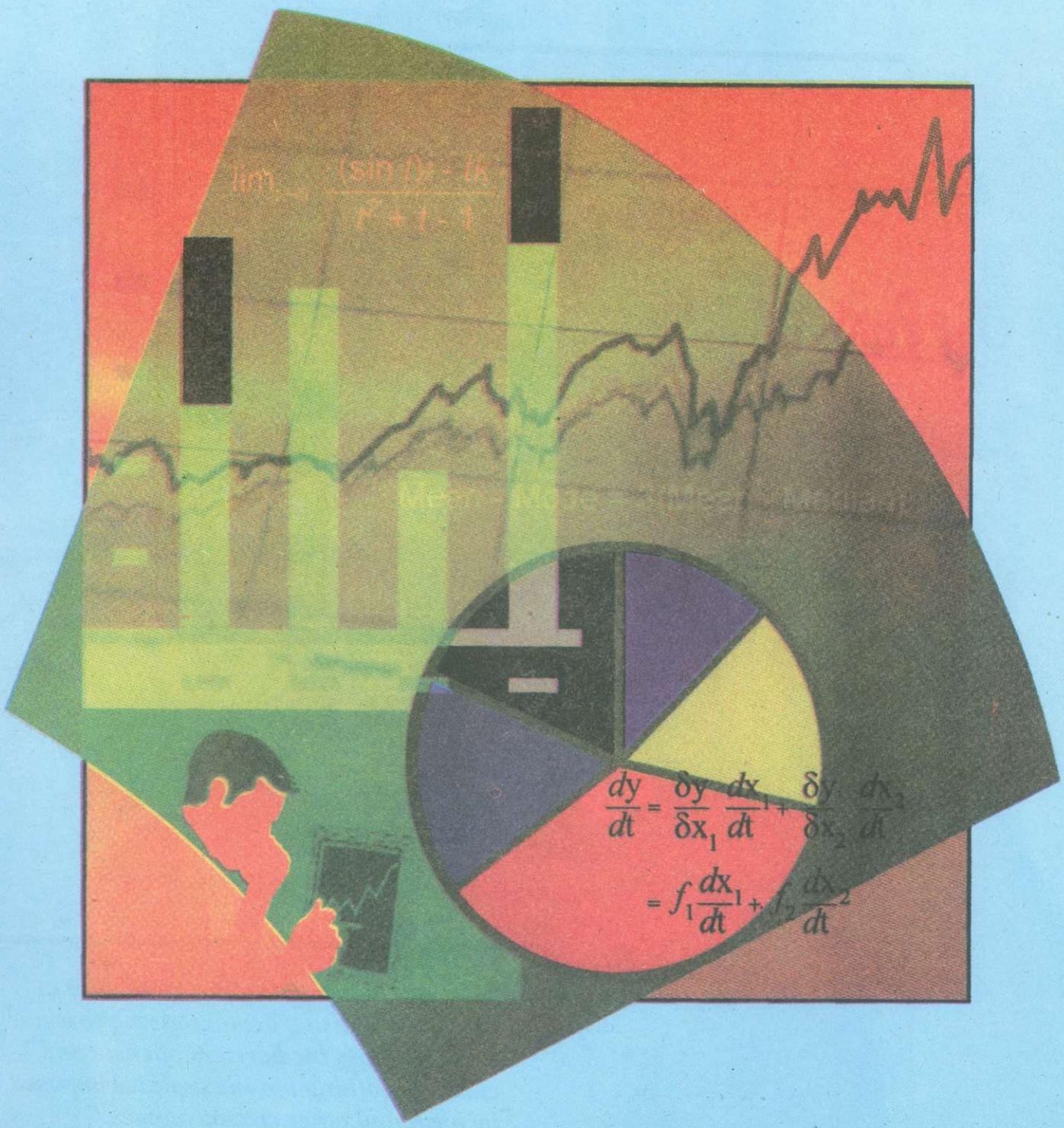
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MEC-103
QUANTITATIVE METHODS



Introduction to Differential Calculus

1

"शिक्षा मानव को बन्धनों से मुक्त करती है और आज के युग में तो यह लोकतंत्रा की भावना का आधार भी है। जन्म तथा अन्य कारणों से उत्पन्न जाति एवं वर्गगत विषमताओं को दूर करते हुए मनुष्य को इन सबसे ऊपर उठाती है।"

- इन्दिरा गाँधी

"Education is a liberating force, and in our age it is also a democratising force, cutting across the barriers of caste and class, smoothing out inequalities imposed by birth and other circumstances."

- Indira Gandhi

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Quantitative Methods

Block

1

INTRODUCTION TO DIFFERENTIAL CALCULUS

UNIT 1
Set Theory 5

UNIT 2
Functions and their Graphical Representation 54

UNIT 3
Differential Calculus: Functions, Limit and Continuity 96

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BLOCK 1 INTRODUCTION TO DIFFERENTIAL CALCULUS

Some of the basic tools of mathematics, which find frequent application in economics are included in this block. **Unit 1** prsents the set theory as a preliminary requirement of language format before going into the differential calculus. Extending the discussion in the set theory. **Unit 2** looks into prsentation of functions and graphs. Technicalities of functions, limit and continuity alongwith a brief review of basic concepts differentiation that facilitates the understanding of the methods used in mathematical analysis are covered in **Unit 3**.

UNIT 1 SET THEORY

Structures

1.0 Objectives

1.1 Introduction

1.2 Set

- 1.2.1 Identity and Cardinality
- 1.2.2 Subsets
  - 1.2.2.1 Power Set of a Set
- 1.2.3 Finite and Infinite Sets
  - 1.2.4.1 Properties of Infinite Sets

1.3 Operations on Sets

- 1.3.1 Properties of Set Theoretical Operations
- 1.3.2 Russell's Paradox
- 1.3.3 Intervals
  - 1.3.3.1 Closed, Open, Bounded and Unbounded Sets
  - 1.3.3.2 Convex Set

1.4 Ordered Pairs and Cartesian Products

- 1.4.1 Ordered Pairs
- 1.4.2 Cartesian Products

1.5 Relations

- 1.5.1 Reflexive, Symmetric and Transitive Relations
- 1.5.2 Domain and Range of Relation
- 1.5.3 Inverse of a Relation
- 1.5.4 Binary Relation

1.6 Equivalence Relations and Partitions

- 1.6.1 Equivalent Relation
- 1.6.2 Partition
- 1.6.3 Equivalent Class

1.7 Order Relation

- 1.7.1 Partial and Total Orders
- 1.7.2 Preorders
- 1.7.3 Lower and Upper Bounds

1.8 Composition of Relations

- 1.8.1 Composition of Binary Relations

1.9 Mapping and Function

- 1.9.1 Mapping
- 1.9.2 Function
  - 1.9.2.1 Functions as Relations
  - 1.9.2.2 Inverse Functions
  - 1.9.2.3 Composition and Function
    - 1.9.2.3.1 Domains and Ranges of Composed Functions
  - 1.9.2.4 Onto and One-to One Functions
- 1.9.3 Cardinality and Bijective Function

1.10 Let Us Sum Up

1.11 Key Words

1.12 Some Useful Books

1.13 Answer or Hints to Check Your Progress

1.14 Exercises and Answers

1.0 OBJECTIVES

After going through this unit, you will be able to:

- Define set and object;
- Identify the elements of a given set;
- Describe conventions used to list sets;
- List the elements of a set using notation;
- List the elements of a set by describing the set and the rules that its elements follows;
- Apply basic set concepts to economic analyses.

1.1 INTRODUCTION

This unit seeks to introduce the fundamental building blocks in mathematics such as sets, relations and functions. Whereas sets are collections of well-defined objects, relations indicate relationships between members of sets (say A and B) and functions are a special type of relation where there is exactly or at most one relationship for each element $a \in A$ with an element in B.

1.2 SET

A set is any collection of objects. For example, set of numbers or the set of all students in a class. We denote set membership by '$\in$'; writing $x \in X$ means that the element x is contained in the set X. If it is not contained in X, we write $x \notin X$. Elements of sets are enclosed in curly brackets $\{\cdot\}$. For example, $\{1, 3, 6\}$ denotes the set with elements 1, 3 and 6. This is called the list form for the set. For example, $1 \in \{1,3,6\}, 3 \in \{1,3,6\}, 6 \in \{1,3,6\}$ *but* $2 \notin \{1,3,6\}$. We usually use capital letters A, B, C, etc., to denote sets.

The set $\{2\}$ is regarded as being different from the number 2. A set of numbers is not a number. $\{2\}$ is a set with only one element which happens to be the number 2. But a set is not the same as the object it contains:$\{2\} \neq 2$. The statement $2 \in \{2\}$ is correct. The statement $\{2\} \in \{2\}$ is wrong.

Example: The set of natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$.

- The integer numbers $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.
- The non-negative integers $\mathbb{Z}^+ = \{0, 1, 2, 3, \ldots\}$.
- The set of prime numbers $= \{2, 3, 5, 7, 11, 13, 17, \ldots\}$.
- The rational numbers is the set of quotients of integers $\mathbb{Q} = \{p/q : q \in \mathbb{Z} \text{ and } q \neq 0\}$.

Two sets are equal if they have the same elements. Thus, if A and B are empty sets we have $A = B$.

1.2.1 Identity and Cardinality

Two sets are identical *if and only if* they have exactly the same members. So $A = B$ iff for every $x, x \in A \Leftrightarrow x \in B$. From the definition of identity follows that there exists only one empty set; its identity is fully determined by its absence of members. Example is empty set (or null set). The special symbol '$\emptyset$' denotes a set without any elements. Thus $\{x: x \text{ is an integer such that } X^2 = -1\} = \emptyset$. Note that empty list notation $\{\}$ is not usually used for the empty set.

The number of elements in a set A is called the *cardinality* of A, written $|A|$ *or* #A. **A set with cardinality 1 is called a singleton.** The cardinality of a finite set is a natural number. Infinite sets also have cardinalities but they are not natural numbers. The cardinality of a null set, i.e., $|\emptyset| = 0$.

Example: If S = {1, 2, 3} and T = {3, 5} then: $|S| = 3$, $|T| = 2$.

1.2.2 Subsets

We say that A is a subset of B and write $A \subseteq B$, or B $\supseteq$A if and only if every element of A is also an element of B, i.e., $x \in A \Rightarrow x \in B$. In this event one also says that A is **included in** B or that B **includes** A. [The double arrow $\Rightarrow$ is an abbreviation for the word "implies"]. A set A is a **proper subset** of a set B, written A$\subset$**B** if and only if $A \subseteq B$ and A$\neq B$.

Note some basic facts:

A $\subseteq$ A for every set A. Every set is a subset of itself.

The empty set is a subset of every set: $\emptyset \subseteq$ A for any set A.

If A $\subseteq$ B and B $\subseteq$ C then A $\subseteq$ C. We say that $\subseteq$ is a transitive relation between sets.

If $A \subseteq B$ and $B \subseteq A$ then $A = B$.

Example: The subsets of $\{1, 2, 3\}$ are

$\emptyset$, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3}.

Thus $\{1, 2, 3\}$ has 8 subsets.

If A is a set with n elements then A has 2^nsub-sets.

Here, $2^n = 2 \times 2 \times ------\times 2$ with n factors.

Let $A = \{a_1, a_2, ---------a_n\}$. How many are there ways to choose a subset in A?

When choosing a subset, we have to decide, for each element, whether we include these elements into our subset or not. We have two choices for the first element: 'include' and 'do not include', two choices for the second element, etc., and finally two choices for the n^{th} element:

$2 \times 2 \times ------\times 2$ choice overall

If A $\subseteq$ B and $A \neq B$ we call A a proper subset of B and write $A \subset B$ to denote this. If $A \subset B$, we also write $B \supset A$.

Example.

Let $A = \{1,3\}, B = \{3,1\}, C = \{1,3,4\}$. Then

$A = B$ true; $A \subset B$ false; $C \subseteq A$ false; $A \subseteq B$ true; $A \subseteq C$ true; $C \subset C$ false; $B \subseteq A$ true; $A \subset C$ true.

Note that a set with n elements contains $2^n - 1$ proper subsets.

For every set A, we have $\emptyset \subseteq A$. That is, we are to prove that every element of $\emptyset$ is an element of A; since there no elements in $\emptyset$, this is automatically satisfied. Alternatively, we may prove this by contradiction. To this end, suppose that $\emptyset$ is not a subset of A. It follows that there is an element $\emptyset$ in that does not belong to A, contradicting the fact that $\emptyset$ has no elements.

It is worth noting that the set $\{\emptyset\}$ is not empty—it is a **singleton**, i.e., a set with one element, which in this case is the empty set. If A is a nonempty set and $a \in A$ then the singleton $\{a\}$ is a subset of A.

For every a, $a \in A$ if and only if $a \subseteq A$.

Note that each of the phrases "for every," "for each," and "for all" is referred to as a **universal quantifier** and is denoted by the symbol $\forall$.

1.2.2.1 Power Set of a Set

Let X be any set. The set of all subsets of X is called the *power set* of X

and is denoted by $\wp(X)$.

That is, we define $\wp(X) = \{A: A \subset X\}$.

Let $S = \{a, b, c\}$. So subsets are:

{a}, {b} and {c}; {a,b}, {a,c} and {b,c}; {a,b,c}; the empty set {};

We list **all** the subsets of $S = \{a, b, c\}$ to get the **Power Set** of $\{a, b, c\}$. So $\wp(S)$ = {{}, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {a, b, c} }

1.2.3 Finite and Infinite Sets

A finite set is a set containing only finite number of elements.

For example, {1, 2, 3} is finite. If A is a finite set, we denote by $|A|$ the number of elements in A. For example, $|\{1, 2, 3\}| = 3$ and $\emptyset = 0$.

A set with infinitely many elements is called an infinite set. The set of all positive integers (also called natural numbers) $\mathbb{N} = \{1,2,3,\cdots\}$ is infinite, the dots indicate that the sequence 1, 2; 3 is to be continued indefinitely.

The set of all non-negative integers is infinite. (There is no consensus about whether to include zero in the set of natural numbers: some define the natural numbers to be the positive integers {1, 2, 3,···, while for others the term designates the non-negative integers {0, 1, 2, 3, …}. More examples of infinite sets:

$\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$	(set of natural numbers)
$\mathbb{Z} = \{\ldots. -2, -1, 0, 1, 2, 3, \ldots\}$	(set of integers)
$\{\cdots. -4, -2, 0, 1, 2, 3, \ldots\}$	(set of even integers)
$\{\cdots. -2, -1, 0, 1, 2, 3, \ldots\}$	(set of odd integers)

$\mathbb{Q}$ denotes the set of all rational numbers (that is, the numbers of the form n/m where n and m are integers and $m \neq 0$, $\mathbb{R}$ the set of all real numbers (in particular, $\sqrt{2} \in \mathbb{R}$ and $\pi \in \mathbb{R}$,

$\mathbb{C}$ the set of all complex numbers (that is, numbers of the form $x + yi$, where x and y are real and i is a square root of $-1, i^2 = -1$. They are all infinite sets and we have the following inclusions: $\mathbb{N} \subset \mathbb{N}_0 \subset \mathbb{Z} \subset \mathbb{C}$.

Check Your Progress 1

1) Are all empty sets equal?

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2) Is predicate a proposition?

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3) Let $U = \{u, v, w, x, y, z\}$.

i) Find the number of subsets of U.

.....

.....

ii) Find the number of proper non-empty subsets of U.

.....

.....

4) List the 8 subsets of $\{a, b, c, d\}$ containing$\{d\}$?

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5) What is the Cardinality of the Power set of the set {0, 1, 2}.

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1.3 OPERATIONS ON SETS

Let X and Y be any two sets. We denote the union of these two sets—i.e., the set of all elements belonging to at least one of the two sets—by $X \cup Y$. Similarly their intersection—i.e., the set of all elements belonging to both sets—by $X \cap Y$ (see Figure 2.1). Both concepts can be

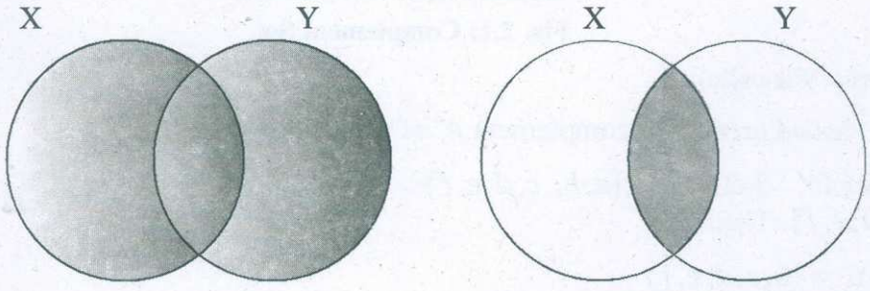


Fig. 2.1: In the left panel the shaded area is the union $X \cup Y$ (left) and in the right panel the intersection $X \cap Y$ (right) of two sets X and Y.

Suppose A and B are sets. Then $A \cap B$ denotes the set of all elements which belong to both A and B:

$A \cap B = \{x: x \in A \text{ and } x \in B\}$.

Example: Let $A = \{1,3,5,7\}$ and $B = \{3,4,5,8\}$, then $A \cap B = \{3,5\}$.

$A \cup B$ denotes the set of all elements which belong to A or to B:

$$A \cup B = \{x: x \in A \text{ or } x \in B\}.$$

$A \cup B$ is called the union of A and B.

Note that "or" is used in the inclusive sense: elements from $A \cup B$ belong to A or to B or to both A and B, or, in brief, to A and/or B. In particular, this means that $A \cap B \subseteq A \cup B$.

Example: Let $A = \{1,3,5,6,7\}, B = \{3,4,5,8\}$, then $A \cup B = \{1,2,3,4,5,6,7,8\}$

If A and B are sets such that $A \cap B = \emptyset$, that is, A and B have no elements in common, we say that A is disjoint from B, or that A and B are disjoint from each other or that A and B do not intersect.

Example: $A = \{1,3,5\}, B = \{2,4,6\}$. Here A and B are disjoint.

Concepts of union and intersection given above can be extended to operate on an arbitrary number of sets X_i, Y_i indexed by $i \in I$ (I is called an index set). The corresponding notation is

$$X = \bigcup_{(i \in I)} = X_i; \qquad Y = \bigcup_{i \in I} Y_i.$$

In any application of set theory all the sets under consideration will be subsets of the universal set. For example, when working with real numbers the universal set is the set R of real numbers. We usually denote the universal set by $\mathbb{U}$.

Let A be a set and $\mathbb{U}$ be the universal set. Then A' (called the complement of A and pronounced "A prime") denotes the set of all elements in $\mathbb{U}$ which do not belong to A:

$$A' = \{x: x \in \mathbb{U} \; and x \notin A\}.$$

Note that the complement A' is sometimes denoted ¬A and pronounced not A", or "A bar"), or A^c ("A compliment").

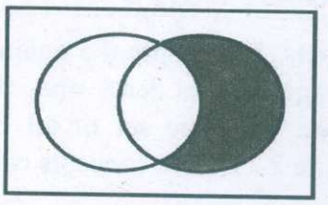


Fig. 2.2: Complement Set

(**Source:** Wikipedia)

The shaded area is the complement A' of the set A.

Example: Let $\mathbb{U} = \{a,b,c,d,e,f\}$, $A = \{a,,c\}$, $B = \{b,c,f\}$, $C = \{b,d,e,f\}$. Then

$B \cup C = \{b,c,d,e,f\}$

$A \cap (B \cup C) = \{c\}$

$A' = \{b,d,e,f\} = C$

$A' \cap (B \cup C) = C \cap (B \cup C) = \{b,d,e,f\} = C$

Example: $\{x \in \mathbb{Z}: x^2 = 4\} = \{2,-2\}$.

Using the symbol $\mathbb{U}$ to denote universal set, an important property of union and intersection we can state is the duality principle

$$\mathbb{U}\setminus\bigcup_i X_i = \bigcap_i (U\setminus X_i)$$

$$\mathbb{U}\setminus\bigcap_i X_i = \bigcup_i (U\setminus X_i)$$

which says that the complement of a union of an arbitrary number of sets equals the intersection of the complements of these sets

The difference of two sets X and Y is expressed as X\Y (the order matters). It means all elements of X that are not contained in Y. The symmetric difference $X\Delta Y$ and Y is defined as (X\Y)∪(Y \X)—both concepts are illustrated in Figure 2.2.

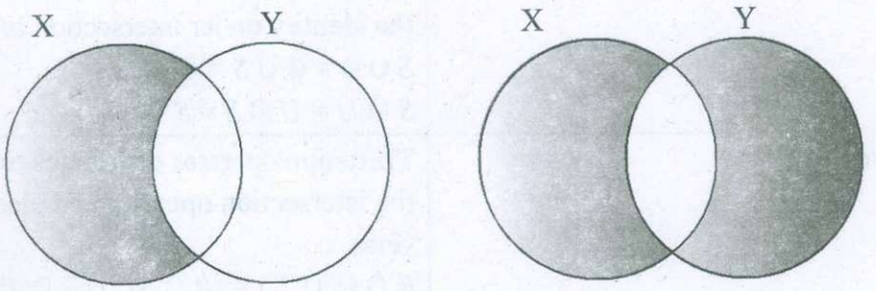


Fig. 2.3: The difference X \Y (left) and symmetric difference $X\Delta Y$ (right) of two sets *X and Y*.

For example, the symmetric difference of the sets $\{1,2,3\}$and$\{3,4\}$ is $\{1,2,4\}$. The symmetric difference of the set of all students and the set of all females consists of all non-female students together with all female non-students.

1.3.1 Properties of Set Theoretical Operations

The set union and set intersection properties are commutative and associative. The properties are listed in Table 2.1.

We give a proof of the distributive property.

Proof of Properties (Distributive Property): To show

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Suppose $x \in A \cap (B \cup C)$ then

$x \in A \wedge x \in (B \cup C)$,

$\Rightarrow x \in A \wedge (x \in B \vee x \in C)$,

$\Rightarrow (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)$,

$\Rightarrow x \in (A \cap B) \vee x \in (A \cap C)$,

$\Rightarrow x \in (A \cap B) \cup (A \cap C)$.

Therefore, $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

Similarly, $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$.

Therefore, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Table 2.1 : Properties of set operations

Property	Description
Commutative	Union and intersection operations are commutative: i.e., $S \cup T = T \cup S$ $S \cap T = T \cap S$
Associative	Union and intersection operations are associative: i.e., $R \cup (S \cup T) = (R \cup S) \cup T$ $R \cap (S \cap T) = (R \cap S) \cap T$
Identity	The identity under set union is $\emptyset$ and the identity under intersection is U. $S \cup \emptyset = \emptyset \cup S = S$ $S \cap U = U \cap S = S$
Distributive	The union operator distributes over the intersection operator and vice versa. $R \cap (S \cup T) = (R \cap S) \cup (R \cap T)$ $R \cup (S \cap T) = (R \cup S) \cap (R \cup T)$
DeMorgan's a law	The complement of $S \cup T$ is given by: $(S \cup T)^c = S^c \cap T^c$. The complement of $S \cap T$ is given by: $(S \cap T)^c = S^c \cup T^c$

1.3.2 Russell's Paradox

Russell posed the question: Is the set of all sets that do not contain themselves as members a set? That is, let S be the set of all sets which are not members of themselves. Or, if

$S = \{S' \mid S' \notin S'$, is S a member of itself?", i.e., $S \in S$?

To understand the paradox non-technically, look at the following illustration: A town that has exactly one barber who is male. *The barber shaves all and only those men in town who do not shave themselves.* The question is, who shaves the barber? If the barber does not shave himself then according to the rule he is shaved by the barber (i.e., himself). If he shaves himself then according to the rule he is not shaved by the barber (i.e., himself). The statement is a contradiction.

Two ways of avoiding the paradox was developed in 1908, viz., Russell's theory of types and Zermelo set theory. Russell's theory of types was a response to the paradox that argued that the set of all sets is ill formed. Russell developed a hierarchy of sets with individual elements at the lowest level, sets of elements at the next level, sets of sets of elements at the next level and so on. It is then prohibited for a set to contain members of different types.

1.3.3 Intervals

Subsets of the real line $\mathbb{R}$ specified by their end points are called intervals. For example, $a \leq b \in \mathbb{R}$, denotes the interval $(a, b) = \{x \in \mathbb{R}: a < x < b\}$ is

called **open**, $[a,b] = x \in \mathbb{R}: a \leq x \leq b$ is called **closed**, and $[a,b) = \{x \in \mathbb{R}: a \leq x < b\}$, $(a,b] = \{x \in \mathbb{R}: a < x \leq b\}$ are neither open nor closed.

Example: Let $A = \{x \in \mathbb{R} : X^2 - 3x + 2 = 0\}$. Prove that $A \subseteq Z$.

Solution: Let $x \in A$. Then

$X^2 - 3x + 2 = 0;$

$(x-1)(x-2) = 0;$

$x = 1 \text{ or } 2$

$x \in Z.$

To prove $A = B$ for particular sets A and B we have to prove $A \subseteq B$ and then $B \subseteq A$.

Recall that a segment $[a,b]$ of the real line $\mathbb{R}$ is defined as the set $[a,b] = \{x \in \mathbb{R} : a \leq x \leq b\}$.

Example: Let $A = [1,2]$ and $B = [0,2] \cap [1,3]$. Prove that $A = B$.

Solution: We first prove that $[1,2] \subseteq [0,2] \cap [1,3]$.

Let $x \in [1,2]$. Then $1 \leq x \leq 2$. Hence $0 \leq x \leq 2$ and $1 \leq x \leq 3$.

Hence $x \in [0,2]$ and $x \in [1,3]$.

Hence $x \in [0,2] \cap [1,3]$, and, since x is an arbitrary element of $[1,2]$, this means that $[1,2] \subseteq [0,2] \cap [1,3]$.

Now we prove that $[0,2] \cap [1,3] \subseteq [1,2]$.

Let $x \in [0,2] \cap [1,3]$. Then $x \in [0; 2]$ and $x \in [1,3]$. Hence $0 < x\ 2 \text{ and } 1 < x < 3$. Therefore $x \geq 1$ and $x \leq 2$.

For this reason $1 \leq x \leq 2$. Consequently, $x \in [1,2]$.

One can immediately see that $[0,2] \cap [1,3] = [1,2]$.

Notice that $[0,1] \cap [1,2] = \{1\}$ while $(0,1) \cap (1,2) = \emptyset$.

Caution: Do not mix notation $\{a,b\}, [a,b], (a,b)$.

1.3.3.1 Closed, Open, Bounded and Unbounded Sets

Extending the above discussion on intervals, we touch upon the closed, open, bounded and unbounded sets. To appreciate these ideas the concept of metric space is required. It is a set S where for every two points $x \text{ and } y$ in S, the distance between them is a nonnegative real number. Let X be a subset of a metric space. Then the set X is open if every point in X has a neighborhood lying in the set. To see take an example of the interval (-3, 3). It is open because if x is any number in (-3, 3), then $-3 < x < 3$ or equivalently, $-3 - x < 0 < 3 - x$. Now let $\varepsilon = min\ (3 + x, 3 - x)$. Then $\varepsilon > 0$, and the interval $(x - \varepsilon/2, x + \varepsilon/2)$ is contained in (-3, 3).

The complement of an open set is a closed set. To understand its meaning consider the whole numbers. They are closed under addition. If you add two whole numbers, you always get another whole number. But the whole numbers are not closed under subtraction, e.g., 2 - 5 = -3. The integers are closed under multiplication (if you multiply two integers, you get another integer), but they are not closed under division. The rationals, however, are closed under addition, subtraction, multiplication, and division.

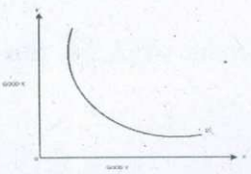
It is also possible for a set to be neither open nor closed, e.g., the half-closed interval(0,1]. The interval (-4, 5] is neither open nor closed. It is not open, because the point $x = 5$ is contained in the set, but every neighborhood of that point is not contained inside the set. It is not closed, because its complement $(-\infty, -4]$ and $(5, \infty)$ is not open at $x = -4$.

We say that a set of numbers is bounded if there is a number M so that the size of every element in the set is no more than M, and unbounded if there is no such number M. For example, the interval $[3,7)$ is bounded and the sequence $1, 2, 3, 4, \ldots$ is unbounded. The set $\{(x, y): 2x + 3y - 6 < 0\}$ is open and $\{(x, y): 2x + 3y - 6 \leq 0\}$ is closed.

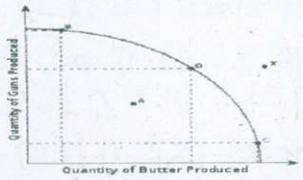
Thus, a set is closed if the set contains all of its boundary points. On the other hand, a set is bounded if the distance between any two points in the set is less than some finite constant. As may be seen $S = \{(x, y) | x^2 + y^2 \leq 1\}$ is closed and bounded.

1.3.3.2 Convex Set

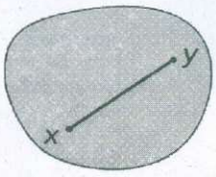
Remember indifference curve and production possibility frontier in economics theory. You can form a broad idea of a convex set. Because a **convex set** is a **set** of points such that, given any two points A, B in that **set**, the line AB joining them lies entirely within that **set** (see the figures below)



Indifference curve

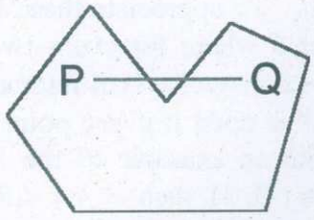


Production possibility frontier

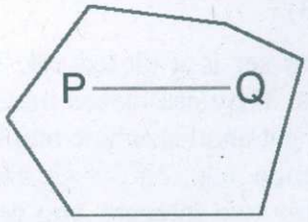


Source: Wikipedia

Thus, a subset S of a space is called convex if and only if for any pair of points p and q in the set S, the line section from p to q is enclosed completely in the set of S, as done on the figure below.



Non-Convex



Convex

Algebraically, if x_1 and x_2 are numbers, than $x_1 x_2$ can be represented as $\lambda x_1 + (1 - \lambda) x_2 \in S$ for any λ such that $0 \leq \lambda \leq 1$.

Check Your Progress 2

1) Let X = {2, 4, 5, 6} Y = {3, 4, 7, 8} Z = {5, 6, 7, 8}, find (X - Y) ∪ (Y - X)

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2) Let ξ = {1, 2, 3, 4, 5, 6, 7} and A = {1, 2, 3, 4, 5} B = {2, 5, 7} show that

a) $(A \cup B)' = A' \cap B'$

b) $(A \cup B) = B \cup A$

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3) Let P = {a, b, c, d} Q = {b, d, f} R = {a, c, e} verify that

$(P \cup Q) \cup R = P \cup (Q \cup R)$

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4) Which interval notation represents the set of all numbers from 2 through 7, exclusive?

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5) Determine whether the following sets are open, closed, both or neither

a) The interval (0,1) as a subset of R

b) The interval (0, 1) as a subset of R^2, that is, $\{(x, 0) \in R^2 | x\ (0,1)\}$

c) R as the subset of R

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1.4 ORDERED PAIRS AND CARTESIAN PRODUCTS

1.4.1 Ordered Pairs

In an **ordered pair** the order in which the elements appear in the pair is significant. In the ordered pair(a, b), the object a is called the *first entry*, and the object b the *second entry* of the pair. Consequently, the pair (a, b) is different from the pair (b, a) unless $a = b$. Alternatively, the objects are called the first and second *coordinates*, or the left and right *projections* of the ordered pair.

The idea of ordered pairs can be extended to ordered triples and in general to ordered n *tuples* for any natural number n. For example, ordered triples are usually defined as (a, b, c) of elements in which a, b and c can be defined either as the ordered pair: $(a,(b, c))$ or as the ordered pair: $((a, b), c)$ where (a, b) and (b, c) are themselves ordered pairs.

The set $\mathbb{R}^3$ is the set of all ordered triples of real numbers; when endowed with appropriate structure, it provides a coordinate representation of the familiar **three-dimensional Euclidean space**. For an arbitrary positive integer$n$, the set is the set of all ordered $n-tuples$ of real numbers and with the addition of appropriate structure, it renders a coordinate representation of ***n*-dimensional Euclidean space.** One of the reasons why ordered $n-tuples$ are important is that they provide a framework for studying and understanding "n-dimensional sets" in mathematics, economics and other disciplines.

1.4.2 Cartesian Product

Suppose we have two sets A and B and we form ordered pairs by taking an element of A as the first member of the pair and an element of B as the second member. The Cartesian product of A and B, written $A \times \mathrm{B}$, (read as A cross B) is the set consisting of all such pairs.

Some examples of Cartesian products are:

Let $K = \{a, b, c\}$ and $L = \{1,2\}$, then

$K \times L = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}$

$\mathrm{L} \times K = \{(1, a), (2, a), (1, b), (2, b), (1, c), (2, c)\}$

$\mathrm{L} \times L = \{(1,1), (1,2), (2,1), (2,2)\}$

For three sets A, B *and* C, the Cartesian product can be defined as

$A \times \mathrm{B} \times \mathrm{C} = ((A \times \mathrm{B}) \times \mathrm{C})$

In the case when $A = \mathrm{B} = \mathrm{C}$ ... a special notation is used:

$A \times \mathrm{A} = \mathrm{A}^2, \mathrm{A} \times \mathrm{A} \times \mathrm{A} = \mathrm{A}^3$, etc.

For the general case, the Cartesian product $X_1 \times X_2 \times X_3 \times \cdots \times X_n$ is sometimes abbreviated as $\prod_{i=1}^{n} X_i$. Again we write X^n (which is read "X n" rather than "X to the power n") in the case where $X_i = X$ for $i = 1,2,3, \ldots, n$.

1.5 RELATIONS

The concept of relation is common in daily life and seems intuitively clear. For instance, let X be the set of all living human females and Y the set of all living human males. The wife-husband relation R can be thought as a relation from X to Y. For a lady $x \in X$ and a gentleman $y \in Y$, we say that x is related to y by R if x is the wife of y, written as xRy. To describe the relation R, we may list

the collection of all ordered pairs (x, y) such that x is related to y by R. The collection of all such related ordered pairs is simply a subset of the product set $X \times Y$. In a mathematical sense, a relation is a subset of a product set. For example consider the

sets X = {2, 3, 4} and Y = {3, 4, 5, 6}, and let us have the propositional function " χ exactly divides y." The product set is then

$$X \times Y$$
$$= \{(2,3),(2,4),(2,5),(2,6),(3,3),(3,4),(3,5),(3,6),(4,3),(4,4),(4,5),(4,6)\}.$$

The set of pairs that satisfy the propositional function, that is, our solution set, is obviously

$$R = \{(2,4),(2,6),(3,3),(3,6),(4,4)\}.$$

The identity relation on A is defined by $i_A a$ for all $a \in A$. Let X be a non-empty set, and R be the set of all order pairs (x, y) in $X \times X$ for which $x = y$, namely, $R = \{(x, y) \in X \times X \mid x = y\}$.

This is the equality relation between elements of X. Here, for any x and y in X, xRy has the same meaning as $x = y$. Such a relation is also known as the ***identity relation*** of X.

1.5.1 Reflexive, Symmetric and Transitive Relations

Let R be a relation on a set X. Then:

- R is said to be **reflexive** if and only if xRx for every $x \in X$.
- R is said to be **symmetric** if and only if xRy implies yRx for every $x, y \in X$.
- R is said to be **transitive** if and only if xRy and yRz implies xRz for every $x, y, z \in X$.

We may also consider R to be **anti-symmetric** if and only if xRy and yRx implies $x = y$ for every $x, y \in X$.

Examples: The weak inequality relation *is greater than or equal to* $(\geq)$ is reflexive, anti-symmetric, and transitive; but it is not symmetric, since $3 \geq 2$ *but not* $2 \geq 3$. The strict inequality relation *is greater than* $>$ on $\mathbb{R}$ is transitive but has none of the other three properties. As a final example, the set inclusion relation $\subseteq$ on the power set $P(U)$ of some universal set U is reflexive, anti-symmetric, and transitive; but it is not symmetric. Why? Since if A is a proper subset of B, we have $A \subseteq B$ but $B \nsubseteq A$.

1.5.2 Domain and Range of Relation

The domain of a relation is the set of all first coordinates of the ordered pairs which are in the relation, while the range of the relation is the set of all second coordinates of the ordered pairs which are in the relation.

If R be an arbitrary relation from a set X to a set Y then it is not necessary that every element of X is related to some element of Y or that every element of Y is related to some element of X. Thus, the domain of a relation $R(A, B)$ is given by $\{ a \in A | \exists b \in B$ and $(a, b) \in R\}$. The domain of the relation $R = \{(a, p), (a,r), (b, q)\}$ is $\{ a, b\}$.

The range of a relation $R(A, B)$ is given by $\{ b \in B | \exists\, a \in A$ and $(a, b) \in R\}$. The range of the relation $R = \{(a, p), (a, r), (b, q)\}$ is $\{ p, q, r \}$.

1.5.3 Inverse of a Relation

Suppose $R \subseteq A \times B$ is a relation between A and B. Then the inverse relation $R^{-1} \subseteq B \times A$ is defined as the relation between B and A and is given by: $bR^{-1}a$ if and only if aRb. That is, $R^{-1} = \{(b,a) \in B \times A : (a,b) \in R\}$.

Example: Let R be the relation between $\mathbb{Z}$ and $\mathbb{Z}^+$ defined by mRn if and only if $m^2 = n$. Then $R = \{(m,n) \in \mathbb{Z} \times \mathbb{Z}^+ : m^2 = n\}$ and $R-1 = \{(n,m) \in \mathbb{Z}^+ \times \mathbb{Z} : m^2 = n\}$.

For example, $-3\ R\ 9, -4\ R\ 16, 0\ R\ 0, 16\ R-1\ 4, 9\ R-1\ 3$, etc.

1.5.4 Binary Relation

A binary relation between two sets X and Y is a subset of $X \times Y$ i.e., a set of ordered pairs $(x,y) \in X \times Y$. For a relation $R \subseteq X \times Y$ we often write xRy instead of $(x,y) \in R$. We write R^c for the complement of R - i.e., xR^cy if and only if $(x,y) \notin R$. If X and Y are the same set, so that the relation R is a subset of $X \times X$, we say that R is a relation on X.

Example 1: X is a set of students, say $X = \{Anu; Ben; Carl; Douglas\}$. Y is a set of courses, say, $Y = \{MEC-01; MEC-02; MEC-03\}$. Then $X \times Y$ has 12 elements. An example of a relation $R \subseteq X \times Y$ is the set of pairs (x,y) for which 'x is enrolled in y'.

When $X = Y$, we say that R is a binary relation on X. Since binary relations from X to Y are subsets of $X \times Y$, we can define intersection, union, and complement for binary relations. The complementary relation of a binary relation $R \subset X \times Y$ is the binary relation $\bar{R} \subset X \times Y$ defined by $x\bar{R}y \Leftrightarrow (x,y) \notin R$

Check Your Progress 3

1) What is $A \times B \times C$ where $A = \{0,1\}; B = \{1,2\}$ and C=$f\{0,1,2\}$.

   Solution:

   $A \times B \times C = \{(0,1,0); (0,1,1); (0,1,2); (0,2,0); (0,2,1); (0,2,2);$

   $(1,1,0); (1,1,1); (1,1,2); (1,2,0); (1,2,1); (1,1,2)$

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2) A binary operation ® on a set of integers is defined as x ®y = $x^2 + y^2$. Which one of the following statements is TRUE about ®?

   a) Commutative but not associative, b) commutative and associative

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3) Let there be a relation $\{(0,1), (3,22), (90,34)\}$. Find the elements of domain and elements of range.

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1.6 EQUIVALENCE RELATIONS AND PARTITIONS

In economics we often need to classify distinct objects as being alike or unalike. For instance, a consumer considers two distinct bundles of goods as being alike if they yield the same level of utility and say that the consumer is *indifferent* between the said bundle of goods. The objects being classified as being alike or unalike with respect to one property or another usually satisfy the following consistency requirements:

- Reflexive: Each object is like itself.
- Symmetry: If one object is like a second, then the second is like the first.
- Transitive: If one object is like a second and the second is like the third, then the first object is like the third.

1.6.1 Equivalent Relation

In general, an equivalence relation results when we wish to "identify" two elements of a set that share a common attribute. The definition is motivated by observing that any process of "identification" must behave somewhat like the equality relation, and the equality relation satisfies the reflexive (x=x for all x), symmetric (x=y implies y=x), and transitive (x=y and y=z implies x=z) properties. An equivalence relation is typically denoted by the symbol ~ (which is pronounced "wiggle").

Example 1: The relation on the set of integers $\mathbb{Z}$ defined by $(a, b) \in R$ if $(a - b) = 2R$ for some $K \in \mathbb{Z}$ is an equivalence relation.

Example 2: Let ~ be a relation defined on $\mathbb{Z}$ by $m \sim n$ if and only if $m - n$ is even. That is, for any $m, n \in \mathbb{Z}$, $m \sim n$ iff $m - n = 2k$ for some $k \in \mathbb{Z}$. Then:

- ~ is reflexive: $m - n = 0 = 2 \,.\, 0$.
- ~ is symmetric: If $m - n = 2k$ then $m - n = 2(-k)$.
- ~ is transitive: For any $n, m\, p \in \mathbb{Z}$, if $m - n = 2k$ and $n - p = 2l$ for some $k, l \in \mathbb{Z}$, then

  $m - p = m - n + (n - p) = 2(k + l)$.

Thus ~ is an equivalence relation on the set $\mathbb{Z}$ of integers, known as *congruence modulo* 2 on $\mathbb{Z}$.

Example 3: Suppose that a consumer lives in a n −commodity world. Then a *consumption bundle* is an ordered n −tuple of real numbers, $x = (x_1, x_2, \ldots x_n)$, with the lth component of the consumption bundle standing for the l amount of commodity consumed; her *consumption set* X, defined as the set of all possible consumption bundles is a subset of $\mathbb{R}^n$, $X \subset \mathbb{R}^n$. For any two consumption bundles $x, y \in X$, define the *indifference relation* ~ on X by: $x \sim y$ (read "x is indifferent to y ") if and only if x and y yield the same level of utility in the "preference scale" of the consumer. Clearly, the indifference relation is an equivalence relation on the consumption set of the consumer.

1.6.2 Partition

An equivalence relation on A leads to a partition of A, and vice versa; for every partition of A there is a corresponding equivalence relation.

Let A be a finite set and let $A_1, A_2, A_3 \ldots, A_n$ be subsets of A such $A_i = \emptyset$ for all i,

$A_i \cap A_j = \emptyset$ if $i \neq j$ and $A = \cup_i^n A_i = A_1 \cup A_2 \ldots \cup A_n$. The sets A_i partition the set A and these sets are called the classes of the partition.

Note that partition of a set is a collection of disjoint sets whose union is the given set. A set $\{A, B, C, \ldots\}$ of non-empty subsets of a set S is a partition of S if

1) $A \cup B \cup C \ldots = S$ and

2) intersection of every pair of distinct subsets is the null set.

Examples:

1) Let S = {1, 2, 3, 4, 5, 6}, A = {1, 5), B = {3, 4, 6} and C = {2}. Then the collection of sets {A, B, C} represents a partition of S.

2) Let N be the set of positive integers, N = {1, 2, 3, ...},

   E be the set E = {2, 4, 6, ...} and F be the set F = {1, 3, 5, ...}.

   Then {E, F} is a partition of N.

1.6.3 Equivalent Class

If R is an equivalence relation and $x \in S$, we define $[x]_R = \{y \in S \mid xRy\}$, that is, the set of all elements of S that are equivalent to x under the relation R. We call $[x]_R$ the **equivalence class** containing x. Often we omit the subscript R.

To find the equivalence classes for a given equivalence relation we use the following simple algorithm: Choose an element x which has not yet been included in an equivalence class (any element will do to begin with). List all the elements that are related to x. These form an equivalence class. Next, choose another element not yet listed and continue until all elements have been included.

Equivalence classes are somewhat like families. Those in the same class are related and those in different classes are not.

Example: Consider the set of railway stations of India with xRy meaning that you can travel by rail from station x to station y with having to change trains one or more times. The equivalence classes are the connected railway networks as we often experience while using the railways of India.

Let us see the following examples to see how we move from equivalent relation to equivalent class.

Example 1: Equals (=) is an equivalence relation because:

1) $a = a$ all the time

2) if $a = b, b = a$

3) if $a = b$ and $b = c, a = c$

Example 2: Let us define $a \sim b$ if a and b have the same sign (let's pretend 0 has positive sign for the purpose of this example). For example, $3 \sim 5, 7 \sim 2, -5 \sim -18$, but -6 is not equivalent to 3.

1) a has the same sign as a, so $a \sim a$

2) if a has the same sign as b, b has the same sign as a

3) if a has the same sign as b, and b has the same sign as c, a and c must have the same sign!

Example 3: We will say $a \sim b$ if a and b have the same remainder when you divide by three.

For example, $3 \sim 6$, because both have remainder 0 when you divide by three.

(Try to check the three rules yourself). Here's the first one:

1) $a \sim a$ because we always get the same remainder when we divide a by three.

Example 4: Let relation R, xRy holds if the integer division $\frac{x}{b}$ leaves the same remainder $m \in N/\phi$ on M as some $y \neq x \in M$. Take, for example, the relation (b = 2) for $x = 5$, $y = 7$. Since $\frac{5}{2}$ leaves the same remainder as $\frac{7}{2}$, namely 1, x *and* y belong to the same 'equivalence class.' Let's check the above properties:

(1) reflexivity holds because $\frac{5}{2}$ always leaves the same remainder, (2) symmetry holds because $x = 5, y = 7$ always leave the same remainder over b = 2, and (3) transitivity holds because if for some z the remainder for $\frac{z}{2}$ is 1, then all three remainders are the same.

Check Your Progress 4

1) Define partition.

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2) It is known that S^= {{a},{b,c}} is a partition of a set A.

a) What is the set A?

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b) What is the equivalence relation whose equivalence classes are the sets in the partition?

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3) Let R and S be relations on X. Determine whether each of the following statements is true or false.

a) If R and S are transitive, then R∪S is transitive.

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b) If R and S are transitive, then R∘S is transitive.

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4) a) Determine whether the following relation is an equivalence relation on {1,2,3,4,5}. {(1,1),(2,2),(3,3),(4,4),(5,5),(1,3),(3,1)}.

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b) Is the following an equivalence relation on the set of all people?

{(x, y) |x is taller than y}.

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c) Is the following an equivalence relation on the set of all people?

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5) A relation R is defined on ordered pairs of integers as follows: (x,y) R(u,v) if $x < u$ and $y > v$. Then is R either a partial order or an equivalent relation?

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6) Consider the binary relation R = {(x, y), (x, z), (z, x), (z, y)} on the set {x, y, z}. It must be either symmetric or anti-symmetric.

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1.7 ORDER RELATION

Intuitively, an ordering relation on a set gives precedence to some elements over others. This is why an ordering on a set cannot be a symmetric relation. In general, "order relations are used to describe how elements of a given set *compare* with respect to such aspects as size, position, or priority".

A few examples of order we encounter in our daily lives are: When there are many people at cashiers of a grocery store, lines are formed and people in these lines are ordered for service; those at the head of a line are served sooner than those at the end. Cars waiting for the signal to change at an intersection are also

ordered similarly. The three types of order viz., partial order, total order, and quasi order are discussed below.

1.7.1 Partial Order and Total Order

A **partially ordered set** (or **poset**) depicts the intuitive concept of an ordering or making arrangement of the elements of a set. A poset consists of a set together with a binary relation. It indicates that for certain pairs of elements in the set, one of the elements precedes the other. Such a relation is called a *partial order* to reflect the fact that not every pair of elements need be related. We may have some pair such that neither element precedes the other in the poset. Not that partial order is referred as such just because it is only a type of ordering of the elements of *a set.* In the total order, in contrast, every pair is related.

Partial order: A binary relation R on a set A is a **partial order** if and only if it is

1) reflexive, i.e., $\forall x \in X, x \geq x$

2) anti-symmetric, i.e., $\forall x, y \in X, x \geq y$ and $y \geq x \Rightarrow x = y$

3) transitive i.e., $\forall x, y, z \in X\ x \geq y$ and $y \geq z \Rightarrow x \geq z$.

Note that an equivalence relation and a partial order are both reflexive and transitive. The only difference between the two is that the property of symmetry in the former has been replaced by the property of anti-symmetry in the latter.

Example: The less-than-or-equal-to relation on the set of integers I is a partial order, and the set I with this relation is a poset.

Total order: When any two elements of a partially ordered set are related in some way by the ordering, then the set is said to be *totally* ordered.

For example, a binary relation R on a set A is a **total order** if and only if it is

1) a partial order, and

2) for any pair of elements a and b of A, $(a, b) \in R$ or $(b, a) \in R$. That is, every element is related with every element one way or the other. A total order is also called a ***linear order***.

Strict Partial Order: A **strict partial order** on a set X is relation on X which is **irreflexive** and **transitive.**

Well Order: A total order R on a set A is a **well order** if every non-empty subset of A has the least element.

Example 1: The set of positive integers in their natural order is a well ordered set since all subsets have a first member. Whereas the set of all integers in their natural order is not well ordered since the set itself has no first member.

Example 2: The set of nonnegative real numbers in their natural order is not well ordered since the subset consisting of numbers greater than, say 3, has no first member.

Example 3: The poset of the set of natural numbers with less-than relation is a well order, because every set of natural numbers has the least element.

1.7.2 Preorders

When the requirement of anti-symmetry is dropped, then a partial order becomes a *partial preorder*; and a total order is converted to a *complete* (or *total*) *preorder or a* **Quasi order**.

Quasi order is a binary relation R on a set A if and only if it is

1) irreflexive, and
2) transitive.

Example 1: The less-than relation on the set of integers I is a quasi order.

Example 2: Let x represent the consumption set of a consumer, with the preference relation $x < y$ (read "x is at most as desired as y"). For two commodity bundles x, y it is possible to have $x \preccurlyeq$ y and y $\preccurlyeq x$, that is, x ~ y (read as "x is *indifferent to y*."

If x is equivalent to y we usually say that, from the point of view of the consumer, x and y are "indifferent." Such a relation on x defines a preordering.

Let $\succcurlyeq$ be a preference relation on a set X. The corresponding **strict preference** relation on X, $\succcurlyeq$ is defined by $\forall x, y \in X, x \succ y \Leftrightarrow x \succcurlyeq y$ but not$y \succcurlyeq x$. If $x \succ y$ is read "x is *preferred to y*", then the above relation can be restated as $\forall x, y \in X, x \succ y \Leftrightarrow x \succcurlyeq y$ and not $y \succcurlyeq x$. The indifference relation, ~, corresponding to $\succcurlyeq$ is defined by

$\forall x, y \in X, x \sim y \Leftrightarrow x \succcurlyeq y$ and $y \succcurlyeq x$.

For any, $x \in X$ the equivalence class $E_X$of x relative to the indifference relation, ~, is called the **indifference class** of x; in the special case where$X \subseteq \mathbb{R}^2$, this is the indifference curve passing through x. Thus the collection of all indifference classes in X is a partition of X. Accordingly, distinct indifference classes are disjoint—a generalisation of the familiar result that indifference curves do not intersect.

Example: Consider the set X = {2, 4, 6, 8, 10, 12} strongly ordered by "x is an exact divisor of y and at most half as large as y" We then have

$R = \{(2,4), (2,6), (2,8), (2,10), (2,12), (4,8), (4,12), (6,12)\}$.

Consider now the following subsets of X

1) $W = \{8, 10, 12\}$.

   We have

   W×W={(8, 8), (8, 10), (8, 12), (10, 8), (10, 10), (10, 12), (12, 8), (12, 10), (12, 12)}.

   Therefore $R^W = R \cap (W \times W) = \emptyset$ and W is not an ordered subset of X.

2) $Z = \{2,4,6\}$.

The direct product is

$ZxZ = \{(2,2), (2,4), (2,6), (4,2), (4,4), (4,6), (6,2), (6,4), (6,6)\}$.

The restriction is now $R^Z = R \cap (ZxZ) = \{(2,4), (2,6)\}$ and Z is partially ordered but not totally ordered for 4 and 6 are not comparable.

3) $Y = \{2,4,8)$.

We have now

$Y \times Y = \{(2,2),(2,4),(2,8),(4,2),(4,4),(4,8),(8,2),(8,4),(8,8)\}$

So that $R^Y = R \cap (Yx\,Y) = \{(2,4),(2,8),(4,8)\}$.

Since all the elements of Y are comparable under R^Y, it is totally ordered by definition.

Note the difference between a preorder and a partial order. The former allows for indifferences, the latter does not

1.7.3 Lower and Upper Bounds

Various types of *extremal elements* which play important roles in partially ordered sets are as follows:

Suppose that $\preceq$ is a partial order on a set S and that $A \subseteq S$.

1) An element $a \in A$ is the *minimum* element of A if and only if $a \preceq x$ for every $x \in A$.
2) An element $a \in A$ is a *minimal* element of A if and only if no $x \in A$ satisfies $x \prec a$.
3) An element $b \in A$ is the *maximum* element of A if and only if $b \succeq x$ for every $x \in A$.
4) An element $b \in A$ is a *maximal* element of A if and only if no $x \in A$ satisfies $x \succ b$.

Suppose again that $\preceq$ is a partial order on a set S and that $A \subseteq S$. Then

1) An element $u \in S$ is a *lower bound* for A if and only if $u \preceq x$ for every $x \in A$.
2) An element $v \in S$ is an *upper bound* for A if and only if $v \succeq x$ for every $x \in A$.
3) The *greatest lower bound* or *infimum* of A, if it exists, is the maximum of the set of lower bounds of A.
4) The *least upper bound* or *supremum* of A, if it exists, is the minimum of the set of upper bounds of A.

Check Your Progress 5

1) What are the *differences between totally* and *partially ordered sets*?

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2) How does an equivalence relation determine a partition?

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3) How does a partition determine an equivalence relation?

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4) Let $R = \{(1,1),(1,2),(1,3),(2,2),(2,3),(3,3)\}$ be a binary relation on the set $S = \{1,2,3\}$. Is it a partial order relation on S? Explain why

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1.8 COMPOSITION OF RELATIONS

The composition of two relations $R_1(A,B)$ and $R_2(B,C)$ is given by $R_2 \circ R_1$ where $(a, c) \in R_2 \circ R_1 c) \in R_2 \circ R_1$ if and only if there exists $b \in B$ such that $(a, b) \in R_1$ and $(b, c) \in R_2$. The composition of relations is associative so that $(R_3 \circ R_2) \circ R_1 = R_3 \circ (R_2 \circ R_1)$.

Example: Consider a library that maintains two files. The first file maintains the serial number s of each book with details of the author of the book. This may be represented by the relation $R_1 = sR_1a$. The second file maintains the library card number c of its borrowers and the serial number s of any books that they have borrowed. This may be represented by the relation $R_2 = cR_2s$.

The library wishes to issue a reminder to its borrowers of the authors of all books currently on loan to them. This may be determined by the composition of $R_1 \circ R_2$, i.e., $cR_1 \circ R_2a$ if there is book with serial number s such that sR_1a and cR_2s.

1.8.1 Composition of Binary Relations

A binary relation R on A is a relation between A and A, and a binary relation can always be composed with itself. Its inverse is a binary relation on the same set. The following are all relations on A:

$R^2 = R \circ R$,

$R^3 = R \circ R \circ R$,

$R^0 = i_A$ (identity relation),

$R^{-2} = R^{-1} \circ R^{-1}$.

Example: Let R be the binary relation on the set of all people P such that $(a, b) \in R$ if a is a parent of b. Then the relation R^n is interpreted as:

R is the parent relationship.

R^2 is the grandparent relationship.

R^3 is the great grandparent relationship.

$R^{-1}$is the child relationship.

$R^{-2}$is the grandchild relationship.

$R^{-3}$is the great grandchild relationship.

This can be generalised to a relation R^n on A where

$R^n = R \circ R \circ ... \circ R$ (n-times).

Example: Combining Relation

Let

A = {1, 2, 3, 4}

B = {1, 2, 3}

R_1 = {(1, 2), (1, 3), (1, 4), (2, 2), (3, 4), (4, 1), (4, 2)}

R_2 = {(1, 1), (1, 2), (1, 3), (2, 3)}

Then

$R_1 \cup R_2$ = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (3, 4), (4, 1), (4, 2)}

$R_1 \cap R_2$ = {(1, 2), (1, 3)}

$R_1 \setminus R_2$ = {(1, 4), (2, 2), (3, 4), (4, 1), (4, 2)}

$R_2 \setminus R_1$ = {(1, 1), (2, 3}

The Sum and Product of Relations

If R and S are relations on the set X, then the **sum** of R and S is the relation $R + S$ defined on X by: $x(R + S)y$ if xRy or xSy.

If R and S are relations on the set X then the **product** of R and S is the relation RS defined on X by: $xRSy$ if there is some z $z \in X$ such that xRz and zSy.

Example: Jack is married to Mary and Mary's mother, Ruby, is therefore Jack's mother-in-law. The relation "mother-in-law of" is built up from the simpler relations of "spouse of" and "mother of". Ruby is the mother-in-law of Jack because there exists a person, Mary, such that Ruby is the mother of Mary and Mary is the spouse of Jack. There is a chain between Ruby and Jack and Mary is the link. If $H + W$ denotes "spouse of" and $M = "mother\ of"$ then $M(H + W)$ denotes the relation "mother-in-law of".

Example: Mary's parents are Ruby and Ted. Ted is the husband of Mary's mother. So with H and M defined above, $HM = "father\ of"$. And $M + HM = "parent\ of"$.

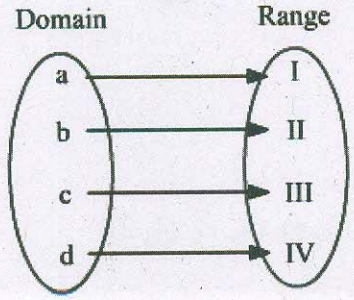
So $(M + HM)^2 = "grandparent\ of"$.

1.9 MAPPING AND FUNCTION

1.9.1 Mapping

A **mapping** is a special kind of binary relation which relates any given element of one set to a *unique* element of another. For example, a **mapping** f **from** S **to** T (or **on** S **into** T), denoted $f: S \to T$, is a relation $f \subseteq S \times T$.

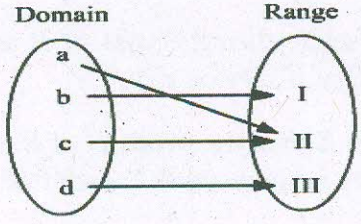
Basically a mapping shows how the elements are paired showing the input and output values. See the following mapping diagrams to form a broad idea on mapping.



(**Source:** Wikipedia)

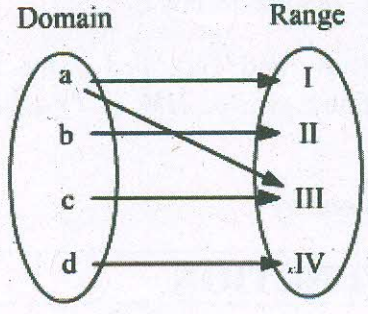
The above mapping diagram consists of two parallel columns. The first column represents the domain of a function f and the other column for its range. Lines or arrows are drawn from domain to range, to represent the relation between any two elements.

A function represented by the mapping above in which each element of the range is paired with exactly one element of the domain is called **one-to-one mapping**.



(**Source:** Wikipedia)

In the mapping, the second element of the range associates with more than one element in the domain. If the element(s) in range that have mapped more than one element in the domain is called **many-to-one mapping.**



(**Source:** Wikipedia)

In this mapping, the first element in the domain has mapped with more than one element in the range. If one element in the domain mapped with more than one element in the range, the mapping is called **one-to-many** relation. One-to-many relations is not a function.

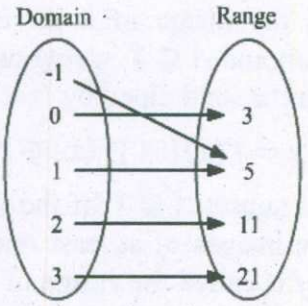
Example

Draw the diagram of mapping for the function $f(x) = 2x^2 + 3$.

First choose some elements from the domain. Then find the corresponding y-values (range) for the chosen x-values.

The domain of the function is all real numbers. Let $x = -1, 0, 1, 2$ and 3. Substitute these values into the function $f(x)$ to find its range.The corresponding y −values (range) are $5, 3, 5, 11$, and 21.

Now draw the diagram of the mapping as given below.



In the context of mapping, for our broad understanding, we may use image or codomian to mean range and pre-image to mean domain. But as we extend these terms to be used function when some subtle differences in the usage are pointed out. For the present discussion we note the following definitions of the terms:

Domain: Let $f: X \to Y$ *be a mapping.* The domain of f is the set X. The domain and the **preimage** of a mapping are the same set.

Codomain: Let $f: X \to Y$ be a mapping. The codomain of f is the set Y.

Image: Let $f: X \to Y$ be a mapping. The image is the set of values taken by f.

Preimage: Let $f: X \to Y$ be a mapping. The inverse of image is preimage. It can be seen as follows: Let $T \subseteq Y$. Then the preimage of T under f is denoted by $f^{-1}(T)$ and the set of all elements of X that map to elements in T under f. That is $f^{-1}(T) = \{x \in X | f(x) \in T\}$.

With these broad understanding, we will use the term **mapping** synonyms for "function"; both are used, especially, when X and Y are arbitrary sets.

1.9.2 Function

A **function** (or **mapping)** from $f: X \to Y$ consists of two sets X and Y together with a *rule* f that assigns to *each* $x \in X$ a *unique* element of Y, denoted by $f(x)$and read "f *of* x." For $x \in X$ we call the element x of X an **argument** of the function, and the corresponding element $f(x) \in Y$ the **image** of x under f (or the **value** of f at x). The set X is referred to as the **domain** (or **source**) and the set Y as the **codomain** (or **target**) of the function.

We will use the word "function" most frequently. In some contexts, the terms **transformation** and **operator** are also used as synonyms. All these alternative names for "function" convey the notation $f(x)$: the rule f takes the element and does something to it to produce the element $f(x) \in Y$.

A function consists of three objects: two sets X and Y and a rule f that assigns to each element x in X a unique element $f(x)$ in Y.

The function, for instance, the Cobb-Douglas utility function of a consumer living in a two-commodity world is $u: \mathbb{R}^2_+ \to \mathbb{R}$ defined by

$u(x_1, x_2) = x_1^{\alpha} x_2^{\beta}$ for all $(x_1, x_2) \in \mathbb{R}_+^2$.

Here x_1 and x_2 represent consumption of good 1 and good 2, respectively; and α and β are positive parameters. It is presumed in this example that consumption set of the consumer is the set $\mathbb{R}_+^2$ of ordered pairs of real numbers with nonnegative components.

Just as seen in case of mapping, the image of a set of elements can be identified. If $f: X \rightarrow Y$ is a function and $A \subseteq X$, we write $f(A)$ to denote the **image of the set** A **under** f, which is the set defined by

$$f(A) := \{y \in Y | \exists x \in A, y = f(x)\} = \{f(x) \in Y | x \in X\}.$$

In other words, the image $f(A)$ of a subset $A \subseteq X$ of the domain is the set of elements of its codomain which are images of at least one element in $\boldsymbol{A}$. The image $f(X)$ of the domain X under f is called the **range** of the function $f: X \rightarrow Y$. While the codomain Y is a set within which the values of f lie, the range of the function $f(X)$ signifies the set of values that f actually takes as its argument sweeps the domain. Evidently, the range is a subset of the codomain: $f(X) \subseteq Y$.

It should be noted at this point that while the range of a function is a very specific set, the codomain is a somewhat arbitrary set—it is any convenient set large enough to contain the range of the mapping.

Given a function $f: X \rightarrow Y$, we may determine not only the image of any subset of the domain X, but also the set of elements of X whose images under f would fall within a prescribed subset $B \subseteq Y$ of the codomain called the **preimage** (or **inverse image**) of B under f. The preimage of is usually denoted by $f^{-1}(B)$. Thus

$f^{-1}(B) = x \in X | f(X) \in B.$

It is a simple exercise in set theory to establish some important facts about images and preimages as summarized in the following:

Consider the mapping $f: X \rightarrow Y$. Let A_1 and A_2 be arbitrary subsets of the domain X, and let B_1 and B_2 be arbitrary subsets of the codomain Y. Then,

a) If $A_1 \subseteq A_2$, $f(A_1) \subseteq f(A_2)$;

b) If $B_1 \subseteq B_2$, $f^{-1}(B_1) \subseteq f^{-1}(B_2)$;

c) for all $A \subseteq X, A \subseteq f^{-1}((A))$;

d) for all B$\subseteq Y, f(f^{-1}(B)) \subseteq B$; and

e) for all B$\subseteq Y, f^{-1}(B^C) = \left(f^{-1}(B)\right)^C$.

Here $B^C = Y \setminus B$ and $\left(f^{-1}(B)\right)^C = X \setminus (f^{-1}(B)$.

1.9.2.1 Functions as Relations

A given function $f: X \rightarrow Y$ depicts a relation determined by $y = f(x)$ over the domain X × Y. Plotting its graph makes the nature of relationship x and y clear.

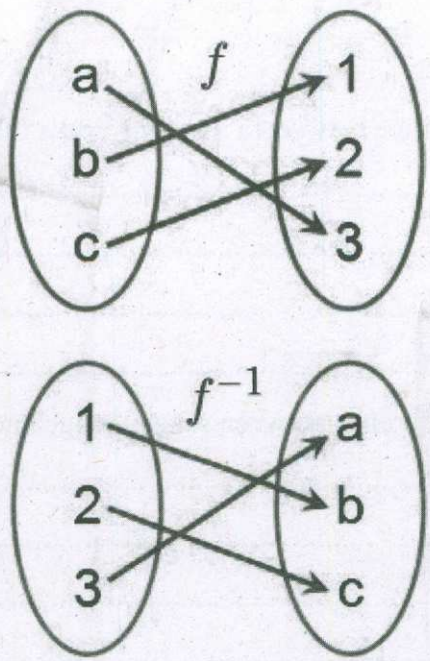
In the one-variable case just as $\rightarrow \sqrt{x}$, we get a graph as a curve by plotting the values of x and y in $x - axis$ and $y - axis$. In the two-variable case, the graphical representation is a *surface* in a three dimensional coordinate

system. Think of plotting the graph of the Cobb-Douglas function $(x_1, x_2) \rightarrow x_1^{\alpha} x_1^{\beta}$ introduced above as an example.

When we take up the function $y = f(x)$ reading it as y is a function f of x, we try to solve the underlying equation for getting the value of y from the given values of x. For that purpose x and y are referred to as **independent variable** and **dependent variable,** respectively. In the present discussion the variables x and y are referred as members of two arbitrary sets X and Y.

1.9.2.2 Inverse Functions

A function that "reverses" another function is an **inverse function**. That is, if f is a function mapping x to y, then the inverse function of f maps y back to x. Diagrams drawn in the following indicate such a process.



(**Source:** Wikipedia)

The function f and its inverse f^{-1} are presented in the diagrams. Whereas f maps a to 3, the inverse f^{-1}maps 3 back to a. Take note not to confuse the symbol f^{-1} with the symbol $1/f$.

Formally, the inverse of a function $f: X \rightarrow Y$ has to have bijection and has the property that every element $y \in$ Y corresponds to exactly one element $x \in X$.

See how it works. Suppose $f: X \rightarrow Y$ is bijective and $y \in Y$. The fact that f is surjective guarantees that $f^{-1}(\{y\}) \neq \emptyset$. Since f is injective, $f^{-1}(\{y\})$ contains only one element, say x, where $f(x) = y$. In this way, we find that f^{-1} is a rule that assigns each element of Y to exactly one element of X; i. e., f^{-1} is a function with domain Y and range X.

Finding Inverse Functions: If $f: X \rightarrow Y$ is a one-to-one and onto, then f^{-1} is a one-to-one and onto function.

Given f , how do we find f^{-1}?

Let $f: R \rightarrow R$ be given by $f(x) = 4x - 1 = y$. Now, swap x and y and solve for

$y: 4y - 1 = x$

or, $4y = x + 1$

or, $y = (x + 1)/4$.

Thus $f^{-1}(x) = (x + 1)/4$.

Let $f: X \to Y$ be a function. Then a function $g: f(X) \to X$ is said to be an **inverse** for f if

$g(f(x)) = x \text{ for all } x \in X$ and $f(g(y)) = y \text{ for all } y \in f(X)$.

Example: Take the sets given under bijection above which are $A = \{1,2,3,4\}$ and $B = \{r, s, t, u\}$. If $f: A \to B$, then define function g such that

$g(r) = 2, g(s) = 4, g(t) = 3$ and $g(u) = 1$. Then f and g are inverses. As can be verified $f(g(r)) = f(2) = r$ and $g(f(3)) = g(t) = 3$.

Note that $(f^{-1})^{-1} = f$.

Check Your Progress 6

1) What is the difference between a mapping and a function?

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2) Note down the difference between range, image and codomain?

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3) Find the inverse of $y = 3x - 2$

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1.9.2.3 Composition and Function

We can build up complicated functions from simple ones by using the process of composition, where the output of one function becomes the input of another. We can also decompose a complicated function into two or more simple functions.

Let g and f be two functions. For composition write down what $f(x)$ is first, and then apply g to the whole of $f(x)$. For example, let f be the function given by $f(x) = x^2$ and let g be the function given by $g(x) = x + 3$, then the composition of g with f is called $g \circ f$ and is worked out as $gf(x) = g(f(x))$

Thus, $gf(x) = g(f(x)) = g(x^2) = x^2 + 3$.

Note that in general $gf(x)$ is not equal to $fg(x)$.

A function can be the composition of two other functions. Such a process is called decomposing the function.

For example, take the function, $h(x) = e^{2x}$. We have already seen that this function may be written as a composite function $h(x) = gf(x)$, where

$g(x) = e^x, f(x) = 2x$.

Take another example. Suppose the function $h(x) = \sqrt[3]{2x + 1}$.

Take x, and first apply the function $f(x) = 2x + 1$. Then take the cube root of the result. So if we let $g(x) = \sqrt{3}\,x$ then $h(x) = \sqrt[3]{2x + 1} = g(2x + 1) = g(f(x))$.

1.9.2.2.1 Domains and Ranges of Composed Functions

Sometimes you will meet pairs of functions that cannot be composed. For example, take two functions $f(x) = -x^2$ and $g(x) = ln(x)$. Since a square is always positive or zero, $f(x) \leq 0$ for any x. Again the natural logarithm $ln\,x$ is defined only for positive numbers. So in this case, $g(f(x))$ is not defined for any values of x. The composed function $g \circ f$ does not exist for these two functions g and f.

There are also functions that cannot be composed for every x, but that can be composed if we restrict the values of x. For example, let us take $f(x) = 4x - 6$, $(x) = \sqrt{x}$.

Now only positive numbers, or zero, have real square roots. So g is defined only for numbers greater than or equal to zero. Therefore $g(f(x))$ can have a value only if $f(x)$ is greater than or equal to zero. You can work out that $f(x) \geq 0$ only when $x \geq 3/2$ So the composed function $gf(x)$ can be defined only for $x \geq 3/2$, and therefore the domain of the function $g \circ f$ is $\geq 3/2$.

In general, the domain of a composed function is either the same as the domain of the first function, or else lies inside it. If x is a valid input for the composed function $g \circ f$ then it must be a valid input for the individual function f.

The range of a function is the set of all values a function can take. For example, the range of the function $f(x) = e^x$ is given by $f(x) > 0$, because e^x is always greater than zero. If we have a composed function gf then its range must lie within the range of the second function g. Here is an example to show this. Take $f(x) = x - 8$, $g(x) = x^2$.

Now the first function $f(x)$ can take any value. But the second function $g(x)$ sends any number to its square, which is greater than or equal to zero. So

$$gf(x) = g(x - 8) = (x - 8)^2 = x^2 - 16x + 64 \geq 0.$$

Whatever $f(x)$ is, $g(f(x))$ must be greater than or equal to zero, because g applied to anything is greater than or equal to zero. So the composed function $gf(x) = x^2 - 16x + 64$ can take only values that are greater than or equal to zero. What happens if we compose the functions the other way round? We shall take g first, and then f. So

$$f(g(x)) = f(x^2) = x^2 - 8 \geq -8.$$

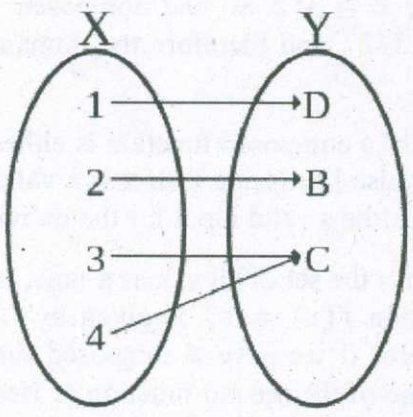
The range of $g \circ f$ is given by $f(g(x)) \geq -8$. So in this case, the range of the composed function $f(g(x))$ is contained in the range of f, but it is not the whole of the range of f. And in general, the range of a composed function is either the same as the range of the second function, or else lies inside it. If a value is a possible output from a composed function then it must be a possible output from the second function.

1.9.2.4 Onto and One-to-One Functions

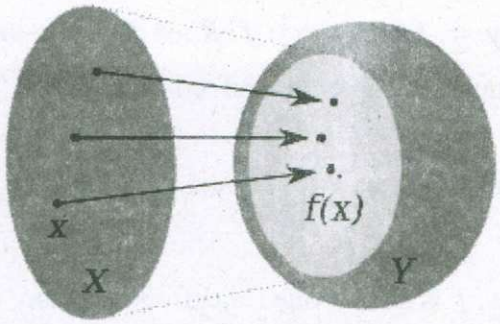
In this section, we discuss three special classes of functions, viz. **surjective**, **injective** and **bijective.** Such classification is done according to their relationship between images and pre-images. To understand the scheme, consider a function $x \rightarrow f(x) = y$ with the domain X and codomain Y. We have seen above that a function takes points from a domain and moves them to points of the range. An "onto" function, also called a "surjection" (which is French for "throwing onto") moves the domain X onto Y; that is, it completely covers Y, so that all of Y is the range of the function. Formally, for every point y in Y, there is some point x in X such that f(x) = y.

An "into" or "one-to-one" function, also called an "injection" (which is French word to describe "throwing into") also moves the domain X into$Y$, in the sense that each point in X retains its identity as a distinct point in Y. That is, no more than one point in X is moved to any given point in$Y$; there are no two points x_1 and x_2 in X such that$f(x_1) = f(x_2)$. Let us see onto and into functions formally.

A function f from X to Y is called **onto** if for all y in Y there is a x in X such that$f(x) = y$. Note that all elements in Y are used. See the diagram given below as an example which represents onto function.



In contrast to the preceding diagram, the following one presents a non-surjective function with domain X and codomain Y. It can be seen that the smaller oval inside Y is the **image** (also called **range**) of f. This function is **not** surjective, because the image does not fill the whole codomain. In other words, Y is colored in a two-step process: First, for every x in X, the point $f(x)$ is colored yellow; Second, all the rest of the points in Y, that are not yellow, are colored blue. The function f is surjective only if there are no blue points.



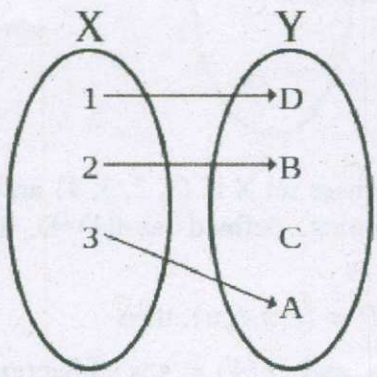
$f: X \to Y$

(**Source:** Wikipedia)

Example : If $X = \{1,2,3,4\}$ and $Y = \{x, y, z\}$

Consider the mapping $f: X \to Y$ defined by $f(1) = x, f(2) = z, f(3) = y$ and $f(4) = x$. Here the range is equal to Y. Hence f is an onto mapping.

An **into** function $f: X \to Y$ is such that f has at least one element y in Y which is not the f-image of X. The function is also called **injective** (**one-to-one**) as every element of the codomain is mapped to by *at most* one element of the domain. We may see from the Venn diagram below an injective function. The element C in Y is not the image of any $x \in X$



Example: If $X = \{1,2,3\}$ and $Y = \{A, B, C, D\}$

Consider the mapping $f: X \to Y$ defined by $f(1) = D, f(2) = B, f(3) = A$. Here the range of X viz., $\{D, B, A\}$ is **proper subset** of Y. Hence f is an into mapping.

Also note that an injective function is **one-to-one function** that preserves distinctness. It never maps distinct elements of its domain to the same element of its codomain. In other words, every element of the function's codomain is the image of *at most* one element of its domain.

The term *one-to-one function* must not be confused with *one-to-one correspondence*, which uniquely maps all elements in both domain and codomain to each other. We will see such a feature under bijection below. To identify an *one-to-one function,* we need to take into account the following notation:

$\forall x, y \in X, f(x) = f(y) \Rightarrow x = y.$

Or equivalently,

$\forall x, y \in X, x \neq y \Rightarrow f(x) \neq f(y).$

We check whether $y = f(x) = x^3$; $f: R \rightarrow R$ is one-one function.

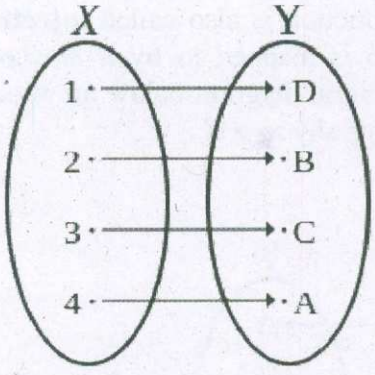
From the given equation, domain and co-domain of 'f' are a set of real numbers.

For one-one function we let $x_1, x_2 \in D_f$ and $f(x_1) = f(x_2)$ which

$\Rightarrow x_1^3 = x_2^3$

$\Rightarrow x_1 = x_2$, i.e., f is one-to-one (injective) function.

The function is bijective between the elements of two sets, where every element of one set is paired with exactly one element of the other set, and every element of the other set is paired with exactly one element of the first set. There are no unpaired elements. Thus it is **one-to-one and onto** or **one-to-one correspondence** and every element of the codomain is mapped to by *exactly* one element of the domain. Such a function is *both* injective and surjective. The diagram below shows a bijective mapping.



A bijective function, $f: X \rightarrow Y$, where set X is {1, 2, 3, 4} and set Y is {A, B, C, D}. It can be seen that the mapping defined as f(1)=D, f(2)=B, f(3)=C and f(4)=D is one-to-one and onto.

Example: If $A = \{1,2,3,4\}$ and $B = \{r, s, t, u\}$, then

$f(1) = u, f(2) = r$, $f(3) = t$ and $f(4) = s$ is a bijection

Example: For any set A, the identity function i_A is a bijection.

From the discussion above, the features of classes of functions we have covered are as follows:

In the definition of a mapping $f: X \rightarrow Y$ we require that one and only element of Y is assigned to each element of X. But we do not require that each element of Y is the image of an element of X under f; nor do we preclude the possibility that f maps more than one element of X to the same element of Y. In effect, we do not preclude the possibility that f maps its entire domain to a single element of Y—the case of a **constant function**. See that if $f: X \rightarrow Y$ is a mapping such that every element in X has the same image in Y. i.e., $f(x) = y$ for every x in X, then f is a said to be a constant mapping. For **example**, let $X = \{1,2,3\}$ and $Y = \{x, y, z\}$ then the mapping $f: X \rightarrow Y$ defined by $f(1) = x, f(2) = x$ and $f(3) = x$. Here every element has image x. Hence f is a constant mapping.

By imposing additional conditions to prevent each or all such possibilities we obtain the special types of functions:

i) ***into*** if $f(X) \subset Y$ and ***onto*** if $f(X) = Y$. Thus every onto mapping is into but not conversely,

ii) $f: X \mapsto Y$ is called ***injective*** if, for every $y \in Y$, there exists *at most* one $x \in X$ such that $f(x) = y$,

iii) $f(X) \subset Y$ is called ***surjective*** if, for every $y \in Y$, there exists *at least* one $x \in X$ such that $f(x) = y$,

iv) $f(X) \subset Y$ is called ***bijective*** if, for every $\in Y$, there exists *exactly* one $x \in X$

such that $f(x) = y$. Hence a bijective mapping is one-to-one and onto.

Check You Progress 7

1) What is real valued function?

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2) What is the meaning of transformation in the context of function?

Ordered lists of real-valued functions of several (real) variables are also referred to as **transformations**—mappings whose domain is a subset of $\mathbb{R}^n$ and codomain is $\mathbb{R}^m$ (where m and n are arbitrary natural numbers).

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3) What is the difference between one to one and onto functions?

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4) What is an identity mapping?

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1.9.3 Cardinality and Bijective Function

We have seen that **cardinality** of a set is the number of elements in it. So see that two sets have the same number of elements, pair the elements up. For example, the sets

$\{a, b, c, d\}$ and $\{1,2,3, C\}$ have the same number of elements because the elements of the first set can be paired with the elements of the second:

a	b	c	d
↓	↓	↓	↓
1	2	3	C

This kind of pairing is called a **bijection** or a **one-to-one correspondence**

Sets S and T have the **same cardinality** if there is a bijection f from S to T. We write $|S| = |T|$ to mean that S and T have the same cardinality.

Example. Prove that the set of natural numbers $\mathbb{N} = \{1,2,3.4, - - -\}$ has the same cardinality as the set $E = \{2,4,6,8, - - -\}$ of positive even integers.

Define $f: \mathbb{N} \to E$ by $f(n) = 2n$.

This function has an inverse $f^{-1}: E \to \mathbb{N}$ given by f^{-1} (m)= $m/2$.

Note that since$m \in E$, m is even, so m is divisible by 2 and $m/2$ is actually a positive integer.

Here's the proof that f and f^{-1} are inverses:

$$f\left(f^{-1}(m)\right) = f(m/2) = 2.m/2 = m$$

$$f^{-1}\left(f(n)\right) = f^{-1}(2n) = 2n/2 = n \quad for\, n \in \mathbb{N}$$

See that although $E \subseteq \mathbb{N}$ the two sets have the same number of elements. Despite the result looks counter intuitive due to our familiar experience in case of finite sets such a feature is not uncommon in infinite sets. If the elements of an infinite set can be *listed* $a_1, a_2, a_3, \cdots$, then the set has the same cardinality as the natural numbers.

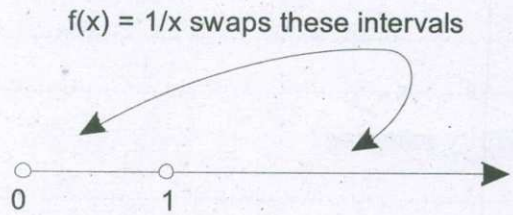
Example: Prove that open interval, $(0,1)$ has the same cardinality as $\mathbb{R}^+ = (0,\infty)$.

Define$f: (0,1) \to (1,\infty)$ by $f(x) = 1/x$.

If $0 < x < 1$, then $1/x > 1$.

Therefore, f *does* map $(0,1)$ to$(1,\infty)$.

$f(x) = 1/x$ swap these intervals



See that$f^{-1}(x) = 1/x$. If$x > 1$, then $0 < 1/x < 1$, sof^{-1} maps $(0,1)$ to$(1,\infty)$. Moreover,

$f\left(f^{-1}(x)\right) = f(1/x) = 1/(1/x)$,

$f^{-1}\left(f(x)\right) = f^{-1}(1/x) = 1/(1/x) = x$.

Thus, f is a bijection.

Define $g: (1,\infty) \to (0,\infty)$ by $g(x) = x - 1$.

If $x > 1$, then $x - 1 > 0$. Therefore, g maps $(1,\infty) to\ (0,\infty)$.

Similarly take $g^{-1} = x + 1$. If $x > 0$, then $x + 1 > 1$, so g^{-1} maps $(0,\infty)$ to $(1,\infty)$. Moreover,

$g(g^{-1}(x)) = g(x+1) = (x+1) - 1 = x,$

$g^{-1}(g(x)) = g^{-1}(x-1) = (x-1) \pm 1 = x$

Therefore, g is a bijection.

With the bijections f and g, we have $|(0,1)| = |(1,\infty)| = |(0,\infty)|$,

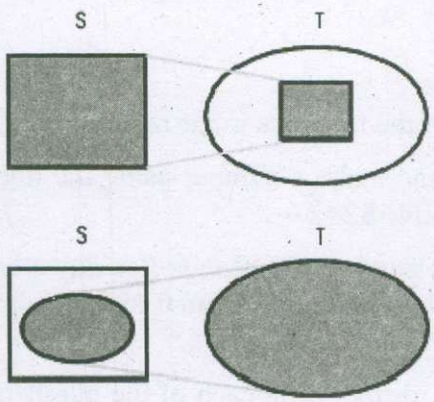
So $(0,1)$ and $(0,\infty)$ have the same cardinality.

In many situations, it's difficult to show that two sets have the same cardinality by actually constructing a bijection between them. An indirect way to show that two sets have the same cardinality is established by Schröder-Bernstein.

Schröder-Bernstein Theorem: Let S and T be sets. Suppose there are injective functions

$f : S \to T$ and $f : T \to S$. Then S and T have the same cardinality.

The Schröder-Bernstein theorem says that if S has the same cardinality as a subset of T, and T has the same cardinality as a subset of S, then S and T must have the same cardinality.



It is a powerful tool for showing that sets have the same cardinality. Here are some examples.

Example: Show that the open interval (0,1) and the closed interval[0,1] have the same cardinality.

The open interval $0 < x < 1$ is a *subset* of the closed interval$0 \le x \le 1$. In this situation, there is an "obvious" injective function $f: (0,1) \to [0,1]$, namely, the function $f(x) = x$ for all$x \in (0,1)$. f is called an **inclusion map**.) If$f(x_1) = f(x_2)$, then$x_1 = x_2$, so f is injective. Next, construct an injective function$g$: [0,1] $\to (0,1)$. The idea is to find a "copy" of [0,1] in $(0,1)$, then do some scaling and translation to map[0,1] onto the copy. Use the interval[0.25, 0.75] as a target in $(0,1)$. The target has length 0.5, so multiply by 0.5 to shrink [0,1] to [0,0.5]. By adding 0.25 we shift [0,0.5] to[0.25,0.75]. Thus we get $g(x) = 0.5x + 0.25$ for $0 \le x \le 1$.

First, if $0 \le x \le 1$, then $0 \le 0.5x \le 0.5$, so $0.25 \le 0.5x + 0.25 \le 0.75$. This proves that g is a function from $[0,1]$ to [0.5, 0.75]. Next, show that g is injective.

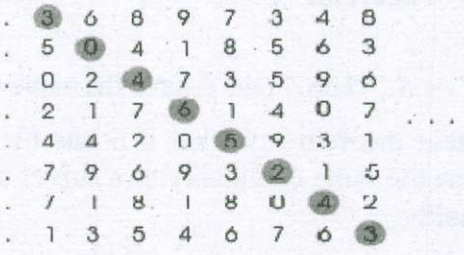
Suppose $g(x_1) = g(x_2)$ and prove that $x_1 = x_2$.

$g(x_1) = g(x_2)$ means that $0.5x_1 + 0.25 = 0.5x_2 + 0.25$, so $0.5x_1 = 0.5x_2$ and hence $x_1 = x_2$; g is injective. In fact, g is bijective, and you could prove injectivity by constructing g^{-1}). With injective functions $(0,1) \to [0,1]$ and [0,1] $\to (0,1)$, by Schröder-Bernstein,$|(0,1)| = |[0,1]|$.

Note that all infinite sets do not have the same cardinality.

A set is **countably infinite** if it has the same cardinality as the natural numbers $\mathbb{N} = \{1, 2, 3, \cdots\}$. A set is **countable** if it is either finite or countably infinite. For example , the open interval $(0, 1)$ is uncountably infinite. To see its proof, consider the diagonalization argument of Georg Cantor (1845--1918).

Proof: Suppose that $(0, 1)$ is countably infinite. Represent numbers in the interval as decimals $0.a_1a_2a_3\cdots$. (If a number ends in an infinite sequence of 9's, rewrite it as a finite decimal. For instance, 0.13499... becomes 0.135.) Since $(0, 1)$ is countably infinite by assumption, arrange the numbers in $(0, 1)$ in a list:



See that the list contains *all* of the numbers in the interval $(0, 1)$.

Now go down the diagonal and make a number using the digits. In the figure above, we get the number $0.30465243\cdots$.

Take each of the digits in this number and *change it* to any other digit except 9. For example, you could add 1 to each digit from 0 to 7 and change 8 or 9 to 0. This would produce the number $0.41576354\cdots$.

The number $0.41576354\cdots$ differs from each of the numbers of the first list. Specifically, the n^{th} digit of $0.41576354\cdots$ is different from the n^{th} digit in the n^{th}number on the list. This means that $0.41576354\cdots$ is not in the first list --- which is a contradiction, because it is assumed that the first list contained *all* of the numbers in the interval $(0, 1)$. Therefore, the interval $(0, 1)$ must be uncountably infinite.

Since the interval $(0, 1)$ has the same cardinality as $\mathbb{R}$, $\mathbb{R}$ is uncountably infinite as well. Notice that $\mathbb{Z}$ (which is countably infinite) is a subset of $\mathbb{R}$. Are there any sets which are "between" $\mathbb{Z}$ and $\mathbb{R}$ in cardinality?

The **Continuum Hypothesis** states that there are *no* sets which are "between" $\mathbb{Z}$ and $\mathbb{R}$ in cardinality; it was first stated by Cantor, and Kurt Gödel proved around 1940.

Definition. Let S be a set. The **power set** $\wp(S)$ of S is the set of all subsets of S.

Example: Let $S = \{a, b, c\}$. The power set of S is

$$\wp(S) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\{a, b, c\}\}.$$

Notice that the power set includes the empty set and the set S itself. Therefore, if S is finite and $|S| = n$, then $|\wp(S)| = 2^n$.

In this example, $|S| = 3$, and $|\wp(S)| = 2^3 = 8$.

Theorem: If S is a set, then S and $\wp(S)$ do not have the same cardinality.

Proof: Suppose first that $S = \emptyset$. Now $\emptyset \subset \emptyset$, so $\wp(\emptyset) = \{\emptyset\}$. Hence, $|\emptyset| = 0$ while $|\wp(\emptyset)| = 1$, and the result is true in this case.

Now suppose that$S \neq \emptyset$. We will prove the result by contradiction. Suppose that $|S| = |\wp(S)|$. This means that there is a bijection$f : S \to \wp(S)$.

Since f is a bijection, every element of the power set, that is, every subset of S is paired up with an element of$S$. For example, there must be an element $s \in S$ for which $f(s) = \emptyset$. Of course, $s \notin \emptyset$. So *s is an element which is paired up with a subset that doesn't contain it.*

Let $T = \{s \in S \mid s \notin f(s)\}$

Since f is bijective, and T is a subset of$S$, so there is an element $s_0 \in S$ such that$(s_0) = T$. Question: *Is* $s_0 \in T$? If so, then by definition of T, $s_0 \notin f(s_0) = T$, which is a contradiction.

If $s_0 \notin T$, then $s_0 \notin T = f(s_0)$, so s_0 satisfies the defining condition for T which means $s_0 \in T$ This is a contradiction.

Since $s_0 \in T$ and $s_0 \notin T$ both lead to contradictions, assumption that $|S| = |\wp(S)|$ is incorrect. Therefore, $|S| \neq |\wp(S)|$.

Check Your Progress 8

1) Show that the set of integers Z and the set of intger multiples of 5, 5 Z, have the same cardinality.

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2) What it means for a set A to be countable?

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3) What does it mean for two sets A and B to have the same cardinality?

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4) Define what it means for a set A to be countable.

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5) Is it true or false that the cardinality of the set of the positive odd integers is the same as the cardinality of the set of all integers?

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1.10 LET US SUM UP

In this unit we discussed notations used in set theory, operation of sets, building blocks of relations and functions. Staring with meaning of a set as one of collection of distinct objects, called elements, which are normally enclosed within brackets and separated by commas, we went on to learn different ways of forming of sets. In the process we saw finite and infinite sets, empty set, subsets, universal set and power sets. Proceeding to operation set, it was seen that union and intersection of two sets means the all elements of both set removing the duplicates and common elements respectively.

The operation on difference, i.e., all elements of *one* that are not elements of *the other* and complement set, viz., all elements in the universal *set* that are not in a given set were covered. Similarly, we have learnt the operation of Cartesian product as an ordered pair taken from elements of two sets. The collection of elements in an ordered pairs that are related is brought in to show the concept of relation. The properties, Reflexive, Symmetric and Transitive of a relation are introduced. Out of many relations, an equivalence relation has been identified which results when we wish to "identify" two elements of a set that share a common attribute. When a set has an equivalence relation defined on its elements, there is a natural grouping of elements that are related to one another, forming what are called equivalence classes. Since an ordering relation on a set gives precedence to some elements over others, the concepts of partial and total ordering of set have been studied.

We move to understand the idea of a mapping which is a special kind of binary relation and it relates any given element of one set to a *unique* element of another. An important concept, function, a rule of defining a relation just like seen in mapping, has been introduced to see the application of set theory. We discussed about three types of functions: one-to-one (injective) onto (surjective), and bijective functions. We have also learnt about inverse function of a bijective function and countably finite and infinite sets .

1.11 KEY WORDS

Antisymmetric Relation: A relation R on a set S where distinct elements are never both related to one another.

Bijective Function: Every element of the codomain is mapped to by *exactly* one element of the domain.

Binary Relation: A collection of ordered pairs of elements on a set.

Cartesian Product: Set of all possible pairs of elements whose components are members of two sets.

Complement : Set containing one set's elements that are not members of the other set.

Disjoint: Sets having no members in common; having an intersection equal to the empty set.

Countably Infinite: Elements of a set can be put in one-to-one correspondence with the set of natural numbers

Element: An object in a set.

Equivalence Class: Any one of the subsets into which an equivalence relation partitions a set, each of these subsets containing all the elements of the set that are equivalent under the equivalence relation.

Equivalence Relation: A binary relation that is reflexive, symmetric and transitive.

Intersection: Set containing all the elements that are common to two or more sets.

Member: An element of a set.

One-to-one (injective) Function: Every element of the function's codomain is the image of *at most* one element of its domain.

Onto (surjective) Function: Every element of the codomain is mapped to by *at least* one element of the domain.

Ordered Pair: A tuple consisting of two elements.

Partition: A collection of non-empty, disjoint subsets of a set whose union is the set itself (i.e. all elements of the set are contained in exactly one of the subsets).

Power Set: The set of all subsets of a set.

Preorder or **quasiorder:** A binary relation that is reflexive and transitive.

Proset: A set which is equipped with a binary relation that is reflexive and transitive is called a **preordered set** (or **proset**).

Reflexive Relation: A binary **relation** on a set for which every element is related to itself.

Relation: A set of ordered tuples.

Set: Collection of objects, disregarding their order and repetition.

Subset: With respect to another set, a set such that each of its elements is also an element of the other set.

Superset: With respect to another set, a set such that each of the elements of the other set is also an element of the set.

Symmetric Relation:A binary relation over a set if whenever a is related to b then b is related to a.

Transitive Relation: A binary relation over a set if whenever an element a is related to an element b, and b is in turn related to an element c, then a is also related to c.

Tuple: Finite sequence of elements; a finite ordered set.

Union: Set containing all of the elements of two or more sets

Venn diagram: Diagram representing sets by circles or ellipses.

1.12 SOME USEFUL BOOKS

Alpha C Chiang and Kevin Wainwright (2005), *Fundamental Methods of Mathematical Economic Analysis*, McGraw-Hill Bank Company London.

Badal Mukherji and V. Pandit (1982), *Mathematical Methods for Economic Analysis*, Allied Publishers Pvt. Ltd. , New Delhi

R.G.D. Allen (2009), *Mathematical Analysis for Economists*, MacMillan, India Limited, Delhi.

Taro Yamane (2012), *Mathematics for Economists: An Elementary Survey*, Prentice Hall of Indian, New Delhi.

Mark Dean (2011): Order Theory, Lecture Notes for Fall 2011 PhD Class in Decision Theory - Brown University (internet)

1.13 ANSWER OR HINTS TO CHECK YOUR PROGRESS

Check Your Progress 1

1) Yes, all empty sets are equal.

2) Yes

3) i) There are $2^6 = 64$

   ii) The answer is 62: there are $2^6 = 64$ subsets in U altogether. We exclude two: U itself (because it is not proper) and the empty set (because it is not non-empty).

4) $\{d\}, \{a, d\}, \{a, b, d\}, \{a, c, d\}, \{a, b, c, d\}, \{b, d\}, \{b, c, d\}, \{c, d\}$.

5) 8

Check Your Progress 2

1) {2, 3, 5, 6, 7, 8}

2) a) L.H.S. = R. H. S = {6}

   b) {1, 2, 3, 4, 5, 7}

3) {a, b, c, d, e, f}

4) (2, 7)

5) a) Open, not closed. Any point is an interior point.

   b) Not open, not closed. None of its points are interior points.

   c) Open and closed

Check Your Progress 3

1) $A \times B \times C = \{(0,1,0); (0,1,1); (0,1,2); (0,2,0); (0,2,1); (0,2,2); (1,1,0); (1,1,1); (1,1,2); (1,2,0); (1,2,1); (1,1,2)$

2) Commutative but not associative

3) Domain: (0, 3, 90) and range: (1, 22, 34)

Check Your Progress 4

1) Define partition. When we "partition a set," we break it down into mutually exclusive and exhaustive regions, i.e., regions with no overlap

2) 1) A={a,b,c}

   2) R={(a,a),(b,b),(c,c),(b,c),(c,b)}

3) a) FALSE. A counterexample is R={(1,2)},S={(2,3)}. Then (1,2) and (2,3) are in R∪S, but (1,3) is not.

   b) FALSE. A counterexample is S={(1,2),(3,4)},R={(2,3),(4,4)}. Then R and S are transitive, but R∘S={(1,3),(3,4)} is not since (1,4) is not in R∘S.

4) a) YES. It is reflexive, symmetric, and transitive.

   b) NO. It is not reflexive.

   c) YES. It is reflexive (clearly), symmetric (if x and y have the same parents, then y and x have the same parents), and transitive (if x and y have the same parents and y and z have the same parents, then x and z have the same parents.)

5) R is neither as it is not reflexive

6) Neither. Since R is not symmetric as (x, y) is present, but (y, x) is not present in R. R is also not antisymmetric as both (x, z) and (z, x) are present in R.

Check Your Progress 5

1) A binary relation $\boldsymbol{R}$ on a set $\boldsymbol{A}$ is a **partial order** if and only if it is (1) reflexive,(2) antisymmetric, and (3) transitive.

   A binary relation $\boldsymbol{R}$ on a set $\boldsymbol{A}$ is a **total order** if and only if it is (1) a partial order, and (2) for any pair of elements $\boldsymbol{a}$ and $\boldsymbol{b}$ of $\boldsymbol{A}$, $(\boldsymbol{a},\boldsymbol{b}) \in \boldsymbol{R}$ *or* $(\boldsymbol{b},\boldsymbol{a}) \in \boldsymbol{R}$.
   That is, every element is related with every element one way or the other.

2) An equivalence relation on a set X is a relation which is reflexive, symmetric and transitive A partition of a set X is a set P of cells or blocks that are subsets of X such that

   1) If $C \in P$ then $C \neq \emptyset$

   2) If $C_1, C_2 \in P$ and $C_1 \neq C_2$ then $C1 \cap C2 = \emptyset$

   3) If $a \in X$ there exists $C \in P$ such that $a \in C$

   4) From any partition P of X, we can define an equivalence relation on X by setting $x \sim y$ precisely when x and y are in the same part in P

   5) This is a partial order. As it satisfy reflexive, antisymmetric, and transitive properties.

Check Your Progress 6

1) Let $f: A \to B$ be a mapping. In it we have the rule that assigns an element from set A to an element from set B. A function is similar except that each element in set A is assigned to a unique element in set B. Mappings do not have this requirement

2) The range and the image are the same thing. Simply put, the range (a.k.a. image) is the set of all "outputs" of a function. The codomain is simply the general set to which the outputs can belong; in general the range is a subset of the codomain. It is possible that the codomain and the range can be equal for a given function. It depends on how that function is defined. If f is the function

$$f(x) = (2x+8)^3$$

then find f^{-1}.

Solution: we must solve the equation $y = (2x + 8)^3$ for x:

$$y = (2x+8)^3$$
$$\sqrt[3]{y} = 2x+8$$
$$\sqrt[3]{y} - 8 = 2x$$
$$\frac{\sqrt[3]{y} - 8}{2} = x.$$

Thus the inverse function f^{-1} is given by the formula

$$f^{-1}(y) = \frac{\sqrt[3]{y}-8}{2}.$$

3) First get $\frac{y+2}{3} = x$.

Switch the location y to x and x to y to have $y = \frac{x+2}{3}$ which is required inverse.

Check Your Progress 7

1) **Real-valued functions** are mappings whose codomain is equal to $\mathbb{R}$. By convention, "functions of one variable" are real-valued functions whose domains are subsets of $\mathbb{R}$; "functions of several (real) variables," on the other hand, are real-valued functions whose domain is a subset of (for some positive integer n).

2) Ordered lists of real-valued functions of several (real) variables are also referred to as **transformations**—mappings whose domain is a subset of $\mathbb{R}^n$ and codomain is $\mathbb{R}^m$ (where m and n are arbitrary natural numbers).

3) A 1-1 into function leaves some (at least one) element in the range with no pre-image, but each element in the domain has a unique image. On the other hand, a 1-1 onto function f has the property has the property that for every t in the range, there is one and only one x in the domain such that f(x) = t.

4) Let X be any set. Then the mapping $f: X \to X$ defined as $f(x) = x$ for every x in X is called the identity Mapping. It is denoted by Ia. The identity function is bijection and is its own inverse.

Check Your Progress 8

1) Let $g: Z \to 5Z$ where g(n) = 5n for all integer n. Then g is seen to be bijection from Z and 5Z have the same cardinality.

2) A set A is said to be countable if A is finite or if there exists a bijection Z $\to$ A.

3) Sets A and B have the same cardinality (written $|A| = |B|$) if there exists a bijection $A \rightarrow B$

4) A set A is said to be countable if A is finite or if there exists a bijection $\mathbb{Z} \rightarrow A$

5) Yes, the cardinalities are the same. In order to prove they are, we need to have a bijective function that maps from one set to the other. The function must be such that it is both one-to-one and onto.

1.14 EXERCISES AND ANSWERS

1) Work out $gf(x)$ for the following pairs of functions:

(a) $f(x) = 3x, g(x) = 2x^2 - 5$, (b) $f(x) = e^{4x}, g(x) = \sqrt{x}$,

2) Work out $fg(x)$ for the following pairs of functions, and compare the results to those you obtained for Exercise 1:

(a) $f(x) = 3x, g(x) = 2x^2 - 5$, (b) $f(x) = e^{4x}, g(x) = \sqrt{x}$,

3) Decompose the following functions into the form $g \circ f$:

(a) $6x + 3$, (b) $4x^2$, (c) $4x^2$ (in a different way), (d) $e^x + 4$, (e) $x^2 + 2x + 1$.

4) For the following pairs of functions, find the domain and range of the composed function gf:

(b) $f(x) = x^2$, $g(x) = e^x$

(c) $f(x) = -x$, $g(x) = \ln x$.

5) Prove that $[-1, 1]$ has the same cardinality as $(1, 3) \cup (4, 6)$.

6) Let $X = \{x \in R: x^4\text{-}1=0\}$. Find the sets equal to X?

7) Let $X = \{x \in R: x^3 = x^2\}$. Find the sets equal to X?

8) Which of the following sets is finite?

(A) $\{1, 2\} \cap R$

(B) $\{x \cap 2\, R : x^2 < 9\}$

(C) $[0, 1] \cap [1/2, 3/2]$

9) Let R_1 and R_2 be equivalence relations on X. Show that

R1∩R2 is an equivalence relation on X.

10) Let

$A = \{x \in R: x^4 + x > 2\}$

$B = \{x \in R: x^3 < 1\}$

and

$C = \{x \in R : x^8 > 1\}$.

ii) Prove that $A \cap B \subseteq C$.

iii) Can you say that A and B are disjoint as they do not meet? And therefore the Empty Set is a subset of C

11) X and Y are sets with the following three properties.

i) X' has 12 elements.

ii) Y' has 7 elements.

iii) $X \cap Y$ 0 has 4 elements.

How many elements in $X' \cap Y$?

12) Which of the following sets is infinite?

(A) $\{0, 1\} \cap R$

(B) $\{x \in R: x^2 < 4\}$

(C) $[0, 1] \cap (4/3, 3/2)$

13) Does the equation $y^2 + x = 1$ represents a function y in terms of x?

14) A survey was made of 25 people to ask about their use of products A and B. The following information was recorded: 14 people used only one of the products; 9 people did not use B; 11 people did not use A.

i) How many people used A?

ii) How many people used both products?

15) In a class of 120 students numbered 1 to 120, all even numbered students opt for Physics, whose numbers are divisible by 5 opt for Chemistry and those whose numbers are divisible by 7 opt for Math. How many opt for none of the three subjects?

16) Of the 200 candidates, who were interviewed for a position at a call center, 100 had a two-wheeler, 70 had a credit card and 140 had a mobile phone. 40 of them had both, a two-wheeler and a credit card, 30 had both, a credit card and a mobile phone and 60 had both, a two wheeler and mobile phone and 10 had all three. How many candidates had none of the three?

17) In a class of 40 students, 12 enrolled for both English and German. 22 enrolled for German. If the students of the class enrolled for at least one of the two subjects, then how many students enrolled for only English and not German?

18) 100 people are asked about three brands of soft drinks called A, B and C. Find how many people like B and how many people don't like any of the drinks given the following information.

i) 18 like A only (not B and not C).

ii) 23 like A but not B (and like C or don't like C).

iii) 26 like A (and like or don't like other drinks).

iv) 8 like B and C (and like A or don't like A).

v) 48 like C (and like or don't like other drinks).

vi) 8 like A and C (and like or don't like B).

vii) 54 like one and only one of the drinks.

Number of candidates who had none of the three = Total number of candidates - number of candidates who had at least one of three devices.

Total number of candidates = 200.

19) In a class 40% of the students enrolled for Math and 70% enrolled for Economics. If 15% of the students enrolled for both Math and Economics, what % of the students of the class did not enroll for either of the two subjects?

20) The domain of f is the set of all non-negative real number; range is the set $\{-1,0,2,7\}$. What are the domain and range of A,B,C if:

$A(x)=f(x+2)$

$B(x)=f(x)+2$

$C(x)=2*f(-x)$

and give a convincing argument that the inverse of is not a function.

Answers to Exercises

1) (a) $18x^2 - 5$ (b) e^{2x}

2) (a) $6x^2 - 15$ (b) $e^{4\sqrt{x}}$

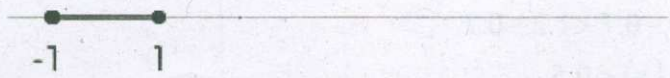
3) (a) f(x)=6x, g(x) = x + 3 (b) $f(x) = x^2$, g(x) = 4x (c) f(x) = 2x, $g(x) = x^2$

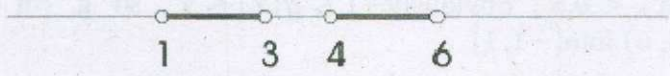
(d) f(x) = x + 4, $g(x) = e^x$ (e) f(x) = x + 1, $g(x) = x^2$

4) (a) domain is all real x, range is $y \geq 1$

(b) domain is $x < 0$, range is all real y.

5) Ans:





The two sets don't "look alike" --- the first set is a single interval which is closed on both ends, while the second set consists of two open intervals. *When two sets don't look alike but you think they have the same cardinality, consider using the Schröder-Bernstein theorem.* Define injective functions going from each set into the other.

The first set is an interval of length 2, which (because of its endpoints) won't fit in either of the intervals that make up the second set. Since the second set's intervals *don't* have endpoints, just slide $[-1,1]$ over, so that its endpoints will stick out of the ends of either$(1,3)$ or $(4,6)$. Thus the idea is to shrink $[-1,1]$ first, then slide it inside either$(1,3)$ or $(4,6)$.

By multiplying $[-1,1]$ by 0.5, we get $[-0.5,0.5]$, an interval of length 1. This will surely fit inside $(1,3)$ (say), and slide $[-0.5,0.5]$ into$(1,3)$ by adding 2. This takes $[-0.5,0.5]$ to $[1.5,\ 2.5]$. Now define $f:[-1,1] \to (1,3) \cup (4,6)$ by $f(x) = 0.5x + 2$.

First, show that this makes sense, that is, that f really takes $[-1,1]$ *into* $(1,3) \cup (4,6)$. Suppose $x \in [-1,1]$. Then

$$-1 \leq x \leq 1$$
$$-0.5 \leq 0.5x \leq 0.5$$
$$-0.5+2 \leq 0.5x+2 \leq 0.5+2$$
$$1.5 \leq f(x) \leq 2.5$$

Since $1.5 \leq f(x) \leq 2.5$, we can say, $1 < f(x) < 3$, so f {\it does} map $[-1,1]$ into$(1,3) \cup (4,6)$.

Next, show that f is injective. Suppose $(a) = f(b)$. Then

$$0.5a + x = 0.5b + 2$$
$$0.5a = 0.5b$$
$$a = b$$

Hence, f is injective.

Now define an injective function $g : (1,3) \cup (4,6) \to [-1,1]$ so that $(1,3) \cup (4,6)$ occupies a total length of $6 - 1 = 5$, whereas the target interval $[-1,1]$ has length 2. If multiplied by $1/5 = 0.2$, $(1,3) \cup (4,6)$ shrinks to $(0.2, 0.6) \cup (0.8, 1.2)$, which has a total length of 1. Next slide $(0.2, 0.6) \cup (0.8, 1.2)$, inside $[-1,1]$ by subtracting 0.7, which should give $(-0.5, -0.1) \cup (0.1, 0.5)$.

Thus, define $g : (1,3) \cup (4,6) \to [-1,1]$ by $g(x) = 0.2x - 0.7$.

We need to check that g maps $(1,3) \cup (4,6)$ into $[-1,1]$. Let $\in (1,3) \cup (4,6)$. Then certainly x is between 1 and 6, i.e. $1 < x < 6$. (Of course, $1 < x < 6$ does *not* imply that $\in (1,3) \cup (4,6)$. Do you see why?) So

$$1 < x < 6$$
$$0.2 < 0.2x < 1.2$$
$$0.2 - 0.7 < 0.2x - 0.7 < 1.2 - 0.7$$
$$-0.5 < g(x) < 0.5$$

Since $0.5 < g(x) < 0.5$, obviously $-1 \le g(x) \le 1$, so g {\it does} map $(1,3) \cup (4,6)$ into $[-1,1]$.

Next, show that g is injective. Suppose $g(a) = g(b)$. Then

$$0.2a - 0.7 = 0.2b - 0.7$$
$$0.2a = 0.2b$$
$$a = b$$

Therefore, g is injective.

Hence, $[-1,1]$ and $(1,3) \cup (4,6)$ have the same cardinality, by the Schröder-Bernstein theorem.

6) {-1, 1}.

7) {0,1}

8) A finite set is a set containing only finite number of elements. For example, 1,2,3 is finite. A set with infinitely many elements is called an infinite set. The set that has been mentioned, {x ∈ R: -3 < x < 3} is infinite: there are infinitely many real numbers between -3 and 3. For example, take a real number which has decimal expansion 1.2345…. No matter how you continue write more digits after the decimal point (and this can be done in infinitely many ways), you will have a number which is bigger than -3 and smaller than 9. Therefore the set {x ∈ R: -3 < x < 3} is not finite. However, the set {x ∈ Z: -3 < x < 3} is finite, it equals {-2,-1, 0, 1, 2} and therefore has 5 elements.

9) $R1 \cap R2$ is reflexive since xR_1x and xR_2x for all x. It is symmetric because, if $(x, y) \in R_1 \cap R_2$ then it is in both R_1 and R_2, and therefore (y, x) is in both. It is transitive because if (x, y) and (y, z) are in both R_1 and R_2 then (x, z) is in both R_1 and R_2.

10) It would be a valid argument if A and B were indeed disjoint. But they are not; one can easily see that -2 belongs to both A and B.A correct solution: Assume $x \in A \cap B$. Then $x \in A$ and $x \in B$.Since $x \in A$, it satisfies $x^4 + x > 2$. Since $x \in B$, it satisfies $x^3 < 1$ which implies $x < 1$ which is the same as $1 > x$. Adding the last inequality to the inequality $x^4 + x > 2$, one gets $x^4 + x + 1 > 2 + x$ which simplifies as $x^4 > 1$. Both parts of this inequality are positive, therefore we can square it and get $x^8 > 1$. But this means that $x \in C$. Hence $A \cap B \subseteq C$.

11) 9. To see how, denote $x = |X \cap Y'|$, $y = |X' \cap Y|$ (this is what we have to find), $z = |X \cap Y|$, $t = |(X \cup Y)'|$ (make a Venn diagram!), then $|X'| = y + t = |Y'| = x + t = 7$; $|X \cap Y'| = x = 4$; Excluding unknowns, we find $t = 3$ and $y = 9$.

12) (B). This set is $\{-2 < x < 2\}$ is infinite. The set A is finite because it is a subset of a finite set $\{0; 1\}$. The set C is empty and therefore finite.

13) Solving the equation for y we get $y^2 = 1 - x$. So $y = \pm\sqrt{(1-x)}$. For one value of x we have two values of y and this is not a function.

14) Denote a number of people using A but not B b number of people using B but not A c number of people using both A and B d number of people not using any product (it is useful to draw a Venn diagram and see that a, b, c, d correspond to its 4 regions). Then _ "14 people used only one of the products" means $a + b = 14$ _ "9 people did not use B" means $a + d = 9$ _ "11 people did not use A" means $b + d = 11$ _ Finally, $a + b + c + d = 25$: Thus you have a system of 4 linear equations with 4 variables: $a + b = 14$; $a + d = 9$; $b + d = 11$; $a + b + c + d = 25$ and it is easy to solve. So $a = 6, b = 8, c = 8, d = 3$.

15) We need to find out the number of students who took at least one of the three subjects and subtract that number from the overall 120 to get the number of students who did not opt for any of the three subjects.

Number of students who took at least one of the three subjects can be found by finding out A U B U C, where A is the set of those who took Physics, B the set of those who took Chemistry and C the set of those who opted for Math.

Now, $A \cup B \cup C$ = A + B + C - (A n B + B n C + C n A) + (A n B n C)
A is the set of those who opted for Physics = 120/2 = 60 students
B is the set of those who opted for Chemistry = 120/5 = 24
C is the set of those who opted for Math = 120/7 = 17.
The 10th, 20th, 30th..... numbered students would have opted for both Physics and Chemistry.

Therefore, A n B = 120/10 = 12

The 14th, 28th, 42nd..... Numbered students would have opted for Physics and Math.

Therefore, C n A = 120/14 = 8

The 35th, 70th.... numbered students would have opted for Chemistry and Math.

Therefore, B n C = 120/35 = 3

And the 70th numbered student would have opted for all three subjects.
Therefore, AUBUC = 60 + 24 + 17 - (12 + 8 + 3) + 1 = 79.

Number of students who opted for none of the three subjects = 120 - 79 = 41.

16) Number of candidates who had at least one of the three = A U B U C, where A is the set of those who have a two wheeler, B the set of those who have a credit card and C the set of those who have a mobile phone. We know that AUBUC = A + B + C - {A n B + B n C + C n A} + A n B n C

Therefore, $A \cup B \cup C = 100 + 70 + 140 - \{40 + 30 + 60\} + 10$ Or AUBUC = 190.

As 190 candidates who attended the interview had at least one of the three gadgets, 200 - 190 = 10 candidates had none of three.

Let A be the set of students who have enrolled for English and B be the set of students who have enrolled for German.

Then, (A U B) is the set of students who have enrolled at least one of the two subjects. As the students of the class have enrolled for at least one of the two subjects, A U B = 40

17) We know A U B = A + B - (A n B) i.e, 40 = A + 22 - 12 or A = 30 which is the set of students who have enrolled for English and includes those who have enrolled for both the subjects.

However, we need to find out the number of students who have enrolled for only English = Students enrolled for English - Students enrolled for both German and English = 30 - 12 = 18.

18) i) 18 like A only (not B and not C).

ii) 23 like A but not B (and like C or don't like C).

iii) 26 like A (and like or don't like other drinks).

iv) 8 like B and C (and like A or don't like A).

v) 48 like C (and like or don't like other drinks).

vi) 8 like A and C (and like or don't like B).

vii) 54 like one and only one of the drinks.

For solution, we draw a Venn diagram.

Let a be number of people liking A only

b be number of people liking B only

c be number of people liking C only

d be number of people liking A and B but not C

e be the number of people who like A and C, but not B.

f be the number of people who like B and C, but not A.

g be the number of people who like all tree products A,

B, and C.

h be number of people liking none of the drinks, as shown on the Venn diagram below.

From (i){(vii) we get

i) $a = 18$

ii) $a + e = 23$

iii) $a + d + e + \} = 26$

iv) $f + g = 8$

v) $c + e + f + g = 48$

vi) $e + g = 8$

vii) $a + b + c = 54$

We also have

(viii) $a + b + c + d + e + f + g + h = 100$

Now (i) gives $a = 18$, (ii) gives $e = 5$, (vi) gives $g = 3$, (iii) gives $d = 0$, (iv) gives $f = 5$, (v) gives $c = 35$, (vii) gives $b = 1$,

(viii) gives $h = 33$.

Therefore the number of people who like B is $b + d + f + g = 9$; and the number of people who like none is $h = 33$.

19) We know that (A U B) = A + B - (A n B), where (A U B) represents the set of people who have enrolled for at least one of the two subjects Math or Economics and (A n B) represents the set of people who have enrolled for both the subjects Math and Economics.

Note
(A U B) = A + B - (A n B) => (A U B) = 40 + 70 - 15 = 95%

That is 95% of the students have enrolled for at least one of the two subjects Math or Economics.

Therefore, the balance (100 - 95)% = 5% of the students have not enrolled for either of the two subjects.

20) Answer: Loosely speaking the domain of $A(x)$ is the set of numbers for which it is possible to compute the result.

In the case of $A(x)$ you should ask: "What numbers can I input to A so that I can compute a result? well since $A(x)=f(x+2)$ and $f(x)$ can only accept non-negative numbers then you reason that $x+2$ should be non negative, i.e. $x+2\geq 0$ which is the same as x must be greater or equal than -2. So the domain of $A(x)$ are all the real numbers that are greater or equal than -2.

With similar reasoning you find that the domain of $B(x)$ is the same as $f(x)$, i.e. the non-negative real numbers. And the domain of C is the non-positive real numbers.

the range of $A(x)$ is the same as $f(x)$. the range of $B(x)$ is the set $\{1,2,4,9\}$ because you add the number 2 to the result of $f(x)$ and the range of $C(x)$ is the set $\{-2,0,4,14\}$ since you multiply by 2 the result of $f(x)$.

$f(x)$ cannot have an inverse because your domain is a much bigger set than its range. Since the range of $f(x)$ has only 4 numbers and the domain of $f(x)$ has much more (infinitely many more).

for each $y\in\{-1,0,2,7\}$ there is an $x\geq 0$ such that $f(x)=y$, and in that case $x-2\geq -2$, so $x-2$ is in the range of A, and $A(x-2)=f((x-2)+2)=f(x)=y$, so y is in the range of A.

UNIT 2 FUNCTIONS AND THEIR GRAPHICAL REPRESENTATION

Structure

2.0 Objectives

2.1 Introduction

2.2 Vertical Line Test of Function

- 2.2.1 Difference between Equation and Function

2.3 Graphical Representation of Single Variable Functions

2.4 Graph of Linear Types of Functions

- 2.4.1 Absolute Value Function
- 2.4.2 Step Function

2.5 Graphing Functions – Curvy Types

- 2.5.1 Even and Odd Functions
- 2.5.2 Quadratic Functions
  - 2.5.2.1 Computation of Vertex
- 2.5.3 Cubic Function

2.6 Graph of Functions – Asymptotic Types

- 2.6.1 Graph of Asymptote
- 2.6.2 Square Root Function
- 2.6.3 Exponential Function
- 2.6.4 Logarithmic Function

2.7 Graphing Functions – Pieces

- 2.7.1 Rational Function
- 2.7.2 Piecewise Functions
- 2.7.3 Discontinuous Function
  - 2.7.3.1 Asymptotic Discontinuities
  - 2.7.3.2 Point Discontinuities
  - 2.7.3.3 Jump Discontinuities

2.8 Graphing Functions – *Curve* with Branches

- 2.8.1 Hyperbola
- 2.8.2 Difference between Hyperbola and Parabola
- 2.8.3 Rectangular Hyperbola

2.9 Graph of Functions with Two Variables: A Brief Introduction

- 2.9.1 Level Curves

210 Let Us Sum Up

2.11 Key Words

2.12 Some Useful Books

2.13 Answer or Hints to Check Your Progress

2.14 Exercises and Answers

MEC-103 1.17F.7K

2.0 OBJECTIVES

After going through this unit, you will be able to:

- differentiate between relation and function;
- apply vertical line test to identify a function;
- graphically represent various types of functions of a single variable; and
- have a preliminary idea of dealing with graphs of two variable functions.

2.1 INTRODUCTION

In the preceding unit we have seen that a **function** is a rule that relates how one quantity depends on other quantities. For example, quantity demanded for a commodity depends on price. Seen in terms of domain and range, the function is a relationship with every element of the domain mapped to a single element in the range. When such a single valued rule is not observed we cannot have a function. Thus, the following diagram giving the relationships between domain and co-domain some of which do not qualify to be functions.

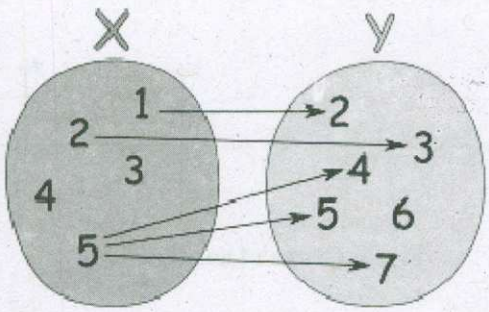


Fig. 1: Relationship and Function

Source: Internet

In the figure elements of X, 3 and 4 have no relation in Y. Another element, 5, is related to more than one element in Y. All the arrows we see in the figure above therefore are not functional relationships between X and Y. Note that element, 6, in Y is not related to any element of X. Such a feature does not affect defining a function.

2.2 VERTICAL LINE TEST OF FUNCTION

The **vertical line test** is a simple method for determining if an equation is indeed a function. Remember that an equation is used to denote the equality between two expressions whereas a function is in much more complex than an equation. A function is used to denote a relation between a set of inputs and a set of corresponding outputs.

The vertical line test is performed by sketching a graph of the equation. Take a vertical line, like a ruler, and pass it over the graph. If the graph crosses the vertical line in only one place at every point, then the equation is a function. If it touches the vertical line at more than one point, then it is not a function (see the figure below).

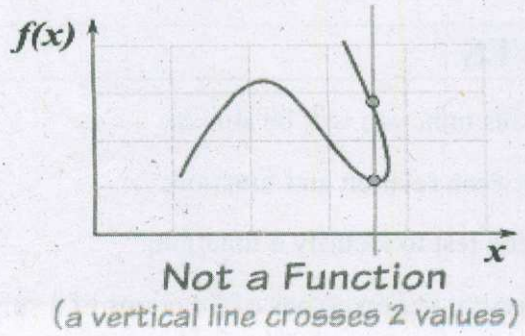


Fig. 2: Graph that is not a Function

Source: Internet

This works because of the definition of a function. For every input (x), there is only one output (y). So, if the graph touches a vertical line in more than one place, there is more than one y value corresponding to a single x value.

On a graph, the idea of **single valued** means that no vertical line ever crosses more than one value. If it **crosses more than once** it is still a valid curve, but is **not a function**. The following graph demonstrates the point.

Thus using vertical line test we can identify the graphs that are not functions. For example you can show that the graphs given below are not functions:

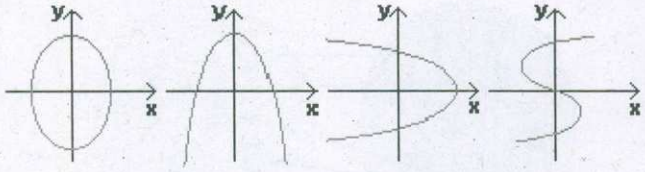


Fig. 3: Graph of not-function

Source: Internet

Thus some of the important equations such as ellipse, circle and parabola are not functions as these not qualify the vertical line test.

Think of the vertical line test this way. The points on the graph of a function f have the form $(x, f(x))$, so once you know the first coordinate, the second is determined. Therefore, there cannot be two points on the graph of a function with the same first coordinate.

2.2.1 Difference between Equation and Function

Before looking at the difference it would be useful to remember that the function concept is a generalisation of the equation concept, allowing us to talk about "curvy lines" without being able to write a specific equation. Not all functions can be expressed in any familiar way as an equation, but all functions can be used to make equations. Thus, both equations and functions use expressions.

- Values of variables in the equations are solved based on the value equated, while values of variables in functions are assigned.
- In a vertical line test, graphs of equations intersect the vertical line at one or two points, while graphs of functions can intersect the vertical line at multiple points.
- Equations always have a graph while some functions cannot be graphed.
- Functions are subsets of equations.

Check Your Progress 1

1) Differentiate between relation and function.

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2) Draw graph of the equation $y^2 = x + 5$ and comment if it is a function.

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3) Differentiate between function and equation.

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2.3 GRAPHICAL REPRESENTATION OF SINGLE VARIABLE FUNCTIONS

A good way of presenting a function is by graphical representation. The most common way to graph a function is to use the **rectangular co-ordinate system**.

It is made up of two number lines:

1) The horizontal number line is the ***x*- axis**.

2) The vertical number line is the ***y*- axis**.

The **origin** is where the two intersect. This is where both number lines are 0. See the following graph for details:

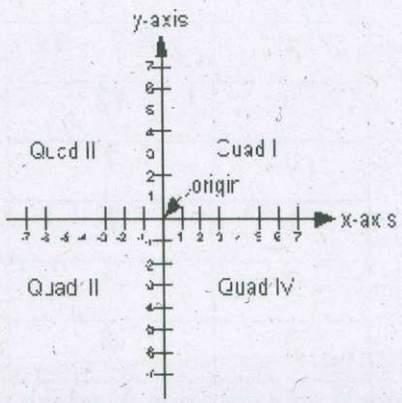


Fig. 4: Graph a Function in Rectangular Co-ordinate System

Source: Internet

It has four **quadrants** marked with Roman numerals.

Each point on the graph is associated with an **ordered pair**. When dealing with an x, y graph, the x **coordinate is** always first and the y **coordinate** is always second in the ordered pair (x, y). It is a solution to an equation in two variables. Even though there are two values in the ordered pair, be careful that it associates to only one point on the graph, **the point lines up with both the x value of the ordered pair (x-axis) and the y value of the ordered pair (y-axis).**

Normally, the values of the **independent** variable (generally the x-values) are placed on the horizontal axis, while the values of the **dependent** variable (generally the y-values) are placed on the vertical axis.

The x-value, called the **abscissa**, is the perpendicular distance of P from the y-axis.

The y-value, called the **ordinate**, is the perpendicular distance of P from the x-axis.

The values of x and y together, written as (x, y) are called the **co-ordinates** of the point P. The center of the coordinate system (where the lines intersect) is called the origin. The axes intersect when both x and y are zero. The coordinates of the origin are (0, 0).

Such a formulation is called the "rectangular" coordinate system because the scale used along the x-axis is evenly spaced, as is the scale along the y-axis. Other systems exist where the scale is not even (e.g., log-log and semi-log graphs) and some are even circular (i.e., Polar Coordinates).

2.4 GRAPH OF LINEAR TYPES OF FUNCTIONS

Writing a function as $y = f(x)$, we take the elements of the domain (x_i) as independent variable and that of range y_i as dependent variable. It helps record the values of x and y while plotting the graphs. For example, consider a linear function depicting the relationship between price(x) and market demand (y) for a commodity. To plot the graph for evaluating the relationship, it is convenient to prepare a table called, t chart, comprising the values of x and y first. If a function takes the form of $\boldsymbol{y = 7 - 5x}$, then t chart is as under:

Table: t-chart of $\boldsymbol{y = 7 - 5x}$

x	$y = 7 - 5x$
-1	12
0	7
1	2
2	-3
3	-8

This equation is an example of a situation in which you want to be particular about the x-values you pick. Because the x is multiplied by a relatively large value, the y-values grow quickly and the graph is:

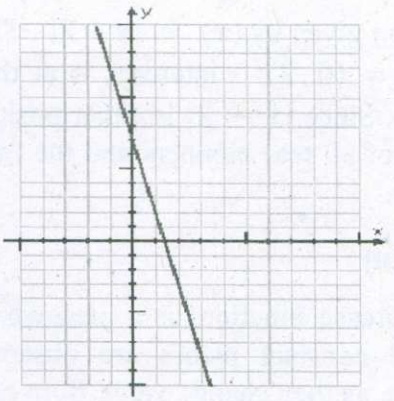


Fig. 5: Graph of Function $y = 7 - 5x$

The graph has x-intercept on a point where y is zero and y-intercept on a point where x is zero. For example to find the x and y intercepts of the equation $3x + 4y = 12$ proceed as follows:

To find the x-intercept, set $y = 0$ and solve for x. That is,

$3x + 4(0) = 12\ or, x = 12/3 = 4$. Similarly to find the y-intercept, set $x = 0$ and solve for y.

That is, $3(0) + 4y = 12\ or,\ y = 12/4 = 3$. Thus the x-intercept is (4, 0) and the y-intercept is (0, 3). The graph of the line is given below.



Fig. 6: Graph of $3x + 4y = 12$

2.4.1 Absolute Value Function

Absolute Value Function is given as f(x) = |x| indicating that we need to consider modulus $|x|$ of a real number keeping it as the non-negative value without regard to its sign. That is, $|x| = x$ for a positive x, $|x| = -x$ for a negative x (in which case $-x$ is positive), and $|0| = 0$. Thus, the absolute value of 2 is 2, and the absolute value of -2 is also taken as 2. The graph of such as function is as follows:

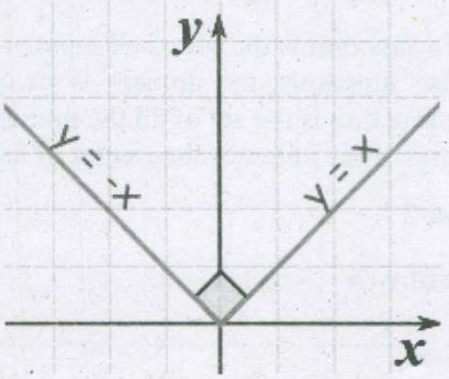


Fig. 7: Graph of Absolute Value

Example: f is a function given by $(x) = |x - 2|$. The y intercept is given by $(0, f(0)) = (0, |-2|) = (0, 2)$; x intercept is at the point $(2, 0)$ since we solve for $|x - 2| = 0$. Since $|x - 2|$ is either positive or zero for x = 2; the domain of f is the set of all real numbers and the range of f is given by the interval $[0, +infinity)$.

2.4.2 Step Function

A step function (or staircase function) is a piecewise function containing all constant "pieces". The constant pieces are observed across the adjacent intervals of the function, as they change value from one interval to the next. A step function is discontinuous (not continuous). You cannot draw a step function without removing your pencil from your paper.

Features of step functions are: open circles and/or closed circles on the graph (open = point not on graph; closed = point is on graph);

- horizontal "pieces"
- discontinuous (cannot be drawn without removing your pencil from the paper)
- may or may be a function.
- Domain: all reals; Range: all integers; y intercept = 0; x-intercept: [0,1)

Consider the function $f(x) = \begin{cases} -3; & x < 2 \\ 0; & -2 \le x \le 1 \\ 3; & x > 1 \end{cases}$. Its graph is given below:

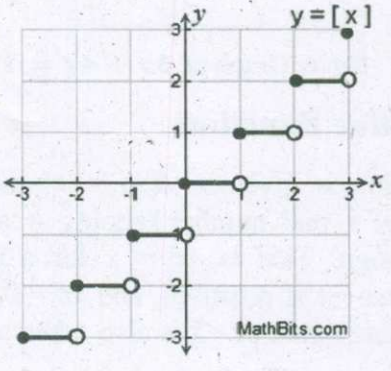


Fig. 8: Graph of Step Function

Note: The *domain* of a function is the set of all inputs for which the function is defined. For piecewise functions, the domain is union of all the individual cases. The *range* of a function is the set of all the possible function outputs. For piecewise functions, this is the union of the ranges of all the individual cases.

Check Your Progress 2

1) Draw the graph of $y = (-5/3)x - 2$.

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2) Find when $f(x) - 2|3x + 2| - 12$ crosses the x-axis.

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3) A truck rental company, charges a rental fee of Rs. 25 plus Rs. 0.30 per mile for driving between 0 and 500 miles for the standard -sized truck. For the same truck, the company charges Rs.100 plus Rs. 0.15 per mile for driving more than 500 miles, but less than 1000 miles. Write a step function that represents the situation.

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2.5 GRAPHING FUNCTIONS – CURVY TYPES

Determining the nature of the function that produces curves requires additional considerations compared to linear types. When you graphed straight lines, you only needed two points to graph the line, though you generally plotted three or more points just to be on the safe side. However, three points will almost certainly *not* be enough points for graphing curves of quadratic or cubic functions. We discuss the caution to be exercised for sketching the graphs of such function.

2.5.1 Even and Odd Functions

A function f is **even** if its graph is symmetric with respect to the $y - axis$. This criterion can be stated algebraically as follows: f is even if $f(-x) = f(x)$ for all x in the domain of f. For example, if you evaluate f at 3 and at -3, then you will get the same value if f is even. Geometrically, such functions are symmetric about the origin.

We have seen that for an even function $f(x) = f(-x)$ for all x and there is symmetry about y-axis. Thus the graph is

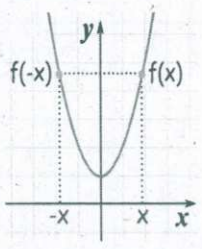


Fig. 9: Graph of Even Function

A function is "odd" when $-f(x) = f(-x)$ for all x and symmetry is observed from the point of origin.

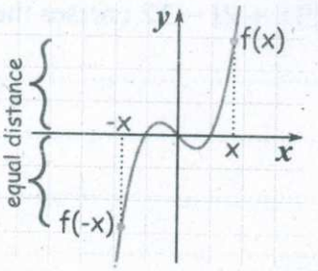


Fig. 10: Graph of Odd Function

Note: The graph based on $f(x) = x^3 - x$

Source: Internet 'Math is Fun'

2.5.2 Quadratic Functions

Quadratic functions help describing demand, cost, revenue and profit which you will find while covering microeconomic analysis. The general technique for graphing quadratics is the same as **for graphing linear equations**. However, since quadratics graph as curvy lines (called "parabolas"), rather than the straight lines generated by linear equations, there are some additional considerations.

The most basic quadratic is $y = x^2$. When you graphed straight lines, you only needed two points to graph your line, though you generally plotted three or more points just to be on the safe side. However, three points will almost certainly *not* be enough points for graphing a quadratic. For example, compute these three points:

Table: T-chart of $y = x^2$

x	$y = x^2$
0	0
1	1
2	4

Then, based only on this experiment the plotted graph will produce a straight line. Thus the graph is not correct presentation of the function. More points need to be considered. By extending the above table with more points we construct the following table.

Table

x	-3	-2	-1	0	1	2	3
y	9	4	1	0	1	4	9

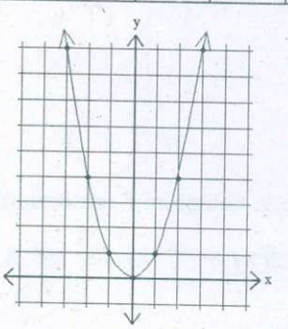
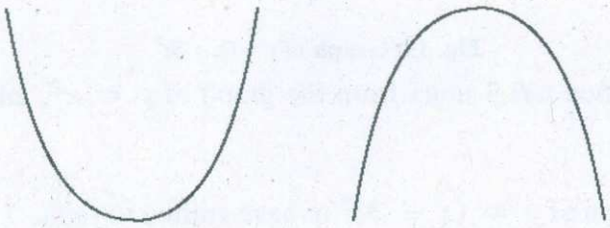


Fig. 11: Graph of Equation $y = x^2$

See that the graph is a parabola.

Note that the parabola does not have a constant slope. In fact, as x increases by 1, starting with $x = 0$, y increases by 1, 3, 5, 7,… . As x decreases by 1, starting with $x = 0$, y again increases by 1, 3, 5, 7,… .

Consider a quadratic function of the form: $f(x) = ax^2 + bx + c$ with $a \neq 0$. The graph of such a function takes one of the two general forms shown in the following figure, depending on the sign of a.



A quadratic function with $a > 0$ | A quadratic function with $a < 0$

Fig. 12: Graph of Quadratic Function

Example: Sketch the graph of function $y = 2x^2 - 8x + 6$. The coefficient of x^2 is positive, so the graph is $U - shaped$. The function is sketched in the following figure.

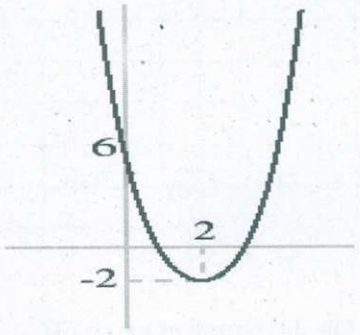


Fig. 13: Graph of Function $y = 2x^2 - 8x + 6$

In the graph of $y = x^2$, the point (0, 0) is called the vertex. The vertex is the minimum point in a parabola that opens upward. In a parabola that opens downward, the vertex is the maximum point. We can also graph a parabola with a different vertex. For example, see the graph of $y = x^2 + 3$:

The graph is shifted up 3 units from the graph of $y = x^2$, and the vertex is (0, 3) and the vertex is (0, - 3).

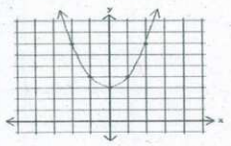


Fig. 14: Graph of $y = x^2 + 3$

We can also shift the vertex left and right. Observe the graph of$y = (x + 3)^2$.

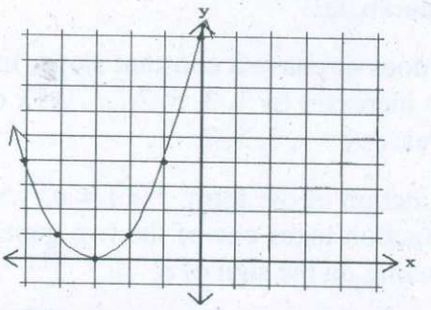


Fig. 15: Graph of $y = (x + 3)^2$

The graph is shifted *left* 3 units from the graph of $y = x^2$, and the vertex is $(-3, 0)$.

Next, see the graph of $y = (x - 3)^2$ to have shifted to right.

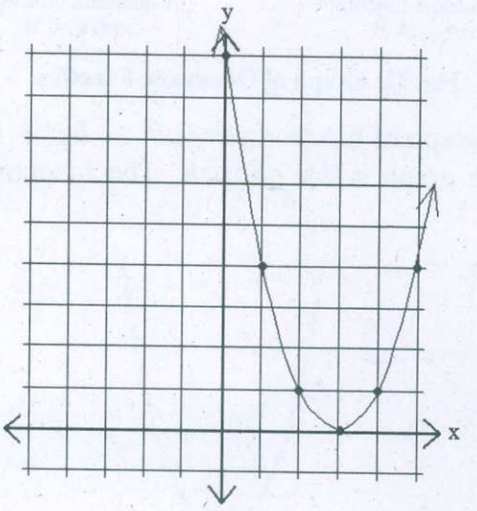


Fig. 16: Graph of $y = (x - 3)^2$

The graph is shifted to the *right* 3 units from the graph of $y = x^2$, and the vertex is $(3, 0)$.

Remember that the axis of symmetry is the line which divides the parabola into two symmetrical halves. It is given by the equation $x = h$. For example, the line of symmetry in the graph of $y = (x - 2)^2 + 1$ is $x = 2$:

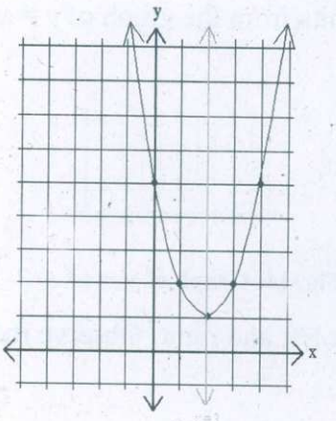


Fig. 17: Axis of Symmetry

2.5.2.1 Computation of Vertex

Consider a generic quadratic expression: $y = ax^2 + bx + c$. Start with **completing the square** on the equation to have $y = a[x^2 + bx/a + c/a]$ or, $y = a[(x + b/2a)^2 - (b/2a)^2 + c/a]$

The expression $-(b/2a)^2 + c/a - (b/2a)^2 + c/a$ is a constant and it does not depend on x. So we can replace it with k. Thus, we write $y = a[(x + b/2a)^2 + k]$

Now, depending on whether a is positive or negative, the parabola given by y will either have a maximum or minimum. Since a and k are fixed, this must occur when $(x + b/2a)^2 = 0$. Hence, $x = -b/2a$ which implies that the function y is at a minimum or a maximum when this is true.

Since parabolas are symmetric over a vertical line through their vertex, let us call that line $x = k$. This means that if the graph crosses the x-axis. That is $ax^2 + bx + c = 0$. To have real solution they must be equidistant from $x = k$, so $(k, 0)$ must be the midpoint of the segment with endpoints at the zeros of the quadratic or k is the average of the zeros. From the quadratic formula, the two zeros of the quadratic are $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ so their sum is $\frac{-b}{a}$ and their average is $k = -\frac{b}{2a}$. So, the $x - coordinate$ of the vertex must be $-\frac{b}{2a}$.

Example: The vertex of $y = 3x^2 + x - 2$ and graph the parabola.

To find the vertex, look at the coefficients a, b, and c. The formula for the vertex gives: $h = \frac{-b}{2a} = \frac{-(1)}{2a} = \frac{-1}{6}$.

Then k is obtained by evaluating y at $h = -1/6$:

$k = 3(-1/6)2 + (-1/6) - 2 = 3/36 - 1/6 - 2 = 1/12 - 2/12 - 24/12 = -25/12$

Since the vertex is at $(-1/6, -25/12$ T-chart can be prepared as follows:

x	$y = 3x^2 + x - 2$
-2	8
-1	0
0	-2
1	2
2	12

The graph can be drawn with the specification vertex as follows:

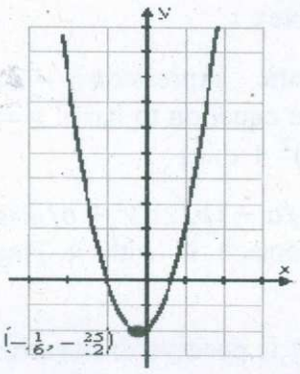


Fig. 18: Graph of $y = 3x^2 + x - 2$

2.5.3 Cubic Function

Cubic functions are encountered in the discussion of total cost in microeconomic analysis. Consider a cubic function of the form $f(x) = ax^3 + bx^2 + cx + d$. Its graph is as follows:

The "basic" cubic function is $f(x) = x^3$. The graph is:

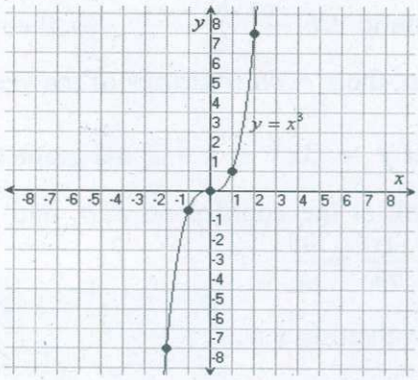


Fig. 19: Graph of Cubic Function

Let f is a cubic function given by $f(x) = x^3$

a) Find the x and y intercepts of the graph of f.

b) Find the domain and range of f.

c) Sketch the graph of f.

a) The y intercept is given by $(0, f(0)) = (0, 0)$. The x coordinates of the x intercepts are the solutions to $x^3 = 0$. The x intercept are at the points (0 , 0). The domain of $f(x)$ is the set of all real numbers.

b) Since the leading coefficient of x^3 is positive, the graph of f is up on the right and down on the left and hence the range of f is the set of all real numbers.

c) Table of values are:

x	-2	-1	0	1	2
$f(x) = x^3$	-8	-1	0	1	8

d) Since f(-x) = - f(x), function f is odd and its graph is symmetric with respect to the origin (0,0).

Check Your Progress 3

1) Graph $y = x^2 - 2x$?

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2) Consider the graph of $y = \beta_0 + \beta_1 x + \beta_2 x^2$ below.

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3) What would be the shape of an odd powered Quadratic Function?

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2.6 GRAPH OF FUNCTIONS – ASYMPTOTIC TYPES

We are familiar with exponential growth models such as population growth and compound interest. The functions representing such behavior will be used to see the graphs that produce.

2.6.1 Graph of Asymptote

An asymptote is, essentially, a line that a graph approaches, but does not intersect. For example, in the following graph of y=1/x, the line approaches the x-axis (y=0), but never touches it. No matter how far we go into infinity, the line will not actually reach y=0, but will always get closer and closer.

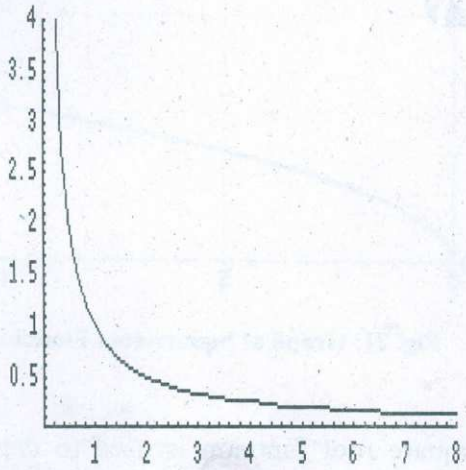
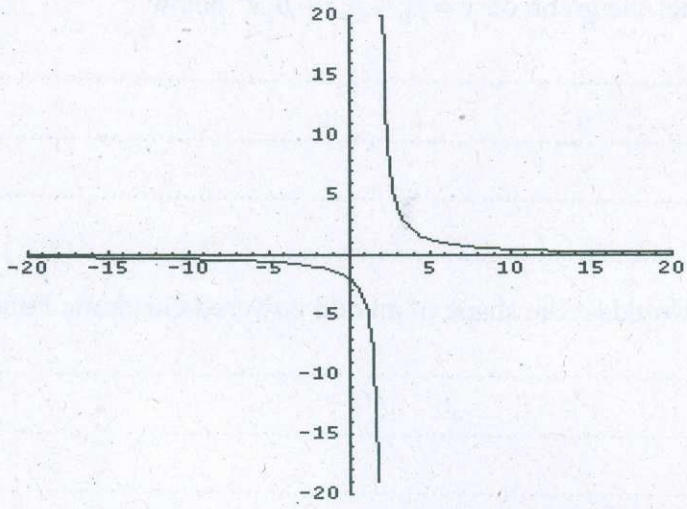


Fig. 20: Graph of Asymptote

This means that the line y=0 is a horizontal asymptote. Horizontal asymptotes occur most often when the function is a fraction where the top remains positive, but the bottom goes to infinity. Going back to the previous example, y=1/x is a fraction. When we go out to infinity on the x-axis, the top of the fraction remains 1, but the bottom gets bigger and bigger. As a result, the entire fraction actually gets smaller, although it will not hit zero. The function will be 1/2, then 1/3, then 1/10, even 1/10000, but never quite 0. Thus, y=0 is a horizontal asymptote for the function y=1/x.

To find vertical asymptotes, look for any circumstance that makes the denominator of a fraction equal zero. For example, consider the equation $\mathbf{y = 4/(x-2)}$.



Note that as the graph approaches x=2 from the left, the curve drops rapidly towards negative infinity. This is because the **numerator** is staying at 4, and the denominator is getting close to 0. That means that the fraction itself is getting very big and negative. When x is exactly 2, the function does not exist because you cannot divide by 0. Immediately after 2 it resumes at positive infinity, because the numerator is 4 and the denominator is again very tiny, but this time it is positive.

2.6.2 Square Root Function

As the name indicates we take a variable x to write the function in the form of $f(x) = \sqrt{x}$. Domain/Range: $[0, \infty)$, intercept (0,0): neither even nor odd; increasing on $(0, \infty)$. Plotting its graph yields the following figure

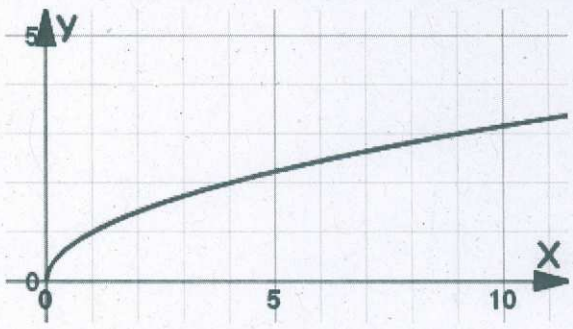


Fig. 21: Graph of Square-root Function

Source: Internet

You may see that square root function is used to depict an *isocline* which provides information on the way different isoquants are related to each other.

In fact the isocline based on square root function converges to a point where output is maximized and marginal product is zero.

2.6.3 Exponential Function

Consider natural exponential of the form $x \rightarrow e^x$ where e is Euler's number, a transcendental number approximately 2.718281828. Since x is an exponent and the exponent can be any real number the domain is all real numbers. The range is the y-values. Since the graph never intersects or goes below the x-axis, the y-values are not zero or negative. Its graph is given below.

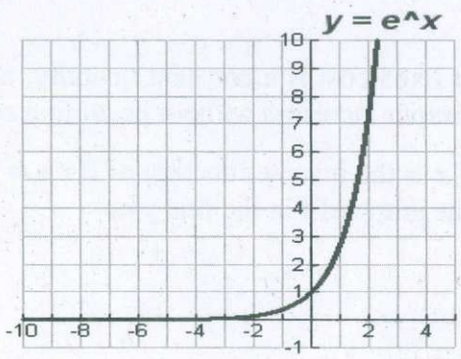


Fig. 22: Graph of Exponential Function

Source: Internet

Three of the most common applications of exponential and logarithmic functions have to do with interest earned on an investment, population growth, and carbon dating. For example, exponential functions are used to describe growth or decay processes with constant growth (or decay rate), population growth, growth of GDP, inflation etc.

2.6.4 Logarithmic Function

The logarithmic function slowly grows to positive infinity as the variable increases and slowly goes to negative infinity as it approaches 0. A function given by $y = log_b x$, where b is any number such that $b > 0, b \neq 1, and\ x > 0$. The function is read "log base b of x". Its graph crosses the x-axis at (1, 0). When $b > 1$, the graph increases; $0 < b < 1$, the graph decreases.

In the logarithmic function, the domain is all positive real numbers (never zero) while the range is all real numbers. The graph is asymptotic to the y-axis - gets very, very close to the y-axis but does not touch it or cross it (see the figure below).

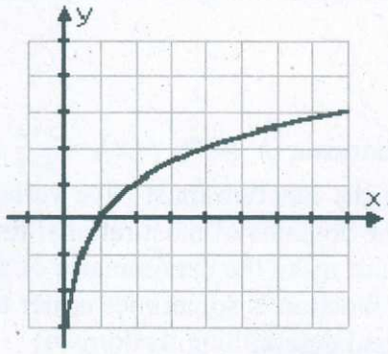


Fig. 23: Graph of Logarithmic Function

Check Your Progress 4

1) Let f be a rational function defined by $f(x) = \frac{x+1}{x-1}$. Find the x and y intercepts of the graph of the function.

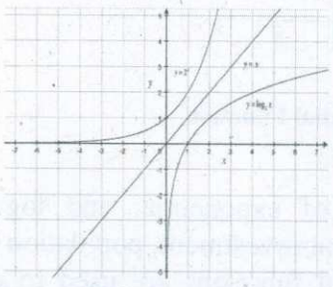
   Ans. The x intercept is found by solving $f(x) = 0$ or, $x + 1 = 0$. The x intercept is at the point $(-1, 0)$. The y intercept is at the point $(0, f(0)) = (0, -1)$.

2) Draw the graph of an average fixed cost curve. Why do you think it is symptotic?

   Ans. Since total fixed cost is a constant quantity, average fixed cost will steadily fall as output increases without becoming zero.

3) If $g(x) = log_2 x$ is the inverse function of $f(x) = 2^x$. The graph of g is a reflection of the graph of f in the line y=x

   Ans.



2.7 GRAPHING FUNCTIONS – PIECES

Up to now in this chapter a function has been represented by a single equation. In many real-life problems, however, functions are represented by a combination of equations, each corresponding to a part of the domain. Such functions are called piecewise functions graph of which will be seen below.

2.7.1 Rational Function

A rational function can be represented as the quotient of polynomials, just as a rational number is a number which can be expressed as a quotient of whole numbers. The graphs of rational functions can be very complicated sporting bumps, holes, and asymptotes.

Consider a rational function that is a quotient of polynomials. It looks like

$$y = \frac{g(x)}{h(x)}$$

where g and h are polynomials ($h \neq 0$). $f(x) = \frac{x+2}{x-5}$ is a rational function. Not that the denominator of the function must have variable to be in this category of function. Note that the domains of most rational functions must be restricted to values of x that will not make the denominator of the function equal to zero. The range of a rational function is sometimes easier to find by first finding the inverse of the function and determining its domain.

Let's sketch the graph of $(x) = \frac{1}{x}$. For drawing the graph we must be careful with the division that produces a 0. In the equation under our consideration we'll have to avoid $x = 0$ since that will give division by zero.

Now, by plugging in some values of x and we get.

Table: T-chart Value

X	f(x)
-4	-.25
-2	-0.5
-1	-1
-.1	-10
-.01	-100
.01	10
1	1
2	0.5
4	0.25

So, as x gets large (positively and negatively) the function keeps the sign of x and gets smaller and smaller. Likewise as we approach $x = 0$ the function again keeps the same sign as x but starts getting quite large. Here is a sketch of this graph

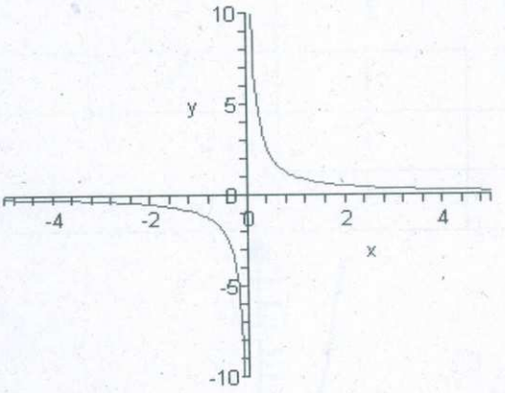


Fig. 23: Graph of Rational Function

You may see that the graph is in two pieces. Most of the rational functions will have graphs in multiple pieces.

As may be seen the graph does not have any intercepts of any kind. Recall that a graph will have a y-intercept at the point $(0, f(0))$. However, in this case we have to avoid $x = 0$ and so this graph will never cross the y-axis. It gets very close to the y-axis, but will never cross or touch it. As a result there will be no y-intercept.

Similarly, we get x-intercepts by solving $f(x) = 0$. In the example above the numerator is one and will never be zero. So the function will have no x-intercepts. Again, the graph will get very close to the x-axis but it will never touch or cross it.

In the graph above as the value of x approaches $x = 0$ the graph starts gets very large on both sides of the line given by $x = 0$. This line is called a **vertical asymptote.**

Also, as x get very large, both positive and negative, the graph approaches the line given by $y = 0$. This line is called a **horizontal asymptote.**

2.7.2 Piecewise Functions

Piecewise functions (split functions) are defined in pieces. So their graphs are drawn in pieces. For instance, suppose we have: $y = \begin{cases} x^2 - 2, x < 1 \\ -2x + 4, x \geq 1 \end{cases}$

Since this has two pieces, you may find it helpful to do two sets of x and y values; if it had more pieces, you could do more charts. The break between the two "halves" of the function (the point at which the function changes rules) is at $x = 1$, so that is where your charts will break. The procedure looks like this:

Table: Chart of $y = \begin{cases} x^2 - 2, x < 1 \\ -2x + 4, x \geq 1 \end{cases}$

x	$y = x^2 - 2$	x	$y = -2x + 4$
-4	14	1	2
-3	7	2	0
-2	2	3	-2
-1	-1	4	-4
0	-2		
1	-1		

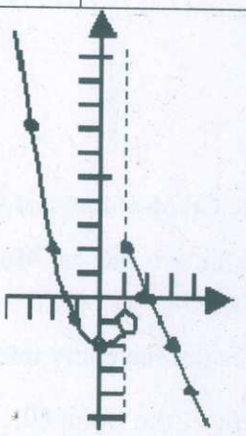


Fig. 24: Graph of $y = \begin{cases} x^2 - 2, x < 1 \\ -2x + 4, x \geq 1 \end{cases}$

2.7.3 Discontinuous Function

A continuous function is a single unbroken curve. You could draw without lifting your pen from the paper.

For example, here is a graph of a continuous function:

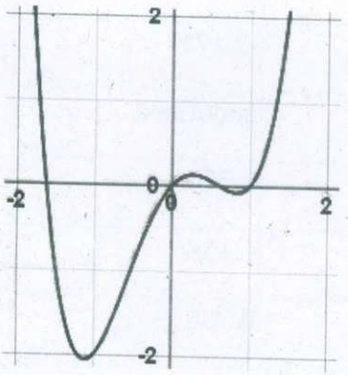


Fig. 25: Graph of Continuous Function

In case of a **discontinuous function** you cannot draw on a graph without lifting your pencil. There are three types of discontinuities:

2.7.3.1 Asymptotic Discontinuities

Consider a function $f(x) = \frac{x+4}{(x-1)(x+8)}$ we know that its domain is limited to all real numbers except 1 and -8. Often, the most interesting points in a function are the problematic points, and indeed, we can see in the graph that the function behaves very strangely at the holes in the domain.

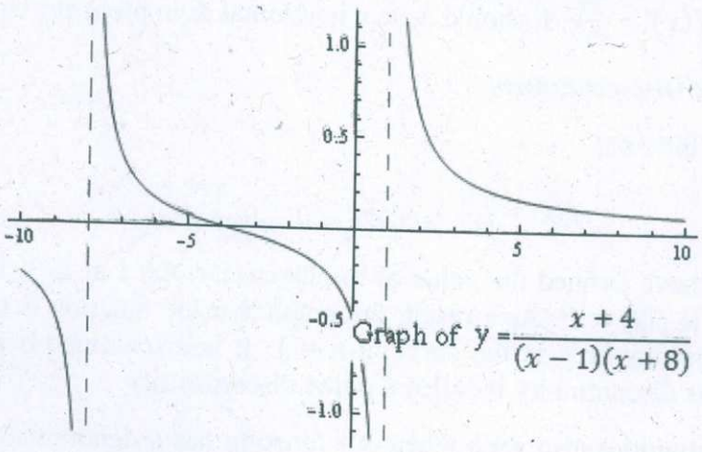


Fig. 26: Graph of Discontinuous Function

Look at the values of the function approaching one of the problematic points, x = 1

Values of y = (x+4)/(x-1)(x+8)	
X	y
0.25	-0.687
0.5	-1.059
0.75	-2.171
1.0	undefined
1.25	2.270
1.5	1.158
1.75	0.786

As we approach x = 1 from the left side, with increasing x values, the y value drops to lower and lower negatives. The graph shows us that the y value seems to approach $-\infty$ from the left as the x-value approaches x = 1, marked by the dotted line. This represents a discontinuity, since the function is not connected over the dotted line. Specifically, this type of discontinuity is called an **asymptotic discontinuity**. The dotted lines represent **asymptotes**; they are values for which the function never takes a value, yet still approaches. The asymptotes we see with this function are called **vertical asymptotes** because they are vertical lines.

There are also **horizontal asymptotes.** To see horizontal asymptotes, we graph the function $f(x) = \frac{2x}{x+4}$. It should have a horizontal asymptote at y = 2.

2.7.3.2 Point Discontinuities

Consider the function

$$f(x) = \begin{cases} 1, & x = 3 \\ x^2, & all\ oter\ real\ values\ of\ x \end{cases}$$

Whereas we have defined the value of the function to be 1 at x=3, in the rest, the function is $f(x) = x^2$. As a result, the graph that the function is continuous except for the tiny hole in the curve at x = 1. It is discontinuous at a single point, and this discontinuity is called a **point discontinuity**.

Point Discontinuities also arise when our function has a denominator that can be equal to zero, but that part of the denominator can also be cancelled out with a like term in the numerator. Consider the function $f(x) = \frac{x^2(x-2)}{x-2}$. If we try to find the value of the function at $x = 2$, we get $f(2) = \frac{2^2(2-2)}{2-2} = \frac{0}{0}$. As $0/0$ represents an undefined number, the function does not exist at that point. However, if we restrict the function to a domain that does not include $= 2$, we can simply cancel out the $\frac{(x-2)}{(x-2)}$ and be left with $f(x) = x^2$. This leaves us to define the function as $f(x) = \frac{x^2(x-2)}{(x-2)} = \begin{cases} x^2, & x \neq 2 \\ undefined, & x = 2 \end{cases}$. See that we

have removed the discontinuity to show that the function behaves exactly like $f(x) = x^2$, except at $x = 2$, where it is undefined.

2.7.3.3 Jump Discontinuities

Consider the function $f(x) = \begin{cases} x^2, & x \leq 1 \\ 2 - x, & x > 1 \end{cases}$. For all intents and purposes, we can consider it two separate functions – one that is defined for x less than or equal to 1 and another function that is defined for x greater than 1. It is useful to think of it this way when we eventually use integration, differentiation, and other such mathematical tools. However, as it is written, f is a single function, called a **piecewise function**, since it is defined piece-by-piece. Note that the function adheres to our definition of being continuous.

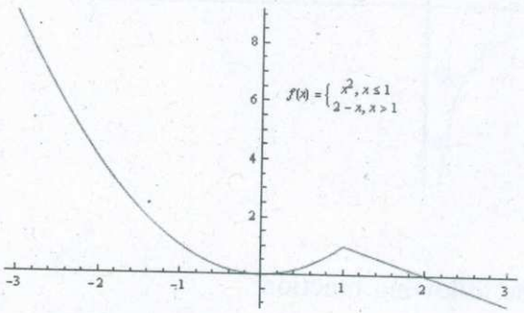
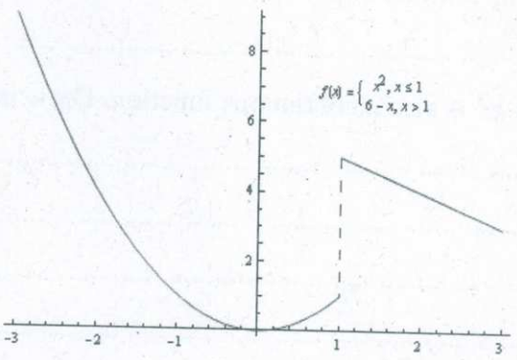


Figure 27: Graph with Discontinuities

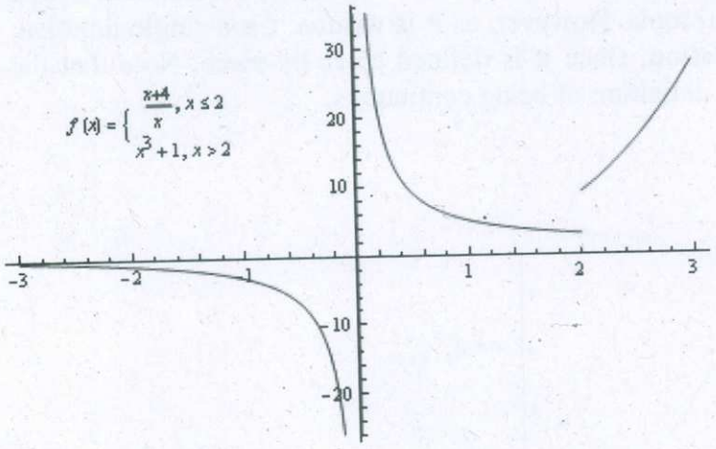
Now, suppose we change our function slightly to $(x) = \begin{cases} x^2, x \leq 1 \\ 6 - x, x > 1 \end{cases}$. The two pieces now have a different value at x = 1, and we can see in the graph that our function f seems to "jump" from one branch to the other. Note this jump makes the function discontinuous. We refer to this as a **jump discontinuity**.



Example 1

We have to check the following function for discontinuities: $f(x) = \begin{cases} \frac{x+4}{x}, & x \leq 2 \\ x^3 + 1, x > 2 \end{cases}$

We begin by looking at the first branch of the function. First, note that it has an x in its denominator. This tells us that it has a problem in its domain at x = 0. As the values for x get very small, the value of the function approaches infinity, so we have an asymptotic discontinuity at x = 0. We also have to check if the branches correspond to the same value at x = 2. The first branch yields $\frac{x+4}{2} = 3$ and the second branch yields $2^3 + 1 = 9$. We can see the functions approach different values at x = 2 so there is a jump discontinuity at x = 2. There are no other discontinuities.



Exercise

Draw the graph of the following function:

$$f(x) = \begin{cases} 2x + 4, & x < 0 \\ -x - 3, & x \geq 0 \end{cases}$$

Check Your Progress 5

1) Draw the graph of $f(x) = -x^2 + 4$.

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2) State why $f(x) = x^3$ is also a continuous function. Draw its graph.

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3) Draw the graph of $f(x) = \frac{1}{x}$. State why it is discontinous.

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2.8 GRAPHING FUNCTIONS – *CURVE* WITH BRANCHES

We put together the functions that have characteristics of curves and pieces. In order to see such graphs we need to see the intersection of a plane with a cone. Graphs of hyperbola and rectangular hyperbola which can be produced from such intersections are presented in the following.

2.8.1 Hyperbola

A hyperbola looks like two mirrored **parabolas**, with the two "halves" being called "branches". The definition of a *hyperbola* is the set of all points where the difference in the distance from two fixed points (called the *foci*) is constant.

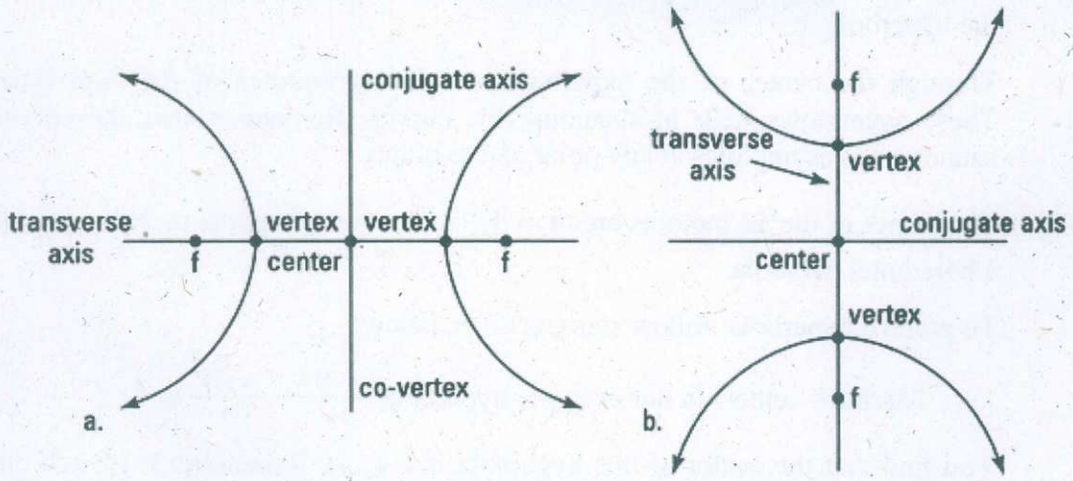
There are two kinds of hyperbolas: horizontal and vertical.

The equation for a horizontal hyperbola is $\frac{(x-h)^2}{a^2} - \frac{(y-v)^2}{b^2} = 1$

The equation for a vertical hyperbola is $\frac{(y-v)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$

Notice that x and y switch places (as well as the h and v with them) to bring in the difference between horizontal and vertical hyperbolas.

The center of a hyperbola is in between the two vertices of the hyperbola. While plotting, the center is taken up first. A hyperbola has two axes of symmetry. The one that passes through the center and the two foci is called the *transverse axis*; the one that's perpendicular to the transverse axis through the center is called the *conjugate axis*. A horizontal hyperbola has its transverse axis at $y = v$ and its conjugate axis at $x = h$; a vertical hyperbola has its transverse axis at $x = h$ and its conjugate axis at $y = v$.



You can see the two types of hyperbolas in the above figure: a horizontal hyperbola on the left, and a vertical one on the right.

If the hyperbola that you are plotting is not in standard form, then you need to complete the square to get it into standard form.

For example, the equation $\frac{(y-3)^2}{16} - \frac{(x+1)^2}{9} = 1$ is a vertical hyperbola. The center (h, v) is $(-1, 3)$.

If $a^2 = 16$, $a = 4$, which means you count in vertical direction the number being under y variable given that $b^2 = 9$ or, $b = 3$. Thus you count horizontally 3 units from the center both to the left and to the right

The distance from the center to the edge of the rectangle marked "a" determines half the length of the transverse axis, and the distance to the edge of the rectangle marked "b" determines the conjugate axis. In a hyperbola, a could be greater than, less than, or equal to b. If you count out a units from the center along the transverse axis, and b units from the center in both directions along the conjugate axis, these four points will be the midpoints of the sides of a very important rectangle. This rectangle has sides that are parallel to the x- and y-axis. This rectangle becomes a useful guide at the time of plotting the graph the hyperbola.

As may be seen from the above figure, hyperbolas contain other important parts. For example a hyperbola has two vertices. There are two different equations — one for horizontal and one for vertical hyperbolas:

- A horizontal hyperbola has vertices at $(h \pm a, v)$.
- A vertical hyperbola has vertices at $(h, v \pm a)$.

The vertices for the above example are at $(-1, 3 \pm 4)$, or $(-1, 7)$ and $(-1, -1)$.

You find the foci of any hyperbola by using the equation $a^2 + b^2 = F^2$

where F is the distance from the center to the foci along the transverse axis, the same axis that the vertices are on. The distance F moves in the same direction as a. As per the example under consideration, $16 + 9 = F^2$ or, $25 = F^2$

To name the foci as points in a horizontal hyperbola, you use $(h \pm F, v)$; to name them in a vertical hyperbola, you use $(h, v \pm F)$. The foci in the example would be $(-1, 3 \pm 5)$, or $(-1, 8)$ and $(-1, -2)$. Note that this places them inside the hyperbola.

Through the center of the hyperbola run the asymptotes of the hyperbola. These asymptotes help in sketching the curves. Remember that the curves cannot cross asymptotes at any point on the graph.

The slopes of the asymptotes are $m = \pm\frac{a}{b}$ for vertical parabola or, $m = \pm\frac{b}{a}$ for a horizontal parabola.

To graph a hyperbola, follow steps as given below:

1) Mark the center: In our example hyperbola $\frac{(y-3)^2}{16} - \frac{(x+1)^2}{9} = 1$

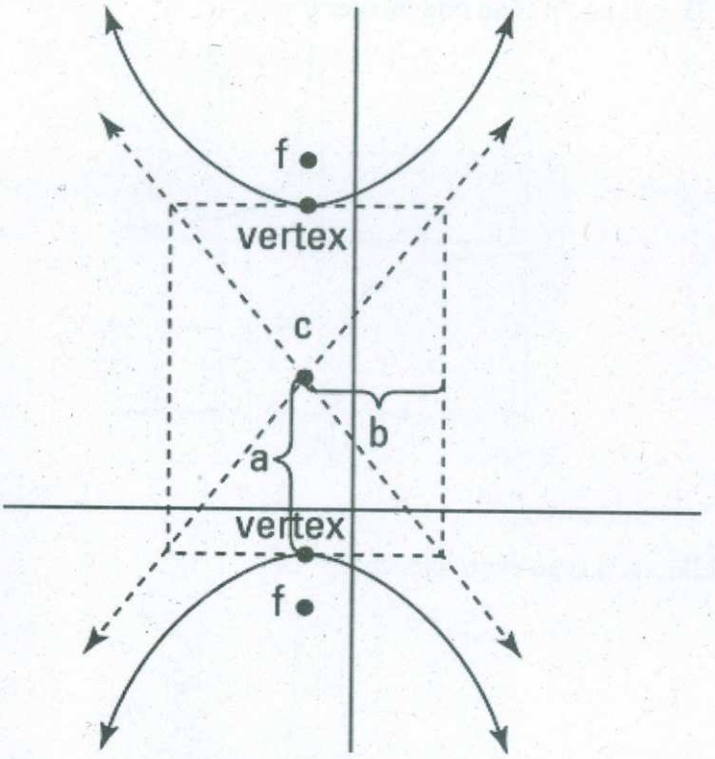
You find that the center of this hyperbola is $(-1, 3)$. Remember to switch the signs of the numbers inside the parentheses, and also remember that h is inside the parentheses with x, and v is inside the parentheses with y.

From the center in Step 1, find the transverse and conjugate axes. Go up and down the transverse axis a distance of 4 (because 4 is under y), and then go right and left 3 (because 3 is under x). The points you marked as a (on the transverse axis) are your vertices.

Use these points to draw a rectangle that will help guide the shape of your hyperbola.

Draw diagonal lines through the center and the corners of the rectangle that extend beyond the rectangle. This gives you two lines that will be your asymptotes.

Sketch the curves.



Draw the curves, beginning at each vertex separately, that hug the asymptotes the farther away from the vertices the curve gets.

Hyperbolas are used extensively in economics and finance (specifically portfolio theory), where they can represent the various combinations of securities, funds, etc., that yield similar risk-return ratios. This is why you often see efficient portfolio frontiers represented as partial hyperbolas. For similar reasons, production frontiers, which represent various combinations of capital and labor that produce a given output, as hyperbolas.

2.8.2 Difference between Hyperbola and Parabola

When a set of points present in a plane are equidistant from the directrix, a given straight line, and are equidistant from the focus, a given point which is fixed, it is called a parabola. When the difference of distances between a set of points present in a plane to two fixed foci or points is a positive constant, it is called a hyperbola.

All parabolas are of the same shape no matter what the size; all hyperbolas are of different shapes

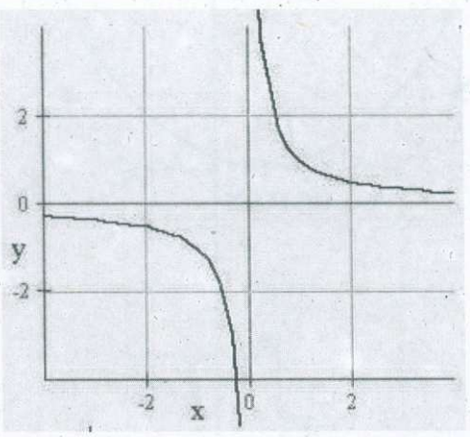
The parabola is given by the equation y2=X; a hyperbola is given by the equation XY=1

In a parabola the two arms become parallel to each other whereas in a hyperbola they do not.

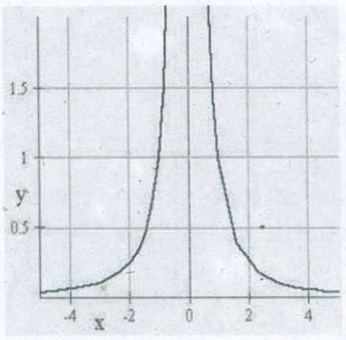
2.8.3 Rectangular Hyperbola

It is a hyperbola with rectangular asymptotes. When the function is in the form $1/x^n$, the graph is a rectangular hyperbola. There are two types:

When n = 2k + 1, i.e., n is an odd integer $y = {}^1/_x$



When n = 2k, i.e. n is an even integer $y = {}^1/x^2$



Properties of the rectangular hyperbola:

1) There is a vertical asymptote at x = 0.
2) There is a horizontal asymptote at y = 0.
3) There are no stationary points.
4) There are no intercepts (since the axes are asymptotes. This is not true when the axes are not asymptotes.)
5) For n = odd integer, the gradient is always decreasing throughout the defined portion of x.
6) For n = even integer, the gradient when x < 0 is always increasing, whilst the gradient when x > 0 is always decreasing.

Example: Draw the graph of $y = \frac{x-1}{x+1}$

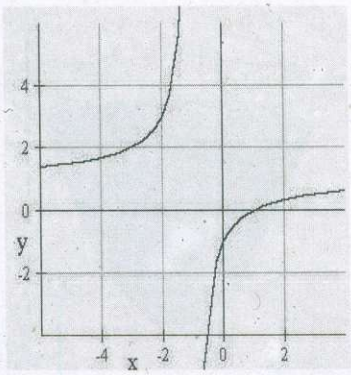
Solution:

1) Intercepts: when x = 0, y = -1 and when y = 0, x = 1
2) Turning point: There are no turning points for this kind of graph.
3) Asymptotes: As x tends to infinity, y tends to 1.

As y tends to infinity, x tends to -1.

x = -1 is a vertical asymptote and y = 1 is a horizontal asymptote.

4) Gradient:The gradient is always increasing.



Rectangular hyperbola is used represent unitary elasticity of demand, average fixed cost, transformation curve and indifference map to name a few examples.

Check Your Progress 6

1) Difference between hyperbola and rectangular hyperbola.

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2) What would be the price elasticity of a rectangular hyperbola shaped demand?

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3) Find the center of this hyperbola of the equation: $\frac{x^2}{81} - \frac{y^2}{9} = 1$

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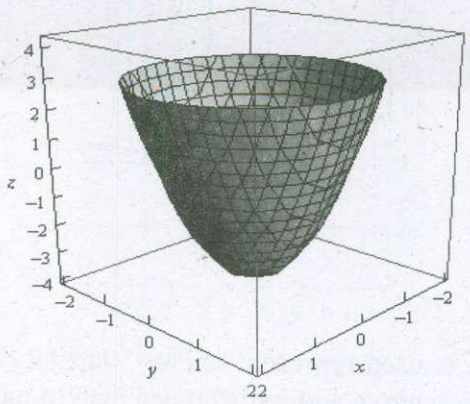
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2.9 GRAPH OF FUNCTIONS WITH TWO VARIABLES: A BRIEF INTRODUCTION

In the above discussion we have seen that for a function of one variable, $y = f(x)$, the graph of f is the set of all points (x, y) in 2-space such that $y = f(x)$. In general, these points lie on a curve in the plane.

Now consider a function of two variables.

The graph of a function of two variables is a surface in three dimensions. Thus we deal with a **three-dimensional graph consisting of** is the graph of a function $f(x, y)$ of two variables, or the graph of a relationship $g(x, y, z)$ among three variables. For example, here is the graph of $z = 2x^2 + 2y^2 - 4$.



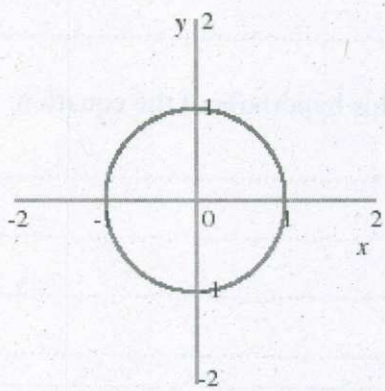
Source: Internet

The information in a three-dimensional graph is often collapsed into a two-dimensional graph with the use of contour lines. The procedure is: The x and y axes are retained, but instead of depicting a z or $f(x, y)$ axis, all x, y combinations giving rise to the same z or $f(x, y)$ value are connected with a contour line; an arbitrary number of these may be shown for various values of z or $f(x, y)$.

2.9.1 Level Curves

Let f be a function of two variables, and c a constant. The set of pairs (x, y) such that $f(x, y) = c$ is called the **level curve** (or, contour lines) of f for the value c.

Example: Let $f(x, y) = x^2 + y^2$ for all (x, y). The level curve of f for value 1 is the set of all pairs (x, y) such that $x^2 + y^2 = 1$, a circle of radius 1. This set is shown in the following figure:



Economists call the level curves of a utility function *indifference curves* and those of a production function *isoquants*. Thus, level curves have algebraic equations of the form $f(x, y) = k$ for all possible values of k. A contour diagram or contour map of a function $f(x, y)$ is a 2-dimensional graph showing several level curves in the xy-plane corresponding to several values of k.

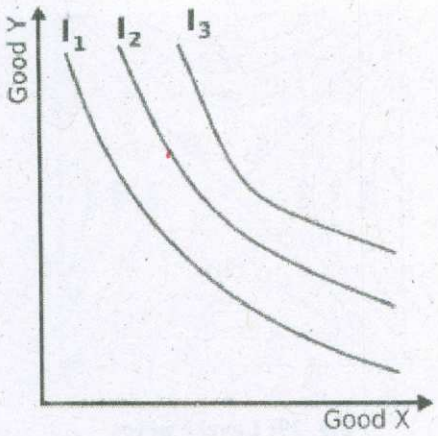


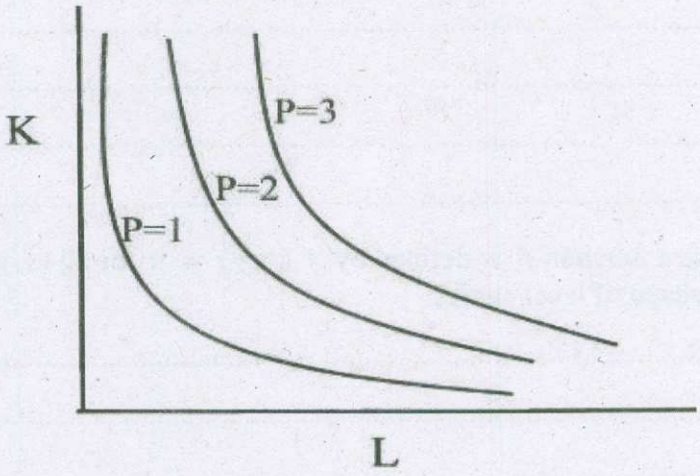
Fig. 28: Indifference Curve Map

Source: Internet wikipedia

Example: Cobb-Douglas Production Model

Given P= the monetary value of all goods produced, K = total capital investment over the same period, and L= the total labor force, the following relationship between these quantities holds: $P = 1.01L^{0.75}K^{0.25}$

The contour diagram shown in figure below shows that reducing labour increases capital investment in order to maintain the same level of production i.e., supposedly more investment is needed in machinery to make up for the lack of available labour. Conversely, increasing labour reduces capital investment since supposedly the work is being done by the labor force.



Example 1

Let $f(x, y) = x^2 - y^2$. We will study the level curves$c = x^2 - y^2$.

First, look at the case c=0. The level curve equation $x^2 - y^2 = 0$ factors to $(x - y)(x + y) = 0$. This equation is satisfied if either $y = x$ or$y = -x$. Both these are equations for lines, so the level curve for $c = 0$ is two lines.

If $c \neq 0$, then we can rewrite the level curve equation $c = x^2 - y^2$as $1 = x^2c - y^2c$.

See that this is the equation for a hyperbola. If c is positive, the hyperbolas open to the left and right. If c is negative, the hyperbolas open up and down.

For example, if $c = 1$, the equation is $x^2 - y^2 = 1$. If $c = -1$, the equation is $x^2 - y^2 = 1$. A number of level curves are plotted below.

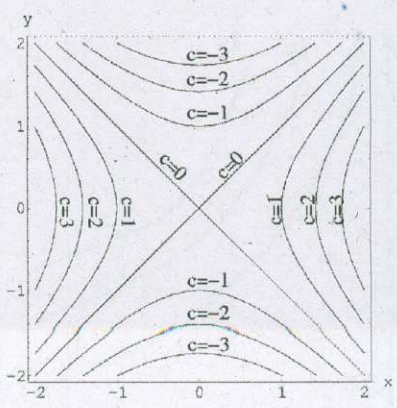


Fig. 29: Level Curves

Check Your Progress 7

1) Explain the meaning of level curve

A curve in two dimensions on which the value of a function f(x,y) is a constant.

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2) Let $f(x, y) = xy$ for all (x, y). Show a collection of level curves of f.

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3) Suppose a function f is defined by $f(x, y) = 1$ for all (x, y). Comment on the shape of level curves.

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2.10 LET US SUM UP

In this unit we discussed the method of sketching graphs of functions. Staring with relationship between elements of domain and co-domain we recalled the basis of functional relationships. An operational method of dealing with function was pointed out to identify dependent and independent variables. Thus we discussed $y = f(x)$, it would be to treat x as the independent variable picked up from among the elements of domain and y to be the dependent variable present as an element in the rage. Such a formulation helped translate the functional relationships into graphical representation in the rectangular co-ordinate system where x-axis would contain the values taken by independent

variables and the y-axis the dependent variables. Before proceeding with producer of sketching graph of a function an attempt was made to point the use of vertical line test which tells some functions like circle and ellipse fail to meet its requirement in the graph and hence could not be considered as functions.

While dealing with the method of drawing the graph, the unit considered functions in five groups, viz., linear, curve, asymptotic, piece-wise and curve with branches. The linear types of graphs were drawn when the functions would be in the nature of linear, absolute value or step-like. In cases of graphs showing curves, functions included were odd and even, quadratic and cubic. Among the graph of asymptotic types, functions included were square root, exponential and logarithmic. Graphs appearing in pieces in the $xy-plane$ comprised rational functions, piecewise functions and discontinuous functions. In the last group of graphs having *curve* with branches functions concerned with hyperbola and rectangular hyperbola were discussed. In the last part of the present unit, graphs of functions with two variables were introduced.

2.11 KEY WORDS

Abscissa: $x-$ (horizontal) coordinate of a point in a two dimensional coordinate system.

Absolute Value Function: Obtaining the *absolute value* (or *magnitude*) of a number.

Asymptote: A line that a curve approaches, as it heads towards infinity.

Asymptotic Discontinuity: Graph approaching a point but never touching the point.

Axis of Symmetry: A line of symmetry for a graph. The two sides of a graph on either side of the axis of symmetry look like mirror images of each other.

Cobb-Douglas Production Function: Production function which is applied to describe how much output two inputs into a production process make.

Contour Diagram (contour map): Graph that provides one way of visualizing functions of two variables.

Co-ordinates: Any of a set of two or more numbers used to determine the position of a point, line, curve, or plane in a space of a given dimension with respect to a system of lines or other fixed references.

Cubic Function: A *polynomial function* of degree 3.

Directrix: A line perpendicular to the axis of symmetry of a parabola.

Discontinuous Function: A function which for certain values or between certain values of the variable does not vary continuously as the variable increases

Even Function: Univariate functionis $f(x)$ is said to be even provided that $f(x)=f(-x)$.

Exponential Function: Function defined by $\exp(z)=z^z$.

Hyperbola: A hyperbola is a curve where the distances of any point from: a fixed point (the focus), and a fixed straight line (the directrix) are always in the same ratio.

Indifference Curve Map: A group of *indifference curves* for two commodities showing different levels of satisfaction.

Isoquants: A graph of all possible combinations of inputs that result in the production of a given level of output.

Jump Discontinuity: A function that jumps from one location to another at a certain point.

Level Curves: A contour diagram or contour map of a function f(x; y) is a 2-dimensional graph showing several level curves in the $xy - plane$.

Logarithmic Function: A function (as $y = \log_a x$ or $y = \ln x$) that is the inverse of an exponential function (as $y = a^x$ or $y = e^x$) so that the independent variable appears in a logarithm.

Log-log Graph: A two-dimensional graph of numerical data that uses logarithmic scales on both the horizontal and vertical axes.

Odd Function: Univariate function $f(x)$is said to be odd provided that $f(-x) = -f(x)$.

Ordinate: Element of an ordered pair which is plotted on the vertical axis of a two-dimensional Cartesian coordinate system.

Piecewise Function (split function): A function can be in Pieces.

Point Discontinuity: Graph of the function has a hole at $x = a$.

Polar Coordinates: A two-dimensional coordinate system in which each point P on a plane is determined by a distance r from a fixed point O that is called the pole (or origin) and an angle $\theta$from a fixed direction.

Quadratic Function: A function that can be described by an equation of the form $f(x) = ax^2 + bx + c$, where $a \neq 0$.

Rational Function: A function that is the ratio of two polynomials.

Rectangular Co-ordinate System: Two real number lines that intersect at a right angle.

Rectangular Hyperbola: A hyperbola for which the asymptotes are perpendicular.

Semi-log Graph: Graph defined by a logarithmic scale on the y-axis, and a linear scale on the x-axis.

Square Root Function: function that maps the set of non-negative real numbers onto itself.

Step Function: A function whose graph looks like a series of steps.

Vertex: A synonym for a node of a graph.

Vertical Line Test of Function: a vertical line anywhere on a graph so that it hits the graph in more than one spot, then the graph is not a function.

2.12 SOME USEFUL BOOKS

Alpha C Chiang and Kevin Wainwright (2005), *Fundamental Methods of Mathematical Economic Analysis*, McGraw-Hill Bank Company London.

Badal Mukherji and V. Pandit (1982), *Mathematical Methods for Economic Analysis*, Allied Publishers Pvt. Ltd. , New Delhi

R.G.D. Allen (2009), *Mathematical Analysis for Economists*, MacMillan, India Limited, Delhi.

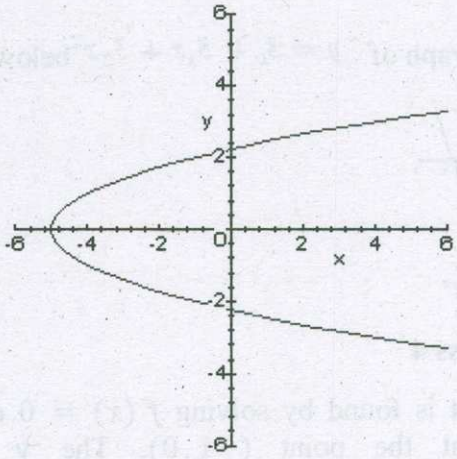
Taro Yamane (2012), *Mathematics for Economists: An Elementary Survey*, Prentice Hall of Indian, New Delhi.

2.13 ANSWER OR HINTS TO CHECK YOUR PROGRESS

Check Your Progress 1

1) A relation is a set of numbers that have a relationship through the use of a domain and a range, while a function is a relation that has a specific set of numbers that causes there to be only be one range of numbers for each domain of numbers.

2) The graph is



By the vertical line test, this graph is not the graph of a function, because there are many vertical lines that hit it more than once.

3) In an equation assignment of a value to one variable will give a value for the other. For example, in $x + y = 7$, if $x = 4$, then $y = 3$. The value of y will change if x is assigned a different value. Whereas a function is an equation that has only one answer for y for every x. It assigns exactly one output to each input of a specified type.

Check Your Progress 2

1) First prepare the t-chart which is

Table: t-chart of $y = (-5/3)x - 2$

x	$y = (-5/3)x - 2$
-6	8
-3	-2
0	2
3	-7

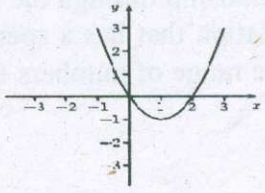
Then you draw the graph.

2) $x - \frac{4}{3}$ or $x - \frac{-8}{3}$

3) $f(x) = \begin{cases} 25 + 0.30x, & 0 < x \leq 500 \\ 100 + 0.15x, & 500 < x < 1000 \end{cases}$

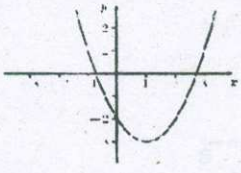
Check Your Progress 3

1)



2)

Consider the graph of $y = \beta_0 + \beta_1 x + \beta_2 x^2$ below.



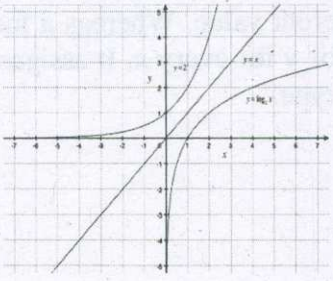
3) A curve like *'s'*.

Check Your Progress 4

1) The x intercept is found by solving $f(x) = 0$ or, $x + 1 = 0$. The x intercept is at the point $(-1, 0)$. The y intercept is at the point $(0, f(0)) = (0, -1)$.

2) Since total fixed cost is a constant quantity, average fixed cost will steadily fall as output increases without becoming zero.

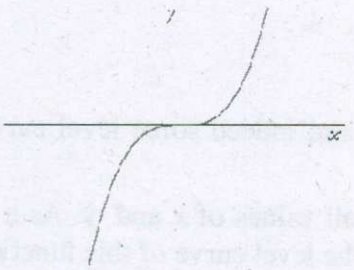
3)



Check Your Progress 5

1) The function can be written $f(x) = ax^2 + c$, where $a = -1$ and c= 4. Since a is -1, the graph is reflected across the x -axis. Since $c = 4$ the graph is translated up 4 units. So, the graph of $y = -x^2 + 4$ is the graph of $y = x^2$ reflected across the x-axis and translated up 4 units. Now you draw the graph.

2)



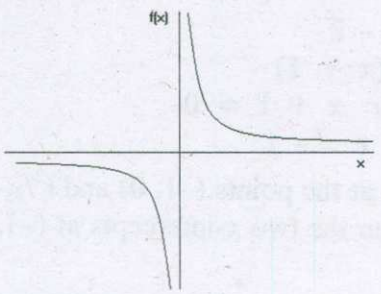
$y = x^3$

The cubic function

3)

$f(x) = \frac{1}{x}$ is a discontinuous function. Either it has a value at which $f(x)$ does not exist or has an asymptote. It is discontinuous at $x=0$ because $f(x)$ is undefined there.

The discontinuity is visible in the graph.



Check Your Progress 6

1) The rectangular hyperbola in general is defined as a hyperbola which has orthogonal asymptotic lines.

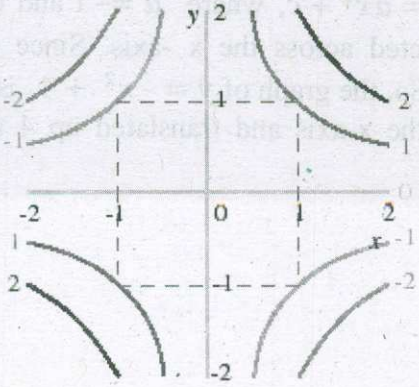
2) Price elasticity equal to unity.

3) Hint: write the standard form of a hyperbola with horizontal transverse axis.

Check Your Progress 7

1) A curve in two dimensions on which the value of a function f(x,y) is a constant.

2) The following figure may be seen. The number beside each curve is the value of the function to which it corresponds. The level curve for the value 0 consists of the axes.



3) Ans. A level curve is defined as a *set*, and indeed some level curves of some functions are not "curves" at all.

In the given function the value of f is 1 for all values of x and y. As a result you get straight lines, not curves. Moreover, the level curve of this function for the value 2 is empty as there are *no* values of (x, y) such that $f(x, y) = 2)$.

2.14 EXERCISES AND ANSWERS

1) Find the vertex and intercepts of $y = 3x^2 + x - 2$ and graph; identify the vertex and the axis of symmetry.

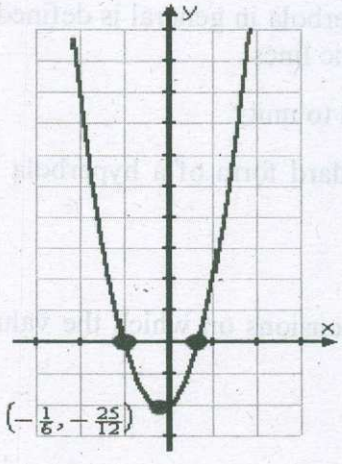
To find the y-intercept, set x equal to zero and solve: $y = 3(0)^2 + (0) - 2 = 0 + 0 - 2 = -2$

Then the y-intercept is the point $(0, -2)$. To find the x-intercept, set y equal to zero, and solve:

$$0 = 3x2 + x - 2$$
$$0 = (3x - 2)(x + 1)$$
$$3x - 2 = 0 \text{ or } x + 1 = 0$$
$$x = 2/3 \text{ or } x = -1$$

From the above, the x-intercepts are at the points $(-1, 0)$ and $(^2/_3, 0)$. The axis of symmetry is halfway between the two x-intercepts at $(-1, 0)$ and at $(^2/_3, 0)$.

The complete answer is a listing of the vertex, the axis of symmetry, and all three intercepts, along with a nice neat graph:



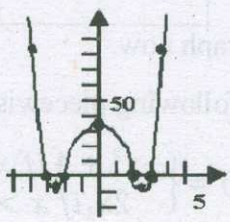
The vertex is at $(^{-1}/_{6}, ^{-25}/_{12})$, the axis of symmetry is the line $x = ^{-1}/_{6}$, and the intercepts are at (0, –2), (–1, 0), and $(^{2}/_{3}, 0)$.

2) Let $y = x^4 - 13x^2 + 36$. Draw its graph.

Ans. This is a positive even power ("to the fourth"), so the graph will be up on both ends. Factoring the polynomial, we get $y = (x + 3)(x - 3)(x + 2)(x - 2)$, so the zeroes ($x$-intercepts) are –3, –2, 2, and 3. If you plot a few other points on your T-chart, it will be no trouble to graph this:

x	$y = x^4 - 13x^2 + 36$
-4	84
-2.5	-6.19
-1	24
0	36
1	24
2.5	-6.19
4	84

Fig.: Graph of $y = x^4 - 13x^2 + 36$



Source: Internet

3) A quadratic function is given as $y = x^2 - 6x + 5$. Draw its graph.

Ans. We need to plot more than three points to get the correct shape. Find the intercepts, which in this case are at (1, 0), (5, 0), and (0, 5); find the vertex, which in this case is at (3, –4. Remember that parabolas look like "smilies", not like straight lines. So we want to plot a few more points:

Table: Chart of $y = x^2 + 6x + 5$

x	$y = x^2 + 6x + 5$
0	5
1	0
3	-4
5	0
6	5

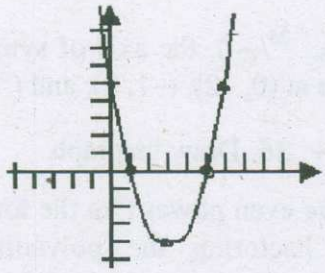


Fig.: Graph of $y = x^2 + 6x + 5$

4) Draw the graph of the following function:

$$f(x) = \begin{cases} 2x + 4, & x < 0 \\ -x - 3, & x \geq 0 \end{cases}$$

5) Sketch the graph of $\boldsymbol{f(x) = (x-1)^3 + 3}$

Ans. We need to pick values of x to plug in. Here are the function evaluations.

x	$f(x)$	(x, y)
-1	-7	(-1,-7)
0	0	(0,0)
1	1	(1,1)
2	2	(2,2)
3	9	(3,9)

You can sketch its the graph now.

6) Sketch the graph of the following piecewise function.

$$g(x) = \begin{cases} -x^2 + 4 \; if \; x < 1 \\ 2x_1 if \; x > 1 \end{cases}$$

Ans.

Recall that graphing piecewise functions are like graphing several functions at once, except we are only going to graph them on very specific intervals. So we will graph the following two functions,

$$-x^2 on \; x < 1$$

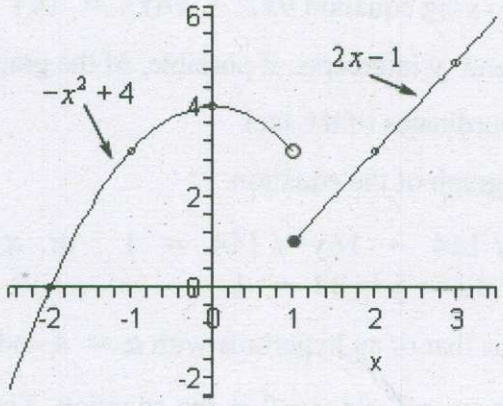
$$2x - 1 \; on \; x \geq 1$$

Use $x = 1$ in both functions not overlooking to the point at $x = 1$ since technically that will only be valid for the bottom function. The table of values for each function on the specified range is:

MEC-103

x	$f(x) = -x^2 + 4$	(x, y)
-2	0	(-2,-0)
-1	3	(-1,3)
0	4	(0,4)
1	3	(1,3)
x	$2x - 1$	(x, y)
1	1	(1,1)
2	3	(2,3)
3	5	(3,5)

Sketch the graph. For the top function we used an open dot for the point at $x = 1$ and for the bottom function we used a closed dot at $x = 1$. In this way we make it clear on the graph that only the bottom function really has a point at $x = 1$.



Note that two graphs didn't meet at $x = 1$ and there a blank space in the graph.

7) Sketch the graph of the following piecewise function.

$$h(x) = \begin{cases} x + 3 \; if \; x < 2 \\ x^2 \; if -2 \leq x < 1 \\ -x + 2 \; if \; x \geq 1 \end{cases}$$

Ans. In this case we will graph three functions on the ranges given above. So, as with the previous example we will get function values for each function in its specified range and we will include the endpoints of each range in each computation. When we graph we will acknowledge which function the endpoint actually belongs with by using a closed dot as we did previously. Also, the top and bottom functions are lines and so we don't need more than two points for these two. We'll get a couple more points for the middle function.

x	$x+3$	(x,y)
-3	0	(-3, 0)
-2	1	(-2,1)
x	x^2	(x,y)
-2	4	(-2,4)
-1	1	(-1,1)
0	0	(0,0)
1	1	(1,1)
x	$-x+2$	(x,y)
1	1	(1,1)
2	0	(2,0)

8) Given the following equation $9x2 - 16y2 = 144$

a) Find the x and y intercepts, if possible, of the graph of the equation.

b) Find the coordinates of the foci.

c) Sketch the graph of the equation.

Ans: a) $9x^2 / 144 - 16y^2 / 144 = 1$ or, $x^2 / 16 - y^2 / 9 = 1$ or, $x^2 / 42 - y^2 / 32 = 1$

The equation is that of an hyperbola with $a = 4$ and $b = 3$.

To get the x intercept, set y = 0 in the equation. Thus, $x^2 / 4^2 = 1$and $x = \mp 4$.

For the y intercept set x = 0. So $-y^2 / 3^2 = 1$. Without any real solution of the equation we do not get y intercept.

b) Find c first.

As $c^2 = a^2 + b^2$, from (a) values of a and b can be put to obtain

$c^2 = 4^2 + 3^2$ or, $c = \mp 5$

Thus foci are F_1 (5 , 0) and F_2 (-5, 0).

c) (i) Find the asymptotes $y = -(b/a)\,x$ and $y = (b/a)\,x$ and plot them.

Thus, $y = -(3/4)\,x$ and $y = (3/4)\,x$

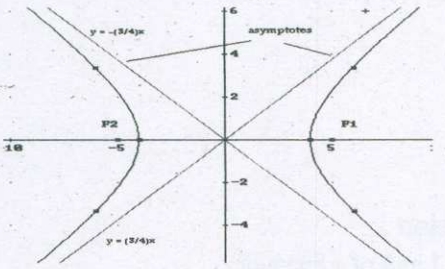
Then plot the x intercepts

(ii) set $x = 6$ and find y such that $9(6)^2 - 16y^2 = 144$ or, $-16y^2 = 144 - 324$

Or, $y^2 = 45 / 4$

Or, $y = 3(5)^{\frac{1}{2}} / 2$ and $y = -3(5)^{\frac{1}{2}} / 2$.

Graph of the hyperbola is,



9) Suppose you have following rational function: $y = \frac{2x^2-18}{x^2-4}$. Draw its graph.

Ans.: Before you draw a T-chart, first find the asymptotes and intercepts. Then choose $x's$ between the x-intercepts and vertical asymptotes, to give you the additional information necessary to graph the function.

The vertical asymptotes are at $x = 2$ and $x = -2$, the horizontal asymptote is at $y = 2$, the x-intercepts are at $x = -3$ and $x = 3$, and the y-intercept is at $y = 4.5$. Now plot a few additional points between these other points:

Table: Chart Value of $y = \frac{2x^2-18}{x^2-4}$

x	$y = \frac{2x^2-18}{x^2-4}$
-5	1.5
-2.5	-2.4
-1	5.3
1	5.3
2.5	-2.4
5	1.5

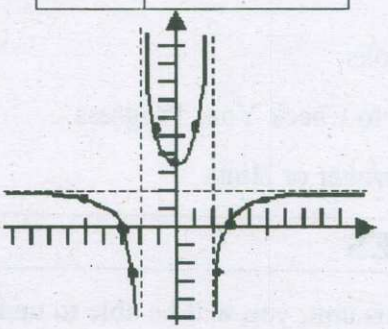


Fig.: Graph of $y = \frac{2x^2-18}{x^2-4}$

UNIT 3 DIFFERENTIAL CALCULUS FUNCTIONS, LIMIT AND CONTINUITY

Structure

3.0 Objectives

3.1 Introduction

3.2 Limit of a Function
- 3.2.1 Right Hand Limit of a Function
- 3.2.2 Left Hand Limit of a Function
- 3.2.3 Functions Tending to Infinity
- 3.2.4 Fundamental Theorems on Limit

3.3 Continuity

3.4 Derivative
- 3.4.1 Geometrical Interpretation of Derivatives
- 3.4.2 Differentials
- 3.4.3 Higher Order Derivatives
- 3.4.4 Application of Simple Derivatives in Economics
- 3.4.5 Slope and Curvature
- 3.4.6 Taylor Series
- 3.4.7 Mean Value Theorem and L'Hospital's Rule

3.5 Partial Differentiation
- 3.5.1 Concept of Partial Derivative
- 3.5.2 Partial Derivative Defined
- 3.5.3 Higher Order Partial Derivatives
- 3.5.4 Cross- Partial Derivatives
- 3.5.5 Total Differential
- 3.5.6 Total Derivative
- 3.5.7 Differentiation of Functions
- 3.5.8 Rules Regarding Implicit Functions
- 3.5.9 Application in Economics
- 3.5.10 Homogeneous Functions and their Properties

3.6 Let Us Sum Up

3.7 Key Words

3.8 Some Useful Books

3.9 Answer or Hints to Check Your Progress

3.10 Exercises and Answer or Hints

3.0 OBJECTIVES

After going through this unit, you will be able to understand the notions of :

- limit and continuity and differentiation of a function; and
- their forms of simple derivation and presentation.

3.1 INTRODUCTION

The present discussion is devoted to introduce you to some of basic concepts, which are adopted widely in the formulation of problems and analysing the results. We will discuss the ideas of limit, continuity and differentiation. We do not intend to enter the technicalities of language as well as derivation of these. To present these themes, therefore, we take the help of examples.

3.2 LIMIT OF A FUNCTION

Limit is the basic concept which allows us to describe and analyse the study of continuous change. Using limit of a function we try to describe the behaviour of the function when the variable is near but is not equal to a specified number. Thus by the phrase 'x tends to the value a', we mean $|x - a|$ is gradually smaller and smaller. Then we can write this as $x \to a$.

If the successive values of a are always greater than a, then we say 'x tends to a from right' and write this as $x \to a + 0$ or $x \to a+$. Similarly, if the successive values of a are always less than a, then we say 'x tends to a from left' and write it as $x \to a - 0$ or $x \to a^-$.

When x approaches a constant quantity a from either side, if there exists a definite finite number t towards which $f(x)$ tends to. The numerical difference of $f(x)$ and t can be made as small as we can think by taking x sufficiently close to a. Then t is defined as the limit of $f(x)$ as x tends to a. This is written as $\lim_{x \to a} f(x) = t$

Analytically, $\lim_{x \to \infty} f(x) = t$ means, given any pre-assigned positive quantity ϵ (as small as we like), we can find another positive quantity δ such that $|f(x) - t| < \epsilon \ \forall$ x such that $0 < |x - a| \leqslant\leqslant \delta$.

[$\forall$ means 'for all']

Example:

i) $\lim_{x \to 1} \dfrac{x^2 - 1}{x - 1} = 2$.

If $x = 1$ ($\delta >$ or < 0) ,then $\frac{x^2-1}{x-1} = 2 + \delta$

Taking δ small enough, the difference between $\dfrac{x^2 - 1}{x - 1}$ and 2 can be made as small as we like. Hence, limit of the function is 2. However, when $x = 1$, the function is non-existent.

ii) $\lim_{x \to 1} (2x - 1) = 1$

$|f(x) - 1| = |2x - 2|$. Let ϵ be a very small positive quantity. Then $|f(x) - 1| < \epsilon$ if $0 < |x - 1| < \dfrac{\varepsilon}{2}$. Take $\dfrac{\varepsilon}{2} = \delta > 0$, then $|f(x) - 1| < \epsilon$ if $0 < |x - 1| < \delta$

3.2.1 Right Hand Limit of a Function

Given any pre-fixed positive quantity ϵ (however small), we can determine a positive quantity δ such that $|f(x) - t_1| < \epsilon$ whenever $0 < x - a \leq \leqslant \delta$, i.e., $a < x \leq \leqslant \delta$.

This **right hand limit** is also denoted by the symbol $f(a + 0)$.

The above limit is so called because as x approaches the value a from the right (i.e., from bigger values), the numerical difference between $f(x)$ and t_1 can be made as small as we please by taking x closer to a, keeping x always greater than a.

3.2.2 Left Hand Limit of a Function

When a quantity t_2 can be obtained such that given any pre-assigned quantity ϵ, however small, we can determine a positive quantity δ, so that $|f(x) - t_2| < \epsilon$ whenever $0 < a - x \leqslant \leq \delta$, i.e., $a - \delta \leq \leqslant x < a$.

The **left hand limit** is also denoted by the symbol $f(a-0)$. When $\lim_{x\to a+0} f(x) = \lim_{x\to a-0} f(x)$, each of these is equal to $\lim_{x\to a} f(x)$, i.e., for $\lim_{x\to a} f(x)$ to exist, each of $\lim_{x\to a+0} f(x)$ and to exist, each of $\lim_{x\to a+0} f(x)$ and $\lim_{x\to a-0} f(x)$ must exist and must be equal.

Example: Does $lim_{x\to 0} \frac{|x|}{x}$ exist?

Take $lim_{x\to 0^+} \frac{|x|}{x} = lim_{x\to 0^+} \frac{|x|}{x} = 1$. This is not equal to $lim_{x\to 0^-} \frac{|x|}{x} = lim_{x\to 0^-} \frac{|x|}{x} = -1$.

Thus, $lim_{x\to 0} \frac{|x|}{x}$ does not exist.

NOTE: $f(0)$ does not exist.

3.2.3 Functions Tending to Infinity

If corresponding to any pre-assigned quantity N, however large, we can determine a positive quantity δ, such that $f(x) > N$, whenever $0 < |x - a| \leqslant \leq \delta$, we say $\lim_{x\to a} f(x) = \infty$.

Similarly, we can define $\lim_{x\to a+0} f(x) = \infty$, $\lim_{x\to a-0} f(x) = \infty$;

$\lim_{x\to a+0} f(x) = -\infty$, $\lim_{x\to a-0} f(x) = -\infty$

Example:

i) $\lim_{x\to 0+0} \frac{1}{x^3} = \infty$, $\lim_{x\to 0-0} \frac{1}{x^3} = -\infty$ Hence, $\lim_{x\to 0+0} \frac{1}{x^3}$ does not exist.

Limit of a function as the variable tends to infinity

$\lim_{x\to\infty} f(x) = t$, provided, given any pre-assigned positive quantity ϵ, however small, we can determine a positive quantity M, such that $|f(x) - t| < \epsilon$ for all values of $x > M$.

Similarly, $\lim_{x\to\infty} f(x) = t$, provided, given any pre-assigned positive quantity ϵ, however small, we can determine a positive quantity M, such that $|f(x) - t| < \epsilon$ whenever $-x > M$. In a similar way, we may define

$$\lim_{x\to\infty} f(x) = \infty, \quad \lim_{x\to-\infty} f(x) = -\infty$$

Example:

$$\lim_{x\to\infty} \frac{1}{x} = 0, \quad \lim_{x\to-\infty} \frac{1}{x} = 0 \quad ; \qquad \lim_{x\to\infty} e^x = \infty, \quad \lim_{x\to-\infty} x^3 = -\infty$$

3.2.4 Fundamental Theorems on Limit

Let $\lim_{x\to a} f(x) = t_1$ and $\lim_{x\to a} \phi(x) = t_2$ where t_1 and t_2 are finite quantities. Then,

i) $\lim_{x\to a}\{f(x) \pm \varnothing(x)\} = t_1 \pm t_2$; ii) $\lim_{x\to a}\{f(x).\varnothing(x)\} = t_1.t_2$

iii) $\lim_{x\to a}\left\{\frac{f(x)}{\varnothing(x)}\right\} = \frac{t_1}{t_2}$ $(t_2 \neq 0)$; iv) $\lim_{x\to a} F[f(x)] = F[\lim_{x\to a} f(x)]$

v) If $\emptyset(x) < f(x) < \Psi(x)$ in a certain neighbourhood of point 'a' and $\lim_{x\to a} \varnothing(x) = t$ and $\lim_{x\to a} \psi(x) = t$ then $\lim_{x\to a} f(x) = t$.

vi) If $\lim_{x\to a} \varnothing(x) = t_1$ and $\lim_{x\to a} \psi(x) = t_2$ and if $\emptyset(x) < \psi(x)$ in a certain neighbourhood of a except a, then $t_1 \leq t_2$.

The first three theorems may be extended for any finite number of functions.

Some important limits:

(Statement without proof)

i) $\lim_{x\to 0} \frac{\sin x}{x} = 1$, x is expressed in radian

ii) a) $\lim_{x\to\pm\infty} \left(1 + \frac{1}{x}\right)^x = e$, b) $\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e$

iii) $\lim_{x\to 0} \frac{1}{x}\log(1+x) = 1$, iv) $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$

v) $\lim_{x\to a} \frac{x^n - a^n}{n - a} = na^{n-1}$, for all rational values of n, provided $a > 0$.

vi) $\lim_{x\to 0} \frac{(1+x)^n - 1}{x} = n$, vii) $\lim_{x\to\infty} \frac{x^n}{\lfloor n} = 0$

Example:

i) $$\lim_{x\to 0} \frac{a_0x^n + a_1x^{n-1} + \ldots + a_n}{b_0x^n + b_1x^{n-1} + \ldots + b_n} = \frac{\lim_{x\to 0}(a_0x^n + a_1x^{n-1} + \ldots + a_n)}{\lim_{x\to 0}(b_0x^n + b_1x^{n-1} + \ldots + b_n)}$$

3.2.1 Right Hand Limit of a Function

Given any pre-fixed positive quantity ϵ (however small), we can determine a positive quantity δ such that $|f(x) - t_1| < \epsilon$ whenever $0 < x - a \leq \leq \delta$, i.e.,

$$= \frac{\lim\limits_{x\to 0} a_0x^n + \lim\limits_{x\to 0} a_1x^{n-1} + \ldots + \lim\limits_{x\to 0} a_n}{\lim\limits_{x\to 0} b_0x^n + \lim\limits_{x\to 0} b_1x^{n-1} + \ldots + \lim\limits_{x\to 0} b_n} = a_n/b_n \text{ [since } \lim_{x\to 0} x^n = 0]$$

ii) $\lim\limits_{x\to\pi} \dfrac{\sin x}{x-\pi}$. Put $x - \pi = z$ as $x \to \pi$, $z \to 0$ so that

$$\lim_{z\to 0} \frac{\sin(z+\pi)}{z} = \lim_{z\to 0} \frac{\sin z\cos\pi - \cos z\sin\pi}{z} =$$

$$\lim_{z\to 0} \frac{\sin z}{z}(-1) = -1$$

iii) $$\lim_{z\to\infty} \frac{1+x^2}{1+x^3} = \lim_{z\to\infty} \frac{\frac{1}{x^3}+\frac{1}{x}}{\frac{1}{x^3}+1} = \frac{\lim\limits_{x\to\infty}\frac{1}{x^3} + \lim\limits_{x\to\infty}\frac{1}{x}}{\lim\limits_{x\to\infty}\frac{1}{x^3}+1} = \frac{0+0}{0+1} = 0$$

iv) $$\lim_{x\to\infty}\left(\frac{1}{n^2}+\frac{2}{n^2}+\ldots+\frac{n}{n^2}\right) = \lim_{x\to\infty}\left(\frac{1+2+\ldots+n}{n^2}\right)$$

$$= \lim_{x\to\infty}\frac{n(n+1)}{2n^2} = \lim_{x\to\infty}\frac{1}{2}\left(1+\frac{1}{n}\right) =$$

$$\frac{1}{2}\left(1+\lim_{x\to\infty}\frac{1}{n}\right) = \frac{1}{2}(1+0) = \frac{1}{2}$$

v) $\lim\limits_{x\to 2}\sqrt{x-2}$ Here $\lim\limits_{x\to 2-0}\sqrt{x-2}$ does not exist. Hence, the limit does not exist.

Check Your Progress 1

1) Evaluate the following limits:

i) $\lim\limits_{x\to 0}\dfrac{\sqrt{a+x}-\sqrt{a-x}}{x}$

ii) $\lim\limits_{x\to 0}\sin x = 0$

iii) $\lim\limits_{x\to\theta}\sin x = \sin\theta$

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2) i) $f(x) = 10 \quad x > 1$

$= 1 \quad x = 1$

$= -10 \quad x < 1$

Find $\lim\limits_{x\to 1} f(x)$

ii) $f(x) = 1 \quad x \neq 1$

$= -1 \quad x = 1$

Find $\lim_{x \to 0} f(x)$

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3) i) $\lim_{x \to 0} \frac{\sin x^2}{x}$

ii) $\lim_{x \to 0} \frac{\text{cosec } x - \cot x}{x}$

iii) $\lim_{x \to \infty} \frac{\sin x}{x}$

iv) $f(x) = ax^2 + bx + c$,

show that $\lim_{n \to 0} \left\{ \frac{f(x+n) - f(x)}{n} \right\} = 2ax + b$

v) $\lim_{n \to \infty} \frac{1^2 + 2^2 + \ldots + n^2}{n^3}$

vi) $\lim_{n \to 0} \frac{\tan^{-1} x}{x}$

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3.3 CONTINUITY

Intuitively, a function $f(x)$ is **continuous** provided its graph is continuous, i.e., a **continuous** function does not have any break at any point of its graph.

More formally, a function $f(x)$ is said to be **continuous** for $x = a$, provided $\lim_{x \to a} f(x)$ exists, finite and is equal to a.

That is, for $f(x)$ to be **continuous** at $x = a$, we should have $\lim_{x \to a} f(x) = \lim_{x \to a-0} f(x) = f(a)$

If $f(x)$ is **continuous** for every value of x in its **domain**, it is said to be **continuous** throughout the interval.

A function, which is not continuous at a point, is said to have a **discontinuity** at that point.

Analytically, the function $f(x)$ is **continuous** at $x = a$, provided $f(a)$ exists and given any pre-assigned positive quantity ε, however small, we can determine a positive quantity δ such that $|f(x) - f(a)| < \varepsilon$ for all values of x such that $\delta \leq \leqslant x \leqslant \leq a + \delta$.

i) If $f(a + 0) \neq f(a - 0)$ then $f(x)$ is said to have an **ordinary discontinuity** at $x = a$.

Example: $\lim_{x \to a} \sqrt{x - a}$. In this case, $f(a)$ may or may not exist.

ii) If $f(a + 0) = f(a - 0) \neq f(a)$, or $f(a)$ does not exist, $f(x)$ is said to have a **removable discontinuity** at x = a.

Example: In $f(x) = \dfrac{x^2 - 1}{x - 1}$, $\lim_{x \to 1} f(x) = 2$, $f(1)$ does not exist.

A function with **removable discontinuity** at a point $x = a$ can be made **continuous** by defining the function at the particular point, e.g., $f(1) = 1$.

i) and

ii) are together called **simple discontinuities**.

iii) If one or both $f(a + 0)$ or $f(a - 0)$ tend to infinity, then $f(a)$ is said to have an **infinite discontinuity** at $x = a$. Here $f(a)$ may or may not exist; e.g., $f(x) = \dfrac{1}{x}$, $\lim_{x \to 0+0} f(x) \to \infty$

iv) There may be **oscillatory discontinuity** at a point where the function may oscillate finitely or oscillate infinitely and does not tend to a limit or $\pm \infty$.

Example:

a) $f(x) = (-1)^x$, $\lim_{x \to \infty} f(x)$ oscillates finitely

b) $f(x) = (-2)^x$, $\lim_{x \to \infty} f(x)$ oscillates infinitely

Some properties of continuous functions

i) The sum or difference of two continuous function is a continuous function. This result is valid for any finite number of functions.

ii) The product of two continuous functions is a continuous function. This result is also valid for any finite number of functions.

iii) The quotient of two continuous functions is a continuous function, provided the denominator is not zero anywhere for the range of values considered.

iv) If $f(x)$ is continuous at $x = a$ and $f(a) \neq 0$, then in the neighbourhood of x = a, f(x) has the same sign as that of $f(a)$, i.e., we can get a positive quantity δ such that $f(x)$ preserves the same sign as that of $f(a)$ for every value of $f(x)$ in the interval $a - \delta < x < a + \delta$.

v) If $f(x)$ is continuous throughout the interval (a, b), and $f(a)$ and $f(b)$ are of opposite signs, then there is at least one value, say ξ of x within the interval for which $f(\xi) = 0$.

vi) Let $f(x)$ is continuous throughout the interval (a, b) and $f(a) \neq f(b)$. Then $f(x)$ assumes every value between $f(a)$ and $f(b)$ at least once in the interval.

vii) A function, which is continuous throughout a closed interval, is bounded therein.

viii) A continuous function in an interval actually attains its upper and lower bounds, at least once each, in the interval.

ix) A function $f(x)$, continuous in a closed interval $[a, b]$, attains every intermediate value between its upper and lower bounds in the interval, at least once.

Example:

i) $f(x) = x^n$, n is any rational number.

$\lim_{x \to a} x^n = a^n$ for all n, except when $a = 0$ and $n < 0$. Thus, x^{n^n} is continuous for all x when n is positive and continuous for all values of x except 0 when n is negative.

ii) $f(x) = \tan x = \dfrac{\sin x}{\cos x}$

$\lim_{x \to a} \sin x = \sin a$ and $\lim_{x \to a} \cos x = \cos a$ – both are continuous.

tan x is also continuous except when $\cos x = 0$, i.e., $x = (2n + 1)\dfrac{x}{2}$.

iii) $f(x) = x$ for $x > 0$

0 for $x = 0$

$-x$ for $x < 0$

Prove that $f(x)$ is continuous at $x = 0$.

iv) $f(x) = \dfrac{x^2 - 4}{x - 2}$. What should the value of $f(2)$ be, so that $f(x)$ is continuous at $x = 2$.

Check Your Progress 2

1) i) $f(x) = x \sin\frac{1}{x}$ for $x \neq 0$. Examine the continuity of $f(x)$at $x = 0$.

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ii) Find the point of discontinuity of the following function: $f(x) = \frac{x^2-2x+4}{x^2-5x+6}$

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iii) $f(x) = \frac{x^2-4}{x-2}$. What should be the value of $f(2)$, so that $f(x)$ is continuous at x=2?

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3.4 DERIVATIVE

The **derivative** of a function measures the function's rate of change. To understand the concept, consider the **increment** of a variable (x) in changing from one value to another is the **difference** obtained by subtracting the first value from the second. An **increment** of x is denoted by Δx (delta x) or h.

If in a function $y = f(x)$, the independent variable x takes an increment Δx(or h), then Δy(or k) denotes the corresponding increment of y, i.e., of $f(x)$.

Thus, we have, $y + \Delta y = f(x + \Delta x)$ i.e., $\Delta y = f(x + \Delta x) - f(x)$

or, $y + k = f(x + h)$ i.e., $k = f(x + h) - f(x)$

Example: Let $y = +\sqrt{x}$

Suppose, x increases from 4 to 4.1, i.e., $\Delta x = 0.1$, then y increases from 2 to 2.025 (approx), i.e., $\Delta y = 0.025$. Suppose x decreases from 4 to 3.9, i.e., $\Delta x = -0.1$, then y decreases from 2 to 1.975, i.e., $\Delta y = 0.025$.

Increments are always reckoned from the arbitrarily fixed initial value of the independent variable x.

From a fixed initial value 4 of x, if x increases successively to 4.1, 4.01, 4.001, etc., then although the corresponding increments $\Delta x = (0.1, 0.01, 0.001, \ldots)$ and $\Delta y = (0.025, 0.2, 0.0002, \ldots)$ are getting smaller, their ratio $\frac{\Delta y}{\Delta x}$ being 0.025, 0.2, 0.02, … (approx.) is approaching a definite number 0.25. Thus, the ratio of the increments can be brought as near to 0.25 as we want by making Δx approaching zero.

Let $y = f(x)$ be a finite function defined in any interval of x and assume x to have any particular value in the interval. Let Δx(or h) be the increment of x, and let Δy (or, k) $= f(x + \Delta x) - f(x)$ be the corresponding increment of y. If the ratio $\frac{\Delta y}{\Delta x}$ of these increments tends to a definite finite limit as Δx tends to zero, then this limit is called the **differential coefficient** (or **derivative**) of

$f(x)$ (or, y) for the particular value of x and is denoted by $f'(x), \frac{d}{dx}\{f(x)\}$, $\frac{dy}{dx}$ or $D\{f(x)\}$ i.e., $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} =$ $\lim_{x \to 0} \frac{f(x+h) - f(x)}{x}$ (provided this limit exists).

If as $\Delta x \to 0, \frac{\Delta y}{\Delta x} \to \pm\infty$, we also say the **derivative** exists.

The process described above is called **differentiation** of the function $f(x)$ **with respect to** the independent variable x.

A function to be **differentiable** at a point must be **continuous** at that point. However, if a function is continuous at a point, it may not be differentiable at that point.

Example: $f(x) = |x|$

This function is continuous at $x = 0$. But $f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$ $= \lim_{h \to 0} \frac{f(h)}{h}$

$\lim_{h \to 0+0} \frac{f(h)}{h} = \lim_{h \to 0+0} \frac{h}{h} = 1$ or, $\lim_{h \to 0-0} \frac{f(h)}{h} = \lim_{h \to 0-0} \frac{-h}{h} = -1$

Thus, $f'(0)$ does not exist at $x = 0$.

A function may be discontinuous at a point, but may have infinite **derivative** at that point.

Example:
$$\begin{cases} f(x) = 0, & when\ 0 \le x < \frac{1}{2} \\ = 1, & when\ x = \frac{1}{2} \\ = 2, & when\ \frac{1}{2} < x \le 1 \end{cases}$$

Now, $\lim_{h \to ½+0} f(x) = 2$ and $\lim_{h \to ½-0} f(x) = 0$

Thus, $f(x)$ is not continuous at $x = ½$. But,

$$f'\left(\frac{1}{2}\right) = \lim_{h \to 0} \frac{f\left(\frac{1}{2}+h\right) - f\left(\frac{1}{2}\right)}{h}$$

$$= \lim_{h \to 0} \frac{f\left(\frac{1}{2}+h\right) - \frac{1}{2}}{h}$$

$$\lim_{h\to 0+0}\frac{f\left(\frac{1}{2}+h\right)-\frac{1}{2}}{h}=\lim_{x\to 0+0}\frac{2-\frac{1}{2}}{h}=\infty\ ;$$

$$\lim_{h\to 0-0}\frac{f\left(\frac{1}{2}+h\right)-\frac{1}{2}}{h}=\lim_{x\to 0-0}\frac{2-\frac{1}{2}}{h}=\infty$$

Thus, $f'(½)$ exists and infinite.

The **right hand limit** $\lim_{h\to 0+0}\frac{f(x+h)-f(x)}{h}$ for any particular value of x (when exists) is known as the **right hand derivative** of $f(x)$ at that point and is denoted by $Rf'(x)$. Similarly, the **left hand limit** $\lim_{h\to 0-0}\frac{f(x+h)-f(x)}{h}$ is known as the **left hand derivative** of $f(x)$ at x and is denoted by $Lf'(x)$. Only when both $Rf'(x)$ and $Lf'(x)$ exist and are equal, $f'(x)$ exists at x.

Now, we illustrate by a simple example how to find the **derivative** when a particular functional form is specified.

Take $f(x)=2x^2$ such that $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\Rightarrow$

$$\lim_{h\to 0}\frac{2\left(x^2+2xh-h^2\right)-2x^2}{h}=\lim_{h\to 0}\frac{2h(2x-h)}{h}=\lim_{h\to 0}(2x-h)$$
$$=4x-2\lim_{h\to 0}h=4x$$

Below are the **derivatives** (without derivation from the 'first principle') for some standard functional specifications.

i) $\frac{d(ax^n)}{dx}=anx^{n-1}$ (for all rational n)

ii) $\frac{d(e^{mx})}{dx}=me^{mx}$ (for all rational m)

iii) $\frac{d(a^x)}{dx}=a^x\log_e a$

iv) $\frac{d}{dx}(\log x)=\frac{1}{x}$

v) $\frac{d}{dx}(\sin x)=\cos x$ (x is radian)

vi) $\frac{d}{dx}(\cos x)=-\sin x$ (x is radian)

vii) $\frac{d}{dx}(\tan x)=\sec^2 x$ ($x\neq ½(2n+1)\pi$)

viii) $\frac{d}{dx}(\cot x)=-\operatorname{cosec}^2 x$ ($x\neq n\pi$)

ix) $\frac{d}{dx}(\sec x) = \sec x \tan x \qquad (x \neq \frac{1}{2}(2n+1)\pi)$

x) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x \qquad (x \neq n\pi)$

Now, we can state some important **rules of differentiation**.

I) $\frac{d}{dx}(c) = 0$, where c is constant.

II) $\frac{d}{dx}[c\phi(x)] = c\frac{d}{dx}\{\phi(x)\}$, where c is constant.

III) $\frac{d}{dx}[\phi(x) \pm \psi(x)]$

$= \frac{d\phi(x)}{dx} \pm \frac{d}{dx}\psi(x)$

The above result can be extended for any finite number of functions.

IV) $\frac{d}{dx}[\phi(x).\psi(x)] = \phi'(x).\psi(x) + \phi(x).\psi'(x)$

This result can also be extended for any finite number of functions.

V) $\frac{d}{dx}\left\{\frac{\phi(x)}{\psi(x)}\right\} = \frac{\phi(x).\psi(x) - \psi(x).\phi(x)}{\{\psi(x)\}^2}$

VI) Let, $y = f(v)$ and $v = \varnothing(x)$. $f(v)$ and $\varnothing(x)$ are continuous (i.e., y is a continuous function of x),

Then $\frac{dy}{dx} = \frac{dy}{dv}\frac{dv}{dx}$

This rule is called **differentiation** of **function of a function**.

In general, let $y = f(v)$; $v = \phi(w)$, $w = \psi(x)$, then, $\frac{dy}{dx} = \frac{dy}{dv}\frac{dv}{dw}\frac{dw}{dx}$.

An implication of the above rule is $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$, provided both the derivatives exist.

Also, let $y = \psi(t)$, $x = \phi(t)$, then, $\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} \left(\frac{dx}{dt} \neq 0\right)$

Here, the third variable t is known as the **parameter**.

We shall illustrate the importance of the above mentioned results with some examples.

Examples: Find $\frac{dy}{dx}$ in the following cases:

i) $y = \sin 2x - 2e^x + 4^4$

ii) $y = 3x^2 \operatorname{cosec} x$

iii) $y = \dfrac{\log x - \sin x}{2x^2 - 1}$

iv) $y = 2\cos v + \tan^2 v$

$v = e^{3x}$

v) $y = a\cos\phi\,,\ x = b\sin\phi$

Some Notes:

i) Algebrical or trigonometrical transformations sometimes prove easier to perform a complicated differentiation.

Example:

a) Let $y = (\operatorname{cosec} x)^{\cot x}$, then $\log y = \cot x \log \operatorname{cosec} x$

Differentiating both sides with respect to x,

$$\frac{1}{y}\cdot\frac{dy}{dx} = -\operatorname{cosec}^2 x \log \operatorname{cosec} x + \frac{\cot x}{\operatorname{cosec} x}(-\operatorname{cosec} x \cot x)$$

$$= -(\operatorname{cosec}^2 x \log \operatorname{cosec} x + \cot^2 x)$$

$$\therefore \frac{dy}{dx} = -(\operatorname{cosec} x)^{\cot x}(\operatorname{cosec}^2 x \log \operatorname{cosec} x + \cot^2 x)$$

b) If $y = \tan^{-1}\dfrac{\sqrt{1+\sin x} - \sqrt{1-\sin x}}{\sqrt{1+\sin x} + \sqrt{1-\sin x}}$ then $y = \tan^{-1}\dfrac{1-\cos x}{\sin x}$,

$$= \tan^{-1}\frac{2\sin^2\frac{1}{2}x}{2\sin\frac{1}{2}x\cos\frac{1}{2}x} = \tan^{-1}(\tan\frac{1}{2}x) = \frac{1}{2}x$$

$$\therefore \frac{dy}{dx} = \frac{1}{2}$$

ii) Derivative of a determinant

Let $\Delta(x) = \begin{vmatrix} f_1(x) & \phi_1(x) & \psi_1(x) \\ f_2(x) & \phi_2(x) & \psi_2(x) \\ f_3(x) & \phi_3(x) & \psi_3(x) \end{vmatrix}$, then

$$\Delta'(x) = \begin{vmatrix} f_1'(x) & \phi_1(x) & \psi_1(x) \\ f_2'(x) & \phi_2(x) & \psi_2(x) \\ f_3'(x) & \phi_3(x) & \psi_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & \phi_1'(x) & \psi_1(x) \\ f_2(x) & \phi_2'(x) & \psi_2(x) \\ f_3(x) & \phi_3'(x) & \psi_3(x) \end{vmatrix} +$$

$$\begin{vmatrix} f_1(x) & \phi_1(x) & \psi_1'(x) \\ f_2(x) & \phi_2(x) & \psi_2'(x) \\ f_3(x) & \phi_3(x) & \psi_3'(x) \end{vmatrix}$$

Example:

$$\Delta(x) = \begin{vmatrix} \sin x & \cos x & \sin x \\ \cos x & -\sin x & \cos x \\ x & 1 & 1 \end{vmatrix}$$

$$\Delta'(x) = \begin{vmatrix} \cos x & \cos x & \sin x \\ -\sin x & -\sin x & \cos x \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} \sin x & -\sin x & \sin x \\ \cos x & -\cos x & \cos x \\ x & 0 & 1 \end{vmatrix} +$$

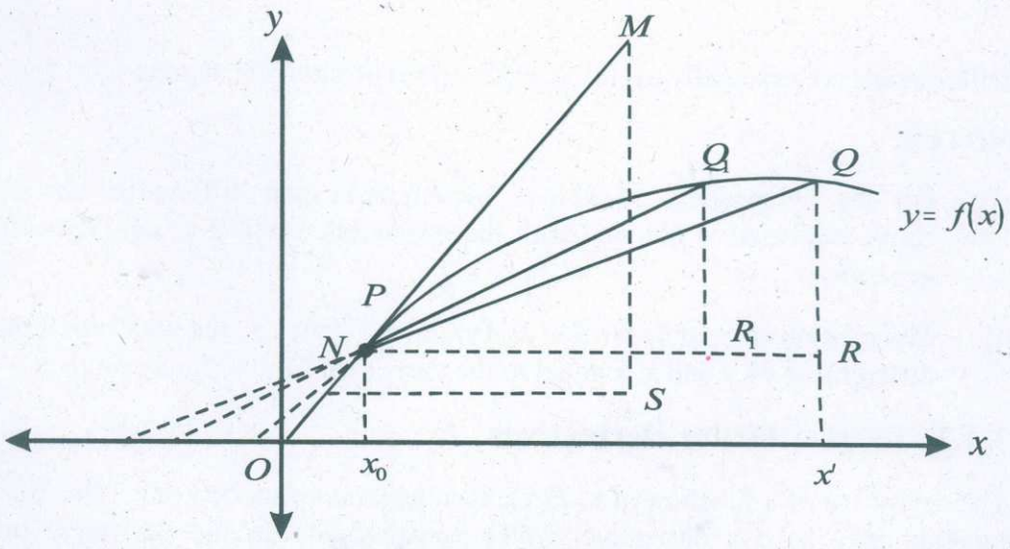
$$\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & -\sin x & -\sin x \\ x & 1 & 0 \end{vmatrix}$$

$$= \cos x\,(-\sin x - \cos x) - \cos x\,(-\sin x - \cos x) + \sin x\,(-\sin x + \sin x) + \sin x(-\cos x) + \sin x(\cos x - x\cos x) + x\sin x\cos x + \sin^2 x - x\sin x\cos x + \cos x\,(\cos x + x\sin x)$$

$$= \sin^2 x + \cos^2 x = 1$$

3.4.1 Geometrical Interpretation of Derivatives

Geometrically, the derivative of a function is the slope of the concerned curve.



The function drawn here is $y = f(x)$, let P and Q be the two points on this function. We start from the point P, where $x = x_0$ Suppose x has risen

from x_0 to x', i.e., $\Delta x = x_0x'$. As a result y has changed from P to Q on the curve and we get $\frac{\Delta y}{\Delta x} = \frac{QR}{PR} = \tan \phi$

This shows as Q approaches P, Δx increases and we get steeper and steeper lines, when $\Delta x \to 0$ the point Q almost coincides with P and the chord PQ ultimately becomes tangent at point P. Let MN be the tangent to $f(x)$ at point P. Thus, the derivative of a function at a point is actually the value of the slope of the function at that point. Thus, the derivative for a linear function is constant, but for a non-linear function the derivatives are different for different values of x.

3.4.2 Differentials

If $f'(x)$ is the derivative of $f(x)$, and Δx is an increment of x, then the differential of $f(x)$, denoted by the symbol $df(x)$, is defined by the relation $df(x) = f'(x)\Delta x$ ... (i)

If $f(x) = x$, then $f'(x) = 1$ and (i) reduces to $dx = \Delta x$

Thus, when x is the independent variable, the differential of $x\ (= dx)$ is identical with Δx. Hence, if $y - f(x)$, then the relation (i) becomes $dy = f'(x)dx$ ... (ii)

i.e., the differential of a function is equal to its derivative multiplied by the differential of the independent variable.

Thus, if $y = tan\, x$, then $dy = sec^2 x\, dx$.

From the definition of the differential of a function, the following formulae are obvious –

$d(c) = 0$ where c is constant

$d(u + v - w) = du + dv - dw$

$d\,(uv) = udv + vdu$

$$d\left(\frac{u}{v}\right) = \frac{vdu - vdu}{v^2}$$

Differentials are especially useful in applications of integral calculus.

NOTES:

i) For the independent variable x, increment Δx and differential dx are equal, but same is not true with the dependent variable y, i.e., $\Delta y \neq dy$ generally.

ii) The relation (ii) can be written as $dy/dx = f'(x)$, i.e., the quotient of the differentials of y and x is equal to the derivative of y with respect to x.

3.4.3 Higher Order Derivatives

The derivative of a function of x, $f(x)$, is in general a function of x. This new function may have a derivative, which is called the second derivative (or second differential coefficient) or, $f(x)$. The original derivative is called the

first derivative (or, first differential coefficient). Similarly, the derivative of the second derivative is called the third derivative and so on for the n^{th} derivative.

Example: Let $y = 4x^5 + 7x^4 + 3x + 9$, then $\frac{dy}{dx} = 20x^4 + 28x^3 + 3$;

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = 80x^3 + 84x^2$; $\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = 240x^2 + 168x$

3.4.4 Application of Simple Derivatives in Economics

The derivatives have wide application in economics. We always deal with functional relations while studying economic problems. Dealing with these functional relations sometimes prompt us to find out the rate of change of one (dependent) variable owing to the change of the other (independent) variable.

Average and Marginal Revenue

The price (p) set by the sellers in an imperfect market depends on the quantity of output (q) the sellers expect to sell, i.e., $p = f(q)$. This is the inverse demand function. The total revenue (R) obtained by the seller is $R = p.q = f(q).q$. Thus, R is a function of q only, i.e., $R = \phi(q)$. This implies R varies in response to a variation in q. We get two concepts – average revenue and marginal revenue. The average value of R, the average (AR) measures the average variation of R over a range of values of q extending from zero upto a certain specified value. The specified value of q is the one at which we seek the average value of R.

Thus, $AR = \frac{R}{q} = \frac{pq}{q} = p = f(q)$

Hence, the function $p = f(q)$ gives us the AR curve when plotted graphically. The marginal value of R or, the marginal revenue (MR) measures the variation of R at the margin.

Thus, $MR = \frac{dR}{dq} = f(q) + q.f'(q)$

We can establish a relation between MR and AR: $MR = AR + q.f'(q)$

The relation describes that the difference between MR and AR will always be equal to $q\,f'(q)$. Since q is always non-negative and since $f'(q) < 0$, we have $MR - AR < 0$ i.e., $MR < AR$ for all q.

Thus, when the AR curve is downwards sloping, the MR curve always lies below the AR curve. However, if AR curve be upward rising $MR > AR$ and the MR curve will lie above the AR curve. When $f'(q) = 0$, the MR curve and the AR curves coincide. This happens when p is a given quantity and fixed for all q. The AR curve will then be a parallel line to the horizontal axis. When $q = 0, AR = f(0)$ and $MR = f(0)$. Thus, at $q = 0$ the two curves intersect each other.

Average and Marginal Cost

The total cost (c) of a firm depends on the total output (q) produced, i.e., $c = c(q)$. Proceeding by the same argument in the last section, average cost $(AC) = \frac{c(q)}{q}$ and marginal cost $(MC) = \frac{dc(q)}{dq}$.

Again, we can establish a relation between AC and MC. Let us differentiate AC function with respect to q.

$$\frac{d}{dq}\left(\frac{c}{q}\right) = \frac{\frac{dc}{dq}.q - c.1}{q^2} = \frac{1}{q}\left(\frac{dc}{dq} - \frac{c}{q}\right) = \frac{1}{q}(MC - AC)$$

If AC curve slopes downwards, then $\frac{d}{dq}\left(\frac{c}{q}\right) < 0$ or, $\frac{1}{q}(MC - AC) < 0$ or, $MC < AC$.

i.e., MC curve lies below the AC curve when AC is falling. When AC slopes upwards, we can conclude MC curve lies above the AC curve. The AC curve intersects the MC curve if the slope of the AC curve is zero.

Elasticity of Demand

Demand is a multivariate function. We simplify our analysis by assuming demand to depend mainly on two factors – price and income. If one or both of these factors change, demand changes accordingly. The elasticity of demand measures the percentage change in quantity demanded due to one percent change in one of these two variables.

Suppose, the demand function is given by $q = f(p)$ where p denotes price and q quantity demanded. Then by definition, price elasticity of demand (e_d) is given by

$e_d = \frac{\frac{\Delta q}{q}\times 100}{\frac{\Delta p}{p}\times 100} = \frac{\Delta q}{\Delta p}.\frac{p}{q}$, where Δp and Δq are the changes in quantity demanded and price respectively. The price elasticity of demand defined in this way is called the arc price elasticity of demand. If we now assume Δp to be infinitesimal we get the concept of point price elasticity of demand (ε_d).

$$\varepsilon_d = \lim_{\Delta p \to 0} \frac{\Delta q}{\Delta p}.\frac{p}{q} = \frac{p}{q}.\lim_{\Delta p \to 0} \frac{\Delta q}{\Delta p} = \frac{p}{q}.\frac{dq}{dp} \qquad \ldots (1)$$

p and q are restricted to non-negative values only. Sine the demand curve is generally downward slopping, i.e., $\frac{dq}{dp} < 0$, the price elasticity of demand is generally negative. The negative value of the right hand expression in (1) gives us absolute elasticity. Thus,

$$|\varepsilon_d| = -\frac{p}{q}.\frac{dq}{dp}$$

This can take any value between zero and infinite. The demand is said to be

i) perfectly inelastic $|\varepsilon_d| = 0$

ii) inelastic if $0 < |\varepsilon_d| < 1$

iii) unity elastic if $|\varepsilon_d| = 1$

iv) elastic if $1 < |\varepsilon_d| < \infty$

v) perfectly elastic if $|\varepsilon_d| \to \infty$

NOTE: Since the value of ε_d depends on initial price-quantity combination, there is no necessary connection between the slope and price elasticity of demand curve. Curve with different slopes may have the same elasticity and curves with different elasticities may have the same slope.

Now, suppose that price remains the same and income changes. The demand function in this case can be written as $q = f(y)$. The income elasticity of demand (η_d), measures the percentage change in quantity demanded due to one percent change in income (y).

Thus, $\eta_d = \dfrac{dq/q \times 100}{dy/y \times 100} = \dfrac{dq}{dy}.\dfrac{y}{q}$

In measuring the income elasticity of demand we assume price to be fixed at a certain level. The demand curve in this case can be written in the following alternative form $q = q(y)$. In this form, the demand curve is called the Engel curve. The income elasticity of demand is thus, the elasticity of the Engel curve.

The shape of the Engel curve depends on the nature of the commodity in question. If the good is a luxurious one its demand will increase with increasing income and the Engel curve slopes upward. Thus, for luxurious goods η_d is positive. Again for luxurious goods demand not only increases with increase in income, but it increases more than proportionately than increase in income. Thus,

$$\frac{dq}{q} > \frac{dy}{y} \to \frac{dq}{dy}\Big/\frac{q}{y} > 1 \to \eta_d > 1$$

For necessities $0 < \eta_d < 1$ and for inferior goods $\eta_d < 0$.

Constant Elasticity Demand Curve

The price elasticity of demand differs at different points on a demand curve. But we may have a special kind of demand curve every point of which shows same constant elasticity. The exponential demand functions are of this nature.

The general form of the constant elasticity demand curve is given by $q = Ap^{\alpha}$ where A, α are constants.

Let us derive the elasticity expression for this function at any point on this curve.

$$\varepsilon_d = \frac{p}{q}.\frac{dq}{dp} = \frac{p}{q}.\frac{d}{dp}(Ap^{\alpha}) = \frac{p}{q}.A\alpha.p^{\alpha-1} = \frac{p}{q}.\frac{q}{p}.Ap^{\alpha} = \alpha$$

Thus, ε_d is a pure number and constant throughout the whole curve. Consider a particular case when $\alpha = -1$. Then, $q = Ap^{-1}$ *or*, $pq = A$.

In this case, $\frac{p}{q}.\frac{dq}{dp} = -1$. Such a term and curve is represented by rectangular hyperbola.

3.4.5 Slope and Curvature

The first and second-order derivatives are of much use in the graphical analysis of economic functions. We know the first derivative of a function at any point measures the slope of the function at that particular point. Consider a function $y = f(x)$. The value of $f'(x)$ at $x = a$ is $f'(a).f'(a) > 0$ implies that at $x = a$ the tangent gradient of the curve is positive, i.e., tangent to the curve at the curve $x = a$ slopes upwards. This implies that $f(x)$ increases through $x = a$ as x increases. Thus, the sign of the first order derivative informs us whether the function increases upwards or decreases downwards at the point in question. If $f'(x) > 0$ for all values of x in the domain of the function, then we can say $f(x)$ rise continuously from left to right.

The function which rises continuously is called monotonically increasing function. Thus, the function to be monotonically increasing implies the first derivative must be positive although. Similarly, a function $y = f(x)$ is said to be monotonically decreasing if $f'(x) < 0$ for all x.

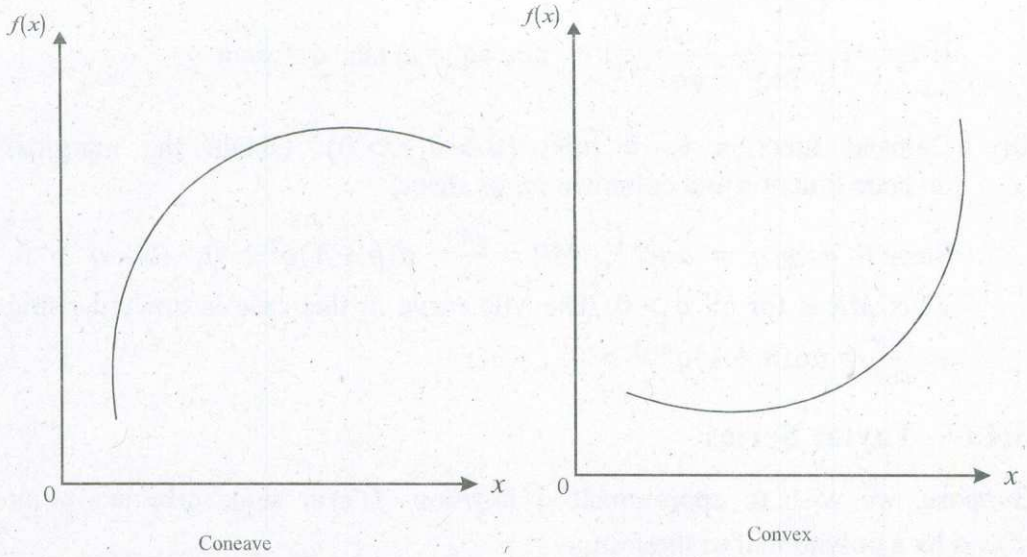
Example:

i) Consider the demand function $q = 3 - 5p$. Here $\frac{dq}{dp} = -5 < 0$. Thus, the function is a monotonically decreasing function.

ii) $c = aq^2 + bq + c$, $(a, b, c > 0)$. This cost function is monotonic because $\frac{dq}{dp} = 2aq + b > 0$ given $q \geq \geqslant 0$. Actually, this function is monotonically increasing.

From the first derivative we learn about the direction of change of $f(x)$ with a change in x. The function $f(x)$ changes in the same direction at which x changes if $f'(x) > 0$ and in the opposite direction if $f'(x) < 0$.

One question immediately follows. What will be the pattern of change of $f(x)$ with a change in x? How can we know whether a function, which is known to be increasing, increases at a constant rate, or at an increasing rate or at a decreasing rate? The second derivative plays its role here. The bending of a curve or curvature is measured by the rate of change of $f'(x)$ or the slope of the curve. If the rate of change of the slope is positive, i.e., $f''(x) > 0$, the curve is said to be convex. For a convex curve $f(x)$ increases or decreases at an increasing rate. If $f''(x) < 0$ the curve increases or decreases at a decreasing rate. This class of curves are known as concave curves. Lastly, the curves which increase or decrease at a constant rate have $f''(x) = 0$, e.g., linear curves.



Below we summarise the nature of the function $y = f(x)$ depending on the values of $f(x)$ and $f''(x)$.

If as x increases through a, The curve at $x = a$ is		the tangent to the curve at $x = a$
$f'(a) > 0$ upward rising $f''(a) > 0$ convex	}	turns anti-clockwise
$f'(a) > 0$ upward rising $f''(a) = 0$ linear	}	does not turn
$f'(a) > 0$ upward rising $f''(a) < 0$ concave	}	turns clockwise
$f'(a) < 0$ downward rising $f''(a) > 0$ convex	}	turns anti-clockwise
$f'(a) < 0$ downward rising $f''(a) = 0$ linear	}	does not turn
$f'(a) < 0$ downward rising $f''(a) < 0$ concave	}	turns clockwise

Example:

i) The function $y = 5x^2 + 6$ is convex because $f''(x) = 5 > 0$.

ii) Consider the demand function $p = aq^2 + bq + c$. Obtain the expression for its price elasticity of demand and tell what restriction should be imposed upon the value of q to make the demand function unitary elastic.

$$\varepsilon_d = \frac{p}{q}\cdot\frac{dq}{dp} = \frac{p}{q}\cdot\frac{1}{2aq+b} = \frac{aq^2+bq+c}{2aq^2+bq}$$

If $\varepsilon_d = 1$, $\frac{aq^2 + bq + c}{2aq^2 + bq} = 1$ i.e., $aq^2 = c$, i.e., $q = \sqrt{c/a}$

iii) Demand function is $= aq^b$, $(a > 0, b > 0)$. Obtain the marginal revenue function and comment on its shape.

Since $R = p.q = aq^{b+1}$, $MR = \frac{dR}{dq} = a(b+1)q^b > 0$; for $q > 0$. Thus, MR is for all $q > 0$. The MR curve in this case is upward rising as $\frac{d^2R}{dq^2} = ab(b+1)q^{b-1} > 0$.

3.4.6 Taylor Series

Suppose, we wish to approximate a function $f(x)$ at some arbitrary point $x = a$ by a polynomial of the form -

$f_n(x) = a_o + a_1(x-a) + a_2(x-a)^2 + ... + a_n(x-a)^n$, where, $f_n(x)$ is the approximate value of $f(x)$.

This exercise will be particularly helpful in finding the conditions for extrema of functions. The functions expressed in the above form are called *analytic functions*.

Our next task is to find appropriate values of the coefficients of the above polynomial. To do this, we proceed as follows:

First, we note that $f(x)$ should equal $f_n(x)$ when $x = a$. Thus, $a_o = f(a)$.

To approximate $f(x)$ even better, let us make the derivatives of $f(x)$ and $f_n(x)$ equal at $x = a$. We know,

$f_n'(x) = a_1 + 2a_2(x-a) + ... + na_n(x-a)^{n-1}$

$f_n''(x) = 2a_2 + 3.2.a_3(x-a) ..... + n(n-1)a_n(x-a)^{n-1}$

....

$f_n^n(x) = n!a_n$

Clearly, when $x = a$, we get

$a_1 = f'(a)$

$a_2 = \frac{f''(a)}{2!}$

.....

Thus, we can write,

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ...$$

This is known as Taylor series.

Note that we have not included the last term (i.e. the $(n+1)^{th}$ term) $\frac{f^n(a)}{n!}(x-a)^n$ in the above expression. This is because $f(x)$ may be *any* type of function. If we take the last term in the above form, then the approximation will not be exact if $f(x)$ itself is not a polynomial.

The power series can be made exact (when $f^{(n-1)}$ is continuous) if the last term is evaluated not at $x = a$, but at some point x^*, such that $x < x^* < a$ (why this is so is beyond the scope of the treatment).

$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^n(x^*)}{n!}(x-a)^n$ where

$x^* = a + \theta(x-a)$, $0 \le \theta \le 1$.

The above series is known as the Taylor series with Lagrange's form of the remainder (Here, remainder $R_n = \frac{f^n(x^*)}{n!}(x-a)^n$ .We can have Taylor series with other form of remainder also. But for our purpose, it is sufficient to know this form of Taylor series).

3.4.7 Mean Value Theorem and L'Hospital's Rule

If we put $n = 1$ in Taylor series, we get $f(x) = f(a) + R_0$

With suitable adjustment (a formal proof is beyond the scope of the course), we can derive that $R_0 = f'(a)(x-a)$.

This is known as *linearization of the function* $f(x)$ (as we are approximating the function f(x) by a polynomial of first degree).

This result is known as the Mean Value theorem. It states that if (1) $f(x)$ is continuous in the closed interval $a \le x \le b$ and (2) $f'(x)$ exists in the open interval $a < x < b$, then there exists at least one value of x, c between a and b such that

$f(b) - f(a) = (b-a)\, f'(c)$.

Since,$a < c < b$, c can be written as $a + \theta\,(b-a)$, where $0 < \theta < 1$. Putting

$b = a + h$, we get another form of the mean value theorem.

$f(a + h) = f(a) + h\, f'(a + \theta h)$, where $0 < \theta < 1$.

L'Hospital's rule

L'Hospital's rule is another useful result (which can be proved by applying Taylor's theorem).

Suppose, $f(x) = g(x)/h(x)$. Now if both $g(x)$ and $h(x)$ approach either zero or infinity as $x \to a$, limit of the function $f(x)$ as $x \to a$ cannot be evaluated.

In that case, L'Hospital's rule states that

$\lim_{x \to a} f(x) = \lim_{x \to a} g(x)/h(x) = \lim_{x \to a} g'(x)/h'(x)$.

(If the latter limit exists).

Examples:

Using L'Hospital's rule, find

a) $\lim_{x\to 0}\frac{e^x-1}{x}$. b) $\lim_{x\to 0}\frac{(1+x)^n-1}{x}$

Here $g(x) = e^x - 1$ and $h(x) = x.g'(x) = e^x$ and $h'(x) = 1$.

Thus, $\lim_{x\to 0}\frac{e^x-1}{x} = \lim_{x\to 0}\frac{e^x}{1} = 1$.

b) Here $g(x) = (1+x)^n - 1$ and $h(x) = x.g'(x) = n(1+x)^{n-1}$ and $h'(x) = 1$.

Thus, $lim_{x\to 0}\frac{(1+x)^n-1}{x} = n$

Check Your Progress 3

Find $\frac{dy}{dx}$ of the following cases

i) $y = sin^2x - e^x + 4^x$
ii) $y = 3x^2 cosec\, x$
iii) $y = \frac{\log x - \sin x}{2x^2-1}$
iv) $y = 2\cos v + tan^2 v,\ v = e^{3x}$,
v) $y = a\cos\theta, x\ x = b\sin\theta$

.....

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3.5 PARTIAL DIFFERENTIATION

So far, we have considered the functions of a single independent variable. However, most of the economic problems deal with the multivariate cases. For example, utility of a consumer depends on the consumption of multiple commodities and the production of a commodity depends on factors like land, labor, and capital and so on.

Like this, other examples can be included to capture more interesting economic problems. Therefore, there is a need to analyse the concept of derivative for the class of functions with more than one independent variables.

3.5.1 Concept of Partial Derivative

For simplicity, we shall start with function of two independent variables. Let the function be $y = f(x_1, x_2)$. When we take such a function $y = f(x_1, x_2)$, with two independent variables, change in y can be examined as result of a change in either x_1 or x_2, or by both by assuming these to be independent.

Thus, if we change only x_1, it will not affect x_2 but only y. The rate of change in y, assuming x_2 to remain constant, can be obtained by finding partial derivative of y with respect to x_1. Similarly, if x_2 is changed alone, the rate of change in y can be obtained from the partial derivative of y with respect to x_2.

Proceeding one-step further and assuming that x_1 and x_2 are related to each other, if we change x_1, there will be change in x_2 (even when x_2 is not changed exogenously), as a result of the change in x_1. Thus, whenever we change the dependent variable, y is affected in two ways: first, directly by the change in x_1 and secondly, indirectly by the change in x_2 via the change in x_1. The rate of change of y due to a change in x_1 in this case is measured by total derivative of y with respect to x_1.

3.5.2 Partial Derivative Defined

Partial derivative is referred to the derivative of a multivariate function when only one of the independent variables is allowed to change, other variables remaining constant.

As an example, consider the bivariate function,

$$U = f(x_1, x_2).$$

If we let only x_1 to change, x_2 remaining constant, the rate of change of U with respect to x_1 is called the partial derivative of U with respect to x_1. Let the change in x_1 is Δx_1, x_2 remaining constant. The corresponding change in U is ΔU. Then the difference quotient is given by

$$\frac{\Delta U}{\Delta x_1} = \frac{f(x_1 + \Delta x_1, x_2^0) - f(x_1, x_2^0)}{(x_1 + \Delta x_1) - x_1}$$

If we try to obtain the limiting value of this difference quotient as $\Delta x_1 \to 0$, we get the partial derivative of U with respect to x_1. This is denoted by $\frac{\partial U}{\partial x_1}, \frac{\partial f}{\partial x_1}$ or simply f_1.

Thus, $f_1 = \lim_{\Delta x_1 \to 0} \frac{\Delta U}{\Delta x_1} = \lim_{\Delta x_1 \to 0} \frac{f(x_1 + \Delta x_1, x_2^0) - f(x_1, x_2^0)}{\Delta x_1}$. Similarly,

$$f_2 = \lim_{\Delta x_2 \to 0} \frac{\Delta U}{\Delta x_2} = \lim_{\Delta x_2 \to 0} \frac{f(x_1^0, x_2 + \Delta x_2) - f(x_1^0, x_2)}{\Delta x_2}$$

Example: Find the partial derivative of the function

$$f(x_1, x_2) = x^3{}_1 + 2x^2{}_1 x_2 + 3 x_1 x_2 + 4 x^2{}_2.$$

To obtain f_1 we have to differentiate the above function with respect to x_1, treating x_2 as constant.

Thus, $f_1 = 3x^2{}_1 + 4x_1 x_2 + 3x_2$.

Similarly, $f_2 = 2x^2{}_1 + 3x_1 + 8x_2$ Note that these partial derivatives are functions of the some dependent variables as in case of the original function f.

we write, If $y = f(x_1, x_2)$, then $f_1 = f_1(x_1, x_2)$, $f_2 = f_2(x_1, x_2)$.

This is because first derivative at any point was not only dependent on the value of the independent variable, but also on the other variables, which are kept constant. First order partial derivative gives us the marginal values. For example, if the function $y = f(x_1, x_2)$,

defines the total utility of the commodities x_1, x_2, then $\frac{\partial U}{\partial x_i}$, gives us additional utility that can be obtained from the use of one additional unit of x_1. Thus, $\frac{\partial U}{\partial x_i}$ is the marginal utility of x_i ($for\ i = 1, 2$).

3.5.3 Higher Order Partial Derivatives

We have already seen, if $y = f(x_1, x_2)$,

$y = f(x_1, x_2)$, then $f_i(x_1, x_2)$ (for $i = 1, 2$) gives the partial derivatives.

So long as this is true, we can repeat the process of partial differentiation and get higher order partial derivatives. However, once the first order partial derivative ceases to be a function of some choice variables as the primitive function, higher order partial derivatives are no longer obtainable.

Example:

i) Differentiate $z = 6x^3+5x^2+10xy$ partially with respect to x twice.

Ans: $z_x = 18x^2 + 10x + 10y$; $z_{xx} = 36x + 10$

ii) Find f_{11}, f_{22} for the function $f(x_1, x_2) = e^{x_1+x_2} + 3x_1 x_2$

Ans:

$f_1 = e^{x_1+x_2} + 3x_2$

$f_2 = e^{x_1+x_2} + 3x_1$

Thus,

$f_{11} = e^{x_1+x_2} + 3$

$f_{22} = e^{x_1+x_2} + 3$

3.5.4 Cross- Partial Derivatives

Consider the function $f(x_1, x_2)$. The partial derivative of the function gives f_{11} and f_2 (defined above) The cross–partial derivatives are defined as follows:

$$f_{12} = \frac{\partial f_1}{\partial x_2} \text{ And } f_{21} = \frac{\partial f_2}{\partial x_1}$$

By Young's theorem, $f_{12} = f_{21}$.

More generally, for a $n - variable$ function $y = f(x_1, - - -, x_n)$.

$f_{ji} = f_{ij}$ (for all $i, j = 1, 2, - - - - - - - -, n$)

Note that such a condition holds when both the partial derivatives exist and one of them is continuous.

Example:

i) $f(x,y) = x^3 + y^5 + 4x^2y^4 + 2xy$, find f_{xy} and f_{yx}.

$f_x = 3x^2 + 8xy^4 + 2y$

$f_y = 5y^4 + 16x^2 y^3 + 2x$

$f_{xy} = 32x\, y^3 + 2 = f_{yx}$

ii) $f(x_1, x_2) = \log (x_1^2 + x_2^2)$. Find the second order partial derivatives.

$f_1 = \frac{2x_1}{x_1^2+x_1^2}$; $f_2 = \frac{2x_2}{x_1^2+x_1^2}$; $f_{11} = \frac{2(x_2^2-x_1^2)}{(x_1^2+x_2^2)^2}$; ; $f_{12} = f_{21} = \frac{4x_1x_2}{(x_1^2+x_2^2)^2}$; and

$f_{22} = \frac{2(x_1^2 - x_2^2)}{(x_1^2 + x_2^2)^2}$

Check Your Progress 4

1) A production function is given by $Q = \frac{1}{2}L^2K^2$

where, Q = level of output; L = Labour input employed;

K = Capital input employed. Find the marginal productivities of labour and capital.

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2) $f(x,y) = x + y$. Find f_{xx}, f_{yy}.

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3) $f(x,y) = y \sin x^2 + x^3 \cos y$. Find f_{xx}, f_{yy}.

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4) $f(x,y) = e^{2x+2y}$, prove that $f_{xy} - f_{yx} = 0$.

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3.5.5 Total Differential

Consider the function $y = f(x_1, x_2)$. By its total differential, we measure the total change in y due to a change in both x_1 and x_2 (where x_1, x_2 are assumed to be independent of each other).

However, the change measured by total differential of y is only a linear approximation of the actual change.

The rate of change of y due to a small change in x_1, keeping x_2 constant is already noted above as f_1. If the volume of change in x_1 is dx_1, then the measured change in y due to a change in x_1 will only be equal to $f_1\ dx_1$. Similarly, if the volume of change in x_2 is dx_2, then the measured change in y due to a change in x_2 will be f_2 and fx_2. Thus, if dy denotes the total change in y when x_1 is changed by dx_1 and x_2 is changed by dx_2 ,we can write,

$$dy = f_1\ dx_1 + f_2\ dx_2\ .$$

This is called the total differential of the function $y = f(x_1, x_2)$.

Example: Find the total differential of the function

$$y = a\ x_1^2 + 2h\ x_1\ x_2 + b\ x_2^2$$

$$dy = 2\ a\ x_1 dx_1 + 2h\ (x_1\ dx_2 + x_2\ dx_1) + 2\ bx_2 dx_2$$

$$= 2\ (ax_1 + hx_2)\ dx_1 + 2\ (bx_2 + h\ x_1)\ dx_2$$

3.5.6 Total Derivative

Through total derivative, we measure the rate of change of the dependent variable owing to any change in a variable on which it depends, when none of the variables is assumed to be constant.

Let $y = f(x_1, x_2)$, such that, $x_1 = g(t)$ and $x_2 = h(t)$.

Then we can write

$$\frac{dy}{dt} = \frac{\partial y}{\partial x_1}.\frac{dx_1}{dt} + \frac{\partial y}{\partial x_2}.\frac{dx_2}{dt} = f_1\frac{dx_1}{dt} + f_2\frac{dx_2}{dt}$$

This is the total derivative of y with respect to t.

The above result can easily be extended for a n- variable case, namely,

$y = f(x_1, x_2, - - - - -, x_n)$.

If $x_1 = x_1(t)$, $x_2 = x_2(t)$, ---, $x_n = x_n(t)$, then

$$\frac{dy}{dt} = \frac{dx_1}{dt} + f_2\frac{dx_2}{dt} + - - + f_n\frac{dx_n}{dt}.$$

Example;

i) $q = 4x_1 + 3x_2$. If $x_1 = t^3 + t^2 + 1$, $x_2 = t^3 - t^2 - t$.

Find $\frac{dq}{dt}$.

$\frac{dq}{dt} = f_1.\frac{dx_1}{dt} + f_2.\frac{dx_2}{dt}$; where, $f_1 = \frac{\partial q}{\partial x_1}, f_2 = \frac{\partial q}{\partial x_2}$. In our example, $f_1 = 4$, $f_2 = 3$. Also, $\frac{dx_1}{dt} = 3t^2 + 2t, \frac{dx_2}{dt} = 3t^2 - 2t - 1$. So $\frac{dq}{dt} = 4(3t^2 + 2t) + 3(3t^2 - 2t - 1)$

$= 21t^2 + 2t - 3$

ii) $u = x_1^2 + x_2^2$ such that $x_1 = 2x_2^2 + 2$; obtain $\frac{du}{dx_2}$. Since $x_1 = g(x_2)$ we get, $\frac{du}{dx_2} = f_1.\frac{du}{dx_2} + f_2; f_1 = 2x_1;$

$\frac{dx_1}{dx_2} = 4x_2 = 2x_2(4x_1 + 1) = 2x_2\{(4(2x_2^2 + 2) + 1\} = 8x_2(2x_2^2 + 2) + 2x_2$

3.5.7 Differentiation of Functions

The formula for total differential can be used to find the derivatives of various types of functions of function. Let us consider a few cases.

Case I: Suppose $z = f(u)$ where u is a function of x and y. Here $dz = f'(u).du$.

$$\frac{dz}{dx} = \frac{dz}{du}.\frac{\partial u}{\partial x} = f'(u).\frac{\partial u}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{dz}{du}.\frac{\partial u}{\partial y} = f'(u).\frac{\partial u}{\partial x}$$

Case II: Suppose, z is a function of x and y which are themselves function of a single independent variable, t.

i) $dz = f(x, y)$ where $x = \phi(t), y = \varphi(t)$. Then, $dz = \frac{\partial z}{\partial x}.dx + \frac{\partial z}{\partial y}.dy$

But, $\frac{dx}{dt} = \phi(t)$ or, $dx = \phi'(t).dt$.

And $\frac{dy}{dt} = \varphi'(t)$ or, $dy = \varphi'(t).dt$

$$\therefore dz = \left\{\frac{\partial z}{\partial x}\Phi'(t) + \frac{\partial z}{\partial y}.\varphi'(t)\right\}dt$$

$$\therefore \frac{dz}{dt} = \frac{\partial z}{\partial x}.\phi'(t) + \frac{\partial z}{\partial y}.\varphi'(t) = \frac{\partial z}{\partial x}.\frac{dx}{dt} + \frac{\partial z}{\partial y}.\frac{dy}{dt}.$$

Case III: If $z = f(x, y)$ where y is a function of x, then

$$dz = \frac{\partial z}{\partial x}.dx + \frac{\partial z}{\partial y}.dy \quad \text{or,} \quad \frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}.\frac{dy}{dx}.$$

Case IV: Suppose z is a function of x and y where x and y are themselves functions of two independent variable u and υ. i.e., z = f(x, y) where $x = \phi(u,\upsilon)$ and $y = \varphi(u,\upsilon)$.

Assume that all functions have known partial derivatives.

Here, $dz = \frac{\partial z}{\partial x}.dx + \frac{\partial z}{\partial y}.dy$, where, $dx = \frac{\partial x}{\partial u}.du + \frac{\partial x}{\partial u}.du$ and $dy = \frac{\partial y}{\partial u}.du + \frac{\partial u}{\partial u}.du$

Putting these values in expression for dz we get

$$dz = \frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial u}.du + \frac{\partial x}{\partial u}.du\right) + \frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial u}.du + \frac{\partial y}{\partial u}.du\right)$$

$$= \left(\frac{\partial z}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial u}\right)du + \left(\frac{\partial z}{\partial x}.\frac{\partial x}{\partial \upsilon} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial \upsilon}\right)d\upsilon.$$

The partial derivative of z with respect to u is the ratio of dz and du when υ is kept constant.

Similarly, the partial derivative of z with respect to υ is the ratio of dz and d υ when u is kept constant. If υ is kept constant, dυ = 0 so that $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial u}$

And $\frac{\partial z}{\partial \upsilon} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial \upsilon} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial \upsilon}$.

Rules for finding differentials are similar to those of finding derivatives.

Suppose, u, υ and w are functions of k and y. Then,

i) $d(u \pm \upsilon) = du \pm du$

ii) $d(u\upsilon) = ud\upsilon + \upsilon du$

iii) $d\left(\frac{u}{\upsilon}\right) = \frac{\upsilon du - ud\upsilon}{\upsilon^2}$

iv) $d(logx) = \frac{dx}{x}$

v) $z = \frac{u\upsilon}{w}, \frac{dz}{z} = \frac{du}{u} + \frac{d\upsilon}{\upsilon} - \frac{dw}{w}$

3.5.8 Rules Regarding Implicit Functions

There is an implicit functional relation between x and y, such that with a given rise of variation, the values that x and y can take are not independent of each other but connected in some definite way.

Thus, if the value of x is fixed, the value (or values) that y can take are fixed and not arbitrary. Converse is also true.

The implicit function can be expressed in the form $f(x, y) = 0$.

e.g., $ax + by + c = 0$, or, $2x^2 + 3xy + 4y^2 = 0$

Some inverse functions can be expressed in terms of one variable. For example, the first function can be written as

$$x = -\frac{c+by}{a} \text{ or, } y = -\frac{c+ax}{b}.$$

These two explicit functions are said to be inverse to each other.

However, all implicit functions may not be expressed in explicit forms where the two variables have been separated. For example, for the second function we cannot express x in terms of y or conversely.

Consider an implicit function $f(x, y) = 0$, where the two variables cannot be separated. Assume that y is a function of x. Here the concept of total differential can be used to find the total derivative $\frac{dy}{dx}$.

The function $f(x, y) = 0$ can be taken as $z = f(x, y)$ where z equals zero.

Now, $dz = fx.dx + fy.dy = 0$ (since z is constant). Thus, $\frac{dy}{dx} = -\frac{fx}{fy}$.

Example:

$f(x, y) = 2x^2 + 3xy + 4y^2 = 0$. Find $\frac{dy}{dx}$.

Here, $f_x = 4x + 3y$, $f_y = 3x + 8y$. So $\frac{dy}{dx} = -\frac{4x+3y}{3x+8y}$.

Similarly, if it is assumed that x is a function of y, the derivative $\frac{dy}{dx}$ will be equal to $-\frac{f_y}{f_x}$. The derivatives $\frac{dy}{dx}$ and $\frac{dx}{dy}$ are reciprocal to each other, since these two functions are reciprocal to each other.

The second derivative of an implicit function is defined and obtained from the first derivative. If $f(x, y) = 0$ gives y as a function of x, then we have seen that $\frac{dy}{dx} = -\frac{f_y}{f_x}$.

Now, $\frac{d^2y}{dx^2} = -\frac{d}{dx}\left(\frac{fx}{fy}\right) = -\left\{\frac{\partial}{\partial x}\left(\frac{fx}{fy}\right) + \frac{\partial}{\partial x}\left(\frac{fx}{fy}\right).\frac{dy}{dx}\right\}$

But, $\frac{\partial}{\partial x}\left(\frac{fx}{fy}\right) = \frac{f_{xx}.f_y - f_x.f_{yx}}{f^2_y}$ and $\frac{\partial}{\partial y}\left(\frac{f_x}{fy}\right) = \frac{f_{xy}.f_y - f_x.f_{yy}}{f_y^2}$. Substituting these values and the value of $\frac{dy}{dx}$ we get,

$$\frac{d^2y}{dx^2} = -\left\{\frac{f_{xx}.f_y - f_x.f_{yx}}{f_y^2} - \frac{f_x}{f_y}\frac{f_{xy}.f_y - f_x f_{yy}}{f_y^2}\right\}$$

$$= -\frac{1}{f_y^3}\left(f_{xx}f_y^2 - 2f_{xy} - f_x - f_y + f_{yy}.f_x^2\right). ----(i)$$

Example:

$f(x, y) = x^2 + xy + y^2$. Find $\frac{d^2y}{dx^2}$.

$$\frac{d^2y}{dx^2} = -\frac{2x+y}{x+2y}$$

$$\frac{d^2y}{dx^2} = \left\{\frac{(x+2y)\frac{d}{dx}(2x+y) - (2x+y)\frac{d}{dx}(x+2y)}{(x+2y)^2}\right\} = -\frac{6x^2+6xy+6y^2}{(x+2y)^3}$$

This can be obtained by putting the values for f_x, f_y, f_{xx}, f_{xy}, f_{yy} in (i).

3.5.9 Application in Economics

Elasticity of Substitution

Consider the production function $q = f(x_1, x_2)$. The elasticity of substation measures the ease with which substitution between the two factors can take place.

The degree of substitutability between two factors increase with higher values of the elasticity of substitution.

If it is zero, it will imply impossibility of substitution.

The elasticity of substitution (σ) is defined as the percentage change in the input ratio in response to percentage change in the marginal rate of technical substitution i.e.,

$$\sigma = \frac{\frac{d\left(\frac{x_2}{x_1}\right)}{\left(\frac{x_2}{x_1}\right)}}{\frac{d\left(\frac{f_1}{f_2}\right)}{\left(\frac{f_1}{f_2}\right)}} = \frac{f_1x_1}{f_2x_2}\cdot\frac{d\left(\frac{x_2}{x_1}\right)}{d\left(\frac{f_1}{f_2}\right)}$$

Now $d\left(\frac{x_2}{x_1}\right) = \frac{x_1dx_2 - x_2dx_1}{x_1^2}$ or, $d\left(\frac{x_2}{x_1}\right) = \frac{x_1\left(\frac{-f_1}{f_2}\right)dx_1 - x_2dx_1}{x_1^2} = -\left(\frac{f_1x_1+f_2x_2}{f_2x_1^2}\right)dx_1$

$\left[\because \frac{dx_2}{dx_1} = -\frac{f_1}{f_2}\right]$

Again, $d\left(\frac{f_1}{f_2}\right) = \frac{df_1\cdot f_2 - df_2\cdot f_1}{f^2{}_2} = \frac{(f_{11}x_1+f_{12}x_2)f_2 - (f_{21}x_1+f_{22}x_2)f_1}{f^2{}_2}$

$$[\because df_1 = f_{11}dx_1 + f_{12}dx_2;\ df_2 = f_{21}dx_1 + f_{22}dx_2]$$

$$= \frac{\left[f_{11}dx_1 + f_{12}\left(-\frac{f_1}{f_2}\right)dx_1\right]f_2 - \left[f_{21}dx_1 + f_{22}\left(-\frac{f_1}{f_2}\right)dx_1\right]f_1}{f^2{}_2}$$

$$= \frac{[f_{11}dx_1 + f_2^2 - f_1f_2f_{12} - f_1f_2f_{21} + f_{22}f_1^2]dx_1}{f^3{}_2}$$

or, $\sigma = \frac{[f_{11}f_2^2 - 2f_1f_2f_{12} + f_1f_2f_{21} + f_{22}f_1^2]}{f^3{}_2}$ [Using $f_{12} = f_{21}$]

Thus,

$$\sigma = \left(\frac{f_1x_1}{f_2x_2}\right).\frac{(f_1x_1+f_2x_2)dx_1}{f_2x_1^2} = \frac{f^3{}_2}{(f_{11}f_2^2 - 2f_1f_2f_{12} + f_{22}f_1^2)dx_1} = \frac{f_1f_2(f_1x_1+f_2x_2)}{x_1x_2(2f_1f_2f_{12} - f_{11}f_2^2 - f_{22}f_1^2)}$$

The convexity of iso-quant ensures $(2f_1f_2f_{12} - f_{11}f_2^2 - f_{22}f_1^2) > 0$ and since $(x_1, x_2, f_1, f_2) > 0$ we have $\sigma > 0$.

The expression of the iso-quant ensures $(2f_1\ f_2\ f_{12} - f_{11}\ f_2^2 - f_{22}\ f_1^2)$ reflects the change of the slope or curvature of an iso-quant. This means that σ is inversely proportional to the degree of convexity of iso-quant. The lower the degree of convexity of the iso-quant, the larger the value of σ and vice versa.

If the two inputs are used only in a fixed proportion, the iso-quant is L-shaped and σ is zero.

In case if the two inputs are perfect substitutes of each other, the iso-quant is a straight line whose MRTS is constant. Here σ is infinity. As σ increases from zero to infinity, substitution between the factors becomes increasingly easier.

Some notes on σ

i) σ varies from one combination of factors to another.

   a) $\sigma = \sigma(x_1, x_2)$

   b) σ is a lower number

ii) $\sigma_{x1\,x2} = \sigma_{x2\,x1}$

iii) If the production function is linear homogeneous, (see below) σ can be written as $\frac{f_1f_2}{f_{12}.q}$. Here, σ is inversely proportional to the cross second order partial derivative of the production function.

3.5.10 Homogeneous Functions and Their Properties

A function is said to be homogeneous if multiplication of all its arguments by some arbitrary constant, say k, multiplies the function by k raised to some power.

Thus, the function $f(x_1, x_2)$ is said to be homogeneous of degree n if

$f(kx_1, kx_2) = k^n f(x_1, x_2)$.

The power of k is called the degree of homogeneity.

For a homogeneous function, the sum of indices for each term of the function is the same.

Thus, the function $f(x_1, x_2) = x_1^2 + x_1x_2 + x_2^2$ is a homogeneous function of degree two. This is because $(kx_1, kx_2) = k^2 f(x_1, x_2)$.

Similarly, $z = x^3y + x^2y^2 + y^4$ is a homogeneous function of degree four.

Property-1: The homogeneous function $z = f(x, y)$ of degree n can be written as $z = x^n\ \phi\left(\frac{y}{x}\right)$ or, $z = y^n \psi\left(\frac{x}{y}\right)$

Proof: Try to prove from the definition by putting $k = \frac{1}{x}$ or, $k = \frac{1}{y}$.

Property-2: The first order partial derivative for the homogeneous function $z = f(x, y)$ of degree n are homogeneous function of degree $(n - 1)$.

Proof: Let $z = f(x, y)$ be a homogeneous function of degree n. This function can be written as $z = x^n \phi\left(\frac{y}{x}\right)$.

Taking partial differential with respect to x,

$$\frac{\partial z}{\partial x} = n x^{n-1} \phi\left(\frac{y}{x}\right) - x^n \phi'\left(\frac{y}{x}\right)\left(\frac{y}{x^2}\right) = nx^{n-1} \phi\left(\frac{y}{x}\right) - yx^{n-2} \phi'\left(\frac{y}{x}\right)$$

$$= x^{n-1} \left[n\phi\left(\frac{y}{x}\right) - \frac{y}{x} \phi'\left(\frac{y}{x}\right)\right] = x^{n-1} . \phi'\left(\frac{y}{x}\right).$$ Thus, $\frac{\partial z}{\partial x}$ is a homogeneous function of degree $(n - 1)$.

Property-3: Let $z = f(x, y)$ be a homogeneous function of degree n. Then we have

$xf_x + yf_y = nf(x, y)$. This result is known as Euler's theorem.

Check Your Progress 5

1) Find the total differential of the function

$y = \sin x_1 + \cos x_2$

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2) Prove Euler's theorem.

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3) The function $q = AL^{\alpha} K^{1-\alpha}$, $A, \alpha > 0$ is known as the Cobb-Douglas production function. q, L and K are the level of output, labor and capital respectively.

Prove the following:

a) Cobb-Douglas production function is homogeneous of degree one.

b) The marginal and average productivities of the two inputs depend on the ratio of the two inputs only.

c) Elasticity of substitution is unity.

d) Iso-quants are downward sloping and convex to the origin.

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3.6 LET US SUM UP

In this unit we have discussed the basic techniques of differential calculus. For that purpose, the background material comprising limit and continuity has been covered at the outset. Limiting value of function and its existence are studied followed by when we treat a function to be continuous.

Concepts like derivative and differential coefficient has been introduced in the next section of limit and continuity. Then we have discussed various rules regarding how to differentiate different types of function. We applied this concept to illustrate average revenue, marginal revenue, average cost, marginal cost and elasticity of demand. We have also discussed slope and curvature of a function. Lastly, we have stated three important theorems to facilitate differentiation of difficult functions.

In the last part of the unit, we have learnt the technique for differentiation for multivariate cases. We have also discussed some pathological cases where differentiation is not immediate and simple. Lastly, we have applied the notion of partial differentiation to illustrate one important class of functions, namely homogeneous function. We have discussed properties of such functions and considered one typical example the Cobb-Douglas production function.

3.7 KEY WORDS

Continuity: A function f(x) is **continuous** provided its graph is continuous. More formally, a function f(x) is said to be **continuous** for x = a, provided $lim_{x \to a} f(x)$

exists, finite and is equal to a.

Derivatives: The limiting value of the ratio of the change in a function to the corresponding change in its independent variable.

Differential coefficient (or **derivative**): measure of the rate of change of given function.

Elasticity of Substitution: Elasticity of the ratio of two inputs to a production (or utility) function with respect to the ratio of their marginal products (or utilities).

Homogeneous Function: A function which satisfies $f(t\,x, t\,y) = t^n(x, y)$.

L'Hospital's Rule: Used to evaluate indeterminate forms of limits. It states that for indeterminate functions/forms (forms where the limits tend to 0/0 or $+ - \infty/\infty$), the limit of that form equals the limit of its derivatives.

Mean Value Theorem: If a function of one variable is continuous on a closed interval and differentiable on the interval minus its endpoints there is at least

one point where the derivative of the function is equal to the slope of the line joining the endpoints of the curve representing the function on the interval.

Partial Derivative: Partial derivative is referred to the derivative of a multivariate function when only one of the independent variable is allowed to change, other variables remaining constant; e.g., the function $U = f(x_1, x_2)$,

Radian: A unit of angular measure defined such that an angle of one radian subtended from the center of a unit circle produces an arc with arc length 1.

Taylor Series: An infinite sum giving the value of a function *f(z)* in the neighbourhood of a point *a* in terms of the derivatives of the function evaluated at *a*.

Total Derivative: Through total derivative, we measure the rate of change of the dependent variable owing to any change in a variable on which it depends, when none of the variables is assumed to be constant.

Total Differential: Rate of change of a variable in terms of another variable, without assuming that all other variables are held constant.

Young's Theorem: When both the partial derivatives exist and one of them is continuous, Young's theorem states that $f_{12} = f_{21}$.

3.8 SOME USEFUL BOOKS

R.G.D.Allen(1937), *Mathematical Analysis for Economists*, London School of Economics, ISBN 10: 1443725226

A.C. Chiang & K. Wainwright (2005), *Fundamental Methods of Mathematical Economics*, 4th edition, McGraw Hill.

Henderson and Quandt(1980) *Microeconomic Theory*, 3rd Edition, McGraw-Hill,ISBN-13: 978-0070281011

3.9 ANSWERS OR HINTS TO CHECK YOUR PROGRESS

Check Your Progress 1

1) i) $\lim_{x\to 0} \dfrac{\sqrt{a+x} - \sqrt{a-x}}{x}$

$$= \lim_{x\to 0} \frac{a+x-a-x}{x(\sqrt{a+x} - \sqrt{a-x})}$$

$$= \lim_{x\to 0} \frac{2x}{x(\sqrt{a+x} - \sqrt{a-x})} = \lim_{x\to 0} \frac{2}{\sqrt{a+x} + \sqrt{a-x}}$$

(Since $x \to 0$ and $x \neq 0$ we can cancel x in the ratio)

Now, as $x \to 0$, $\sqrt{a \pm x} - \sqrt{a}$ (you may check it by applying (δ, ε) definition of limit). Thus, the limit is $\frac{2}{2\sqrt{a}} = \frac{1}{\sqrt{a}}$.

ii) $|\sin x - 0| = |\sin x|$ can be made arbitrarily small by making $|x|$ arbitrarily small.

Thus, $\lim_{x\to 0} \sin x = 0$

iii) $\sin x - \sin\theta = 2\sin\frac{1}{2}(x-\theta)\cos\frac{1}{2}(x+\theta)$

As, $x \to 0, sin\frac{1}{2}(x-\theta)$. Also $\left|\cos\frac{1}{2}(x+0)\right| \leq$ 1. Thus, $lim_{x\to 0}(\sin x - \sin\theta) = 0$, i.e., $\lim_{x\to\theta} \sin x = \sin\theta$

2) i) $\lim_{x\to 1+0} f(x) = 10$, $\lim_{x\to 1-0} f(x) = -10$

Thus, $\lim_{x\to 1} f(x) = 10$ does not exist; whereas $f(1)$ exists = 1.

ii) $\lim_{x\to 0+0} f(x) = 10$, $\lim_{x\to 0-0} f(x) = 1$, which is not equal to $f(0) = -1$.

3) i) Put $x^2 = z$ as $x \to 0, z \to 0$

$$= \lim_{z\to 0} \frac{\sin z}{z} \cdot \sqrt{z} = \left(\lim_{z\to 0} \frac{\sin z}{z}\right)\left(\lim_{x\to\infty} \sqrt{z}\right) = 1.0 = 0.$$

ii) $$\lim_{x\to 0} \frac{1-\cos x}{x \sin x} = \lim_{x\to 0} \frac{2\sin^2\frac{x}{2}}{x.2\sin\frac{x}{2}\cos\frac{x}{2}} =$$

$$\lim_{x\to 0} \frac{\sin\frac{x}{2}}{x\cos\frac{x}{2}} = \frac{1}{2}\left(\lim_{\frac{x}{2}\to 0} \frac{\sin\frac{x}{2}}{\frac{x}{2}}\right) \cdot \frac{1}{\left(\lim_{\frac{x}{2}\to 0} \cos\frac{x}{2}\right)} = \frac{1}{2} \cdot 1 \cdot \frac{1}{1} = \frac{1}{2}$$

iii) Put $\frac{1}{x} = z$ as $x \to \infty, z \to 0$, $lim_{z\to 0} z \sin\frac{1}{z}$. Since z is not actually zero, value of $\sin\frac{1}{z}$ is a finite quantity between $[-1, 1]$. Thus, by taking z as small as we like we can make $z \sin\frac{1}{z}$ as small as we like, i.e., $\left|z \sin\frac{1}{z} - 0\right|$ is less than any pre-assigned positive quantity. Thus, the limit is zero.

iv) $$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{a(x+h)^2 + b(x+h) + c - ax^2 - bx - c}{h}$$

$$= \lim_{h\to 0} \frac{a(x^2 + 2xh + h^2) + bh - ax^2}{h} = \lim_{h\to 0} \left(\frac{2axh + h^2) + bh}{h}\right)$$

$$= \lim_{n\to 0} (2ax + b) = 2ax + b \ [\because h \neq 0]$$

v) $$\lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6n^3} = \lim_{n\to\infty} \frac{1}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) = \frac{1}{6}.1.2 = \frac{1}{3}$$

vi) Put $\tan^{-1} x = \theta$, i.e., $\tan\theta = x$. As $x \to 0, \theta \to 0$. Now,

$$\lim_{\theta\to 0}\frac{\theta}{\tan\theta}=\left(\lim_{\theta\to 0}\frac{\theta}{\sin\theta}\right).\left(\lim_{\theta\to\infty}\cos\theta\right)=1.1=1.$$

Check Your Progress 2

1) i) $\lim_{x\to 0} f(x)=0$ (It has already been proved in the problem for limit). Thus,

$\lim_{x\to 0} f(x)=f(0)$ and $f(x)$ is continuous at $x = 0$.

ii) $f(x)=\dfrac{x^2-2x+4}{x^2-5x+6}$ is the ratio of two continuous functions. (Polynomials are continuous can be verified easily). Thereby the property – III of the continuous function, $f(x)$ will be continuous at all values of x except when $x^2 - 5x + 6$ equals zero, i.e., the point of discontinuity of $f(x)$ are $x = 2, 3$.

iii) $\lim_{x\to 2} f(x)=\lim_{x\to 2}\dfrac{(x+2)(x-2)}{(x-2)}=4$. Thus, $f(2) = 4$ is the requirement for $f(x)$ to be continuous at $x = 2$.

Check Your Progress 3

1) $\dfrac{dy}{dx}=2\sin x\cos x-2e^x+4^x\log_e 4=sin^2x-2e^x+4^x\log_e 4$

NOTE: Put $v = \sin x$. If $T = \sin^2 x = v^2$, then $\dfrac{dT}{dx}=\dfrac{dT}{dv}.\dfrac{dv}{dx}=2v.\cos x$

$= 2\sin x \cos x$. Thus, we are using the results II, III, VI to differentiate this simple looking expression.

2) $\dfrac{dy}{dx}=3(2x\operatorname{cosec} x-\operatorname{cosec} x\cot x)\quad=3\operatorname{cosec} x(2x-\cot x)$

3) Here, take $\phi(x) = \log x - \sin x$ and $\Psi(x) = 2x^2 - 1$. So,

$\phi'(x)=\dfrac{1}{x}-\cos x$, $\psi'(x)=4x$. Thus,

$$\frac{dy}{dx}=\frac{(2x^2-1)(\frac{1}{x}-\cos x)-4x(\log x-\sin x)}{(2x^2-1)^2}$$

$$=\frac{(2x-\frac{1}{x}-2x^2\cos x+4x\sin x+\cos x-4x\log x}{(2x^2-1)^2}$$

4) $\dfrac{dy}{dv}=-2\sin v+2\tan v\sec^2 v$ and $\dfrac{dv}{dx}=3e^{3x}$. Therefore,

$$\frac{dy}{dx}=\frac{dy}{dv}.\frac{dv}{dx}=6e^{3x}[\tan(e^{3x})\sec^2(e^{3x})-2\sin(e^{3x})]$$

5) $\frac{dy}{dx} = \frac{dy}{d\phi}.\frac{d\phi}{dx} = -a\sin\phi.\frac{1}{b\cos\phi} = -\frac{a}{b}\tan\phi$

Check Your Progress 4

1) $Q_L = LK^2$, $Q_K = L^2K$.

2) $f_x = 1 = f_y$, Thus $f_{xx} = f_{yy} = 0$

$f(x, y) = y \sin x^2 + x^3 \cos y$. Find f_{xx}, f_{yy}.

3) $f_x = 2yx\cos x^2 + 3x^2\cos y$

$f_y = \sin x^2 - x^3\sin y$

$f_{xx} = 2y\cos x^2 - 4x^2 y\sin x^2 + 6x\cos y$

$f_{yy} = -x^3\cos y$

4) By Young's theorem, $f_{yx} = f_{xy}$, hence, $f_{xy} = f_{yx} = 4e^{2x+2y}$.

Check Your Progress 5

1) $dy = \cos x_1\, dx_1 - \sin x_2\, dx_2$.

2) Hint: Proceed along the lines for the proof of Property 2.

3) The function $q = AL^{\alpha}K^{1-\alpha}$, $A, \alpha > 0$ is known as the Cobb-Douglas production function. q, L and K are the level of output, labor and capital respectively. To answer the questions follow the hints:

The equation of the iso-quant is given by $\bar{q} = AL^{\propto}K^{1-\propto}$. Take that $\bar{q}$ is fixed level of output for different possible combinations of L and K. The slope is given by $\frac{d^2k}{dL^2} > 0$. Thus, the iso-quant is convex.

3.10 EXERCISES AND ANSWERS OR HINTS

i) prove that if $f'(a)$ is finite, $f(x)$ must be continuous at $x = a$.

ii) $f'(a) = \lim_{h\to 0}\frac{f(a+h) - f(a)}{h}$

$f(a+h) - f(a) = \frac{f(a+h) - f(a)}{h} \times h$

i.e., $\lim_{h\to 0}\{f(a+h) - f(a)\} = f'(a) \times 0 = 0$.

i.e., $\lim_{h\to 0} f(a+h) = f(a)$

Thus, $f(x)$ is continuous at $x = a$.

iii) The average cost (AC) of a firm is $AC = q^2 - 2q + 5$. The maximum capacity of the firm is 30 units. Find the ranges of the output for which AC is decreasing and for which it is increasing.

Ans. As is decreasing when $\frac{d(AC)}{dq} < 0$. So $2q - 2 < 0$ *and* $q < 1$,

Thus, AC decreases for $0 < q < 1$ and AC increases for $1 < q < 30$.

iv) The toal cost function is given by $c = ae^{bq}$ (a, b are constants). Find the value of q for which marginal and average cost for this function is equal.

Ans. $AC = \frac{a}{q}e^{bq}$; $MC = \frac{dc}{dq} = bae^{bq}$ and

$AC = MC \rightarrow \frac{a}{q}e^{bq} = bae^{bq}$; i.e., $= \frac{1}{b}$.

v) The production function is given by $q = -\frac{L^3}{3} + 2L^2 + 12L$, where L is the amount of labour employed. Find the maximum amount of labour employed beyond which average return from labour starts diminishing.

Ans. Average return from labour $= \frac{q}{L} = -\frac{L^2}{3} + 2L + 12$; we have $\frac{d}{dL}\left(\frac{q}{L}\right) = -\frac{2L}{3} + 2$, when the above equals zero, we get a value of L below which it will turn negative, i.e., L = 3.

vi) The demand curve for a consumer is α. k and α are constants. Find the price elasticity of demand.

Ans. $-k \log p = \log \alpha - \log q$, i.e., $-k\frac{dp}{p} = -\frac{dq}{q}$. Thus,

$$\varepsilon_d = \frac{dq}{q} \Big/ \frac{dp}{p} = k$$

vii) Total cost $c = 3q\left(\frac{q+7}{q+5}\right) + 5$. Comment on the nature of MC.

Ans. $MC = \frac{dc}{dq}$. Then derive $\frac{dMC}{dq}$. Show that for continually right q, MC falls, i.e., $\frac{dMC}{dq} < 0$.

NOTES

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