

Block

8

NON-COOPERATIVE GAME THEORY-II

UNIT 23**Static Games of Incomplete Information
(with Application to Auction)****5**

UNIT 24**Dynamic Games with Incomplete Information:
Perfect Bayesian Equilibrium****41**

UNIT 25**Signaling Games and their Application****75**

UNIT 26**Refinements of Perfect Bayesian Equilibrium****105**

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September 2021 (Reprint)

© Indira Gandhi National Open University 1993

ISBN: 978-93-86375-56-0

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Printed and Published on behalf of the Indira Gandhi National Open University, by Registrar
MPDD, IGNOU New Delhi.

Printed at : M/s Nova Publications and Printers (Pvt.) Ltd., Plot No. 09-10, Sector-59, Phase-2 Faridabad-121004,
Phone : 0129-4317645.

BLOCK 8 NON-COOPERATIVE GAME THEORY-II

Introduction

This block takes up for discussion the non-cooperative game theory with incomplete information. Solution to static games is given in **Unit 23** by highlighting Harsanyi Formulation, which transforms incomplete information to imperfect information. Bayesian Nash Equilibrium, pure strategy Bayes-Nash equilibria (BNE) and computation of BNE pure and mixed strategies are presented along with applications of this class of games. **Unit 24** moves to dynamic games with incomplete information and introduces the solution concept of perfect Bayesian equilibrium. The idea of sequential rationality equilibrium and role of belief in taking decision with strategy profiles are presented. Consistent assessment requirement of off-the-path equilibrium is presented to help compute the perfect Bayesian equilibrium solution. Signaling games, a class of two-player games of incomplete information, is presented in **Unit 25**. Applying the solution of perfect Bayesian Nash equilibrium and computing pooling and separating equilibria form the major themes of the unit. The last, **Unit 26**, of the block deliberates on the refinement criteria, intuitive and divinity, to overcome the multiple equilibria encountered with perfect Bayesian equilibrium to improve the predictive power of the solution obtained. The signaling games remain the benchmark for such refinements in the unit.

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UNIT 23 STATIC GAMES OF INCOMPLETE INFORMATION

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23.0 OBJECTIVES

After going through this unit, you will be able to:

- identify the elements of an incomplete information game and represent them in a game form;
- understand a Bayesian game as an extensive form game with imperfect information;
- find Bayesian Nash Equilibria (BNE); and
- understand how to apply these concepts in the case of auctions.

23.1 INTRODUCTION

We explore the solution mechanism of games of incomplete information in this unit. Broadly speaking, in a game of incomplete information players do

not have common knowledge of the game being played and there is at least one player who does not know his rival's payoffs as she has some private information that is not to known others.

Recall that in a game of imperfect information, players are simply unaware of the actions chosen by other players. However, they know who the other players are, what their possible strategies/actions are, and the preferences/payoffs of these other players. Hence, information about the other players in imperfect information is complete.

In **incomplete information** games, however, players may or may not know some information about the other players, e.g., their "*type*", their strategies, payoffs or their preferences.

A search for making decision under such a situation would take us back to simultaneous move games where a player does not know what action his opponents are taking. Nevertheless, he forms an idea to choose his best response according to his belief on the possible move of the rivals.

To find solutions in such games, John Harsanyi in the 1960s developed an operational way to capture the ideas of beliefs over not only the action's of one's opponents, but also over their other characteristics, or their *types*. Posing the requirement of beliefs to be consistent, he could reach the equilibrium solution of the games with incomplete information.

2.3 THE IDEA OF INCOMPLETE INFORMATION

Like a game of complete information, a normal or strategic form game of incomplete information retains the following two components of an n -player simultaneous move game:

Players: the set of players $N = \{1, 2, \dots, n\}$;

Actions: the action space A_i of each player $i \in N$.

We still assume common knowledge of both these items. But we allow uncertainty about players' preferences.

In a game of complete information each player has a single utility function (more precisely, one equivalence class of utility functions) that maps action or strategy profiles into payoffs. But in a game of incomplete information each player may have one of many possible utility functions. So players' preferences may not be common knowledge and player i 's type determines i 's preferences/payoffs. To appreciate the underlying idea see the following example.

Recall the battle of the sexes game. It is a complete information "coordination" game. Both parties want to meet, but they have different preferences on "Ballet" and "Football". Suppose that the payoff matrix of the game is as follows:

Table 23.1: Battle of Sexes Game

Player 1	Player 2	
	B	F
B	2,1	0,0
F	0,0	1,2

In this game there are two pure strategy equilibria (one of them better for player 1 and the other one better for player 2), and a mixed strategy equilibrium.

To translate this game into one of incomplete information game, imagine that player 1 does not know whether player 2 wishes to meet or wishes to avoid her.

We represent this by thinking as player 2 having two different *types*, one *type* that wishes to meet player 1 and the other wishes to avoid her. That is to say player 2 knows what he wants (the **state of the world** is known to her), but player 1 does not have this piece of information. So we call it incomplete information.

23.2.1 Normal Form Representation of Static Games of Incomplete Information

Just as in case of static games of complete information we can represent the static games of incomplete information in the normal form with the components of a list of the number of players, the action space of each of the players, the strategies of the players and the payoffs of the-players. This representation is done in a matrix form for standard two-player games. The additional consideration in case of a static game of incomplete information is the need to show the incomplete nature of information present in the game. For that purpose, we use the terminology of “types” and “type space”.

Suppose the game is between a boy and a girl, where the boy does not know whether the girl wants to meet him or not. Such lack of information on the part of the boy is dealt with by saying that the girl has two possible *types* (wants to meet, t_1 and wants to avoid, t_2). So, the boy does not know exactly which type of girl (t_1 or t_2) he is facing. Note that the opponents’ type space is denoted by T_{-i} , from which an opponent takes a type t_{-i} .

With n players, each player with n possible actions $\{a_1, a_2, a_3, \dots, a_n\}$, the payoff for player i is represented as: $u_i = u_i(a_1, a_2, a_3, \dots, a_n; t_i)$ where, t_i is the type of player i and is an element of the space of possible *types* called type space T_i .

In that event strategies of the game can be formulated as follows:

From player 1’s point of view, player 2 has two possible types (or equivalently, the world has two possible states). Thus we write the payoff matrices in Table 23.2 such that the left matrix represents payoff of one state and the right matrix payoff for the other.

Table 23.2: Payoff matrices in Battle Sexes Games when Player 2 has Two Types

Player 1	Player 2			Player 1	Player 2		
		B	F			B	F
	B	2,1	0,0		B	2,0	0,2
	F	0,0	1,2		F	0,1	1,0
Payoff when player 2 is type 1				Payoff when player 2 is type 2			

In the game in Table 23.2, player 2 knows her type in so far as she knows whether she wishes to meet or avoid player 1. On the contrary, player 1's choice depends on his expectation concerning the unknown type of Player 2.

23.2.2 Nash Equilibrium of the Game

To understand the strategies of the game, assume that two types of player 2 have probability $1/2$ each. The payoff matrix for each type can be constructed in the normal form representation of the game. In the Table 23.2 left and right matrix has the chance of being chosen for each type with probability $1/2$.

Let us use the concept of Nash Equilibrium in an expanded form of the game, where each different type of player 2 has a different strategy. Each player form conjectures about other player's actions in each state (type) and acts optimally given these conjectures.

We have assumed that player 1 does not know player 2's type. Therefore, to choose an action rationally he needs to form a belief about the action of each type of the rival player. Given these beliefs and his belief about the likelihood of each type, he can calculate his expected payoff to each of her actions. For example, if he thinks that the type who wishes to meet him will choose B and the type who wishes to avoid him will choose F , then he could anticipate that B will yield him a payoff of 2 with probability $\frac{1}{2}$ and a payoff of 0 with probability $\frac{1}{2}$. Consequently, his expected payoff is

$$\frac{1}{2} \times 2 + \frac{1}{2} \times 0 = 1.$$

Choice of selecting F on the other hand will yield him an expected payoff of

$$\frac{1}{2} \times 0 + \frac{1}{2} \times 1 = \frac{1}{2}.$$

Similarly, calculations for the other combinations of actions for the two types of player 2 yield the expected payoffs as given in Table 23.3.

Table 23.3: Expected Payoff Matrix in Battle Sexes Games when Player 2 has Two Types

	(B,B)	(B,F)	(F,B)	(F,F)
B	2	1	1	0
F	1	$\frac{1}{2}$	$\frac{1}{2}$	1

The expected payoffs of player 1 for the four possible pairs of actions of the two types of player 2

It may be seen that each column of the table records the pair of actions for the two types of player 2. The first member of each pair gives the action of the type who wishes to meet player 1 and the second member gives the action of the type who wishes to avoid player 1.

Using the above presentation a pure strategy *Nash equilibrium can be seen as* a triple of actions, one for player 1 and one for each type of player 2. Also we find that

- the action of player 1 is optimal, given the actions of the two types of player 2 (and player 1's belief about the state); and
- the action of each type of player 2 is optimal, given the action of player 1.

To understand the above inferences we treat the two types of player 2 as separate players, and analyze the situation as a three-player strategic game in

which player 1's payoffs as a function of the actions of the two other players (i.e. the two types of player 2). Such payoffs are given in Table 23.2. Note that the payoff of each type of player 2 is independent of the actions of the other type and depends on the action of player 1 as given in Table 23.3 (the left table for the type who wishes to meet player 1, and the right table for the type who wishes to avoid player 1). In a Nash equilibrium, player 1's action is a best response to the pair of actions of the two types of player 2. Again, the action of the type of player 2 who wishes to meet player 1 is a best response in the left table of Table 23.2 to the action of player 1. Lastly, the action of the type of player 2 who wishes to avoid player 1 is a best response in the right part of Table 23.2 to the action of player 1.

It can be seen that $(B, (B, F))$, where the first component is the action of player 1 and the other component is the pair of actions of the two types of player 2, is a Nash equilibrium. Given that the actions of the two types of player 2 are (B, F) , player 1's action B is optimal (see Table 23.3); given that player 1 chooses B , B is optimal for the type who wishes to meet player 2 and F is optimal for the type who wishes to avoid player 2 (see payoff Table 23.2). Suppose that player 2 wishes to meet player 1. Then both player 1 and player 2 choose B ; player 1, who does not know if player 2 wants to meet him or avoid him, believes that if player 2 wishes to meet him she will choose B , and if she wishes to avoid him she will choose F .

To handle the games of incomplete information we have found a way out. We can start with the framework of finding solution of games with perfect information. Add supplementary considerations to that method when game is with incomplete information. Since players may have different types, we need to address the issue of what players know when they play the game. Next, we need to determine the optimal behaviour of players. Here again players maximize their payoffs given their beliefs about the situation they are in, just as we did with the analysis of games of complete information. Thus, we have to assume that players know their own preferences, which in turn will allow us to analyze a player's best response given her assumption on the behaviour of her opponents.

Check Your Progress 1

- 1) Define incomplete information game.

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- 2) What's the difference between imperfect information and incomplete information?

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- 3) How do you explain the term state of the world.

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23.3 BELIEFS

In game theoretic environment the consequence of a choice also depends on the choices made by others. In such situations it is natural to reason about the possible choices rivals. To predict about the opponents' choices, it may be helpful to reason about the possible desires and beliefs of opponents. Such a process leads to the emergence of belief hierarchies which do not only describe what one believes about the others' choices and desires, but also what one believes about the beliefs that others have about their opponents' choices and desires, and so on.

23.3.1 Harsanyi Formulation of Belief

Harsanyi (1967) stated that incomplete information in a game might force us to consider an infinite hierarchy of beliefs. Suppose there was some parameter in a game that player-1 does not know. Then player-1 could have beliefs about this parameter which, according to Bayesian decision theory, could be mathematically described by a probability distribution over the possible values of the parameter. Such a probability distribution, which might be called i 's first-order belief, could be important for predicting i 's behavior. However, it might be asked whether everybody knew i 's first-order beliefs. If not, then any other player j might have beliefs about i 's first-order beliefs, which could be described by a probability distribution over the space over the original parameter space. This second-order belief of player j could be important to predict j 's behavior, and so it must also be included in while modeling the game. Third- and higher-order beliefs might be defined similarly.

It may be important to note that as long as a model includes any parameters that are unknown to some players, we will always find more parameters, the players' subjective probability distributions over the unknowns, which are relevant to predicting players' behavior and are measurable (in principle). So Harsanyi defined a Bayesian game with incomplete information to be a mathematical model consisting of (1) a set of players, (2) a set of possible actions for each player, (3) a set of possible types for each player, (4) a payoff function that specifies each player's expected payoff for every possible combination of all players' actions and types, and (5) a probability function that specifies, for each possible type of each player, a probability distribution over the other players' possible types, describing what each type of each player would believe about the other players' types.

Working in the lines suggested by Harsanyi, it is not unreasonable assume that players know their own preferences, but they do not know the preferences, or types of their opponents. So we proceed with a set of tools that will allow for equilibrium-like analysis. In particular, we allow the players to form rational conjectures on the preferences and types of their opponents. As consequence, they themselves predict about their opponents behavior. Why? Because, as Harsanyi (1967) pointed out the game can begin with a move by Nature (Chance) which selects the different players' preferences or their types. The type profile determines not only players' payoffs but also their beliefs about other players' types. Thus, although each player does not necessarily know the actual preferences of her opponents, she knows the precise way in which the nature chooses these preferences. When taking a two-player game, we assume that both players have the same prior beliefs about the probability distribution on nature's moves. Player 1 observes his own type (i.e., he learns

the move by Nature) but player 2 does not. See that after player 1 learns his type, he has private information. Player 2, however, knows is that probability of him being of one type or another. Note that player 2's beliefs are common knowledge. That is, player 1 knows what she believes his type to be, and she knows that he knows, and so on. It is important because player 1 will be optimizing given what he thinks player 2 will do, and her behavior depends on these beliefs.

23.3.2 The Nature of "Nature"

While adding nature as a player, we simply add a proxy for saying there is some randomness in the *type* of player with whom we play a game. The probabilities associated with nature's move are the *subjective probabilities* of the player facing the uncertainty about the other player's type.

When thinking about player types, two features can be noted, viz.,

- the identity of a player is known, but his *preferences are unknown*.

"I know I am playing against Tina, but I do not know whether he is selfish or nice." Nature whispers to Tina his type, and I, the other player, have to figure it out.

- Nature selects from a population of *potential player types*. I am going to play against another player, but I do not know if she is smart or dumb, forgiving or unforgiving, rich or poor, etc. Nature decides.

To appreciate the underlying process, consider the following example of entry game:

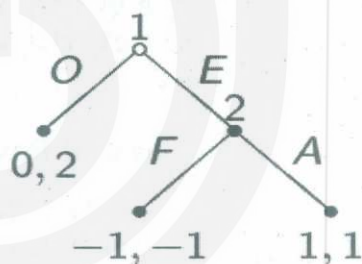


Fig. 23.1: Entry Game in Extensive Form

In the game Firm 1 is a potential entrant to an industry, which decides whether or to enter a market (E) or stay out (O). If player 1 does enter, the incumbent player 2, responds by either fighting (F) or accommodating (A) entry. The specified payoffs imply that (E,A) is a unique subgame perfect equilibrium.

Now extend the game such that instead of only one type of player 1, there are two types of player 2. One "rational" type labelled $2r$ has the payoff originally specified; the other "crazy" type labelled $2c$ enjoys fighting for its own sake, so its payoff from (Entry, Fight) is 2 instead of -1.

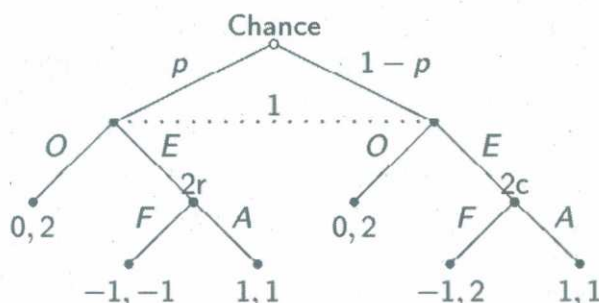


Fig. 23.2: Entry Game in Extensive Form: One Player with Two Types

The initial node of the game has Chance which moves with p or $1 - p$. Let p

Denote the exogenously specified and commonly known probability that Chance chooses "rational" (r) as player 2's type. Note that this information allows us to depict the extensive form of this entry game with incomplete information.

We assume that players know their own preferences, and so can find best responses given what they assume about other players' behaviour. This explains the dashed information sets in the extensive form entry game. The information sets show that firm 2 knows its own preferences and takes the decision. Player 1, however, remains unsure about player 2's preferences.

In the extensive form of the entry game, when we specify that p is common knowledge and it is represented by player 1's information set.

Because player 2 has a unique dominant strategy at each information set which makes the behaviour player 2 is independent of p .

Another important feature of the extensive form entry game above is that player 1 has one information set, whereas player 2 has two. This is because player 2 knows its own type, while player 1 only has a (correct) belief about player 2's type. We have specified the belief of player 1 with probabilities p and $1 - p$. Seen in terms of type of player 2 these two probabilities are related to "rational" (r) or "crazy" (c).

Following the terminology of Bayesian statistics, which Harsanyi himself adopted, this is often referred to as the common prior assumption. It means that all the players, before discovering their own types, share a common prior probabilistic theory of how Chance (Nature) chooses the players' random type profile.

23.3.4 Type Contingent Strategies and Payoffs

As has already done above, we can translate the entry game into its normal form. In this form, player 1 must have four pure strategies: at each of the two information sets there are two actions to choose from.

A strategy of player 2 can be expressed as a pair $xy \in \{A_r A_c, A_r F_c, F_r A_c, F_r F_c\} = \{A_r, A_c\} \times \{F_r A_c\}$, where $x \in \{A_r, F_r\}$ describes player 2's action when he is "rational" (r) type, and $y \in \{A_c, F_c\}$ what it does if its type is "crazy" (c). More generally: when information is incomplete, a player's strategy must prescribe what each type of that player should do. See that each player's strategy is contingent on that player's type. Of course, player 1's strategy set is simply $\{E, O\}$.

Expected Payoffs

In two-player game discussed above, each pair of pure strategies gives rise to a path of play that starts with Nature's choice, and follows with the simultaneous actions of both players. In this example, suppose player 2 plays

$A_r F_c$ i.e., A if "rational" (type r), but F if "crazy" (type c).

Suppose player 1 plays E. The outcome will yield payoffs of:

- (1,1) with probability p ;
- (-1,2) with probability $1 - p$
- Expected payoffs from the strategy pair $(E, A_r F_c)$ are

$$u_1 = p \cdot 1 + (1 - p) \cdot (-1) = 2p - 1 \text{ and } u_2 = p \cdot 1 + (1 - p) \cdot 2 = 2 - p$$

The Entry Game in Normal Form

When $P = \frac{2}{3}$ these payoff s will be $(u_1, u_2) = (\frac{1}{3}, \frac{4}{3})$.

Similarly, the payoffs from the other seven pure strategy pairs are as indicated in the table below:

		P_2			
		$A_r A_c$	$A_r F_c$	$F_r A_c$	$F_r F_c$
P_1	O	0 2*	0 2*	0* 2*	0* 2*
	E	1* 1	$\frac{1}{3}$ $\frac{4}{3}$ *	$-\frac{1}{3}$ $-\frac{1}{3}$	-1 0

This is like any other normal form game. Its origins as a game of incomplete information are irrelevant. Standard tools can find best responses and Nash equilibria.

Nash Equilibria in the Entry Game when $p = \frac{2}{3}$

		P_2			
		$A_r A_c$	$A_r F_c$	$F_r A_c$	$F_r F_c$
P_1	O	0 2	0 2	0 2	0 2
	E	1 1	$2p - 1$ $2 - p$	$1 - 2p$ $1 - 2p$	-1 $2 - p$

There are three Nash equilibria in pure strategies: $(O, F_r A_c)$, $(O, F_r F_c)$, and $(E, A_r F_c)$.

In the extensive form, only $(E, A_r F_c)$ is subgame perfect. Also, $A_r F_c$ weakly dominates all player 2's other strategies. The only mixed strategy equilibria involve 1 playing O, and 2 mixing $F_r A_c$ and $F_r F_c$ arbitrarily. (To deter entry, firm 2 must fight if "rational") Nash Equilibria in the Entry Game for General p

		P_2			
		$A_r A_c$	$A_r F_c$	$F_r A_c$	$F_r F_c$
P_1	O	0 2	0 2	0 2	0 2
	E	1 1	$2p - 1$ $2 - p$	$1 - 2p$ $1 - 2p$	-1 $2 - p$

In case $p > \frac{1}{2}$, the best responses are as for $p = \frac{2}{3}$.

The Nash equilibria are $(O, F_r A_c)$, $(O, F_r F_c)$, and $(E, A_r F_c)$, of which only $(E, A_r F_c)$ is subgame perfect in the extensive form. But in case $p < \frac{1}{2}$, the Nash equilibria are $(O, A_r F_c)$ and $(O, F_r F_c)$, of which only $(O, A_r F_c)$ is subgame perfect in the extensive form.

Example 1: Prisoner's Dilemma Game

Consider Prisoner's Dilemma Game. Assume that player 1 has the standard selfish preferences but player 2 has either selfish preferences or nice preferences. Suppose player 1's preferences now depend on whether player 2 is nice or selfish (or vice versa).

The payoff matrices the game are as follows:

		Player 2			Player 2	
		C	D		C	D
Player 1	C	4,4	0,6		6,6	2,4
	D	6,0	2,2		4,2	0,0
		Player 2 selfish			Player 2 nice	

It may be seen that

- if player 2 is selfish then player 1 will want to be selfish and choose D, but if player 2 is nice, player 1's best response is to play C;
- be nicer to those who play nice, mean to those who play mean.

Analysis of the Example

Player 2 knows his type, and plays his dominant strategy: D if selfish, C if nice.

Player 1's choice depends on her expectation concerning the unknown type of player 2.

- If player 2 is selfish, player 1's best response is to play D.
- If player 2 is nice, player 1's best response is to play C.

Suppose player 1 attaches probability p to player 2 being selfish, so $1 - p$ is the probability that player 2 is nice. Then player 1's expected payoff from C is $0p + 6(1 - p)$ and payoff from D is $2p + 4(1 - p)$.

So, $0p + 6(1 - p) = 2p + 4(1 - p)$, or, $6 - 6p = 4 - 2p$, $\rightarrow 2 = 4p$, $p = 1/2$. Thus,

- player 1's best response is to play C if $p < 1/2$, D otherwise. In first version, $p = 1/3$, play C; in second, $p = 2/3$, play D.

Example 2: Elina and the Two Faces of John

		J			J	
		Dancing	Party		Dancing	Party
M	Dancing	2,1	0,0		2,0	0,2
	Party	0,0	1,2		0,1	1,0
		John likes company			John is a loner	

Assume that John knows his true type, and therefore, which of the two games are being played. Further assume that Elina attaches probability p to John liking company and $1 - p$ to John being a loner. It is important to note that John knows Elina's estimate of p (assumption of a common prior).

Check Your Progress 2

- 1) How does Harsanyi's help find solution to games of incomplete information?

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2) What is a player type?

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3) What would you interpret players' beliefs in game theory?

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4) Explain your understanding on common prior and consistent beliefs.

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5) Differentiate between first-order and second-order beliefs in game theory.

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6) What is infinite hierarchy of beliefs?

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23.4 BAYESIAN GAMES

A Bayesian Game is a game in normal form with incomplete information that consists of:

- 1) Players $i \in \{1, 2, \dots, I\}$
- 2) Finite action set for each player $a_i \in A_i$
- 3) Finite type set for each player $\theta_i \in \Theta_i$
- 4) A probability distribution over types $p(\theta)$ (common prior beliefs about the players' types)
- 5) Utilities $u_i: A_1 \times A_2 \times \dots \times A_I \times \Theta_1 \times \Theta_2 \times \dots \times \Theta_I \rightarrow R$

In the above formulation players' types contain all relevant information about certain player's private characteristics. The type θ_i is only observed by player i , who uses this information both to make decisions and to update his beliefs about the likelihood of opponents' types (using the conditional probability $p(\theta_{-i}|\theta_i)$).

Note that the name Bayesian game is due to the way beliefs are modeled using conditional probabilities calculated using Bayes law. When a player does not know exactly the type information about its opponent, we have to specify **beliefs** that the player has about its opponent to complete the normal form depiction of the game. Saying that player i does not know the type of its opponents models the incompleteness of information in the game, but is not enough to solve the game. Player i must be able to say something more about its opponents. Here comes the beliefs that player i has about the types of its opponents. We denote this belief by the probability distribution $p(t_{-i}|t_i)$, which gives the probability with which player i of type t_i believes that its opponent is of type t_{-i} . The calculation of the conditional probability follows Bayes law.

Let us do a quick recall of this law. Given two events A and B, the probability that both these events A and B occur together is given by $p(A \cap B)$ and the probability that event B occurs alone is $p(B)$. Hence the conditional probability (occurrence of one event A conditional on another event B) is $p(A|B)$.

By Bayes law, the relationship between $p(A|B)$, $p(A \cap B)$ and the unconditional probability $p(B)$ is: $p(A|B) = p(A \cap B)/p(B)$ as long as $p(B)$ is greater than zero.

Similarly, the conditional probability $p(B|A) = p(A \cap B)/p(A)$ as long as $p(A)$ is greater than zero. By using this definition, beliefs of player i is the conditional probability that opponent's type is t_{-i} given that the player's type is t_i is given by:

$$p(t_{-i}|t_i) = p(t_1, t_2, \dots, t_n) / p(t_i); p(t_i) > 0$$

If the joint distribution $p(t_1, t_2, \dots, t_n)$ is independently distributed across types, then $p(t_{-i}|t_i) = p(t_{-i})$.

Using the Bayesian law how beliefs are updated will be taken up in the next unit while discussing dynamic games. In the static games of incomplete information we do not require such updating.

23.4.1 Bayesian Nash Equilibrium (BNE)

Bayes-Nash equilibria is generalisation of Nash equilibrium for an incomplete information game. To solve the game we need to convert the game into a game of imperfect information as a first step. Then use the Nash equilibria of imperfect information game as the solution concept.

Apply this technique to the Elina and John Game discussed above in Example 2. Elina's pure strategy choices are Dancing, D, or Party P. She can also play a mixed strategy, D with probability m . John's strategy is a *pair*, one for each "type" of John: the first component is for the John who likes company (John type 1) and the second component is for John the loner (John type 2). Pure strategies for John are thus (D,D), (D,P), (P,D), and (P,P). John also has a *pair* of mixed strategies g_1 and g_2 indicating the probability John plays D if type 1 or if type 2.

23.4.1.1 Pure Strategy Bayes-Nash Equilibria

Suppose Elina plays D for certain. Type 1 John plays D. Type 2 John plays P: John (D,P). Does Elina maximize her payoffs by playing D against the Johns' pure strategy of (D,P)? With probability p , she gets the D,D payoff 2, and with probability $1 - p$ she gets the D,P payoff, 0. So expected payoff from D against John (D,P) is $2p$. If instead, she played P against John (D,P), she would get with probability p , the P,D payoff, 0 and with probability $1 - p$ she gets the P,P payoff, 1. So expected payoff from P against John (D,P) is $1 - p$. Thus playing D against John (D,P) is a best response if $2p > 1 - p$, or if $3p > 1$, or if $p > 1/3$.

If $p > 1/3$, it is a Bayes-Nash equilibrium for Elina to play D, while the Johns play (D,P).

Next, suppose Elina plays P for certain, while type 1 John plays P and type 2 John plays D: John (P,D). Does Elina maximize her payoffs by playing P against the Johns' pure strategy of (P,D)?

With probability p , she gets the P,P payoff 1, and with probability $1 - p$ she gets the P,D payoff, 0. So expected payoff from P against John (P,D) is p .

If she instead played D against John (P,D), she would get with probability p , the D,P payoff, 0 and with probability $1 - p$ she gets the D,D payoff, 2. So the expected payoff from D against John (P,D) is $2(1 - p)$.

- Finally, playing P against John (P,D) is a best response if $p > 2(1 - p)$, or if $3p > 2$, or if $p > 2/3$.
- If $p > 2/3$, it is a Bayes-Nash equilibrium for Elina to play P, while the Johns play (P,D).

Results of the game are summarized as follows:

If $p > 2/3$, there are 2 pure-strategy Bayes-Nash equilibria

- 1) Elina plays D, the Johns play (D,P).
- 2) Elina plays P, the Johns play (P,D).

If $2/3 > p > 1/3$ there is just 1 pure strategy Bayes Nash equilibrium, #1 above, where Elina plays D and the Johns play (D,P). In our example, $p = 1/2$, so Elina should play D and the two Johns play (D,P). If $p < 1/3$ there is no pure strategy Bayes-Nash equilibrium.

We find that the normal form representation for a static game of incomplete information (or Bayesian games) with n -players is a well-defined collection of action spaces $(A_1, A_2, A_3, \dots, A_n)$, type spaces $(T_1, T_2, T_3, \dots, T_n)$, beliefs

$\{p(t_{-1}|t_1), p(t_{-2}|t_2), p(t_{-3}|t_3), \dots, p(t_{-n}|t_n)\}$ and payoffs $\{\pi_1, \pi_2, \pi_3, \dots, \pi_n\}$.

The BNE for this game is a Nash equilibrium of the game, given type spaces, beliefs and payoffs as defined above, such that the action taken by a player is the best response to the opponent's action and vice versa. No player has an incentive to deviate from such an equilibrium, as it gives the highest expected payoffs for each player.

Example 3: Consider a Cournot duopoly game in Example 1 with two firms. The two firms engage simultaneously in quantity competition with the inverse demand function given by $P(Q) = a - Q$, where $Q = q_1 + q_2$. Player 2's cost being private information (as described in Example 1) is given by $C_2(q_2) = c_H q_2$, with probability p_1 and $C_2(q_2) = c_L q_2$, with probability p_2 . Cost function of

firm 1: $C_1(q_1) = c_1 q_1$ is common knowledge. Solve for the unique Bayes Nash equilibrium (BNE) of this game.

Both the firms simultaneously maximize their profits. Firm 2's maximization is:

$\text{Max}_{\{q_2(c_H)\}} (a - q_1 - q_2(c_H) - c_H) q_2(c_H)$ when its cost function is c_H and;

$\text{Max}_{\{q_2(c_L)\}} (a - q_1 - q_2(c_L) - c_L) q_2(c_L)$ when its cost function is c_L .

In either case, the first order condition with respect to q_2 is:

$$q_2^*(c_H) = \frac{a - c_H - q_1^*}{2} \dots\dots\dots(1)$$

$$q_2^*(c_L) = \frac{a - c_L - q_1^*}{2} \dots\dots\dots(2)$$

For firm 1's maximization, its knowledge about firm 2's cost type matters as that will determine firm 2's output q_2 .

$\text{Max}_{\{q_1\}} p_1(a - q_1 - q_2(c_H) - c_1) q_1 + p_2(a - q_1 - q_2(c_L) - c_1) q_1$; where $p_1 + p_2 = 1$

The first order condition for this maximization is:

$$q_1^* = \frac{p_1[a - c_1 - q_2^*(c_H)] + p_2[a - c_1 - q_2^*(c_L)]}{2} \dots\dots\dots(3)$$

Simultaneously solving equations (1), (2) and (3), we get the unique equilibrium outputs as: $q_2^*(c_H) = \frac{[a - 2c_H + c_1]}{3} + \left[\frac{1 - p_1}{6}\right][c_H - c_L]$ and

$$q_2^*(c_L) = \frac{[a - 2c_L + c_1]}{3} - \left[\frac{p_1}{6}\right][c_H - c_L].$$

$$q_1^* = \frac{[a - 2c_1 + p_1 c_H + p_2 c_L]}{3}$$

This is the unique BNE of the game. There is an interesting comparison of outputs in the complete version Cournot game and this game. Suppose firm 2's cost was commonly known to be c_2 and firm 1's cost was c_1 , as in this case. The Cournot competition output for either firm i would be: $q_i^* = \frac{[a - 2c_i + c_j]}{3}$

Where $i \neq j$. This is the case with perfect information. Due to imperfect information, with cost c_H , firm 2's output $q_2^*(c_H) > \frac{[a - 2c_H + c_1]}{3}$ whereas firm 2's output with cost c_L , firm 2's output $q_2^*(c_L) < \frac{[a - 2c_L + c_1]}{3}$. Firm 2 reacts to firm 1's output and as it knows that firm 1 does not know 2's cost exactly, it overproduces when its costs are high to reduce output competition from firm 1. This is how firm 2 strategically use private information. Similarly, firm 2 produces less (compared to perfect information) when its costs are low reducing competition from firm 1.

Example 4. Consider the following game:

	L	R
T	1, 1	0, 0
B	0, 0	0, 0

Game A

	L	R
T	0, 0	0, 0
B	0, 0	2, 2

Game B

Step 1. Nature chooses one of the two following games with equal probability.

Step 2. Player 1 does not know which game is selected by nature, but player 2 knows.

Find all the BNE of this game.

This becomes a Bayesian game, as the beliefs of the man about the type of the woman matters for the outcome of the game. Compute the BNE of this game.

Solution: Let us characterise the elements of the game first.

Players: $n=2$ {player 1, player 2}

Actions: {T, B} for player 1; {L, R} for player 2.

Strategies: Player 1 has two strategies: {T, B}. As far as player 1 is concerned, there are two types of player 2, type 1 knowing that nature has chosen game A and playing it and type 2 knowing about and playing game B. Each of these types occur with equal probability ($1/2$). Given that there are two types of player 2, there are four possible strategies for each of the possible actions of man:

- Given that player 1 chooses T, both Type I and Type II of player 2 play L; else if player 1 chooses B, both Type I and Type II of player 2 play L.
- Given that player 1 chooses T, player 2 of Type I plays L and Type II plays R; else if player 1 chooses B, player 2 of Type I plays L and Type II goes to R.
- Given that player 1 chooses T, player 2 of Type I plays R and Type II plays L; else if player 1 chooses B, player 2 of Type I plays R and Type II plays L.
- Given that player 1 chooses T, both Type I and Type II of player 2 play R; else if player 1 chooses B, both Type I and Type II of player 2 play R.

Thus (T, (L, R)) is the strategy where player 1 chooses T and type 1 of player 2 chooses L in game A and type 2 of player 2 chooses R in game B.

Expected payoffs: Table 23.4 below shows the expected payoffs for player 1 and all the types of player 2 in the game. Here, we use the ex-ante probabilities $1/2$ for either Game A or B to weight the payoffs of either player to find out their expected payoff.

Table 23.4: Expected Payoffs for Player 1 and Both the Types of Player 2

	(L, L)	(L, R)	(R, L)	(R, R)
T	$1/2^*, 1/2^*$	$1/2, 1/2^*$	$0^*, 0$	$0, 0$
B	$0, 0$	$1^*, 1^*$	$0^*, 0$	$1^*, 1^*$

Equilibrium: We compare across strategies T and B for player 1 and put a star mark on the best responses of the both types of player 2. Similarly, we compare across all possible strategies of player 2 and star-mark the best responses of player 1. We observe that (T, (L, L)), (B, (L, R)) and (B, (R, R)) are all BNE of the game, as these are mutual best responses. For instance if player 1 plays T, it is a best response of both types of players to play L and vice versa. On the other hand, if player 1 plays B and player 2 of type 2 plays R, then player 2 of type 1 playing either L or R are best responses and vice versa. Hence, there are three pure strategy BNE in the game.

23.4.1.2 Computing BNE Pure and Mixed Strategies

Computation Example 1

Consider the following Bayesian game:

- 1) Nature decides whether the payoffs are as in Matrix I or Matrix II, with equal probabilities
- 2) P-1 is informed of the choice of Nature, P-2 is not
- 3) P-1 chooses U or D, P-2 chooses L or R (choices are made simultaneously)
- 4) Payoffs are as in the Matrix chosen by Nature

For each of these games, we will find all the Bayesian Nash equilibria.

Matrix I		
	L	R
U	1, 1	0, 0
D	0, 0	0, 0

Matrix II		
	L	R
U	0, 0	0, 0
D	0, 0	2, 2

Pure Strategy BNE

The first step is write the incomplete information problem as a static extended game with all the possible strategies.

Strategies for each player in the game are:

P-2 has only two strategies (L and R) because he does not know in which matrix the game is played.

P-1 knows in which Matrix the game occurs, and the strategies are UU (play U in case of being in Matrix I and U in case he is in Matrix II), UD, DU and DD.

Knowing the probability ($\frac{1}{2}$) the Nature locate the game in each matrix the new extended game can be written as:

	L	R
UU	$\frac{1}{2}, \frac{1}{2}$	0, 0
UD	$\frac{1}{2}, \frac{1}{2}$	1, 1
DU	0, 0	0, 0
DD	0, 0	1, 1

Recall DU is a dominated strategy for P-1. After eliminating that possibility, the game has three pure Nash Equilibrium shown by squares in the Matrix above as $\{(UU, L); (UD, R); (DD, R)\}$.

Mixed Strategy BNE

Assume the probabilities of playing each action are as shown in the matrices below (y is the probability P-2 plays L (not conditional on the matrix since this is an information P-2 does not have), x is the probability P-1 plays U if the game is in Matrix I and z is the probability P-1 plays U if the game is in Matrix II).

Matrix I

		$y \quad (1-y)$	
		L	R
x $(1-x)$	U	1, 1	0, 0
	D	0, 0	0, 0

Matrix II

		$y \quad (1-y)$	
		L	R
z $(1-z)$	U	0, 0	0, 0
	D	0, 0	2, 2

Players' best responses:

P-1's best response in Matrix I:

P-1 would play U instead of D ($x = 1$) if $1y + 0(1-y) > 0 \Rightarrow y > 0$
which can be summed up as:

- i) $y > 0 \Rightarrow x = 1$
- ii) $y = 0 \Rightarrow x \in [0, 1]$

P-1's best response in Matrix II:

P-1 would play D instead of U ($z = 0$) if $0 < 2(1-y) \Rightarrow y < 1$
which can be summarized as:

- iii) $y < 1 \Rightarrow z = 0$
- iv) $y = 1 \Rightarrow z \in [0, 1]$

P-2's best response

Recall P-2 does not know in which Matrix he or she is playing, assigning a probability of $\frac{1}{2}$ to be in one or the other.

P-2 would play L instead of R ($y = 1$) if

$$\begin{aligned} \frac{1}{2}[1x + 0(1-x)] + \frac{1}{2}[0z + 0(1-z)] \\ > \frac{1}{2}[0x + 0(1-x)] + \frac{1}{2}[0z + 2(1-z)] \end{aligned}$$

$\Rightarrow \frac{x}{2} > (1-z) \Rightarrow x > 2(1-z)$ which can be summed up as:

- v) $x = 2(1-z) \Rightarrow y \in [0, 1]$
- vi) $x > 2(1-z) \Rightarrow y = 1$
- vii) $x < 2(1-z) \Rightarrow y = 0$

Now, we can check all the possibilities in order to find the Nash Equilibria, i.e., those strategies consistent for all players. We check P-2's strategies since there are less combinations.

Mixed Equilibria

Possible Case 1

If $y = 0$, then from ii) $x \in [0, 1]$ and from iii) $z = 0$.

So we need to check this is an equilibrium from P-2's point of view. From vii) we can see that when $z = 0$, then $x < 2$ which always hold and that $y = 0$. This Nash Equilibrium supports two of the three pure BNE found before:

(DD, R) , which is the same as $y = 0, x = 0$ and $z = 0$ and (UD, R) which is the same as $y = 0, x = 1$ and $z = 0$.

Thus, we have Nash Equilibria of the form $y = 0, x \in [0, 1]$ and $z = 0$.

There are many BNE in which P-2 plays R and P-1 plays

$xU + (1 - x)D$, where $x \in [0, 1]$ if Matrix I occurs and D if Matrix II occurs.

Possible Case 2

If $y = 1$, then from iv) $z \in [0, 1]$ and from i) $x = 1$.

From vi) we can see that when $x = 1$, then it should be the case that $z > \frac{1}{2}$ in order to be true that $y = 1$. See that such BNE are restricted to $y = 1, z \in (\frac{1}{2}, 1]$ and $x = 1$.

The third pure Nash Equilibrium, (UU, L) , found before with $y = 1, x = 1$ and $z = 1$ is same with the BNE obtained now.

There are many BNE in which P-2 plays L and P-1 plays U if Matrix I occurs and $zU + (1 - z)D$, where $z \in [\frac{1}{2}, 1]$ if Matrix II occurs.

Possible Case 3

If $y \in (0, 1)$, then from i) $x = 1$ and from iii) $z = 0$.

From v) it can be seen that for y to be in interval between 0 and 1, the value of x should be equal to $2(1 - z)$. However, this equality will not hold when both $z = 0$ and $x = 1$.

Hence, the case when $y \in (0, 1)$ is not a BNE.

Computation Example 2

Consider the following Bayesian game:

- 1) Nature decides whether the payoffs are as in Matrix I or Matrix II, with equal probabilities;
- 2) P-1 is informed of the choice of Nature, P-2 is not;
- 3) P-1 chooses U or D, P-2 chooses L or R (choices are made simultaneously);
- 4) Payoffs are as in the Matrix chosen by Nature

For each of these games, find all the Bayesian Nash equilibria. Write each equilibrium in mixed behavioural strategies.

Matrix I		
	L	R
U	1, 1	0, 2
D	0, 2	1, 1

Matrix II		
	L	R
U	2, 2	0, 1
D	4, 4	2, 3

Pure Strategy BNE

Again we will collapse the incomplete information problem as a static extended game with all possible strategies.

As before P-2 has only two strategies (L and R) and P-1's strategies are UU (play U in Matrix I and U in Matrix II), UD , DU and DD .

Thus, the payoff matrix is

	L	R
UU	$\frac{3}{2}, \frac{3}{2}$	$0, \frac{3}{2}$
UD	$\frac{5}{2}, \frac{5}{2}$	$1, \frac{5}{2}$
DU	$1, 2$	$\frac{1}{2}, 1$
DD	$2, 3$	$\frac{3}{2}, 2$

It is important to see that we can eliminate two dominated strategies for P-1, UU and DU. Hence, P-1 will not play U if he knows he is in Matrix II. The game has only one pure Bayesian Nash Equilibrium given by (UD, L)

Mixed Strategy BNE

We considered the payoff matrices of the players as before to have,

Matrix I

	y	$(1-y)$
	L	R
x		
(1-x)	U $1, 1$	D $0, 2$
	D $0, 2$	D $1, 1$

Matrix II

	y	$(1-y)$
	L	R
z		
(1-z)	U $2, 2$	D $0, 1$
	D $4, 4$	D $2, 3$

Best responses of players

P-1's best response in Matrix I:

P-1 would play U instead of D ($x = 1$) if $1y + 0(1-y) > 0y + 1(1-y) \rightarrow y > 1-y$

which can be summed up as:

- i) $y > \frac{1}{2} \Rightarrow x = 1$
- ii) $y < \frac{1}{2} \Rightarrow x = 0$
- iii) $y = \frac{1}{2} \Rightarrow x \in [0, 1]$

Check for P-1's best response in Matrix II:

- iv) When P-1 observes Matrix II she will always play D, since U is a strictly dominated strategy. In other words, always $z = 0$.

P-2's best response

It may be recalled that P-2 does not know in which Matrix he is playing.

Assigning a probability $\frac{1}{2}$ we can compute the payoff.

P-2 would play L instead of R ($y = 1$) if $\frac{1}{2}[1x + 2(1-x)] + \frac{1}{2} \cdot 4 > \frac{1}{2}[2x + 1(1-x)] + \frac{1}{2} \cdot 3$, which can be summed up as:

- v) $x = 1 \Rightarrow y \in [0, 1]$
- vi) $x < 1 \Rightarrow y = 1$.

Mixed Equilibria

Possible Case 1

If $y = \frac{1}{2}$, then from iii) $x \in [0, 1]$ and from iv) $z = 0$.

We need to check if this is an Equilibrium from P-2's point of view. From vi) it can be seen that, when $x < 1$, then $y = 1$. Of course this is not the case

here. Hence the only possibility is, by v), $x = 1$, in which case $y \in [0, 1]$ (and consequently contains the value $\frac{1}{2}$). Hence we have only one BNE: $y = \frac{1}{2}$, $x = 1$ and $z = 0$.

The conclusion is there is one BNE in which P-2 plays $\frac{1}{2}L + \frac{1}{2}R$ and P-1 plays U if Matrix I occurs and D if Matrix II occurs.

Possible case 2

If $y > \frac{1}{2}$, then from i) $x = 1$ and from iv) $z = 0$.

From v) when $x = 1$, $y \in [0, 1]$. So, Nash Equilibria are: $y \in (\frac{1}{2}, 1]$, $x = 1$ and $z = 0$. These results support the unique pure Nash Equilibrium found before, i.e., (UD, L) , and these findings are the same when $y = 1$, $x = 1$ and $z = 0$.

The preceding discussion indicates that there are many BNE in which P-2 plays $yU + (1 - y)D$, where $y \in (\frac{1}{2}, 1]$ and P-1 plays U if Matrix I occurs and D if Matrix II occurs.

Possible case 3

If $y < \frac{1}{2}$, then from ii) $x = 0$ and from iv) $z = 0$.

From vi), if $x = 0$, then y should be 1 which is not possible. So, when $y \in [0, \frac{1}{2})$ it is not a BNE.

Check Your Progress 3

- 1) Give example of a Bayesian game.

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- 2) What is Bayesian equilibrium?

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- 3) You are asked to show that asymmetric information in the Market for Lemon causes inefficiency. Given the following data how would you present problem for a Bayesian Nash equilibrium. A seller and a buyer meet and decide whether to trade a used car (T) or not (N) for given price p . Trade occurs if and only if both choose T. The used car can be Peach (good): with probability $\frac{1}{2}$ or Lemon (bad): with probability $\frac{1}{2}$.

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23.5 APPLICATION TO AUCTIONS

Auctions are an important mechanism for discovering the price of an item that is to be traded between a buyer and a seller, when the underlying valuation of the buyer is unknown to the seller. In most cases, where the seller is confident about buyer valuations, price formation takes the form of posted prices (prices pre-announced by the seller; as is common in most retail markets). However, there are many situations where this is not true, as in the case of normal and “reverse auctions”. The latter occurs when the buyer has to procure a commodity and is unsure about the price at which it can be purchased. This is observed in most procurement auctions, e.g., public auctions to procure agricultural produce. We consider auctions where only a single item has to be traded.

Auctions are divided into four main categories: (a) static sealed bid: (1) first price auctions and (2) second price or Vickrey auctions (named after the economist William Vickrey who analyzed this auction in detail in 1961) and (b) dynamic open bid: (3) ascending-bid or open/oral English auctions and (4) descending-bid auctions.

For the sealed bid static auctions, participants submit their willingness to pay in sealed bids which are not revealed to the other participants. In first price sealed bid auctions, the winner is the highest bidder who pays what she bid. Sale of mineral rights in government-owned land, art and real estate auctions are often first price sealed-bid in nature. In second price auctions, the winner is the highest bidder who pays the price of the second highest bid. This is less popular, with some stamps and internet auctions being conducted in this manner.

In ascending auctions, the auctioneer conducting the auction openly raises the price of the object until only one bidder remains who wins the object by paying what she bid. This auction can also be conducted by the bidders themselves calling out the bids or the bidders simultaneously submitting their bids electronically. As the price rises continuously, existing bidders drop out from the auction. Once a bidder drops out, he or she is not allowed re-entry into the auction. Antique auctions are mostly open ascending auctions.

The descending price auction works in the opposite manner, with the auctioneer starting with a very high price, and the price is lowered continuously. The first bidder to accept the current bid wins and pays her bid. Dutch flower auctions, fish sale in Israel and tobacco trade in Canada are auctioned in this manner.

We shall be concerned with static sealed bid auctions which can be modeled as static Bayesian games.

23.5.1 Independent Private Values Model Assumptions

We assume that the underlying valuations of the **single** object to be auctioned are independently distributed for all agents (independent private values model).

Elements of the game are:

- 1) Bidders i from 1 to n with independent private values for the object.
- 2) Bidders' valuations for the object is distributed independently and identically over the space $[v_l, v_h]$ according to the cumulative probability

distribution function $F(v)$, where $F(v)$ gives the probability that the buyer's value is less than or equal to v . Let $f(v)$ be the probability density function.

- 3) Action space of the players is the bids they place $\{b_1, b_2, \dots, b_n\}$.

Let $P(b)$ be the price paid by the winner who bids b . In the first price auction, this price is the same as the highest bid and the second price auction is one where the winner pays the second highest bid.

Given that b is the winning highest bid, the expected payoff of player i is:

$$U_i(b_i, v_1, \dots, v_n) = v_i - P(b), \text{ if } b_i = b \text{ and } U_i(b_i, v_1, \dots, v_n) = 0, \text{ if } b_i < b$$

23.5.2 Second Price Sealed Bid Auctions

Auctions can be conducted under complete information or under incomplete information.

Under complete information, each of the bidders knows the value that the other bidder has for the object. While it seems unrealistic, such an assumption serves continuity of the analysis where we state that all players know the payoffs of all other players.

However, under incomplete information, each of the bidders knows its own value for the object, but does not know the value of the other bidder. However, we assume that a bidder knows about the other bidder's value is from which distribution it is drawn. For simplicity, consider that the values are drawn from the uniform distribution from 0 to 1.

In a second price auction, the player with the highest bid is the winner (ties are broken with a coin flip). The price that the winner then pays for the object is equal to the bid of the other player. That is, the price is not equal to the highest bid (the bid from the winner), but is equal to the second-highest bid (the bid from her opponent). Hence, the name of the auction is the "second-price" auction.

Clearly you would suspect that the outcome of the game will differ depending according as whether or not the game is played under complete information or under incomplete information. We will discuss in this section that the dominant strategy in a second price auction is for a player to set her bid equal to her value: $b_i = v_i$. Recall that a dominant strategy provides the (weakly) highest payoff no matter what strategy is chosen by the other player. If a strategy is dominant, then one bidder does not need to consider the payoffs and strategies of the other bidder. All a player needs to do is place a bid equal to this dominant strategy.

To prove the assertion suppose v_i is the true valuation of player i , let s_i is any bid made by player i and β be the highest bid made among all the other $(n-1)$ opponents of player i .

There are three possible outcomes for player i :

- 1) $\beta > s_i, v_i$: In this case, i does not win the object in any case and cannot do better than bid $s_i = v_i$.
- 2) $s_i > \beta > v_i$: In this case, i will win the auction but will end up paying more than her own valuation v_i . Hence, clearly, bidding any valuation above v_i is not in the best interest of player i .
- 3) $\beta < s_i, v_i$: In this case, i will win the auction and have to pay β . Hence, player i cannot better than bid v_i .

This proposition characterises the unique symmetric BNE in the second price auction, where all the players bid in a similar fashion.

23.5.3 First Price Sealed Bid Auctions

In the first price auction, the winner of the auction is the player who submits the highest bid (coin flip if tie). However, the price that the winner must pay for the object equals its own bid. Thus, the price is the highest or the “first price” bid.

Bidding one's true value is not a dominant strategy for the first price sealed bid auctions. By bidding true value, a player's payoff is zero whether or not she wins the object (as she has to pay her valuation). Therefore, the optimal strategy is to lower the bid slightly downward, while balancing two trade-offs: (i) If the bid is very close to true valuation, then the payoff is small though chance of winning is high. (ii) If the bid is very far away from true valuation, then the payoff is larger but chance of winning becomes very low. Determining the optimal trade-off depends on what opponents believe and the distribution of valuations.

We now characterise the symmetric equilibrium bidding strategy using the “first order conditions” approach.

In the symmetric equilibrium of the first price sealed bid auction, the bidders use a strictly increasing, continuous and differentiable function $b(v)$, which is given by:

$$b(v) = v - \frac{\int_{v_l}^v F^{n-1}(x) dx}{F^{n-1}(v)} < v$$

Proof: Suppose that bidders $j \neq i$ use identical bidding strategies $b_j = b(v_j)$ which are strictly increasing, continuous and differentiable functions of true valuation v_j . Now, consider bidder i 's problem. Bidder i 's expected payoff, as a function of his bid b_i and valuation v_i is: $U_i(b_i, v_i) = (v_i - b_i) \cdot \Pr[b_j = b(v_j) \leq b_i, \forall j \neq i] = (v_i - b_i) F^{n-1}(b^{-1}(b_i))$

Note that:

$$\Pr[b_j = b(v_j) \leq b_i, \forall j \neq i] = \prod_{j \neq i} \Pr[v_j \leq (b^{-1}(b_i))] = \prod_{j \neq i} F(b^{-1}(b_i)) = F^{n-1}(b^{-1}(b_i)).$$

Thus, bidder i chooses b to solve: $\max_{b_i} (v_i - b_i) F^{n-1}(b^{-1}(b_i))$

The first order condition is:

$$(v_i - b_i) (n - 1) F^{n-2}(b^{-1}(b_i)) f(b^{-1}(b_i)) \frac{1}{b'(b^{-1}(b_i))} - F^{n-1}(b^{-1}(b_i)) = 0.$$

At a symmetric equilibrium, $b_i = b(v_i)$, so the first order condition reduces to a differential equation: $b'(v) = (v - b(v)) (n - 1) f(v) / F(v)$. Using the boundary condition that $b(v_l) = v_l$, this equation can be solved to yield: $b(v) = v - \frac{\int_{v_l}^v F^{n-1}(x) dx}{F^{n-1}(v)} \dots \dots \dots (I)$

This proposition is easily understood with an example.

Example 5. Suppose the actual valuations are distributed according to the uniform distribution over $[0,1]$ in an auction with only two bidders ($n=2$). We know that the cumulative distribution function with this distribution is $F(x)=x$.

Hence, by formula (I), the actual symmetric bids made in equilibrium are:

$$b(v) = v - \frac{\int_0^v x^{n-1} dx}{v^{n-1}} = \left[\frac{n-1}{n} \right] v = v/2 \dots (II)$$

Therefore, in equilibrium, each bidder bids half his true valuation.

23.5.4 Revenue Equivalence Theorem

We have until now considered the game from the point of view of the bidders (the buyers of the object). However, in any auction, there exists a seller on the other side of the market. The seller has an objective function and prefers to sell the object at a high price. So an interesting property of auctions relates to the fact that the expected revenue received by the seller in a first price auction is identical to the expected revenue in a second price auction. We now compare the revenue of sellers in the first and second price auctions.

There are two trade-offs here as well. Whereas in the first price auction there is underbidding by buyers reducing the total collection from the sale, the seller herself commits to a lower purchase price in the second price auction by asking the winner to pay the second highest price. The balance of these trade-offs show which of these two auction formats will be preferred by the seller.

The surprising result here is that both the auctions yield the same expected revenue for the seller. The distribution of the buyers' valuation does not matter as long as the buyers follow their equilibrium strategies. This result is called the Revenue Equivalence Theorem.

The expected revenue of the seller is the same under the first and second price auctions.

Demonstration 1. To understand this result, perform a small experiment. Consider n independently drawn numbers from the uniform distribution on the interval $[0,1]$. Rank these numbers from the smallest to the largest. The expected value of the number in the i^{th} position in the list is $\frac{i}{n+1}$.

In the second price auction, the seller's revenue is the expected value of the second highest bid. With n players and bids ranked from the smallest to the largest, this bid is the $(n-1)^{\text{th}}$ bid. According to the formula given above, the expected value of this bid is $\frac{n-1}{n+1}$.

Now consider the first price auction. For the uniform distribution over $[0, 1]$, formula (II) in Example 5 shows that the buyer will bid $\left[\frac{n-1}{n} \right]$ times the valuation of the bidder. The bidder has to pay the highest bid n in the uniform distribution, whose expected value is $\left[\frac{n}{n+1} \right]$ as per formula (I). Hence the expected revenue for the seller is $\left[\frac{n-1}{n} \right] \times \left[\frac{n}{n+1} \right] = \frac{n-1}{n+1}$.

Therefore, for the uniform distribution over $[0,1]$, we have demonstrated the Revenue Equivalence Theorem. Note that the theorem holds generally for other distributions as well.

Question: Suppose there are n bidders in a first price auction with valuations independently and uniformly distributed over $[0,10]$. Find the BNE (in symmetric and linear) and the seller's expected revenue from this auction.

Solution: Given the problem, the cumulative density function is $F(v) = \frac{1}{10} v$ and the probability density function is $f(v) = \frac{1}{10}$ where $v \in [0,10]$.

We start with a guess of a symmetric linear bid used by all the bidders. Let this bid function be: $b(v) = \theta v$.

Expected payoff for the bidder with valuation v and bid b is given by (assuming all the others also use the same bidding function):

$$E(u_i) = \Pr(\text{win} | b) * [v - b] = \Pr(\text{all the other } (n-1) \text{ bids are lower than bid } b) * [v - b]$$

$$= \Pr(\theta v < b)^{n-1} * [v - b] = \Pr(v < b/\theta)^{n-1} * [v - b] = \left[\frac{1}{10} * \frac{b}{\theta} \right]^{(n-1)} [v - b]$$

The first order condition of maximizing $E(u_i)$ with respect to b leads to:

$$(n-1) \left[\frac{1}{10} * \frac{b}{\theta} \right] \left[\frac{1}{10} * \frac{b}{\theta} \right]^{(n-2)} [v - b] - \left[\frac{1}{10} * \frac{b}{\theta} \right]^{(n-1)} = 0$$

which, on simplification, yields, $b = \left[\frac{n-1}{n} \right] v$

This is true for all bidders, so that equation (III) gives the BNE bidding function.

Now, expected payment to the seller with the bidding function given above is:

$$\Pr(\text{bidder wins auction}) * (\text{bid amount } \left[\frac{n-1}{n} \right] v) = \Pr(\text{other bids are lower than } \left[\frac{n-1}{n} \right] v) * (\text{bid amount } \left[\frac{n-1}{n} \right] v) = \left[\frac{1}{10} v \right]^{n-1} * \left[\frac{n-1}{n} \right] v$$

As the seller does not know the exact value of v , she computes the expected revenue:

$$\int_0^{10} \text{expected payment } f(v) dv = 10 * \frac{n-1}{n(n+1)}$$

With n bidders in the auction, the above expression has to be multiplied by n to get the total expected revenue of the seller. Hence, the expected revenue of the auction is $10 * \frac{n-1}{(n+1)}$.

23.5.5 Common Values Model: Assumptions and Winner's Curse

Up until now, we were considering the independent private values model. Another modeling scenario is common values for the object by the bidders. This applies mostly in cases where the bidder plans to resell the object after purchase in the auction. Hence, there is a common final value of the object once it is resold, but this common value need not be known to the bidders a priori. Individual values of the object are no longer independent, and this introduces a new source of complication. This added complexity results in what is popularly known as the **winner's curse** in auctions. The curse is that the winner places a bid higher than her actual valuation of the object in order to win the object, leading to losses for the winner.

In this case, it is no longer optimal to bid one's true valuation even in a second price auction. The primary reason for this is that the winner's bid of the object is an overestimate of the common value of the object. It is common in auctions of oil-drilling tracts (for which various bidders have a common value) in the petroleum industry and spectrum auctions for mobile telecommunications.

The more the number of bidders, the extent of losses and therefore, the curse increases.

Example 6: Suppose $n=3$. The value of the object is either 53, 54 or 55 million INR, with bidders independently bidding for each of these numbers with equal probability (1/3). There is a common valuation of 54 million INR for the object, which is not common knowledge. What is the expected value of the winning bid and is there any winner's curse?

To check for the expected value of the winning bid, find out the expected value with which each of the possible values are bid for. This is shown in Table 23.5 below.

Table 23.5: Probability for Each Bid to Win

BID	Probability that this bid wins	
53 million INR	Pr (every bidder bids 53 million INR) = (A)	$= (1/3) * (1/3) * (1/3) = 1/27$
54 million INR	$= 1 - (A) - (B)$	$= 1 - 1/27 - 19/27 = 7/27$
55 million INR	$1 - \text{Pr}(\text{every bidder bids less than 55 million INR}) = (B)$	$= 1 - (2/3)^* (2/3) = 19/27$

The expected value of the winning bid is $(1/27)*53 + (7/27)*54 + (19/27)*55 = 54.67$ million INR. Therefore, there is a winner's curse as the winning bid is 0.67 million INR higher than the common value for the object. In 19/27 or about 70.4 % of the cases, the actual bid (55 million INR) is higher than the common valuation at 54 million INR.

Typically, bidders anticipate the loss and shade their bids downwards.

Example 7: Two companies are bidding for an item worth either (with probabilities): 10 ($p=1/4$), 20 ($p=1/2$) or 30 ($p=1/4$). The information given is as follows:

- If item worth 10: both bidders see "Low"
- If item worth 20: one bidder sees "Low", other sees "High" (equal chance on either)
- If item worth 30: both bidders see "High",

The Bayesian Game

- $A_i = [0, 30]$ possible bids
- $\Theta_i = \{L, H\}$
- $u_i(a_i, a_{-i}; t_i, t_{-i}) = 0$ if lose; otherwise:
$$= 30 - a_j \text{ if } \theta_i = H$$

$$= 10 - a_j \text{ if } \theta_i = L$$

$$\begin{aligned} \text{Common Values} &= 20 - a_j \text{ if } \theta_i \neq \theta_j \\ &= 10 - a_j \text{ if } \theta_i = \theta_j \end{aligned}$$

Consider a second-price auction. Get the equilibrium from the private values case: bid your expected value
See "Low" two possibilities:

- Item worth 10 (probability $1/4$)
- Item worth 20, and then i happened to be the one seeing the Low signal (probability $1/2$ times $1/2$ = probability $1/4$)
- So if see Low, expected value is 15. If other player sees Low, then actually worth 10. Other also bids 15, expected payoff = $(10-15)/2 = -2.5$. If other sees High, then lose the auction. Expected payoff when Low is $-2.5/2 = -1.25$. Better off not bidding.
- Similarly, if see High then expected value is 25

Equilibrium in this case

- If see Low: bid 10. If other sees "Low" then it is worth 10, and are bidding/paying 10, no benefit from deviating
- If other sees "High" then it is worth 20, and other is bidding 30. In order to win would have to bid at least 30, and pay 30, not worth it
- If see High: bid 30. If other sees "Low", then it is worth 20, and other is bidding 10, so you are winning paying 10, wonderful, no benefit from deviating
- If other sees "High", then it is worth 30, and are tied at bidding 30, no gain from increasing or decreasing.
- Strategy should take into account what you learn from winning
- Equilibrium can differ substantially from bidding expected value
- Results for IPV case no longer valid
- English auction no longer equivalent to second-price auction
- Revenue Equivalence no longer holds.

Check Your Progress 4

1) Differentiate between first-price auction and second-price auction.

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2) What is common value auctions?

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3) State the meaning of winner's curse

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4) What is revenue equivalence in auction theory?

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- 5) Consider the auction for a single good between three bidders. For simplicity, each bidder has value $v_1 = v_2 = v_3 = v$ for the object. Consider a third price sealed-bid auction. That is, the winner of the auction is the one with the highest bid (ties broken randomly) and the winner must pay the third highest bid for the object (with only three bidders, the third highest bid is also the lowest submitted bid). Is it a dominant strategy for a bidder to bid its value v ? Why or why not?

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- 6) Suppose you are one of two bidders in a first price sealed bid auction for Super Bowl tickets. Each bidder (including you) knows its own value for the Super Bowl tickets, but not the values of the other bidders. All a bidder knows about the values of the other bidders is that they are uniformly distributed between Rs.200 and Rs.300. Your value for the tickets is Rs.280. What is the optimal bid that you should place?

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23.6 LET US SUM UP

Static games of incomplete information have many common applications, most important of these being auctions. In these games, we convert incompleteness of information to imperfection of information by introducing the construct of beliefs which the agent without private information holds about that piece of information relevant for outcomes.

Bayes Nash method is commonly used for exploring equilibria in such games. BNE is a solution technique which requires that each player in the game gives their best response given their beliefs and strategies of opponents.

Auctions are a mechanism for allocating good(s) between a buyer and seller, particularly when the buyer's valuation is not known to the seller. Buyers bid truthfully in a second price auction with independent buyer values, but under-report valuation in a first price auction. For the seller, expected revenue from the first and second price auctions are the same. Winner's curse occurs in a common values model, where the winner bids more than her valuation.

23.7 KEY WORDS

Auction: A mechanism for allocating a good to buyers, mostly when valuation of the good by the buyers is not known to the seller

Bayesian Equilibrium: A Nash equilibrium of a Bayesian game: A list of behaviour and beliefs such that each player is doing his best to maximize his payoff, according to his beliefs about the behaviour of the other players.

Bayesian Game: An interactive decision situation involving several decision makers (players) in which each player has beliefs about (i. e. assigns probability distribution

Bayesian Nash Equilibrium (BNE): An equilibrium concept used for static games of incomplete information. It is a collection of the set of players, their actions, strategies, payoffs and beliefs such that each player gives a best response to any action of other players. No player has an incentive to deviate from the BNE.

Common Prior and Consistent Beliefs: The beliefs of players in a game with incomplete information are said to be consistent if they are derived from the same probability distribution (the common prior) by conditioning on each player's private information. That is, difference in beliefs is difference in information.

Correlated Equilibrium: A Nash equilibrium in an extension of the game in which there is a chance move, and each player has only partial information about its outcome.

First Price Sealed Bid Auction: An auction format where the winner of the auction pays the actual amount that she bid for the object.

Revenue Equivalence Theorem: The expected revenue realised from the first and second price auctions (in the independent private values model) is the same for the seller.

Second Price Sealed Bid Auction: An auction format where the winner of the auction pays the bid amount of the second highest bidder.

State of Nature: Payoff relevant data of the game such as payoff functions, value of a random variable, etc. It is convenient to think of a state of nature as a full description of a 'game-form' (actions and payoff functions).

State of the World: A specification of the state of nature (payoff relevant parameters) and the players' types (belief of all levels). That is, a state of the world is a state of nature and a list of the states of mind of all players.

Type: Also known as *state of mind*, is a full description of player's beliefs (about the state of nature), beliefs about beliefs of the other players, beliefs about the beliefs about his beliefs, etc. ad infinitum.

Winner's Curse: In a common values model, the winner bids more than her actual valuation, leading to losses despite winning the auction.

23.8 SOME USEFUL BOOKS

Gibbons, Robert (1992), *A Primer in Game Theory*, Harvester Wheatsheaf.

Osborne, M.J.(2004), *An Introduction to Game Theory*, Oxford University Press.

23.9 ANSWERS TO CHECK YOUR PROGRESS

Check Your Progress 1

- 1) Generally, players may not possess full information about their opponents. In particular, players may possess private information that others should take into account when forming expectations about how a player would behave.
- 2) In case of imperfect information, you are at a decision point and you don't know what strategy the other player has taken. But you still know that which game you are in, and also know the other players' strategy sets. Under incomplete information you might not even know what game you are into.
- 3) It is a specification of the state of nature (payoff relevant parameters) and the players' types (belief of all levels). That is, a state of the world is a state of nature and a list of the states of mind of all players.
- 4) A state of nature and a list of the states of mind of all players.

Check Your Progress 2

- 1) Harsanyi's insight transforms a game of incomplete information among players in a game of imperfect information among types.
- 2) A full description of player's beliefs (about the state of nature), beliefs about beliefs of the other players, beliefs about the beliefs about his beliefs, etc. ad infinitum.
- 3) Beliefs are probabilities that some players assign to the event of tackling this or that type of some other players, while the distribution of probability that each player has about the types of her competitors is called her system of beliefs about the other players.
- 4) The beliefs of players in a game with incomplete information are said to be consistent if they are derived from the same probability distribution (the common prior) by conditioning on each player's private information. In other words, if the beliefs are consistent, the only source of differences in beliefs is difference in information.
- 5) A player's first-order beliefs are her beliefs about the underlying uncertainty; her second-order beliefs are her beliefs about the other players' first-order beliefs. Thus, believing something is a first-order belief; believing that you believe it is a second-order belief.
- 6) Each player believes that the other player believes about what he believes... is the actual payoff associated with a certain outcome.

Check Your Progress 3

- 1) In a game the players do not have complete information on the other players (e.g., on their available strategies or payoffs), but, they have beliefs with known probability distribution. Consider a sealed bid auction where an item is to be traded between a buyer and a seller. Here the underlying valuation of the buyer is unknown to the seller.
- 2) A list of behaviour and beliefs such that each player is doing his best to maximize his payoff, according to his beliefs about the behaviour of the other players.

- Only seller knows the actual type (value) of the car. So the value of the lemon for the buyer (/seller) is 1000 (/0); the value of the peach for the buyer (/seller) is 3000 (/1500). Strategies for each player are:
 - only lemon is traded (if $p < 1000$);
 - neither types are traded. (if $1000 < p < 1500$);
 - both types are traded (if $1500 < p < 2000$);
 - neither types are traded (if $2000 < p$).

So Bayesian Nash equilibrium varies as price changes (or, games are different when prices are different). Trade is always profitable since the buyer's value is higher than the seller's for both types of the car. Asymmetric information causes inefficiency.

Check Your Progress 4

- 1) A First-Price Auction is one in which bidders submit bids, the highest bidder wins and pays the winning bid. Whereas a Second-Price Auction is one in which bidders submit bids, the highest bidder wins and pays a price equal to the second highest bid.
- 2) The value of the item for sale is identical amongst bidders, but bidders have different information about the item's value. This stands in contrast to a private value auction where each bidder's private valuation of the item is different and independent of peers' valuations.
- 3) The value of the item for sale is identical amongst bidders, but bidders have different information about the item's value. In contrast a private value auction is characterised by each bidder's private valuation of the item. It is different and independent of peers' valuations.
- 4) Winner's curse is a phenomenon that may occur in common value auctions with incomplete information. In short, the winner's curse says that in such an auction, the winner will tend to overpay.
- 5) Given certain conditions, any mechanism that results in the same outcomes (i.e., allocates items to the same bidders) also has the same expected revenue.
- 6) Let the bids for all three bidders be given by b_1, b_2 , and b_3 : A dominant strategy means that for all strategies by the other bidders, no other strategy does strictly better. Consider whether $b_1 = v$ is a dominant strategy. Let $b_2 < v$ and $b_3 > v$: The strategy $b_1 = v$ is worse than $b_1^* > b_3$. With $b_1^* > b_3$, bidder 1 wins the auction and has payoff $v - p = v - b_2 > 0$. The strategy $b_1 = v$ results in a payoff of 0 as bidder 1 loses the auction.
- 7) Suppose that you are bidder 1. The optimal bid is such that

$$b_1 = \frac{1}{2}v_1 + \frac{1}{2}(Rs.200) = \frac{1}{2}(Rs.280) + Rs.100 = Rs.240:$$

23.10 EXERCISES AND ANSWER/HINTS

- 1) Consider the first price sealed bid auction with n bidders, where valuations are independently and uniformly distributed over $[0, 1]$. Show that the Bayesian Nash equilibrium involves players bidding $\left[\frac{n-1}{n}\right]$ times their true valuation.

- 2) There are two firms in some industry: an incumbent (player 1) and a potential entrant (player 2). Player 1 decides whether to build a plant, and simultaneously player 2 decides whether to enter. Suppose that player 2 is uncertain whether player 1's building cost is 1.5 or 0, while player 1 knows his own cost. The payoffs are shown in Fig. 1. Player 2's payoff depends on whether player 1 builds or not. Entering for player 2 is profitable only if player 1 does not build.

	Enter	Don't		Enter	Don't
Build	0, -1	2, 0	Build	1.5, -1	3.5, 0
Don't	2, 1	3, 0	Don't	2, 1	3, 0
	High-cost			Low-cost	

Fig. 23.1: The Two Firm Game

- i) Denote the probability that player 2 enters with y , show the entry condition of a low-cost player 1. Write the extensive-form of the game that converts the incomplete information game into imperfect information game.
- ii) Use the extensive form of the game to show the payer 1's pure strategies, payoff matrix and best responses of the player of the game.
- 3) You may be familiar with the complete version of this coordination game, where a man and a woman simultaneously decide whether to go to the theatre (T) or to the boxing match (B). Both the players want to be together, but the woman has preference for T, whereas the man has a preference for B. In the incomplete version of the game, assume that there are two types of the woman: one type wants to go together with the man whereas the other type wants to avoid the man. The nature of asymmetry of information is that the man is unsure as to whether or not the woman wants to go with him or not. The woman, however, knows her preferences.

We model this situation by using the two following normal form representations of the game, each of which we assume to be equally likely as far as the man is concerned.

	Theatre (T)	Boxing (B)
Theatre (T)	2, 1	0, 0
Boxing (B)	0, 0	1, 2

Table A. Type I woman: wants to meet man

	Theatre (T)	Boxing (B)
Theatre (T)	2, 0	0, 2
Boxing (B)	0, 1	1, 0

Table B. Type II woman: does not want to meet man

Solution. Let us characterise the elements of the game first.

Players: $n=2$ {man, woman}

Actions: {T, B} for both the players.

Strategies: The man has two strategies: {T, B}. Given that there are two types of woman, there are four possible strategies for each of the possible actions of man:

- i) Given that man chooses T, woman of both types I and II go to T; else if man chooses B, woman of both types I and II go to T.
- ii) Given that man chooses T, woman Type I goes to T and woman Type II goes to B; else if man chooses B, woman Type I goes to T and woman Type II goes to B.
- iii) Given that man chooses T, woman Type I goes to B and woman Type II goes to T; else if man chooses B, woman Type I goes to B and woman Type II goes to T.
- iv) Given that man chooses T, woman of both types I and II go to B; else if man chooses B, woman of both types I and II go to B.

Thus (T, (T, B)) is the strategy profile where man chooses T and type I of woman chooses T and type II woman chooses B. In this manner, there are 8 possible strategy profiles, as shown in Table C below.

Expected payoffs: Table C below shows the expected payoffs for man and all the types of woman in the game. Here, we use the ex-ante probabilities $\frac{1}{2}$ for either Table A or B to weight the payoffs of either player to find out their expected payoff.

	(B,T)			
	(T, T)	(T, B)	(B, T)	(B, B)
T	2*, $\frac{1}{2}$	1*, $\frac{3}{2}$ *	1*, 0	0, 1
B	0, $\frac{1}{2}$	$\frac{1}{2}$, 0	$\frac{1}{2}$, $\frac{3}{2}$ *	1*, 1

Table C. Expected payoffs for man and both the types of woman

Note the calculation of the expected payoff for each strategy profile.

For (T, (T,T)), the payoff for man choosing T is $= \frac{1}{2} * 2$ [from Table A, 2 is the payoff is both types of woman choose (T,T) when man plays T] + $\frac{1}{2} * 2$ [from Table B, 2 is the payoff is both types of woman choose (T,T) when man plays T] $= \frac{1}{2} * (2+2) = 2$.

For the woman, the expected payoff across types is $\frac{1}{2}$ as it is $\frac{1}{2} * 1$ [from Table A, 1 is the payoff to type I woman when man plays T] + $\frac{1}{2} * 0$ [from Table B, 0 is the payoff to type II woman when man plays T] $(= \frac{1}{2})$.

Therefore, (2, ($\frac{1}{2}$ for either type of woman)) is the expected payoff for the strategy profile (T, (T, T)).

Equilibrium: We compare across strategies T and B for and put a star mark on the best responses of man for any strategy of both types of woman (using expected payoffs). Similarly, we compare across all possible strategies of both types of woman and star-mark the best responses of man. We observe that (T, (T, B)) is the equilibrium strategy profile with man choosing T, type I woman choosing T and type II woman choosing B, as it is a mutual best response. As long as man chooses T, it is a mutually best response for woman type I to choose T and type II to choose B. Therefore, it is the unique BNE of the game.

This result makes intuitive sense as well. Since, man has an innate preference for going to T (with the woman), he chooses it in equilibrium leaving to the type of woman who wants to accompany him to choose T as well. This shows that not having private information about the type of woman is better for the man. In the standard battle-of-the-sexes game with complete information, there are two equilibria: both man and woman choose T and one where both choose B.

5) Consider the game given below:

P-1 and P-2 play a game in which P-1 knows his type but P-2 does not. With probability $\frac{3}{5}$ P-1 is a person that plays considering Matrix I and with probability $\frac{2}{5}$ the game is played in Matrix II.

Matrix I		
	L	R
U	2, 4	0, 0
D	0, 0	4, -10

Matrix II		
	L	R
U	4, -10	0, 0
D	0, 0	4, 4

It is clear that if both P-1 and P-2 knew everything (specifically in which Matrix the game occurs), in each case there would be a pure strategy Nash Equilibrium (namely (U, L) in Matrix I and (D, R) in Matrix II. In the presence of incomplete information, there is no BNE in which P-2 plays a pure strategy. Can you show this?

Answer: There is a unique BNE where P-2 plays $\frac{2}{3}L + \frac{1}{5}R$, P-1 plays

$\frac{5}{9}U + \frac{4}{9}D$ if the true game is played in Matrix I and U if the true game is in Matrix II.

4) Consider the following strategic situation. Two opposed armies are poised to seize an island. Each army's general can choose either "attack" or "not attack". In addition, each army is either "strong" or "weak" with equal probability (the draws for each army are independent), and an army's type is known only to its general. Payoffs are as follows: The island is worth M if captured. An army can capture the island either by attacking when its rival does not or by attacking when its rival does if it is strong and its rival is weak. If two armies of equal strength both attack, neither captures the island. An army also has a "cost" of fighting, which is s if it is strong and w if it is weak, where $s < w$. There is no cost of attacking if its rival does not. Identify all pure strategy Bayesian Nash equilibria of this game.

Solution: There are four pure strategies for each player:

- Attack if either weak or strong type (AA);
- Attack if strong and Not Attack if weak (AN);
- Not Attack if strong and Attack if weak (NA);
- Never attack (NN).

Get normal form of the game as:

AA, AA	AA, AN	AA, NA	AA, NN
AN, AA	AN, AN	AN, NA	AN, NN
NA, AA	NA, AN	NA, NA	NA, NN
NN, AA	NN, AN	NN, NA	NN, NN

There exist four equiprobable eventualities:

Player 1 is Strong, Player 2 is Weak;

Player 1 is Strong, Player 2 is Strong;

Player 1 is Weak, Player 2 is Weak;

Player 1 is Weak, Player 2 is Strong.

Let us consider now the top-left box (AA,AA), in which the reasoning is the same for both players. We must evaluate the expected benefits and expected costs in the four cases. The players obtain the island with its value M only in one of the four cases (i.e., when he is Strong and the rival is Weak). So the expected benefit is $\frac{M}{4}$.

The costs for each of the players are equal to s when they are Strong and w when they are Weak. So, the expected costs are $\frac{s}{2} + \frac{w}{2} = 2$. Then, the expected payoffs are both given by: $\frac{M}{4} - \frac{s+w}{2}$. Consider now the box AN,AA. Now we need to perform two different reasoning for the two players. Let us start with player 1. Again he conquers the island with its value M in one of the four cases: when he is Strong and the other is Weak. So the expected benefits are $\frac{M}{4}$. Now he only sustains a cost when he is Strong, that is, in two (of four) cases. The expected costs are given by $\frac{s}{2}$.

For player 2 these are the expected benefits and costs:

He conquers the island in two cases: when the other is Weak. In this case in fact player 1 does not Attack, so his Attack will be successful. So, the expected benefit is $M/2$.

The costs for Player 2 are present only when Player 1 Attacks, that is when he is Strong, in two cases (of four). In one of this cases the cost will be equal to s and the other equal to w . The expected costs are: $s/4 + w/4$. The expected payoffs are given by: $\frac{M}{4} - \frac{s}{2}$; $\frac{M}{2} - \frac{s+w}{4}$. Continuing with the other strategy combinations Bayesian Nash equilibria would be (AA,NN); (AN,AN) and (NN,AA).

- 5) Consider a duopoly playing Cournot competition. Let firm's profit be quadratic:

$u_i = q_i(\theta_i - q_i - q_j)$ where θ_i is the difference between the intercept of linear demand curve and firm- i 's constant unit costs ($i = 1,2$) and where q_i is the quantity chosen by firm i . It is common knowledge that for firm-1, $\theta_1 = 1$ (firm-2 has complete information about firm-1, or firm-1 has only one potential type). Firm-2, however, has private information about its unit cost. Firm-1 believes that $\theta_2 = \frac{3}{4}$ with probability $\frac{1}{2}$ and $\theta_2 = \frac{5}{4}$ with probability $\frac{1}{2}$, and this belief is common knowledge. We

can call this potential types for firm-2 a low-cost type ($\theta_2 = \frac{5}{4}$) and high-cost type ($\theta_2 = \frac{3}{4}$). If the two firms choose their outputs simultaneously, find the pure-strategy Bayesian NE.

Solution: Firm- 2's equilibrium choice $q_2^*(\theta_2)$ must satisfy: $q_2(\theta_2) = \operatorname{argmax}_{q_2} q_2(\theta_2) (\theta_2 - q_1 - q_2)$ from which we get $q_2^*(\theta_2) = \frac{\theta_2 - q_1}{2}$.

Firm-1 does not know which type of firm-2 it faces, so its payoff is the expected value over firm 2's types:

$$q_1^* = \operatorname{argmax}_{q_1} \left[\frac{1}{2} q_1 (1 - q_1 - q_2^H) + \frac{1}{2} q_1 (1 - q_1 - q_2^L) \right] \text{ from which: } q_1^* = \frac{1 - q_2^H - q_2^L}{4}.$$

Now, by using $q_2^*(\theta_2)$ we obtain the unique Bayesian NE: $q_1 = \frac{1}{3}$; $q_2^H = \frac{5}{24}$; $q_2^L = \frac{11}{24}$.

UNIT 24 DYNAMIC GAMES WITH INCOMPLETE INFORMATION: PERFECT BAYESIAN EQUILIBRIUM

Structure

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- 24.1 Introduction
- 24.2 Problem with SPE
- 24.3 Requirements of Perfect Bayesian Equilibrium
- 24.4 Beliefs
- 24.5 Sequential Rationality
 - 24.5.1 Check for Sequential Rationality
 - 24.5.2 Some Worked Out Examples
- 24.6 Assessment and Perfect Equilibrium
- 24.7 Weak Sequential Equilibrium
- 24.8 Consistent Assessment Off-the-Path Equilibrium
 - 24.8.1 Trembling Hand Perfect Equilibrium
- 24.9 Let US Sum Up
- 24.10 Key Words
- 24.11 Some Useful Books
- 24.12 Answer or Hints to Check Your Progress
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24.0 OBJECTIVES

After going through this unit, you will be able to:

- understand the notion of dynamic games in a scenario of incomplete information; and
- find the central solution concept of Perfect Bayesian Equilibrium (PBNE).

24.1 INTRODUCTION

We continue with incomplete information scenario discussed in the preceding unit where a player does not know another player's characteristics (in particular, preferences). By bringing in the dynamic elements into the game, the present unit attempts to work out the perfect Bayesian equilibrium (PBE) as a solution concept.

One reason that we consider the new equilibrium concept of PBE is to rule out "implausible" strategies that satisfy the requirements for Nash equilibrium and sub-game perfect Nash equilibrium. This is not the only reason, however. The key addition with a PBE is the beliefs, which are useful when we consider games of incomplete information. Moreover, this is not to suggest that the

number of PBE is always less than or equal to the number of Nash equilibria. It only suggests that in many examples, out of the multiple Nash equilibria, the “implausible” Nash equilibria are ruled out by introducing the PBE concept.

24.2 PROBLEM WITH SPE

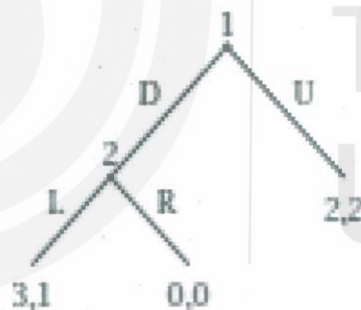
Let us recall games of sequential moves with complete information. The solution concept discussed for that class of games is sub-game perfect equilibrium (SPE). The solution consists of a strategy profile such that each sub-game of the game is a Nash equilibrium. With the games of incomplete information, however, such a solution is not adequate. To illustrate the main problem in the sub-game perfect equilibrium as a solution concept, consider the following games:

Example 1: The Little Horsey Game

Consider the following game. Player 1 gets to choose between U , M , or D . If he chooses D , the game ends. If he chooses either U or M , Player 2 gets to choose between L and R without knowing what action Player 1 has taken except that it was not D .

Example 2: Selten's Game

Some of the results obtained in the static games have important drawbacks because they ignore the dynamic nature of the extensive-form. Selten, in 1965, has shown that some Nash equilibria are “more reasonable” than others when time is allowed with the moves of the players in the games. Let us take following example in Figure 1 to see his discussion of the issue.



Selten's Example		
	L	R
U	2, 2	2, 2
D	3, 1	0, 0

Fig. 24.1: Selten's Game

From the strategic form representation of the game, it is easy to determine two pure-strategy Nash equilibria, (D, L) and (U, R) . However a close look at the Nash equilibrium (U, R) would indicate that it is somewhat less reasonable. Since Player 2's information set is never reached in the game, she loses nothing by playing R . If Player 2's information set is reached, then she would be better off by choosing L instead of R . Thus, the choice of Player 2 is only a threat meant for Player 1 which is not credible. For, given the chance, Player 2 will always play L . Therefore, Player 1 would expect her to play this if he chooses D . We get the other equilibrium, (D, L) .

The problem with (U, R) is that it specifies the incredible action at an information set that is off the equilibrium path of play. Since Player 2's information set is not reached when Player 1 chooses U , Nash equilibrium failed to track the optimality of the action at that information set. In fact, the strategy profile of the player is such that all information sets are reached with positive probability. The reason for this is, if the Nash equilibrium profile reaches all information sets with positive probability, then it will also reach all

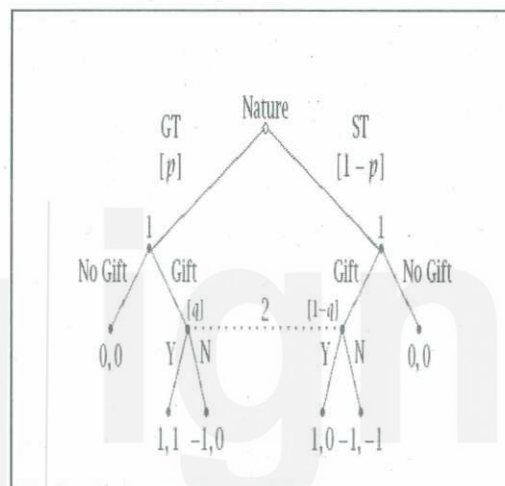
outcomes with positive probability. Consequently, no player would profit by deviating from his Nash strategy. In other words, his actions at all information sets are credible. If, on the other hand, the strategies leave some information sets off the path of play, then the Nash equilibrium is not plausible.

Example 3: Giving Gifts Game

In the game given below there are two players. Player 1 receives a book which, with probability p , is on game theory and with probability $1 - p$ a Star Trek data manual. The player wonders if it can be given as a gift to Player 2. The type of Player 2 is such that she hates Star Trek. However, she is facing problem in her game theory course. So she would prefer to have a game theory book but not the other. As she has no knowledge of what gift is being offered, her liking or otherwise of the gift can be known only when she receives it.

Strategic Form of Gift Giving Game

	Y	N
GG	1, p	$-1, p - 1$
GN	p, p	$-p, 0$
NG	$1 - p, 0$	$p - 1, p - 1$
NN	0, 0	0, 0



Player 1 observes Nature's move and offers the wrapped gift to Player 2. If the gift is accepted, then Player 1 derives a positive payoff due to satisfaction that the gift is accepted.

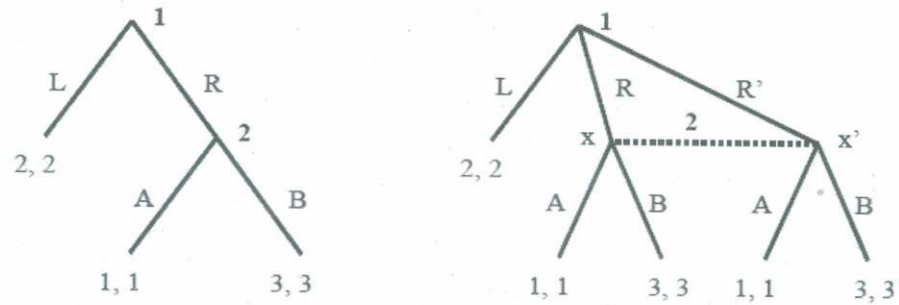
On the contrary, his payoff is -1 in the event of rejection of the gift. Player 2 prefers accepting the game theory book to not accepting it; she is indifferent between not accepting this book and accepting the Star Trek manual, but hates rejecting the Star Trek manual more than the game theory book. The strategic form of this game is as follows:

The action GN: refers to "give if Game Theory, not give if Star Trek"; (GG, Y) is a Nash equilibrium for any value of p because $p - 1 < 0 < p < 1$. Player 1 offers the gift regardless of its type, and Player 2 accepts always. In addition, (NN, N) is a Nash equilibrium. Player 1 never offers any gifts, and Player 2 refuses any gifts if offered.

Two Nash equilibria of the game are (GG, Y) and (NN, N). See that (NN, N) is irrational as accepting a gift strictly dominates non-accepting a gift, regardless of the gift, if the game ever reaches Player 2's information set.

As seen above the requirement of sub-game perfection does not work well when it fails to rule out actions that are sub-optimal given any "beliefs" about uncertainty. To understand the assertion, see the following example due to Janathan Levin, 1982.

Example 4: Consider the following games:



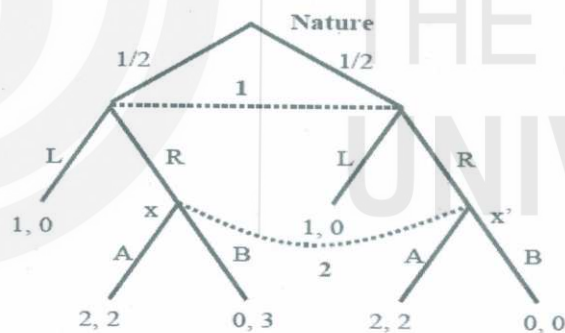
It may be seen that above two games are very similar. However, in the game on the left, (R, B) is the only SPE. However, in the game on the right, there are other SPEs, viz., $(pR + (1 - p)R', B)$ for any p , and $(L, qA + (1 - q)B)$ for any $q \geq 1/2$. A problem in the game is that it has no sub-games other than the game itself. So SPE does not have any significance and we need to find other ways for solution.

One way is to solve the problem by requiring the players to choose optimally at all information sets. To make sense of this, we need to introduce the idea of beliefs.

Example 4 (cont.): Suppose at the information set $h = \{x, x'\}$, we require player two to choose the action that maximizes her expected payoff given some belief assigning probability $\mu(x)$ to being at x and $\mu(x')$ to being at x' , with $\mu(x) + \mu(x') = 1$. Then for any belief player two might have, choosing B becomes optimal.

Another issue with the problem of SPE is that of subgame perfection not allowing actions that are possible only with beliefs that are "unreasonable".

Example 5: Consider the following game:



As may be seen that in the above game, (L, B) is a SPE and just as in the previous example, (L, B) was a SPE. Note the feature that this game has no subgames. Further, see that there are beliefs for which B is an optimal choice for Player 2. If Player 2 assigns the probability of $2/3$ on being at x and his information set $h = \{x, x'\}$, then B is an optimal choice for her. However, such beliefs seem not unreasonable. Why? Because if Player 1 chooses R, then Player 2 should place equal probability on being at either x or x' .

Check Your Progress 1

- 1) Consider the following game: Player 1 moves first and chooses among L, M, R. If he chooses R the game ends. If he plays L or M Player 2 is called upon to move. He knows that R was not chosen but he cannot distinguish between L and M. Player 2 chooses L' or M' and then the game ends.

- i) Write the normal form of the game and find the Nash equilibria
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- ii) Write the game in extensive-form representation and find out the sub-games, if any.
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- iii) Which of the equilibria of the game is based on non-credible threat? Give reasons to support your answer.
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24.3 REQUIREMENTS OF PERFECT BAYESIAN EQUILIBRIUM

To analyse dynamic games with incomplete information we introduce a new equilibrium concept, **perfect Bayesian equilibrium**.

A crucial feature of the concept is that beliefs are elevated to the level of importance of strategies. That is, the definition of equilibrium includes, in addition to the strategies, beliefs, for each player whenever the player has to move but is uncertain about the history of prior play. Previously, while discussing sub-game perfect equilibrium, players have to choose credible strategies. Now they have to hold reasonable beliefs.

To get a **perfect Bayesian equilibrium** we need to have strategies and beliefs satisfying four requirements given below:

Requirement 1. At each information set, the player with the move must have a *belief* about which node has been reached.

Requirement 2. Given their beliefs, the players' strategies must be *sequentially rational*, i.e., at each information set the action taken by the player that has to move must be optimal given:

- the beliefs at that information set;
- the other players' subsequent strategies.

Requirement 3: At information sets on the equilibrium path, beliefs are determined by Bayes' rule and equilibrium strategies.

Requirement 4: At information sets off- the-equilibrium path, beliefs are determined by Bayes' rule and equilibrium strategies when possible.

We discuss these requirements before proceeding to derive the perfect Bayesian equilibrium. To have a broad idea on perfect Bayesian equilibrium, let us start with the following description of perfect Bayesian equilibrium drawn from the lecture notes of Muhamet Yildiz available in internet.

In the game depicted in Figure 2, a firm is to decide whether to hire a worker, who can be hard-working (High) or lazy (Low). If the worker is hard-working, then working is better for the worker, and the firm makes profit of 1 when the worker works. If the worker is lazy, then shirking is better for him, and the firm will lose 1 when the worker shirks. If the worker is sequentially rational, then he will work if he is hard-working and shirk if he is lazy. Since the firm finds the worker more likely to be hard-working, the firm will hire the worker. But there is another Bayesian Nash equilibrium: the worker always shirks (independent of his type), and therefore, the firm does not hire the worker. This equilibrium is indicated by the bold lines in the figure. It shows sequential irrationality of the worker who shirks when he is hard-working. It takes place off-the-equilibrium path. Therefore, such irrationality is ignored in the Bayesian Nash equilibrium.

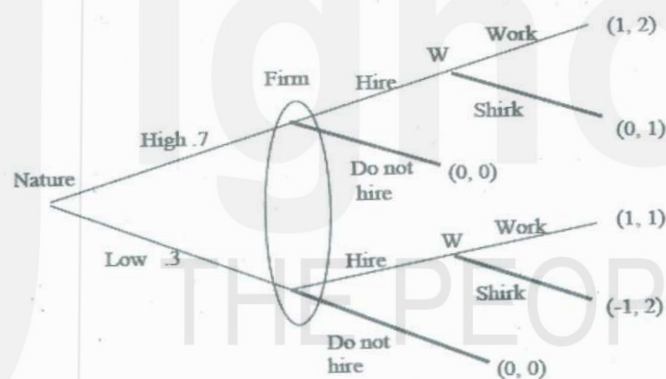


Fig. 24.2: Worker Hiring Game

To get the perfect Bayesian Nash equilibrium of the game, we require sequential rationality at each information set. To examine this, we need take up each information set and specify the beliefs of the player who moves at that information set.

Consider the game in Figure 3 for process of accomplishing such a task. In the figure, consider Player 2's information set. His beliefs are summarized by μ , which is the probability that he assigns to the event that Player 1 played T and Player 2 gets the move. The assumption of rationality requires Player 2 to choose R as it dominates L.

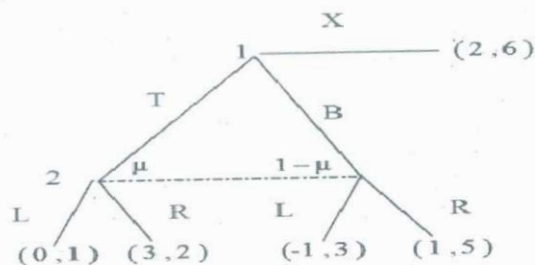


Fig. 24.3: Belief in Information Set

In Figure 4, we depict a situation in which player 1 plays T while player 2 plays R. However, such a choice of Player 2 is not rationalisable. Since Player 2 assigns probability 0.9 to the event that player 1 plays B. Why?

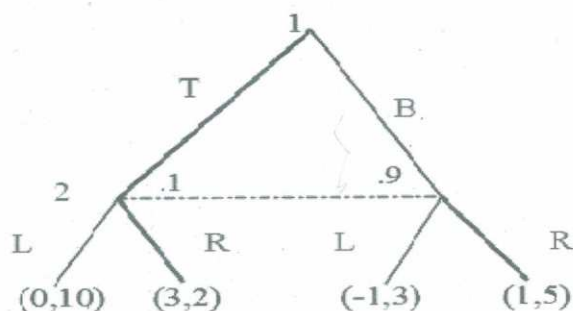


Fig. 24.4: When Choice Not Rationalisable

Given his beliefs, Player 2's move is sequentially rational. Player 1 plays his dominant strategy, therefore his move is sequentially rational. However, the problem with solution is that Player 2's beliefs are not consistent with Player 1's strategy. To arrive at equilibrium, Player 2 has to maximize his expected payoff given the other players' strategies.

To be consistent, in Figure 3, Player 2 has to assign probability 1 to the event that Player 1 plays T.

Next, we turn to the problem of information set that is reached off-the-equilibrium path. To appreciate this, consider the game in Figure 5. In this game, after player 1 plays E, there is a sub-game with a unique rationalisable strategy profile: Player 2 plays T and Player 3 plays R.

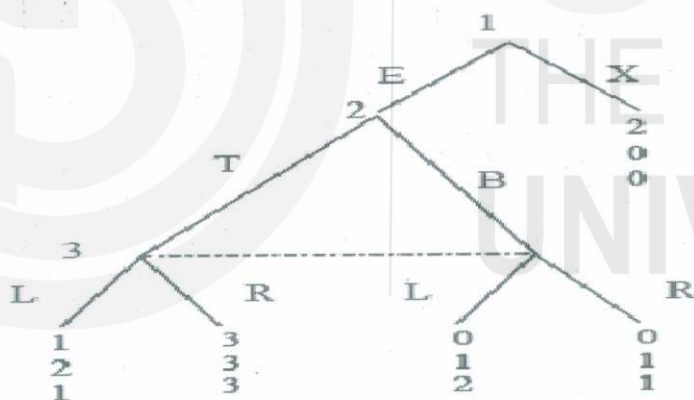


Fig. 24.5: Information Set Reached Off-the-equilibrium Path

Anticipating this, Player 1 must play E.

However, consider the strategy profile (X,T,L), in which players 1, 2 and 3 play X, T, and L, respectively and assume that, at his information set, Player 3 assigns probability 1 to the event that Player 2 plays B. See that players' moves are all sequentially rational, but Player 3's beliefs are not consistent with what the other players play. In order to reach an equilibrium, we also need beliefs of the players to be specified consistently with the other players' strategies.

Thus, the only perfect Bayesian equilibrium in Figure 5 is (E,T,R) as in it the strategy profile is sequentially rational given beliefs and belief system is consistent with the strategy profile of player.

Check Your Progress 2

- 1) List the requirements of a Bayesian equilibrium.

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- 2) When a choice is not a rationalisable.

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- 3) What is the meaning of sequentially rational move?

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- 4) Why do you want to consider perfect Bayesian equilibrium?

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24.4 BELIEFS

Note that belief is used as a technical term depicting the idea about the probability distribution of a subjective assessment of likelihood of an event. For the example, we have μ and $1 - \mu$ as the belief in the example of Battle of Sexes game given below:

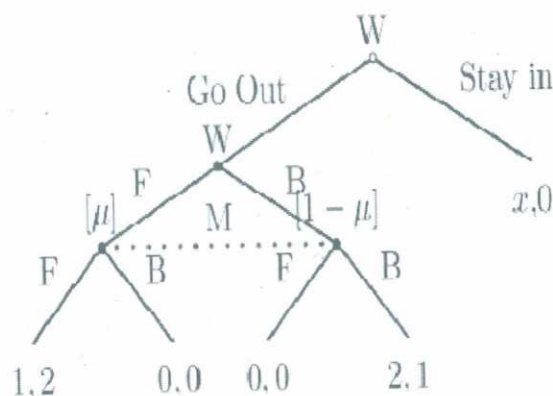


Fig. 24.6: Belief in Battle of Sexes Game

See that μ is the probability that W would choose F , given that she went out.

Note that the probability distribution expresses the individual's *beliefs*, while the information set represents what the individual *knows*. When forming new beliefs in response to new information, a rational player proceeds to modify the initial probabilities by using the rule of conditional probability. And this can only be done only if the new information is represented by an event E , which given the initial beliefs, has positive probability ($P(E) > 0$).

Thus, '*belief updating*' or '*Bayesian updating*' refers to the modification of initial beliefs expressed by an initial probability distribution P . This is done by applying the conditional probability rule and based on the assumption that the belief change is prompted by the arrival of new information, represented by an event E such that $P(E) > 0$.

The conditional probability formula can be applied to events as follows: Let A and B be two events, with $P(B) > 0$. Then the *probability of A conditional on B* , denoted by $P(A|B)$, is defined by $P(A|B) = \frac{P(AB)}{P(B)}$. It is easy to check that an equivalent way of defining $P(A|B)$ is $P(A|B) = \sum_{w \in (AB)} P(w|B)$.

Similarly, the probability of B conditional on A is given by $P(B|A) = \frac{P(BA)}{P(A)}$. Since $(AB) = (BA)$, we write $P(AB) = P(B|A)P(A)$ and $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$ which is known as *Bayes' rule*.

Check Your Progress 3

- 1) State the meaning of belief used in dynamic games with incomplete information.

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- 2) Differentiate between probability distribution of belief and information set.

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- 3) What is belief updating?

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24.5 SEQUENTIAL RATIONALITY

Recall that a player, in an extensive form game, is sequentially rational, if she uses strategy that is optimal at every node in the game tree. We call this principle *sequential rationality*, since it implies that players are playing rationally at every stage in the sequence of play. In other words, a strategy is **sequentially rational** for player i at the information set h if player i would actually want to choose the action prescribed by the strategy if h is reached. Because we require players to have beliefs for each information set, we can “split” each information set and evaluate optimal behaviour at all of its nodes. Remember, we need to include the nodes in information sets that are not reached along the equilibrium path. That is, players have to be rational not only in very sub-game but also in every continuation game.

We note some terminology here.

A **continuation game** Γ_h of an extensive form game Γ is a part of the game tree such that

- i) it starts at an information set h ;
- ii) it contains every successors of all such x that $x \in h$;
- iii) if it contains a node in an information set, then it contains all nodes in that information set.

Thus, a continuation game is composed of an information set I and the nodes that follow from that information set. Notice that the continuation game does not start with a single decision node, and hence it is not a sub-game. Broadly, the optimal action at an information set may depend on which node in the information set the play has reached. Consider the following game.

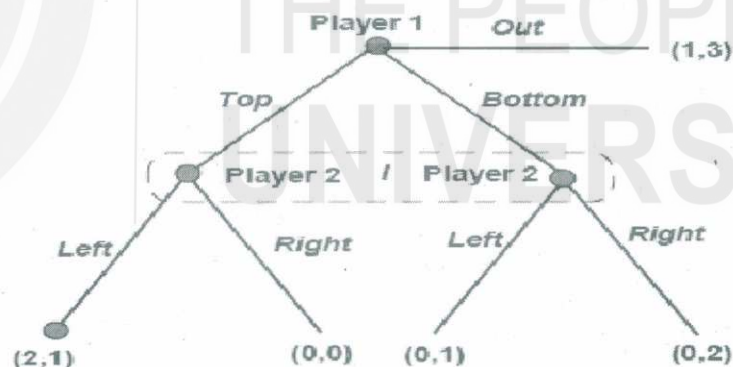


Fig. 24.7: Game Tree Depicting Continuation

It may be seen that 2's decision at the information set I depends on Player 1's option of *Top* or *Bottom*. Player 2 has no way to know the decision of Player 1. Consequently, Player 2 has to form beliefs regarding which decision node he is at.

24.5.1 Check for Sequential Rationality

We check the sequential rationality with the help of game given in Figure 8. In the figure, Player 2 assigns beliefs to the two decision nodes at the information set I . Let the probability assigned to the node that follows *Top* be $\mu \in [0,1]$ and the one assigned to the node that follows *Bottom* be $1 - \mu$. Given these beliefs, the expected payoff to action *Left* is

$\mu \cdot 1 + (1 - \mu) \cdot 2 = 2 - \mu$, whereas the expected payoff to *Right* is $\mu \cdot 0 + (1 - \mu) \cdot 1 = 1 - \mu$.

Note that $2 - \mu > 1 - \mu$ for any $\mu \in [0, 1]$. Therefore, sequential rationality requires that in equilibrium Player 2 never plays *Right* with positive probability. This eliminates the sub-game perfect equilibrium (*Out*, *Right*), which is implausible.

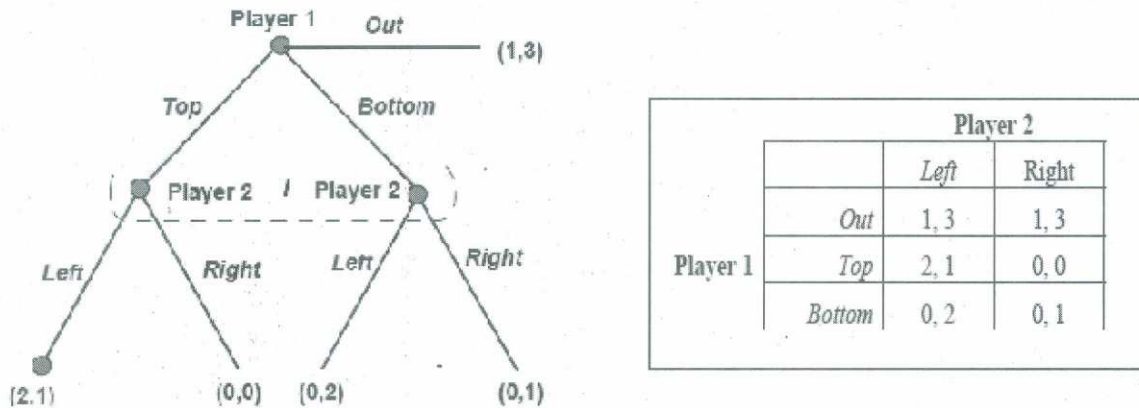


Fig. 24.8: Game Tree to Check Sequential Rationality

For reaching an equilibrium, we require the beliefs to be consistent with the players' strategies (for discussion on consistent belief see concept of assessment below.)

For example, consider the game given in Figure 24.8 above. Suppose Player 1 plays with strategies of *Out*, *Top*, and *Bottom* with probabilities $\alpha_1(\text{Out})$, $\alpha_1(\text{Top})$, $\alpha_1(\text{Bottom})$, respectively. Further, let μ be the belief assigned to node that follows *Top* in the information set *I*. If $\alpha_1(\text{Top}) = 1$ and $\mu = 0$, then inconsistency arises between Player 1's strategy and Player 2's beliefs. The only consistent belief in this case would be $\mu = 1$. We invoke Bayes' Rule, whenever possible to attain consistency and find $\mu = \frac{\alpha_1(\text{Top})}{\alpha_1(\text{Top}) + \alpha_1(\text{Bottom})}$ provided $\alpha_1(\text{Top}) + \alpha_1(\text{Bottom}) \neq 0$. In words, Player 1 plays action *Out* with probability but Player 2 does not obtain any information on his decision nodes that has been reached from the fact that the play has reached *I*.

24.5.2 Some Worked Out Examples

To calculate the expected payoff of an action at information set *h*, we have to consider the continuation game.

Example 1. Consider 'The Little Horsey Game' given above. Let *p* denote Player 2's belief that she is at the left node in her information set conditional on this set having been reached. Find her unique sequentially rational strategy.

The expected payoff from choosing *L*: $p \times (1) + (1 - p) \times (2) = 2 - p$; and the expected payoff from choosing *R*: $p \times (0) + (1 - p) \times (1) = 1 - p$. She would choose *L* whenever $2 - p > 1 - p$.

Hence, the unique sequentially rational strategy for Player 2 at her information set is to choose *L* with certainty.

Example 2: Using the Gift Giving Game above, compute the unique sequentially rational strategy of Player 2.

Expected payoff of Player 2 from choosing $Y = q(1) + (1 - q)(0) = q$ and the expected payoff from choosing $N = q(0) + (1 - q)(-1) = q - 1$. Since $q > q - 1$ for all values of q , it is not optimal for Player 2 to choose N regardless of the beliefs Player 2 might hold. Therefore, the strategy N is not sequentially rational because there is no belief that Player 2 can have that will make it optimal at her information set. So, the unique sequentially rational strategy is to choose Y .

Check Your Progress 4

- 1) What is strategy of sequentially rational?

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- 2) State the meaning of continuation of a game.

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- 3) When you cannot use Bayes' Rule.

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- 4) While working with games of incomplete information when do you need to endow players with beliefs in every part of the game?

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24.6 ASSESSMENT AND PERFECT EQUILIBRIUM

An assessment in an extensive game is a pair consisting of (1) a profile of behavioural strategies and (2) a belief system. It constitutes an equilibrium if it satisfies the following two conditions:

- i) **Sequential rationality:** each player's strategy is optimal whenever she has to move, given her own belief and the other players' strategies.

- The strategy has to be optimal in every information set, regardless of whether that information set is reached if the players follow their strategies.
- ii) **Consistency of beliefs** with strategies: each player's belief is consistent with the strategy profile.
- Each player's belief must be correct in equilibrium.

We start with elaboration on the concept of assessment. Given an extensive-form game G , an *assessment* for G is a pair (σ, μ) , where σ is a profile of behavioral strategies and μ is a list of probability distributions of beliefs, one for every information set, over the nodes in that information set. We call μ a *system of beliefs*.

The system of beliefs μ specifies, for every information set, the beliefs –aboutpast moves – that the relevant player would have if informed that her information set had been reached. Consider, for example, the extensive-form game of Figure 9.

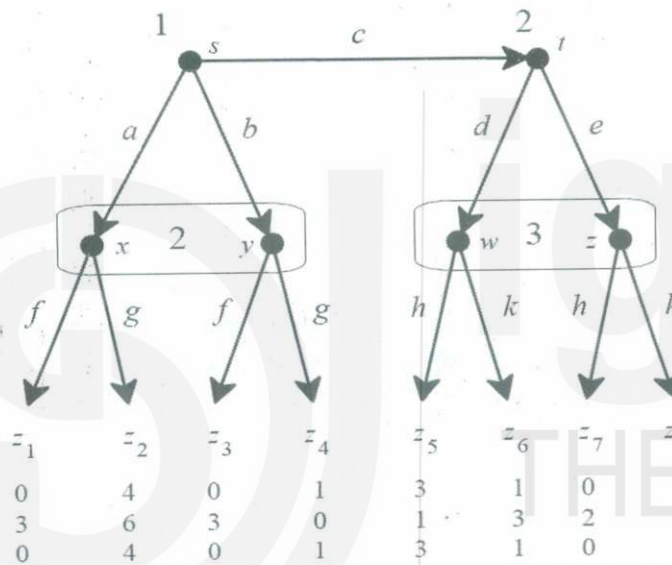


Fig. 24.9: Assessment in Extensive Form Game

A possible assessment for this game is (σ, μ) with the set of strategies $(a, b, c; f, g; d, e; h, k)$ such that σ = probabilities assigned to each strategy and information set (x, y, w, z) to contain μ = belief system. The interpretation of this assessment is that Player 1 plans to play a with probability $\frac{1}{8}$, b with probability $\frac{3}{8}$ and c with probability $\frac{4}{8}$; Player 2 plans to play f if her information set $\{x, y\}$ is reached and to mix between d and e with probabilities $\frac{3}{4}$ and $\frac{1}{4}$, respectively, if her decision node t is reached; Player 3 plans to mix between h and k with probabilities $\frac{1}{5}$ and $\frac{4}{5}$, respectively, if his information set $\{w, z\}$ is reached. Furthermore, Player 2 –if informed that her information set $\{x, y\}$ had been reached – would attach probability $\frac{2}{3}$ to Player 1 having played a (node x) and probability $\frac{1}{3}$ to Player 1 having played b (node y); Player 3 – if informed that his information set $\{w, z\}$ had been reached – would attach probability $\frac{1}{2}$ to node w (that is, to the sequence of moves cd) and probability $\frac{1}{2}$ to node z (that is, to the sequence of moves ce).

An assessment (σ, μ) is “reasonable” when it meets two requirements:

- The choices specified by σ should be optimal given the beliefs specified by μ . We call this requirement *sequential rationality*.
- The beliefs specified by μ should be consistent with the strategy profile σ . We call this requirement *Bayesian updating*.

For example, assessment (σ, μ) in the game of Figure 9 fails to satisfy sequential rationality. Because at Player 3's information set $\{w, z\}$ the planned mixed strategy (h, k) yields Player 3 – given her beliefs (wz) – an expected payoff of $\frac{1}{2} \left[\frac{1}{5}(3) + \frac{4}{5}(1) \right] + \frac{1}{2} \left[\frac{1}{5}(0) + \frac{4}{5}(1) \right] = \frac{10}{11}$ while she could get a higher expected payoff, viz., $\frac{1}{2} \left[\frac{1}{2}(3) + \frac{1}{2}(0) \right]$, with the pure strategy h . This assessment also fails in respect of belief updating. For example, given σ , the prior probability that node x is reached is $P(x) = \frac{1}{8}$ (it is the probability with which Player 1 plays a) and the prior probability that node y is reached is $P(y) = \frac{3}{8}$ (the probability with which Player 1 plays b). So updating on information $\{x, y\}$ we get,

$$P(x | \{x, y\}) = \frac{P(x)}{P(x) + P(y)} = \frac{\frac{1}{8}}{\frac{1}{8} + \frac{3}{8}} = \frac{1}{4} \quad \text{and} \quad P(y | \{x, y\}) = \frac{P(y)}{P(x) + P(y)} = \frac{3}{4}.$$

That is, Player 2's beliefs should be $(\frac{1}{4}, \frac{3}{4})$ in order to be consistent with the rule for belief updating.

In general, we say that σ is *sequentially rational* if, for every Player i , σ_i is sequentially rational. Note that for Player i 's strategy σ_i to be sequential rationality, we need to check if Player i could improve her payoff by changing her choice(s) at H and possibly also at her information sets that follow H . To see this, consider the game of Figure 10 below.

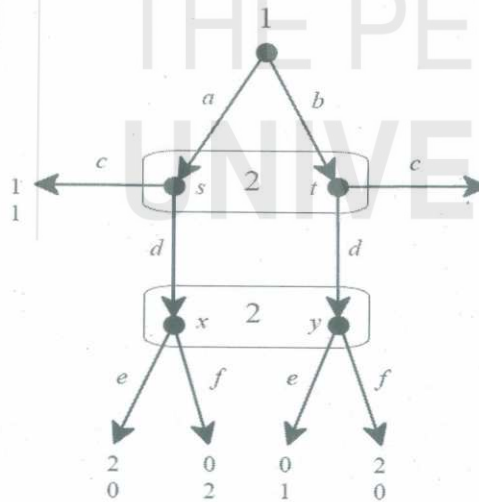


Fig. 24.10: Assessment Examination-i

Let (σ, μ) be the assessment where σ is the pure-strategy profile $(a, (c, e))$ and belief system of $(s, t; x, y)$ with $\mu = (1, 0; 0, 1)$. For Player 2, e is rational at information set $\{x, y\}$ because, given her belief that she is making her choice at node y , e gives her a payoff of 1 while f would give her a payoff of 0; furthermore, at information set $\{s, t\}$ – given her belief that she is making her choice at node s and given her future choice of e at $\{x, y\}$ – choice c is better than choice d because the former gives her a payoff of 1 while the latter a payoff of 0. However, the strategy (c, e) (while sequentially rational at $\{x, y\}$) is not so at $\{s, t\}$. This is because of her belief that she is making her choice at

node s – with (c,e) , she gets a payoff of 1; but if she switched to (d,f) she would get a payoff of 2. In other words, Player 2 can improve her payoff by changing her choice at $\{s,t\}$ from c to d and also changing her future planned choice at $\{x,y\}$ from e to f . Note that the pure-strategy profile $(a, (c,e))$ is not a Nash equilibrium, for Player 2, the unique best reply to a is (d,f) .

An example of a sequentially rational assessment for the game in Figure 11 is strategy profile $(L,M,R; A,B; c,d)$ and belief system $(x,y; u,v,w)$ with $\sigma = (0,0,1; \frac{1}{2}, \frac{1}{2}; 1,0)$ and $\mu = (\frac{1}{4}, \frac{3}{4}; \frac{1}{5}, \frac{3}{5}, \frac{1}{5})$.

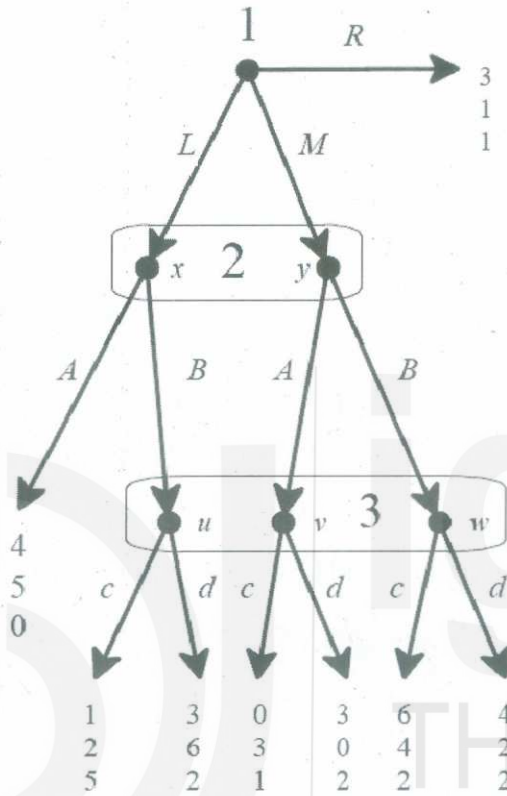


Fig. 24.11: Assessment Examination-ii

First, let us examine the sequential rationality of Player 3's strategy. At her information set $\{u, v, w\}$, given her beliefs, c has the payoff at $c = \frac{1}{5}(5) + \frac{3}{5}(1) + \frac{1}{5}(2) = 2$, while at d it is equal to $\frac{1}{5}(2) + \frac{3}{5}(2) + \frac{1}{5}(2) = 2$. Thus, both c and d , including any randomization over c and d are optimal.

Consider the optimal choice of Player 2. At his information set $\{x, y\}$, given his beliefs and given the strategy of Player 3, payoff at due to $A = \frac{1}{4}(5) + \frac{3}{4}(3) = 3.5$ while it is equal to $\frac{1}{4}(2) + \frac{3}{4}(4) = 3.5$ for a move with B . Also, note that any mixture of A and B is optimal for player 2. Here also the mixture $(A, B) = (\frac{1}{2}, \frac{1}{2})$ is optimal. Finally, at the root, R the payoff of gives Player 1 is 3; a move with L has a payoff of $\frac{1}{2}(4) + \frac{1}{2}(1) = 2.5$ and that with M is equal to $\frac{1}{2}(0) + \frac{1}{2}(6) = 3$. Therefore, R including any mixture over M and R is optimal.

The second requirement for an assessment to be "reasonable" is that the beliefs profile μ should be consistent with the behaviour postulated by σ . That is, the rule for belief updating has to be applied to form those beliefs, whenever it is applicable. To understand the rule, which is called Bayesian updating, we consider the following:

Let x be a node that belongs to information set H and let $P_{root,\sigma}(x)$ be the probability that node x is reached from the root of the tree. If σ is implemented, i.e., if $(a_1 \dots a_m)$ is the sequence of choices that leads from the root to x , then $P_{root,\sigma}(x) = \sigma(a_1) \times \dots \times \sigma(a_m)$. Seen in terms of an extensive-form game and a behavioural strategy σ , we say that an information set H is reached by σ if $P_{root,\sigma}(H) > 0$. For example, taking σ and μ from the game of Figure 9 we see that $P_{root,\sigma}(x) = \frac{1}{3}$ and $P_{root,\sigma}(y) = \frac{2}{3}$. So $P_{root,\sigma}(x, y) = \frac{1}{3} + \frac{2}{3} = 1$. That is, $\{x, y\}$ is reached by σ . On the other hand, (w, z) is not reached since $P_{root,\sigma}(w) = \sigma(c)\sigma(d) = 0 \left(\frac{3}{4}\right) = 0$ and $P_{root,\sigma}(z) = \sigma(c)\sigma(e) = 0 \left(\frac{3}{4}\right) = 0$, resulting in $P_{root,\sigma}(w, z) = 0$.

Given an extensive-form game and an assessment (σ, μ) , we say that (σ, μ) satisfies *Bayesian updating at reached information sets* if, for every information set H and every node $x \in H$, if H is reached by σ (that is, if $P_{root,\sigma}(H) > 0$), then $\mu(x) = \frac{P_{root,\sigma}(x)}{P_{root,\sigma}(H)}$.

For example, in Figure 10.1 when we consider game after modification of strategy profiles $(a, b, c; f, g; d, e; h, k)$ and belief and belief system (x, y, w, z) . If $\sigma = \left(\frac{1}{9}, \frac{5}{9}, \frac{3}{9}; 0, 1; \frac{3}{4}, \frac{1}{4}; 1, 0\right)$ and $\mu = \left(\frac{1}{6}, \frac{5}{6}, \frac{3}{4}, \frac{1}{4}\right)$, then Bayesian updating at reached information sets is satisfied. See that

$$\begin{aligned} P_{root,\sigma}(x) &= \frac{1}{9} \quad \text{and} \quad P_{root,\sigma}(y) = \frac{5}{9} \quad \text{giving} \quad P_{root,\sigma}(x, y) = \frac{1}{9} + \frac{5}{9} = \frac{6}{9}; \\ P_{root,\sigma}(w) &= \sigma(c)\sigma(d) = \left(\frac{3}{9}\right)\left(\frac{3}{4}\right) = \frac{9}{36} \quad \text{and} \quad P_{root,\sigma}(z) = \sigma(c)\sigma(e) = \frac{3}{9}\left(\frac{1}{4}\right) = \frac{3}{36} \quad \text{giving} \quad P_{root,\sigma}(w, z) = \frac{9}{36} + \frac{3}{36} = \frac{12}{36}. \end{aligned}$$

From these results we get, $\mu(x) = \frac{P_{root,\sigma}(x)}{P_{root,\sigma}(x,y)} = \frac{1}{6}$; $\mu(y) = \frac{P_{root,\sigma}(y)}{P_{root,\sigma}(x,y)} = \frac{5}{6}$; $\mu(w) = \frac{P_{root,\sigma}(w)}{P_{root,\sigma}(w,z)} = \frac{3}{4}$; and $\mu(z) = \frac{P_{root,\sigma}(z)}{P_{root,\sigma}(w,z)} = \frac{1}{4}$.

Thus perfect Bayesian equilibrium incorporates sequential rationality and consistency of beliefs. That is, in equilibrium (i) each player's strategy specifies optimal actions, given her belief and strategies of other players, and (ii) beliefs are consistent with Bayes' rule wherever possible.

Check Your Progress 5

- 1) What are two conditions of defining a consistent belief?

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- 2) Define the term assessment in an extensive game.

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3) What do you mean by a behavioural strategy?

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4) Define sequential rationality for a strategy profile.

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5) While working with games of incomplete information when do you need to endow players with beliefs in every part of the game?

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24.7 WEAK SEQUENTIAL EQUILIBRIUM

Note that the “weak” in “weak perfect Bayesian” refers to the lack of restrictions on off-the-equilibrium path beliefs. Such beliefs may induce off equilibrium actions. That in turn may influence behaviour on-equilibrium path. We need restrict these beliefs.

A *weak sequential equilibrium* is an assessment (σ, μ) which satisfies two requirements, viz., (1) sequential rationality and (2) Bayesian updating at information sets reached.

We illustrate the type of reasoning that one needs to compute weak sequential equilibria. Consider first the game of Figure 24.12.

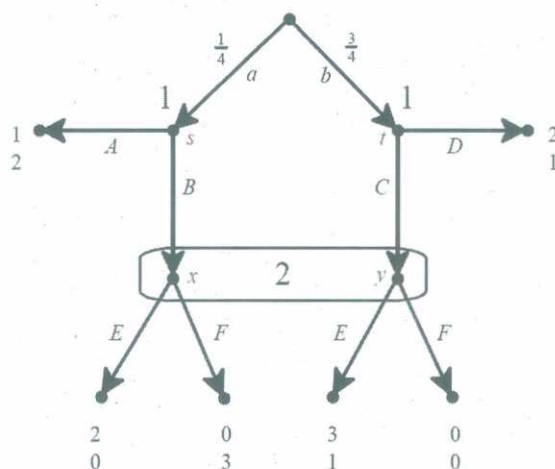


Fig. 24.12: Weak Sequential Equilibrium Examination

Let us see if there is a weak sequential equilibrium (σ, μ) where Player 1's strategy in σ is a pure strategy. The set of pure strategies of Player 1 is $S_1 = \{(A, C), (B, D), (B, C), (A, D)\}$

The strategy of Player 1 determines the beliefs of Player 2 at her information set $\{x, y\}$. Let us consider the four possibilities.

- If Player 1's strategy is (A, C) , then Player 2's information set $\{x, y\}$ is reached with positive probability and the only beliefs that are consistent with Bayesian updating are $(0, 1)$. So, by sequential rationality, Player 2 must choose E . However, if Player 2's strategy is E , then at node s it is not sequentially rational for Player 1 to choose A . Thus there is no weak sequential equilibrium where Player 1's strategy is (A, C) .
- If Player 1's strategy is (B, D) , then Player 2's information set $\{x, y\}$ is reached with positive probability and the only beliefs that are consistent with Bayesian updating are $(0, 1)$, so that —by sequential rationality—Player 2 must choose F . However, if Player 2's strategy is F then at node s it is not sequentially rational for Player 1 to choose B . Thus there is no weak sequential equilibrium where Player 1's strategy is (B, D) .
- If Player 1's strategy is (B, C) , then Player 2's information set $\{x, y\}$ is reached (with probability 1) and the only beliefs that are consistent with Bayesian updating are $(\frac{1}{4}, \frac{3}{4})$. Given these beliefs, Player 2's payoff from playing E is $\frac{1}{4}(0) + \frac{3}{4}(1) = \frac{3}{4}$ and her payoff from playing F is $\frac{1}{4}(3) + \frac{3}{4}(0) = \frac{3}{4}$. Thus, any mixed strategy is sequentially rational for Player 2, that is, for any $\varepsilon[0, 1]$, playing at (E, F) with probability p and $1 - p$ is sequentially rational. At node s choice B is sequentially rational for Player 1 if and only if the expected payoff from playing B is at least 1 (which is the payoff from playing A): $2p + 0(1 - p) \geq 1$, that is, $p \geq \frac{1}{2}$. At node t choice C is sequentially rational for Player 1 if and only if the expected payoff from playing C is at least 2 (which is the payoff from playing D): $3p + 0(1 - p) \geq 2$, that is, $p \geq \frac{2}{3}$. Hence, if $p \geq \frac{2}{3}$ then both B and C are sequentially rational. Thus we have an infinite number of weak sequential equilibria at which Player 1's strategy is (B, C) : for every $p \in [\frac{2}{3}, 1]$, (σ, μ) is a weak sequential equilibrium, where the set of strategies $(A, B; C, D; E, F)$ has profile $\sigma = (0, 1; 1, 0; p, 1 - p)$ and information set (x, y) is reached with $\mu = (\frac{1}{4}, \frac{3}{4})$.
- If Player 1's strategy is (A, D) , then Player 2's information set $\{x, y\}$ is not reached and thus, according to the notion of weak sequential equilibrium, any beliefs are allowed there. Let Player 2's strategy at E, F be assigned probabilities $(p, 1 - p)$. From previous calculations we have that both A and D are sequentially rational if and only if $p \leq \frac{1}{2}$. One possibility is to set $p = 0$; this means that Player 2 chooses the pure strategy F and this is sequentially rational if and only if her beliefs at (x, y) are $(q, 1 - q)$ with payoff from F greater than payoff from E , i.e., $3q + 0(1 - q) \geq 0q + 1(1 - q)$ or, $q = \frac{1}{4}$. Thus, we have an infinite number of weak sequential equilibria: (σ, μ) is a weak sequential equilibrium, where with strategies $(A, B; C, D; E, F)$, $\sigma = (0, 1; 1, 0; 0, 1)$ and information set (x, y) is reached with beliefs $\mu = (q, 1 - q)$ for every $q \in [\frac{1}{4}, 1]$.

Next consider the case where $0 < p \leq \frac{1}{2}$, so that Player 2 uses a mixed strategy. This can be sequentially rational if and only if the payoff from E is equal to the payoff from F , that is, if and only if her beliefs at information set (x, y) computed above are $(\frac{1}{4}, \frac{3}{4})$. So, again for every $p \in [0, \frac{1}{2}]$ we have an infinite number of weak sequential equilibria.

Check Your Progress 6

- 1) What do you mean by weak in weak perfect Bayesian?

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- 2) State the requirements of a *weak sequential equilibrium*.

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- 3) Why do we need to be so careful to specify beliefs properly in a weak sequential equilibrium?

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- 4) Define a Weak Perfect Bayesian Equilibrium.

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- 5) Why do off-equilibrium beliefs matter?

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24.8 CONSISTENT ASSESSMENT OFF-THE-PATH EQUILIBRIUM

There are problems with the weak perfect Bayesian equilibrium. To see it, consider assessments (σ, μ) where σ is such that some information sets are not reached. In those cases, you decide, whether μ is consistent with σ . Kreps and Wilson (1982) proposed the following criterion: the assessment (σ, μ) is consistent if there is a completely mixed strategy profile σ' which is arbitrarily close to σ and whose associated unique system of beliefs μ' (obtained by applying Bayesian updating) is arbitrarily close to μ . So given an extensive game, an assessment (σ, μ) is *consistent* if there is a sequence of completely mixed strategy profiles $\sigma_1, \sigma_2, \dots, \sigma_n$ such that: (1) the sequence converges to σ as n tends to infinity, that is, $\lim_{n \rightarrow \infty} \sigma_n = \sigma$ and (2) letting μ_n be the unique system of beliefs obtained from σ_n by using Bayesian updating, the sequence $(\mu_1, \mu_2, \dots, \mu_n)$ converges to μ as n tends to infinity, that is $\lim_{n \rightarrow \infty} \mu_n = \mu$. For example, consider the extensive form of Figure 11.2 below and the following assessment $\sigma = (c, d, f)$ and $\mu = (\frac{3}{8}, \frac{5}{8})$ is consistent. To see this take the strategy profile $\sigma_n = (a, b, c, d, e, f, g)$ with probabilities assigned as $(\frac{3}{n}, \frac{5}{n}, 1 - \frac{8}{n}, 1 - \frac{1}{n}, \frac{1}{n}, 1 - \frac{1}{n}, \frac{1}{n})$. Then, as n tends to infinity, all of $\frac{3}{n}, \frac{5}{n}$ and $\frac{1}{n}$ tend to 0; both $(1 - \frac{8}{n})$ and $(1 - \frac{1}{n})$ tend to 1, which results in $\lim_{n \rightarrow \infty} \sigma_n$ the strategy file (a, b, c, d, e, f, g) with their respective probabilities $(0, 0, 1, 1, 0, 1, 0)$ tend to σ . Moreover, $\mu_n(s) = \frac{\frac{3}{n}}{\frac{3}{n} + \frac{5}{n}(1 - \frac{1}{n})} = \frac{3}{8 - \frac{5}{n}}$ which tends to $\frac{3}{8}$ as $n \rightarrow \infty$ and $\mu_n(t) = \frac{\frac{5}{n}(1 - \frac{1}{n})}{\frac{3}{n} + \frac{5}{n}(1 - \frac{1}{n})} = \frac{5 - \frac{5}{n}}{8 - \frac{5}{n}}$ which tends to $\frac{5}{8}$ as $n \rightarrow \infty$. So $\lim_{n \rightarrow \infty} \mu_n = \mu$ (as s and t are $\frac{3}{8}$ and $\frac{5}{8}$).

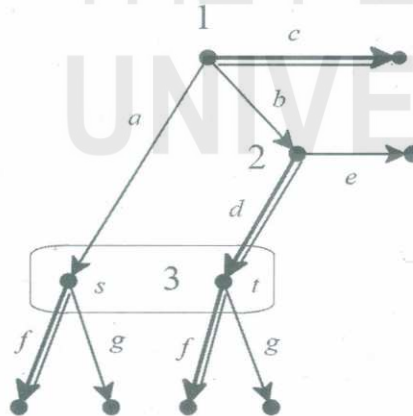


Fig. 24.12: Off-the-path Equilibrium Examination

24.8.1 Trembling Hand Perfect Equilibrium

We know that the information sets off the path of equilibrium are reached with probability 0. Hence, Bayes' formula cannot be applied to compute the beliefs. In that event we might make the players "tremble" a little bit so that every information set is reached with positive probability. Such a move will help apply Bayes rule to compute the conditional probabilities in such a perturbed strategy profile.

The idea trembling hand perfection introduced in Selten (1975), is that if your opponents deviate from their intended strategy (i.e., their hands tremble when pushing a button thereby selecting the wrong strategy), you should prepare for that in choosing your strategy.

Formally, for any $\varepsilon > 0$, replace the original game with a new game in which the set of mixed strategies is constrained to contain only those strategies that give weight at least to every pure strategy. A Nash equilibrium of this constrained game is called an ε -perfect equilibrium.

Consistency requires that the players' beliefs must be close to the probabilities that are derived using Bayes' rule for some such small tremble (as the size of the tremble goes to 0). In Figure 13, for any small tremble (for player 1 and 2), the Bayes rule yields a probability close to 1 for the event that player 2 plays T. In that case, consistency requires that player 3 assigns probability 1 to this event. Consistency is required both on and off the equilibrium path.

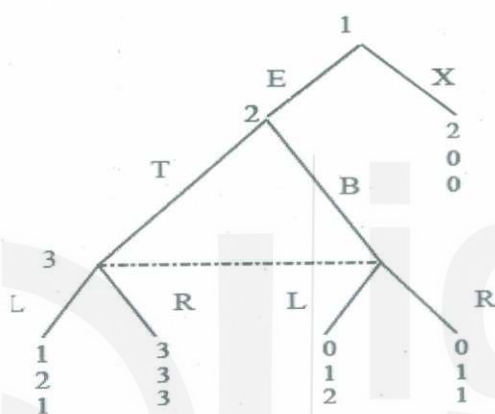


Fig. 24.13: Consistency Requirement Examination

Check Your Progress 7

- 1) Discuss if there is difference between sequential equilibrium and perfect Bayesian equilibrium.

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- 2) List the strategy-belief pair that constitutes perfect Bayesian equilibrium.

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- 3) What do you mean by consistent assessment in perfect Bayesian equilibrium when the information set is off the path?

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24.9 LET US SUM UP

In this unit we have discussed dynamic games with incomplete information. We examine the need for introducing a new equilibrium concept to find solution to this class of games. Thus, a perfect Bayesian equilibrium has been studied as a *refinement* of a Bayesian Nash Equilibrium in the sense that it considered explicitly the players' beliefs. We have examined the requirements implied by a perfect Bayesian equilibrium, viz., (i) beliefs, (ii) sequential rationality and (iii) consistency of beliefs with equilibrium strategies.

So the equilibrium includes, in addition to the strategies, beliefs, for each player whenever the player has to move but is uncertain about the history of prior play. In the process it rules out "implausible" strategies. To achieve this, we have examined the *sequential rationality*, requiring players to move rationally at every stage in the sequence of play. Considering extensive game, it is seen that the equilibrium reached by taking into account a pair consisting of (1) a profile of behavioural strategies and (2) a belief system. Accordingly, the strategy has to be optimal in every information set and each player's belief is consistent with the strategy profile. The perfect Bayesian equilibrium could be weak if no restrictions are imposed on off-the-equilibrium path beliefs. Since off-equilibrium actions may influence behaviour on-equilibrium path, we have spelt out consistent belief system obtained by Bayesian updating in a mixed strategy profile for those information sets that are not reached.

24.10 KEY WORDS

Assessment: a pair of beliefs and strategies in perfect Bayesian equilibrium.

Behavioural Strategy: a strategy which assigns probabilities over actions to a player at each information set at when called upon to move.

Belief Updating (Bayesian updating): revision of prior belief after the receipt of new information

Belief: probability distribution of a subjective assessment of likelihood of an event.

Beliefs on-the-equilibrium path: an information set is on the equilibrium path if it is reached with positive probability.

Consistency of Beliefs: game where a player must be correct about her opponents' beliefs in some manner and player must believe her opponents are correct about her beliefs.

Consistent belief: beliefs in information sets reached with positive probability

Continuation Game: refers to the information set and all nodes that follow from that information set in a game.

Off-the-Path Equilibrium: equilibrium on an information set which is not reached by the player.

Perfect Bayesian Equilibrium: equilibrium that incorporates sequential rationality and consistent belief system.

Sequential Rationality Strategy: behaviour strategy profiles, where the player has to be rational not only on the whole game but also on parts of the game tree.

Trembling Hand Perfect Equilibrium: if one's opponents deviate from their intended strategy (i.e., their hands tremble when pushing a button thereby

selecting the wrong strategy), the player should prepare for that in choosing her strategy.

Weak Sequential Equilibrium: sequential rationality, without any restriction imposed on belief.

24.11 SOME USEFUL BOOKS

Mas-Colell, A., Whinston, M., and Green, G. (1995) *Microeconomic Theory*, Oxford University Press.

Gibbons, R. (1992) *A Primer in Game Theory*, Prentice Hall.

Osborne, M.J. (2004) *An Introduction to Game Theory*, Oxford University Press.

24.12 ANSWER OR HINTS TO CHECK YOUR PROGRESS

Check Your Progress 1

- i) Two Nash equilibria in this game are (L, L') and (R, R') .
- ii) As a sub-game begins from a decision node that is a singleton information set but is not the first decision node, there is no sub-games in this game.
- iii) (R, R') . Both the equilibria are satisfy subgame perfect condition. If player 2 gets to move he has no incentive to play R' . R' is dominated by L' .

However if player 1 believes that player 2 will play R' , then he prefers to choose R with a payoff of 1 because otherwise he gets 0.

Check Your Progress 2

- 1) Belief of the player with the move about the node reached; sequential rationality of the strategy; beliefs determination at on the path equilibrium; and belief determination at off-the-path equilibrium.
- 2) Failure to pick up preferred strategy given the choice of a rival.
- 3) Structure of analysis that will cause the elimination of equilibria that involve such non-credible threats.
- 4) PBNE rules out “implausible” strategies that satisfy the requirements for Nash equilibrium and sub-game perfect Nash equilibrium. Other reason is addition of beliefs of players with their strategy profiles. Consequently, out of the multiple Nash equilibria, the “implausible” Nash equilibria are ruled out with the concept.

Check Your Progress 3

- 1) Probability distribution of a subjective assessment of likelihood of an event.
- 2) Probability distribution expresses the individual's *beliefs*, while the information set represents what the individual *knows*.
- 3) ‘*Belief updating*’ or ‘*Bayesian updating*’ refers to the modification of initial beliefs expressed by an initial probability distribution.

Check Your Progress 4

- 1) A strategy is **sequentially rational** for player i at the information set h if player i would actually want to choose the action prescribed by the strategy if h is reached.
- 2) A continuation game refers to the information set and all nodes that follow from that information set.
- 3) When the denominator term of the Bayes rule becomes equal to zero.
- 4) To introduce sequential rationality at information sets that are not singletons, beliefs in every part of the game must be such that these will include information sets that are reached through the proposed actions of the players (the profile of strategies that we are scrutinizing as equilibrium behaviour).

Check Your Progress 5

- 1) Consistent belief is defined by two conditions. First, players must be correct about their opponents' beliefs in some manner and second, players must believe their opponents are correct about her beliefs.
- 2) An assessment in an extensive game is a pair (σ, μ) , where σ is a profile of behavioral strategies and μ is a function that assigns to each information set a probability measure on the set of histories in the information set.
- 3) A behaviour strategy is a strategy which assigns probabilities over actions to a player at each information set at which it is her turn to move. That is to say, a player's (probabilistic) choice at each node is independent of his/her choices at other nodes.
- 4) For a given pair (σ, μ) of strategy profile σ and belief assessment μ , strategy profile σ is said to be sequentially rational iff, at each information set H , the player who is to move at H maximizes his expected utility.
- 5) To introduce sequential rationality at information sets that are not singletons, beliefs in every part of the game must be such that these will include information sets that are reached through the proposed actions of the players (the profile of strategies that we are scrutinizing as equilibrium behaviour).

Check Your Progress 6

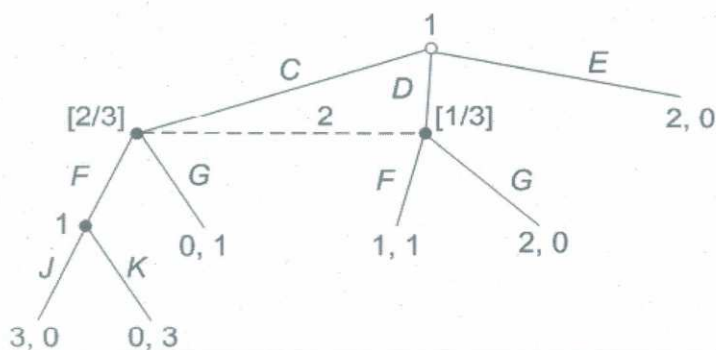
- 1) Refers to the lack of restrictions on off-the-equilibrium path beliefs.
- 2) Beliefs to be consistent with Bayes rule **whenever it is applicable**, and places no restriction on cases where Bayes rule does not apply (as when an information set is reached with probability zero).
- 3) Belief must be consistent with strategy of other players.
- 4) Assessment (σ, μ) is a Weak Perfect Bayesian Equilibrium (WPBE) iff it is (a) belief consistent and (b) sequentially rational at every information set.
- 5) Because they may induce off equilibrium actions that in turn influence behaviour on-equilibrium path. To make the concept of PBE more useful in applications, additional restrictions for off-equilibrium beliefs have been introduced.

Check Your Progress 7

- 1) They differ in the restrictions they impose on players' beliefs off the equilibrium path, i.e. after histories that occur with probability zero given the equilibrium strategies. Both solution concepts require that each player behaves sequentially rationally given his or her beliefs. And both require that players update their beliefs according to Bayes rule whenever possible. Sequential equilibrium further requires that off the equilibrium path, at a zero-probability history, beliefs be consistent with Bayes rule applied to some small departure (or "tremble") from the equilibrium strategies. Perfect Bayesian equilibrium, in its weakest form, doesn't restrict off-equilibrium path beliefs. (so SE $\Rightarrow$ PBE but not vice versa), although there are variants of PBE that do impose restrictions, and even restore equivalence with sequential equilibrium.
- 2) A strategy-belief pair, (σ, μ) is a perfect Bayesian equilibrium if
 - i) (Beliefs) At every information set of player i , the player has beliefs about the node that he is located given that the information set is reached.
 - ii) (Sequential Rationality) At any information set of player i , the restriction of (σ, μ) to the continuation game must be a Bayesian Nash equilibrium.
 - iii) (On-the-path beliefs) The beliefs for any on-the-equilibrium-path information set must be derived from the strategy profile using Bayes' Rule.
 - iv) (Off-the-path beliefs) The beliefs at any off-the-equilibrium-path information set must be determined from the strategy profile according to Bayes Rule whenever possible.
- 3) An assessment (σ, μ) is consistent if there is a sequence $((\sigma^n, \mu^n))_{n=1}^{\infty}$ of assessments that converges to (σ, μ) and has the properties that each strategy profile σ^n is completely mixed and that each belief system μ^n is derived from σ^n using Bayes rule. (Note: If behavioral strategy profile σ assigns a strictly positive probability to every action, we say σ is completely mixed and write $\sigma \geq 0$).

24.13 EXERCISES AND ANSWERS

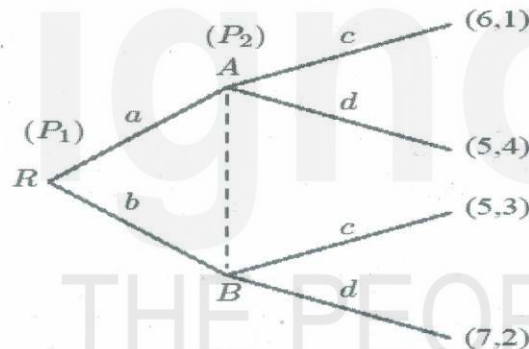
- 1) Given the following game find
 - i) Consistency of beliefs and
 - ii) Weak sequential equilibrium.



- i) If player 1's strategy is EJ, player 2 can hold any belief at her information set. If player 1's strategy at her first information set is to choose C with

probability p , D with probability q , and E with probability $1 - p - q$, then player 2 must assign probability $\frac{p}{p+q}$ to history C and $\frac{q}{p+q}$ to history D. That implied if player 1 chooses D with probability 1, then player 2's belief must assign probability 0 to C and 1 to D.

- ii) Take the strategy profile (EJ; G) and player 2's belief that assigns $\frac{2}{3}$ to history C and $\frac{1}{3}$ to history D. Find that Player 1's strategy EJ is sequentially rational given Player 2's G. Player 2's strategy G is also sequentially rational given her beliefs described in the figure and player 2's strategy EJ. Player 2's belief is (weakly) consistent with the strategy profile (EJ;G). But there is no weak sequential equilibrium in which the strategy profile is (DJ; G). Player 1's strategy DJ is sequentially rational. But given player 1's strategy, player 2 should believe the history is D with probability 1 at her information set, and should therefore choose F rather than G.
- 2) Consider the game tree given below. Player P_2 cannot choose between c and d on the information set $I = \{A, B\}$ because d is better starting at A and c is better starting at B . Show that there is no pure strategy Nash equilibrium in the whole game.



Ans. If P_1 chooses a then P_2 would choose d . Note that P_1 has an incentive to switch to b , payoff $7 > 5$. However, if P_1 chooses b then P_2 would choose c . In that case, P_1 has an incentive to switch to a since $6 > 5$. Therefore, there is no pure strategy profile which gives a Nash equilibrium on the whole game.

We could use a mixed strategy, which is a probability on the choices of strategy profiles. However, in some games, some of the mixed strategy profiles are hard to interpret. Instead, we take a probabilistic choice of edges at nodes, which are called a behavior strategy profile.

Returning to the preceding example, assume that player P_1 chooses a with probability p , with $0 \leq p \leq 1$, and b with probability $1 - p$.

In the same way, assume player P_2 chooses c with probability q , with $0 \leq q \leq 1$, and d with probability $1 - q$. The combination (p, q) is called a **behaviour strategy profile** for the game.

For these choices, the expected payoff of player P_1 at the root R is given as follows:

$$\begin{aligned}
 E_1(p, q) &= pq(6) + p(1 - q)(5) + (1 - p)q(5) + (1 - p)(1 - q)(7) \\
 &= pq(6 - 5 - 5 + 7) + p(5 - 7) + q(5 - 7) + 7 \\
 &= 3pq - 2p - 2q + 7.
 \end{aligned}$$

In the same way, the expected payoff of player P2 at the root R is given as follows:

$$\begin{aligned} E_2(p, q) &= pq(1) + p(1 - q)(4) + (1 - p)q(3) + (1 - p)(1 - q)(2) \\ &= pq(1 - 4 - 3 + 2) + p(4 - 2) + q(3 - 2) + 2 \\ &= -4pq - 2p + q + 2. \end{aligned}$$

Player P1 can change the behaviour strategy p , and $\frac{\partial E_1}{\partial p} = 3q - 2$ which equals zero when $q = 2/3$. If P1 maximizes $E_1(p, q)$, then

$$p = \begin{cases} 0 & \text{If } q < 2/3 \\ \text{arbitrary} & \text{If } q = 2/3 \\ 1 & \text{If } q > 2/3 \end{cases}$$

Player P_2 can change the behaviour strategy q , and $\frac{\partial E_2(p, q)}{\partial q} = -4p + 1$, which equals zero when $p = 1/4$. If P_2 maximizes $E_2(p, q)$ then

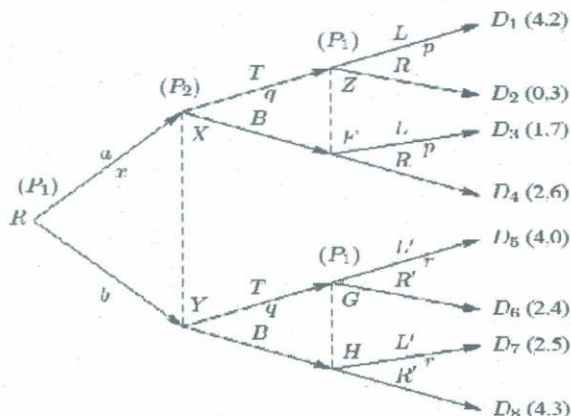
$$q = \begin{cases} 1 & \text{If } p < 1/4 \\ \text{arbitrary} & \text{If } p = 1/4 \\ 0 & \text{If } p > 1/4 \end{cases}$$

The only simultaneous solutions of the two criteria occurs for $p^* = \frac{1}{4}$ and $q^* = \frac{2}{3}$. See **Figure 2** which depicts a Nash equilibrium for the whole game in terms of a behaviour strategy.



Fig. 1a: Nash Equilibrium

- 3) Discuss the relationship between Strategy profile, Beliefs and payoffs in the following game.



In the game given above strategies are:

- 1) Let $x = \sigma(a)$ be the weight of the behaviour strategy on the edge a and $1 - x = \sigma(b)$ be the weight on the edge b . These choices are made by P_1 at the node R .
- 2) Let $q = \sigma(T)$ be the weight of the behavior strategy on the edge T and $1 - q = \sigma(B)$ be the weight of the behaviour strategy on the edge B . These choices are made on the information set $I_1 = \{X, Y\}$ by P_2 . The same weights must be used at both nodes X and Y .
- 3) Let $p = \sigma(L)$ be the weight of the behaviour strategy on the edge L and $1 - p = \sigma(R)$ be the weight of the behavior strategy on the edge R . These choices are made by P_1 on the information set $I_2 = \{Z, F\}$.
- 4) Let $r = \sigma(L')$ be the weight of the behavior strategy on the edge L' and $1 - r = \sigma(R')$ be the weight of the behavior strategy on the edge R' . These choices are made by P_1 on the information set $I_3 = \{G, H\}$.

To calculate the payoffs at each node, we use the behaviour strategy to determine the probabilities of taking different paths to terminal nodes and add them up. At node X , the probabilities of reaching the various terminal nodes is $\sigma(X, D_1) = \sigma(XZ)\sigma(ZD_1) = qp$, $\sigma(X, D_2) = \sigma(XZ)\sigma(ZD_2) = q(1 - p)$,

$\sigma(X, D_3) = (1 - q)p$ and $\sigma(X, D_4) = (1 - q)(1 - p)$. The payoff vectors are follows:

$$E(Z, \sigma) = p(4, 2) + (1 - p)(0, 3) = (4p, 3 - p)$$

$$E(F, \sigma) = p(1, 7) + (1 - p)(2, 6) = (2 - p, 6 + p)$$

$$E(G, \sigma) = r(4, 0) + (1 - r)(2, 4) = (2 + 2r, 4 - 4r)$$

$$E(H, \sigma) = r(2, 5) + (1 - r)(4, 3) = (4 - 2r, 3 + 2r)$$

$$E(X, \sigma) = q(4p, 3 - p) + (1 - q)(2 - p, 6 + p) = (2 - 2q - p + 5pq, 6 - 3q + p - 2pq)$$

$$E(Y, \sigma) = q(2 + 2r, 4 - 4r) + (1 - q)(4 - 2r, 3 + 2r) = (4 - 2r - 2q + 4qr, 3 + q + 2r - 6qr)$$

$$E(R, \sigma) = x(2 - 2q - p + 3pq, 6 - 3q + p - 2pq) + (1 - x)(4 - 2r - 2q + 4qr, 3 + q + 2r - 6qr).$$

For a node N , we write $\sigma(N)$ to mean $\sigma(RN)$. Thus $\sigma(Z) = xq$ and $\sigma(F) = x(1 - q)$. If $x > 0$ and P_1 knows he is in the information set $I_2 = \{Z, F\}$, then P_1 should rationally expect that the probability of being at node Z is

$$\frac{\sigma(Z)}{\sigma(Z) + \sigma(F)} = \frac{xq}{xq + x(1 - q)} = q.$$

We call this expectation by P_1 a belief, and denote it by $\mu_1(Z)$. Similarly, $\mu_1(F) = 1 - q$.

If $x = \sigma(a) = 0$, then the probability of reaching I_2 is zero, $\sigma(Z) + \sigma(F) = xq + x(1 - q) = 0$, and the strategy profile does not immediately induce a system of beliefs on I_2 . However, if we modify the strategy profile slightly and let $\sigma^n(a) = \frac{1}{n}$ and $\sigma^n(b) = 1 - \frac{1}{n}$, then the induced belief is $\mu^n(Z) =$

$$\frac{\frac{1}{n}\sigma(T)}{\frac{1}{n}\sigma(T) + \frac{1}{n}\sigma(B)} = \frac{\sigma(T)}{\sigma(T) + \sigma(B)} = \frac{q}{q + (1 - q)} = q, \text{ which is the same for all } n, \text{ so } \mu_1(Z) =$$

q . Similarly, $\mu_1(F) = 1 - q$. Therefore, in this example, the beliefs on X and F must be q and $1-q$ respectively.

- 4) Show that in the example of Question no. 3 above the belief system on an information set is arbitrary.

Ans. On the information set I_2 , the payoff for P_1 is

$$\begin{aligned} E_1(I_2, \sigma, \mu^\sigma) &= qE_1(Z, \sigma) + (1 - q)E_1(F, \sigma) = E_1(X, \sigma) \\ &= 2 - 2q - p + 5pq. \end{aligned}$$

Similarly, on the information set $I_3 = \{G, H\}$, the payoff for P_1 is

$$E_1(I_3, \sigma, \mu^\sigma) = E_1(Y, \sigma) = 4 - 2r - 2q + 4qr.$$

Also, on the information set $I_1 = \{X, Y\}$, the payoff for P_2 is

$$\begin{aligned} E_2(I_1, \sigma, \mu^\sigma) &= E_2(R, \sigma) \\ &= x(6 - 3q + p - 2pq) + (1 - x)(3 + q + 2r - 6qr). \end{aligned}$$

- 5) Consider Selten's game from Fig. 1 in the text and suppose $\sigma_1 = (1/3, 2/3)$, and

$\sigma_2 = (0.5, 0.5)$. The continuation game from Player 1's information set includes the entire

game tree. What is Player 1's expected payoff? Let z_1 denote the outcome (D, L) , z_2 denote

the outcome (D, R) , and z_3 denote the outcome (U) . The expected payoff then is:

$$\begin{aligned} U_1(\sigma|\theta) &= P(z_1|\sigma, \theta)u_1(z_1) + P(z_2|\sigma, \theta)u_1(z_2) + P(z_3|\sigma, \theta)u_1(z_3) \\ &= \sigma_1(D)\sigma_2(L)(3) + \sigma_1(D)\sigma_2(R)(0) + \sigma_1(U)(2) \\ &= 1/3 \times 1/2 \times 3 + 2/3 \times 2 = 11/6. \end{aligned}$$

Similarly, the payoffs in the continuation game starting at Player 2's information set is

$$\begin{aligned} U_1(\sigma|D) &= P(z_1|\sigma, D)u_1(z_1) + P(z_2|\sigma, D)u_1(z_2) \\ &= \sigma_2(L)(3) + \sigma_2(R)(0) = 1/2 \times 3 = 3/2. \end{aligned}$$

We can also calculate Player 2's payoff in both continuation games in an analogous manner:

$$\begin{aligned} U_2(\sigma|\theta) &= \\ &P(z_1|\sigma, \theta)u_2(z_1) + P(z_2|\sigma, \theta)u_2(z_2) + P(z_3|\sigma, \theta)u_2(z_3) = \\ &\sigma_1(D)\sigma_2(L)(1) + \sigma_1(D)\sigma_2(R)(0) + \sigma_1(U)(2) = 1/3 \times 1/2 \times 1 + \\ &2/3 \times 2 = 3/2 \text{ and } U_2(\sigma|D) = \\ &P(z_1|\sigma, D)u_2(z_1) + P(z_2|\sigma, D)u_2(z_2) = \sigma_2(L)(1) + \sigma_2(R)(0) = \\ &1/2 \times 1 = 1/2. \end{aligned}$$

Player 2 evaluates the optimality of her behavior using her beliefs as follows:

If the information set h is a singleton, the behaviour strategy profile σ is sequentially rational for i at h if, and only if, Player i 's strategy σ_i maximized Player i 's expected payoff at h if all other players behaved according to σ .

Player 2 has only one sequentially rational strategy at her information set D in the Selten's game. The expected payoff in the continuation game following this node is maximized if she chooses L with certainty. For

Player 1, the calculation involves Player 2's strategy. Player 1 would choose D .

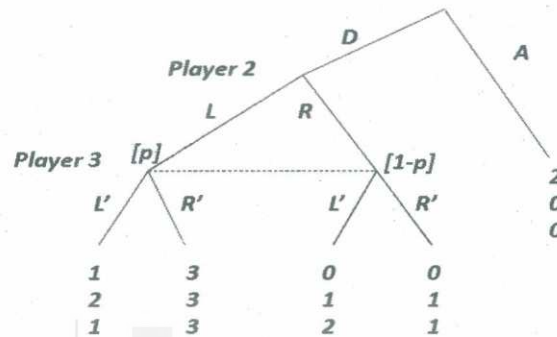
$$\text{if: } U_1(D, \sigma_2) \geq U_1(U, \sigma_2) \rightarrow \sigma_2(L)(3) + \sigma_2(R)(0) \geq 2 \cdot \sigma_2(L) \geq 2/3.$$

So, strategy D is sequentially rational for any $\sigma_2(L) > 2/3$ whereas

strategy U is sequentially rational for any $\sigma_2(L) < 2/3$. If $\sigma_2(L) = \frac{2}{3}$, then

both D and U are sequentially rational.

6) Consider the following game:



- Find the Nash Equilibria of the game from the strategic representation of the game.
- Show that subgame perfect Nash equilibrium $\{D, L, R\}$ satisfies the requirements of 1, 2 and 3 of perfect Bayesian equilibrium. Why would you say that requirement 4 is met by this equilibrium?
- Consider the NE: $\{A, L, L'\}$. show that with belief $p = 0$ requirements 1, 2, 3 are satisfied but requirement 4 is not satisfied.

Ans. (i)

		Player 1			
		D		A	
		Player 2		Player 2	
Player 3		L	R	L	R
	L'	1, 2, 1	2, 1, 0	0, 0, 2	0, 0, 2
	R'	3, 3, 3	1, 1, 0	0, 0, 2	0, 0, 2

Four Nash equilibria: $\{D, L, R'\}; \{A, L, L'\}; \{A, R, L'\}; \{A, R, R'\}$

One subgame that begins after Player plays D.

This game has only one NE: $\{L, R'\}$

One SPNE: $\{D, L, R'\}$

Ans. (ii)

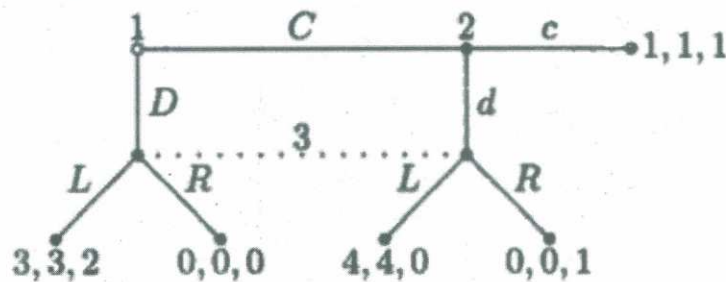
R1 and R3: Player 3's belief is $p=1$. R2: R' is a best response given the beliefs and others strategies. R4 is satisfied because there are no information sets off the equilibrium path.

Ans. (iii)

Requirement 1, 2, 3 are satisfied. But $p = 0$ is inconsistent with player 2's strategy.

Then Requirement 4 is not satisfied.

- 7) You are given the game of Selten Horse below. Find i) sequentially rational strategies and ii) pure strategy perfect Bayesian equilibrium.



Ans. (Hints) If we look at sequential rationality, the second of these equilibria will be ruled out. Suppose we have (D, c, L) . The belief of player 3 will be $\mu_3(\text{left}) = 1$. Player 2, if he gets a chance to play, will then never play c , since d has a payoff of 4, while c would give him 1. If he were to play d , then player 1 would prefer C , but (C, d, L) is not an equilibrium, because then we would have $\mu_3(\text{left}) = 0$ and player 3 would prefer R .

Therefore, there is a unique pure strategy Perfect Bayesian Equilibrium outcome (C, c, R) . The belief system that supports this could be any $\mu_3(\text{left}) \in [0, \frac{1}{3}]$.

- 8) Discuss on your understanding of belief assessment.

A belief assessment is a list σ of probability distributions on information sets; for each information set H , σ gives a probability distribution $\sigma(\cdot | I)$ on H .

For any information set H , the player who moves at H believes that he is at node $n \in H$ with probability $\sigma(n | I)$. For example, for the game in Figure 2a, in order to define a belief assessment, we need to assign a probability μ on the node after T and a probability $1 - \mu$ on the node after B . When Player 2 moves, she believes that Player 1 played T with probability μ and B with probability $1 - \mu$.

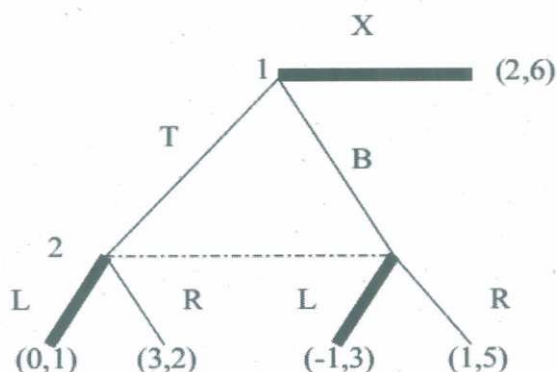


Fig. 2a: Player 2' Irrational Strategy in SPE

- 9) How would you check the consistency requirement in extensive form game of weak perfect Bayesian Nash equilibrium?

Ans. Let σ be a profile of strategies and μ a system of beliefs.

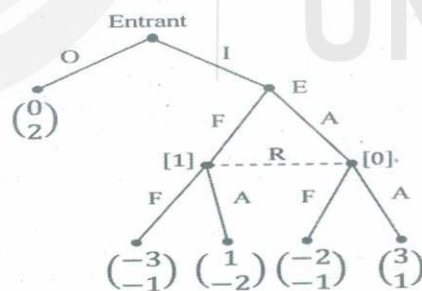
A pair (σ, μ) is a **weak perfect Bayesian equilibrium (WPBE)** if σ is sequentially rational given μ , and μ is derived from σ using Bayes rule **whenever applicable**.

A pair (σ, μ) is a **sequential equilibrium (SE)** if σ is sequentially rational given μ and μ and σ are consistent in the following sense: There exists a sequence of completely mixed strategies $\sigma^k \rightarrow \sigma$ as $k \rightarrow \infty$ such that, for each k μ^k is derived from σ^k using Bayes rule (**which is always applicable**), and that μ is the limit of $\{\mu^k\}$.

From the definitions, we can see that the two concepts differ in their restrictions of beliefs: WPBE requires beliefs to be consistent with Bayes rule **whenever it is applicable**, and places no restriction on cases where Bayes rule does not apply (as when an information set is reached with probability zero). In contrast, the consistency requirement imposed by SE **forces Bayes rule to apply everywhere** in the tree by demanding that the SE-belief is the limit of a sequence of beliefs derived from a profile of completely mixed strategies.

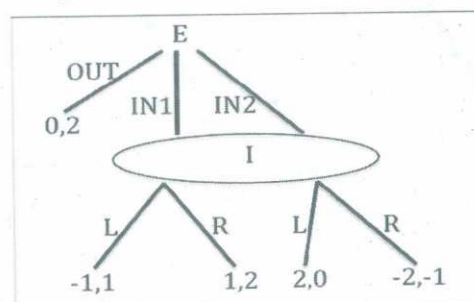
To illustrate, consider the following example of market entry game. The entrant first decides to enter (I) or stay out (O), and if entered, the entrant (E) and the resident (R) simultaneously choose whether to fight (F) or abstain (A)

The equilibrium where entrant chooses (O , A if I) and the resident chooses F , together with the resident believing that the entrant will choose F when entered constitute a WPBE. Clearly the resident's belief is wrong, but that is allowed in WPBE because the information set at which the resident makes his decision is reached with probability zero. Thus Bayes rule does not apply, and he is free to have any belief. SE will not allow this to happen, because as soon as I is played with a positive (albeit tiny) probability, Bayes rule applies at the resident's information set, and given the entrant's strategy, the only consistent belief is to assign probability 1 to the node after A . Thus the strategy profile $((O, A \text{ if } I), F)$ is not a SE.



Therefore we have the following theorem relating SE and SPE (subgame perfect equilibrium): If (σ, μ) is a SE, then σ is a SPE.

- 10) Determine all weak perfect Bayesian-Nash equilibria of the following game:



Let μ denote the probability that I assigns to being at the left hand node in his information set. We first determine the values of μ that support I's choice of L over R:

$$\begin{aligned} L: 1 \cdot \mu + 0 \cdot (1 - \mu) &= \mu \\ R: 2\mu + -1 \cdot (1 - \mu) &= 3\mu - 1 \end{aligned}$$

L is preferred to R if $\mu > 3\mu - 1 \Leftrightarrow 1 > 2\mu \Leftrightarrow \mu < \frac{1}{2}$

We first try to construct an equilibrium in which I chooses L. This leads E to choose IN2, in which case $\mu = 0$. We have our first WPBNE: E chooses IN2, I chooses L, $\mu = 0$.

We next try to construct an equilibrium in which I chooses R. In this case, E chooses IN1, resulting in $\mu = 1$. We therefore have a second WPBNE: E chooses IN1, I chooses R, $\mu = 1$.

We consider $\mu = \frac{1}{2}$ so that I is indifferent between L and R. Let μ denote the probability that I chooses L. If E is to choose IN1 and IN2 with positive probability, we must have

$$IN1: -\sigma + (1 - \sigma) = 2 \cdot \sigma + -2 \cdot (1 - \sigma) : IN2$$

$$IN1: -2\sigma + 1 = 4\sigma - 2 : IN2 \Leftrightarrow \sigma = \frac{1}{2}$$

With σ , E is indifferent among OUT, IN1 and IN2, as each gives him an expected payoff of zero. We thus have a family of WPBNE: For any $\sigma \in [0, 1]$, E plays OUT with probability $1 - 2\sigma$ and IN1 and IN2 each with probability σ . We have $\sigma = 1$, and I plays each of L, R with probability $\frac{1}{2}$.

Are there any more WPBNE? We're still working in the case of $\mu = \frac{1}{2}$

For $\neq \frac{1}{2}$, one of IN1, IN2 produces a positive expected payoff for E and the other produces a negative expected payoff. I's best response is therefore exactly one of IN1, IN2, which contradicts $\mu = \frac{1}{2}$. We are therefore done.

11) Discuss the requirements imposed for a perfect bayesian equilibrium

Formally, the concept of a *perfect Bayesian equilibrium (PBE)* imposes the following three requirements:

- 1) At each information set the player with the move must have a *belief* regarding which node has been reached.
- 2) Given their beliefs at each information set the current move and subsequent strategies must be optimal given the beliefs and subsequent strategies of the other players.
- 3) Beliefs are determined by Bayes' rule and players' equilibrium strategies.

A 4th requirement that distinguishes between information set which are *on the equilibrium paths* (and is reached with positive probability if the game is played according to the equilibrium strategies) and information sets *off the equilibrium path* which are certain *not* to be reached if equilibrium strategies are played. However requirements 1 to 3 capture the essence of the Bayesian Nash equilibrium concept.

In the game depicted in Figure 3a do these requirements eliminate the bad equilibrium?

The answer is that requirements 1 and 2 alone, which insist that players have beliefs and act optimally given these beliefs, are sufficient to rule out (R, R') with probability p and M with probability $1 - p$. Then her expected payoff from playing L' is $p \times 1 + (1 - p) \times 2 = 2 - p$. Her expected payoff from playing R' is $p \times 0 + (1 - p) \times 1 = 1 - p$. Since $2 - p > 1 - p$ for all p , player, 2 will *never* play R' . Hence player 1 will play L and the only perfect Bayesian equilibrium (PBE) is (L, L') .

Requirement 3 insists that these beliefs are also optimal which means they must be updated using Bayes' rule. To illustrate this aspect suppose that there were a *mixed-strategy* equilibrium in which player 1 plays L with probability q_1 , M with probability q_2 and R with probability $1 - q_1 - q_2$. (In fact (L, L') is the only PBE so it turns out that $q_1 = 1, q_2 = 0$).

However given this equilibrium strategy, player 2 would form a belief following a move not R by player 1 given by

$$p(L|\text{not } R) = \frac{p(\text{not } R|L)p(L)}{p(\text{not } R)} = \frac{1 \times q_1}{q_1 + q_2} = \frac{q_1}{q_1 + q_2} > q_1 \text{ because } q_1 + q_2 < 1$$

Suppose following player 1's move which is not R , player 2 believes 1 has played L .

	L'	R'
L	$(2, 1)$	$(0, 0)$
M	$(0, 2)$	$(1, 0)$
R	$(1, 3)$	$(1, 3)$

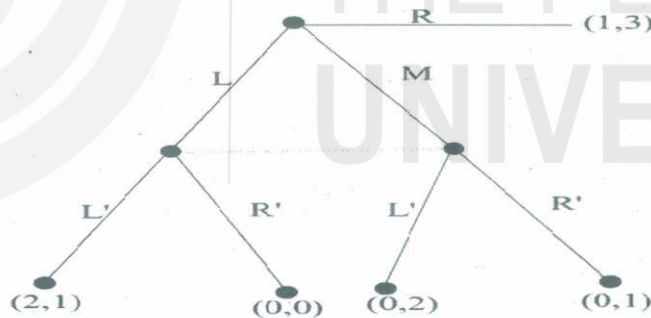


Fig. 3a

UNIT 25 SIGNALING GAMES AND THEIR APPLICATIONS

Structure

- 25.0 Objectives
- 25.1 Introduction
- 25.2 Modeling Signaling Games
 - 25.2.1 Structure of Signaling Games
 - 25.2.2 Extensive Form Representation
 - 25.2.3 Example of Quiche-Beer Game
 - 25.2.3.1 Bayesian Normal Form and Pure Strategy Equilibria of Beer-Quiche Game
- 25.3 A Second Approach to Equilibrium Analysis: Pooling and Separating Equilibria
 - 25.3.1 Pooling Strategies
 - 25.3.2 Separating Strategies
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- 25.4 Application: Job Market Signaling
 - 25.4.1 Complete Information: Benchmark Scenario
 - 25.4.2 Incomplete Information: Separating Equilibria
 - 25.4.2.1 Separating PBNE: No Mimicry
 - 25.4.2.2 Separating PBNE: Mimicry Possible
 - 25.4.3 Incomplete Information: Pooling Equilibria
- 25.5 Let Us Sum Up
- 25.6 Key Words
- 25.7 Some Useful Books
- 25.8 Answer or Hints to Check Your Progress
- 25.9 Exercises and Answers/Hints

25.0 OBJECTIVES

After going through this unit, you will be able to:

- understand the structure of signaling games as a particular kind of dynamic game of incomplete information;
- apply Perfect Bayesian Nash Equilibrium (PBNE)
- differentiate between pooling and separating equilibria; and
- discuss signaling games in job markets.

25.1 INTRODUCTION

Signaling games refer to a class of two-player games of incomplete information in which one player is informed and the other is not. The informed player's strategy set consists of signals contingent on information

and the uninformed player's strategy set consists of actions contingent on signals. More generally, a signaling game includes any strategic setting in which players can use the actions of their opponents to make inferences about hidden information.

25.2 MODELING SIGNALING GAMES

In a signalling game, there are two players who we call the Sender and the Receiver. Nature moves first and chooses randomly one of a finite number of outcomes. The Sender sees the outcome, which for this reason we often call the Type of the Sender. The Sender then chooses from a finite number of possibilities, which are often called Messages. The Receiver sees the Message sent, but not the Type of the Sender and chooses from another finite number of possibilities, which are often called Responses. The payoff to each player can depend on the Type of the Sender, the chosen Message, and the chosen Response.

To appreciate the modeling of signaling games see, two examples, viz., the job-market signaling model of Spence (1973) and corporate capital structure Myers and Majluf (1984).

The first example is on the model formulated by Spence, in which the labour market is taken as the sender by the job market applicant. Nature chooses the type of this worker, which cannot be observed directly by the employer (the receiver). The worker may be high productivity or low productivity. The exact type is known to the worker alone. Hence, the sender (worker) uses education as a signal (m_j) to communicate her type to the employer. The action of the employer is to choose an appropriate wage (a_k), upon observing the sign of education.

In the second example corporate mentioned above, capital structure has been studied using a signaling model, where the sender s is a firm requiring finance for a new project and the receiver r is a potential investor. The type of the sender is identified by the type of the project that s wants financed (risky or safe) but the message that s sends for r is a proposal to share the profits of the project. The action of r is the decision of how much finance to provide the firm.

The reason these games are called signaling games is because of the potential role that player 1's actions can play. Namely, if in equilibrium each type of player 1 is playing a different choice then in equilibrium, the action of player 1 will reveal player 1's type to player 2 through the system of beliefs. Because of the signaling potential of player 1's strategies, these games have two important classes of equilibria:

- 1) **Pooling Equilibria:** Equilibria in which all the types of player 1 choose the same action, thus revealing nothing to player 2. Player 2's beliefs must be derived from Bayes rule only in the information set that is reached with probability 1, and all other information sets are reached with probability 0. In those information sets, player 2 must hold beliefs that support his strategy, and the strategy of player 2 that is sequentially rational given his beliefs is what keeps player 1 from deviating from her "pooling" strategy.

- 2) **Separating Equilibria:** Equilibria in which each type of player 1 chooses a different action, thus revealing herself in equilibrium to player 2. Player 2's beliefs are thus well defined by Bayes rule in all the information sets that are reached. If there are more actions than types for player 1, then player 2 has to have beliefs for the information sets that are not reached (the actions that no type of player 1 chooses), which in turn must support the strategy of player 2, and player 2's strategy supports the strategy of player 1.

In a pooling equilibrium all the types of player 1 pool together in the action set, and thus player 2 can learn nothing from the action of player 1. That is, his posterior belief after player 1 moves must be equal to his prior belief that comes immediately from the distribution of Nature's choices of types for player 1. In a separating equilibrium each type of player 1 separates from the others by choosing a unique action that no other type chooses. Thus, after observing what player 1 did, player 2 can infer exactly what type player 1 is.

Hybrid equilibria: There is a third class of equilibria called hybrid equilibria. In these, different types choose different mixed strategies, and the same information set (action) can be reached by different types. Thus, by Bayes rule, player 2 can learn something about player 1, but cannot infer exactly which type he is (we have not discussed this idea in the present unit. See Fudenberg and Tirole (1990), Chapter X for more on this).

25.2.1 Structure of Signaling Games

Signaling games are two-stage games where Player 1 (with private information) moves first. His move is observed by Player 2 and undertakes his move with no knowledge of Player 1's private information. Then payoffs are realised.

Formally, the game starts with *Nature* choosing a random variable t_1 , observable only to Player 1, from a distribution $p(t_1)$. We start with **stage 1** thereafter where Player 1 chooses an action a_1 from the set A_1 . Player 2, after observing the action ("message.") sent by Player 1, chooses an action a_2 from the set A_2 . In **stage 2**, payoffs are realized as $\Pi_1(a_1, a_2; t_1); \Pi_2(a_1, a_2; t_1)$.

You may have observed that the description above follows the modeling approach of Harsanyi seen in static Bayesian games (Unit 23). Note that Player 2's payoff depends on the type of player 1.

(When Player 2 moves first, and Player 1 moves second, (it is called a *screening game*.)

25.2.2 Extensive Form Representation

Suppose there are two types for Player 1, and two actions for each player. So we have

$$t_1 = H \text{ or } t_1 = L$$

$$\text{Let } p = P(t_1 = H)$$

$$\bullet A_1 = \{a, b\}$$

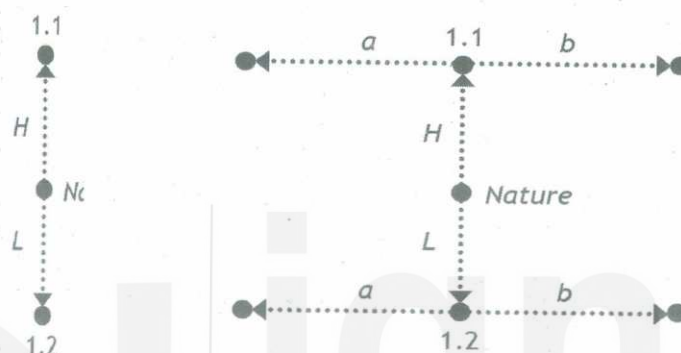
$$\bullet A_2 = \{A, B\}$$

In the game, Nature moves first, so we have

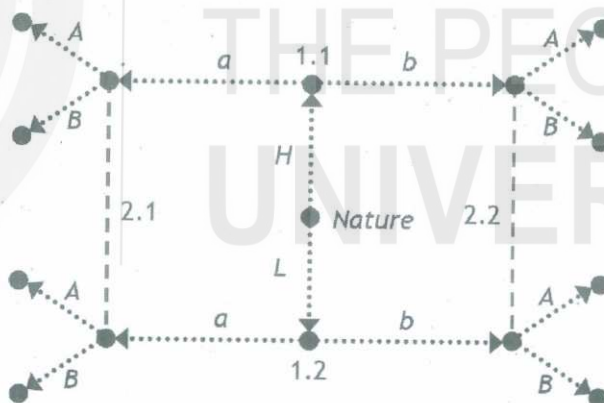
1) Nature moves



2) Player 1 moves second giving us



3) Player 2 observes Player 1's action and moves



4) Payoffs are realized: $\Pi_i(a_1, a_2; t_1)$

In the game above each player has 2 information sets, and 2 actions in each, so there are 4 strategies.

So, we summarize the signaling game as follows:

Nature moves first and assigns a type t_i to the sender s from a feasible set of types, $T = \{t_i, \dots, t_I\}$, according to the probability distribution $p(t_i)$ such that $\sum_{t_i \in T} p(t_i) = 1$

The sender s alone observes t_i and chooses a message m_j from the set of feasible message $M = \{m_1, \dots, m_J\}$

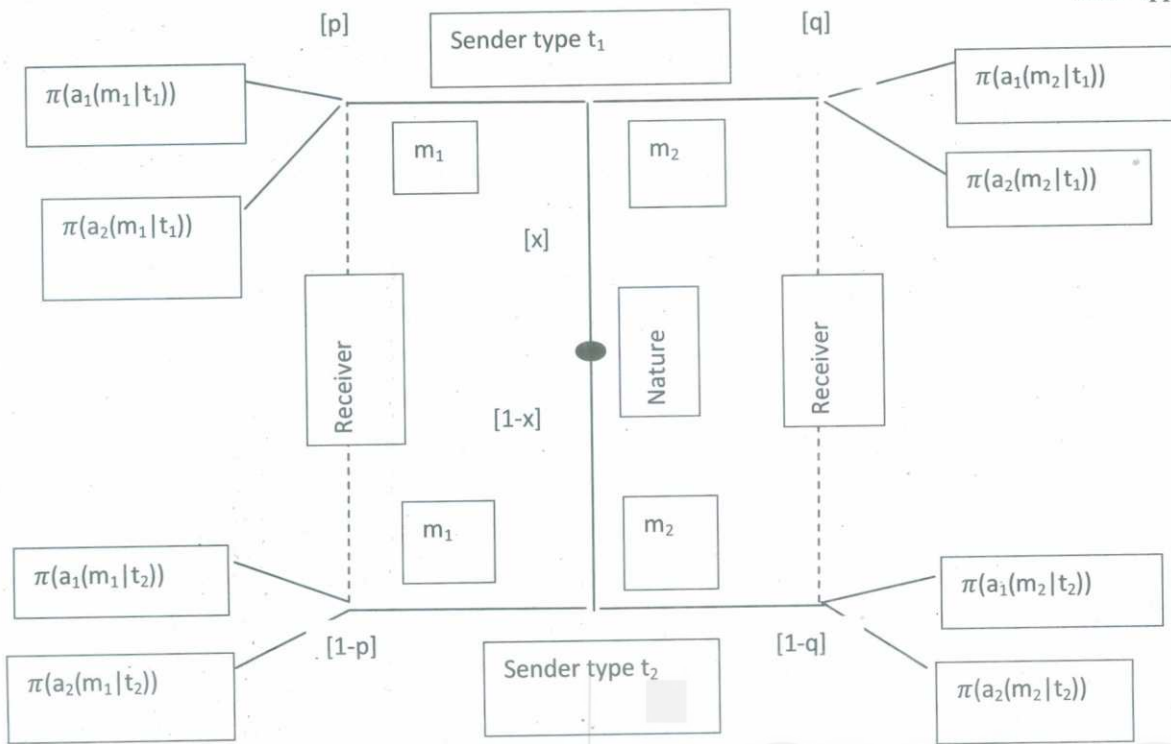


Fig. 25.1: Normal Form Representation

The receiver r observes the message m_j (but not the type of the sender t_i) and chooses an action $a_k \in A = \{a_1, \dots, a_K\}$.

Payoffs are determined by $U_S(t_i, m_j, a_k)$ for the sender and $U_R(t_i, m_j, a_k)$ for the receiver.

Seen in terms of a two player signaling game, we have two types $\{t_1, t_2\}$ of the sender s , each of whom has a choice of sending one of two messages $\{m_1, m_2\}$ from the message space. Taking that the sender has private information about her type, Figure 25.1 above may be drawn.

Note that the collection of the pair of payoffs $\{U_S, U_R\}$ is shown as $\pi(\cdot)$ in the figure. The central black node shows the first move by Nature (or chance), which picks sender type t_1 with probability x and type t_2 with probability $[1-x]$. These are the prior or ex-ante probabilities/ beliefs about the sender types held by the receiver before receiving the messages from the sender. After this, either type of sender chooses either m_1 or m_2 .

Depending upon the message sent by the sender of either type, the receiver updates his beliefs about the type of the sender. These updated beliefs are shown as $\{[p], [1-p]\}$ on the left branch and $\{[q], [1-q]\}$ on the right branch of the extensive form. For the left branch, $[p]$ is the updated probability that the message m_1 has been sent by type t_1 and $[1-p]$ is the updated probability that the message m_1 has been sent by type t_2 . A similar interpretation is for $\{[q], [1-q]\}$. If the message sent is on the equilibrium path, then the receiver has to update his belief about the sender's type in a Bayesian manner. Then, the receiver takes an action from among $\{a_1, a_2\}$ on the basis of his updated beliefs.

25.2.3 Example of Quiche-Beer Game

Beer-Quiche Game: In Beer-Quiche game, player 1 is either strong (surly or tough) or weak (wimpy). Player 2, the bully, can only infer player 1's type by observing the player 1's choice of breakfast quiche or beer. Player 2 chooses either to duel (fight) with him or not (concede) after taking into account the choice of player 1. Thus, there are two possible actions, beer or quiche, for player 1. The bully's possible actions are: duel or not duel with player 1. Player 2 wins the duel if and only if he has challenged the wimp type. Player 1 prefers not to be challenged. The game is given in Figure 25.2.

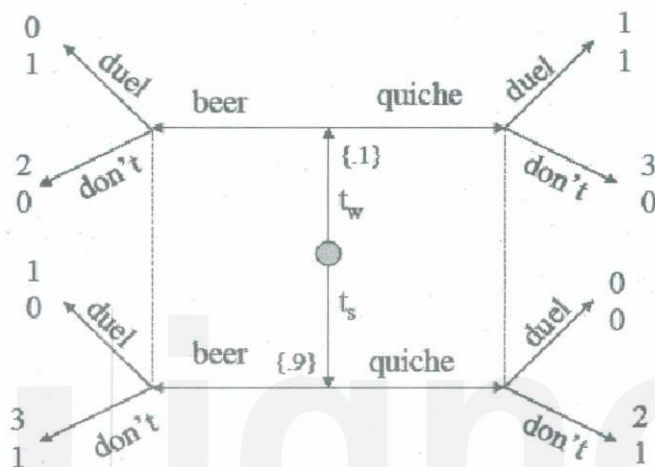


Fig. 25.2: Beer-Quiche Game

25.2.3.1 Bayesian Normal Form and Pure Strategy Equilibria of Beer-Quiche Game

As there are two possible types of player 1, let us find the pure strategy equilibria. So we convert the game to Bayesian normal form and identify pure strategy equilibria in the normal form game. Table 25.1 shows the Bayesian normal form. In this table, Player 1's strategies are an ordered pair, with the first entry representing his action if he is the "Surly" type and the second entry representing his action if he is the "Wimpy" type. Similarly, Player 2's strategies are an ordered pair, with the first entry representing his action if player 1 serves "Beer" and the second entry representing his action if Player 1 serves "Quiche". Table 25.1 Payoffs of Beer-Quiche Game

Player 1's Action	Player 2's Action	Player 2's Action	Payoffs
Strong	Beer	Fight	(1,0)
Strong	Beer	Not Fight	(3,1)
Strong	Quiche	Fight	(0,0)
Strong	Quiche	Not Fight	(2,1)
Weak	Beer	Fight	(0,1)
Weak	Beer	Not Fight	(2,0)
Weak	Quiche	Fight	(1,1)
Weak	Quiche	Not Fight	(3,0)

It may be seen from the table that player 1's payoff starts at 0 and increases by 2 if player 2 chooses to "Concede". In addition, player 1 increases his payoff by +1 if he chooses the menu preferred by his type. Player 2's payoff starts at 0 and increases by 1 if he responds appropriately to player 1's type: player 2 gets a payoff of +1 if he chooses "Fight" and player 1 is "Weak" or if he chooses "Concede" and player 1 is "Strong". Otherwise, player 2 gets a payoff of 0 if he chooses "Fight" and player 1 is "Strong" or if he chooses "Concede" and player 1 is "Weak".

Bayes-Nash equilibrium of the game can be derived considering the best response functions for player 2. To work out the equilibrium, we have changed the probability assigned to player 1 being strong as p and that of weak $1 - p$ from 0.9 and 0.1 as given in the Figure 25.2.

Table 25.1a: Best Response Functions for Beer-Quiche if $\lambda > \frac{1}{2}$

	Concede, Concede	Concede, Fight	Fight, Concede	Fight, Fight
Beer, Beer	$2 + p, p$	$2 + p, p$	$p, 1 - p,$	$p, 1 - p$
Beer, Quiche	$3, p$	$1 + 2p, 1$	$3 - 2p, 0$	$1, 1 - p$
Quiche, Beer	$2, p$	$2 - 2p, 0$	$2p, 1$	$0, 1 - p$
Quiche, Quiche	$3 - p, p$	$1 - p, 1 - p$	$3 - p, p$	$1 - p, 1 - p$

Table 25.1b: Best Response Functions for Beer-Quiche if $\lambda < \frac{1}{2}$

	Concede, Concede	Concede, Fight	Fight, Concede	Fight, Fight
Beer, Beer	$2 + p, p$	$2 + p, p$	$p, 1 - p,$	$p, 1 - p$
Beer, Quiche	$3, p$	$1 + 2p, 1$	$3 - 2p, 0$	$1, 1 - p$
Quiche, Beer	$2, p$	$2 - 2p, 0$	$2p, 1$	$0, 1 - p$
Quiche, Quiche	$3 - p, p$	$1 - p, 1 - p$	$3 - p, p$	$1 - p, 1 - p$

The best response to (Beer, Beer) by player 1 depends on whether $p \geq \frac{1}{2}$, and similarly, the best response to (Quiche, Quiche) depends on whether $p \geq \frac{1}{2}$. The best responses for player 1 and the best responses by player 2 to (Beer, Quiche) and (Quiche, Beer) do not depend on the value of p , but the best responses to (Beer, Beer) and (Quiche, Quiche) depend on the value of p , as shown in Tables 25.1a and 25.1b.

Comparing the best responses for players 1 and 2 in Tables 25.1a and 25.1b, we find that there are no pure strategy equilibria if $p < \frac{1}{2}$, while there are two pure strategy Bayes – Nash equilibria if $p \geq \frac{1}{2}$:

[(Beer, Beer), (Concede, Fight)] and [(Quiche, Quiche), (Fight, Concede)]. In each of these pure strategy equilibria, both types of player 1 select the same action. Such a feature indicates that one of player 1's actions is off-the-path of equilibrium, so these Bayes-Nash equilibria only satisfy the requirements of perfect Bayesian equilibrium if player 2's strategy is a best response for some beliefs for player 1's action off- the-path of equilibrium.

It is found from the results of the example that both pure strategy Bayes-Nash equilibria are also perfect Bayesian equilibria for $p \geq \frac{1}{2}$. As may be seen, if player 1's strategy is (Beer, Beer) and $p \geq \frac{1}{2}$, then option Quiche is off-the-path of equilibrium.

Suppose $\mu_q =$

$P(\text{Surly} \mid \text{Quiche})$, player 2's best response to Quiche is Fight if $\mu_q < \frac{1}{2}$, and Concede if $\mu_q > \frac{1}{2}$. Since either of the pure strategies is a best response for some possible beliefs for player 2 after Quiche, a pure strategy Bayes-Nash equilibrium is also a perfect Bayesian equilibrium. This can be said as a best response for some possible beliefs for Player 2 after Beer holds good for either of the pure strategies.

We rework Bayesian equilibrium assigning probabilities 0.9 and 0.1 to strong and weak types of player 1. The extensive format of Beer-Quiche game is presented in Figure 25.3.

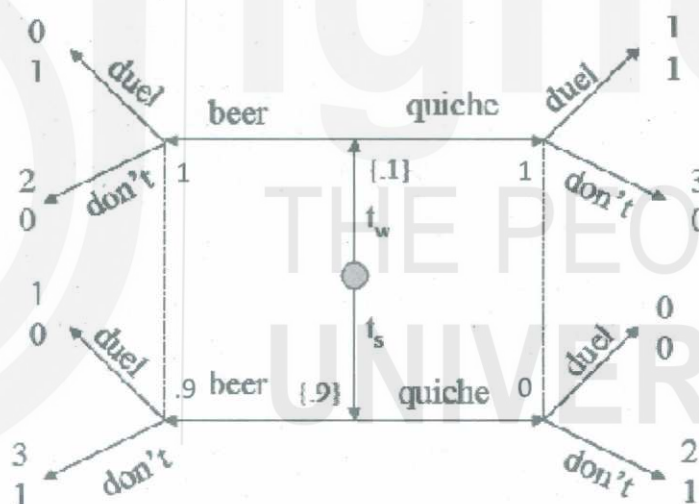


Fig. 25.3: Beer-Quiche Game

In Figure 25.3 we have $t_1 = \text{Wimpy}$ (with probability 0.1) or $t_2 = \text{Surly}$ (with probability 0.9). Both Sender types then have the same choice of messages-Quiche or Beer (as alternative breakfasts). The Receiver observes the message but not the type. See that the dashed line connecting two of the Receiver's two decision nodes indicates that the Receiver knows that one of the nodes in this "information set" was reached, but does not know which node is. So the Receiver observes the Sender's breakfast but not his type. Finally, following each message, the Receiver chooses between two actions: duel or not with the Sender. We need to consider the perfect Bayesian Nash equilibrium depicted in Figure 25.3. For that purpose, you need to check sequential rationality and consistency requirements as discussed in Unit 24.

It may be seen from the strategy profile of player 2 that his move is sequentially rational. This can be said due to the fact that in his information set on the right, player 2 is sure that player 1 is weak. So he chooses to duel. Seeing that player 1 is having beer for his breakfast he assigns probability 0.9

to the event that player 1 is strong. Hence, his expected payoff from duel is $0.9 \times 1 = 0.9$ whereas without duel his expected payoff is 0.1. That means his moves are sequentially rational.

If the player 1 is strong-type and he chooses beer, then he gets 3 while by choosing quiche, gets 0. So he chooses beer. Now we consider the weak type. If he chooses beer, he gets 2, while he gets only 1 if he chooses quiche. So he chooses beer establishing player 1's moves are sequentially rational.

Let's now check the consistency. The information set after the beer is on the path of equilibrium; hence we need to use the Bayes' rule. The probabilities 0.9 and 0.1 are the results when computed through Bayes' rule. The information set after the quiche is off-the-path of equilibrium. In this game, any belief off-the-path of equilibrium path is consistent. For, the present belief, which puts probability 1 to the weak type, consider a perturbation in which player 1 trembles and orders quiche with probability ε if he is weak, and he does not tremble if he is strong. Consequently, Bayes' rule yields $\varepsilon/\varepsilon = 1$ as the conditional probability of being weak given quiche. Therefore, the players beliefs are consistent, and we have a perfect Bayesian Nash equilibrium.

Note that we have a continuum of equilibria in which player 1 orders beer. After the quiche, player 2 assigns equal probabilities to each node and mixes between duel and not duel, where the probability of duel is at least 0.5. Check also that there is a perfect Bayesian Nash equilibrium in which player 1 orders quiche independent of this type, and player 2 fights when he observes a beer.

Check Your Progress 1

- 1) A new kid moves to town. Other kids don't know if he is tough or weak. Class bully likes to beat up weak kids, but doesn't like to fight tough kids. Bully gets to see what new kid eats for breakfast. New kid can choose either beer or quiche.

Preferences: Tough kids get utility of 1 from beer and 0 from quiche. Weak kids get utility of 1 from quiche and 0 from beer. Bully gets payoff of 1 from fighting a weak kid, -1 from fighting a tough kid, and 0 from not fighting. New kid's total utility is his utility from breakfast minus w if the bully fights him and he is weak and utility from breakfast plus s if he is strong and bully fights him.

- i) Draw the game tree
- ii) See the game given in Question 1. How many possible strategies are there for the bully? Name these.
- iii) See the game given in Question 1. What are possible strategies for New Kid? Name these.

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25.3 A SECOND APPROACH TO EQUILIBRIUM ANALYSIS: POOLING AND SEPARATING EQUILIBRIA

We consider a second approach to equilibrium analysis that uses the “Guess and Verify” approach to address the signaling nature of the game. There are two kinds of pure strategies for player 1: separating strategies (Beer, Quiche) and (Quiche, Beer) where each type of player 1 plays a different pure strategy, and pooling strategies (Beer, Beer) and (Quiche, Quiche) where each type of player 1 plays the same pure strategy.

Before we proceed to describe the PBNE of signaling game, consider the strategies of the players (s and r) in the game described in Figure 25.4.

There are two kinds of strategies: pooling and separating. Pooling strategy is one in which both types of the sender send the same message to the receiver, whereas in separating strategy different types of the sender send different messages.

25.3.1 Pooling Strategies

Sender decides to send message m_1 if Nature selects its type to be either t_1 or t_2 : pooling strategy on m_1 .

Sender decides to send message m_2 if Nature selects its type to be either t_1 or t_2 : pooling strategy on m_2 .

25.3.2 Separating Strategies

Sender decides to send message m_1 if Nature selects its type to be t_1 and message m_2 if Nature selects type t_2 .

Sender decides to send message m_1 if Nature selects its type to be t_2 and message m_2 if Nature selects type t_1 .

Definition of Perfect Bayesian Nash Equilibrium (PBNE): A pure strategy PBNE (Perfect Bayesian Nash Equilibrium) in the signaling game is a pair of strategies ($m^*(t_i)$ and $a^*(m_j)$) and an updated belief $p(t_i|m_j)$ such that the following conditions are satisfied:

Upon receiving a message m_j (an element of the message space M), the receiver updates his belief about the type of the sender (t_i) who has sent the message m_j . We denote this by the probability distribution $p(t_i|m_j)$, where

$$p(t_i|m_j) \geq 0 \text{ and } \sum_{t_i \in T} p(t_i|m_j) = 1.$$

If a message m_j falls on the equilibrium path, then the receiver's belief about the node in the information set corresponding to the message must be updated using the Bayes' rule: $p(t_i|m_j) = \frac{x(t_i)}{\sum_{t_i \in T} x(t_i)}$, where, $x(t_i)$ is the ex-ante or prior probability distribution, as shown in Figure 25.1.

If the message is off the equilibrium path, then it is updated by Baye's rule wherever possible.

(For Receiver): For each m_j in M and updated belief $p(t_i|m_j)$, the receiver's action $a^*(m_j)$ must maximize his expected payoff across all possible types of

the sender. Therefore, the problem for the receiver is to find that action $a^*(m_j)$ that maximizes $\sum_{t_i \in T} p(t_i|m_j)U_R(t_i, m_j, a_k)$

(For Sender): Given the receiver's strategy $a^*(m_j)$, for each type of the sender $t_i \in T$, the sender's message must maximize her payoff, i.e., $\max_{m_j \in M} U_S(t_i, m_j, a^*(m_j))$

25.3.3 Pooling and Separating Equilibria

Somewhat similar to the description of pooling and separating strategies in the earlier section, we can classify PBNE as either pooling or separating.

Pooling PBNE arise when all sender types play pooling strategies. Thus, in equilibrium, it is not possible to distinguish from the message which type sent the signal.

Separating PBNE is more informative than a pooling equilibrium, as each sender type sends a different message. Therefore, in equilibrium, the receiver can fully infer which sender type sent which message.

Example 1. Consider the signaling game given in Figure 25.4 below, where the payoff of the sender is written first followed by the payoff of the receiver in the terminal nodes. It is part of the solution to the game how r understands the underlying type of s from the message m sent by s .

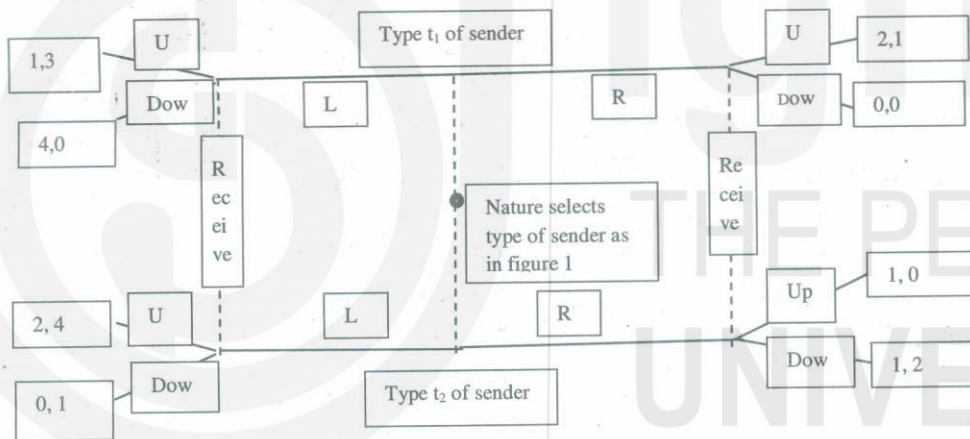


Fig. 25.4: Example of a Signaling Game

The timing of the game is:

Nature selects the type of the sender $\{t_1, t_2\}$ with equal probability $\{[x], [1-x]\} = \{1/2, 1/2\}$

The sender (of either type) sends a message from the set $\{L, R\}$

Upon receiving the message, the receiver updates his belief about the sender's type as follows:

Probability that the message L is sent by $t_1 = p(t_1|m_j=L)$

Probability that the message L is sent by $t_2 = p(t_2|m_j=L) = 1 - p(t_1|m_j=L)$

Probability that the message R is sent by $t_1 = q(t_1|m_j=R)$

Probability that the message R is sent by $t_2 = q(t_2|m_j=R) = 1 - q(t_1|m_j=R)$

Given his belief and the message sent by the sender, the receiver chooses an action from the set $\{Up, Down\}$.

To find out the PBNE of this game, we first note the strategies of the sender and the receiver.

As mentioned earlier, the sender has two classes of strategies: pooling and separating.

Pooling Strategies:

L if t_1 and L if t_2

R if t_1 and R if t_2

Separating Strategies:

L if t_1 and R if t_2

R if t_1 and L if t_2

A pure strategy PBNE (Perfect Bayesian Nash Equilibrium) in this game is defined as the collection $\{m^*(t_i), a^*(m_j), p(t_i|m_j)\}$ which satisfies the four conditions mentioned in the earlier sub-section.

We now work out the pooling and separating equilibria of this game.

Pooling equilibrium on L: For this equilibrium, the message L is on the equilibrium path and R is off the equilibrium path. Now, on the equilibrium path, since both the types send the same message L, the ex-ante probability $x = (1-x) = 0.5$ does not change. Thus, the updated belief about either type sending the message L does not change and it is $p(t_1|m_j=L) = 0.5$ and $[1 - p(t_1|m_j=L)] = 0.5$.

Given this belief about the sender types and the message L sent by either sender type, the receiver's best response is to play Up. On the equilibrium path, the expected payoff to the receiver from playing Up is $[3p + 4(1-p)]$, which is strictly greater than the expected payoff from playing down i.e. $[0p + 1(1-p)]$.

Now, we check that this equilibrium is indeed a mutual best response; i.e. no player has an incentive to deviate from his/her chosen action. Given that the receiver plays Up, we first check for the sender's best response. Note that her payoff from playing L is 1 if she is of type t_1 and 2 if she is of type t_2 . Whereas, if the sender deviates and plays R, then type t_1 will get 2 if the receiver continues to play Up. Thus this type has an incentive to deviate to R, only if the receiver keeps his strategy unchanged. Now, what the receiver will do in the face of a deviation depends upon his beliefs off the equilibrium path.

In order for either type of sender not to want to deviate from this pooling equilibrium on L, there must be a condition on the belief of the receiver off the equilibrium path, such that he prefers to play Down rather than stick to Up.

Noting that the updated belief off the equilibrium path is q (the probability with which the receiver plays Up when he receives the signal R from the sender), the receiver will play Down on the off-the-equilibrium path only when expected payoffs from Down are greater than playing Up; i.e. $0 \cdot q + 2 \cdot (1-q) \geq 1 \cdot q + 0 \cdot (1-q)$.

This gives us $q \leq 2/3$. With this off the equilibrium path belief, the receiver plays Down following a deviation by the sender which ensures that the sender has no incentive to deviate either.

Therefore, pooling PBNE here is $\{(L, L), (Up, Down), (p=0.5, q \leq 2/3)\}$, which should be read as both the sender types send the message L, the receiver reacts with Up on the equilibrium path and Down off the equilibrium

path, the updated belief on the equilibrium path is $p=0.5$ whereas off the equilibrium path, it is $q \leq 2/3$.

Pooling equilibrium on R: As discussed in the first pooling equilibrium, here the ex-ante probability $x=(1-x)=0.5$ does not change and hence, on the equilibrium path, $p(t_1|m_j=R)=0.5$ and $[1-p(t_1|m_j=R)]=0.5$.

Given this updated belief and the message sent by the sender, the optimal response of the receiver is to play Down. This is because, for the receiver, the expected payoff from playing Up is $1 \cdot q + 0 \cdot (1-q) = 0.5$ (since $q=0.5$), whereas the expected payoff from playing Down is $0 \cdot q + 2 \cdot (1-q) = 1$. Thus, by sending a pooling message R, the sender gets 0 if she is of type t_1 and 1 if she is of type t_2 .

Again, we have to check whether the sender of either type has an incentive to deviate from playing R. As before, the optimality of this deviation depends on the receiver's off the equilibrium path belief and action. Note that off the equilibrium path, the receiver always plays Up independent of the value of p . This is because playing Up (payoff of $[3 \cdot p + 4 \cdot (1-p)]$ as shown before) strictly dominates playing Down (payoff of $[0 \cdot p + 1 \cdot (1-p)]$) for all values of p for the receiver. There is no feasible value of p (which lies within 0 and 1) such that off the equilibrium path, the receiver will play Down rather than Up.

Given this off the equilibrium path action of the receiver, it is obvious that type t_2 would prefer to deviate from sending message R to sending message L. Doing so, fetches her a payoff of 2 (with receiver playing Up) rather than 1 on the equilibrium path. Similarly, type t_1 also has an incentive to deviate and play L. By doing so, she gets 1 rather than 0 on the equilibrium path.

Since both senders have an incentive to deviate (as there are no feasible off the equilibrium path beliefs for the receiver not to play Up), it is not possible to support (R, R) as a PBNE. *There is no pooling equilibrium on R in this game.*

Separating Equilibrium

Case 1: Type t_1 plays L and type t_2 plays R: Now both the messages are on the equilibrium path. Hence, by Bayesian updation, the updated beliefs are $p(t_1|m_j=L)=1$ and $q(t_1|m_j=R)=0$ (as type t_1 sends the message L with certainty).

Given these beliefs and the actions of the sender, the optimal action of the receiver is to play Up in response to the action L of t_1 and Down in response to the action R of t_2 . This is because Up is the dominant strategy for the receiver given action L of either type of sender. Given message R sent by t_2 , Down dominates Up for the receiver as shown in case B. above.

As before, we now check whether the sender of either type has an incentive to deviate or not. We see that type t_2 has an incentive to deviate to L (receiver's best response in that case will be Up as discussed earlier) giving her a payoff of 2 rather than 1 by sticking to R. Therefore, *this strategy does not qualify as a separating PBNE.*

Separating Equilibrium Case 2: Type t_1 plays R and type t_2 plays L:

We leave this as an exercise

Check Your Progress 2

- 1) What is the difference between a separating and pooling equilibrium?

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25.4 APPLICATION: JOB MARKET SIGNALING

This application is based on the seminal model of Spence (1973), where education is modeled as an unproductive signal sent by the privately informed worker (sender) to the uninformed employer (receiver).

The worker is privately informed about her type (either high productivity H , or low productivity L) by Nature. The employer receives the signal of the worker and updates his belief about the type of the worker and then offers wages to the worker. As we have set up the story, we see that it is line with the structure of a standard signaling game.

The timing of the game is given below:

Nature moves first and determines the workers' type, $x=\{H, L\}$ with ex-ante probability of the type being H being q and L being $(1-q)$, with $0 < q < 1$.

Worker learns ability and chooses education level $e \geq 0$.

The employer observes workers' education (but not their type) and simultaneously make wage offers to the workers.

The worker accepts the higher of the two offers. Let w be the accepted wage.

Note here that the signal of education e measures the difficulty of academic courses pursued by the students or even the challenges in acquiring distinction grades in similar courses. Effort of sending education signal e is easier to put in by a worker with high innate ability (type $x=H$) than a low ability worker (type $x=L$). We model the cost of acquiring education as $C(x, e)$ for worker type x . Hence, this cost is the psychological costs or cost of mental effort in acquiring education rather than monetary cost of education itself. These costs are higher for a low ability worker as he has to put in more effort e to acquire the same level of education as the high ability worker.

The interesting result in the paper of Spence is that wages might increase with education, even if education does not affect productivity at all. Education here is a wasteful signal, that workers are forced to pick up only because there is no way for the employer to distinguish between worker types by any other mechanism. Apart from this extreme result, the paper shows the more general case where the increase in wages due to education are larger than the positive contribution of education on innate productivity. In the presence of perfect information, the worker would settle for a lower level of education (no education in the extreme case where education has no effect on productivity and is an entirely wasteful signal). Education is used as a (somewhat useless but costly) signal by workers for the employer in the presence of asymmetric information.

We note that the signal of education is continuous (zero indicating the lowest possible education). What makes education as a valid signal? The signal must

be correlated with the worker type (as acquiring education is less costly for the high ability type than the low ability type of worker), but it should not directly reveal the type of the worker (it is possible for both workers to choose the same level of education in a pooling strategy). We are only considering costly signals here. Note that the output that can be produced by the worker of type x is $y(x, e)$, where $x = \{H, L\}$.

Assumptions

$y(H, e) > y(L, e)$ for all levels of e : this implies that for a given level of education, the high ability worker produces higher output than a low ability worker.

$\frac{\partial y(x, e)}{\partial e} \geq 0$ for all x and e : this implies that the marginal impact of education on output is non-negative for either type of worker for all levels of education.

$\frac{\partial C(L, e)}{\partial e} > \frac{\partial C(H, e)}{\partial e}$ for all levels of e : the marginal cost of effort in acquiring education is higher for the low type than the high type for all levels of education.

We assume that there is a single employer making a wage offer $w(e)$ and receiving a payoff $-(y(x, e) - w(e))^2$. Given the signal e , this single employer will form updated beliefs $p(H|e)$ and $(1-p(H|e))$, where $p(H|e)$ is the conditional probability that the type is H given signal e and its complement $(1-p(H|e))$ is the conditional probability that the type is L given signal e . The wage offer made is :

$w(e) = p(H|e) \cdot y(H, e) + (1 - p(H|e)) \cdot y(L, e)$ Here, the wage offer is made on the basis of expected productivity of the worker. The worker's payoff is: $w(e) - C(x, e)$.

The employer makes zero profits. Consider the indifference map of the worker. It is upward sloping (as effort is costly, but wage positively contributes to utility) and convex in the wage-effort plane. Given our assumptions, the high type worker needs to be compensated with lower wages to acquire the same level of education (e') than the low type worker, as shown in Figure 25.5. Note also that a *single crossing property* holds: the indifference curve of the high type and the low type can intersect at most once.

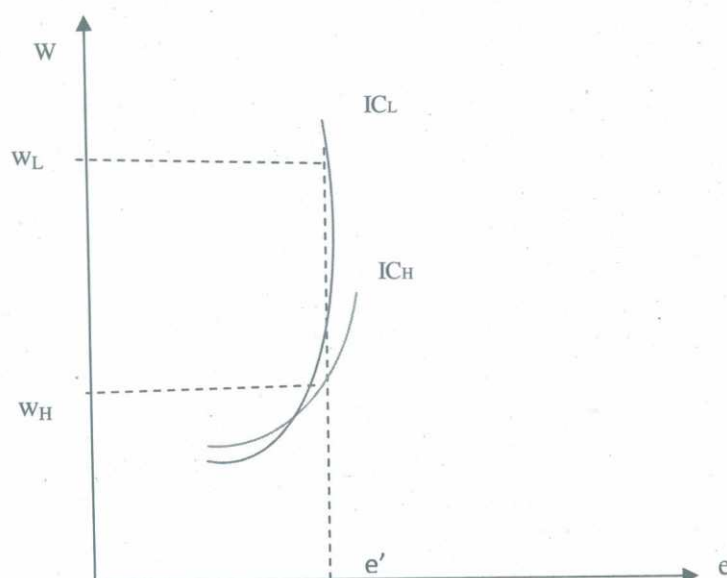


Fig. 25.5: Worker's Indifference Map: Single Crossing Property

25.4.1 Complete Information: Benchmark Scenario

This is the scenario in which the employer has complete information. Therefore, education as a signal is redundant. The employer gets zero profit and hence passes on the entire output as wages depending on the worker's type.

$$\text{Employer's strategy: } w(e) = \begin{cases} y(H, e) & \text{if } x = H \\ y(L, e) & \text{if } x = L \end{cases}$$

Worker's maximization problem is to solve the following for either type:

$$\text{Max}_{\{e\}} y(x, e) - c(x, e) \quad \text{for all } x = \{H, L\}$$

The equilibrium wage offer is: $w^*(x, e) = y[x, e^*(x)]$ for $x = \{H, L\}$.

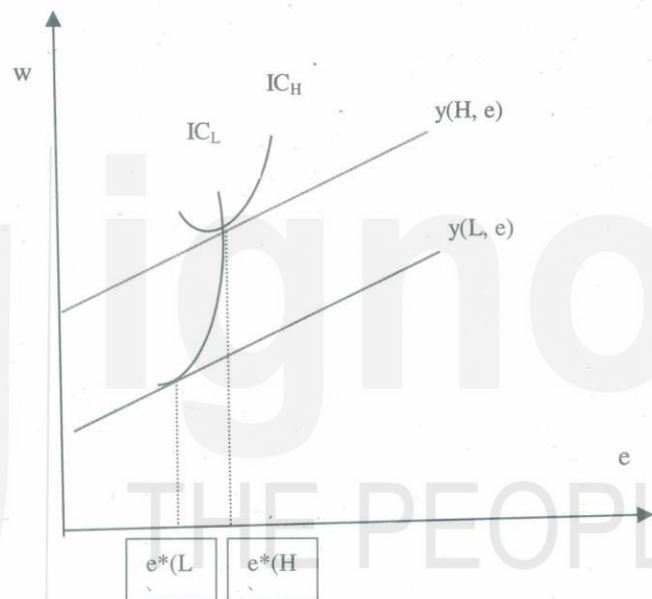


Fig. 25.6: Complete Information Benchmark Case

The result of the individual optimization problems gives us two independent wage schedules, one for each type of worker, w_H for the high type and w_L for the low type as shown in Figure 25.6 above.

25.4.2 Incomplete Information: Separating Equilibria

Under this class of equilibria, the high type worker and the low type worker sends distinct education signals to the employer. We first ask ourselves whether we can replicate the benchmark levels of education signals $e^*(H)$ and $e^*(L)$ corresponding to wage levels w_H and w_L respectively. The answer to this question depends on whether or not there is an incentive for any type of worker to mimic the education level of the other type and deviate from the benchmark levels of education, which we derived under complete information. The problem with incomplete information is that there is an incentive for a particular worker type to send a signal such that the employer believes that she is of the other type. The only reason to carry out such an action of 'mimicry' or untruthful signaling is that it gives a higher payoff to that type of worker.

25.4.2.1 Separating PBNE: No Mimicry

Now, consider the equilibria when it is not profitable for the low type to mimic the high type by acquiring an education level $e^*(H)$ that is meant for the high type rather than the lower type $e^*(L)$. Here, we can replicate the benchmark levels of signaling even in the incomplete information scenario. Note that the high type has no incentive to deviate to the low type's level of education. Moreover, the low type will be on a lower IC, say, IC_L^* which is to the right of IC_L , if he tries to acquire a higher level of education than $e^*(L)$. Therefore, he will have no incentive to deviate from education level $e^*(L)$.

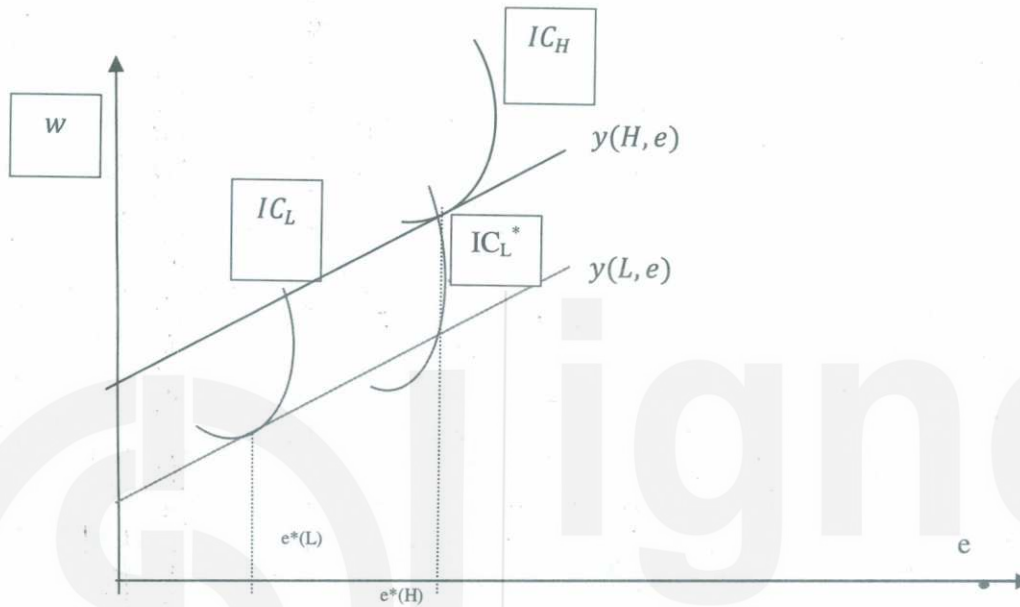


Fig. 25.7: Separating Equilibrium in the No Mimicry Case

In equilibrium, the low type chooses $e^*(L)$ and high type chooses $e^*(H)$. The employer's beliefs are updated on observing these education levels as follows: $p(H|e^*(L))=0$ and $p(H|e^*(H))=1$ [employer believes that the type of the worker is H with certainty or probability 1 if she receives an education signal of $e^*(H)$] and the corresponding strategy of the receiver is: $w(e^*(L))=w^*(L)$ and $w(e^*(H))=w^*(H)$.

In this no-envy (or no-mimicry) case, we know that: $w^*(L) - c(L, e^*(L)) > w^*(H) - c(L, e^*(H))$. This means that if the low type mimics the high type's education level to attain the wage offered to the high type, the low type will be worse off.

To determine the separating equilibrium, we need to find:

The receivers' beliefs at off the equilibrium path levels of education and the receivers' strategy $w(e)$.

Suppose, off the equilibrium path, the receiver believes that the worker is of high type if e is atleast education is at least $e^*(H)$ but is otherwise of low type then:

$$\hat{p}(H|e) = \begin{cases} 1, & e \geq e^*(H) \\ 0 & e < e^*(H) \end{cases}$$

The receiver's strategy is:

$$w(e) = \begin{cases} y(L, e) & \text{for } e < e^*(H) \\ y(H, e) & \text{for } e \geq e^*(H) \end{cases}$$

We have to prove that the best response of the worker of type x to the firms' strategy $w(e)$ mentioned above is to choose $e = e^*(x)$ for $x = \{L, H\}$.

At $e^*(L)$, the low type has no incentive to deviate and mimic the high type because he will be at a lower IC, i.e. IC_L^* .

At $e^*(H)$, the high type does not have any incentive to deviate and acquire more education because he will not be compensated for it. Further, if he acquires $e < e^*(H)$ then he will be paid $y(L, e) < y(H, e)$. Thus, he doesn't have any incentive to acquire $e < e^*(H)$ either.

The worker of low type will maximize: $\max_{e | e < e^*(L)} w(e) - c(L, e)$

The first order condition for this maximization is: $\frac{\partial}{\partial e} [w(e^*(L))] = \frac{\partial}{\partial e} [c(L, e^*(L))]$, which gives the worker's best response level of education, $w^*(L)$. Similarly, for the worker of high type, we will get $w^*(H)$.

Therefore, the separating PBNE is: [Low type selects education $e^*(L)$, high type selects education $e^*(H)$]; [low type is offered $w^*(L)$, high type is offered $w^*(H)$]; on the equilibrium path beliefs are $p(H|e^*(L))=0$ and $p(H|e^*(H))=1$ with off the equilibrium path beliefs given by: $\hat{p}(H|e) = \begin{cases} 1, & e \geq e^*(H) \\ 0, & e < e^*(H) \end{cases}$

25.4.2.2 Separating PBNE: Mimicry Possible

Again, there is asymmetric information between sender and receiver. However, as opposed to the no-envy/no-mimicry case, suppose now that $w^*(L) - c(L, e^*(L)) \leq w^*(H) - c(L, e^*(H))$. Hence, it is in the interest of the low type to mimic the education level of the high type and get the wages offered to the high type as that will increase his welfare. Deviation now leads the low type to a higher IC.

Now we characterise the PBNE for the case when mimicry is possible.

Least Cost Separating PBNE with $e(H)=e_s > e^*(H)$ and $e(L)=e^*(L) < e_s$

Fix the IC of the low type at the equilibrium level as in the benchmark case (IC_L is tangent to $y(L, e)$ giving the realisation of $\{w^*(L), e^*(L)\}$). Note the point where this IC_L crosses the $y(H, e)$ line. Given the single crossing property, it is obvious that the IC_H through this point, e_s , is lower than the one relevant for the benchmark case (which is tangent to $y(H, e)$ giving realisation of $\{w^*(H), e^*(H)\}$).

We claim that $\{e(H)=e_s, e(L)=e^*(L)\}$ is a separating PBNE in this case.

Note that, as shown in Figure 25.8 below, if high type acquires an education level higher than or equal to e_s , the low type of worker will have no incentive to mimic the high type and we will get a separating equilibrium. For education levels which lie between $e^*(H)$ and e_s , the low type will have incentive to mimic, as the low type is on the same indifference curve. Therefore, any education level less than e_s , will not be valid as a candidate for separating equilibrium.

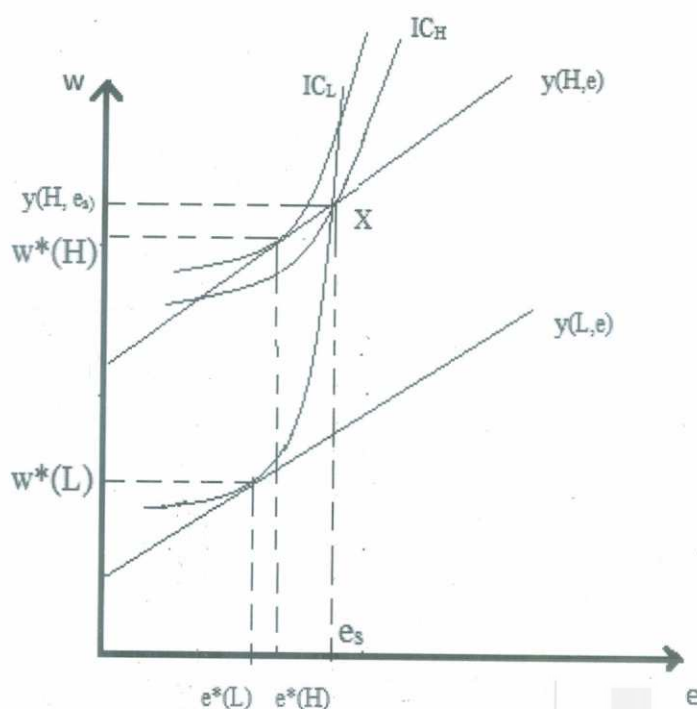


Fig. 25.8: Separating PBNE with Mimicry: Least Cost
(with $\{e(H)=e_s, e(L)=e^*(L)\}$)

We now ask the question: What beliefs does the receiver have to form off the equilibrium path so that we obtain a separating PBNE with the following education levels $[e^*(L), e_s]$?

On the equilibrium path, the beliefs are determined by Bayes' Law and strategy of the sender, so that: $p(H|e_s) = 1$ and $p(H|e^*(L)) = 0$

One possible specification for the off the equilibrium path beliefs is:

$$\hat{p}(H|e) = \begin{cases} 0 & \text{for } e < e_s \\ 1 & \text{for } e \geq e_s \end{cases}$$

Given these beliefs, the employer's strategy is:

$$w(e) = \begin{cases} y(L, e) & \text{for } e < e_s \\ y(H, e) & \text{for } e \geq e_s \end{cases}$$

Given the receiver's strategy, the low type worker is indifferent between the following two responses:

Choose $e^*(L)$ and earn $w^*(L)$

Choose e_s and earn $y(H, e_s)$

Given indifference between these choices, we can safely assume that the low type will choose $e^*(L)$ and earns $w^*(L)$.

Given the employer's belief and strategy, the high type workers' best response is to choose e_s , as:

Any level of education less than e_s will not be able to identify him as a high type to the employer (he will be worse off with the wages $y(L, e|e < e_s)$ meant for the low type as compensation). This is shown in 8, as the high type's indifference curve through X lies above the wage function $w = y(L, e)$ for $e <$

e_s . Any level of education greater than e_s will only increase his costs of acquiring education.

As e_s is the lowest level of education that the high type can take such that the low type will not mimic him in equilibrium, this equilibrium is called the *least cost separating PBNE*.

Separating PBNE with $e(H) > e_s$ and $e(L) = e^*(L) < e_s$

Apart from the least cost separating equilibrium, there are other separating PBNE, where the high type takes up education levels such as $e_s' > e_s$. Rather than be clubbed with the low type worker, the high type prefers to acquire a level of education higher than e_s , so that the low type does not find it profitable to send this signal. We leave this as an exercise.

25.4.3 Incomplete Information: Pooling Equilibria

There are several pooling equilibria in this game.

Pooling PBNE with $e = e_p$ for both types of workers

Since both types of workers are choosing the same $e = e_p$, the receiver's belief after e_p must be same as the prior belief, that is, $p(H|e_p) = q$. Therefore, the receiver offers wages which are average of the wages he would have offered a high wage worker and a low wage worker if he had complete information. This indicates that the wage schedule: $w_p = q \cdot y(H, e_p) + (1-q) \cdot y(L, e_p)$

To show that we have a PBNE with workers pooling at e_p , we need to

Specify the receivers' belief $\hat{p}(H|e_p)$ on the off-the-equilibrium path when e is not equal to e_p , which is then used to specify the firm's strategy when e is not equal to e_p .

Suppose that the employer believes that off the equilibrium path:

$$\hat{p}(H|e) = \begin{cases} 0, & e \neq e_p \\ q, & e = e_p \end{cases}$$

Given this off the equilibrium path belief, we have to prove that both workers' best response to the firm's wage schedule, w_p , is to choose $e = e_p$ on the equilibrium path.

A worker of type x chooses e such that

$$\max_e w(e) - c(x, e) \quad \forall x = \{L, H\}$$

Consider the employer's strategy, given the aforementioned off the equilibrium path belief:

$$w(e) = \begin{cases} y(L, e), & e \neq e_p \\ w_p = q \cdot y(H, e_p) + (1-q) \cdot y(L, e_p), & e = e_p \end{cases}$$

Noting this strategy of the employer, it is obvious that one solution to the worker's problem would be to choose e_p which gives the worker of either type w_p as wage.

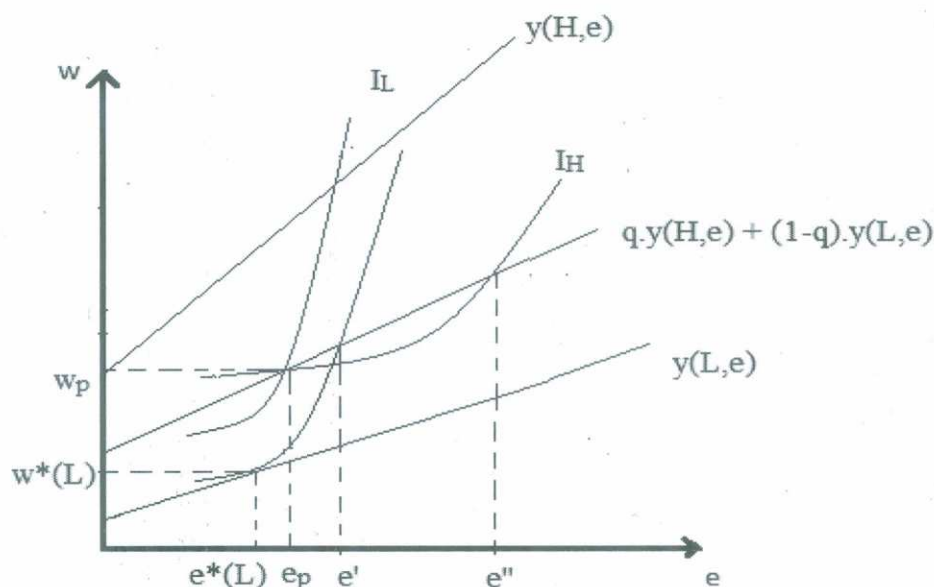


Fig. 25.9: Possible Pooling Equilibria

As shown in Figure 25.9, the low type is on a higher IC than with the education level $e^*(L)$ and has no incentive to deviate, while the high type has no incentive to deviate because his IC_H is above $y(L, e)$.

Therefore, a possible pooling PBNE with $e = e_p$ can be defined as follows:

{[L chooses e_p , H chooses e_p]; employer offers wage $w_p = q.y(H, e_p) + (1 - q).y(L, e_p)$ to either type of worker; on the equilibrium path belief is $p(H|e)=q$ with off the equilibrium path belief given by

$$\hat{p}(H|e) = \begin{cases} 0, & e \neq e_p \\ q, & e = e_p \end{cases}$$

Note that the e_p is the lowest education level that the high type can signal in order to pool with the low type. For all other pooling equilibria, the high type as well as the low type acquire a higher level of education.

Pooling PBNE with $e \in (e_p, e')$

Note that e' is the level of education at which the low type's education level through the benchmark level of education and wage ($e^*(L), w^*(L)$) intersects the average compensation line w_p . Between e_p and e' there exist many pooling equilibria. Consider the pooling level of education $\hat{e} \in (e_p, e')$. Note that with $\hat{e}$ level of education, the high type of worker does better than either at $e^*(L)$ or at e_p . The low type of worker is indifferent between $e^*(L)$ and e' .

On the equilibrium path, the ex-ante beliefs do not get updated as this is a pooling equilibrium.

The off the equilibrium path belief supporting this equilibrium is:

$$\hat{p}(H|e) = \begin{cases} 0, & e \neq e_p \\ q, & e = e_p \end{cases}$$

where $e \in [e_p, e']$

The wage offer of the employer is:

$$w(e) = \begin{cases} y(L, e), & e \neq e_p \\ w_p = q.y(H, e_p) + (1 - q).y(L, e_p), & e = e_p \end{cases}$$

Therefore, $\{e(L)=e(H)=\hat{e}\}$ is a pooling PBNE with the equilibrium wage offer and beliefs characterised above.

Consider now the remaining set of education levels which are $e \in (e', e'')$, where e'' is the education level where the indifference curve of the high type through e_p intersects the average wage line w_p in Figure 25.9. No pooling equilibria can exist here because the low type worker will want to reveal his type and acquire only $e^*(L)$ level of education.

This indicates that e' is the highest level of education for which a pooling equilibrium can exist.

The pooling equilibrium $\{e_p, e_p\}$ can be sustained with off the equilibrium path beliefs, apart from the one considered in case I. An alternative is:

$$\hat{p}(H|e) = \begin{cases} 0, & e \leq e'' \text{ except for } e = e_p \\ q, & e = e_p \\ q, & e > e'' \end{cases}$$

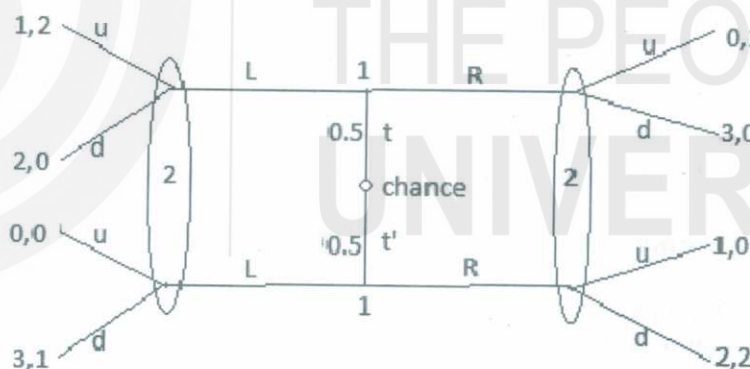
where, e'' is defined in case III. The wage offer now is:

$$w(e) = \begin{cases} y(L, e), & e \leq e'' \text{ except for } e = e_p \\ w_p, & e = e_p \\ w_p, & e > e'' \end{cases}$$

No sender type has an incentive to deviate from $\{e_p, e_p\}$ in equilibrium.

Check Your Progress 3

1)



In the signalling game above:

- Find the corresponding strategic form game and its pure-strategy Nash equilibria.
- Determine (if any) the game's separating equilibria.
- Determine (if any) the game's pooling equilibria.

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25.5 LET US SUM UP

Signaling games are a special kind of game of dynamic games of incomplete information. In these games, the typical structure involves two active agents, a sender and a receiver and a neutral third agent called nature. The sender is the player with private information about her type while the receiver is the uninformed agent.

The timing of this game is typically:

Nature selects the type of the sender with an ex-ante probability.

The sender privately observes her actual type, but not the receiver.

The sender selects a particular message to 'signal' to the receiver.

In order to work out the equilibrium of these games, we have to ensure that the receiver responds optimally to the signal sent by the sender (after updating beliefs on the equilibrium path using Bayes' Law) and that no type of sender has an incentive to deviate from the equilibrium signal, given the response of the receiver. These actions and belief on the equilibrium path of the receiver, along with the latter's off the equilibrium path (for which there are no restrictions) constitute the Perfect Bayesian Nash Equilibrium (PBNE) of these games.

Interestingly, depending on how much information the signal conveys to the receiver, there are two classes of pure strategy PBNE for signaling games. Pooling PBNE are equilibria where all types of the sender send the same signal, whereas separating PBNE are equilibria where different types of the sender send different signals.

Application of signaling game presents Spence model of job markets in the unit and discusses the derivations of pooling and separating equilibria.

25.6 KEY WORDS

Pooling PBNE: a PBNE where the equilibrium strategy for all the sender types is to send the same signal.

Receiver: the agent without private information about the type of the sender. The action of the receiver is based on the signal received from the sender and the updated belief of the receiver.

Sender: the agent with private information, who wants to convey the private information to the uninformed agent using the signal. The action of the sender is the choice of the signal to send to the receiver.

Separating PBNE: a PBNE where different types send different signals in equilibrium.

Signal: The message sent by the informed agent to the uninformed agent about the type of the former.

Signaling Game: a two-player game of incomplete information in which one player signals from disjoint subsets of the set of available signals.

Hybrid Equilibria: sender randomizes between pooling and separating equilibria.

25.7 SOME USEFUL BOOKS

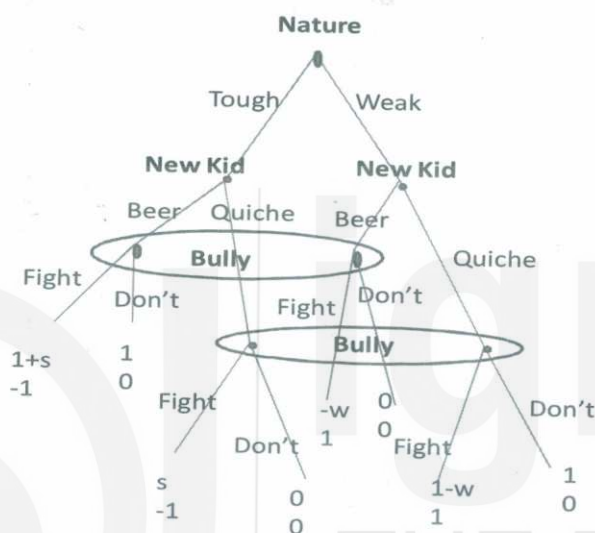
Gibbons, (1992), *A Primer in Theory*, Harvester Wheatsheaf.

Osborne, M.J., (2002), *An Introduction to Game Theory*, Oxford University Press

25.8 ANSWERS/ HINTS TO CHECK YOUR PROGRESS

Check Your progress 1

1) Game Tree



- 2) (4). (Fight if quiche, Fight if beer; Fight if quiche, Don't if beer; Fight if beer, Don't if quiche; Don't if beer, Don't if quiche).
- 3) (4). (Beer if tough, Beer if weak; Beer if tough, Quiche if weak; Quiche if tough, Beer if weak; Quiche if tough, Quiche if weak)

Check Your Progress 2

- 1) In an equilibrium of the game, the first player will condition her actions on her "type". If all types of the same player play the same strategy, we call it a pooling equilibrium. If all types of the same player play different strategies, we call it a separating equilibrium (because the actions of that player reveal/separate her different types to the other player(s)).

Check Your Progress 3

- a) P 1's pure strategies are pairs in $(L, R) \times (L, R)$, denoting the action after t and t' , respectively. P 2's pure strategies are pairs in $(u, d) \times (u, d)$, denoting the action after message L and R, respectively.

Strategic form:

Signaling Games and
their Applications

	u, u	u, d	d, u	d, d
L, L	$\frac{1}{2}, 1^*$	$\frac{1}{2}, 1^*$	$\frac{5}{2}, \frac{1}{2}$	$\frac{5}{2}, \frac{1}{2}$
L, R	$1^*, 1$	$\frac{3}{2}, 2^*$	$\frac{3}{2}, 0$	$2, 1$
R, L	$0, \frac{1}{2}$	$\frac{3}{2}, 0$	$\frac{3}{2}, 1^*$	$\frac{3}{2}, \frac{1}{2}$
R, R	$\frac{1}{2}, \frac{1}{2}$	$\frac{5}{2}, 1^*$	$\frac{1}{2}, \frac{1}{2}$	$\frac{5}{2}, 1^*$

Payoffs corresponding with best replies are starred, so there is an unique pure-strategy Nash equilibrium $((R, R); (u, d))$.

- b) Separating equilibria must be Nash equilibria; but the only candidate $((R, R); (u, d))$ is of the pooling type: P 1 sends the same message R for both types. Conclude: no separating equilibria.
- c) In (a), we found the candidate strategy profile $((R, R); (u, d))$. But what should the belief system be? Let $\alpha_1, \alpha_2 \in [0, 1]$ denote the probability assigned to the top node in the left and right info set, respectively.

Bayesian consistency: requires

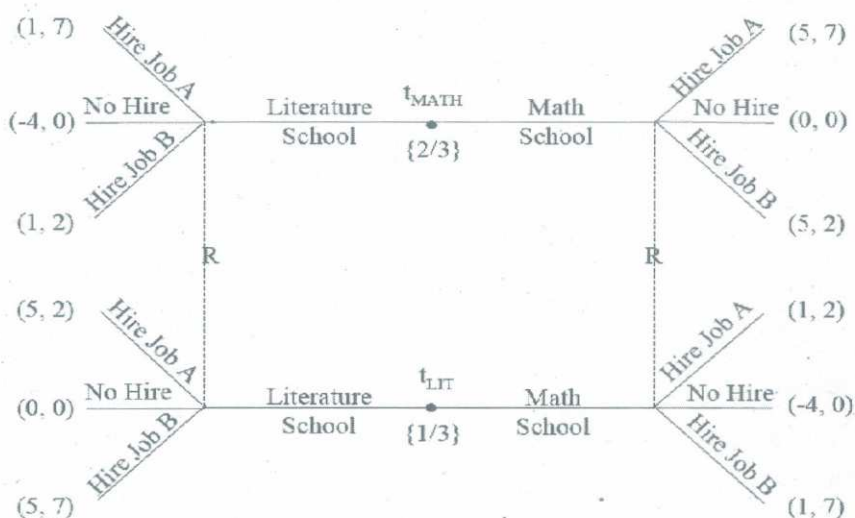
that $\alpha_2 = \frac{1}{2}$, but imposes no constraints on α_1 .

Sequential rationality: Both info sets of P1 and the right info set of P2 are reached with positive probability. Since $((R, R); (u, d))$ is a NE, the players choose a best reply in those information sets. The left information set of P2 is reached with zero probability. But the beliefs should be such that 2's action u is a best reply there. P 2's payoff from u is $2\alpha_1 + 0(1 - \alpha_1)$ and from d is $0\alpha_1 + 1(1 - \alpha_1)$, so sequential rationality requires $\alpha_1 = 1/3$.

Conclude: Assessments (s_1, s_2, β) with strategies $(s_1, s_2) = ((R, R); (u, d))$ and belief system $\beta = (\alpha_1, \alpha_2) \in \left[\frac{1}{3}, 1\right] \times \left(\frac{1}{2}\right)$ are the game's pooling equilibria.

25.9 EXERCISES AND ANSWERS/HINTS

- 1) Consider the following Beer-Quiche game



The first player is called A, and A either is a wimp (t_w) or is surly (t_s). At the start of the game, A knows his type, and is faced with the choice of what breakfast to have: beer or quiche. After breakfast, A meets with B. B, at this meeting, chooses whether to duel A.

- i) What are all of the (Bayesian) Nash equilibria in the beer-quiche game?
- ii) What are all of the Perfect Bayesian equilibria in the beer-quiche game?
- iii) What are all of the PBE meeting Signaling Refinement A in the beer-quiche game? (This is the weaker of the two signaling refinements, that relies on restricting beliefs based on the sending of some messages by some types being dominated.)
- iv) What are all of the PBE meeting Signaling Refinement B in the beer-quiche game? (This is the stronger of the two signaling refinements, equivalent here to Cho-Kreps's "Intuitive Criterion," restricting beliefs based on the sending of some messages by some types being dominated by their equilibrium payoffs.)

Ans.

- i) First note that there is no BNE in which the different types of A choose different actions. To see this, suppose that A plays Beer if wimpy and Quiche if surly. Then B must believe that he is facing t_w if he observes Beer and t_s if he observes Quiche. Then B's best response is to play Duel if Beer and Don't duel if Quiche. But then t_w would want to deviate to Quiche. Similarly, suppose that A plays Quiche if wimpy and Beer if surly. Then B must believe that he is facing t_w if he observes Quiche and t_s if he observes Beer. Then B's best response is to play Duel if Quiche and Don't duel if Beer. But then t_w would want to deviate to Beer.

Consider next BNE where both types of A play the same action. Suppose first that A chooses Beer regardless of her type. Then B chooses Don't Duel if Beer, since B's belief is that A is surly with (prior) probability 0.9 if Beer. To sustain this as a BNE, we must make sure that no type wants to deviate. Note that the type that could have the most profitable deviation is t_w . For this type not to deviate, we need B to choose Duel with probability $p \geq 0.5$ upon observing Quiche.

Finally, suppose that A chooses Quiche regardless of her type. Then B chooses Don't Duel if Quiche (since B's belief is that A is surly with (prior) probability 0.9 if Quiche). To sustain this as a BNE, we must make sure that no type wants to deviate. In this case, the type that could have the most profitable deviation is t_s . For this type not to deviate, we need B to choose Duel with probability $p \geq 0.5$ upon observing Beer.

Hence, there are two classes of BNE. The first class is (σ_A^*, σ_B^*) with σ_A^* : Both types of A choose Beer.

σ_B^* : B chooses Don't duel if Beer and Duel with probability $p \geq 0.5$ if Quiche.

The second class of BNE is $(\sigma_A^{**}, \sigma_B^{**})$ with σ_A^{**} : Both types of A choose Quiche.

σ_B^{**} : B chooses Don't duel if Quiche and Duel with probability $p \geq 0.5$ if Beer.

- ii) The set of PBE is the same as the set of BNE characterized in part (i). To show this, note that for both classes of equilibria, there exist beliefs at the out-of-equilibrium information sets that rationalize B's dueling. Consider the first class of BNE where both types of A choose Beer.

If B believes following Quiche that $\text{prob}(t = t_w) \geq 0.5$, then playing Duel with probability ≥ 0.5 is an optimal response. (If he believes that $\text{prob}(t = t_w) = 0.5$ if Quiche, then playing Duel with any probability is an optimal response, and if he believes that $\text{prob}(t = t_w) > 0.5$ if Quiche, then playing Duel with probability one is an optimal response.) So these equilibria are perfect Bayesian. Similarly, consider the second class of BNE where both types of A choose Quiche. If B believes following Beer that $\text{prob}(t = t_w) \geq 0.5$, then playing Duel with probability ≥ 0.5 is an optimal response. So these equilibria are also perfect Bayesian.

- iii) The set of PBE that meet Signaling Refinement A is the same as the set of PBE. This refinement does not rule out any PBE since there are no actions that are dominated for some types. The wimpy type prefers Beer to Quiche if B chooses Don't Duel if Beer and Duel if Quiche, and he prefers Quiche to Beer if B chooses Don't Duel if Quiche and Duel if Beer. The same is true for the surly type.
- iv) The set of PBE that meet Signaling Refinement B is the PBE where both types of A choose Beer, B chooses Don't Duel if Beer and Duel if Quiche. This refinement rules out the class of PBE where both types of A choose Quiche.

We first show that the PBE where both types of A choose Beer survives this refinement. In these equilibria, the out-of-equilibrium message is Quiche. The surly type of A would never deviate to Quiche, since he gets 3 in equilibrium and could get at most 2 by deviating. The wimpy type of A would deviate if B chose Don't Duel if Quiche, as he would then get a payoff of 3, which is greater than the payoff of 2 he obtains in the equilibrium. Signaling Refinement B then says that B's belief should be restricted to put no weight on A being surly if Quiche is observed. Then B believes that A is a wimp if Quiche, and hence his best response is to Duel if Quiche. But this supports the equilibrium outcome described above. (More precisely, the PBE that meets Signaling Refinement B is that in which both types of A play Beer, B plays Don't Duel if Beer, and Duel with probability one if Quiche.)

We next show that the PBE where both types of A choose Quiche fail this refinement. In these equilibria, the wimpy type obtains a payoff of 3 at the equilibrium. By deviating to Beer, the best payoff he could obtain is 2. Thus, deviating to Beer is never profitable for this type, no matter what B believes upon observing Beer. But this is different for the surly type.

At the equilibrium, this type obtains a payoff of 2. If he deviates to Beer and B believes that he is facing the surly type if he observes Beer, then this type obtains $3 > 2$ (since B chooses Don't Duel). Refinement B then says that we should restrict B's beliefs to put no weight on the wimpy type of A if he observes Beer. Thus, the only beliefs B could hold if he observes Beer are that A must be surly, and B would not duel. Anticipating this, the surly type of A wants to deviate to Beer.

- 2) Check that $\{(R, L), (Up, Up), p=0, q=1\}$ is a valid separating PBNE for the game in Example 1.

Ans. We need to prove that $\{(R, L), (Up, Up), p=0, q=1\}$ is a valid separating PBNE for this game.

In this separating PBNE, t_1 sends R and t_2 sends L. Both the information sets are on the equilibrium path. Updating by Bayes' Law, we get $p=0$ and $q=1$ for the beliefs of the receiver.

Given the strategy of the sender and the beliefs of the receiver, the latter's best response is to play Up in response to R (for t_1) and Up in response to L (for t_2) giving him a payoff of 1 and 4 respectively.

Given this strategy of the receiver, we check that the sender type has no incentive to deviate. As the receiver responds with Up if t_1 deviates to L, she gets a lower payoff of 1 rather than 2. Similarly, if t_2 deviates to R, with the receiver responding with Up following deviation, the payoff is lower at 1 rather than 2 without deviation.

Therefore, $\{(R, L), (Up, Up), p=0, q=1\}$ is a valid separating PBNE for this game.

- 3) Show that there exist separating PBNE in the Spence signaling game, with education level higher than that under least cost signaling (e_s).

Ans. Consider any education level $\tilde{e} > e_s$ for the high type. Note that in this separating equilibrium, the levels of education signals are $\{e(L) = e^*(L), e(H) = \tilde{e} > e_s\}$

This level of education for the high type is such that the high ability worker would rather signal this level $\tilde{e}$ of education rather than be pooled together with the low ability. In this case, $y(H, \tilde{e}) - c(H, \tilde{e}) > y(L, e) - c(H, e)$ for all levels of education e . This indicates that the high type is willing to undertake the cost of acquiring $\tilde{e}$ and be compensated with a $y(H, \tilde{e})$ rather than get the compensation of $y(L, e)$ meant for the low type for any other level of education e , with a lower cost $c(H, e)$. $\tilde{e}$ of education rather than be pooled together with the low ability. On the equilibrium path, the employer updates beliefs using Bayes Law and the sender's signal as: $p(H|\tilde{e}) = 1$ and $p(H|e^*(L)) = 0$. Out of equilibrium beliefs of the employer supporting this equilibrium would be: $\hat{p}(H|e) = \begin{cases} 0 & \text{for } e < \tilde{e} \\ 1 & \text{for } e \geq \tilde{e} \end{cases}$

The employer's wage offer would be: $w(e) = \begin{cases} y(L, e) & \text{for } e < \tilde{e} \\ y(H, e) & \text{for } e \geq \tilde{e} \end{cases}$

Neither type of sender has an incentive to deviate from $\{e(L)=e^*(L), e(H)=\tilde{e} > e_s\}$ given these beliefs.

- 3) **Entry as a signaling game.** A new entrant (E) threatens to enter the market of an incumbent (I). Player E is Strong with probability p and Weak with probability $(1-p)$, with $0 < p < 1$.

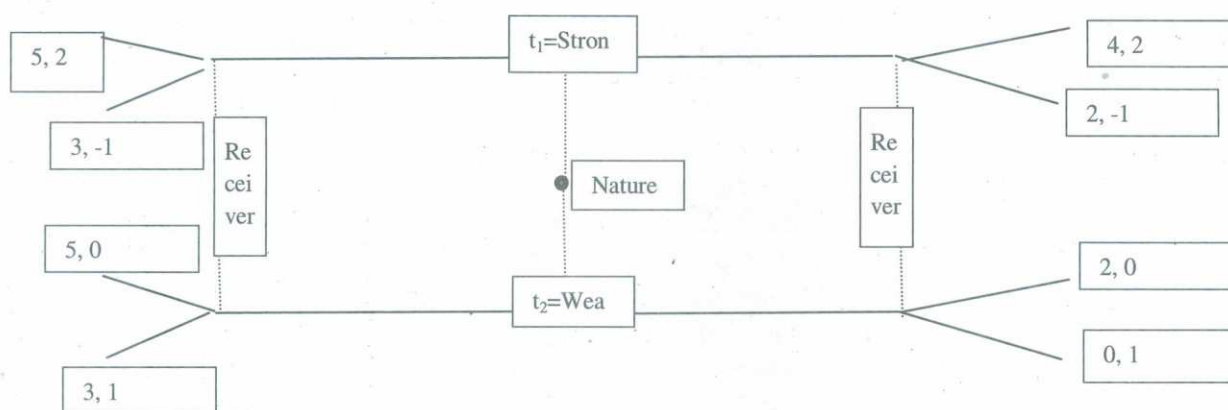


Fig. 25.10: Entry as a Signaling Game

While player E knows its exact type (Strong or Weak), but player I does not. The signal that player E can send to player I is its readiness to fight (either ready (R) or unready (U)). Given the belief of player I and the action chosen by player E, player I decides whether to fight (F) or acquiesce (A). The extensive form of this game, with payoffs in the terminal nodes is shown in Figure 25.10 above.

The receiver updates the ex-ante belief about the type of the sender after receiving the signal and then chooses an appropriate action to respond to the sender. Find the equilibria of the game.

Answer. We characterise the pooling and separating equilibria in this example.

The set of pooling strategies for the sender (E) are:

Both types of player E (Weak (W) and Strong (S)) play U

Both types of player E play R

The set of separating strategies for the sender (E) are:

Type W of player E plays U and type S plays R

Type W of player E plays R while type S plays U

First note that in this problem, playing U is a dominant strategy for the Weak type of Entrant (player E), irrespective of the action of the other player Incumbent (I). By playing U, player E gets either 5 or 3 depending upon the action of player I. However, playing R gives player E either 0 or 1. Hence pooling or separating equilibria where player E of type W plays Rare ruled out. Hence, we can rule out strategies 2. and 4. as candidates for equilibrium.

Pooling equilibrium with both types of player E sending signal U

As type W of player E has no incentive to deviate from this strategy, we have to check only whether type S has an incentive to deviate.

For this, consider the beliefs and best response of the receiver player I.

With a signal of U from both types of player E, player I updates his belief on the equilibrium path by Bayes' Law. In this case, $q_1 = p$ (the ex-ante probability) as no additional information is revealed by this pooling strategy.

The expected payoff to player E, given this belief, is:

$$E\pi_I(A|E \text{ plays } U) = 2 \cdot q_1 + 0 \cdot (1 - q_1)$$

$$E\pi_I(F|E \text{ plays } U) = (-1) \cdot q_1 + 1 \cdot (1 - q_1)$$

Note that these equations indicate that the expected payoff to I from playing A is greater than that from playing F only if $2 \cdot q_1 + 0 \cdot (1 - q_1) \geq (-1) \cdot q_1 + 1 \cdot (1 - q_1)$, i.e. $q_1 \leq 1/4$. For $q_1 > 1/4$, playing F is better than A for player I.

Given this strategy of the receiver E, does type S of player E have an incentive to deviate to playing R rather than U? Note that if player I continues to believe that player I is of type S with significantly high probability so that he plays F rather than A if S deviates to play R, then type S of E has no incentive to deviate from U. We work out the out of equilibrium path belief q_2 such that this happens. For this case:

$$E\pi_1(A| \text{ Type S of E deviate to R}) = 2 \cdot q_2 + 0 \cdot (1 - q_2) \leq E\pi_1(F| \text{ Type S of E deviates to R}) = (-1) \cdot q_2 + 1 \cdot (1 - q_2)$$

This implies that $q_2 \leq 1/4$.

Hence, if player E has the off the equilibrium path belief as given above, he will continue to play F with a deviation to R by type S of E. This deviation will yield either 2 or 0 to type S, whereas playing U gives a payoff of 3 to type S of E.

The pooling equilibrium is {both S and W types of E play U, receiver I plays F on the equilibrium path only if $q_1 > 1/4$ and continues to play F off the equilibrium path only if $q_2 \leq 1/4$ }.

Separating equilibrium with type W of player E playing U and type S of player E playing R. Now both the information sets are on the equilibrium path.

Given the strategy of the sender E, the updated beliefs of the receiver I is $q_1=0$ and $q_2=1$.

Given these beliefs, the best response of the sender E is to: play F for a signal of U from type W of sender E (giving I a payoff of 1 rather than 0 from playing A) play A for a signal of R from type S of sender E (giving I a payoff of 2 rather than -1 from playing F).

Given the beliefs and the actions of the receiver, the sender of either type has no incentive to deviate. If type S of player E deviates to U, then as long as player I continues to play F, the deviation gives a lower payoff of 3 rather than 4 from playing R. For $q_1=0$ and $(1-q_1)=1$, it is optimal for the receiver to continue to play F. Hence, no deviation is profitable (for player of type W, it is a dominant strategy to play U and not deviate).

UNIT 26 REFINEMENTS OF PERFECT BAYESIAN EQUILIBRIUM

Structure

- 26.0 Objectives
- 26.1 Introduction
- 26.2 Sequential Equilibrium is not Stringent Enough
- 26.3 Signaling Games
- 26.4 The Intuitive Criterion
 - 26.4.1 Intuitive Criterion with Two Types of Agents and only Two Responses
- 26.5 The Divinity Criterion
 - 26.5.1 Similarity and Difference between Intuitive Criterion and D_1 -Criterion
- 26.6 Spence's Labour Market Signaling Game
- 26.7 When Do We Need to Apply the D_1 -Criterion?
- 26.8 Let Us Sum UP
- 26.9 Key Words
- 26.10 Some Useful Books
- 26.11 Answer/Hints to Check Your Progress
- 26.12 Exercises with Answer/Hints

[This unit is prepared drawing mostly from Felix Munoz-Garcia and Ana Espinola-Arredondo (2010), The Intuitive and Divinity Criterion: Interpretation and Step-by-Step Examples, Journal of Industrial Organization, Education, Volume 5, Issue 1. No claim is made on contribution to the theme in the present write up. Credit goes to the authors for making the article available in the public domain for the benefit of learners.]

26.0 OBJECTIVES

After going through this unit, you will be able to:

- understand refinement requirements in signaling games;
- examine intuitive criterion as a technique of refinement;
- discuss divinity criterion as a stronger condition over intuitive criterion; and
- appreciate the application of intuitive criterion and divinity criterion to Spence's Labour Market Signaling Game.

26.1 INTRODUCTION

We continue with signaling games discussed in the preceding Unit 25. One of the main drawbacks of this class of games is that the set of strategy profiles that can be supported as perfect Bayesian equilibria is usually relatively large. Such a feature limits the predictive power that could be derived from results of the games. Moreover, there is another disadvantage that can be traced to

equilibria of signaling games. That is, in equilibrium we get insensible behaviour of the players. Such a result shows up due to the fact that the perfect Bayesian solution concept does not specify what the receiver's beliefs must be when the sender deviates. To help overcome these potential disadvantages, we focus on two refinement criteria viz., Intuitive Criterion put forth by Cho and Kreps (1987) and Universal Divinity Criterion formulated by Banks and Sobel (1987) in this unit.

To understand the main stages involved in both of these refinement criteria, our analysis focuses on the Spence's labour signaling model, assuming two types of workers. It will be shown Intuitive Criterion is sufficient to eliminate all but one equilibria. We then consider the result under more than two types of workers where the intuitive criterion does not eliminate any equilibria. So we go over to a more powerful refinement criterion, such as the Divinity Criterion, to restrict the set of equilibria in the signaling game.

We begin with the **intuitive** criterion (Cho and Kreps, 1987) which requires that there cannot be some type of player B that surely gains from deviating in every continuation for which player A responds with a strategy that is optimal based on a belief that assigns zero probability to those types of B that cannot gain from the deviation. That is, an equilibrium fails the intuitive criterion if A's belief fails to recognize that B's deviation is a credible signal about his type. These authors also define an alternative version, called the **equilibrium domination** criterion that requires for each continuation in which A responds with a strategy that is optimal based on a belief that assigns zero probability to those types of B that cannot gain from the deviation, there cannot be some type of B that gains from deviating. More restrictive is the criterion D_1 (Banks and Sobel, 1987), also called 'divinity' when it is applied iteratively, which requires that if the set of A's responses for which one type of B gains from deviating is larger than the set for which a second type gains, then A's beliefs must assign zero probability to the second type. The criterion D_2 is similar except that some (rather than one) type of B gains. All these criteria are weaker than the **never weak best reply** criterion that requires an equilibrium to survive deletion of a player's strategy that is not an optimal reply to any equilibrium with the same outcome. Our discussion in the unit has not covered criterion D_2 and **never weak best reply** criterion which may have been seen by interested learners from Banks and Sobel, 1987.

26.2 SEQUENTIAL EQUILIBRIUM IS NOT STRINGENT ENOUGH

Sequential Equilibrium (S.E) leaves room to some non-sensible equilibrium outcomes. Let's take a look at the game given in Figure 26.1 to see the outcome. It can be seen that the game has two pooling S.E., namely, (R,D) and (L,U).

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26.3 SIGNALING GAMES

Consider a sequential-move game with the following time structure:

- 1) Nature reveals to player i the private information. We denote this information as player i 's type θ_i where $\theta_i \in \Theta$.

Then, player i , who privately observes θ_i , chooses an action (or, message m) which is observed by other player j .

Player j observes message m , but does not know player i 's type. He knows the prior probability distribution that nature selects a given type θ_i from $\mu(\theta_i) \in [0, 1]$.

For example, the prior probability for $\Theta = (q_L, q_H)$ can be

$$\mu(q_L) = p \text{ and } \mu(q_H) = 1 - p.$$

- 2) After observing player i , player j updates his beliefs about player i 's type. For this reason, this action is normally referred to as message m . Let $\mu(\theta_i|m)$ denote player j 's beliefs about player i 's type being exactly $\theta = \theta_i$ after observing a particular message m .

- 3) Given his beliefs about player i 's type $\mu(\theta_i|m)$, player j selects an optimal action, a , as a best response to player i 's message, m .

In a Perfect Bayesian equilibrium of this signaling game, given equilibrium message m^* chosen by the sender, equilibrium action a^* chosen by the receiver, and the sender's type being θ_i , player i 's equilibrium payoff is $u_i(m^*, a^*, \theta_i)$, where for convenience $u_i^*(\theta) \equiv u_i(m^*, a^*, \theta_i)$. Similarly, player j 's utility when player i 's type is θ_i is $u_j(m^*, a^*, \theta_i)$. Finally, the action that the receiver optimally selects after observing message m from the sender, and given that the set of possible types which can potentially send message m is denoted as $a \in A^*(\Theta, m)$.

26.4 THE INTUITIVE CRITERION

The intuitive criterion is used for equilibrium refinement in signaling games. It aims to reduce possible outcome scenarios by first restricting the type group to types of agents who could obtain higher utility levels by deviating to off-the-equilibrium messages and second by considering in this sub-set of types the types for which the off-the-equilibrium message is not equilibrium dominated.

In order to apply the intuitive criterion we proceed through two steps:

In the First Step, we ask which type of senders could benefit by deviating from their equilibrium message?

As a second step we look into the following: If deviations can only come from the senders identified in the First Step, is the lowest payoff from deviating higher than their equilibrium payoff?

Thus, the First Step focuses on those types of senders who can obtain a higher utility level by deviating (i.e., when they send off-the-equilibrium messages) than by keeping their equilibrium message m^* unaltered. Specifically, this set of agents is taken as the subset of types for which a given off-the-equilibrium message is not equilibrium dominated (i.e., for which the equilibrium payoff does not dominate the highest payoff then could obtain by

sending such an off – the – equilibrium message). Formally, for any off – the – equilibrium message m , we construct a subset of types $\theta^{**}(m) \subset \theta$ for which m cannot be equilibrium dominated.

That is,

$$\theta^{**}(m) = \{\theta \in \theta \mid u^*(\theta) \leq \max_{a \in A^*(\theta, m)} u_i(m, a, \theta)\} \quad (1)$$

In other words, expression (1) states that, from all types in θ , attention is paid only on those types of agents for which sending the off-the-equilibrium message could give them a utility level higher than that in equilibrium, $u_i^*(\theta)$. Note that $\max_{a \in A^*(\theta, m)} u_i(m, a, \theta)$ gives the highest payoff that a θ - type can have by sending the off-the-equilibrium message m . Essentially, $\theta^{**}(m)$ is the subset of senders who could achieve a higher utility level by sending the off-the-equilibrium message m rather than their equilibrium message m^* .

The Second Step of the Intuitive Criterion considers the subset of types for which the off-the-equilibrium message m is not equilibrium dominated, $\theta^{**}(m)$. It checks if the equilibrium strategy profile (m^*, a^*) , is related with equilibrium payoff for the sender $u_i^*(\theta)$ satisfies.

$$\min_{a \in A^*(\theta^{**}(m), m)} u_i(m, a, \theta) > u_i^*(\theta) \text{ for some } \theta \in \theta^{**}(m) \quad (2)$$

What the Second Step does is as follows: If deviations can only come from the senders identified in the First Step, is the lowest payoff from deviating higher than their equilibrium payoff? So it seeks an answer: If it is yes, then the equilibrium violates the Intuitive Criterion. When the answer is no, the equilibrium survives the Intuitive Criterion.

That is, there is at least one type of sender who prefers to deviate to a message m which provides him with a higher utility level than his equilibrium message m^* , regardless of the response of the receiver.

Thus, an equilibrium strategy profile (m^*, a^*) violates the Intuitive Criterion if there is a type of agent θ and an action he can take m such that condition (2) is satisfied. Otherwise, it survives the Intuitive Criterion.

The deviation by this type of agent can be conceived as if he explains the following to the receiver:

It is clear that my type is in $\theta^{**}(m)$. If my type was outside $\theta^{**}(m)$ I would have no chance of improving my payoff over what I can obtain at the equilibrium (condition (1)). We can therefore agree that my type is in $\theta^{**}(m)$. Hence, update your beliefs as you wish, but restricting my type to be in $\theta^{**}(m)$. Given these beliefs, any of your best responses to my message improves my payoff over what I would obtain with my equilibrium strategy (condition (2)). For this reason, I am sending you such off-the-equilibrium message.

The underlying idea behind the Intuitive Criterion is: if there exists a type who will never benefit from taking a particular deviation, then the uninformed parties should deduce that this deviation is very unlikely to come from this type.

Such an idea falls within the category of forward induction where rather than solving the game simply backwards, we think about what type of inferences will others derive from a deviation.

26.4.1 Intuitive Criterion with Two Types of Agents and only Two Responses

Consider a game with two types of agents and only two responses for the receiver. We assume discrete messages for the analysis.

Take the sequential-move game with incomplete information with,

- a monetary authority privately observes its real degree of commitment with maintaining low inflation levels.
- After knowing its type (either Strong or Weak), the monetary authority decides whether to announce that the expectation for inflation is High or Low.
- A labour union, observing the message sent by the monetary authority, decides whether to ask for high or low salary raises (denoted as H or L, respectively).

For simplicity, it is assumed that the monetary authority is Strong with probability 0.6 or Weak with probability 0.4, where this prior probability distribution is common knowledge among all players. For convenience, we denote by μ the labour union's beliefs that the monetary authority is strong after observing a high inflation announcement, and let γ denote these beliefs after observing that the monetary authority announced a low inflation forecast (see Figure 26.2)

You can check that only two strategy profiles can be supported as a Perfect Bayesian Equilibria (PBE) in this signaling game: a pooling PBE with both types of monetary authorities announcing a high level of inflation (*High, High*); and a separating PBE in which the strong monetary authority announces low inflation, while the weak monetary authority announces high inflation (*Low, High*).

The figure gives the pooling equilibrium in which both types of monetary authorities send a message of high inflation.

We start with checking if (*High, High*) survives the Intuitive Criterion.

First Step: Which types of monetary authority have incentives to deviate towards Low inflation?

Note that Low inflation is an off-the-equilibrium message.

Applying condition (1) to the Strong type, we examine the inequality

$$\underbrace{u_{Mon}^*(High | Strong)}_{\text{Equil. Payoff}} < \underbrace{\max_{a_{labour}} u_{Mon}(low | Strong)}_{\text{Highest payoff from deviating to Low}}.$$

We will see that the *Strong* type of monetary authority has incentives to deviate towards *Low* inflation in the following.

Remember that the Intuitive Criterion eliminates those off-the-equilibrium messages that are equilibrium dominated. In this case, a message of *Low* inflation is an off-the-equilibrium message. In order to check if *Low* is equilibrium dominated, we need to find what types of monetary authorities prefer to keep sending *High* (in equilibrium) rather than deviating by sending *Low* (off-the-equilibrium message).

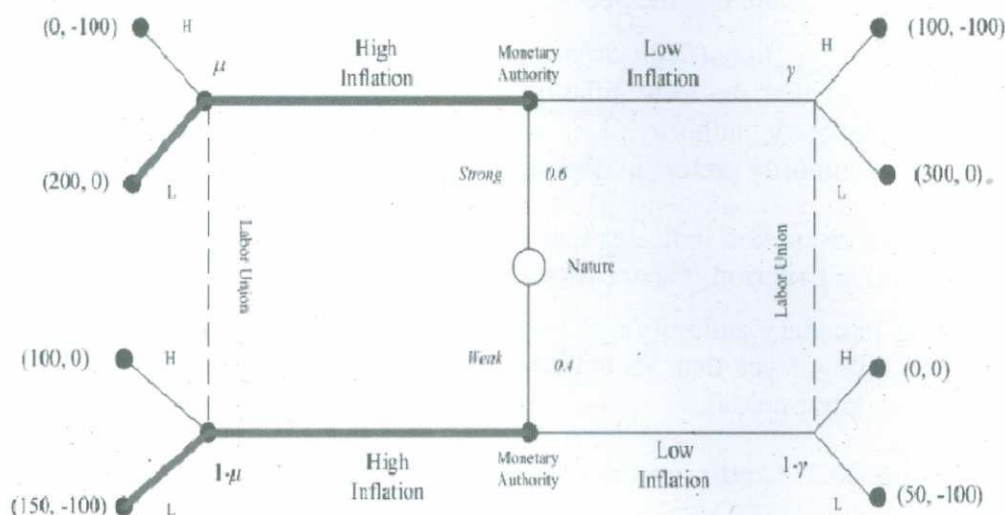


Fig. 26.2: Monetary Authority Announcements Game

Particularly,

$$200 = u_{Mon}^*(High|Strong) < \max_{a_{Labor}} u_{Mon}(Low|Strong) = 300$$

The first inequality is indeed satisfied since $200 < 300$ when the monetary authority is *Strong*.

Thus, the strong monetary authority obtains a payoff of 200 in equilibrium (by sending a high inflation announcement, which is responded with *Low*). But it could obtain a higher payoff by deviating towards a *Low* inflation announcement, 300, which arises when the labor union responds with *Low* wage demands.

Next, check if the Weak type also has incentives to deviate towards *Low*:

$$\text{See that, } 150 = u_{Mon}^*(High|Strong) > \max_{a_{Labor}} u_{Mon}(Low|Strong) = 50$$

When the monetary authority is *Weak*, its equilibrium payoff in the pooling PBE, 150, is higher than the maximum that it could obtain by deviating, 50 (which also occurs when the labour union responds to a Low inflation announcement choosing Low wage demands). Hence, the Strong monetary authority could indeed deviate to Low announcements of inflation but the Weak type could not. As a consequence, the subset of types for which the off-the-equilibrium message (Low inflation) is not equilibrium dominated is $\theta^*((Low) = (Strong))$, since such a message can only come from the Strong monetary authority. Hence, the labour union's beliefs when observing such a message are $\gamma = 1$ (at the upper right-hand corner of the figure). We examine next Study if there is a type of monetary authority and a message it could send such that condition (2) is satisfied as the sending step.

Second Step: The above restriction on beliefs $\gamma = 1$ is used to study if there is a type of monetary authority and a message it could send such that condition (2) is satisfied (i.e., obtaining a higher utility than in equilibrium, regardless of the labour union's response). First, when the labor union observes the off-the-equilibrium message of Low inflation, it responds with Low wage demands, since it concentrates all its beliefs in the node at the upper right-hand side corner of the game tree, i.e., $\gamma = 1$. By sequential rationality, and given this labour union response, the Strong monetary authority prefers to make an announcement of Low inflation levels. Indeed, this announcement is responded by the labour union with Low, providing a

payoff of 300 to the monetary authority, which is higher than its equilibrium payoff of 200. Note that the Second Step of the Intuitive Criterion involves

$300 = \min_{a_{Labor}} u_{Mon}(High\ Strong) > u_{Mon}^*(Low|Strong) = 200$ conditional on the belief that the Low inflation announcement can only come from the Strong monetary authority, i.e., $a_{Labor} \in A^*(Strong, Low)$. Then, the Strong monetary authority prefers to deviate from the pooling PBE of (High, High).

Thus, our discussion indicates that the pooling PBE of (High, High) violates the Intuitive Criterion. There exists a type of sender

(Strong monetary authority) and a message (Low) which gives that sender a higher utility level than in equilibrium, regardless of the response of the follower (labor union).

Example 26.1: Continuous messages

Analyse the traditional Spence's (1973) signaling game with two types of workers, one with a high productivity level, and the other with a low productivity, $\theta = \{\theta_H, \theta_L\}$ and a continuum of wage offers $w \in [0, 1]$. The worker acts as the sender in this game because he acquires a particular education level that is observed by the firm which is potentially interested in hiring him. Education is, nonetheless, not enhancing the worker's productivity, and hence it serves only as a signal about the worker's productivity level.

The firm's profit function is $\pi(w, \theta) = \theta - w$, and the worker's utility function is $u_i(e, w, \theta_K) = w - c(e, \theta_K)$, where $c(e, \theta_K)$ is the worker's cost of acquiring education level e . Of course acquiring no education is costless, $c(0, \theta_K) = 0$ for both types of workers. Assuming that the marginal cost of acquiring an additional year of education, $c_e(e, \theta_K)$, is decreasing in the worker's productivity, i.e., $c_e(e, \theta_H) \leq c_e(e, \theta_L)$, worker's indifference curves satisfy the single-crossing property.

In the game first, nature selects the type of a worker, either θ_H (high productivity) or θ_L (low productivity), such that $\theta_H > \theta_L$. The worker observes his own productivity level, but the firm does not. Observing his type, the worker chooses an education level, ≥ 0 . Observing the education level of the worker, e , the firm offers a wage $w(e)$. The worker's utility function is $u_{worker}(e|\theta) = w - \frac{e}{2\theta}$, if he accepts a wage offer, and zero if he rejects. (Note that q only affects the worker's cost of acquiring education).

Separating Perfect Bayesian equilibria of the game:

Figure 26.3 gives separating equilibria where $e_L^* = 0$ and $e_H^* \in [e_1, e_2]$ and $w(e_L^*) = \theta_L$ and $w(e_H^*) = \theta_H$ where the θ_L -type of worker sends a message of $e_L = 0$ years of education, while the θ_H -type of worker acquires $e_H^* = e_2$ years of education. In present case, education fully reveals the worker's type, since the firm can perfectly infer the worker's productivity level by observing the education he acquires. As a consequence, the firm offers a low wage offer to workers who acquire no education, $w(e_L^*) = \theta_L$, and a high wage to workers with e_2 years of education, $w(e_H^*) = \theta_H$. In Figure 26.3, IC_L and IC_H denote the indifference curves for the low and high-productivity workers, respectively, in this equilibrium. Since higher wages increase worker's utility and education is costly to acquire, indifference curves to the northwest (higher wages and less education) are related to higher utility levels.

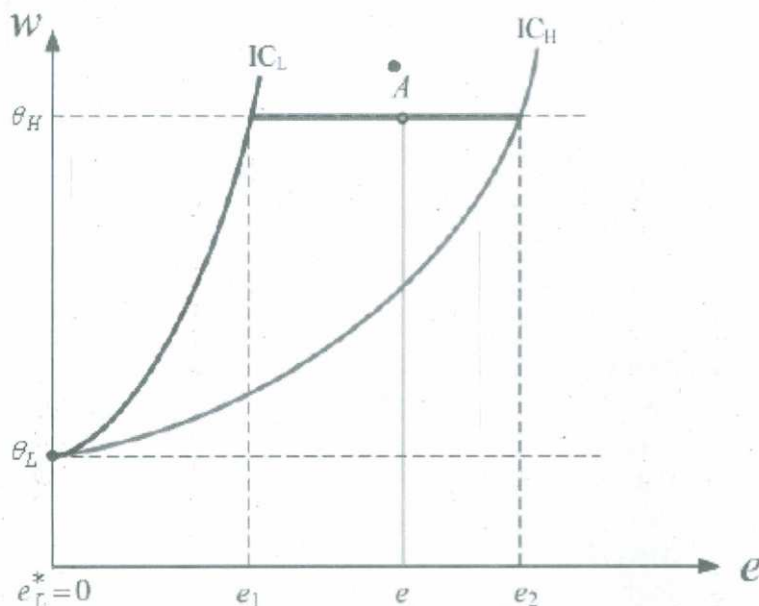


Figure 26.3: Labor Market Game with Two Types

First Step: Check if the separating equilibrium $e_L^* = 0$ and $\theta_H = e_2$ survives the Intuitive Criterion. For that we consider any off-the-equilibrium message $e \in (e_1, e_2)$ in Figure 26.3 (Green color).

In order to study what type of worker might have sent such a message, apply the previous analysis of equilibrium dominance. In particular, for the θ_L -type of worker, we have that $u_L^*(\theta_L) > \max_{w \in W^*(\theta, m)} u_L(e, w, \theta_L)$.

That is, his equilibrium payoff $u_L^*(\theta_L)$ is higher than the maximal utility he could obtain if the firm offered him the highest possible salary. In other words, his equilibrium payoff from sending $e_L^* = 0$, $e_L^*(\theta_L) = \theta_L - c(0, \theta_L) = \theta_L$, is higher than the highest payoff he could obtain by sending the off-the-equilibrium message $e = \theta_H - c(e, \theta_L)$ (when the firm believes that the worker is a θ_H -type and pays him a salary of $w(e) = \theta_H$). Therefore, the above inequality implies that for any off-the-equilibrium message $e \in (e_1, e_2)$, $c(e, \theta_L) > \theta_H - \theta_L$.

It is not difficult to see that if the cost from acquiring e years of education for the low-productivity worker, $c(e, \theta_L)$, exceeds the wage increase, $\theta_H - \theta_L$. If the firm believes that, because of acquiring education level e , he must be a high productivity worker, pays him $w(e) = \theta_H$. Seen graphically, the θ_L -worker's indifference curve when he acquires the equilibrium education level $e_L^* = 0$ is given by IC_L , and the indifference curve from acquiring the (off-the-equilibrium) education level e and receiving a salary of $w(e) = \theta_H$ would cross point A. Clearly, the indifference curve of an education level e_L^* implies a higher utility level than that can be found with e , even when the salary the worker receives is $w(e) = \theta_H$. Such a process can then be repeated for any off-the-equilibrium message $e \in (e_1, e_2)$, concluding that the θ_L -type of worker does not send such a message, because it is *equilibrium dominated*.

Thus, we say the θ_L -type of worker doesn't have incentives to deviate towards e since:

$\underbrace{u_L^*(\theta_L)}_{\text{Equil. Payoff}} > \underbrace{\max_{w \in W^*(\theta, m)} u_L(e, w, \theta_L)}_{\text{Highest payoff from deviating towards } e}$ and education levels in $e \in (e_1, e_2)$, can only come from the θ_H -worker.

Next, apply the same analysis of equilibrium dominance to the θ_H -worker. This type of worker can send the off-the-equilibrium message e since:

$$u_i^*(\theta_H) < \max_{w \in W^*(\theta, e)} u_H(e, w, \theta_H)$$

$$\theta_H - c(e_2, \theta_H) < \theta_H - c(e, \theta_H)$$

That means, he receives the same salary as in equilibrium ($w(e) = \theta_H$) but incurs lower costs because of acquiring a lower education level, i.e., $c(e_2, \theta_H) > c(e, \theta_H)$ since $e_2 > e$. Hence, the equilibrium payoff of such a worker is lower than the maximal payoff he could obtain if the firm manager offers him a salary of $w(e) = \theta_H$ after observing education level e . Seen graphically, indifference curves through point A (if he receives the high salary) indicate higher utility levels than that in equilibrium. It is shown by IC_H . So, off-the-equilibrium message e is not equilibrium dominated for the θ_H -worker, while it is for the θ_L -worker. We can now state which is the subset of types that the receiver (firm) considers after observing the off-the-equilibrium message e . In particular, the firm focuses its beliefs on the θ_H -type of worker, since he is the only type whose utility can increase by deviating from his equilibrium message. Thus, the subset of types for which message e is not equilibrium dominated is given by $\theta^{**}(e) = (\theta_H)$.

Second Step:

If e comes from θ_H , the firm offers a wage $w(e) = \theta_H$ after observing e .

Note that the minimal utility level that the worker can achieve from sending the off-the-equilibrium message e is

$\min_{w \in W^*(\theta^{**}(e))} u_H(e, w, \theta_H) = \theta_H - c(e, \theta_H)$ and the equilibrium payoff for the equilibrium education level e_2 is $u_H^*(\theta_H) = \theta_H - c(e_2, \theta_H)$.

Since $c(e_2, \theta_H) > c(e, \theta_H)$, we have $\theta_H - c(e, \theta_H) > \theta_H - c(e_2, \theta_H)$. Hence, $\min_{w \in W^*(\theta^{**}(e))} u_H(e, w, \theta_H) > u_H^*(\theta_H)$.

Therefore, the separating PBE where workers acquire education levels $\{e_L^*(\theta_L), e_H^*(\theta_H)\} = (0, e_2)$ violates the Intuition Criterion. For, there exists a type of worker, θ_H , and an off-the-equilibrium message $e \in (e_1, e_2)$, in which the above inequality is satisfied.

In view of such a result we may observe that the θ_H -worker can signal his type (productivity level) to the firm by acquiring less education than in the separating equilibrium where he acquires $e_H^* = e_2$.

It can be verified that all separating equilibria can be eliminated following the above procedure, except for the equilibrium in which the low-type acquires zero education and the high-type acquires education level e_1 . The surviving separating equilibria is usually referred to as the efficient equilibrium outcome, since at that workers spend the least amount of resources in signaling to the firm their different productivity levels. Particularly, the θ_L -type acquires an education level of $e_L^* = 0$ and the θ_H -type acquires the minimal education level that allows him to separate himself from the θ_L -type, $e_H^* = e_1$. We illustrate such an equilibrium in the following figure.

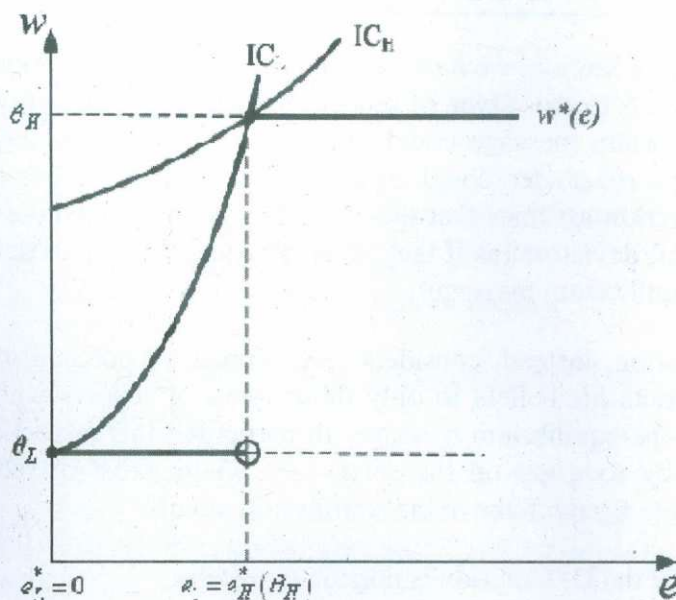


Figure 26.4: Efficient Separating Equilibrium

Check Your Progress 1

- 1) Under Intuitive Criterion a player defects in which belief system?
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- 2) Differentiate between backward and forward induction
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- 3) What would you expect to happen if all separating equilibria in which the θ_H -worker sendse $e \in (e_1, e_2)$?
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- 4) To which equilibria would you refer as the efficient outcome?
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26.5 THE DIVINITY CRITERION

In the preceding section, we have seen that the Intuitive Criterion restricts the receiver's beliefs to those type of senders for which deviating towards a given off-the-equilibrium message could improve his equilibrium payoff. If more than one type of sender could benefit from such deviation, however, the Intuitive Criterion assumes that the receiver's beliefs assign the same weight to all potential deviators (as if they were all equally likely to deviate towards the off-the-equilibrium message).

The D_1 Criterion, instead, considers that, among all potential deviators, the receiver restricts his beliefs to only those types of senders who most likely send the off-the-equilibrium message. In particular, this restriction on beliefs is analysed by focusing on the sender for whom most of the responder's actions provide a payoff above his equilibrium payoff.

Application of the D_1 Criterion is done in two steps:

First Step is to ask which type of senders are more likely to deviate from their equilibrium message? In particular, for which type of senders are most of the responder's actions beneficial?

Second Step: If deviations can only come from the senders identified in the First Step, is the lowest payoff from deviating higher than their equilibrium payoff?

If the answer is yes, then the equilibrium violates the D_1 Criterion.

If the answer is no, then the equilibrium survives the D_1 Criterion.

We introduce some notation for analysing the issue as under:

For any off-the-equilibrium message m , define

$$D(\theta, \hat{\theta}, m) = \bigcup_{\mu: \mu(\hat{\theta}|m)=1} \{a \in MBR(\mu, m) | u_i^*(\theta) < u_i(m, a, \theta)\} \quad (3)$$

where $D(\theta, \hat{\theta}, m)$ is the set of mixed best responses (MBR) of the receiver such that the θ -type of sender is better-off by sending message m than the equilibrium message m^* .

Note that $(\hat{\theta}|m) = 1$ shows the belief of the receiver that message m only comes from types in the subset $\hat{\theta} \in \theta$

Similarly, for MBR that make the sender indifferent

$$D^o(\theta, \hat{\theta}, m) := \bigcup_{\mu: \mu(\hat{\theta}|m)=1} \{a \in MBR(\mu, m) | u_i^*(\theta) < u_i(m, a, \theta)\} = u_i(m, a, \theta) \quad (4)$$

Next, identify which type of senders are more likely to deviate from their equilibrium message:

$$[D(\theta, \hat{\theta}, m) \cup D^o(\theta, \hat{\theta}, m)] \subset D(\theta', \hat{\theta}, m) \quad (5)$$

That is, for a given message m , the set of receiver's actions which make the θ' -type of sender better off (relative to equilibrium) $D(\theta', \hat{\theta}, m)$ is larger than those actions making the θ -type of sender strictly better off, $D(\theta, \hat{\theta}, m)$ or indifferent, $D^o(\theta, \hat{\theta}, m)$.

The set of types that cannot be deleted after using the above procedure is denoted by $\theta^{**}(m)$.

As discussed in the previous section, the Second Step of both the Intuitive and the D_1 -Criterion, analyses the subset of types for which the off-the-equilibrium message m is not equilibrium dominated, $\theta^{**}(m)$, and checks if

the equilibrium strategy profile (m^*, a^*) , yields the equilibrium payoff for the sender $u_i^*(\theta)$, satisfies

$$\underbrace{\min_{a \in A^*(\theta^{**}(m), m)} u_i(m, a, \theta)}_{\text{lowest payoff from deviating to } m} > \underbrace{u_i^*(\theta)}_{\text{equilibrium payoff}} \quad \text{for some } (\theta) \in \theta^{**}(m) \quad (6)$$

The inequality implies if deviations can only come from the senders identified in the First Step, is the lowest payoff from deviating higher than their equilibrium payoff?

See that it coincide with the 2nd step of Intuitive Criterion

26.5.1 Similarity and Difference between Intuitive Criterion and D₁-Criterion

Both refinement criteria coincide in their 2nd step. The 1st step of the D₁-Criterion and the Intuitive Criterion are different. In particular, they differ in how they determine the set of senders who can benefit by deviating from their equilibrium message:

Intuitive Criterion: For which senders there is at least one action of the responder that is beneficial?

D₁-Criterion: For which senders most of the actions of the responder are beneficial?

Hence, the types of senders who benefit from deviating according to the D₁-Criterion are a subset of those who benefit from deviating according to the Intuitive Criterion. Therefore, the set of equilibrium surviving the D₁-Criterion are a subset of those surviving the Intuitive Criterion.

Check Your Progress 2

- 1) Why would you consider divinity criterion over intuitive criterion?

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- 2) Point out the similarities and differences between Intuitive Criterion and divine Criterion.

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26.6 SPENCE'S LABOUR MARKET SIGNALING GAME

Apply the D₁-Criterion in the Spence's (1973) labour market signaling game. As seen above it has two types of workers, one with a high productivity level,

and the other with a low productivity, $\theta = (\theta_H, \theta_L)$, and a continuum of wage offers $w \in [0, 1]$. The worker acts as the sender in this game because he acquires a particular education level that is observed by the firm which is potentially interested in hiring him. Education is, nonetheless, not enhancing the worker's productivity, and hence it serves only as a signal about the worker's productivity level.

Now, check if the separating equilibrium $e_L^* = 0$ and $e_H^* = e_2$ survives the D_1 -Criterion. We start with the off-the-equilibrium message ? $e'' \in (e_1, e_2)$ (in green colour).

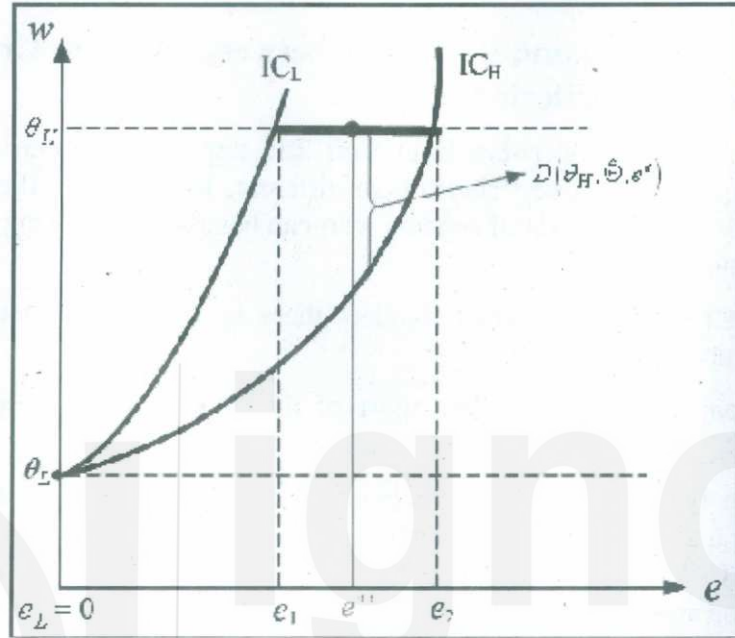


Fig. 26.5: Applying the D_1 -criterion to the labour market game

First Step: First, after sending an off-the-equilibrium message e' , the set of wage offers that improve the equilibrium payoff of the low-productivity worker, $D(\theta_L, e')$, is smaller than that for the high-productivity worker, $D(\theta_H, e')$, i.e., $D(\theta_L, e') \subset D(\theta_H, e')$. Thus $\theta^{**}(e') = (\theta_H)$. Repeating this process for any off-the-equilibrium message,

Firm's beliefs are restricted to $\theta^{**}(e'') = (\theta_H)$.

Second Step: After restricting the subset of types who could have sent any off-the-equilibrium message e to $\theta^{**}(e) = (\theta_H)$, we have $w(e'') = \theta_H$. Thus, the minimal utility level that the worker can achieve by sending the off-the-equilibrium message e is

$$\frac{\min_{w \in W^*(\theta^{**}(e''), e'')} u_H(e'', w, \theta_H)}{\theta_H - c(e'', \theta_H)} > \frac{u_H^*(\theta_H)}{\theta_H - c(e_2, \theta_H)}$$

Since it is given that $e_2 > e''$ and $c_e(e'', \theta_H) > 0$, we have $c(e_2, \theta_H) > c(e'', \theta_H)$ which ultimately implies $\theta_H - c(e'', \theta_H) > \theta_H - c(e_2, \theta_H)$. Therefore, the separating PBE where workers acquire education levels $(e_L^*(\theta_L), e_H^*(\theta_H)) = \{0, e_2\}$ violates the D_1 -Criterion. We can say this as there exists a type of sender (θ_H -worker) and an off-the-equilibrium message, e , for which the above inequality is satisfied. Just as in case of the Intuitive Criterion, we can show that all separating equilibria in this game can be eliminated using the D_1 -Criterion, except for the efficient outcome, where the low-productivity worker acquires an education level of $e_L^* = 0$ and the high-productivity worker only acquires education $e_H^* = e_1$.

Check Your Progress 3

- 1) Is the Perfect Bayesian Equilibrium reasonable in Spence model? Give reasons in support of your answer.

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- 2) In Spence model separating PBE does support D_1 Criterion?

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- 3) Comment on the main finding of Spence model.

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26.7 WHEN DO WE NEED TO APPLY THE D_1 -CRITERION?

In the previous section, we described the Intuitive and D_1 -Criterion, and examined that, when there are only $n = 2$ types of senders, the equilibria that survive these two equilibrium refinement coincide. However, as we will show in this section, it might not be the case when there are $n > 2$ types of senders (for instance, more than two types of workers in the labor market signaling game).

First, we describe how the application of the Intuitive Criterion to signaling games with more than two senders might not help us restrict the set of equilibria, and then we show that the D_1 -Criterion reduces the set of equilibria in this class of games.

Example 26.2: Continuous messages with $n = 3$ types of workers.

Intuitive Criterion: In the following we analyse if the separating PBE $\{e_L^*(\theta_L), e_M^*(\theta_M), e_H^*(\theta_H)\} = (0, e_M, e_H)$ would survive the Intuitive Criterion. Figure 26.6 used for the Spence's signaling game with three types of workers game presents multiple separating equilibria. To do that check if this separating equilibrium survives the Intuitive Criterion, by choosing an off-the-equilibrium message $e \in (\hat{e}, e_H^*)$.

First Step: We need to construct the subset of types $\theta^{**}(e) \subset \theta$ for which the off-the-equilibrium message, $e \in (\hat{e}, e_H)$, is never equilibrium dominated (see message $e \in (\hat{e}, e_H)$ in Figure 26.6). That is,

$$\theta^{**}(e) = \{\theta \in \theta | u_i^*(\theta) \leq \max_{w \in W^*(\theta, e)} u_i(e, w, \theta)\}$$

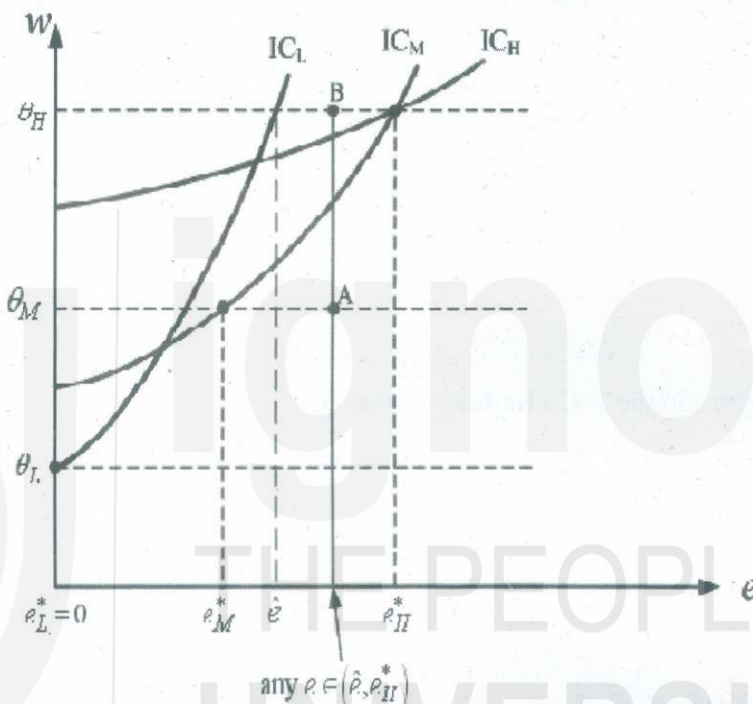


Fig. 26.6: Intuitive Criterion with 3 Types of Workers

We start checking this condition for the L-type. In particular, the note that θ_L -type sending a message $u_L^*(\theta_L) > \max_{w \in W^*(\theta, e)} u_L(e, w, \theta_L)$ indicates that equilibrium payoff is greater than the highest payoff from deviating to e since $u_{\theta_L} = \theta_L - c(0, \theta_L) = \theta_L$ and $\max_{w \in W^*(\theta, e)} u_L(e, w, \theta_L) = \theta_H - c(e, \theta_L)$. That is, the above condition implies $c(e, \theta_L) > \theta_H - \theta_L$, indicating that the cost of acquiring e years of education for the L-type of worker exceeds his potential salary gain, $\theta_H - \theta_L$. Graphically, his equilibrium utility level, $u_L^*(\theta_L)$, is given by the indifference curve IC_L , and $\max_{w \in W^*(\theta, e)} u_L(e, w, \theta_L)$ would correspond to the downward shift of the indifference curve IC_L , that passes through point B (when the worker is paid the high-productivity wage $w(e) = \theta_H$). So, θ_L does not send a message $e \in (\hat{e}, e_H)$. In contrast, θ_M -workers could send such a message $e \in (\hat{e}, e_H)$ because $u_M(\theta_M) < \max_{w \in W^*(\theta, e)} u_M(e, w, \theta_M)$ since $\theta_M - c(e_M, \theta_M) < \theta_H - c(e, \theta_M)$, or alternatively, $c(e, \theta_M) - c(e_M, \theta_M) < \theta_H - \theta_M$, reflecting that the cost of acquiring $e - e_M$ additional years of education is offset by the increase in salary that the M-type of worker can obtain if the firm offers him a high-productivity wage $w(e) = \theta_H$.

Seen graphically, $\max_{w \in W^*(\theta, e)} u_M(e, w, \theta_M)$ is given by the indifference curve of the M-type of worker, passing through point B, depicts a higher utility level than his equilibrium utility (see, IC_M). Similarly, for the H-type of worker, $u_H^*(\theta_H) < \max_{w \in W^*(\theta, e)} u_H(e, w, \theta_H)$.

since $\theta_H - c(e_H, \theta_H) < \theta_H - c(e, \theta_H)$, given that $c(e_H, \theta_H) > c(e, \theta_H)$. Intuitively, by deviating towards e the H-type of worker does not modify his salary (if the firm maintains a wage offer of $w(e) = \theta_H$), but he does not incur so much education costs. In Figure 26.6, $\max_{w \in W^*(\theta, e)} u_H(e, w, \theta_H)$ is illustrated by the indifference curve of the H-type of worker passing through point B. This curve gives a utility level which is above the equilibrium payoff of the worker (see IC_H). Hence, education levels in the interval $e \in (\hat{e}, e_H)$ are not equilibrium dominated for the θ_M and θ_H workers, since they both have incentives to deviate from their equilibrium messages. Therefore, when firms observe $e \in (\hat{e}, e_H)$ they will concentrate their beliefs on those types of workers for which these education levels are not equilibrium dominated:

$$\theta^{**}(e) = (\theta_M, \theta_H) \text{ for all } e \in (\hat{e}, e_H)$$

Second Step: Once we have determined $\theta^{**}(e) = (\theta_M, \theta_H)$ for all $e \in (\hat{e}, e_H)$, we need to find a type θ that can be tempted to send an education level in $e \in (\hat{e}, e_H)$ anticipating that firms' best response to this education level will be a wage offer somewhere in between $w(e) = \theta_M$ and $w(e) = \theta_H$. First, for the M -worker, if he thinks pessimistically, he can consider the case in which his deviation towards message $e \in (\hat{e}, e_H)$ is interpreted by firms as coming from a θ_M -worker.

Hence, the firm will offer $w(e) = \theta_M$ and θ_M -workers indifference curve will pass through point A, being below the indifference curve corresponding to his equilibrium payoff. Therefore, $\min_{w \in W^*(\theta^{**}(e), e)} u_M(u, w, \theta) < u_M^*(\theta)$ and the M-type's equilibrium payoff exceeds the lowest payoff he can obtain from deviating towards e . Similarly for the H -worker, if he thinks pessimistically, he can consider the same situation described above. That is, firms believe that any message $e \in (\hat{e}, e_H)$ must come from a θ_M -worker, and as a consequence they offer $w(e) = \theta_M$. Therefore, θ_H -worker's indifference curve through point A is also below his indifference curve at the equilibrium payoff. So,

$$\min_{w \in W^*(\theta^{**}(e), e)} u_H(u, w, \theta) < u_H^*(\theta)$$

Therefore, there does not exist any type of worker in the set $\theta^{**}(e) = (\theta_M, \theta_H)$ who would deviate towards the off-the-equilibrium message $e \in (\hat{e}, e_H)$. Hence, the separating PBE specified in Figure 26.6 does not violate the Intuitive Criterion. Thus, the application of the Intuitive Criterion does not necessarily eliminate separating PBE with $n > 2$ types of senders.

Example 26.2b: Continuous messages with $n = 3$ types of workers. D_1 -Criterion.

We now show that the D_1 -Criterion restricts the set of equilibria, even if the set of senders is strictly larger than $n = 2$.

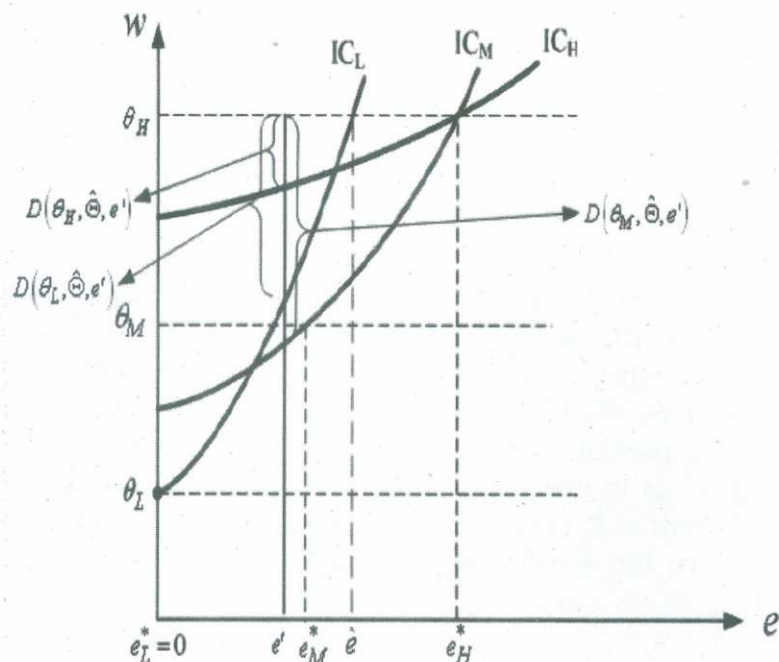


Fig0 26.7: D_1 -Criterion with 3 Types of Workers

First Step: We reduce the firm's beliefs by considering who is the type of worker who most probably sent message e' . We next define the set of wage offers for which a worker of type $i = \{L, M, H\}$ can improve his equilibrium utility level, $u_i^*(\theta_i)$, by acquiring education level e' rather than his equilibrium education of e_i^*

(see Figure 26.7). $D(\theta_i, \hat{\theta}, e') = \bigcup_{\mu: \mu(\theta|m)=1} \{a \in MBR(\mu(e'), e') | u_i^*(\theta_i) < u_i(e', w, \theta_i)\}$

Applying this concept to the L and M-types of workers, we have

$$[D(\theta_L, \hat{\theta}, e') \cup D^o(\theta_L, \hat{\theta}, e')] \subset D(\theta_M, \hat{\theta}, e')$$

Intuitively, the set of wage offers for which the M-type of worker improves his equilibrium utility is larger than those for which the L-type of worker improves his (see Figure 26.7), making the former more likely to deviate towards e' than the latter. So, applying the D_1 -Criterion, we can eliminate θ_L -type worker from having sent e' . Similarly, $[D(\theta_H, \hat{\theta}, e') \cup D^o(\theta_H, \hat{\theta}, e')] \subset D(\theta_M, \hat{\theta}, e')$ and the M-type of worker is more likely to deviate towards education level e' than the H-type of worker. So, applying the D_1 -Criterion, we can eliminate θ_H -type worker from having sent e' .

Hence, firms beliefs when observing an education level of e_0 can be restricted to only the M-type of worker, $\theta^{**}(e') = (\theta_M)$:

Second Step: Given $\theta^{**}(e') = (\theta_M)$, firms offer a wage $w(e') = \theta_M$ when observing an education level of e' . Therefore, for the θ_M -worker we have that

$$\min_{w \in W^*(\theta^{**}(e'), e')} u_H(e', w, \theta_M) = w(e') - c(e', \theta_M) = \theta_M - c(e', \theta_M)$$

And his equilibrium payoff is $u_M^*(\theta_M) = w(e_M) - c(e_M, \theta_M) = e_M - c(e_M, \theta_M)$ and given that $e' < e_M$ and $c(e, \theta) > 0$, we then have $c(e', \theta_M) < c(e_M, \theta_M)$; which implies

$$\theta_M - c(e', \theta_M) > \theta_M - c(e_M, \theta_M)$$

That is, $\min_{w \in W^*(\theta^{**}(e'), e')} u_M(e', w, \theta_M) > u_M^*(\theta_M)$

Hence, we have found a type of worker, θ_M , for whom deviating towards e_0 improves his equilibrium payoff, $u_M^*(\theta_M)$. Therefore, the separating PBE described in Figure 26.7 above violates the D_1 -Criterion. Repeating this process for all off-the-equilibrium messages, we can delete all separating PBEs, except for the efficient equilibrium outcome described in the following figure.

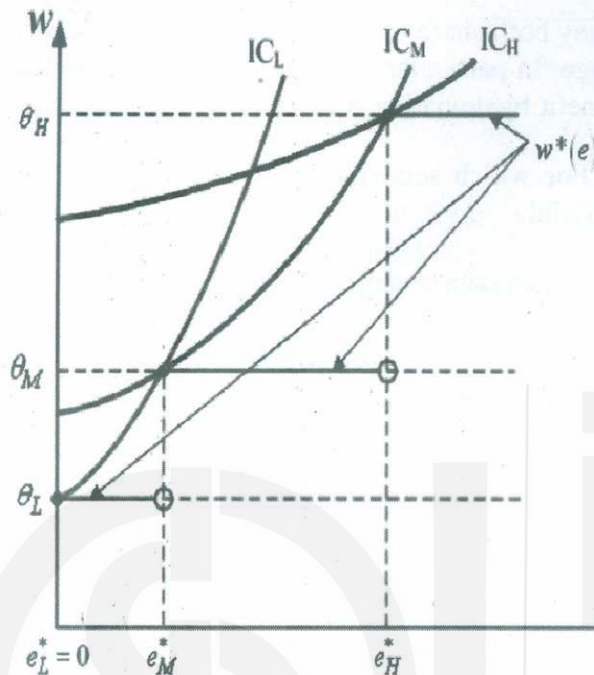


Fig. 26.8: Efficient Separating Equilibrium with 3 Types of Workers

Check Your Progress 4

- 1) Interpret the “single-crossing” assumption in the context of education and cost of low ability workers.

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- 2) What insight is gained on the choice of high and low ability workers regarding education level in terms of separating and pooling equilibria?

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26.8 LET US SUM UP

This unit discusses the refinement introduced in signaling games. The major focus is tackling the problem when the set of strategy profiles that can be supported as PBE in a Signaling games is usually very large, and contains equilibria predicting insensible behaviors. The Intuitive- and D_1 -Criteria are useful to eliminate multiple equilibria.

In their application, they both share a common feature in the second stage, but differ in their first stage. In particular, they differ in how to restrict the set of senders who could benefit by deviating from their equilibrium message:

Intuitive Criterion: For which sender/s there is at least one action of the responder that is beneficial.

D_1 -Criterion: For which sender/s most of the actions of the responder are beneficial.

The set of equilibria surviving the Intuitive Criterion might coincide with those surviving the D_1 -Criterion, but generally, the latter is a subset of the former.

Applying such refinement criterion to Spence model of labour market signaling game the unit evaluates when the pooling and separating equilibria survive the refining criteria.

26.9 KEY WORDS

Divinity D_1 : an equilibrium refinement that requires out-of-equilibrium beliefs to be supported on types that have the most to gain from deviating from a fixed equilibrium. Moreover, in D_1 requirement that there is a single player type t' that always strictly wishes to defect whenever player type t does.

Divinity D_2 : a requirement poses the weaker requirement that for any response that causes player type t to defect and there is some type (which may change with the particular response) that wishes strictly to do so. Note that D_2 will permit, in general, more type-message pairs to be struck than will D_1 .

Divinity Criterion: a restriction of receiver's beliefs to the type of senders for whom deviating towards an off-the-equilibrium message could improve the outcome compared to the equilibrium payoff.

Divinity: an equilibrium refinement that requires out-of-equilibrium beliefs to place relatively more weight on types that gain more from deviating from a fixed equilibrium.

Equilibrium Dominated: payoff does not dominate the highest payoff that could be obtained by sending an off-the-equilibrium message.

Forward Induction: players forming an assessment about the future play of the game and should believe in the rationality of their opponents even after observing deviations.

Intuitive Criterion: a refinement that reduces possible outcome scenarios by first restricting the type group to types of agents who could obtain higher utility levels by deviating to off-the-equilibrium messages and second by

considering in this sub-set of types the types for which the off-the-equilibrium message is not equilibrium dominated.

Never-a-Weak-Best-Response: a strategy for one of the players is said to be never-a-weak-best-response for the player, relative to the set of equilibria, if no equilibrium from the set is the strategy in question as good for the player as the strategy prescribed by the equilibrium

26.10 SOME USEFUL BOOKS

Gibbons, (1992), *A Primer in Theory*, Harvester Wheatsheaf.

Osborne, M.J., (2002), *An Introduction to Game Theory*, Oxford University Press.

Also see, In-Koo Cho and David M. Kreps (1987), Signaling Games and Stable Equilibria, *The Quarterly Journal of Economics*, Vol. CII, Issue 2; and Jeffrey S. Banks and Joel Sobel (1987), Equilibrium Selection in Signaling Games, *Econometrica*, Vol. 55, No. 3.

26.11 ANSWER/HINTS TO CHECK YOUR PROGRESS

Check Your Progress 1

- 1) Under all possible beliefs.
- 2) Backward induction represents a pattern of reasoning in which a player, at every stage of the game, only reasons about the opponents' future behaviour and beliefs, and not about choices that have been made in the past. So, he takes the opponents' past choices for granted, but does not draw any new conclusions from these. In contrast, forward induction requires a player, at every stage, to think critically about the observed past choices by his opponents. He should always try to find a plausible reason for why his opponents have made precisely these choices in the past, and he should use this to possibly reconsider his belief about the opponents' future, present, and unobserved past choices.
- 3) Violate the Intuitive Criterion.
- 4) The unique separating equilibrium surviving the Intuitive Criterion is that in which the θ_H -worker sends $e = e_1$.

Check Your Progress 2

- 1) In addition to the restriction suggested by the Intuitive Criterion, the Divinity Criterion considers only those types which are most likely to send the off-the-equilibrium message. Further, if more than one sender could benefit from the deviation, the Intuition Criterion assigns equal probabilities for all the senders, whereas the D_1 -Criterion considers different probabilities.
- 2) Both refinement criteria coincide in their 2nd step.

Check Your Progress 3

- 1) No. This equilibrium is being supported by the belief that the worker who gets education is no better than a worker who does not. But education is more costly for low ability workers, so they should be less likely to deviate to obtaining education. This can be ruled out by various different refinements of equilibria.

- 2) No. It violates D_1 criterion.
- 3) Benefits of education are only greater than the costs for workers with a high level of ability, so only workers with a high ability will get an education.

Check Your Progress 4

- 1) Education is more costly for low ability workers. This is often referred to as the “single-crossing” assumption, since it makes sure that in the space of education and wages, the indifference curves of high and low types intersect only once.
- 2) In Separating equilibria high and low ability workers choose different levels of schooling whereas in pooling equilibria they choose the same level of education.

26.12 EXERCISES AND ANSWER HINTS

- 1) Explain with an example how the intuitive criterion used as a refinement to minimise the set of perfect Bayesian equilibria in signalling games.

Ans: (Taken from internet)

The intuitive criterion rules-out any out-of-equilibrium beliefs that can only be correct if some player acts irrationally. In many signalling games (that is, games in which one player – the sender – can communicate information to another – the receiver), there are often a lot of implausible equilibria. This happens because the Perfect Bayesian solution concept does not specify what the receiver’s beliefs must be when the sender deviates; we can therefore support a lot of equilibria simply by saying that if the sender deviates from those equilibria then he will be “punished” with very bad beliefs. Such punishment will usually be enough to make the sender play a strategy that would otherwise not be a best response.

For example, in Spence’s classic job market signalling paper there is an equilibrium in which high-ability individuals invest in education (learning is easy for them) whilst low-ability individuals do not (because they find it too costly to do so). Education is then a signal of ability. We might ask: is there also an equilibrium of this game in which nobody chooses to get an education and no information is transmitted to the receiver? The answer is ‘yes’. We can support such an equilibrium by saying that a deviation in which a sender is educated causes the receiver to adopt the belief that the sender is certainly low-ability. If education has the effect of signalling low-ability then, of course, everyone is happy to play along with the putative equilibrium and not get educated.

It is also clear that this equilibrium is not very plausible: the receiver knows that it is less costly for a high-ability agent to get an education than a low-ability one, so it doesn’t make much sense for him to think of an education as signaling low-ability. The intuitive criterion rules out this kind of equilibrium by requiring beliefs to be “reasonable” in the following sense:

Suppose the receiver observes a deviation from the equilibrium. The receiver should not believe that the sender is of type t_{bad} if both of the following are true:

- 1) the deviation would result in type t_{bad} being worse off, then if he has stuck to the equilibrium for any beliefs.
- 2) There is some type t_{good} who is better off by playing the deviation than by sticking to the equilibrium for some belief other than t_{bad} .

Returning to the education signalling model: Suppose that the equilibrium is that nobody gets an education and that the receiver believes that a deviation to getting education signals low ability. Anticipating these beliefs, a low ability worker is made worse-off by deviating because he not only incurs the cost of the education but is then thought of as a bad type as a result. Thus, condition 1. is satisfied.

Can we find some alternative belief such that the high-ability worker *would* like to deviate to getting education? The answer is yes: if the receiver believes that education signals high ability then this deviation is indeed profitable for the high-type. Thus, condition 2 is also satisfied.

Since both conditions are satisfied, the intuitive criterion rules-out the implausible pooling equilibrium.

- 2) Explain by constructing a model how the intuitive criterion used as a refinement to minimise the set of perfect Bayesian equilibria in signalling games.

Ans: (Taken from internet)

A worker is of (privately known) type H or L , each with probability $1/2$. The marginal product of the two types is $\pi_H > \pi_L$. The labour market is competitive so that workers are paid their (expected) marginal product. The worker can invest in education; doing so costs type- i , c_i , with $\pi_H - c_L < \pi_L$ and $\pi_H - c_H > (\pi_H/2) + (\pi_L/2)$.

The game is as follows: the worker observes his type and decides whether to invest in education. Employers then observe whether the worker invested or not and make competitive wage offers based on their beliefs about his productivity.

Consider the following two perfect Bayesian equilibria (PBE) of the game.

- 1) (Separating equilibrium) Type H invests; type L does not invest. If employers observe investment then they update their beliefs to $\Pr(H)=1$ and offer a wage of π_H . If they observe no investment then they update to $\Pr(H)=0$ and offer wage π_L .

We can check that this is an equilibrium: type H 's payoff is $\pi_H - c_L$. If he deviates to no education then his payoff is π_L , which is lower. Type L 's payoff is π_L . If he deviates to getting education then his payoff is $\pi_H - c_L < \pi_L$, which is lower. Thus neither type wants to deviate. The wage offers are (trivially) best responses given the beliefs because the labour market is competitive. Lastly, note that the beliefs are consistent with Bayes' rule and the equilibrium play of the game.

- 2) Pooling equilibrium: Neither type invests. The employer updates beliefs to $\Pr(H)=0$ if education is observed and offers wage π_L . The employer sticks with the prior belief of $\Pr(H)=1/2$ and offers wage $(\pi_H/2) + (\pi_L)/2$ if education is not observed.

Let's check that this is also an equilibrium. Since education is costly but adversely affects the employer's beliefs in equilibrium, it is optimal for neither type to get education. Given beliefs and the competitiveness of the labour market, the putative wage offers are optimal. The belief $\Pr(H)=1/2$ is consistent with Bayes' rule if no education is observed (because this observation contains no new information about the worker's type). Lastly, Bayes' rule does not pin down beliefs in the event of (out of equilibrium) investment in education so, according to the definition of a PBE, we are free to specify whatever beliefs we like.

The intuitive criterion rules out equilibrium number 2. Firstly, if type L deviates to getting education then the best payoff he can get is $\pi_H - c_L < \pi_L$ so such a deviation is dominated. Secondly, suppose type H deviates to getting education and the employers adopt some posterior belief $\Pr(H)=1$. The payoff of the deviating H -type is then $\pi_H - c_L > \pi_L$. So that the deviation would be profitable. The intuitive criterion therefore rules that the beliefs $\Pr(H)=0$ are not reasonable for a deviation to investing in education and we cannot have any equilibrium that depends of such beliefs.

In fact, this game has other pooling equilibria. For example, there is a pooling equilibrium in which the employer sticks with his prior belief irrespective of whether he observes education or not. This (and all other pooling equilibria) is also ruled-out by the intuitive criterion. The reason is that any deviation from an equilibrium in which nobody is educated is dominated for the L -type so the intuitive criterion is going to require that the employer never associates education with L -types. Given that education is linked to H -types, it is profitable for H -types to deviate from the no education equilibrium.

- 3) Consider the Spence job market signaling model. Assume $c_L < c_H$. You know that separating equilibria in the model do not exist. What are the pooling equilibria? Do they satisfy the intuitive criterion? Why or why not?

Ans. Do yourself after reading the discussion given in the unit

- 3) Consider the Spence job market signaling model with three types, $\theta = \{\theta_H, \theta_M, \theta_L\}$, where $\theta_H > \theta_M > \theta_L$. Assume their cost of education satisfies $c_H < c_M < c_L$.
 - a) Characterize the separating equilibria of the game.
 - b) Identify which separating equilibria satisfy the intuitive criterion.

Ans. Do yourself after reading the discussion given in the unit.