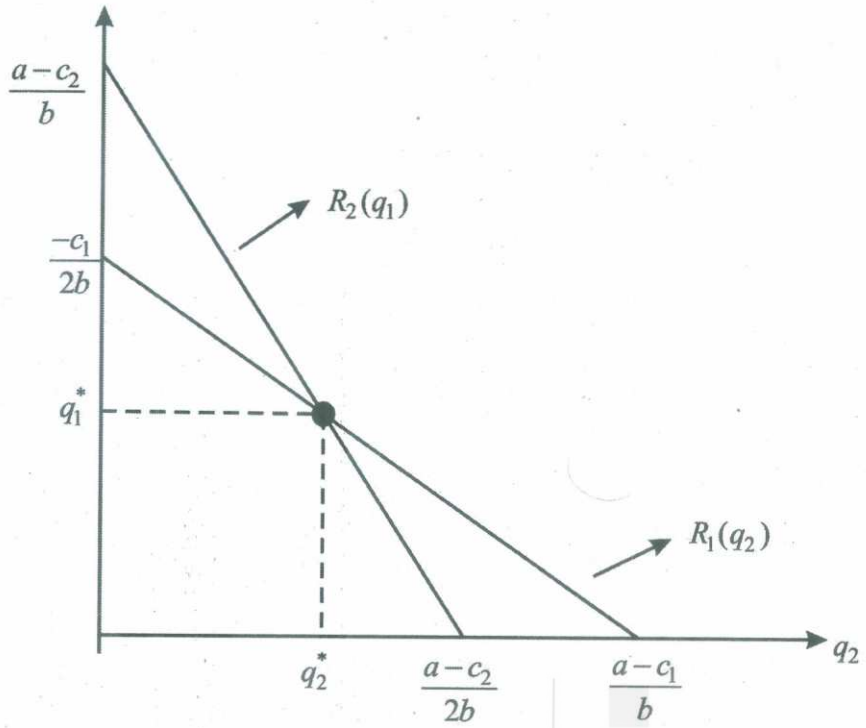


**Fig. 21.2: Nash Equilibrium Under Cournot Competition**

The equilibrium industry output:

$$Q = (q_1^* + q_2^*) = \frac{a-2c_1+c_2}{3b} + \frac{a+c_1-c_2}{3b} = \frac{2a-c_1-c_2}{3b} \quad \ldots\ldots\ldots (21.9)$$

Equilibrium price: $p^* = a - bQ = a - b\frac{2a-c_1-c_2}{3b} = \frac{a+c_1+c_2}{3}$ ..... (21.10)

Suppose the firm 1 is more efficient than firm 2 $with c_1 < c_2$. Then the output of the low cost efficient firms 1 is greater than the output of the high cost inefficient firm 2.

Plugging the equilibrium values from equation 21.8, 21.9, 21.10 in the profit function of the firm 1, we get the two equilibrium profits of the firm 1 and 2 respective as shown below:

$$\pi_1^*(q_1^*, q_2^*) = \left(\frac{a+c_1+c_2}{3}\right)\left(\frac{a-2c_1+c_2}{3b}\right) - c_1\left(\frac{a-2c_1+c_2}{3b}\right)$$

$$= \frac{(a-2c_1+c_2)^2}{9b} \quad \ldots\ldots\ldots (21.11a)$$

and $\pi_2^*(q_1^*, q_2^*) = \frac{(a+c_1-2c_2)^2}{9b}$ ..... (21.11b).

### 21.4.1.1 Comparative Statics

Suppose the firm 1 becomes more efficient. That means $c_1$ falls. Other things remaining unchanged, the intercept term $\frac{(a-c_1)}{2b}$ falls. Equilibrium output of firm 1 increases (from $q_1^*$ *to* $q_1^{**}$) and that of firm 2 falls (from $q_2^*$ *to* $q_2^{**}$). See Figure 21.2. From equation 21.8 you may see the fall in $c_1$ *will lead to fall in* equilibrium price $p^*$. The equilibrium profit of the firm 1 will increase and that of the firm to will fall (you may check $\frac{\partial \pi_1^*(q_1^*, q_2^*)}{\partial c_1} < 0$ and $\frac{\partial \pi_2^*(q_1^*, q_2^*)}{\partial c_1} > 0$). See Figure 21.3.

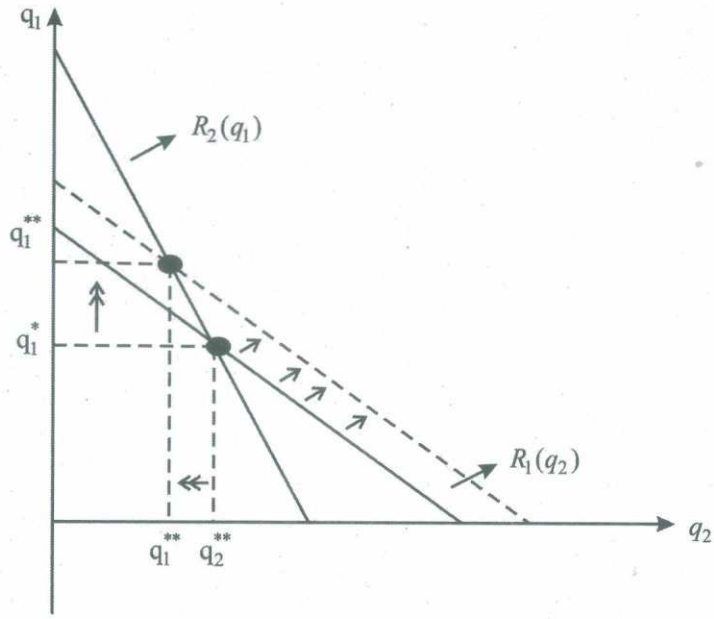


**Fig. 21.3: Comparative Static (in $C_1$) Under Cournot Competition**

### 21.4.1.2 Cournot Competition vs. Collusion

Let us consider two symmetric firms (i.e., $c_1 = c_2$) under Cournot competition. Then from the best response functions of the duopolist, $R_1(q_2)$ *and* $R_2(q_1)$ from equation 21.4 and 21.5, the equilibrium output of the firm 1 and 2 are

$$q_1^{c*} = \frac{(a-c)}{3b}, \quad q_2^{c*} = \frac{(a-c)}{3b} \quad \ldots\ldots\ldots (21.12)$$

Given the firm 2's output is 0, the firm 1's best response is to produce monopoly output, i.e., $R_1(0) = q_1^M \big|_{q_2=0} = \frac{(a-c)}{2b} = q_1^M$; similarly, $R_2(0) = q_2^M \big|_{q_1=0} = \frac{(a-c)}{2b} = q^M$, ..... (21.13)

[Note: You may check the equilibrium monopoly output under the linear demand function $p(q) = a - bq$, and constant marginal cost $C(q) = cq$. The profit function of the monopolist is $\pi^M = pq - cq = (a - bq)q - cq]$ .

FOC of the profit maximization of the monopolist: $\frac{d\pi^M}{dq} = (a - 2bq - c) = 0 \Longrightarrow q^M = \frac{(a-c)}{2b}$ is the equilibrium monopoly output. The monopoly profit will be $= \frac{(a-c)^2}{4b}$.

Suppose the two symmetric duopoly firms agree to collude together into a single entity to take output decision jointly and thereby share the monopoly profit equally. Then the colluded firms behave just like a monopolist. If the marginal costs are constant, any division of monopoly output $q^M$ (any division of profit) between the 2-firms will give industry profit equal to the monopoly profit. All possible division of monopoly output between the two duopolist is shown by the line segment $q_1^M q_2^M$. Any equal division of monopoly output and profit corresponds to point M. (See Fig 21.3).

We have seen under symmetric condition,Cournot equilibrium output (from equation 21.12)is$q_1^{c*} = q_2^{c*} = q^c = \frac{(a-c)}{3b}$. Each firm would have a profit: $\pi_1^{*C} = \pi_2^{*C} = \pi^{*C} = \frac{(a-c)^2}{9b}$.

Hence

$q^M = \frac{(a-c)}{2b} > q^c = \frac{(a-c)}{3b} \Longrightarrow$

Monopoly output is greater than Cournot output at the firm level.

Also, the monopoly profit = $\frac{(a-c)^2}{4b}$>(Cournot profit of the individual firm) of $\frac{(a-c)^2}{9b} \Longrightarrow \frac{1}{2} \cdot \frac{(a-c)^2}{4b} > \frac{(a-c)^2}{9b} \Longrightarrow \frac{1}{2}$ Monopoly profit>Cournot profit of the individual firm.

So, if both the firm under duopoly forms a collusion and produce and share the combined monopoly profit, then they will get higher profit under collusion as compared to their duopoly profit. That implies both the firms have an incentive to collude. If each of them restricts its output to half of monopoly level of output $\frac{q^M}{2}$ (instead of Cournot Nash equilibrium output) the profit of each firm will be higher. In fact, any point inside the lens of two iso-profit curves reflects Parato superior point compared to Cournot equilibrium.

Now the question is, if both the firms agree to collude, will the agreement sustainable?

Under collusion, although both the firm decided to produce $(\frac{q^M}{2}, \frac{q^M}{2})$. However,each firm (given the other firm is producing $(\frac{q^M}{2})$ has an incentive to deviate according to its best response function or the reaction function and realize even higher profit by producing $q_i^d$ (i=1, 2) instead of $\frac{q^M}{2}$, where $q_i^d = R_i(\frac{q_j^M}{2})$, i, j = 1, 2. Point M $(\frac{q^M}{2}, \frac{q^M}{2})$ is not the equilibrium as it is not profit maximizing for any of the firm. This is because it is not in the reaction function (or, the best response function) of any of the firm. The collusive agreement is not sustainable as it is not a Nash equilibrium. See Figure 21.4.

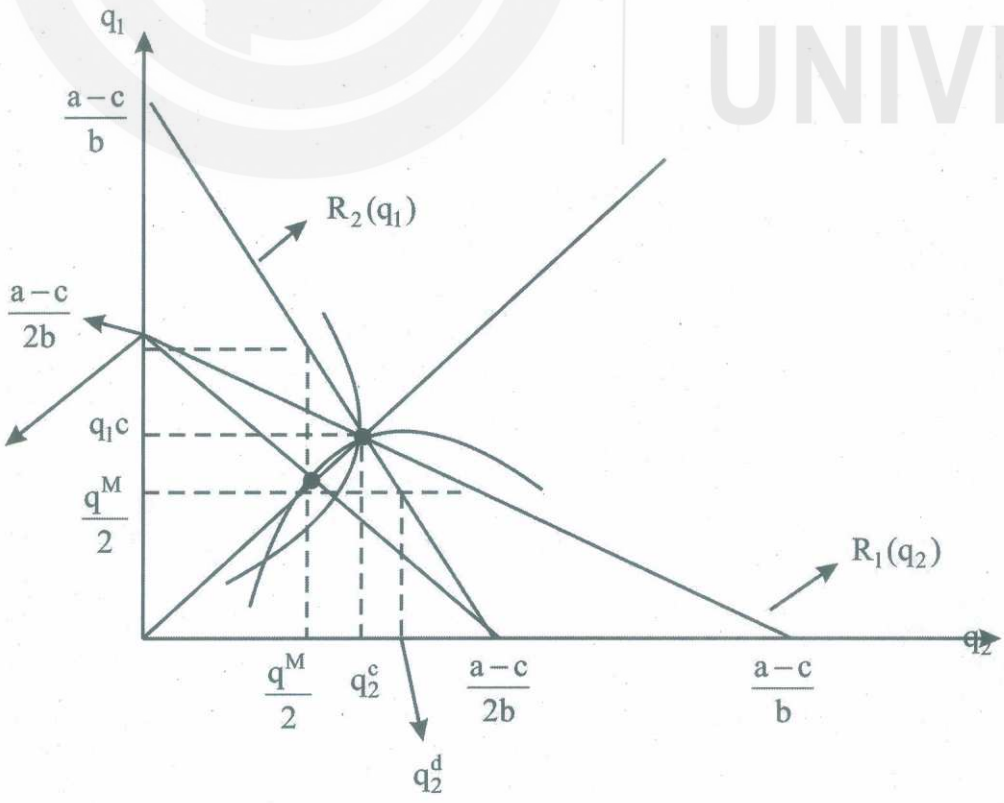


**Fig. 21.4: Cournot vs. Collusion**

Duopolists will be better off if they collude. But given the other firm producing the collusive output, each has an incentive to deviate. The firm which respects the agreement will be worse off (compared to its Cournot outcome) while the firm which ditches the partner firm by deviating will be better off. So there is a credible threat for cooperation for each of the firm because the other may ditch. As a result both the firms are worse off when each responds to its own private incentives. Hence a Cournotduopolist faces a situation of *Prisoner's Dilemma.*

### 21.4.1.3 Numerical Example

We are considering a duopoly market structure with two symmetric firms, with the following market demand function and identical cost function.

Inverse demand function:
$p(Q) = 130 - Q = 130 - 1.(q_1 + q_2), where, Q = q_1 + q_2,$

Constant marginal costs of each firm: $c = 10 \Longrightarrow TC_1 = 10q_1,\ TC_2 = 10q_2$

Profit function of the firm 1 and 2:

$\pi_1(q_1, q_2) = p(Q)\, q_1 - cq_1 = \{130 - (q_1 + q_2)\}q_1 - 10q_1;$

$\pi_2(q_1, q_2) = p(Q)\, q_2 - cq_2 = \{130 - (q_1 + q_2)\}q_2 - 10q_2.$

FOC of profit maximization of the firm 1

$\frac{\partial \pi_1(q_1, q_2)}{\partial q_1} = 130 - (q_1 + q_2) - q_1 - 10 = 0$ or $2q_1 = 120 - q_2$

Reaction function of the firm 1is $R_1(q_2) = \frac{120 - q_2}{2} = 60 - \frac{1}{2}q_2$ ..... (a)

Similarly, FOC of profit maximization of the firm 2 is

$\frac{\partial \pi_2(q_1, q_2)}{\partial q_2} = 130 - (q_1 + q_2) - q_2 - 10 = 0$or $2q_2 = 120 - q_1.$

Reaction function of the firm 2 is$R_2(q_1) = \frac{120 - q_1}{2} = 60 - \frac{1}{2}q_1$ ..... (b)

Solving equation (a) and (b), Cournot Nash equilibrium output of each firm is

$q_1^{C*} = q_2^{C*} = q^C = 40.$

Therefore, the Cournot industry level output: $Q^{C*} = q_1^{C*} + q_2^{C*} = 40 + 40 = 80$

Equilibrium Cournot Price=$p(Q^{C*}) = 130 - Q^{C*} = 130 - 80 = 50$

Cournot firm level profit of the furm1=

$\boldsymbol{\pi_1^{C*}}(q_1^{C*}, q_2^{C*}) = p(Q^{C*})q_1^{C*} - cq_1^{C*} = 40(130 - 80 - 10) = 40.40 = 1600.$

Similarly, equilibrium profit of the firm 2=$\boldsymbol{\pi_2^{C*}}(q_1^{C*}, q_2^{C*}) = 40.40 = 1600$

We get the industry level profit as 1600 + 1600 = 3200.

Now consider the two firms collude together by jointly producing the monopoly level of output and share the profit equally.

Monopoly profit (or, collusive profit) of the industry=$\pi^M = (130 - q^M)q^M - 10q^M$

Monopoly output (or, collusive output) of the industry can be obtained by maximizing $\pi^M$w.r.t.$q^M$.TheFOC of the profit maximization of the monopolist=$\frac{d\pi^M}{dq} = (130 - 2q^M - 10) = 0 \Longrightarrow 2q^M = 120 \Longrightarrow q^M = 60.$

So the collusive output of the each firm $(\frac{q^M}{2}, \frac{q^M}{2}) = (30, 30)$.

Equilibrium price: $p^M = 130 - q^M = 130 - 60 = 70$.

Equilibrium profit under collusion=$\pi^M = (130 - q^M)q^M - 10q^M = 60(70 - 10) = 60.60 = 3600$. Given the firms share the output $(\frac{q^M}{2}, \frac{q^M}{2})$, the individual firm share of the profit: $(\frac{\pi^M}{2}, \frac{\pi^M}{2}) = (1800, 1800)$. Thus the firms are better off by colluding because they get higher profit share: $(\frac{\pi^M}{2}, \frac{\pi^M}{2}) = (1800, 1800) > (\pi_1^{C*}, \pi_2^{C*}) = (1600, 1600)$. That means,each firm has an incentive to collude.

However, each firm has an incentive to deviate given the partner firm is committing to collusion by producing $\frac{q^M}{2}$.

Thus, the output of the firm 1 under defection (when firm 2 sticks to collusion by producing $\frac{q^M}{2} = 30 = R_1(q_2 = \frac{q^M}{2}) = R_1(30) = 60 - \frac{1}{2}q_2 = 60 - 15 = 45$ (from equation (a).

Similarly, the output of the firm 2 under defection (when firm 1 sticks to collusion by producing $\frac{q^M}{2} = 30 = R_2(q_1 = \frac{q^M}{2}) = R_2(30) = 60 - \frac{1}{2}q_2 = 60 - 15 = 45$ (from equation (b).

Profit of firm 1 under defection (when firm 2 sticks to collusion)= $\pi_1(q_1 = 45, q_2 = 30) \Rightarrow (130 - 45 - 30)45 - 10(45) = 45(55 - 10) = 45(45) = 2025$.

*The* profit of the firm 2 when firm 1 defects=$\pi_2(q_1 = 45, q_2 = 30)$

$= (130 - 45 - 30)30 - 10(30) = 30(55 - 10) = 45(30) = 1350.$

Since, 2025 > 1800, the firm which defects (given the partner firm commits to cooperation) has an incentive to do so, because it becomes substantially better off from the collusive outcome. However the firm which sticks to the collusive commitment is really worse off. That is, $1350 < 1800 < 1600$.The firm is not only worse off that of than the collusive outcome, it looses further as compared to the Cournot outcome. So given the other cooperate there is an incentive for defection for each firm. There is a credible threat also because if any of the firms defects the partner who is cooperating will worse off. This credible threat actually drives the firm to play Cournot strategy.

### 21.4.2 Bertrand Price Competition

Let there be two firms producing a *homogenous commodity*. Each firm chooses a price simultaneously and produces enough output to meet the demand (no *capacity constraint*), given the prices chosen by other firms ($p_i > 0$).

Payoff functions (or the profit functions) are: $\pi_1(p_1, p_2)$ , $\pi_2(p_1, p_2)$ .

Firms are symmetric in cost. Cost function=$C_i(q_i) = c.q_i$, i = 1, 2, c = Constant marginal cost.

Demand function faced by the firms=

$D_i(p_1, p_2) = \alpha - p, \ \forall p \leq \alpha$and

$D_i(p_1, p_2) = 0, \forall p > \alpha \ and \ c < \alpha.$

Rationing Rule: $D_i(p_1, p_2) = \begin{cases} \alpha - p_i, & andif\ p_i < p_j, p_i \leq \alpha \\ \frac{\alpha - p_i}{2}, & andif\ p_i = p_j, p_i \leq \alpha \\ 0 & if\ p_i > p_j, \end{cases}$

Here is the rationing rule.

1) If $p_i < c$ (for either i=1, 2), $p_j > c\ i.e., p_i < p_j$, Firm i will capture the entire market. But Firm i earns negative profit ($\pi_i < 0$). In this situation firm i can raise the profit by increasing $p_i$ till it is equal to c, i.e., $p_i = c$.

2) If $p_i = c$ and $p_j > c\ i.e., p_i < p_j$ ,then firm i is charging lower price than firm j. The consumers will be purchasing from firm i only. So the entire market will be captured by firm i and firm j will be outcompeted $i.e., q_j = 0$. Then firm j can slash the price till $p_j = c$ in order to capture the market share.

3) If $p_i = p_j$, then the market will be equally shared by the two firms.

4) If $p_i > c, p_j > c, and\ p_i > p_j$ thenthe entire market will be captured by firm j and firm i's market share will be zero. But the firm j is earning positive profit since $p_j > c$. In that case firm i can reduce the price below $p_j$ in order to capture the entire market. This price undercutting will continue till $p_i = p_j = c$.

Profit or payoff function of firm i under the rationing rule is

$$\pi_i(p_1, p_2) = \begin{cases} (\alpha - p_i)(p_i - c), andif\ p_i < p_j, p_i \leq \alpha \\ (\frac{\alpha - p_i}{2})(p_i - c), & andif\ p_i = p_j, p_i \leq \alpha \\ 0 & if\ p_i > p_j \end{cases}$$

In contrast to the Cournot game (where the profit function is continuous with respect to the strategic variable i.e., quantity), there is a discontinuity feature in the payoff functions under Bertrand price competition. The profit of a firm can sharply drops to zero for charging a very little higher price. A slight undercutting would enable it to grab 100% market share.

Nash equilibrium under Bertrand price competition (under homogenous product industry) is a set of price $(p_i^*, p_j^*)$ such that

$\pi_i(p_i^*, p_j^*) \geq \pi_i(p_1, p_j^*)$ and $\pi_j(p_i^*, p_j^*) \geq \pi_j(p_i^*, p_j)$

The Nash equilibrium under Bertrand price competition (under homogenous product industry with symmetric firms) is the pair of price strategy $(p_i^*, p_j^*)$ where, both the firms will charge competitive price and therefore make zero profit i.e., $p_i^* = p_j^* = (c, c)$.In this situation the reciprocal undercutting reduces the price to marginal cost. This is called *Bertrand Paradox.*

1) Two firms are sufficient for competitive outcome.

2) Competition between 2 firms result in complete dissipation of profits.

Consider a situation when

1) If $p_i^* > p_j^* >$ c. Firm i makes zero profit but it can do better by charging a price $p_j^* - \epsilon$, where, where $p_j^* - \epsilon > c$. It then captures the entire market and makes a positive profit $\{(p_j^* - \epsilon - c).D(p_j^* - \epsilon)\}$. It further implies firm j can retaliate by undercutting its price below $(p_j^* - \epsilon)$ and make positive profit.

2) If $p_i^* = p_j^* = p^* > c$. Both the firms are making a profit $\{\frac{1}{2}(p^* - \epsilon - c).D(p^* - \epsilon)\}$. If any of the firm charges a little lower price $(p^* - \epsilon)$ it can capture the entire market and makes a positive profit $((p^* - \epsilon - c).D(p^* - \epsilon)$, which is higher than before. But that induces the rival firm to retaliate for further price cutting.

3) If $p_i^* > p_j^* = c$. Firm j is making zero profit even through it serves the entire market. Firm i's market share is zero but can capture half the market by undercutting its price to $p_j^* = c$.

Hence under homogenous product industry with symmetric firms under price competition th reciprocal undercutting reduces the price to marginal cost (c) at the equilibrium.

Following diagram (Figure 21.5 a, b, c) is showing the firm i's profit for different price strategies of the firm i and j.

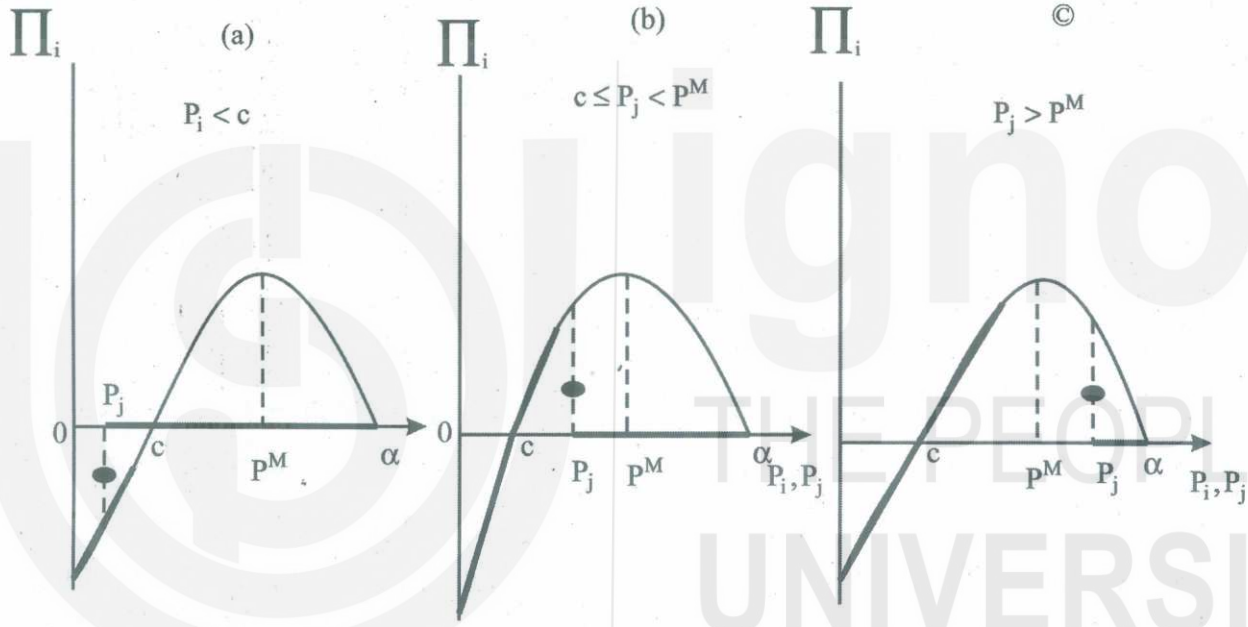


**Fig. 21.5 (a, b, c): Profit Function of the ith Firm**

Now we have to derive the best response function for the firm i and j, i.e., we have to derive those strategies which give the firm i (and firm j) maximum payoffs.

1) If $p_j < c, \Longrightarrow$ Firm i's profit is negative if $p_i \leq p_j$ and zero if $p_i > p_j$, So any price $p_i > p_j$ is the best response to $p_j . \Longrightarrow B_i(p_j) = \{p_i : p_i > p_j\}$

   Thus in Figure 21.5 **(a),** the dark segment of the firm i's profit function shows the relevant profit, when$p_j < c$. When $p_i = p_j < c$, both the form will earn negative profit, as the market will be equally shared between the two firms. So ith firm profit is a discrete point shown in the negative axis.

2) If $p_j = c$, then analysis is similar to the previous case except that $p_i \geq p_j$ yields $\pi_i = 0$ and hence $B_i(p_j) = \{p_i : p_i \geq p_j\}$

3) If $< p_j \leq p_m$,then Firm i can set price $p_i \leq p_j$ and earn positive profit. $\pi_i(p_1, p_j)$is increasing in $p_i$ till $p_i \leq p_j$ and $\pi_i(p_i, p_j) = 0$ at $p_i > p_j$. So the best response function is $B_i(p_j) = \{p_i : c < p_i \leq p_j\}$

See Figure 21.5(b), the dark segment of the firm i's profit function shows that for a given $p_j$, if the firm i charges$p_i < c$, it will capture the entire market but earn negative profit. For any $p_i : c < p_i < p_j$, firm i will earn positive profit and enjoy entire market. When $p_i = p_j, the$ market will be shared between the two firm, so the firm i's profit turned out to be a discrete point, For any price $p_i > p_j$, firm $i$ will loose the entire market share and earn zero profit.

4) If $p_j > p_m$, then charging the monopoly price $(p_m)$ is the unique best response of firm $ii.e., B_i(p_j) = p_m$.

See Figure 21.5(c), the dark segment of the firm $i$'s profit function shows that for a given $p_j > p_m$, for any price less than $p_j$, firm $i$ will enjoy the entire market share and is increasing in $p_i$. For $p_i = p_j$, firm i's profit drop to a discrete point and becomes zero for any price $p_i > p_j$.

Thus firm i's best response function is

$$B_i(p_j) = \begin{cases} \{p_i : p_i > p_j\}, andif\ p_j < c \\ \{p_i : p_i \geq p_j\}, andif\ p_j = c \\ \{p_i : c < p_i \leq p_j\},\ if\ c < p_j \leq p_m \\ \{p_m\},\quad if\ p_m < p_j \end{cases}$$

Similarly, the firm j's best response function is

$$B_j(p_i) = \begin{cases} \{p_j : p_j > p_i\}, andif\ p_i < c \\ \{p_j : p_j \geq p_j\}, andif\ p_i = c \\ \{p_j : c < p_j \leq p_i\},\ if\ c < p_i \leq p_m \\ \{p_m\},\quad if\ p_m < p_i \end{cases}$$

Thus the best response function of the Bertrand game is qualitatively different from Cournot game. Because of the discontinuity feature, for some actions of the rival $j$, firm i has no best response and for some action it has multiple best response (see Figure 21.5). The point of intersection of the two best response functions shows that $p_i^* = p_j^* = (c, c)$ as the Nash equilibrium.

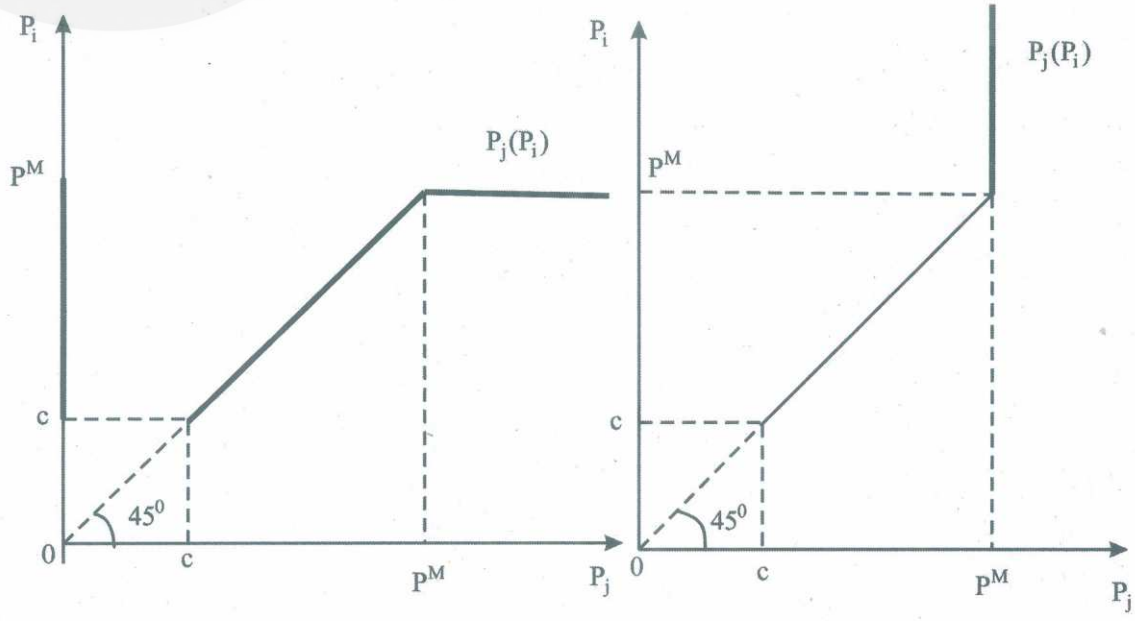


**Fig. 21.6: Best Response Functions of the Bertrand Competitors**

## Check Your Progress 3

1) What do you mean by tacit collusion?

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2) What does Bertrand paradox refer to?

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3) In a Cournot duopoly when does a Nash equilibrium occur?

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4) Consider the case of Cournot Duopoly (n = 2) when the market demand is $p = 10 - Q$ and the firms are symmetric in costs with $C(q_i) = 1.q_i, where\ i = 1,2$. Show that the Cournot equilibrium quantity $(q_i, q_i) = (3,3)$, equilibrium price $p = 4$ and profit at the firm level = 9.

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5) Consider the case of Cournot Duopoly (n = 2) when the market demand is $p = 10 - Q$ and the firms are symmetric in costs with $C(q_i) = 1.q_i, where\ i = 1,2$. Show that if both firms colludes and shares the profit equally, then the equilibrium output of each firm (under collusion)= (2.25, 2.25), Price = 5.5, Each firm's share of profit = 10.125.

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6) Show that by cheating on the Cartel (Collusion) a firm increases its own profits but necessarily lowers its rival's profit by an even larger amount.

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## 21.5 WAR OF ATTRITION

Wars of attrition often arise in business besides in other settings. The game arises when two (or more) firms compete with each other, each one losing money but hoping that the competitor will eventually give up and exit the industry. When playing the game, each firm must decide whether to cut its losses and exit, or alternatively, tough it out in the hope that the competitor will soon exit.

**A Simple Example**

For our understanding the War of Attrition game, let us consider the following simple example:

Suppose two companies, A and B, must decide each month whether to spend Rs.1000 lakh. If in the first month one company spends the Rs.1000 lakh and the other does not, the game is over – the first company becomes a monopolist worth Rs.10000 lakh, and the second company must exit from the race. If neither invests Rs.1000 lakh in the first month, the game is also over, with neither company losing nor making money. However, if both companies spend Rs.1000 lakh in the first month, neither one wins anything.

Now move to the second month, where again each company has to decide whether to spend Rs.1000 lakh. If both companies again spend Rs.1000 lakh, the game moves to the third month, and so on. If, at the start of some month, one of the companies spends Rs.1000 lakh and the other does not, the first company wins the Rs.10000 lakh prize. But of course many months (and much money) could go by before this happens.

Suppose you are Company A, and one of your classmates is Company B. What should you do in this situation? Think carefully about the following questions:

1) Is it rational to spend Rs.1000 lakh in the first round of this game? Why or why not?

   How would you decide whether or not to spend the Rs.1000 lakh?

2) Would it be rational to start playing the game with a plan to exit if, after three or four rounds, your opponent has not yet exited? Why or why not?

3) Is there anything that you could say to your opponent, or that your opponent could say to you, that would affect your decision to start playing the game? Is there anything you or your opponent could say that would affect your decision to continue playing if, after two or three rounds, neither of you has dropped out?

Again, think carefully about these questions before proceeding further.

Initial conceptualization of the gamehas come from strategies related to the conflict between two animals fighting over a prey. Each one chooses the time at which it intends to give up. Fighting involves efforts. So, it is costly. Naturally, each animal prefers the fight as short as possible. If both animals intend to give up at the same time, then each has an equal chance of obtaining the prey. Let the fighting (or waiting time) is continuous which starts at 0 and continues infinitely. Also, let there be two players $i$and $j$. Each player's set of action is the set of possible concession times. Let $v_i$ be the ith individual's value of the object of dispute ($v_i > 0$).

1) If the player $i$ concedes first at the time$t_i$, his payoff is $(-t_i)$, as he spent $t_i$ units of time and does not obtain the object.
2) On the other hand, if the other player concedes first at the time$t_j$, player $i$ gets the object. So ith player's payoff: $(v_i - t_j)$.
3) If both the players concede at the same time $t_i = t_j then$player $i$'s payoff $= \frac{1}{2}(v_i - t_i)$.

   Thus the payoffs of the ith player:

$$u_i(t_i, t_j) = \begin{cases} -t_i, & and if \ -t_i < t_j \\ \frac{1}{2}(v_i - t_i), & and \ if \ t_i = t_j \\ (v_i - t_j), & if \ t_i > t_j \end{cases}$$

For Nash equilibrium we need to derive the best response functions.

Intution: If $t_j$is small, i.e., player $j$'s intended concession time is early enough then it is optimal for player $i$ to wait for the player $j$ to concede. In that case player $i$'s optimal concession time will be later than $t_j$.

If $t_j$is large, i.e., player $j$'s intended concession time is long then it is optimal for player $i$ to concede immediately (if $t_j > v_i$).

Following diagram (21.6 a, b, c) is shows the player $i$'s payoff in three cases:

1. $t_j < v_i$; 2. $t_j = v_i$;3. $t_j > v_i$.

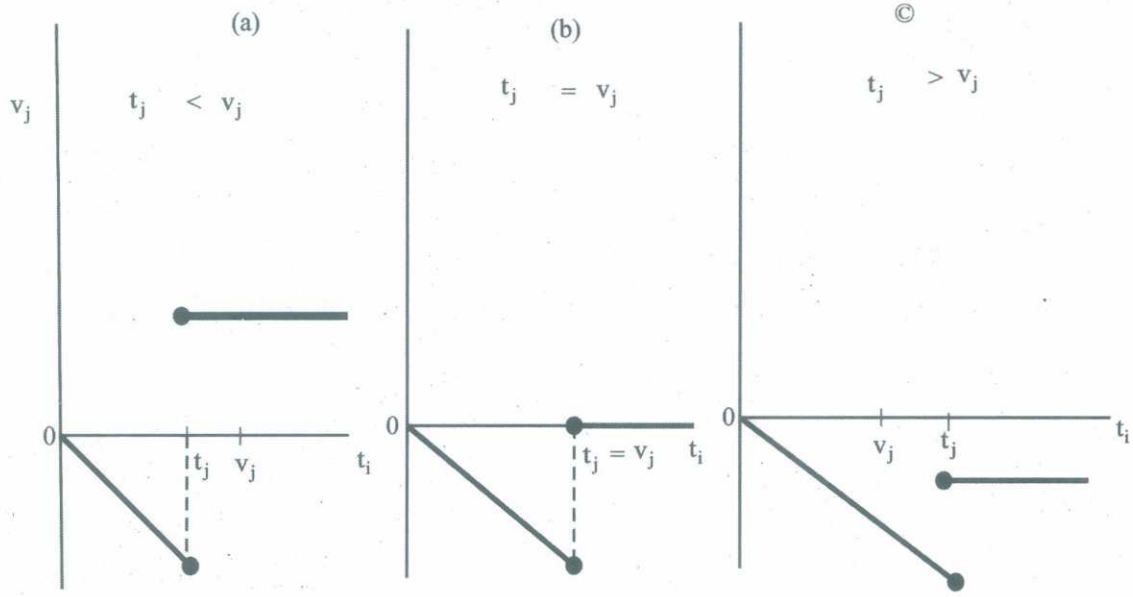


**Fig. 21.7 (a, b, c): Player $i$'s Payoff Function**

Best response function of the player i (for $v_i > v_j$):

$$B_i(t_j) = \begin{cases} \{t_i : t_i > t_j\}, & and if\ t_j < v_i \\ \{t_i : t_i = 0\ or\ t_i > t_j\}, & and\ if\ t_j = v_i \\ 0, & if\ t_j > v_i \end{cases}$$

Best response function of the player j (for $v_i > v_j$):

$$B_j(t_i) = \begin{cases} \{t_j : t_j > t_i\}, & and\ if\ t_i < v_j \\ \{t_j : t_j = 0\ or\ t_i > t_j\}, & and\ if\ t_i = v_j \\ 0, & if\ t_i > v_j \end{cases}$$

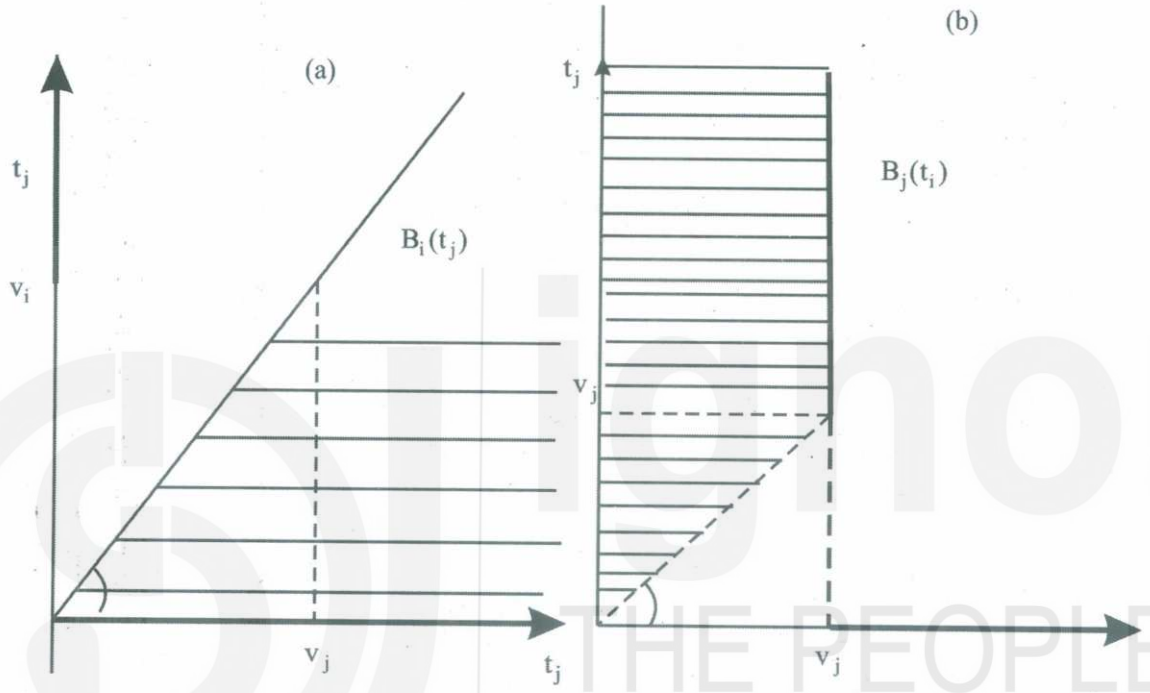


**Fig. 21.8 (a, b) : Player $i$ and $j$'s Best Response Functions ($B_i(t_j), B_j(t_i)$).**

Superimposing the player $i$ and $j$'s best response functions, we see there are two areas of intersection: 1. Vertical axis $\geq v_i$ 2. Horizontal axis at and to the right of $v_j$.

Nash equilibrium is $(t_i^*, t_j^*)$: 1. $t_i = 0, t_j \geq v_i$,

2. $t_j = 0,\ t_i \geq v_j$

*That is,* player $i$ concedes immediately and player $j$ concedes at time $v_i$ or later *and* Player $j$ concedes immediately and player $i$ concedes at time $v_j$ or later.

### Check Your Progress 4

1) Discuss the similarities between chicken game and war of attrition game.

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2) What is the easiest way of finding the Nah equilibrium in the war of attrition game?

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3) Write the properties of War of Attrition.

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## 21.6 HOTELLING'S LOCATION GAME

Hotelling's linear city model has introduced the notions of locational equilibrium in a *duopoly* in which two firms have to choose their location taking into consideration consumers' distribution and transportation costs.

Initially the model was developed as a *game* in which firms first chose a location and after a selling price for their products. In order to set their business in the best location to maximize profits, the firms will have to evaluate three key variables: competitors' location, customers' distribution and transportation costs. We can consider static model where firms choose their location and prices simultaneously.

The basic idea is that two ice cream vendors are on a beach that stretches the 0-1 interval. Customers are uniformly distributed along that interval. The vendors simultaneously select a position. Customers go to the closest vendor and split themselves evenly if the vendors choose an identical position. Each vendor wants to maximize its number of customers.

The Nash equilibrium is for both vendors to select the median location (0.5); doing this guarantees each vendor half the business, but deviating to any other point generates strictly less.

We reframe the question as two candidates placing themselves along an ideological spectrum, with citizens voting for whichever one is closest. This is seen in the following:

Consider the case of electoral competition where there are two political parties 1 and 2, viz., (n = 2) contesting for votes. Suppose there exists a linear city [0, 1], where voters are uniformly distributed over the interval [0,1] with a unit density i.e., each points represents one voter. It is assumed that the political parties have no political conviction but wants to maximize their vote share by strategically locating themselves across the linear city. On the other hand, voters have the political conviction and therefore the voters try to locate themselves nearer to their political ideology. What would be the Nash equilibrium voting configuration?

Case 1: $x_1 < x_2 \neq \frac{1}{2}$

**Fig. 21.9 (a)**

Party 1 and 2 are located at a distance $x_1$ and $x_2$, $(x_1 < x_2 \neq \frac{1}{2})$ distance from the origin (0 at the extreme left). All the voters to the left of $x_1$ *will* will all vote to Party 1. However the voters to the right of $x_1$, $(x_1 < x_2)$ will be splitted between Party 1 and 2. Thus, the vote share of Party 1: $x_1 + \frac{1}{2}(x_2 - x_1) = \frac{1}{2}(x_1 + x_2)$. In order to understand whether this configuration is a Nash equilibrium, we have to justify whether it is a mutually best response strategy.

Suppose the Party 1 deviates from $x_1$ distance to the right of $x_1$, to say, $x_1'$, where $x_1' = x_1 + \varepsilon$, where $\varepsilon > 0$.

Now the vote share of the Party 1 is: $x_1' + \frac{1}{2}(x_2 - x_1') = \frac{1}{2}(x_1' + x_2) = \frac{1}{2}(x_1 + \varepsilon + x_2) = \frac{1}{2}(x_1 + x_2) + \frac{1}{2}\varepsilon$. So the Party 1 gains by relocating itself to the right. So the Party 1 has an incentive to deviate. Hence this is not the Nash equilibrium strategy.

Case 2: $x_1 = x_2 \neq \frac{1}{2}$

**Figure 21.9 (b)**

In this configuration both the parties are located together but not in the centre. Suppose P is the point where both are located. Each party will have equal vote share, i.e., $\frac{1}{2}x_1 + \frac{1}{2}(1 - x_1) = \frac{1}{2}$.

Now if any party gains from deviation we can claim that this is not a Nash equilibrium. Suppose the Party 1 deviates from $x_1$ distance to the right of $x_1$, to say $x_1'$, where $x_1' = x_1 + \varepsilon$, where $\varepsilon > 0$. So the Party 1's vote share $= (1 - x_1') + \frac{1}{2}(x_1' - x_1)$.

Let the initial distance of both the parties was less than the centre of the linear city *i.e.*, $(1 - x_1') > \frac{1}{2} \Rightarrow (1 - x_1') + \frac{1}{2}(x_1' - x_1) > \frac{1}{2}$. So, if any of the parties deviates (to the right in this example) there are gains from deviation. Such a configuration is also not the Nash equilibrium.

Case 3: $x_1 = x_2 = \frac{1}{2}$

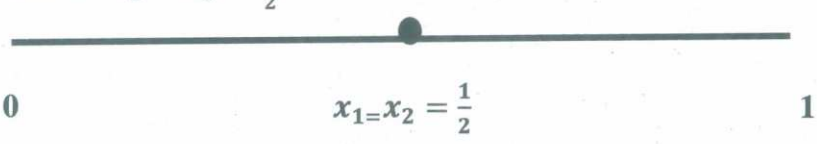


**Fig. 21.9 (c)**

Here, each of the Party is getting exactly half vote share.

We will see that as any of the parties deviates (to its left or right), it looses some vote share. So deviation by either party won't be gainful.

Suppose the Party 1 deviates to its left at $x_1'$, where $x_1' = x_1 - \varepsilon$.

So the Party 1's vote share after deviation: $x_1' + \frac{1}{2}(x_1 - x_1') = [(x_1 - \varepsilon) + \frac{1}{2}(x_1 - x_1 + \varepsilon)] = [(x_1 - \varepsilon) + \frac{1}{2}\varepsilon] = (x_1 - \frac{1}{2}\varepsilon) < \frac{1}{2}$. See that the deviation by either party is not gainful. Hence $x_1 = x_2 = \frac{1}{2}$ is the Nash equilibrium. So both the parties would prefer to locate themselves at the centre in order to maximize their vote share.

**Check Your Progress 5**

1) Suppose consumers are located uniformly along the line segment from 0 to 1. If firm A and B are located somewhere within the line segment, discuss the Nash equilibrium you will get.

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2) If two candidates choose their positions before the election in the Hotelling's linear city formulation, why would they tend to run to the centre as the election progresses?

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3) In linear city model of Hotelling two firms share — transport costs with their customers.

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## 21.7 LET US SUM UP

In this unit we extend some of the basic ideas in game theory, particularly, static games with complete information. We begin by discussing problems encountered in common property resources and brought in prisoners' dilemma game, assurance game and chicken game to find solutions. Then we will turn to models of duopolistic competition. We first consider the choice of strategic variable when duopolists compete in the sale of homogenous products — in particular, implications of choosing quantities instead of prices. The War of Attrition game is seen to draw inferences on what it means to play this game rationally.

Towards the concluding part of the unit we have included Hotelling's linear city model to examine the locational equilibrium in a *duopoly* in which two firms have to choose their location taking into consideration consumers' distribution and transportation costs. A basic conclusion of the model is that of firms choosing to be in the extremes i.e., they choose maximum differentiation.

## 21.8 KEY WORDS

**Assurance Game:** A situation where there is a conflict between safety and cooperation.

**Capacity Constraint:** A firm is unable to increase its output production due to the increase in demand at least in the short run. The supply of the firm remains insensitive to price.

**Cartel:** An association of manufacturers or suppliers with the purpose of maintaining prices at a high level and restricting competition.

**Chicken Game** (or, Hawk-dove game, or snowdrift game): A situation where each person prefers not to yield to the other, but the outcome where neither person yields is the worst for both.

**Common Property Resource:** Natural resources owned and managed collectively by a community or society rather than by individuals.

**Coordination Game**: A situation where two persons ought to choose the same strategy.

**Homogenous Product:** Product that cannot be distinguished from competing products from different suppliers.

**Monopoly:** Market structure characterized by a single seller, selling a unique product in the market.

**Strategic Complement:** Decisions of two or more players are called strategic complements if they mutually reinforce one another.

**Strategic Substitute:** Decisions of two or more players mutually offsetting one another.

## 21.9 SOME USEFUL BOOKS

1) Avinash K. Dixit(2015), Susan Skeathand David H. Reiley, *Games of Strategy*, W. W. Norton and Company, 4th edition.

2) Drew Fudenberg and Jean Tirole(1991), *Game Theory*, MIT Press.

3) Drew Fudenberg and David Levine (1998), *The Theory of Learning in Games*, MIT Press,

4) Robert Gibbons (1992), *A Primer in Game Theory*, Harvester Wheatsheaf.

5) Martin J. Osborne and Ariel Rubinstein (1994), *A Course in Game Theory*, MIT Press.

6) Michael Maschler, Eilon Solan, and Shmuel Zamir (1992), *Game Theory*, Cambridge University Press.

## 21.10 CHECK YOUR PROGRESS ANSWERS/HINTS

### Check Your Progress 1

1) A common resource is a resource, such as water or pasture, that provides users with tangible benefits. A major concern with common resources is overuse, especially when there are poor social-management systems in place to protect the core resource. Common resources that are not owned by anyone are called open-access resources.

2) Most resources are congestible (rivalrous). This means that if some of the resource is used for one purpose the same chunk cannot be used for another purpose: one use precludes another use.Consider congestible resources that have a zero price, i.e., everyone can take as much of the resource as is available without having to pay for it. Such resources are called common-property resources.

3) Each of the villagers faces a credible threat that if he participates, his rivals may defects and push him to zero payoff.

4) A solution that people tend to use because it seems natural, special, or relevant to them in the absence of communication.

5) Coordination games are linked to the economic concept of positive externalities. That is, benefit reaped from being in the adopting the same strategy as that of other agents. However, under negative externalities choosing the same action creates a cost rather than a benefit. This class of game, Chicken game falls in that, is anti-coordination game.

### Check Your Progress 2

1) Strategy that produces the most favourable outcome for a player, taking other players' strategies as given.

2) Decisions of two or more players are called **strategic complements** if they mutually reinforce one another, and they are called **strategic substitutes** if they mutually offset one another.

3) If two individuals devote more effort, both are better off.

### Check Your Progress 3

1) No formal agreements on market sharing.

2) The possibility that all that is necessary for prices to be the same as in competitive markets is for there to be two firms competing in a market.

3) When the firms' reaction curves intersect.

4) Do yourself as per the clue included in the question.

5) Do yourself as per the clue included in the question.

6) Do yourself as per the clue included in the question.

### Check Your Progress 4

1) The War of Attrition is a game something like Chicken stretched out over time, where both players start with Continue, and the game ends when the first one picks Swerve. Until the game ends, both earn a negative amount per period, and when one exits, he earns zero and the other player earns a reward for outlasting him.

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2) By getting the best response functions and plotting them.

3) Wars once started can last for a long time with no definite end; even winner may lose money; random termination; reduce cost of fighting induces lengthening of the war.

**Check Your Progress 5**

1) A variety of Nash equilibria are possible depending on the assumption about the cost of travel for consumers.

2) Due to the fact that the candidates locate in the centre in the Nash equilibrium of the game.

3) Linear

## 21.11 EXERCISES

1) Write a short note on the CPR management dilemma.

2) Derive the Nash equilibrium under the Cournot competition in the case of 3 symmetric firms with constant marginal costs (c). The market demand function is $p(Q) = a - bQ = a - b(q_1 + q_2 + q_3)$. Derive the equilibrium price and profit at the firm level as well as industry level.

3) Consider a Bertrand duopoly market structure. Define Bertrand Paradox.

4) Derive the best response function of the War of Attrition game. Show graphically and explain the Nash equilibrium.

5) Discuss the Nash equilibrium under Hotelling's location game in the context of electoral competition with two political parties.

6) Consider an example of the Tragedy of Commons. Show how the phenomena of over fishing can be represented as a Prisoners' Dilemma. You may set up the game with two players, each of which can undertake low or high fishing activity.

**Solution:**

The case of over fishing should be set up in a manner similar to this:

| | | India | |
|---|---|---|---|
| U.S | | High fishing activity | Low fishing activity |
| | High fishing activity | 1,1 | 3,0 |
| | Low fishing activity | 0,3 | 2,2 |

The sustainable fishing catch is higher when both nations undertake low fishing activity.

However, there is then an incentive for both to increase fishing. In fact, high fishing is a

dominant strategy for both players. We therefore, end up with the worst outcome.

7) There is competition that conveys the following information. Prize worth Rs.1, cost $c$ to fight per period. Each player chooses probability p of exiting per period. If both exit, there is a split of the prize. Best response requires $½p = p - c + (1 - p)\,½p$. Value of exit = value of staying in.

Solve the game to get how long the the war would last, expected profits for competitors.

**Solution:**

Get $= 1 - \sqrt{1-2c}$. Thus, $c = (1/2)c$. That means for $c = 0.1$, war will last five years on average.

Expected profits are½ $p$. As profits are increasingin c, the effect on competitor outweighs own effect.

Market demand is given by $p(Q) = \begin{cases} 140 - Q & if\ Q < 140 \\ 0 & ot'\ rwise \end{cases}$

8) There are two firms, each with unit costs = Rs.20. Firms can choose any quantity.

i) Define the reaction functions of the firms;

ii) Find the Cournot equilibrium;

iii) Compare the Cournot equilibrium to the perfectly competitive outcome and to the monopoly outcome.

iv) One possible strategy for each firm is to produce half of the monopolist quantity. Would the resulting outcome be better for both firms (Pareto dominant)? Explain why this is not the equilibrium outcome of the Cournot game.

**Solution:**

i) In a Cournot duopoly the reaction function of Firm A identifies its optimal response to any quantity produced by Firm B. In the presence of private firms, the optimal quantity is the one that maximizes $\Pi_A$, Firm A's profit, where

$$\Pi_A = P(Q)q_A - cq_A$$
$$= (140 - q_A - q_B)q_A - 20q_A$$

The first order condition for profit maximization is:

$$\frac{\partial \Pi_A}{\partial q_A} = 140 - 2q_A - q_B - 20 = 0$$

$$q_A(q_B) = \frac{120 - q_B}{2}$$

Since the game is symmetric (firms have identical cost functions), the reaction function of firm B is:

$$\hat{q}_B(q_A) = \frac{120 - q_A}{2}$$

ii) Cournot equilibrium is identified by the quantities that are mutually best responses for both firms; so, they are obtained by the solution of the following two-equation system:

$$\begin{cases} q_A = \dfrac{120 - q_B}{2} \\ q_B = \dfrac{120 - q_A}{2} \end{cases}$$

The equilibrium quantities are

$q_A^* = q_B^* = 40$

$Q^* = 80$and the equilibrium price is

$P(Q^*) = 140 - 80 = 60$

and firms' profits are:

$\Pi_B^* = \Pi_A^* = 60q_A^* - 20q_A^* = 1600$

iii) The competitive equilibrium outcome is characterized by P(Q) = c = 20. So total quantity should be:

$$P(Q) = 140 - Q = 20, \quad or, Q = 120$$

In such a case firms' profits are zero.

The quantity produced by a monopoly is obtained by the usual first order condition:

The price under monopoly is $P(q_M) = 140 - 60 = 80$ and profits are

$$\Pi_M = 80 \times 60 - 20 \times 60 = 3600$$

So, profits under monopoly are higher than the sum of firms' profits under Cournot competition; i.e., $\Pi_M > \Pi_A > \Pi_B$

iv) If the each firm agreed to produce half of the monopolist quantity $\frac{q_M}{2} = 40$, their profits would be $\frac{\Pi_M}{2} = 1800$, larger than the Cournot profits. So, a Pareto improvement with respect to Cournot equilibrium would be possible. However, this cannot be an equilibrium since firms' strategies are not mutually consistent best response; that is

$$\hat{q}_i\left(\frac{q_M}{2}\right) \neq \frac{q_M}{2} \; for \; all \; i = A, B$$

9) See the following problem:

Market demand is given by

$$P(Q) = \begin{cases} 100 - Q & \text{if } Q < 100 \\ 0 & \text{otherwise} \end{cases}$$

Suppose that two firms both have average variable cost c = \$50. Assuming that firms complete in prices, then:

1) Define the reaction functions of the firms;
2) Find the Bertrand equilibrium;
3) Would your answer change if there were three firms? Why?

**Solution**

1) The construction of the reaction function in a competition a la Bertrand proceeds in the following manner.

Since firms compete in prices, we need to use the direct demand function where Q is a function of P. From the inverse demand:

$$Q(P) = \begin{cases} 100 - P & \text{if } P < 100 \\ 0 & \text{otherwise} \end{cases}$$

There are two firms, firm 1 and 2. Consider now the effect of the price choice of any firm I on its own profits for any given price chosen by firm j with i, j = 1, 2 and $i \neq j$

a) $P_i > P_j : \Pi_i = 0$

In this case, firm 1 sells no output and therefore gets zero profits.

b) $P_i > P_j : \Pi_i = \frac{1}{2}\ [(P_i - 50)\ (100 - P_j)]$

This is where firm i shares the total profits in the industry. The total profits of the industry are here determined in the following way. Price minus average cost gives the profit per unit of output $(P_i - 50)$ and this is multiplied by the total output $(100 - P_j)$.

c) $P_i < P_j : \Pi_i = [(P_i - 50)\ (100 - P_j)]$

Here firm i gets the whole profits of the industry.

In order to firm the reaction function of firm I we can distinguish 3 different cases.

a) $P_i\ (P_j) = 50 \quad \text{if } P_j \leq 50$

In such a case, any price $P_i < P_j$ will make negative profits, so firm i will prefer to lose the race rather than beating j on price. More precisely we can say that firm i will never set a price below c = 50. This is usually portrayed as the strategy that if $P_j < 50$, $P_i = 50$.

b) $P_i\ (P_j) = P_j - \varepsilon \quad \text{if } 50 < P_j \leq 75 = P^M$

Where $P^M$ is the monopolistic price. If $P_j > 50$, there is potential for positive profits, as long as $P_j < 100$, at which point quantity demanded falls to zero, so that profits would be zero too. The first issue is, should firm i match the price of firm j, or attempt to undereut its rival? Intuition suggests that the firm should undercut its rival rather than match $P_j$. This can be shown formally by comparing the profits for firm i if it matches $P_j$, $\Pi_i^A$, with the profits, $\Pi_i^B$, it gets if undercuts $P_j$ by a small amount $\varepsilon$ so that $P_i = P_j - \varepsilon$.

$$\Pi_i^A = \frac{1}{2}\left[\left(P_j - 50\right)\left(100 - P_j\right)\right]$$

$$\Pi_i^B = \left[\left(P_j - \varepsilon - 500\right)\left(100 - P_j + \varepsilon\right)\right]$$

$$\Pi_i^B = 2\Pi_i^A - \varepsilon\left(150 - 2P_j - 2\varepsilon\right)$$

This means:

As $\varepsilon \to 0$, that is as the $P_i$, gets closer and closer to $P_{ij}$, $\Pi_i^A \to 2\Pi_i^A$: if firm i sets its price just below $P_{ij}$, its profit will be roughly double the profit from matching $P_j$. However, the fact that firm i will do better by just undercutting firm j than it does by matching firm j does not mean that just undercutting is always the best strategy. When the firm has the whole market, its optimal price is the monopoly price. Therefore, when $P_{ij}$ is greater than the monopoly price, firm j should set the monopoly price. In this example, the monopoly price $P^M = 75$. Following this discussion:

c) $P_i\ (P_j) = 75 \quad \text{if } P_j > P^M$

The reaction function of any firm is then the following:

$$P_i(P_j) = \begin{cases} 50 & \text{if } P_j \leq 50 \\ P_j - \varepsilon & \text{if } 50 < P_j \leq 75 \\ 75 & \text{if } P_j > 75 \end{cases}$$

2) The Nash equilibrium is where the two reaction functions interest, which is where $P_1 = P_2 = 50$. In such a case the best response of the two firms are mutually consistent. This is actually the only point in which this is so. (Please check this result by your own Can you find another Nash equilibrium?). This is the same as the competitive solution. Price is set equal to average (= marginal) cost and zero profits are made.

3) No because an increase in competition does not change the price set by the firms that cannot be lower than marginal cost. This is the essence of the so-called Bertrand paradox: two firms are enough to achieve the competitive outcome.

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# UNIT 22 DYNAMIC GAMES WITH COMPLETE INFORMATION

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## 22.0 OBJECTIVES

After going through this unit, you will be able to:

- understand the idea behind dynamic games;
- present the dynamic games in extensive form keeping in view complete, perfect and imperfect information;
- use backward induction as a solution concept to arrive at subgame perfect Nash equilibrium; and
- analyse repeated games.

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## 22.1 INTRODUCTION

We have seen in Unit 20 the presentation of a normal form game. In the discussion there players choose their strategies simultaneously, without knowing the choices of the other players. So these are static games. In this unit we take up dynamic games by allowing the players have the choice moving sequentially. That is, players make moves at different times or in turn. So, one player chooses her action before the others choose theirs. That means players who move later in the game have additional information about the actions of other players.

## 22.2 EXTENSIVE-FORM REPRESENTATION OF DYNAMIC GAMES

In the extensive form representation of the game we get the following features:

- Decision points: The dots, or nodes, represent a decision that must be made by some player. The name of the player is usually placed above or next to the dot.
- Actions: The lines, or branches, represent different actions that a player can take at each decision point. The label is the name of the action, like "enter" or "exit", etc.
- Payoffs: The numbers at the bottom of the tree represent the payoffs to the players. The convention is that the payoffs are given in order of the timing at which that player moved first. So if Player 1 moves first, Player 2 moves second, Player 3 moves third, and so on, the payoffs are given in the order: Player 1, Player 2, Player 3...

Such a method depicts the tree form of the game. Thus, game trees are made up of nodes and branches, which are used to represent the sequence of moves and the available actions, respectively. To appreciate the idea, consider two players, Black Smith and White Elena, who are playing a sequential game. Black moves first and has the option of Up or Down. White then observes his action. Regardless of what Black chooses, she then has the option of High or Low. The tree form of the game is follows:

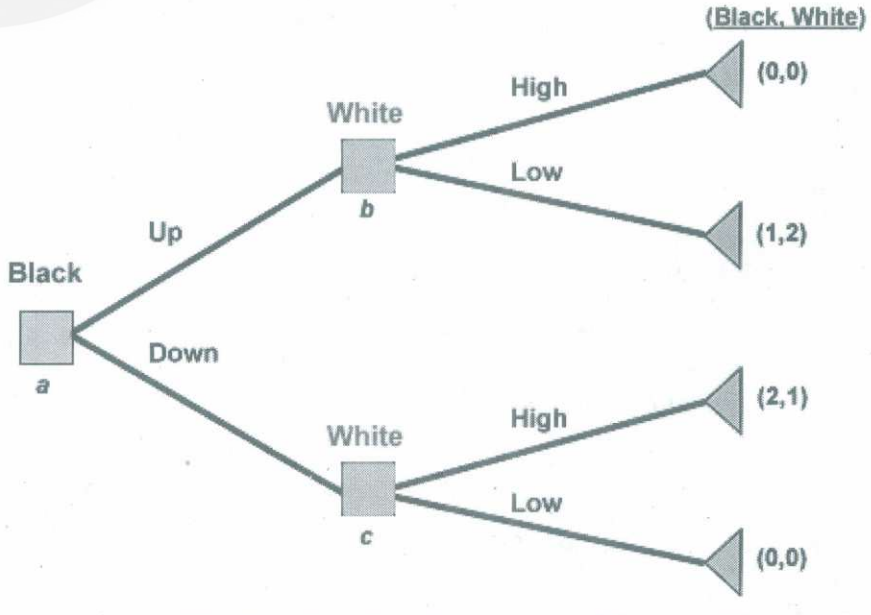


**Fig. 22.1: Tree Form Presentation of Sequential Games**

In the game tree (Figure 22.1), decisions of the players are depicted by square nodes. Note that node $a$ is the decision node where player Black chooses between Up and Down. Since node $a$ is the first node, it is also known as the initial node. Nodes $b$ and $c$ are the decision nodes at which Player White chooses between High and Low. The triangle-shaped ending nodes on the right are the terminal nodes, which also have the payoffs for each player associated with each outcome listed beside them.

### 22.2.1 Strategies in Extensive-Form

In sequential games, it is important to clearly define what is meant by strategy. Game theorists define a strategy as a complete contingent plan of actions. In other words, a strategy specifies what action a player will take at each decision node. Consider the game given above between players Black and White.

Player Black has two strategies available — Up and Down. Player White, however, actually has four strategies available since there are two nodes to consider — $b$ and $c$— and two possible actions at each node — High and Low. Thus, strategies available to player White summarised in Table 22.1.

**Table 22.1: Strategies of the Player**

| | **Strategies of Player Black** | | |
|---|---|---|---|
| | | **Up** | **Down** |
| **Strategies of Player White** | Strategy 1 | High | High |
| | Strategy 2 | High | Low |
| | Strategy 3 | Low | High |
| | Strategy 4 | Low | Low |

Because actions always lead to reactions, an important aspect of strategy in sequential games is that players must consider — and plan for — their opponent's reactions. In the example above, if player Black wants to maximise his payoff, he must consider how his rival player White will react if he moves Up and how she will react if he moves Down.

**Check Your Progress 1**

1) Differentiate between static and dynamic game.

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2) What is extensive form representation of a game?

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3) What is meant by strategy in an extensive form representation of a game?

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## 22.3 DYNAMIC GAMES OF COMPLETE AND PERFECT INFORMATION

We have seen above that in perfect information games, the history is common knowledge. However, there is a class of games where players move without necessarily knowing the past. Such games are categorized as imperfect information games.

In the present unit our analysis of dynamic games is restricted to complete information by which we mean that the term is used to describe a game in which knowledge about other players is available to all participants. We also discuss the class of games in which the player with the move knows the full history of the play of the game thus far. Therefore, we call these games with perfect information. For example, Chess is a game with perfect information as each player sees all of the pieces on the board at all times.

Note that perfect information is different from complete information. In complete information game, the structure of the game and the payoff functions of the players are commonly known. In games of perfect information, on the other hand, every player observes other players' moves, but may lack some information on others' payoffs, or on the structure of the game. Thus, a game with perfect information may or may not have complete information. However, when a game is described as having perfect information, complete information is often assumed, unless specified otherwise.

### 22.3.1 Backward Induction

Extensive games with perfect information can be analyzed by backward induction. A backward induction is the process of reasoning backwards in time, from the end of a problem or situation, to determine a sequence of optimal actions. It proceeds by first considering the last time a decision might be made and choosing what to do in any situation at that time.

To understand the process of applying backward induction consider the quality choice game where consumers' choice of buying or not depends on the quality of services (high or low) (see,Figure 22.2).The service provider, player I, makes the first move, choosing High or Low quality of service. Then the customer, player II, is informed about that choice. Player II can then decide separately between buy and don't buy in each case. The resulting payoffs are also given in the figure.

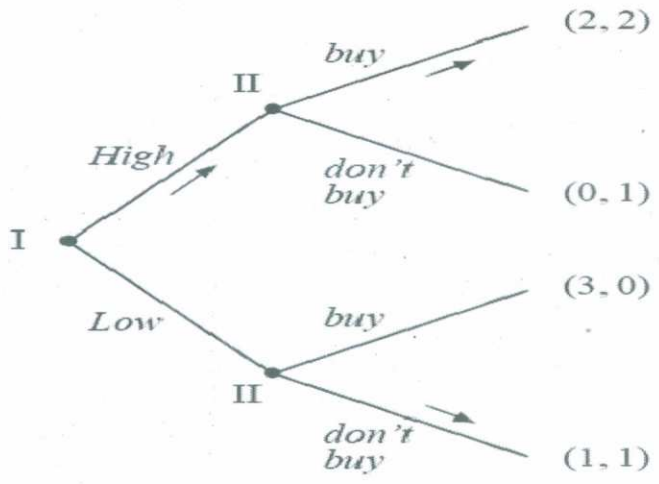



**Fig. 22.2: Quality Choice Game**

Here, player II moves last. Since she knows the play will end after her move, she can safely select the action which is best for her. If player I has chosen to provide high quality service, then the customer prefers to buy, since her resulting payoff of 2 is larger than 1 when not buying. If the provider has chosen Low, then the customer prefers not to purchase. The choices available to player II are indicated by arrows in figure.

When the last move is decided, backward induction proceeds to the player making the next-to-last moves (and the moves continue). In Figure 22.2, player I makes the next-to-last move, which in this case is the first move in the game. Being rational, he anticipates the subsequent choices by the customer. He therefore realizes that his decision between High and Low is effectively between the outcomes with payoffs(2,2) or (1,1). Clearly, his choice for High results in a payoff of 2 for him. In case of a selection of Low, the outcome is payoff 1. So the unique solution to the game, as determined by backward induction, is that player I offers high quality service, and player II responds by buying the service.

See that the concept of backwards induction corresponds to the assumption that it is a common knowledge that each player will act rationally at each node where she moves — even if her rationality would imply that such a node will not be reached. More precisely at each node *i* the player is certain that all the players will act rationally at all nodes j that follow node*i*; and at each node *i* the player is certain that at each node *j* that follows node *i* the player who moves at j will be certain that all the players will act rationally at all no desk that follow node j,...ad infinitum.

### 22.3.2 Strategies in Dynamic Games with Complete Information

In an extensive game with perfect information, backward induction usually yields unique choices at the players' decision nodes. However, there is an exception to it. For, when a player is indifferent between two or more moves at a node the selection of the option is blurred.

The solution through backward induction specifies the way the game will be played. Starting from the root of the tree, play proceeds along a path to an outcome. Note that the analysis yields more than the choices along the path. Because backward induction looks at every node in the tree, it specifies for

every player a complete plan of what to do at every point in the game where the player can make a move. Note that a player's plan includes a node even when that point may never arise in the course of play. Such a plan is called a strategy of the player. For example, a strategy of player II in Figure 22.2 is "buy if offered high-quality service, don't buy if offered low quality service." Such a strategy for player II's is obtained by backward induction.

With strategies defined as complete plan of move by the player, one can obtain the strategic form representation of the extensive games. Recall that the strategic form game tabulates all strategies of the players. In the game tree, any strategy combination results into an outcome of the game, which can be determined by tracing out the path of play arising from the players adopting the strategy combination. The payoffs to the players are then entered into the corresponding cell in the strategic form. Figure 22.3 shows the strategic form for quality choice example. The second column is player II's backward induction strategy, where "buy if offered high-quality service, don't buy if offered low-quality service" is abbreviated as H: buy, L: don't.

| | H:buy; L:buy | H:buy; L: don't | H: don't; L:buy | H: don't;L: don't |
|---|---|---|---|---|
| High | 2,2 | 2,2 | 1,0 | 1,0 |
| Low | 0,3 | 1,1 | 0,3 | 1,1 |

**Fig. 22.3: Strategic Form of the Extensive Game in Fig. 22.2.**

A game tree can therefore be analyzed in terms of the strategic form. It is not difficult to see that backward induction defines a Nash equilibrium. In Figure 22.3, it is the strategy combination (High; H: buy, L: don't).

The sequential games converted into normal form typically contain redundancies. Figure 22.3, for example, has eight cells, but the game tree in Figure 22.2 has only four outcomes. Every outcome appears twice, because there are two strategies of player II which differ only in the move that is not reached after the move made by player I. All the combinations of moves by player II must be distinguished as strategies. Otherwise, any two of them may produce to different outcomes, depending on the action of player I.

See that in Figure 22.3, cell with (Low; H: don't, L: don't) is an equilibrium. Here the customer never buys. So Low is the best response of the service provider such an expected behaviour of player II. Although H: don't is not an optimal choice, player II has no occasion to make that move. Therefore, player II is not better off by changing her strategy. Thus, while we have an equilibrium, it a suboptimal move. We require additional tool of subgame for reaching a perfect equilibrium.

### 22.3.3 Subgames

A subgame is a subset of any game that includes an initial node. To understand subgames use the extensive form representation. In the following game tree (Figure 22.4) there are six subgames other than the game itself, two of them containing two subgames each.

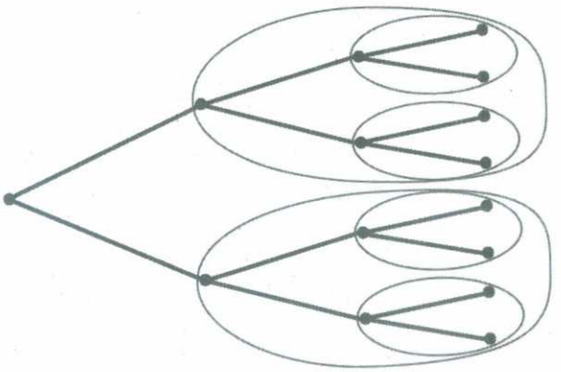


**Fig. 22.4: Subgames in a Game Tree**

To understand the concept of subgame, it will be useful remember the following points:

- Any game is a subgame of itself. Subgames other than the original game itself are called proper subgames.
- For games of perfect information, every node other than a terminal node defines a subgame.
- Any subgame is a game in its own right, satisfying all of our rules for game trees.
- A strategy for the original game also defines a strategy for each of its subgames, sometimes called a continuation strategy

### 22.3.4 Subgame Perfect Equilibrium

A subgame-perfect equilibrium is a solution of the game where not only overall, but also each subgame is a Nash equilibrium. To understand the concept consider the following game: player 1 has to decide between going up or down (U/D), while player 2 has to decide between going left or right (L/R). We represent the game using the strategic form specifying all the possible strategies for player 2:

- go Right if player 1 goes Up, go Left otherwise;
- go Left if player 1 goes Up, go Right otherwise;
- go Right no matter what;
- go Left no matter what.

The extensive form of the game is given in Figure 22.5 (game tree on the left). For its *strategic form see the presentation in the right part of the figure.* Since it is a *sequential game*, we describe all possible outcomes depending on player 2 decisions, as seen in the game matrix.

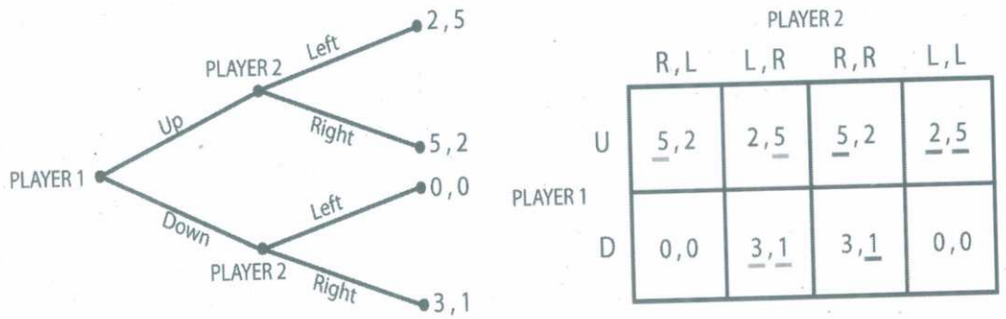



**Fig. 22.5: Extensive-Form Game Converted to Normal Form**

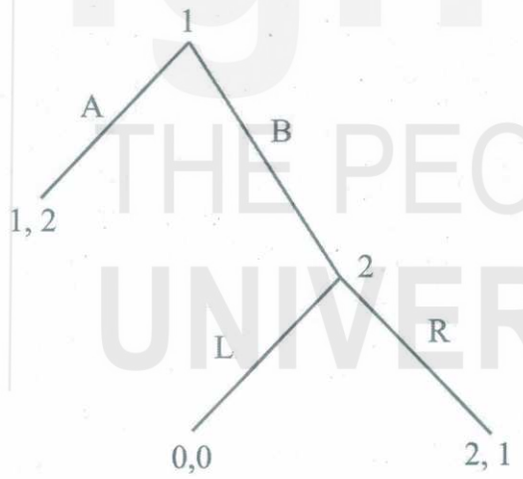
The equilibrium of the game can be seen by considering both the parts of the figure. It may be seen that Up-Left is a Nash equilibrium. However, it's not a perfect equilibrium. In order to find the subgame-perfect equilibrium, we need to solve the game through backwards induction. Starting at the last move of the game, we proceed to the second to last move, and so on. In the game player 2 will choose Left if player 1 goes Up, and Right if player 1 goes Down, since these are the moves that maximise his utility. Because there is complete information, each player's payoffs are known. So, player 1 knows these choices in advance, and will therefore choose to go Down, because the payoff will be greater. Therefore, Down-Right is the subgameperfect equilibrium.

### 22.3.5 Backward Induction and Subgame Perfection

In extensive-form games, we can have a Nash equilibrium profile of strategies where player 2's strategy is a best response to player 1's strategy, but where she will not want to carry out her plan at some nodes of the game tree.

For example, consider the following game, given in both normal-form and extensive-form.

| | | Player 2 | |
|---|---|---|---|
| | | L | R |
| Player 1 | A | 1,2 | 1,2 |
| | B | 0,0 | 2, 1 |




This game has two Nash equilibria: (A,L) and (B,R). If the players have planned to select right move before the game starts, then the NE profile (A,L) makes sense. If player 2 is convinced that player 1 will play A, then playing L is a best response. In fact, she would do everything in her power to convince player 1 that she will play L, so that he plays A.

While player 2 might "threaten" to play L before the game starts, if the game proceeds and it is actually her turn to move, then L is not rational. The threat is not credible, and player 1 should realize this and play B.

Thus, L might be a reasonable plan for player 2 if she thinks player 1 will play A, but it is not rational to carryout the plan after player 1 plays B. It is not sequentially rational for player 2 to play L.

If sequential rationality is common knowledge, then players will look ahead and realize that participants of the game will not take actions down the tree that are not rational at that point.

For games of perfect information, backward induction is the process of "looking ahead and working backwards" to solve a game based on common knowledge of sequential rationality. The process is as given below:

In Figure 22.6 by looking ahead to player 2's decision node, we find that her optimal choice is R. So we can convert her decision node into a terminal node with payoffs (2,1).

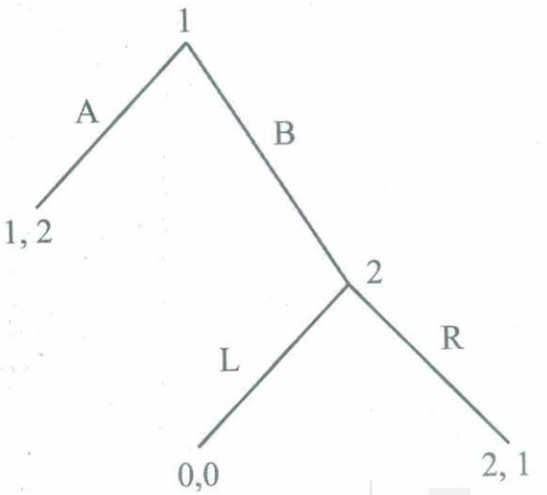



**Fig. 22.6: Optimal Choice in Extensive-Form Game**

Next, examine the smaller games. In Figure 22.7, player 1 can either choose A and reach the terminal node with payoffs (1,2), or B and reach the terminal node with payoffs (2,1).

**Fig. 22.7: Extensive-Form Game with Optimal Choice BR**

We find that player 1's optimal choice is B. So, backward induction leads to the strategy profile (B,R), with payoffs (2,1).

Note that every finite game of perfect information can be solved using backward induction. If each player has a strict preference over his possible terminal node payoffs (no ties), then backward induction gives a unique sequentially rational strategy profile. The resulting strategy profile is a Nash equilibrium.

The following points may be useful in while dealing with the solution concept subgame perfect equilibrium:

- A strategy profile for an extensive-form game is a subgame perfect Nash equilibrium (SPNE) if it specifies a Nash equilibrium in each of its subgames.

- The idea behind SPNE is that even if a NE strategy profile dictates that certain subgames are not reached, we require that what the players would do conditional on reaching those subgames should constitute a NE. The "off-the-equilibrium-path" behaviour can be important, because it affects the incentives of players to follow the equilibrium.
- Notice that every SPNE must also be a NE, because the full game is also a subgame.
- For finite games of perfect information, any back ward induction solution is a SPNE and vice-versa. The advantage of SPNE is that it can be applied to games of imperfect information too.

### 22.3.6 Application 1: Stackelberg Competition

With quantity competition, the timing of production matters. In the Stackelberg Game, firm 1 produces its output first. Then firm 2 observes $q_1$before deciding its ownoutput.

Let us adapt the Cournot model covered earlier. Marginal production cost is equal to 100, and market inverse demand is given by $p = 1000 - q_1 - q_2$. (If the expressionis negative, we take the price to be zero.)

Firm 1's strategy set is$S_1 = \{q_1 : q_1 \geq 0\}$.

Firm 2's strategy set is more complicated, reflecting the extensive-form nature of the game. Firm 2 can let q2 depend on q1.

$S_2 =$ {functions $q_2(q_1) : q_2(q_1) \geq 0$ for each$q_1$}.

A Nash equilibrium is a profile of strategies, $(q_1^*, q_2^*(q_1))$.In any SPNE, the subgame following any $q_1$must be aNE. In other words, $q_2^*(q_1)$ must be a best-response to $q_1$.

$q_2^*(q_1) = BR_2(q_2^*(q_1)) = 450 - \frac{q_1}{2}$.

Notice the distinction: Under simultaneous-move Cournot competition, firm 2 chooses a quantity, which in the NE is a best response to firm 1's quantity $q_1^*$.

Under Stackelberg competition, firm 2 chooses a function describing firm 2's choice at each information set, and in the SPNE this function is $BR_2(q_1)$. Firm 2 must bestrespond to every possible output from firm 1, not just $q_1^*$.

Now that we have found the strategy of firm 2 satisfying sequential rationality, we can work backwards and find$q_1^*$. Firm 1's payoff function is

$u_1(q_1, q_2) = [1000 - q_1 - q_2]q_1 - 100q_1$.

In a SPNE, firm 1 chooses a best response to firm 2'sstrategy function, which we can substitute into the payoff function:

$$u_1(q_1, q_2^*(q_1)) = \left[1000 - q_1 - \left(450 - \frac{q_1}{2}\right)\right] q_1 - 100q_1$$
$$= \left[550 - \frac{q_1}{2}\right] q_1 - 100q_1 = \left[450 - \frac{q_1}{2}\right] q_1 - 100\, q_1$$

Taking the derivative with respect to q1, setting it equal to zero, and solving for $q_1$yields$450 - q_1 = 0$

$q_1^* = 450$.

The SPNE is$(450, 450 - \frac{q_1}{2})$.

Along the equilibrium path, we see that firm 1 chooses output of 450, and firm 2 chooses an output of 225. The corresponding price is 325. This is very different from the Cournot quantities, (300, 300), and price, 400.

You can check that there is a first mover advantage. Firm1 receives a higher payoff than firm 2, and a higher payoff than it would receive under simultaneous play.

Although the game has one SPNE, there are many NE that violate sequential rationality, for example:

$q_1 = 64\ q_2(q_1) = 418 \text{if} q_1 = 64 = 1000 \text{otherwise.}$

### 22.3.7 Application 2: Sequential Bargaining

How will people bargain over a pie of a certain size? A "reasonable" or "fair" division of it could be $(1/2,1/2)$ there number of agents involve is two. However, bargaining can involve bluffing and posturing, there is no certainty that an agreement will be reached. A paper by Rubinstein (1982), however, showed that a sequential bargaining would yield a unique subgame perfect equilibrium. We consider this model.

1) **The Model**

Take two players, one and two, playing the game sequentially and making offers about how to divide a pie of size one. Time runs from $t = 0, 1, 2, \ldots$. At time 0, playerone can propose a split $(x_0, 1 - x_0)$(with $x_0 \in [0, 1]$), which player 2 canaccept or reject. If player 2 accepts, the game ends and the pie is consumed. If player two rejects, the game continues to time $t = 1$. She now proposes a split$(y_1, 1 - y_1)$. Such a proposal of player two may be accepted or rejected by player one.The sequence of the game goes to ad infinitum.

Let us assume that both players want a larger slice, and also that they both dislike delay. Thus, if agreement to split the pie $(x, 1 - x)$is reached at timet, the payoff for player one is $\delta_1^t x$ and the payoff for player two is $\delta_2^t(1 - x)$, for some $\delta_1, \delta_2 \in (0, 1)$.

2) **A Finite-Horizon Version**

We consider a variant of the game where there is some finite number of offers $N$that can be made. If we solve for the subgame perfect equilibrium through backward induction, we need to start from the final offer.

For simplification assume $N = 2$. Then at date 1, player two will be able to make a final take-it-or-leave-it offer. Given that the game is about to end, player one will accept any split. Therefore, player two can offer$y = 0$.

Consider its implication for date zero. Player two expects that if she rejects player one's offer, she can get the whole pie in the next period and the total payoff is $\delta_2$. Thus, to get her offer accepted, player one must offer player two at least $\delta_2$. It follows that player one will offer a split $(1 - \delta_2, \delta_2)$,and player two will accept.

See that in the $N = 2$ offer of the sequential bargaining game, the unique SPE involves an immediate $(1 - \delta_2, \delta_2)$ split.

### 3) Solving Rubinstein Model

We consider infinite time horizon to solve the bargaining game examined above for two periods. The problem here is we cannot apply backward induction. Fortunately Rubinstein (1982), showed that the solution has a remarkably simple form.

Suppose player one makes an offer at a given date t. then player two acceptance of it depends on her belief about what she will get if she rejects. This in turn depends on what sort of offer player one will accept in the next period, and so on. Even with such a time horizon we will show there is a unique subgame perfect equilibrium in the sequential bargaining game.

When player one proposes, she suggests a split $(x, 1-x)$with$x = (1-\delta_2) / (1-\delta_1\delta_2)$. Player two accepts any division giving her at least $1-x$. When player two proposes, she suggests a split $(y, 1-y)$with $y = \delta_1(1-\delta_2) / (1-\delta_1\delta_2)$. Player one accepts any division giving her at least y. Thus, bargaining ends immediately with a split$(x, 1-x)$.

We start by examining that proposed equilibrium is actually an SPE. By a classic dynamic programming argument of the one-step deviation principle, we know that it suffices to check that no player can make a profitable deviation from her equilibrium strategy in one single period.

Now consider a period when player one offers. Player one has no profitable deviation. She cannot make an acceptable offer that will get her more than$x$. Because when the offer made by her is rejected she will get $y = \delta_1 x$ the next period, or $\delta_1^2 x$ in present terms, which is worse than $x$. Player two also has no profitable deviation. If she accepts, she gets $1-x$. If she rejects, in the next period she gets $1-y$, or in present terms $\delta_2(1-x) = \delta_2(1-\delta_1 x)$. A similar argument applies to periods when player two offers.

We now show that the equilibrium is unique. To do this, let $\underline{v}_1$, $\bar{v}_1$denote the lowest and highest payoffs that player one could conceivably get in any subgame perfect equilibrium starting at a date where he gets to make an offer.

To begin, consider a date where player two makes an offer. Player one will certainly accept any offer greater than $\delta_1\bar{v}_1$ and reject any offer lessthan $\delta_1\bar{v}_1$. Thus, starting from a period in which she offers, player two cansecure at least $1-\delta_1\bar{v}_1$by proposing a split $(\delta_1\bar{v}_1, 1-\delta_1\bar{v}_1)$. On the otherhand, she can secure at most $1-\delta_1\underline{v}_1$.

Now, consider a period when player one makes an offer. To get player two to accept, he must offer her at least $\delta_2(1-\delta_1\bar{v}_1)$to get agreement.

Thus:

$\bar{v}_1 \leq 1-\delta_2(1-\delta_1\bar{v}_1)$.

At the same time, player two will certainly accept if offered more than

$\delta_2(1-\delta_1\bar{v}_1)$. Thus:

$\underline{v}_1 \geq 1-\delta_2(1-\delta_1\underline{v}_1)$.

It follows that: $\underline{v}_1 \geq \frac{1-\delta_2}{1-\delta_1\delta_2} \geq \bar{v}_1$. Since $\bar{v}_1 \geq \underline{v}_1$by definition, we know that in any subgame perfect equilibrium, player one receives $v_1 = (1-\delta_2) / (1-\delta_1\delta_2)$. We can make the same argument for player two.

*Few Comments on Rubinstein Model of Bargaining*

1) It helps to be patient. Note that player one's payoff, $(1 - \delta_2) / (1 - \delta_1\delta_2)$, is increasing in $\delta_1$ and decreasing in $\delta_2$. The reason is that if you are more patient, you can afford to wait until you have the bargaining power (i.e. get to make the offer).

2) The first player to make an offer has an advantage. With identical discount factors $\delta$, the model predicts a split $\left(\frac{1}{1+\delta}, \frac{\delta}{1+\delta}\right)$ which is better for player one. However, as $\delta \rightarrow 1$, this first mover advantage goes away. The limiting split is (1/2, 1/2).

3) There is no delay. Player two accepts player one's first offer.

4) The details of the model depend a lot on there being no immediate counter-offers. With immediate counter-offers, it turns out that there are many many equilibria.

**Check Your Progress 2**

1) Differentiate between games with complete information and perfect information.

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2) What is backward induction?

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3) Define a subgame.

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4) Define the concept of Subgame Perfect Equilibrium.

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5) Explain the relationship between Backward Induction and Subgame Perfection.

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6) What are the three important components a price leader in the Stackelberg model is assumed to know?

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7) Point out the main insight of Rubinstein'sbargaining model.

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8) Sequential Bargaining problem is solved using: i) subgame or ii) backward induction.

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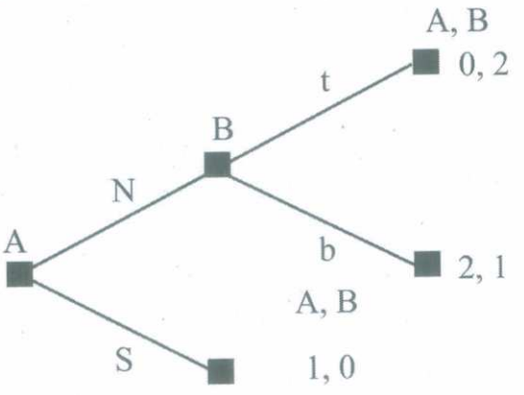
9) Use backward induction to determine the subgame perfect equilibrium of the following game:

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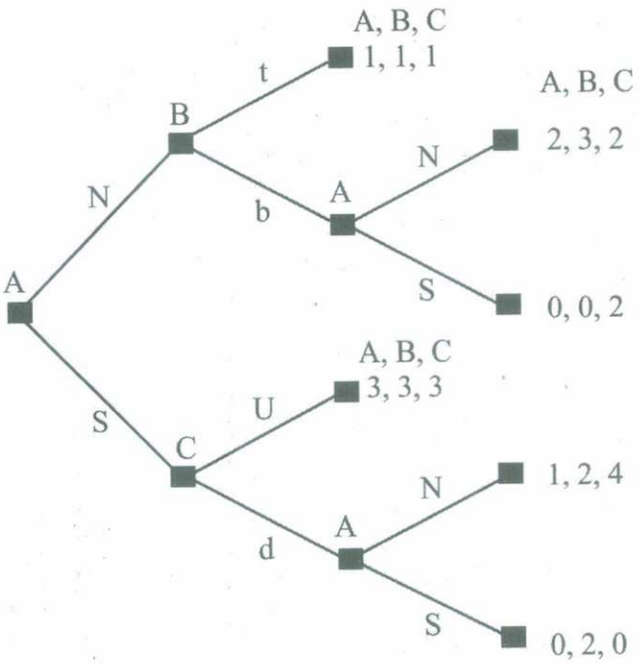
10) Use backward induction to determine the subgame perfect equilibrium of the following game:

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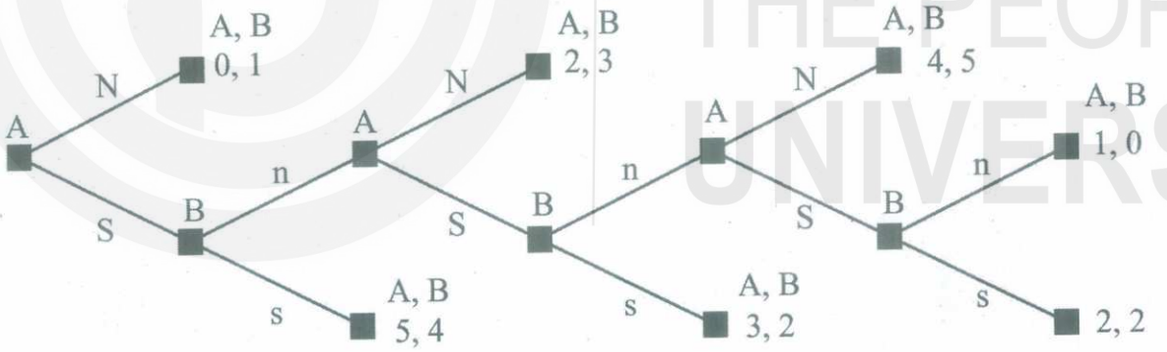
11) Use backward induction to determine the subgame perfect equilibrium of the following game:

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## 22.4 DYNAMIC GAMES OF IMPERFECT INFORMATION

Often players do not have full access to all the information which is relevant to their choices. Such games with imperfect information model exactly which information is available to the players when they make a move.

A dynamic game of complete but imperfect information can be represented in extensive form by using non-singleton information sets to indicate what each player knows (and does not know) when he or she has to the move. Thus, there are three analytically equivalent definitions involving the rules, viz.,

In dynamic games of imperfect information,

1) at some stage of the game some players do not know its entire development before the current move;
2) players may move simultaneously; and
3) some players at some stages may move while not observed by the others.

These games can be represented in extensive (game tree) form by using non-singleton information sets to indicate what each player knows (and does not know) when she has to the move. Since each player does not know all earlier moves, we must partition the set of player's decision nodes into subsets, called information set. We connect the nodes are connected with a dashed line if the moving player does not know from which node of the game tree he is making the move (see Figure 22.8).Sometimes instead of dotted line we can use oval format to represent information set.

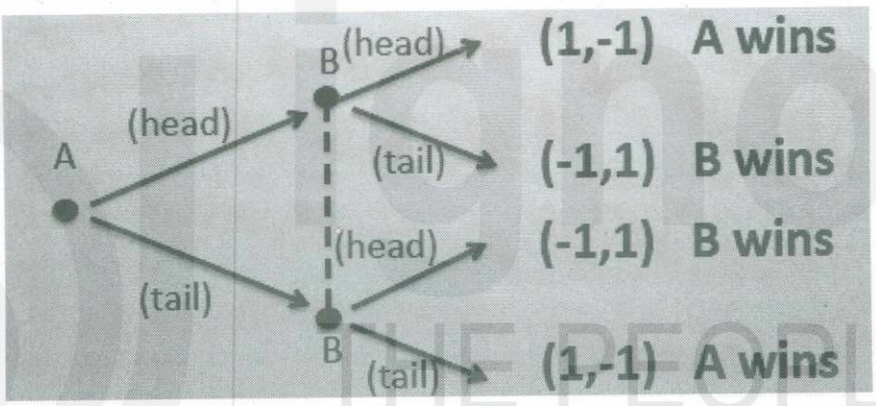
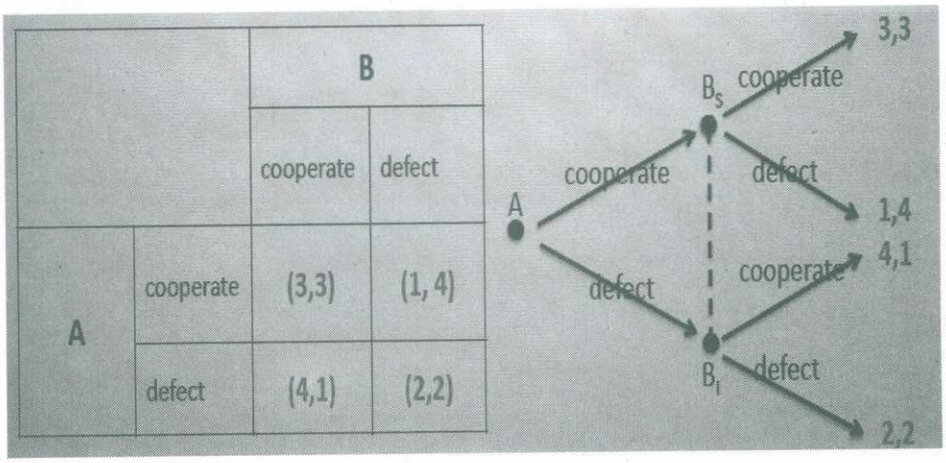



**Fig. 22.8: Extensive-form Game with Information Set**

When information sets are all singleton (made of a single node) the game is one of perfect information. In games of imperfect information however, some information sets includes more than one node.

Example: Prisoner dilemma in extensive form.



| | | B | |
|---|---|---|---|
| | | cooperate | defect |
| A | cooperate | (3,3) | (1, 4) |
| | defect | (4,1) | (2,2) |



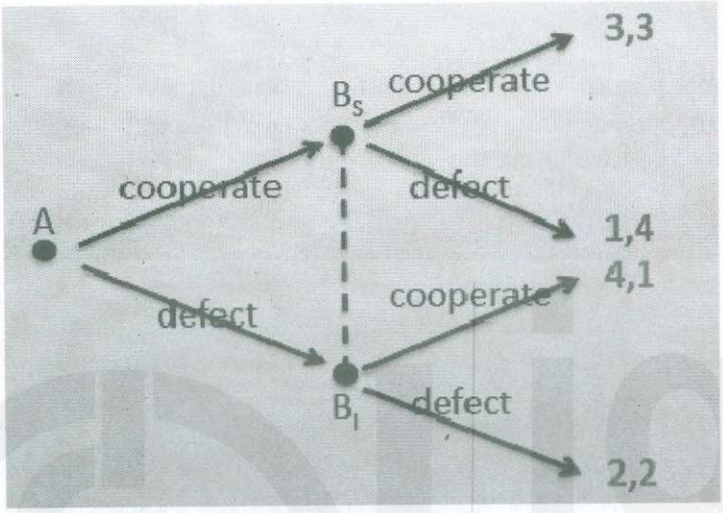
The game is symmetric: rows and columns can be exchanged in normal form. So, first and second move can be exchanged in extensive form.

### 22.4.1 Imperfect Information and Backward Induction

In games of imperfect information backward induction procedure has to be considered when

- a player does not know which node he is moving from he cannot single out the best action;
- preceding player into the game tree cannot anticipate what she will do;
- common knowledge is not able to give the usual information;
- a dominated strategy not available.

**A special case: prisoner dilemma**




In tree above given as a special case of prisoner dilemma, even if B does not know if she is in BS or in BI she can observe that, whatever her starting node, "defect" gives her more utility then "cooperate":

- 4 instead of 3 if A has done "cooperate";
- 2 instead of 1 if A has chosen "defect" This is because "cooperate" is a dominated strategy.

### 22.4.2 Subgames with Imperfect Information

A subgame is a subset of the extensive form that satisfies the following criteria:

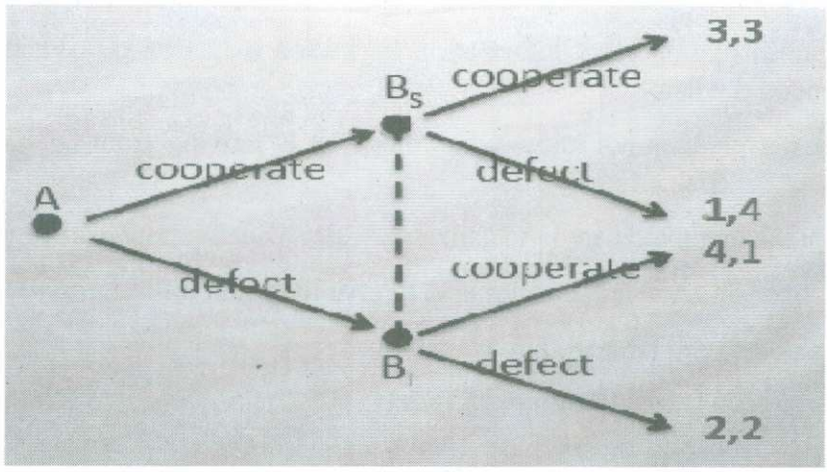
- begins at a node (singleton);
- includes all nodes following this initial node and no others;
- does not cut any information sets (if two nodes are part of the same information set they belong to the same subgame).

The characteristics of subgames are:

- self-contained extensive forms, meaningful trees on their own;
- subgames that start from nodes other than the initial node are called proper subgames; and
- in a game of perfect information, every node initiates a subgame.

Figure 22.9(1) presents these characteristics.

Figure 22.9(2): *Information Sets and Subgames*




In the figure player A has one information set that is a singleton. Player B has one information set with two nodes (none of which is a singleton). See that no subgames exist for PD but the whole game.

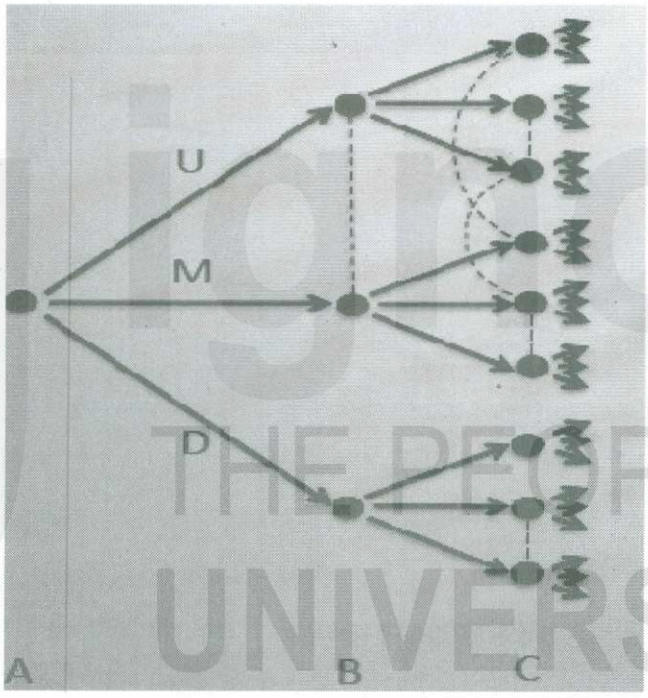



**Fig. 22.9(2): Information Sets and Subgames**

In the figure above each player A, B, C has three moves: {U, M, D} ; {u, m, d} ; {υ, μ, δ}.

- At the first stage A has one information set (singleton).
- At the second stage B knows that A either chooses {D} or {U,M} that are the two information sets of B.
- At the third stage C knows that B has chosen among {Dd,Dm} , {Du}, {Md,Mm,Ud,Um}, {Mu,Uu} that are four C's information sets.

Therefore, the game has three subgames (two proper subgames): i) the original game, ii) a subgame following D, iii) a subgame following the path D-u.

Through the concept of information set game theory can face strategic interactions that are partly sequential and partly simultaneous moves.

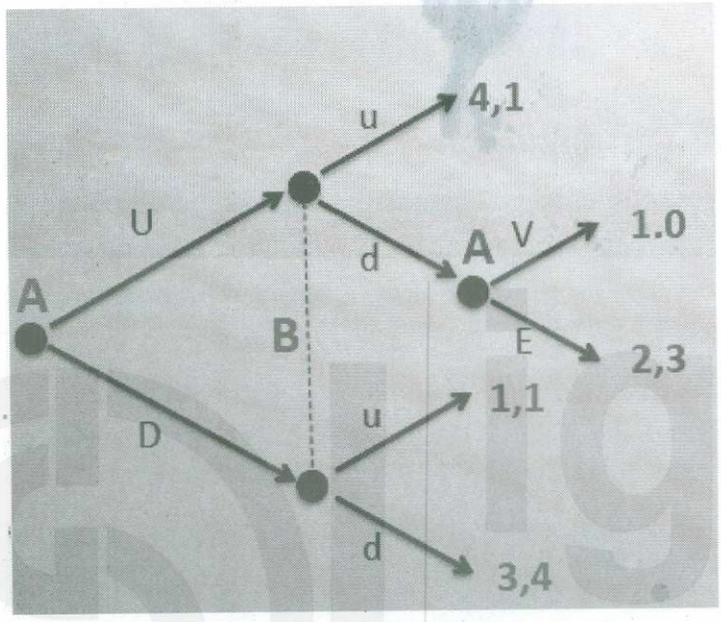
### 22.4.3 Strategies with Imperfect Information

To determine NE, it is necessary to know the strategies of each player i (i = 1, 2, …, n). A definition of strategy (a complete plan of action) is needed for games in extensive form. For any given player i a strategy specifies what that player should do at any information set.

### 22.4.4 Finding SPNE with Imperfect Information

First solve each proper subgame and then go back to the whole game or viceversa. It should always produce the same result.

*Example: strategies in the tree-1*

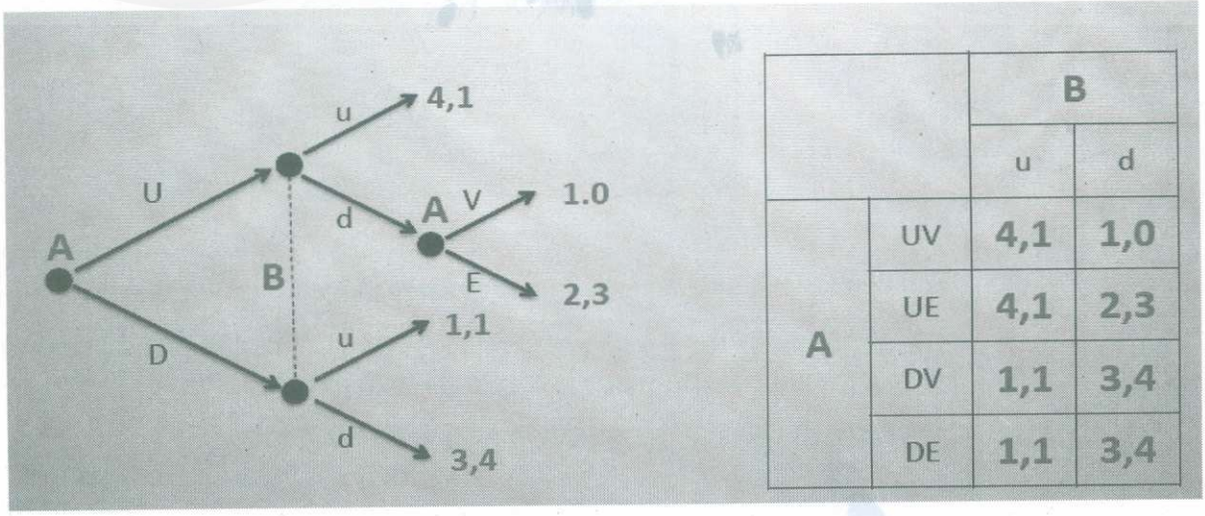



By definition

- A has four strategies: $\{UV, UE, DV, DE\}$
- B has two strategies: $\{u, d\}$

Although DV and DE are sequences of moves never played by A, they cannot be discarded.

*Convertion of Extensive Games to Normal Form*

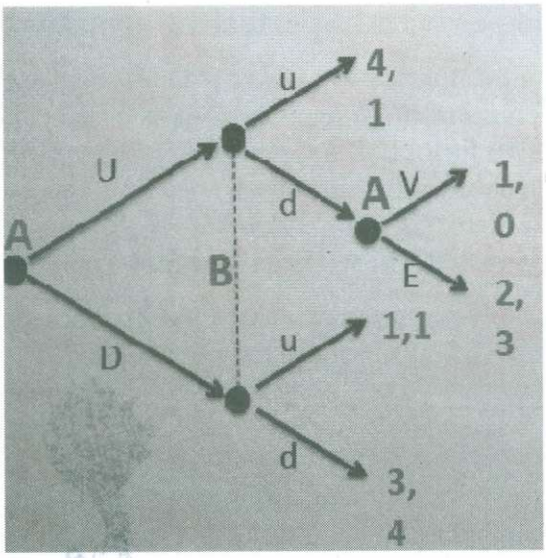



| | | B | |
|---|---|---|---|
| | | u | d |
| A | UV | 4,1 | 1,0 |
| | UE | 4,1 | 2,3 |
| | DV | 1,1 | 3,4 |
| | DE | 1,1 | 3,4 |

The matrix shows three NE: {UV,u}, {DV,d}, {DE,d}

… and back to the tree.

*Example: strategies in the tree-2*



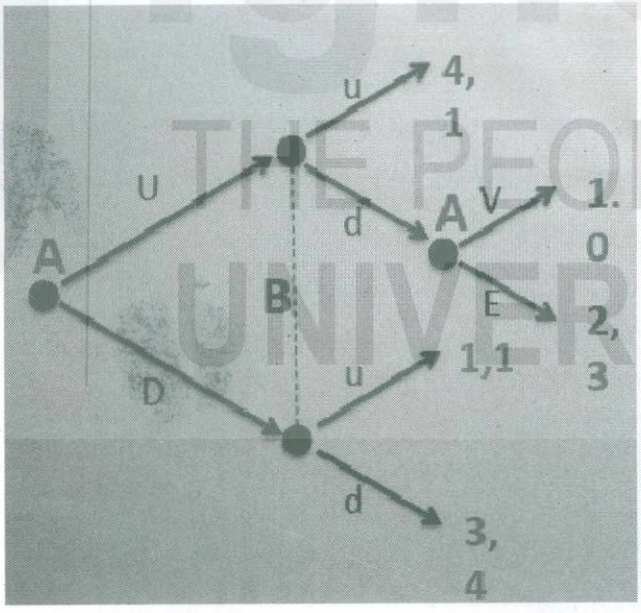

NE: {UV,u}, {DV,d}, {DE,d}

Are all those equilibria coherent with backward induction?

By backward induction if A has the opportunity to make her second move, she will choose E (2>1). So the only strategy profile coherent with backward induction is {DE,d} with payoff(3,4).

Again refer back to the tree -2.

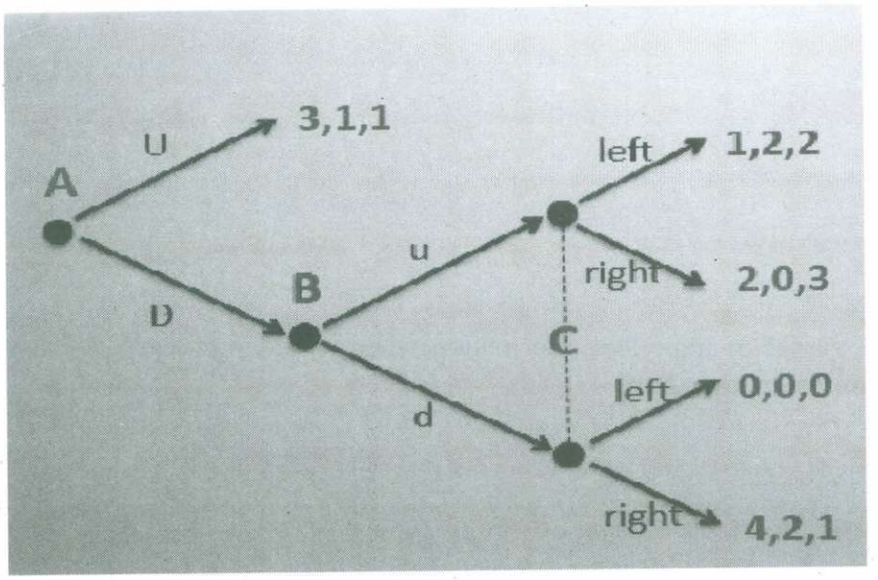
You will find the NE: {UV,u}, {DV,d}, {DE,d}. The same result is more quickly achievable through the concept of subgames.




There are two subgames:

- the first starts at the node where A moves for the second time;
- the second is the whole game {UV,d} and {DV,d} do not satisfy subgame perfection. The only SPNE is {DE,d} with payoff(3,4).

Example: Game Representation with Three Players




With three players we represent the game in normal form. See that the representation of the game is done with two matrices as given below:

| A plays U | | | |
|---|---|---|---|
| | | C | |
| | | **Left** | **Right** |
| B | u | 3, 1, 1 | 3, 1, 1 |
| | d | 3, 1, 1 | 3, 1, 1 |

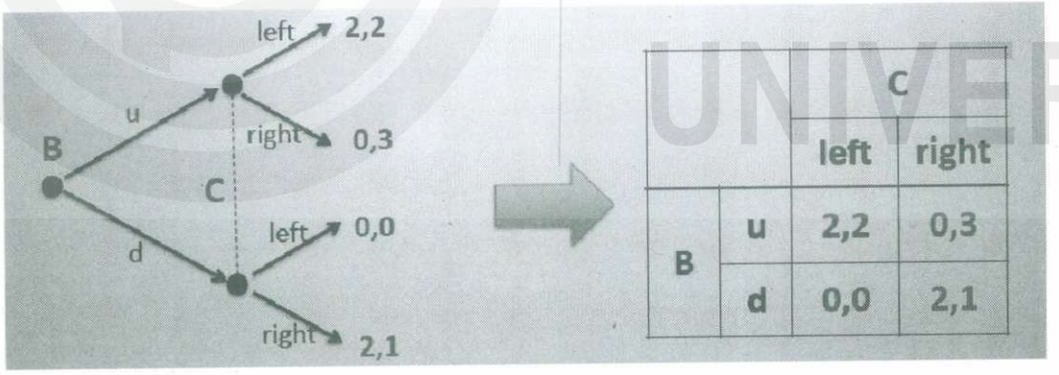
| A plays D | | | |
|---|---|---|---|
| | | C | |
| | | Left | Right |
| B | u | 1, 2, 2 | 2, 0, 3 |
| | d | 0, 0, 0 | 4, 2, 1 |

A: matrix player; B: row player; C: column player

The game has the following NE: {U,u,left}, {U,u,right}, {U,d,left}, {D,d,right}.

For instance {U,u,left} yields payoff (3,1,1) is one NE.

Consider the subgame where only players B and C play (See figure below).




| | | C | |
|---|---|---|---|
| | | **left** | **right** |
| **B** | **u** | **2,2** | **0,3** |
| | **d** | **0,0** | **2,1** |

It can be seen that "right" is a dominant strategy for player C and the matrix of the subgame has the only NE {d,right}. That excludes {U,u,left} from being a SPNE. The only SPNE of the original game is {D,d,right} leading to the payoff (4,2,1).

**Check Your Progress 3**

1) What the difference between perfect and imperfect information?

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2) What is an Information set?

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3) In games of imperfect information when would you consider backward induction procedure?

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4) What is a subgame perfect Nash equilibrium?

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5) A "credible" threat is a threat of an action that a player would be willing to undertake if---------

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6) A subgame perfect equilibrium is a Nash equilibrium that involves only--

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## 22.5 REPEATED GAMES

In this section we analyse whether threats and promises about future can affect the current behaviour when a game is played repeatedly. Also we will define the concept of sub-game perfect Nash equilibrium for repeated games

### 22.5.1 Two-Stage Repeated Games

In two-stage repeated game, players play the game once again after the payoffs of the game is collected by them. Consider the Prisoners' Dilemma given in normal form in the following table

| | | Player 2 | |
|---|---|---|---|
| | | L2 | M2 |
| Player 1 | L1 | (1, 1) | (5, 0) |
| | M1 | (0, 5) | (4, 4) |

| | | Player2 | |
|---|---|---|---|
| | | L2 | M 2 |
| Player 1 | L1 | (2, 2) | (6, 1) |
| | M1 | (1, 6) | (5, 5) |

Suppose the players play the game twice, and they observe the outcome of the first game before playing it once again. We further assume that the payoff of the entire game, taking two stages together is simply the sum of the payoffs from the two stages (we assume there is no discounting, that is, the same amount of payoff in any period gives the same amount of utility. Generally, if we think in monetary terms we will prefer Rs. 100 today than Rs. 100 tomorrow. This is because there is a discount rate, which is embedded in our inter-temporal choice decisions).

It is easy to see that the Nash equilibrium of the first stage of the game is (L1, L2) with payoff (1, 1); we have added the equilibrium payoff of the first stage to the payoffs of the game in the next stage. In the second stage also there is a unique Nash equilibrium of the two-stage prisoners' dilemma.

### 22.5.2 Finitely Repeated Games

The argument stated above holds more generally. We generalise the concept of repeated games to games with finite number of repetitions. Let $G = \{A_1, A_2..., A_n; u_1, u_2,..., u_n\}$ denote a static game of complete information in which players 1 through n simultaneously choose their strategies $a_1$ to $a_n$ respectively from their strategy spaces $S_1$ through $S_n$. This game is repeated T times after collecting and observing the outcomes of each game. The game G is called the stage game of the repeated game.

Definition: Given a stage game G, let G(T) denote the finitely repeated game in which G is played T times, with the outcome of all the previous games observed before the next play begins. The payoff for G(T) is simply the sum of the payoffs from the T stage game.

Proposition: If a stage game G has a unique Nash equilibrium then, for any finite T, the repeated game G(T) has a unique sub-game perfect Nash equilibrium: the Nash equilibrium of G being played in every stage of the game.

It is interesting to investigate if there is more than one Nash equilibrium in the stage game itself. What will be the nature of the equilibrium when the game is repeated? We consider a simple two period game, which is a simple extension of the prisoners' dilemma such as there are two Nash equilibria in it. In addition to the strategies L1 and M1, we add another strategy R1 at the disposal of player 1. Similarly, we add the strategy R2 to the strategy space of player 2. The game is described below in normal form.

| | | Player 2 | | |
|---|---|---|---|---|
| | | L2 | M2 | R2 |
| Player 1 | L1 | (1,1) | (5,0) | (0,0) |
| | M1 | (0,5) | (4,4) | (0,0) |
| | R1 | (0,0) | (0,0) | (3,3) |

As a result of adding the two strategies and the distribution of payoffs now there are two pure strategy Nash equilibrium, namely, (L1, L2) and (R1, R2). Suppose the above stage game is played twice with the first stage outcome being observed before the second stage begins.

Since the stage game has more than one Nash equilibrium, it is now possible for the players to anticipate that different first stage outcomes will be followed by different stage game equilibria in the second stage. Let us suppose for example the players anticipate that (R1, R2) will be the second stage outcome if the first stage outcome is (M1, M2), but (L1, L2) will be the second stage outcome if any one of the eight other first stage outcomes occur. Thus, the game reduces to the following one shot game, where (3, 3) has been added to the (M1, M2) cell and (1, 1) has been added to rest of the cells.

| | | Player 2 | | |
|---|---|---|---|---|
| | | L2 | M2 | R2 |
| Player 1 | L1 | (2,2) | (6,1) | (1,1) |
| | M1 | (1,6) | (7,7) | (1,1) |
| | R1 | (1,1) | (1,1) | (4,4) |

There are three pure strategy Nash equilibrium of the above game (L1, L2), (M1, M2) and (R1, R2). Let us denote the outcome of the repeated game as [(w, x): (y, z)]; where (w, x) is the first stage outcome and (y, z) is the second stage outcome. Therefore, the Nash equilibria namely, (L1, L2), (M1, M2) and (R1, R2) can be achieved in the simplified one shot game if the outcomes of the repeated game are [(L1, L2), (L1, L2)], [(M1, M2); (R1, R2)] and [(R1, R2), (L1, L2)] respectively (if the first stage outcome is (L1, L2), the second stage outcome has to be (L1, L2) according to the players' anticipation and so on for each of the one shot game's Nash equilibria). The first and the last Nash equilibria of the repeated game are sub-game perfect and simply concatenate the Nash equilibrium outcome of the stage game. But the Nash equilibrium (M1, M2) of the one shot game is possible if the outcome of the repeated game has the sub-game perfect outcome [(M1, M2); (R1, R2]], which means in the first stage, the players chose (M1, M2) which is not a Nash equilibrium of the stage game. We can conclude that cooperation can be achieved in the first stage of a sub-game perfect outcome of a repeated game.

Proposition: We extend this idea to a stage game being played T times, where T is any finite number. If $G = \{A_1, A_2..., A_n; u_1, u_{2,...}, u_n\}$ is a static game of complete information with multiple Nash equilibria, then there may be sub-game perfect outcomes of the repeated game G(T) in which for any $t < T$, the outcome in stage t is not a Nash equilibrium of the stage game G.

The main points to extract from the above example is that credible threats or promises about the future behaviour can affect current behavior sub-game perfection (as we described in the previous unit) may not be a definition strong enough to embody credibility.

### 22.5.3 Infinitely Repeated Games

In the finite horizon case, the main focus was that credible threats or promises about future behaviour can influence current behaviour and if there are multiple Nash equilibria of the stage game, then there may be sub-game perfect outcomes of the repeated game G (T) in which, for any t < T, the outcome of a stage game is not a Nash equilibrium of G. Whereas in case of infinitely repeated games a stronger result is true. Even if the stage game has a unique Nash equilibrium, there could be sub-game perfect outcome of the infinitely repeated game in which no stage's outcome is a Nash equilibrium of the stage game G.

| | | Player 2 | |
|---|---|---|---|
| | | L2 | M2 |
| Player 1 | L1 | (1, 1) | (5, 0) |
| | M1 | (0, 5) | (4, 4) |

An infinitely repeated game is an extension of a finitely repeated game, it being played infinitely. Suppose the prisoners' dilemma game is to be repeated infinitely and for each t, the outcome of the t-1 preceding plays of the stage game is observed before the $t^{th}$stage begins. Simple summation of the payoffs from this infinite sequence of stage games does not provide a useful measure of a player's payoff in the infinitely repeated game. This is because receiving a payoff 4 is better than receiving a payoff 1 in every stage but a summation of the payoff 1, repeated till infinity and that of 4 is same, which is infinity. To tackle this problem, we introduce the concept of discount rate. As we have argued earlier, Rs. 100 today is not the same as Rs. 100 tomorrow. If the rate of interest is 'r', one can earn (100×r) one-year later in addition of the principle Rs. 100. Therefore, Rs. 100 today is worth Rs. 100(1+r) tomorrow. To find the present value of a future income or future stream of income, we must discount it to get the present value of the future income. To get the present value of future income to be received t years later we multiply, it with $\frac{1}{(1+r)^t}$. This fraction $\frac{1}{(1+r)}$ is called the discount factor; it is generally denoted by $\delta$. We can apply this method of calculating the present value of an income stream to calculate the present value of the payoffs of an infinitely repeated game.

Definition: Given the discount factor $\delta$, the present value of the infinite sequence of payoffs $\pi_1, \pi_2, \pi_3, \cdots$is given by

$$\pi_1 + \delta\pi_2 + \delta^2\pi_3 + \cdots = \sum_{t=1}^{\infty} \partial^{t-1} \cdot \pi_t$$

Let us consider the infinitely repeated prisoners' dilemma, where the payoff of the players is the present value of the player's payoff from the stage games. The discount factor is $\partial$ for both the players. We want to show that cooperation, that is, (M1, M2) can occur in every stage of a sub-game-perfect outcome of an infinitely repeated game, even though the only Nash equilibrium in the stage game is cooperation, which is (L1, L2).

Suppose the $i^{th}$ player begins the infinitely repeated game by cooperation and then cooperates in rest of the periods if and only if both the players have cooperated in the previous stages of the game. Formally, the strategy of the $i^{th}$ player is Play Mi in the first stage.

In the $t^{th}$stage if the outcome of all the preceding (t-1) has been (M1, M2) then play Mi in the $t^{th}$stage otherwise play Li, (i = 1, 2) (This type of strategy is called a trigger strategy). If both the player follows this trigger strategy, then the outcome of the infinitely repeated game would be (M1, M2) in every stage of the game. Now we will prove that given some conditions on the value of $\partial$, the above trigger strategy is a Nash equilibrium of the infinitely repeated game, and such an equilibrium is sub-game-perfect.

To show that the above described trigger strategy is a Nash equilibrium of the game, we have to show that if the player i adopts the trigger strategy, the best response of the player j is to adopt the same strategy. The present value of the payoffs that will be generated, for the $j^{th}$player, if both of the players stick to the trigger strategy is given by

$$V = 4 + 4\partial + 4\partial^2 + 4\partial^3 \ldots = \frac{4}{1-\partial^*}$$

If the $j^{th}$player deviates from the trigger strategy, that is, she plays Lj in the first stage, which will eventually lead to non-cooperation from player i in the second stage (Li) and consequently from player j (Lj) also, the discounted payoff of the $j^{th}$ player is given by (the payoff of the first stage for the $j^{th}$ player would be 5, as the $i^{th}$ player in the first period following her trigger strategy will play Ri, and for the remaining periods the payoff of the $j^{th}$ player would be 1)$V' = 5 + 1\partial + 1\partial^2 + 1\partial^3 \cdots = 5 + \frac{\partial}{1-\partial}$

Therefore, playing Mj, or following the trigger strategy is optimal for the $j^{th}$ player given that the $i^{th}$ player sticks to her trigger strategy if and only if $\frac{4}{1-\partial} \geq 5 + \frac{\partial}{1-\partial} \rightarrow \partial \geq \frac{1}{4}$. Therefore, the trigger strategy is the Nash equilibrium for both the players if and only if$\partial \geq \frac{1}{4}$.

Now we are in a position to formally define an infinitely repeated game, history, strategy and sub games of an infinitely repeated game.

Definition of Infinitely Repeated Game: Given a stage game G, let G ($\infty$, $\partial$) denote the infinitely repeated game in which G is repeated forever and all the players have the same discounting factor denoted by $\partial$. For each t (t is a positive integer), the outcomes of the t-1 previous plays are observed before the $t^{th}$ stage begins. Each player's payoff in G ($\infty$, $\partial$) is the present value of the player's payoff from the infinite sequence of the stage game.

History of an Infinitely Repeated Game: In the finitely repeated game G(T) or the infinitely repeated game G ($\infty$, $\partial$), the history of play through stage t is the record of the player's choices in the stage 1 through t.

Strategy of an Infinitely Repeated Game: Notice while defining the repeated game in section we used the notation G = {$A_1$, $A_2$…, $A_n$; $u_1$, $u_{2,\ldots}$, $u_n$} while in previous units we described a static game of complete information as G = {$S_1$, $S_2$…, $S_n$; $u_1$, $u_{2,\ldots}$, $u_n$}; remember Si is in a static game of complete information implies strategy spaces whereas the $A_i$ s in a dynamic or repeated game of complete information implies the action space. The players might have chosen ($a_{11}$, $a_{21}$, $a_{31}$,… $a_{n1}$) in stage 1 (suppose there are n players) and ($a_{12}$, $a_{22}$, $a_{32}$,… $a_{n2}$) and so on. For each player i and for each stage s, the action $a_{is}$belongs to the Action space $A_i$.

Thus, in an infinitely repeated game there is a difference between strategy and actions. In a finitely repeated game G(T) or an infinitely repeated game G ($\infty$, $\partial$), a player's strategy specifies the action the player will take in every stage of

the game, for each possible history throughout the game. So, strategy is a complete plan of action. It is defined for each possible circumstance and for every stage of an infinitely repeated game.

Sub-game of an Infinitely Repeated Game: In the finitely repeated game G(T), a sub-game beginning at the stage t+1 is the repeated game in which G is played T-t times, denoted by G(T-t). There are many sub-games that begin at the stage t+1, one for each possible history of the game through stage t. Whereas for an infinitely repeated game G ($\infty$, $\partial$), each sub-game is identical to the original game G ($\infty$, $\partial$). As in the finitely repeated game, there are as many sub-games beginning at stage t+1 of G($\infty$, $\partial$) as there are possible histories of play through stage t. In the case of two stage prisoners' dilemma (with two strategies available to each player), there are four sub-games, corresponding to the second stage game that will follow the four possible first stage outcomes. Similarly, the extension of the two stage prisoners' dilemma where each player had three strategies at their disposal has nine sub-games. A sub-game is not only a piece of the game which starts at a point where the history of the game thus far is common knowledge among the players but also includes all the moves that follow this point in the original game.

Sub-game Perfect Nash Equilibrium of an Infinitely Repeated Game (Selton 1965): A Nash equilibrium is sub-game perfect if the players' strategies constitute a Nash equilibrium in every sub-game of the infinitely repeated game. Sub-game perfect Nash equilibrium is refinement of the concept of Nash equilibrium, which means for a strategy profile to be sub-game perfect, it must be a Nash equilibrium first and then it must satisfy an additional test, that is, it must be a Nash equilibrium in every sub-game of the game.

Our objective of describing all the definition above was to show that the trigger strategy that we have already defined in the game of infinitely repeated prisoners' dilemma is sub-game perfect. Therefore, we need to show that the trigger strategy constitute a Nash equilibrium on every sub-game of that infinitely repeated game. Recall that every sub-game of an infinitely repeated game is identical to the game as a whole. In the trigger strategy, Nash equilibrium of the infinitely repeated prisoners' dilemma sub-games could be grouped into two classes:

- subgames in which all the outcomes of the earlier stages have been (R1, R2)
- sub-games in which the outcome of at least one stage differs from (R1, R2).

If the players adopt the trigger strategy for the game as a whole, then (i) the players' strategies in a sub-game in the first case are again the trigger strategy, which we have already shown to be the Nash equilibrium of the game as a whole.

The players' strategies in a sub-game in the second case are simply to repeat the stage game equilibrium ($L_1$, $L_2$) forever, which is also Nash equilibrium of the game as a whole. Therefore, the trigger strategy Nash equilibrium of the infinitely repeated prisoners' dilemma is sub-game-perfect.

### 22.5.4 Application 3: Collusion between Cournot Duopolists

Friedman was first to show that cooperation could be achieved in an infinitely repeated game by using trigger strategies that switch forever to the stage game Nash equilibrium following any deviation. The original application was to collusion in a Cournot oligopoly.

Take the case of Cournot oligopoly: if the aggregate quantity in the market is $Q = (q_1 + q_2)$, and the market clearing price is P = a – Q. Assuming Q < a, and each firm has a marginal cost c, if the firms' choose their quantities simultaneously, then the unique Nash equilibrium of the game is both firm producing $\frac{(a-c)}{3}$, which we call the Cournot quantity and denote it by $q_c$. Since the equilibrium aggregate quantity$2\frac{(a-c)}{3}$ , exceeds the monopoly quantity $q_m = \frac{(a-c)}{2}$. Clearly, both the firms would be better off if each firm produced half of $q_m$, the monopoly quantity$\frac{(a-c)}{4}$.

We will consider an infinitely repeated game based on this Cournot stage game when both the firms have the discount factor $\partial$. We will seek for a sub-game perfect Nash equilibrium in which both the firms collude and their payoffs are more than Cournot payoff.

Let us consider the following trigger strategy:

Produce half of the monopoly output ($q_m/2$) in the first period. Continue to produce the same in the $t^{th}$ period if both the firms produced ($q_m/2$) in the $t\text{-}1^{th}$ period; otherwise produce the Cournot output $q_c$.

The profit to one firm when both the firms produce ($q_m/2$) is denoted by$\frac{\pi_m}{2}$ i.e.,$\frac{\pi_m}{2} = \frac{q_m}{2}(a - q_m - c) = \frac{(a-c)^2}{8}$

[Check $q_m$= (a - c)/2]

Whereas the profit accruing to each firm when both produce $q_c$ is denoted by$\pi_c$. So, $\pi_c = q_c(a - 2q_c - c) = \frac{(a-c)^2}{9}$

Finally, if firm 1 is going to produce ($q_m/2$) this period, then the quantity that maximises j's profit in this period, is obtained solving the following simple maximisation problem: $max_{q_j}\left(\frac{a-\frac{q_m}{2}-c}{2}\right)q_j$. The solution of the problem can be obtained from the first order condition of profit maximisation, that is, $\left(a - 2q_j - \frac{q_m}{2} - c\right) = 0$. So, $q_j = \frac{a-\frac{q_m}{2}-c}{2} Or, q_j = \frac{a-\frac{(a-c)}{2.2}-c}{2} = \frac{3(a-c)}{8}$ with associated profit$\frac{9(a-c)^2}{64}$. We will denote this profit by $\pi_d$(d stands for deviation).

Therefore, it is a Nash equilibrium for both the firms to play the trigger strategy, given earlier provided that,present value of payoff from the trigger strategy ≥ present value of the payoffs deviated in the first period.

Or, $\frac{\pi^m}{2} + \partial\frac{\pi^m}{2} + \partial^2\frac{\pi^m}{2} + \partial^3\frac{\pi^m}{2} + \cdots \geq \pi_d + \partial\pi_c + \partial^2\pi_c + \cdots to\ \infty = \frac{\left(\frac{\pi^m}{2}\right)}{1-\partial} = \frac{\pi^m}{2(1-\partial)}$

Or, $\frac{1}{1-\partial}\cdot\frac{1}{2}\pi^m \geq \pi_d + \frac{\delta}{1-\delta}\pi_c$

Substituting the values of $\pi_d$, and $\pi_m$ into the above equation, we get if $\delta \geq \frac{9}{17}$, then the inequality will hold and the trigger strategy will be sub-game perfect.

Thus, we see that collusion in infinitely repeated games can fetch extra payoffs to the firms.

## Check Your Progress 4

1) What is a strategy in a repeated game?

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2) What do mean by a stage game?

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3) An infinitely repeated game promising A per period with continuing cooperation, B for the one period gain from cheating, and C per period for the non-cooperative solution (B > A > C) will have a subgame perfect Nash equilibrium in trigger strategies providing-----

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4) A cartel-like collusive solution can be a Nash equilibrium only in price-setting games with -------

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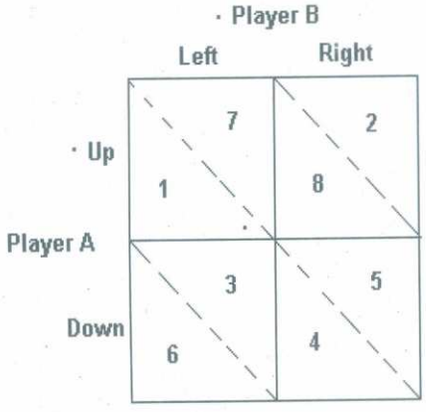
5) a) What is the equilibrium of the following game if it is played as a sequential game with Player A going first?

b) If this game is played as a sequential game with Player B going first, the equilibrium will be:



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6) What is a behavioural strategy?

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## 22.6 LET US SUM UP

In this unit we have discussed the dynamic games with complete information. The dynamic game is a sequential game where the players move after observing the decision taken by the rivals. The complete information is further divided as perfect information and imperfect information. Perfect information refers to the fact that each player has the same information that would be available at the end of the game whereas imperfect information only reduces uncertainty but (unlike perfect information) does not eliminate it. We proceed to discuss the solution concept in dynamic games with its representation in extensive form and invoking backward induction. Such a technique has helped refined the Nash equilibrium with multiple equlibria by elimination the incredible threat. It is also seen that the decision trees with non-singleton information sets as found in case of game with imperfect information. The backward induction does not work in such cases. Therefore such games are solved with subgame perfect equilibrium. In the process we have learnt that subgame is a subset of any game that includes an initial node (which has to be independent from any information set) and all its successor node. With identification of subgamesit is possible to have a *subgame perfect equilibrium that representsNash equilibrium if every subgame of the original game*is also a Nash equilibrium.Using such solution concepts we have seen their applications in Stackelberg Competition and sequential bargaining.

The unit discussed repeated games that analyses whether threats and promises about future can affect the current behavior. Besides learning the presentation of these games, we have been exposed to their solution mechanismsby considering finite and infinite time horizons. In the process we have seen the change that occurs in one shot games like prisoner's dilemma. The use oftrigger strategy where a player initially cooperates but punishes the opponent if observed a certain level of defection. Bilateral bargaining is seen as an attempt to resolve a bargaining situation, i.e., to determine the particular form of cooperation and the corresponding payoffs for two players involved. Using the framework of infinitely repeated games we have seen the process of collusion fetching extra payoff.

## 22.7 KEY WORDS

**Complete Information**: *Knowledge about other players is available to all participants.*

**Incredible Threat**: A threat made by a player in a Sequential Game which would not be in the best interest for the player to carry out.

**Backward Induction**: Backward induction is a technique to solve a game of perfect information. It first considersthe moves that are the last in the game, and determines the best move for the playerin each case.

**Common Knowledge**: A fact which all players know.All know that they all know it, andso on.

**Dominating Strategy**: A strategy dominates another strategy of a player if it always gives a better payoff tothat player, regardless of what the other players are doing.

**Extensive Game**: A game that describes with a tree how a game is played.**Game**: A game is a formal description of a strategic situation.

**Mixed Strategy**: An active randomization, with given probabilities, that determines theplayer's decision.

**Perfect Information**: At any point in time in the game only one player makes a move,and knows all the actions that have been made until then.

**Rationality**: A player seeks to play in a manner which maximizes her ownpayoff.

**Subgame Perfect Equilibrium:** An equilibrium in which all actions are Nash equilibria for all subgames.

**Perfect Information**: Each information set consists of onlyone node.

**Perfect Recall**: Each player recalls exactly what he did in thepast.

**Discount Factor**: If the rate of return is r (which is generally positive), then the ratio $\frac{1}{1+r}$ is called the discount factor. The discount factor in game theory literature is generally denoted by $\partial$.

**Finitely Repeated Games**: When a game is repeated finite number of times it is called a finitely repeated game.

**Infinitely Repeated Games**: When a game is repeated infinite number of times it is called a finitely repeated game.

**Present Value**: A future asset evaluated at present time is called the present value of the asset. If t periods later an asset is worth A, then it's present value now is given by the formula $\frac{A}{(1+r)^t}$.

**Stage Game**: In a repeated game the initial game, which is repeated in every time period is called the stage game.

**Stackelberg Competition:** A strategic game in which the leader firm moves first and then the follower firms move sequentially.

## 22.8 SOME USEFUL BOOKS

1) Fundenberg, Derw and Jean Tirole (1980), *Game Theory*, MIT Press.

2) Tirole Jean (1989), *The Theory of Industrial Organisation*, MIT Press, Cambridge Massachusetts, London, England.

3) Andreu Mas-Colell, Michael D. Whinston and Jerry R. Green (2005), Microeconomic Theory, Oxford University Press.

4) Robert Gibbons (1992), *A Primer in Game Theory,* Harvester Wheatsheaf

5) Martin J. Osborne (2003), *An Introduction to Game Theory,* Oxford University Press.

## 22. 11 ANSWER OR HINTS TO CHECK YOUR PROGRESS/HINTS

**Check Your Progress 1**

1) In a static or simultaneous move game players move one time simultaneously. Any game with sequential moves falls under the category of dynamic games. A key difference is that in a static game no new information is revealed to any of the players during the game before they make their play.

2) The extensive form is a way of describing a game using a game tree. It's simply a diagram that shows that choices are made at different points in time (corresponding to each node). The payoffs are represented at the end of each branch.

3) A complete contingent plan of actions of players as they move sequentially.

**Check Your Progress 2**

1) In a game of complete information, the structure of the game and the payoff functions of the players are commonly known but players may not see all of the moves made by other players, there may also be a chance element (as in most card games). Conversely, in games of perfect information, every player observes other players' moves, but may lack some information on others' payoffs, or on the structure of the game.

2) *The process of reasoning backwards in time from the end of a problem to determine a sequence of optimal actions.*

3) A subset of any game that includes an initial node (which has to be independent from any information set) and all its successor nodes.

4) *A subgame perfect equilibrium is a refinement of a Nashequilibrium. A strategy profile is a subgame perfect equilibrium if it represents a Nash equilibrium of every subgame of the original game.*

5) If the players played any smaller game that consisted of only one part of the larger game and arrived at a Nash equilibrium of that smaller game, then they have a subgame perfect equilibrium of the larger game. A common method for determining subgame perfect equilibria in the case of a finite game is backward induction. However, backward induction cannot be applied to games of imperfect or incomplete information because this entails cutting through non-singleton information sets.

6) a) the market demand curve, b) its own cost function, c) its rival's reaction function.

7) Under complete information, equilibrium strategies are determined by the relative impatience of the bargaining parties. In equilibrium, the proposer makes an offer so that the responder is indifferent between accepting the offer and rejecting it, given the cost of waiting; and the responder accepts the offer.

8) Backward induction.

9) { S ; t } with payoffs of (1,0). Player A's equilibrium strategy is S; B's equilibrium strategy is "t if N." To characterise a subgame perfect equilibrium, one must find the optimal strategy for a player, even if the player is never called upon to use it. In this case, although player B never has to select between "t" and "b," the fact that the player would select "t" is what makes playing "S" an equilibrium for player A.

10) { N, N, N ; b ; d } with payoffs (2,3,2). Starting at the end, Player A would select "N" in either of the last moves he may have to make. If player B is asked to make a decision, he selects b (knowing that player A will then select A). If player C is asked to make a decision, he selects d, knowing that player A will then select N. Now considering the first period, player A chooses N.

11) { S,S,N ; n,n,s } with payoffs of (4,5). Start with the last decision and work backwards to the root of the tree. At each step, be careful to concentrate only on the payoffs of the player making the decision.

**Check Your Progress 3**

1) Under perfect information each player has the same information that would be available at the end of the game. So each player knows or can see other player's moves. Take for good example the game of chess, where each player sees the other player's pieces on the board.Imperfect information, on the other hand, appears when decisions have to be made simultaneously, and players need to balance all possible outcomes when making a decision. For example, in a card game each player's card are hidden from the rest of the players.

2) In the extensive form, an information set is a set of decision nodes such that: Every node in the set belongs to one player. When play reaches the information set, the player with the move cannot differentiate between nodes within the information set, i.e. if the information set contains more than one node, the player to whom that set belongs does not know which node in the set has been reached.

3) When (i) a player does not know which node he is moving from he cannot single out the best action; (ii) preceding player into the game tree cannot anticipate what she will do; (iii) common knowledge is not able to give the usual information; and (iv) a dominated strategy not available.

4) A subgame perfect Nash equilibrium is an equilibrium such that players' strategies constitute a Nash equilibrium in every subgame of the original game.

5) It may be found by backward induction, an iterative process for solving finite extensive form or sequential games. First, one determines the optimal strategy of the player who makes the last move of the game. Then, the optimal action of the next-to-last moving player is determined taking the last player's action as given. The process continues in this way backwards in time until all players' actions have been determined.

6) In a position to do so.

7) Credible threats.

**Check Your Progress 4**

1) Strategy is a complete plan of action. A player's strategy specifies the action the player will take in every stage of the game.

2) In *a* repeated game *an extensive form* game *which consists in some number of repetitions of some base* game *called a stage* game.

3) $i > (C - A)/(A - B)$.

4) Infinite replications.

5) a) Down, Right; b) Down, Left

6) Independent assignment of strategy in each move is called the behavioural strategy. That is, you resort to independent coin toss every time an information set is encountered

## 22.12 EXERCISES

1) The simultaneous move game is played twice with the outcome of the first stage being observed before playing the second stage. There is no discounting factor. Can the payoff (4, 4) be achieved in the first stage in a pure strategy sub-game perfect Nash equilibrium? If so, give strategies that do so. If not then, why not?

| | | Player 2 | | |
|---|---|---|---|---|
| | | Left | Center | Right |
| Player 1 | Top | 3,1 | 0,0 | 5,0 |
| | Middle | 2,1 | 1,2 | 3,1 |
| | Bottom | 1,2 | 0,1 | 4,4 |

2) Suppose there are n firms in a Cournot oligopoly. Inverse demand function is given by P (Q) = a - Q, where $Q = \sum_{i=1}^{n} q_i$ . Consider the infinitely repeated game based on this stage game. What is the lowest value of $\partial$ such that the firms can use trigger strategies to sustain the monopoly output level in a sub-game perfect Nash equilibrium? How does the answer vary with n?

3) Explain with a suitable example that in an infinitely repeated game a feasible payoff can be obtained as a sub-game perfect Nash equilibrium of the game, for appropriate discounting factor.

4) Does the concept of sub-game change when we talk about infinitely repeated games.

5) Suppose in a Cournot model the discount factor $\partial < 9/17$. Now, clearly the firms cannot support a quantity as low as half of the monopoly quantity. (see Section 25.5). But for any value of $\partial$ it is sub-game perfect Nash equilibrium to repeat the Cournot quantity forever. Therefore, the most profitable quantity that trigger strategy can support is between half of monopoly output and Cournot output. Consider the following trigger strategy to compute that level of output.

**Hint: Produce q* in the first period. In the $t^{th}$ period produce q* if both firms have produced q* in each of the t-1 periods. Otherwise produce the Cournot quantity.**

6) Consider the Bertrand Duopoly model with constant marginal cost. We have seen in the previous unit that the unique Nash equilibrium of this game is ($p_1$*= $p_2$*=c). Consider the following trigger strategy:

Player i at period t=0, chooses $p^m$ (monopoly price) and at period t>0, will choose $p_i = p^m$ if and only if outcome of all earlier stages have been $(p^m, p^m)$; otherwise choose c forever.

Ans. Let us consider that player 2 strictly follows the trigger strategy. If player 1 also follows the strategy, the present value of her future payoff stream *is* $V = \frac{\pi^m}{2} + \partial\frac{\pi^m}{2} + \partial^2\frac{\pi^m}{2} + \partial^3\frac{\pi^m}{2} + \cdots to\ \infty = \frac{\left(\frac{\pi^m}{2}\right)}{1-\partial} = \frac{\pi^m}{2(1-\partial)}$

But if she deviates from her strategy, the present value of her future payoff stream is given by $V' = \pi^m + \partial(0 + 0 + 0 \cdots to\infty) = \pi^m$. Therefore, there is no point for him to deviate from the trigger strategy if

$V > V'$or, $\frac{\pi^m}{2(1-\partial)} > \pi^m or, \partial > \frac{1}{2}$.

Thus, the trigger strategy is a Nash equilibrium, but to show that it is sub game perfect Nash equilibrium, you need to show that the trigger strategy induces Nash equilibrium in every sub-game of the infinitely repeated Bertrand duopoly.

There are two possible situations

Case I: No one deviates at any point. Then the sub-game will look like the game itself. $(p^m, p^m)$ is chosen in all earlier stages and the strategy profile induces $(p^m, p^m)$ in the subsequent games, which we have shown to be Nash equilibrium provided the restriction on $\partial$.

Case II: If something else happens in the game, then the strategy will induce (c, c) at all subsequent games, which we have shown to be a Nash equilibrium in the previous unit.

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