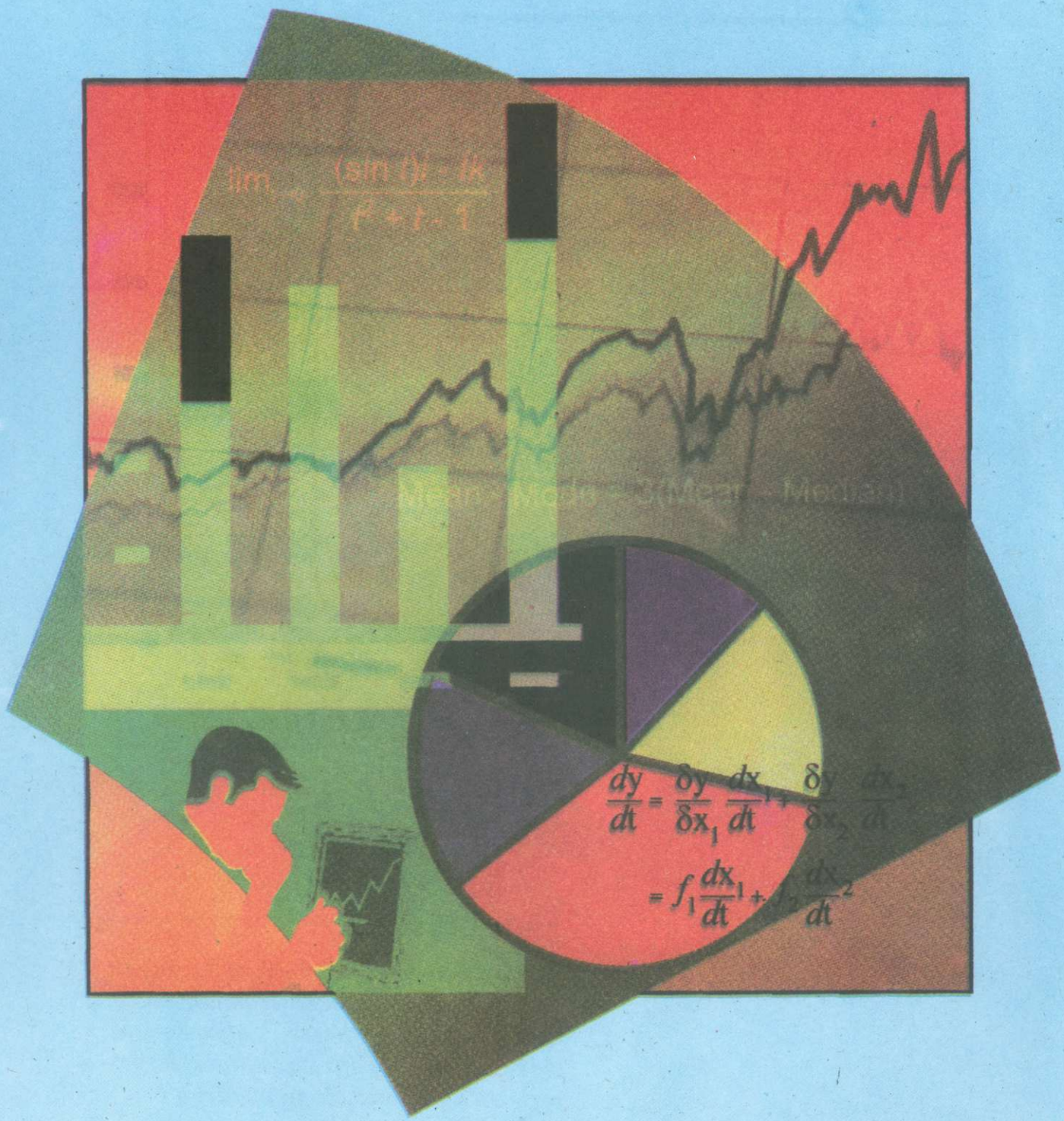
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MEC-103
QUANTITATIVE METHODS



Matrices and Vectors

4

"शिक्षा मानव को बन्धनों से मुक्त करती है और आज के युग में तो यह लोकतंत्रा की भावना का आधार भी है। जन्म तथा अन्य कारणों से उत्पन्न जाति एवं वर्गगत विषमताओं को दूर करते हुए मनुष्य को इन सबसे ऊपर उठाती है।"

- इन्दिरा गाँधी

"Education is a liberating force, and in our age it is also a democratising force, cutting across the barriers of caste and class, smoothing out inequalities imposed by birth and other circumstances."

- Indira Gandhi

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MEC-103
Quantitative Methods

Block

4

MATRICES AND VECTORS

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Expert Committee

Prof. Bhaswar Moitra
Department of Economics
Jadavpur University
Kolkata

Dr. Naresh Kumar Sharma
University of Hyderabad
School of Economics, Hyderabad

Dr. Anirban Kar
Deptt. of Economics
Delhi School of Economics
University of Delhi

Dr. Indrani Roy Chowdhury
Economics Faculty, Jamia Millia Islamia
New Delhi

Prof. Sudhir Shah
Deptt. of Economics
Delhi School of Economics
University of Delhi

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NIPFP
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New Delhi

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School of Social Sciences
Indira Gandhi National Open University
New Delhi

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School of Social Sciences
Indira Gandhi National Open University
New Delhi

Prof. B.S. Prakash
School of Social Sciences
Indira Gandhi National Open University
New Delhi

Mr. Saugato Sen
School of Social Sciences
Indira Gandhi National Open University
New Delhi

Course Editor : Prof. Gopinath Pradhan

Programme Coordinators	Course Coordinator
Prof. Gopinath Pradhan, IGNOU, New Delhi	Prof. Gopinath Pradhan
Prof. Kaustuva Barik, IGNOU, New Delhi	

Block Preparation Team

Unit	Writer
10,11,12 and 13	Sunando Basu

PRINT PRODUCTION TEAM

Mr. S. Burman	**Mr. Tilak Raj**	**Mr. Yashpal Sharma**
D.R.(P) MPDD-IGNOU,	A.R.(P) MPDD-IGNOU,	S.O.(P) MPDD-IGNOU,
Maidan Garhi, Delhi-110068	Maidan Garhi Delhi-110068	Maidan Garhi Delhi-110068

November 2018 (Reprint)

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ISBN : 978-93-86375-15-5

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Further information on the Indira Gandhi National Open University Courses may be obtained from the University's Office at Maidan Garhi, New Delhi-110 068 or visit University's http:// www.ignou.ac.in.

Printed and Published on behalf of the Indira Gandhi National Open University, New Delhi by Registrar, MPDD. IGNOU

Printed at: A-One Offset Printers, 5/34, Kirti Nagar Indl. Area, New Delhi-110015

BLOCK 3 MATRICES AND VECTORS

Introduction

This block is a continuation of the themes on quantitative techniques taken up earlier, we discuss the variables to be presented through linear relations and focus on writing these in compact and concise forms. After familiarising ourselves with their basic operations, we extend the coverage to give two important areas of operation. **Unit 10,** under the title, Vector Analysis, deals with basic operations of variables when their magnitude and direction is considered. Presentation of variables in an array of rows and colums (determinates-and matrices) and understanding their forms, properties and operations are included in **Unit 11.** Development of input-output analysis as a major applications tools necessities special treatment of the theme within the area of linear economics. As a result, features and techniques of input-output analysis including themes such a input-output table, technological coefficient matrix, closed and open input-output models an feasibility conditions of the system are discussed in **Unit 12.** The last unit of the block **(Unit 13)** takes up the basic technique of linear programming which again has witnessed wider application. Keeping in view its application dimension, we have put greater emphasis on its formulation and solution methods, sidelining the theoretical considerations.

UNIT 10 VECTOR ANALYSIS

Structure

10.0 Objectives

10.1 Introduction

10.2 Vector Addition and Scalar Multiplication

10.3 Geometrical and Physical Interpretations

10.4 Norm and Inner Product

10.5 Vector Spaces and Subspaces

10.6 Linear Dependence of Vectors

10.7 Generators and Basis

10.8 Vectors and Matrices

10.9 Characteristic Value Problem

- 10.9.1 Characteristic Equation
- 10.9.2 Sum and Product of Roots
- 10.9.3 Characteristic Vector
- 10.9.4 Diagonalisation

10.10 Linear Independence of Eigen Vectors

10.11 Quadratic Forms

10.12 Definiteness and Eigen Values

10.13 Vector Differentiation

10.14 Let Us Sum Up

10.15 Key Words

10.16 Some Useful Books

10.17 Answer or Hints to Check Your Progress

10.18 Exercises

10.0 OBJECTIVES

After going through this unit, you will be able to:

- understand the presentation of variables with magnitude and direction;
- carry on the basic operations of addition, subtraction multiplication and division of vectors;
- deal with the problems in quadratic forms; and
- do the differentiation of vectors.

10.1 INTRODUCTION

Vectors as ordered sets

Generally elements of a set can be arranged in any manner. It does not matter whether a three element set is written as (a, b, c) or (a, c, b) or (b, c, a). But for certain purposes this is not helpful and we need to introduce the consideration of order. Sets are distinguished not only by their members but also by the order in which they appear. Thus, as an '*ordered*' set the pair (a, b) is different from (b, a) unless b ≡ a. An ordered set of this kind is called a '*vector*'.

Here we shall consider only vectors with a finite number of elements. Further, unless stated otherwise, we shall assume that each element in a vector is a real number. Thus, we are typically concerned with the n-type $(x_1, x_2, \ldots\ldots, x_n)$ of real number. A vector all of whose elements are zero is called the '*null vector*'. In economic analysis, quantities of various commodities consumed by an individual can be represented as a vector.

10.2 VECTOR ADDITION AND SCALAR MULTIPLICATION

A given vector, i.e., ordered set of finite number of elements, can be written as a row, e.g., (a, b, c...) or, as a column, e.g., $\begin{pmatrix} a \\ b \\ c \\ . \\ . \end{pmatrix}$.

These are referred as row and column vectors respectively.

Let us consider $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ in which each $a_i \in \mathbb{R}$. **a** is then said to be a vector over the field $\mathbb{R}$. Like ordinary number, sum and difference of two vectors can be defined provided they have the same number of elements.

Thus, if, $\mathbf{a} = (a_1, a_2, \ldots, a_n)$; $\mathbf{b} = (b_1, b_2, \ldots, b_n)$

then $\mathbf{a} + \mathbf{b} = (a_1+b_1, a_2+b_2, \ldots, a_n+b_n)$ and

$$\mathbf{a} - \mathbf{b} = (a_1-b_1, a_2-b_2, \ldots, a_n-b_n)$$

Note that here **a**=**b** if and only if $a_1=b_1, a_2=b_2, \ldots, a_n=b_n$.

Further, if λ is a real number, we can define the scalar product (λ.**a**) as follows: $(\lambda.\mathbf{a}) = (\lambda a_1, \lambda a_2, \ldots, \lambda a_n)$

Example 1: $\mathbf{a} = (1\ \ 3\ \ -2)$, $\mathbf{b} = (4\ \ 0\ \ 6)$

Then $\mathbf{a} + \mathbf{b} = (1+4\ \ 3+0\ \ -2+6) = (5\ \ 3\ \ 4)$;

$\mathbf{a} - \mathbf{b} = (1-4\ \ 3-0\ \ -2-6) = (-3\ \ 3\ \ -8)$ and $3\mathbf{a} = (3\ \ 9\ \ -6)$

Check Your Progress 1

1) List the Difference between vector and scalar presentation.

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2) If $\mathbf{a}_1 = (2\ 3\ 4\ 7)$, $\mathbf{a}_2 = (0\ 0\ 0\ 1)$ and $\mathbf{a}_3 = (1\ 0\ 1\ 0)$, find

$\mathbf{a}_1 + 2\mathbf{a}_2 + 3\mathbf{a}_3$.

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3) If $\mathbf{x}_1 = (2\ 9\ 8)$, $\mathbf{x}_2 = (0\ 1\ 0)$ and $\mathbf{x}_3 = (1\ 0\ 1)$, find $2\mathbf{x}_2 + 5\mathbf{x}_3 - \mathbf{x}_1$.

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10.3 GEOMETRICAL AND PHYSICAL INTERPRETATIONS

Vector is often represented geometrically by a line with an arrowhead on it $\left(say, \overrightarrow{x}\right)$. The length of the line indicates the magnitude of the vector, and the arrow denotes its direction. It should be noted that a vector is not a number. If we are considering vector lying in a plane, then two numbers are needed to describe any vector: one for its magnitude and another giving its direction (the angle it makes with one of the coordinate axes). If vectors in three-dimensional spaces are being studied, three numbers are needed to describe any vector: one number for its magnitude, and two to denote its orientation with respect to some coordinate system.

In general, a vector may originate at any point in space and terminate at any point. Physical quantities such as force are, of course, independent of where one places the vector in a coordinate system. They depend only on the magnitude and direction of the vector. For this reason, it is convenient to have a vector start always at the origin of the coordinate system as in Figure 10.1. Here we shall, for simplicity adopt the convention that all vectors begin at the origin of the coordinate system rather than at some other point in space.

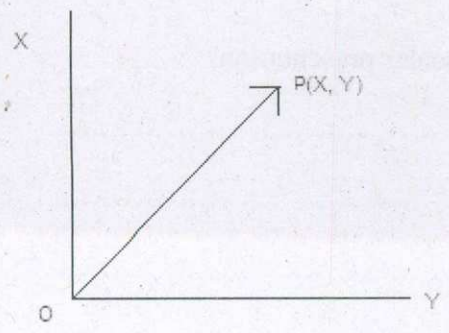


Fig. 10.1

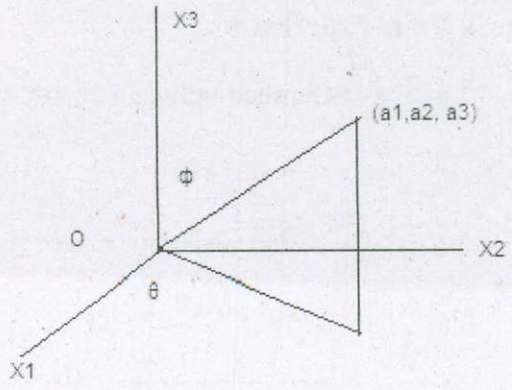


Fig. 10.2

Consider a vector in Figure 10.2. It will be observed that by specifying point (a_1 a_2 a_3), that is, the point where the head of the vector terminates, we have completely characterised the vector. Its magnitude (in some physical units) is $\sqrt{a_1^2+a_2^2+a_3^2}$ and its direction is characterised by the two angles θ and ϕ, where

$$\tan\theta = \frac{a_2}{a_1} \quad \text{and} \quad \cos\phi = \frac{a_3}{a_1^{\,2}+a_2^{\,2}+a_3^{\,2}}.$$

Thus, there is a one-to-one correspondence between all points in space and all vectors, which emanate from the origin. For any given point (a_1, a_2, a_3), a corresponding unique vector can be drawn from the origin to this point. Conversely, for any given vector **a**, there is a unique point (a_1, a_2, a_3), which is the point where the vector terminates. Because of this correspondence between vectors and points we can write **a** = (a_1, a_2, a_3).

10.4 NORM AND INNER PRODUCT

Both scalar multiplication and vector addition transform one or a pair of vectors into another vector. We can also define certain functions on one or a pair of vectors, which are useful in defining their lengths and angles between them. Unlike the geometrical representations defined in the preceding section, we are now dealing with functions, which transform a vector into a scalar, in the present case a real number. These are as follows: consider a vector **a** = (a_1, a_2, …, a_n).

We can define the '*norm*' of **a** as $\|\mathbf{a}\| = a_1^{\,2} + a_2^{\,2} + \ldots + a_n^{\,2}$

Example 2: If **a** = (5 -2 3) then $\|a\| = 5^2+(-2)^2+3^2=38$

The positive square root of the norm is called the length (or magnitude, as stated earlier) of **a**, and denoted by $\ell(\mathbf{a})$.

Thus, $\ell(\mathbf{a}) = \|\mathbf{x}\|^{1/2} = \left[\sum_{i=1}^{n} x_i^{\,2}\right]^{1/2}$

For a pair of vectors (consisting of the same number of elements n):

$\mathbf{a} = (a_1, a_2, ..., a_n)$ and $\mathbf{b} = (b_1, b_2, ..., b_n)$. We can define their inner product as

$$< a, b > = a_1b_1 + a_2b_2 + ... + a_nb_n = \sum_{i-1}^{n} a_ib_i$$

Example 3: If $\mathbf{a} = (1, -1, 2)$; $\mathbf{b} = (-2, 3, 6)$,

then $<\mathbf{a}, \mathbf{b}> = 7$

Clearly, the norm defined above can be regarded as the '*inner product*' of vector **a** with itself.

Two vectors are said to be orthogonal to each other if their inner product is zero, i.e., $<\mathbf{a}, \mathbf{b}> = 0$

This would hold if, e.g., $\mathbf{a} = (1, 3)$, $\mathbf{b} = (-3, 1)$. It is easy to see that that null vector **0** is orthogonal to every other vector.

Using the distances and the inner product, we can now define

$$\cos\theta = \frac{<\mathbf{a},\mathbf{b}>}{\{\|\mathbf{a}\|.\ \|\mathbf{b}\|\}^{1/2}}$$

By Schwartz's inequality, it is shown that $\left(\sum_{i=1}^{n} a_ib_i\right)^2 \leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2$

$\Rightarrow \cos^2\theta \leq 1$ as it should be.

Two results follow right away from the above definition.

a) If $\mathbf{b} = \lambda\mathbf{a}$ i.e. **b** is **a** scalar multiple of a,

$\|\mathbf{b}\| = \lambda^2 \|\mathbf{a}\|$ and $<\mathbf{a}, \mathbf{b}> = \lambda\|\mathbf{a}\|$

$\Rightarrow \cos\theta = 1$ or $\theta = 0$

b) If $<\mathbf{a}, \mathbf{b}> = 0$ i.e., **a** and **b** are orthogonal,

then $\cos\theta = 0 \Rightarrow \theta = \pi/2$. In other words, **a** and **b** are perpendicular to each other.

Check Your Progress 2

1) Given $x = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ and $y = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$, find 2y and x-y graphically.

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2) Find the norm of the following vectors: (i) (2, 5); (ii) (-2, 2).

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3) Find the inner product of the following pairs of vectors:

(i) (2, 3, 4) and (4, 5, 5); (ii) (-2, -3, 4) and (4, 5, -6).

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10.5 VECTOR SPACES AND SUBSPACES

Let us now define a set $\vee$ such that it contains, as its elements, all n – component vectors that can be generated from the field of real numbers, i.e.,

$$\vee=\left\{(x_1, x_2, \ldots\ldots, x_n) \,\middle|\, x_i \in \mathbb{R} \; ; i = 1, 2, \ldots, n\right\}$$

Properties:

i) If $\mathbf{x}, \mathbf{y}, \mathbf{z}, \in \vee$, then $\mathbf{x} + \mathbf{y} \in \vee$

such that $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$

and, $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{Y}) + \mathbf{z}$.

$\vee$ contains a vector 0 such that $x + 0 = x$

ii) For any $\lambda, \theta, \delta \in \mathbb{R}$, it must be true that $\lambda . \mathbf{x} \in \vee$

such that $\lambda.(\delta.\mathbf{x}) = (\lambda.\delta).\mathbf{x}$, $\lambda.(\delta.\mathbf{x}) = \delta\,(\lambda.\mathbf{x})$ and $1.\mathbf{x} = \mathbf{x}$

iii) $\lambda(\theta + \delta)\mathbf{x} = (\lambda.\theta + \lambda.\delta)\mathbf{x}$ and $\lambda(\mathbf{x} + \mathbf{y}) = \lambda.\mathbf{x} + \lambda.\mathbf{y}$

The set $\vee$ satisfying i, ii and iii is said to be a vector space or a linear space over the field $\mathbb{R}$. The aforesaid properties imply that $\vee$ is closed under vector addition as well as under scalar multiplication of vectors. Both operations are commutative and associative. Scalar multiplication is distributive over vector addition. And $\vee$ contains an element 0, the null vector, in which each component is equal to zero.

w is said to be a subspace of $\vee$ if

a) w is a subset of $\vee$ and,

b) w is a vector space in its own right, i.e., it satisfies all the axioms specified above for a vector space.

Example 4: Consider a two dimensional plane, $\vee_2$, and suppose the set w consisting of all points lying along the horizontal (or the vertical) axis. w is a subset of $\vee_2$ and it satisfies all the properties listed above. In fact, all points on any line drawn through the origin constitute a subspace. Algebraically,

$$w = \{(x_1, x_2) \mid x_1 \in \mathbb{R}, x_2 = 0\}$$

$$\text{or, } w = \{(x_1, x_2) \mid x_2 \in \mathbb{R}, x_1 = 0\}$$

$$\text{or, } w = \{(x_1, x_2) \mid c_1x_1 + c_2x_2 = 0\}$$

are subspaces of $\vee_2 = \{(x_1, x_2) \mid x_1 \in \mathbb{R}, x_2 \in \mathbb{R}\}$

10.6 LINEAR DEPENDENCE OF VECTORS

A set of vectors $\mathbf{x}^1, \mathbf{x}^2, \ldots \mathbf{x}^n$ are said to be linearly independent if there exists scalars $c_1, c_2, \ldots, c_n$ not all zero such that $c_1\mathbf{x}^1 + c_2\mathbf{x}^2 + \ldots + c_n\mathbf{x}^n = 0$

On the contrary, if no such scalars exist, then the vectors $\mathbf{x}^1, \ldots, \mathbf{x}^n$ are linearly independent.

The term '*linearly*' in the above definition is important because only linear operations, i.e., scalar multiplication and vector addition, are permitted in obtaining the null vector.

Two important results regarding linear independence or lack of it are as follows:

a) If $\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^k$, a subset of a set of vectors $\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^n$ $(n>k)$, are linearly dependent, then the entire set is linearly dependent.

b) If a set of vectors $x^1, x^2, \ldots, x^n$ are linearly independent, then any subset of vectors, say, $x^1, x^2, \ldots, x^k$ $(k<n)$ are also linearly independent.

Example 5: Consider two vectors $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} 3 \\ 0 \\ 9 \end{pmatrix}$

To put differently, $3\mathbf{x} - \mathbf{y} = 0$.

In other case, $\mathbf{x} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $\mathbf{y} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$, and $\mathbf{z} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

so, $\mathbf{x} - \mathbf{y} - \mathbf{z} = 0$

i.e., in each of these cases any of the given vectors can be obtained from the remaining vectors through scalar multiplication and vector addition (or subtraction); hence, vectors are linearly dependent.

10.7 GENERATORS AND BASIS

Consider the case of two-dimensional Euclidian space (E^2).

[Note: An n-dimensional Euclidian space (Euclidian vector space) is defined as the collection of all vectors (points) $a = [a_1, a_2,\ldots,a_n]$ and it holds the addition and multiplication property and the concept of distance between the vectors is also applicable here]

Now take any vector $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. It may be generated from vectors such as $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ etc., as follows:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + c_2 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + c_3 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

By specifying all possible values for c_1,c_2 and c_3 we may be able to obtain all possible two element vectors $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ belonging to the two dimensional space. If this is the case, then we say that u, v and w are the '*generators*' of the two dimensional spaces (E^2).

Example 6: Consider the vector $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$. This can be expressed as

$$2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 5\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \text{ or } 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \text{ or } \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 4\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\text{or } 2\begin{pmatrix} 1 \\ 2 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Thus, we can generate the vector $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$ in a variety of ways by taking a set of vectors such as $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, or $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, or $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, or $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Then after a suitable choice of scalars c_1, c_2, etc., get the scalar product of these vectors and finally obtain the vector $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$ as the sum of vectors so obtained. So, we can say that $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$ is generated by vectors $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and so on.

Let us discuss some important points that have to be kept in mind. First, an arbitrary choice of u, v and w may not be good enough to generate all vectors

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Second, we need at last two vectors for this purpose. No single vector is, by itself, capable of giving us all elements of E^2. Third, the choice of vectors such as u, v and w can be made in a large number of alternative ways. In other words, such a set of vectors is not unique. Any set of such vectors is said to be a set of generators for E^2.

Now, it is clear that we need at least two vectors to generate the vectors in a plane. If these two vectors are linearly independent we can also assert that any additional vector would be unnecessary for our purpose. In fact, we need exactly two linearly independent vectors to generate all vectors $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

Any (two) vectors, which are linearly independent and are used to generate the elements of E^2 are said to constitute the '*basis*' of E^2.

To sum up, in general, we can say that a set of vectors $a_1, a_2,\ldots, a_k$ from E^n is said to span or generate E^n if every vector in E^n can be written as a linear combination of $a_1, a_2,\ldots, a_k$ and a basis for E^n is a linearly independent subset of vectors from E^n which spans the entire space.

Example 7: The vector **a** = [2, 3, 4,] can be written uniquely in terms of the vectors

$e_1 = [1, 0, 0]$, $e_2 = [0, 1, 0]$, and $e_3 = [0, 0, 1]$ which form a basis for E^3: $a = 2e_1 + 3e_2 + 4e_3$

10.8 VECTORS AND MATRICES

Before discussing the relationship between vectors and matrices, let us first introduce the concept of matrix. Though detail discussion will be given in the next unit, here we'll just introduce the basic concepts of matrices.

A system of 'm' linear equations in 'n' variable $(x_1, x_2,\ldots, x_n)$ can be arranged into the following format:

$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = d_1$

$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = d_2$

.....

$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = d_m$

Here variable x_1 appears only within the leftmost column, and in general, the variable x_j appears only in the j^{th} column on the left side of the equals sign. The double – subscript parameter symbol a_{ij} represents the coefficient appearing in the i^{th} variable. For example, a_{21} is the coefficient in the second equation, attached to the variable x_1. The parameter $d_{i,\ which}$ is unattached to any variable, on the other hand, represents the constant term in the i^{th} equation. For instance, d_1 is the constant term in the first equation. All subscripts are, therefore, keyed to the specific locations of the variables and parameters in the above set of equations.

There are three types of elements in the above equation system. The first is the set of coefficients a_{ij} ; the second is the set of variables $x_1, x_2, ..., x_n$ and the last is the set of constant terms $d_1, ..., d_m$. If we arrange the three sets as three rectangular arrays and label them, respectively, as A, X and d (without subscripts), then we have

$$A = \begin{bmatrix} a_{11} & a_{12} & ..... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & ... & a_{mn} \end{bmatrix}_{mxn} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{nx1} \quad d = \begin{bmatrix} d \\ d \\ \vdots \\ d_m \end{bmatrix}_{mx1}$$

Example 8: Consider the following linear – equation system

$6x_1 + 3x_2 + x_3 = 22$

$x_1 + 4x_2 - 2x_3 = 12$

$4x_1 - x_2 + 5x_3 = 10$

We can write,

$$A = \begin{bmatrix} 6 & 3 & 1 \\ 1 & 4 & -2 \\ 4 & -1 & 5 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad d = \begin{bmatrix} 22 \\ 12 \\ 10 \end{bmatrix}$$

Each of the term given within [ ] constitutes a matrix. In particular, a matrix is defined as a rectangular arrow of numbers, parameters or variables. The members of the array, referred to as the elements of the matrix and the location of each element in a matrix is unequivocally fixed by the subscript, every matrix is an ordered set. The number of rows and the number of columns in a matrix together define the dimension of the matrix. Since matrix A contains m row and n columns, it is said to be of dimension m x n (read: 'm by n'). Matrix with only one column is called column vector and with only one row is called row vector.

So, there exists pronounced similarity between matrices of one row or column and vector. Indeed, even the notation is the same. Let us point out the complete equivalence between vectors and matrices having a single row or column. First, if one examines the definition of equality, addition and multiplication by a scalar for both vectors and matrices, it will be observed that they are equivalent when a matrix has a single row or column. Furthermore, the scalar product of a row vector and a column vector corresponds precisely to matrix multiplication, when the pre-multiplier is a row matrix, and the post multiplier a column matrix. We shall see later that the notation indicating the scalar product of two column vectors or two row vectors will again be identical with the appropriate matrix notation. Hence, here is a complete equivalence between matrices of one row or one column and vectors.

10.9 CHARACTERISTIC VALUE PROBLEM

A problem which arises frequently in application of linear algebra is that of finding values of a scalar parameter λ for which there exists vector $x \neq 0$ (i.e.,

non-null vector) satisfying $Ax = \lambda x$, where A is a given nth order square matrix. Such a problem is called a characteristic value ('*Eigen Value*', or proper value) problem. If $x \neq 0$ satisfies the equation $Ax = \lambda x$ for a given x, then A operating on x yields a vector which is a scalar multiple of x.

Thus, λ is called the characteristic value (or equivalently, characteristic root, Eigen value or proper value) and x is called the characteristic vector (or '*Eigen Vector*') associated with A if $Ax = \lambda x$

10.9.1 Characteristic Equation

To solve simultaneously for n and for λ, the system of equation $Ax = \lambda x$ can be written as $(A - \lambda I)x = 0$, where λ is a unit matrix of order n. Now, if the system has to have a non-trivial solution for x then rank$[A - \lambda I] < n$.

[Note: the concept of rank of the matrix will be discussed in detail in the next chapter.]

See that $[A - \lambda I] = 0$

i.e, if and only if $\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0$

This gives us an n^{th} degree polynomial equation in λ. This is called the '*characteristic equation*' of A and the left hand side is called the '*characteristic polynomial*'.

Example 9: Characteristic polynomial for second order matrix is

$$\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} = 0 \Rightarrow (a_{11}-\lambda)(a_{22}-\lambda) - a_{12}a_{21} = 0$$
$$\Rightarrow \lambda^2 - \lambda(a_{11}-\lambda) + (a_{11}a_{22} - a_{12}a_{21}) = 0$$

which is a quadratic equation in λ.

[Note: if A is of order n, then the equation $[A - \lambda I] = 0$ has n roots which may be real or complex, distinct or multiple, and zero or non-zero.]

10.9.2 Sum and Product of Roots

Consider the equation $\lambda^2 - \lambda(a_{11} - \lambda) + (a_{11}a_{22} - a_{12}a_{21}) = 0$ and let us write it as $c_0\lambda^2 + c_1\lambda + c_2 = 0$ where, $c_0 = 1$, $c_1 = -(a_{11} + a_{22})$, $c_2 = a_{11}a_{22} - a_{12}a_{21}$.

Applying the knowledge of determining roots of the quadratic equation, we can write,

$$\lambda_1 + \lambda_2 = -\frac{c_1}{c_0} = a_{11} + a_{22}$$

and $\lambda_1 \lambda_2 = \frac{c_2}{c_0} = a_{11}a_{22} - a_{21}a_{12} = |A|$

In general, if A is of order n, then the characteristic equation $[A - \lambda I] = 0$ can be written as $c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + ... + c_{n-1}\lambda + c_n = 0$

where $c_0 = 1, c_1 = -\sum_{I=1}^{n} a_{ii}$, and $c_n = (-1)^n |A|$

C_2 = Sum of all principal minors of A of order i.

$C_j = (-1)^j \sum$ all principal minor or order j

Thus, if characteristic roots are $\lambda_1, \lambda_2, ..., \lambda_n$,

then $\sum_{i=1}^{n} \lambda i = \sum_{i=1}^{n} a_{ii} = \text{Trace}(A)$ and $\lambda_1.\lambda_2. .... .\lambda_n = \prod^{n}_{i=1} \lambda_i = |A|$

Example 10: Let, $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

The characteristic equation is

$$[A - \lambda I] = \begin{vmatrix} 2 - \lambda & 1 \\ 1 & 2 - \lambda \end{vmatrix} = (2 - \lambda)^2 - 1 = 0$$

or, $4 - 4\lambda + \lambda^2 - 1 = 0$

or, $\lambda^2 - 4\lambda + 3 = 0$

$$\lambda = \frac{4 \pm \sqrt{16 - 12}}{2} = \frac{4 \pm 2}{2} = 3, 1 \quad \text{i.e. } \lambda_1 = 3, \lambda_2 = 1$$

Properties:

i) If A is singular, i.e., |A| = 0, then at least one characteristic root of A is zero.

ii) If A is symmetric all roots are real, i.e., a real symmetric matrix cannot have complex characteristic roots.

iii) If $\lambda_1, \lambda_2, ..., \lambda_n$ are the roots of a non-singular matrix A, then the roots of A^{-1} are $1/\lambda_1, 1/\lambda_2, ..., 1/\lambda_n$

iv) If A and B are two square matrices of the same order such that AB=BA and their roots are $\lambda_1, \lambda_2, ..., \lambda_n$ and $\mu_1, \mu_2, ..., \mu_n$, then the roots of AB are $\lambda_i \mu_i$; i =1, 2...., n.

v) The eigen values of a diagonal or a triangular matrix coincide with the diagonal elements.

vi) If $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$ are the roots of A and k is an integer, then the roots of A^k are $\lambda_1^k, \lambda_2^k, \ldots\ldots, \lambda_n^k$.

10.9.3 Characteristic Vector

Let us now consider the determination of **x** of the equation $\mathbf{Ax} = \lambda \mathbf{x}$. Note that λ is chosen in such a way that $[\mathbf{A} - \lambda\mathbf{I}]$ is singular, i.e., of rank less than n. This ensures that $[\mathbf{A} - \lambda\mathbf{I}]\,\mathbf{X} = 0$ is a homogeneous system of equations having a non-trivial solution. But this solution will be non-unique. We may make it unique through some additional restrictions. Further, one set of non-trivial solution x^i will correspond root of λ_i.

Example 11: Let us find the characteristics vector for the matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

As we derived earlier, $\lambda_1 = 3$, $\lambda_2 = 1$.

$\therefore$ For $\lambda_1 = 3$, $(A - \lambda_i I) = \begin{bmatrix} 2-3 & 1 \\ 1 & 2-3 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$

To get $x^1 = \begin{bmatrix} x_1^1 \\ x_2^1 \end{bmatrix}$ we have $\left.\begin{aligned} -x_1^1 + x_2^1 &= 0 \\ x_1^1 - x_2^1 &= 0 \end{aligned}\right\} \Rightarrow x_1^1 = x_2^1$

So, the number of solutions is infinite. Let us choose one by setting $x_1^1 = 1$, so that $x^1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Next, take $\lambda_2 = 1$, and proceeding in the same way, we get, $x_2^2 = -x_1^2$

Again, setting $x_2^2 = 1$, we get $x^2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

we could normalize x^1 and x^2, i.e., define these such that $\|x^1\| = \|x^2\|$. This would give us $x^1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ and $x^2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

Note: x^1 and x^2 are orthogonal to each other.

10.9.4 Diagonalisation

It is a general property of real symmetric matrices that their roots are always real and that the associated characteristic vectors are mutually orthogonal. This

implies that such a matrix can always be diagonalized by P, the matrix of n characteristic vectors.

It we define a 2 x 2 matrix as $P \equiv (x^1\, x^2) = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

Then P'P = I, i.e., P is an orthogonal matrix.

$$\text{and } P'AP = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \Lambda$$

i.e., P'AP =Λ, where Λ is the diagonal matrix consisting of the characteristic values of A as its diagonal elements.

10.10 LINEAR INDEPENDENCE OF EIGEN VECTORS

Another important property in this context is that, if $\lambda_1, \lambda_2, \ldots\ldots \lambda_n$, are distinct, then the corresponding characteristic vectors of **A**, $\mathbf{x}_1$, $\mathbf{x}_2$, …, $\mathbf{x}_n$, are linearly independent. The condition (that roots are distinct) is sufficient but not necessary. Even when some root (or roots) is repeated, it may be possible to find a set of n characteristic vectors, which are linearly independent.

Moreover, if two roots are distinct then the characteristic vectors corresponding to them will be orthogonal.

10.11 QUADRATIC FORMS

Consider the expression $a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2$ where a_{11}, a_{12} and a_{22} are given constants and x_1 and x_2 are real variables. Expressions such as this, being homogeneous of degree 2 in x_1 and x_2, is called quadratic forms.

Generally, the sign of the expression would depend not only on a's but also on the values of x_1 and x_2. But some times given the values of a's, it is possible to determine the sign of the expression, whatever x_1 and x_2 may be.

For example, $Q(x_1, x_2) = x_1^2 - 4x_1x_2 + 4x_2^2 = (x_1 - 2x_2)^2$

So, the sign of this is positive for all x_1 and x_2.

Similarly, $Q\ (x_1x_2) = -10x_1^2 + 6x_1x_2 - x_2^2 = -x_1^2 - (3x_1 - x_2)^2$

Therefore, the sign is negative for all (x_1, x_2). Such quadratic forms, which have a definite sign for all (x_1, x_2) are said to be either positive definite or negative definite depending on the sign. Others, which could vary in sign, are said to be indefinite.

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In general, with variables $x_1, x_2, ..., x_n$, we may have a quadratic form as:

$$Q(x_1, x_2, ..., x_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$$

Thus, Q(.) is positive definite if $Q(x) > 0 \; \forall \; x \in \mathbb{R}_n$

Or, positive semi-definite if $Q(x) \geq 0 \; \forall \; x \in \mathbb{R}_n$

Identical definitions hold for negative definite and negative semi-definite quadratic forms.

Consider the equation: $Q(x_1, x_2) = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2$

$$= a_{11}\left[\left(x_1 + \frac{a_{12}}{a_{11}} x_2\right)^2 + \left(\frac{a_{22}}{a_{11}} - \frac{a_{12}^2}{a_{11}^2} \right) . x_2^2 \right]$$

$$= a_{11}\left(x_1 + \frac{a_{12}}{a_{11}} x_2 \right)^2 + \left(\frac{a_{11}a_{22} - a_{12}^2}{a_{11}} \right) x_2^2$$

Clearly, $Q > 0$ for all (x_1, x_2) if $a_{11} > 0$ and $\dfrac{a_{11}a_{22} - a_{12}^2}{a_{11}} > 0$

Thus, $a_{11} > 0$, $a_{11}a_{22} - a_{12}^2 > 0$ are sufficient conditions for $Q > 0$. These are necessary also, as for $a_{11} \leq 0$, Q can be less than zero with high x_1 and low x_2 or as for $(a_{11}a_{22} - a_{12}^2) < 0$. Q can be less than zero if $a_{11} < 0$.

10.12 DEFINITENESS AND EIGEN VALUES

A necessary and sufficient condition for

$Q \geq 0$ is that $|A| = 0$ and $a_{11} \geq 0$.

If $a_{11} > 0$, then $a_{22} = 0$.

$$\left[Here\ A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \right]$$

In terms of characteristic roots, the necessary and sufficient conditions are that $[A - \lambda I] = 0$ $\lambda_1 \geq 0$, $\lambda_2 \geq 0$. At least one root must be zero.

For Q to be negative semi definite, the corresponding conditions are

$a_{11} \leq 0$ and $|A| = 0$

And in terms of characteristic roots $\lambda_1 \leq 0$, $\lambda_2 \leq 0$, equality must hold in at least one case.

10.13 VECTOR DIFFERENTIATION

In some maximisation and minimisation problems, expressions involving vectors and matrices must be differentiated. The derivatives of simple expressions involving vectors and matrices can be obtained directly from the definitions. The vector derivatives of linear functions, quadratic forms and bilinear forms are defined and illustrated in this section. Second and higher order derivatives are not discussed explicitly, but can be obtained by successive differentiation in the usual manner.

Vector Differentiation of a Linear Function

An expression of the form **a'x**, where **a** is nx1 and **x** is nx1 is a linear function.

For the linear function $\mathbf{a'x} = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

$$= a_1x_1 + a_2x_2 + \cdots + a_nx_n$$

the partial derivative of **a'x** with respect to the scalar x_i, i= 1, 2, ... , n, are

$$\frac{\partial}{\partial z}(x'Bz) = B'x$$

that is, the partial derivatives are the elements of the vector **a**. Thus, if the n partial derivatives are arranged as a vector **a**, the process of vector differentiation is defined by $\frac{\partial(a'x)}{\partial x} = a$ where $\frac{\partial(a'x)}{\partial x}$ indicate the operation of differentiating **a'x** with respect to the elements of the vector **x**.

Check Your Progress 3

1) Define the terms: eigen value, eigen vector and characteristic equations.

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2) If $a = \begin{bmatrix} 2a \\ -a \\ 3a \\ a \end{bmatrix}$ and $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ then prove that $\frac{\partial}{\partial x}(a'x) = a$

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Vector Differentiation of a Vector of Functions

If **y** is an n dimensional column vector each element of which is a function of the m elements of **x**, that is, if $y_i = f(x_1, x_2, \ldots, x_m)$, i=1, 2, … , n, then each y_i can be differentiated partially with respect to each x_j and these partial derivatives can be arranged in an m x n matrix as follows:

$$\frac{\partial y}{\partial x} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_2}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_1} \\ \frac{\partial y_1}{\partial x_2} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_n}{\partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_1}{\partial x_m} & \frac{\partial y_2}{\partial x_m} & \cdots & \frac{\partial y_n}{\partial x_m} \end{vmatrix}$$

Example: If $y = \begin{bmatrix} x_1^2 + 2x_2^2 - 3x_1x_3 \\ 3x_1x_2^2 - x_2^2 + 4x_2x_3^2 \end{bmatrix}$,

then $\frac{\partial y}{\partial x} = \begin{bmatrix} 2x_1 - 3x_3 & 3x_2^2 \\ 4x_2 & 6x_1x_2 - 2x_2 + 4x_3^2 \\ -3x_1 & 8x_2x_3 \end{bmatrix}$

Vector Differentiation of a Quadratic Form

An expression of the form **x'Ax**, where **a** is an n x n symmetric matrix, is a quadratic form. The quadratic form **x'Ax** can be expanded as follows

$$\mathbf{x'Ax} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= a_{11}x_1^2 + a_{12}x_1x_2 + 2a_{13}x_1x_3 + \ldots + 2a_{1n}x_1x_n + a_{22}x_2^2 + 2a_{23}x_2x_3 + \ldots +$$

$$2a_{2n}x_2x_n + \ldots + a_{nn}x_n^2.$$

Taking partial derivatives with respect to the elements of **x**

$$\frac{\partial}{\partial x_1}(x'Ax) = 2(a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n)$$

$$\frac{\partial}{\partial x_2}(x'Ax) = 2(a_{12}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n)$$

.....

$$\frac{\partial}{\partial x_n}(x'Ax) = 2(a_{1n}x_1 + a_{2n}x_2 + \cdots + a_{nn}x_n)$$

Apart from the factor 2, the right hand side of the set of equations contains the elements of **Ax**, which is an n dimensional column vector; alternatively. The right hand side contains the elements of **x'A**, which is an n dimensional row vector. Thus, $\frac{\partial}{\partial x}(\mathbf{x'Ax}) = 2\mathbf{Ax}$ or $\frac{\partial}{\partial x}(\mathbf{x'Ax}) = 2\mathbf{x'A}$

In practice, the choice between these two forms usually depends on the context in which differentiation occurs, since matrices can be equated only if they are of the same order and, specifically, a row vector cannot be set equal to a column vector.

Check Your Progress 4

1) If $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $A = \begin{bmatrix} 3 & 1 & -2 \\ 1 & 0 & 3 \\ -2 & 3 & 2 \end{bmatrix}$ then show that $\frac{\partial}{\partial x}(x'Ax) = 2x'A$

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2) If $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $B = \begin{bmatrix} 2b_1 & b_1 + b_2 \\ b_2 + 2b_3 & 3b_2 \\ b_1 + b_3 & 4b_3 \end{bmatrix}$ $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$

then $\frac{\partial}{\partial x}(x'Bz) = Bz$ and $\frac{\partial}{\partial z}(x'Bz) = B'x$

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Vector Differentiation of a Bi-linear Form

An expression of the form **x'Bz**, where **x'** is 1xm, **B** is mxn, and **z** is nx1, is a bilinear form. The following derivatives for a bilinear form can be verified using the procedures above.

$\frac{\partial}{\partial x}(x'Bz) = Bz$ and $\frac{\partial}{\partial z}(x'Bz) = B'x$.

10.14 LET US SUM UP

This unit tells us the way of presenting a variable with magnitude and direction. It also gives the idea about the basic operations of vector such as addition, subtraction, multiplication and division along with the concepts such as norm, inner product, generator, basis, characteristic equation, eigen value, and eigen vector. The relationship between vector and matrices is covered through this unit. We have learnt the way of testing the definiteness of an equation of quadratic forms. It gives a method of finding the solutions of an equation system by applying differentiation of vectors.

10.15 KEY WORDS

Basis: Any (two) vectors, which are linearly independent and are used to generate the elements of E^2 are said to constitute the basis of E^2.

Characteristic Equation: To solve simultaneously for n and for λ, the system of equation $Ax = \lambda x$ can be written as $(A - \lambda I)\,x = 0$, where λ is a unit matrix of order n. Now, if the system has to have a non-trivial solution for x then rank $[A - \lambda I] < n$. $\Rightarrow [A - \lambda I] = 0$ i.e., if and only if $\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0$ This gives us an n^{th} degree polynomial equation in λ. This is called the 'characteristic equation' of A and the left hand side is called the 'characteristic polynomial'.

Eigen Value and Eigen Vector: If there exist a vector $\mathbf{X} \neq \mathbf{0}$ (i.e. non-null vector) satisfying $\mathbf{Ax} = \lambda\mathbf{x}$, where **A** is a given nth order square matrix and λ is a scalar parameter, then such a problem is called a eigen value problem and x is called the eigen vector associated with A if $Ax = \lambda x$.

Generator: Any vector $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ can be written as

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + c_2 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + c_3 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \text{ where}$$

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},\ v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},\ w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \text{ etc.,}$$

By specifying al possible values for c_1,c_2 and c_3 we may be able to obtain all possible two element vectors $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ belonging to the two dimensional space. If this is the case, then we say that u, v and w are the '*generators*' of the two dimensional spaces (E^2).

Inner Product: For a pair of vectors (consisting of the same number of elements n): $\mathbf{a} = (a_1, a_2, ..., a_n)$ and $\mathbf{b} = (b_1, b_2, ..., b_{an})$, their inner product can be defined as $< a, b > = a_1b_1 + a_2b_2 + ... + a_nb_n = \sum_{i-1}^{n} a_ib_i$

Norm: For a single vector $\mathbf{a} = (a_1, a_2, ..., a_n)$, the 'norm' of **a** can be defined as $\|a\| = a_1^2 + a_2^2 + ... + a_n^2$

Quadratic Forms: The expression $a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2$ where a_{11}, a_{12} and a_{22} are given constants and x_1 and x_2 are real variables, being homogeneous of degree 2 in x_1 and x_2, is called quadratic forms.

Sub Space: w is said to be a subspace of $\vee$ if

a) w is a subset of $\vee$ and,

b) w is a vector space in its own right, i.e., it satisfies all the axioms specified above for a vector space.

Vector Space: Define a set $\vee$ such that it contains, as its elements, all n – component vectors that can be generated from the field of real numbers, i.e., $\vee = \{(x_1, x_2, ..., x_n) \mid x_i \in \mathbb{R} ; i = 1, 2, ..., n\}$. Now if $\vee$ satisfies the following properties

i) x, y, z, $\in \vee$, then $x + y \in \vee$

such that $x + y = y + x$ and,

$x + (y + z) = (x + Y) + z$.

$\vee$ contains a vector 0 such that $x + 0 = x$

ii) For any $\lambda, \theta, \delta \in \mathbb{R}$,

it must be true that $\lambda . x \in \vee$

Such that $\lambda.(\delta.x) = (\lambda.\delta).x$, $\lambda.(\delta.x) = \delta(\lambda.x)$ and $1.x = x$

iii) $\lambda(\theta + \delta)x = (\lambda.\theta + \lambda.\delta)x$ and $\lambda(x + y) = \lambda.x + \lambda.y$

Then the set $\vee$ is said to be a vector space or a linear space over the field $\mathbb{R}$.

Vector: An ordered set expressed in a particular form like (a, b) with a, b as its elements is called a 'vector'.

10.16 SOME USEFUL BOOKS

Chiang Alpha C.(1984), *Fundamental Methods of Mathematical Economics*, Third edition, McGraw-Hill Book Company.

Chakravorty, J.G & P.R. Ghosh (1996), *Higher Algebra Including Modern Algebra*, Twelfth Edition, U.N. Dhur & Sons Pvt. Ltd. Calcutta.

Hadley, G. (2000), *Linear Algebra*, Narosa Publishing House, New Delhi, 2000.

Mukherji, B. & V. Pandit (1982), *Mathematical Methods for Economic Analysis*, Allied Publishers Pvt. Ltd., New Delhi.

10.17 ANSWER OR HINTS TO CHECK YOUR PROGRESS

Check Your Progress 1

1) $\mathbf{a}_1 + 2\mathbf{a}_2 + 3\mathbf{a}_3 = (5\ 3\ 7\ 9)$

2) $2\mathbf{x}_2 + 5\mathbf{x}_3 - \mathbf{x}_1 = (3\ -11\ -3)$

Check Your Progress 2

1) Do yourself

2) Answer: (i) 9; (ii) 8.

3) Answer: (i) 43; (ii) -47.

Check Your Progress 3

1) See the text and answer

2) [Hint: $\mathbf{a'x} = 2ax_1 - ax_2 + 3ax_3 + ax_4$, then find

$$\frac{\partial}{\partial x_1}(a'x) = 2a,\ \frac{\partial}{\partial x_2}(a'x) = -a,\ \frac{\partial}{\partial x_3}(a'x) = 3a \text{ and } \frac{\partial}{\partial x_4}(a'x) = a\ ]$$

Check Your Progress 4

1) [Hint: $\mathbf{x'Ax} = 3x_1^2 + 2x_3^2 + 2x_1x_2 - 4x_1x_3 + 6x_2x_3$ then find

$$\frac{\partial}{\partial x_1}(x'Ax) = 6x_1 + 2x_2 - 4x_3,\ \frac{\partial}{\partial x_2}(x'Ax) = 2x_1 + 6x_3 \text{ and}$$

$$\frac{\partial}{\partial x_3}(x'Ax) = -4x_1 + 6x_2 + 4x_3\ ]$$

2) Follow the same logic as given above.

10.18 EXERCISES

1) Derive the norm of the following vectors if

(i) $\mathbf{a}=(0\ 2\ 1)$; $\mathbf{b}=(6\ 3\ 2)$; $\mathbf{c}=(-5\ 4\ 3)$; $\mathbf{d}=(0\ 0\ 1)$; $\mathbf{e}=(-3\ -4\ -2)$.

[Answer: (i) 5; (ii) 49; (iii) 50; (iv) 1; (v) 29.]

2) If $b = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ and $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, find $\frac{\partial}{\partial x}(b'x)$. [Answer: $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$]

3) If $c = \begin{bmatrix} c^2 + 1 \\ 3c \\ 4c - 5 \\ c^3 \end{bmatrix}$ and $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, find $\frac{\partial}{\partial x}(c'x)$. [Answer: $\begin{bmatrix} c^2 + 1 \\ 3c \\ 4c - 5 \\ c^3 \end{bmatrix}$]

4) If $y = \begin{bmatrix} x_1^2 + 3x_2 \\ 2x_1x_2 - x_2 \\ 2x_1 + x_1x_2 - 3x_2^2 \\ 3x_1^2 - x_1^2x_2 - x_2^3 \end{bmatrix}$, find $\frac{\partial y}{\partial x}$.

[Answer: $\begin{bmatrix} 2x_1 & 2x_2 & 2 + x_2 & 6x_1 - 2x_1x_2 \\ 3 & 2x_1 - 1 & x_1 - 6x_2 & -x_1 - 3x_2^2 \end{bmatrix}$]

5) If $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $A = \begin{bmatrix} 3 & 0 & -1 \\ 2 & 1 & 4 \\ -1 & 0 & 3 \end{bmatrix}$ and $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$,

find $\frac{\partial}{\partial x}(x'Ay)$ and $\frac{\partial}{\partial y}(x'Ay)$.

[Answer: $y = \begin{bmatrix} 3y_1 - y_3 \\ 2y_1 + y_2 + 4y_3 \\ -y_1 + 3y_3 \end{bmatrix}$]

UNIT 11 LINEAR ALGEBRA

Structure

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11.0 OBJECTIVES

After studying this unit, you will be able to:

- explain the concepts and presentation of equations in the form of determinants and matrices;
- evaluate their properties; and
- know the techniques of solution offered by determinants and matrix operations.

11.1 INTRODUCTION

Analysis of economic relationships often requires inclusion of a large number of interrelated variables into the model. Consequently, finding solution to must of problems under investigation becomes cumbersome. Linear algebra helps in various ways providing means for their concise presentation and solution mechanism. In the present unit we discuss the tools of solving simultaneous equations with the aid of matrices and determinants. The following discussion starts with determinants, their main features and solution techniques. The concept and operational methods of matrices are taken up subsequently.

11.2 DETERMINANTS

11.2.1 Definition and Concepts

If the linear equation $a_1x+b_1=0$ and $a_2x+b_2=0$ have the same solution, then $\frac{a_1}{a_2}=\frac{b_1}{b_2}$, i.e., $\frac{a_1}{b_1}=\frac{a_2}{b_2}$, i.e., a_1b_2-a_2b_1=0. The expression $(a_1b_2-a_2b_1)$ is called a determinant and is denoted by the symbol $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$.

Note that a_1, a_2, b_1, b_2 are called the '*elements of the determinant*'. The elements in the horizontal direction form 'rows' and those in the vertical direction form '*columns*'. The determinant $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ has two rows and two columns. So it is called a 'determinant of the second order' and it has 2! = 2 terms in its expansion of which one is positive and another is negative. The '*diagonal term*' or the '*leading term*' of the determinant is (a_1, b_2) whose sign is 'positive'.

Again, if the linear equations $a_1x+b_1y+c_1=0$, $a_2x+b_2y+c_2=0$ and $a_3x+b_3y+c_3=0$ have the same solutions, then we have from the last two equations by cross multiplication.

$$\frac{x}{b_2c_3-b_3c_2}=\frac{y}{c_2a_3-c_3a_2}=\frac{1}{a_2b_3-a_3b_2}$$

or, $x = \dfrac{b_2c_3 - b_3c_2}{a_2b_3 - a_3b_2}$, $\quad y = \dfrac{c_2a_3 - c_3a_2}{a_2b_3 - a_3b_2}$

where $(a_2b_3 - a_3b_2) \neq 0$

These values of x and y must satisfy the first equation. Hence $a_1(b_2c_3\text{-}b_3c_2) + b_1(c_2a_3\text{-}c_3a_2) + c_1(a_2b_3\text{-}a_3b_2) = 0$.

The expression $a_1b_2c_3 - a_1b_3c_2 + a_3b_1c_2 - a_2b_1c_3 + a_2b_3c_1 - a_3b_2c_1$ is denoted by the symbol $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and has three rows and three columns.

Therefore, it is called a 'determinant of the third order' and it has 3! =6 terms of which three terms are positive and rest are negative. These terms may be obtained by writing a, b, c in the natural order and arranging the suffixes 1, 2, 3 in all possible orders. The 'diagonal term' or the 'leading term' of the determinant is a_1, b_2, c_3 whose sign is positive. So far a 'determinant of n^{th} order' is concerned there will be n! terms in its expansion and out of them half of the terms would be positive and rest would be negative.

Moreover, a third order determinant can be written as:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1(b_2c_3\text{-}b_3c_2) - b_1(a_2c_3\text{-}a_3c_2) + c_1(a_2b_3\text{-}a_3b_2)$$

A determinant of the form $\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial y} - \dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial x}$, where u and v are two given functions of x and y, is termed as 'Jacobean' of u and v with respect to x and y and is often written as $\dfrac{\partial(u,v)}{\partial(x,y)}$.

Similarly, $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$ is a 3rd order Jacobean of u, v and w as functions of x, y and z.

11.2.2 Minors and Cofactors

In the last part, it is to be noted that a_1 is multiplied by a lower order determinant (here the order is 2). $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$ is obtained by deleting the column and row containing a_1. The determinant $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$ is termed as 'minor' of a_1 in

the original matrix. Same procedure will be applied for other terms also. Thus, the minor of b_2 will be obtained by deleting the 2nd row and 2nd column.

$$\text{In } \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ minor of } b_2 = \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$$

So, the minor of any element in a n^{th} order determinant is thus a $(n-1)^{th}$ order determinant where n=1, 2, 3, … any finite positive number.

The '*cofactor*' of any element in a determinant is its coefficient in the expansion of the determinant. It is therefore equal to the corresponding minor with a proper sign. The sign of the cofactor of an element in the i^{th} row and j^{th} column is $(-1)^{i+j}$.

Now, a_1 lies in the 1st row and 1st column, hence its cofactor is

$$(-1)^{1+1}\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}.$$

Similarly b_3 lies in 3rd row and 2nd column,

hence its cofactor is $(-1)^{3+2}\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$.

$$\text{Thus, } \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1A_1 + b_1B_1 + c_1C_1 = a_1A_1 + a_2A_2 + a_3A_3 \text{ (Do yourself)}$$

where A_i, B_i, C_i (i=1, 2, 3) are respective cofactors of a_i, b_i and c_i

Example 1: Find the cofactors and the value of the following determinant

$$\begin{vmatrix} 3 & 4 & 7 \\ 2 & 1 & 3 \\ 7 & 2 & 1 \end{vmatrix}.$$

Solution: Here, $a_1 = 3$, $b_1 = 4$, $c_1 = 7$, $a_2 = 2$, $b_2 = 1$, $c_2 = 3$, $a_3 = 7$, $b_3 = 2$, $c_3 = 1$

$$\text{Here the cofactors are } A_1 = (-1)^{1+1}\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}$$

$$A_2 = (-1)^{1+2}\begin{vmatrix} 2 & 3 \\ 7 & 1 \end{vmatrix} = -\begin{vmatrix} 2 & 3 \\ 7 & 1 \end{vmatrix}$$

$$A_3 = (-1)^{1+3}\begin{vmatrix} 2 & 1 \\ 7 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 7 & 2 \end{vmatrix}$$

and so on.

So, expanding the determinant by 1st row, we have its value

$$= 3\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} - 4\begin{vmatrix} 2 & 3 \\ 7 & 1 \end{vmatrix} + 7\begin{vmatrix} 2 & 1 \\ 7 & 2 \end{vmatrix}$$

$$= 3(1-6) - 4(2-21) + 7(4-7)$$

$$= -15 + 76 - 21 = 40$$

Similarly, we can expand the determinant by any row or column. If we expand it by the 1st column, then, value of the determinant will be

$$= 3\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} - 2\begin{vmatrix} 4 & 7 \\ 2 & 1 \end{vmatrix} + 7\begin{vmatrix} 4 & 7 \\ 1 & 3 \end{vmatrix}$$

$$= 3(1-6) - 2(4-14) + 7(12-7)$$

$$= -15 + 20 - 35 = 40$$

11.2.3 Properties of Determinants

Property I → A determinant is unchanged in value, if all the rows of the determinant are changed into columns or vice versa, i.e.,

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{vmatrix}$$

Example 2: $\begin{vmatrix} 1 & 3 & 0 \\ 2 & 4 & 0 \\ 1 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 3 & 4 & 0 \\ 0 & 0 & 3 \end{vmatrix}$ [following property (1)]

Property II → The value of a determinant is unaltered numerically but changed in sign if two rows (or two columns) are interchanged.

Example 3: $\begin{vmatrix} 1 & 3 & 0 \\ 2 & 4 & 0 \\ 1 & 0 & 3 \end{vmatrix} = -\begin{vmatrix} 3 & 1 & 0 \\ 4 & 2 & 0 \\ 0 & 1 & 3 \end{vmatrix}$ [by interchanging column (1) and (2)]

$= \begin{vmatrix} 4 & 2 & 0 \\ 3 & 1 & 0 \\ 0 & 1 & 3 \end{vmatrix}$ [by interchanging row (1) and (2)]

Property III → A determinant has zero value if the elements of one row (or of one column) are equal or proportional to the corresponding elements of a second row (or of a second column).

Example 4: $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{vmatrix} = 0$ [since row (1) and (3) are identical]

Example 5: $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 1 & 2 & 4 \end{vmatrix} = 2\begin{vmatrix} 1 & 1 & 3 \\ 2 & 2 & 1 \\ 1 & 1 & 4 \end{vmatrix} = 0$ [since column (1) and (2) are identical]

Property IV → If all the elements of any one row (or column) are multiplied (or divided) by K, then value of the determinant is also multiplied (or divided) by K.

Example 6: $\begin{vmatrix} Ka_{11} & Ka_{12} & Ka_{1n} \\ a_{21} & a_{22} & a_{2n} \\ \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{nn} \end{vmatrix} = K\begin{vmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{nn} \end{vmatrix}$

It can also be numerically shown by considering a 3rd order determinant

$\begin{vmatrix} 2 & 10 & 16 \\ 3 & 7 & 1 \\ 5 & 4 & 11 \end{vmatrix} = 2\begin{vmatrix} 1 & 5 & 8 \\ 3 & 7 & 1 \\ 5 & 4 & 11 \end{vmatrix}$ [since all the elements in the first row have a common factor 2]

Property V → If all the elements of any row (or column) of a determinant can be expressed as the sum of (or difference between) two numbers, then the determinant can be expressed as the sum of (or difference between) two determinants.

Example 7: $\begin{vmatrix} a_1+\alpha_1 & b_1+\beta_1 & c_1+\delta_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Numerically it can be shown as follow

$$\begin{vmatrix} 8+2 & 2+5 & 5+4 \\ 4 & 9 & 1 \\ 5 & 11 & 7 \end{vmatrix} = \Delta(\text{say}) = \begin{vmatrix} 8 & 2 & 5 \\ 4 & 9 & 1 \\ 5 & 11 & 7 \end{vmatrix} + \begin{vmatrix} 2 & 5 & 4 \\ 4 & 9 & 1 \\ 5 & 11 & 7 \end{vmatrix} = \Delta_1 + \Delta_2 \quad (\text{say})$$

$$\therefore \Delta_1 = 8(63-11) - 2(28-5) + 5(44-45)$$

$$= 416 - 46 - 5 = 365$$

$\Delta_2 = 2(63-11)-5(28-5)+4(44-45) = 104-115-4 = -15$

$\therefore \Delta_1 + \Delta_2 = 365+(-15) = 365-15 = 350$

and $\Delta = 10(63-11)-7(28-5)+9(44-45) = 520-161-9 = 350$

$\therefore \Delta = \Delta_1 + \Delta_2$ Hence, the proposition is proved.

Property VI → The value of a determinant remains unchanged by adding (or subtracting) k times the elements of any row (or column) to (or from) the corresponding elements of any other row (or column), where k is any given number.

Example 8:

$$\begin{vmatrix} a_1+kb_1 & b_1 & c_1 \\ a_2+kb_2 & b_2 & c_2 \\ a_3+kb_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} kb_1 & b_1 & c_1 \\ kb_2 & b_2 & c_2 \\ kb_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + k\begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

[since the value of the second determinant is zero as its 1st and 2nd columns are identical]

It can also be numerically shown as follow:

$$\begin{vmatrix} 2+6 & 3 & 6 \\ 3+4 & 2 & 1 \\ 4+2 & 1 & 7 \end{vmatrix} = \Delta(say) \qquad = \begin{vmatrix} 2 & 3 & 6 \\ 3 & 2 & 1 \\ 4 & 1 & 7 \end{vmatrix} + \begin{vmatrix} 6 & 3 & 6 \\ 4 & 2 & 1 \\ 2 & 1 & 7 \end{vmatrix} =$$

$$\begin{vmatrix} 2 & 3 & 6 \\ 3 & 2 & 1 \\ 4 & 1 & 7 \end{vmatrix} + 2\begin{vmatrix} 3 & 3 & 6 \\ 2 & 2 & 1 \\ 1 & 2 & 7 \end{vmatrix} = \Delta_1 + \Delta_2 \text{ (say)}$$

$\therefore \Delta_1 = 2(14-1)-3(21-4)+6(3-8) = 26-51-30 = -55$

$\Delta_2 = 2[3(14-1)-3(14-1)+6(2-2)] = -55$

$\Delta = 8(14-1)-3(49-6)+6(7-12) = 104-129-30 = -55$

$\therefore \Delta = \Delta_1 = \Delta_2$; Hence, the proposition is proved.

11.2.4 Product of Two Determinants

The product of two determinants of same order is another determinant of the same order.

Example 9: $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} c_1 & d_1 \\ c_2 & d_2 \end{vmatrix} = \begin{vmatrix} a_1c_1 + b_1d_1 & a_1c_2 + b_1d_2 \\ a_2c_1 + b_2d_1 & a_2c_2 + b_2d_2 \end{vmatrix}$

Example 10: $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} =$

$$\begin{vmatrix} a_1x_1+b_1y_1+c_1z_1 & a_1x_2+b_1y_2+c_1z_2 & a_1x_3+b_1y_3+c_1z_3 \\ a_2x_1+b_2y_1+c_2z_1 & a_2x_2+b_2y_2+c_2z_2 & a_2x_3+b_2y_3+c_2z_3 \\ a_3x_1+b_3y_1+c_3z_1 & a_3x_2+b_3y_2+c_3z_2 & a_3x_3+b_3y_3+c_3z_3 \end{vmatrix}$$

Numerically, $\begin{vmatrix} 4 & 5 & 9 \\ 1 & 2 & 3 \\ 3 & 0 & 1 \end{vmatrix} \times \begin{vmatrix} 2 & 1 & 5 \\ 3 & 2 & 1 \\ 1 & 6 & 4 \end{vmatrix}$

$$= \begin{vmatrix} 4x2+5x3+9x1 & 4x1+5x2+9x6 & 4x5+5x1+9x4 \\ 1x2+2x3+3x1 & 1x1+2x2+3x6 & 1x5+2x1+3x4 \\ 3x2+0x3+2x1 & 3x1+0x2+2x6 & 3x5+0x1+2x4 \end{vmatrix} = \begin{vmatrix} 32 & 68 & 61 \\ 11 & 23 & 19 \\ 8 & 15 & 23 \end{vmatrix}$$

Check Your Progress 1

1) Find the value of the determinant $\begin{vmatrix} 2 & 1 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$

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2) Your are given a determinant $\begin{vmatrix} 2 & 0 & -1 \\ 1 & 1 & 7 \\ 3 & 3 & 9 \end{vmatrix}$. Work on it to verify important properties.

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3) Evaluate $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$

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4) Find the value of $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$

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11.2.5 Adjoint and Reciprocal Determinants

The '*adjoint*' or '*adjugate*' of a determinant Δ is the determinant Δ' whose elements are the cofactors of the corresponding elements of Δ

Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ then, $\Delta' = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$

Where $A_1, B_1, C_1, \ldots$ are the respective cofactors of $a_1, b_1, c_1, \ldots$ of Δ.

Note: Here $\Delta = \Delta^2$ when $\Delta \neq 0$.

Example 11: Let $\Delta = \begin{vmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{vmatrix}$

Here, $a_1 = 0, b_1 = 1, c_1 = 2$

$a_2 = 2, b_2 = 0, c_2 = 1$

$$a_3 = 3, b_3 = 2, c_3 = 0$$

$$\therefore \Delta_1 = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 2 & 0 \end{vmatrix} = 0 - 2 = -2$$

$$B_1 = (-1)\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} = -(0-1) = +1$$

$$\text{Similarly}, C_1 = 4, A_2 = 4, B_2 = -2, C_2 = 1,$$
$$A_3 = 1, B_3 = 4, C_3 = -2$$

$$\therefore \text{Adjoint of } \Delta \text{ is } \Delta' = \begin{vmatrix} -2 & 1 & 4 \\ 4 & -2 & 1 \\ 1 & 4 & -2 \end{vmatrix}$$

The '*reciprocal*' or '*inverse*' of a determinant $\Delta(\neq 0)$ is the determinant, which is formed by dividing every element of the adjoint of Δ by Δ. Thus, the reciprocal of Δ is

$$\begin{vmatrix} \frac{A_1}{\Delta} & \frac{B_1}{\Delta} & \frac{C_1}{\Delta} \\ \frac{A_2}{\Delta} & \frac{B_2}{\Delta} & \frac{C_2}{\Delta} \\ \frac{A_3}{\Delta} & \frac{B_3}{\Delta} & \frac{C_3}{\Delta} \end{vmatrix} = \frac{1}{\Delta^3}\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \frac{1}{\Delta^3}.\Delta' = = \frac{1}{\Delta^3}.\Delta^2 = \Delta^{-1}$$

Example 12: Following the last example (i.e., Example 11), we get,

$$\Delta = 0(0-2) + 1(1-0) + 2(4-0) = 9 \neq 0$$

$$\therefore \text{The reciprocal of } \Delta \text{ is } \begin{vmatrix} -\frac{2}{9} & \frac{1}{9} & \frac{4}{9} \\ 4/9 & -2/9 & 1/9 \\ 1/9 & 4/9 & -2/9 \end{vmatrix}$$

11.2.6 Symmetric Determinants

If a determinant is written in the form $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

in which the suffix of any element indicates its position (e.g., the suffix 12 of the element a_{12} indicates that the element is in the 1st row and in the 2nd column) and if $a_{ij} = a_{ji}$ for all i,j = 1, 2, 3; then the determinant is said to be symmetric.

Thus $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$ is a symmetric determinant.

Properties: <a>The square of any determinant is a symmetric determinant.

<b>The adjoint of symmetric determinant is symmetric.

Example 13: Let $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 7 \end{vmatrix}$

Here Δ is a symmetric determinant

Now, $\Delta^2 = \Delta.\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 7 \end{vmatrix} \text{ x } \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 7 \end{vmatrix}$

$$= \begin{vmatrix} 1+4+9 & 2+8+15 & 3+10+21 \\ 2+8+15 & 4+16+25 & 6+20+35 \\ 3+10+21 & 6+20+35 & 9+25+49 \end{vmatrix}$$

$$= \begin{vmatrix} 14 & 25 & 34 \\ 25 & 45 & 61 \\ 34 & 61 & 83 \end{vmatrix}$$, which is also a symmetric determinant.

Now adj. $\Delta = \begin{vmatrix} 3 & 1 & -2 \\ 1 & -2 & 1 \\ -2 & 1 & 0 \end{vmatrix}$, which is also a symmetric determinant

11.2.7 Skew and Skew-Symmetric Determinants

The determinant $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

is said to be 'skew', if $a_{ij}=-a_{ji}$ $\forall$ i,j=1,2,3 and $i \neq j$

Thus, $\begin{vmatrix} a & -b & -g \\ h & b & -f \\ g & f & c \end{vmatrix}$ is a skew determinant.

And if $a_{ij}=-a_{ji}$ $\forall$ i,j=1,2,3 and $i \neq j$ with $aij = 0 \forall i = j$,

then the determinant is said to be skew-symmetric.

Thus, $\begin{vmatrix} o & a & b \\ -a & o & c \\ -b & -c & o \end{vmatrix}$ is a skew-symmetric determinant.

Properties: <a>Every skew-symmetric determinant of third order (more generally of odd order) is zero.

<b>Every skew-symmetric determinant of second order (more generally of even order) is a perfect square.

Example 14: Let $\Delta = \begin{vmatrix} 0 & 5 & 2 \\ -5 & 0 & -3 \\ -2 & 3 & 0 \end{vmatrix} = (-1)^3 \begin{vmatrix} 0 & -5 & -2 \\ 5 & 0 & 3 \\ 2 & -3 & 0 \end{vmatrix}$

[taking (-1) common from each column]

$= -\Delta$

Hence, $2\Delta = 0 \Rightarrow \Delta = 0$

Again, let $\Delta_1 = \begin{vmatrix} 0 & 2 \\ -2 & 0 \end{vmatrix}$

$\therefore \Delta_1 = 0 - (-4) = 4 = 2^2$ which is a perfect square.

11.2.8 Solution of Simultaneous Equations by Cramer's Rule

Gabriel Cramer obtained a simple method of solution of linear simultaneous equations by using the properties of determinants.

In particular, let the system of equations be

$a_i x + b_i y + c_i z = k_i \quad (i = 1,2,3)$ ..... (1)

and $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$

Then $x\Delta = \begin{vmatrix} xa_1 & b_1 & c_1 \\ xa_2 & b_2 & c_2 \\ xa_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1x + b_1y + c_1z & b_1 & c_1 \\ a_2x + b_2y + c_2z & b_2 & c_2 \\ a_3x + b_3y + c_3z & b_3 & c_3 \end{vmatrix}$ $[c_i' = c_1 + yc_2 + zc_3]$

$= \begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix}$ [from (1)] $= \Delta_1$ (say)

$\therefore\ x = \frac{\Delta_1}{\Delta}$

Similarly, $y = \frac{\Delta_2}{\Delta}$ and $z = \frac{\Delta_3}{\Delta}$ where Δ_2 and Δ_3 are the determinants obtained from Δ substituting the columns (k_1, k_2, k_3) for the second and third column respectively.

Example 15: Suppose that the numbers x_1 and x_2 satisfy the equations

$x_1 - 2x_2 = 3$ and $3x_1 + 5x_2 = 20$.

Find x_1 and x_2 by using Cramer's rule.

Solution: $x_1 - 2x_2 = 3$, $3x_1 + 5x_2 = 20$

It may be written as $AX = B$

where, $A = \begin{pmatrix} 1 & -2 \\ 3 & 5 \end{pmatrix}$, $AX = B$ $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $B = \begin{pmatrix} 3 \\ 20 \end{pmatrix}$

Cramer's rule tells us that

$$x_1 = \frac{\begin{vmatrix} 3 & -2 \\ 20 & 5 \end{vmatrix}}{\begin{vmatrix} 1 & -2 \\ 3 & 5 \end{vmatrix}} = \frac{3(5) - (-2)(20)}{1(5) - (-2)3} = \frac{55}{11} = 5$$

$$\text{and } x_2 = \frac{\begin{vmatrix} 1 & 3 \\ 3 & 20 \end{vmatrix}}{\begin{vmatrix} 1 & -2 \\ 3 & 5 \end{vmatrix}} = \frac{1(20) - 3(3)}{11} = \frac{11}{11} = 1$$

Thus, if we have the value of the original determinant along with the values obtained from substituting the column of constants (i.e., k_i) in different places, then by Cramer's rule we can have the value of the corresponding variable.

11.2.9 Consistency of Equations

When a system of linear equations has one or more solutions, the equations are said to be 'consistent' or 'compatible'; otherwise they are said to be 'inconsistent' or 'incompatible'.

For the system of equations $a_i x + b_i y + c_i z = k_i$ $(i=1, 2, 3)$

we have, by Cramer's Rule, the unique solution

$x = \frac{\Delta_1}{\Delta}$, $y = \frac{\Delta_2}{\Delta}$, $z = \frac{\Delta_3}{\Delta}$ provided $\Delta \neq 0$ and Δ_i (i=1, 2, 3) is the determinant obtained by replacing the i^{th} column in the coefficient determinant by the column consisting of elements k_1, k_2, k_3.

Thus, the above equations are consistent, if $\Delta \neq 0$.

If $\Delta = 0$ and at least one of the determinants $\Delta_1, \Delta_2, \Delta_3$ be non-zero, then these equations do not posses any solution and are therefore inconsistent.

On the other hand, if $\Delta = 0$ and all of $\Delta_1, \Delta_2, \Delta_3$ are also zero, the system admits of an infinite number of solutions.

Note: If $k_1=k_2=k_3=0$, then the equations reduce to the set of homogeneous linear equations $a_ix + b_iy + c_iz = k_i$ (i=1, 2, 3)

By Cramer's rule, in this case, we have $x = y = z = \frac{0}{\Delta}$

If $\Delta \neq 0$, then the equations have no other but obvious solution x = y = z = 0, which is called a '*trivial solution*'.

On the other hand, such solutions in which x, y and z are not all zero are called a '*non-trivial solution*'.

Check Your Progress 2

1) Determine the signs to be attached to relevant minors if you are given the co-factors of a determinant as $|C_{13}|$, $|C_{23}|$, $|C_{33}|$, $|C_{41}|$ and $|C_{34}|$.

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2) Solve by Cramer's rule $x + 24 + 3z = 6$

$2x + 4y + z = 7$

$3x + 24 + 9z = 14$

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11.3 MATRIX

11.3.1 Concept of Matrix

In real life, we do not have to deal with a single magnitude but with a set of magnitudes. In analysis we treat such a set of magnitudes as a single entity or a single object of thought, abstracted by numbers. Let us consider a problem where we treat a set as single entity and a number of such sets as a single system. We will then examine how the fundamental operations of arithmetic can be suitably defined to deal with such a system.

Suppose there are 3 individuals A, B, C in a family and that

A has a set of 3 pants, 3 shirts and 1 tie.

B has a set of 5 pants, 5 shirts and 2 ties.

C has a set of 6 pants, 8 shirts and no tie.

We can arrange the data in the following convenient system:

Individuals		*Pants* (*P*)	*Shirts* (*S*)	*Ties* (*T*)		
↓						
A	→	3	3	1	→	1st Row
B	→	5	5	2	→	2nd Row
C	→	6	8	0	→	3rd Row
		↓	↓	↓		
		1st Col.	2nd Col.	3rd Col.		

This system comprises 3 rows and 3 columns (note that, by convention, rows are written first and columns afterwards). We have written belongings of A in the first row, belongings of B in the second row and the belongings of C in the third row. Clearly, therefore, 1st column gives us the total member of pants that A, B and C together have while second and third columns enumerate respectively the number of shirts and ties that three individuals together have.

Number written in such a particular form of rows and columns and enclosed by [ ] → square brackets or ( ) → large parentheses are called a '*matrix*'.

A '*matrix*' is defined as a rectangular array of elements arranged in rows and columns and '*matrix algebra*' basically provides methods for dealing with sets of simultaneous relationships involving many variables.

In general form, $A=\begin{bmatrix} a_{11} & a_{12} & \dots a_{1n} \\ a_{21} & a_{22} & \dots a_{2n} \\ \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots a_{mn} \end{bmatrix}$

This is a matrix of m rows and n columns. It is denoted by (a_{ij}) and is read as m by n. The first subscript refers to the number of row and the second to the number of column to which the element belongs. Thus, a_{13} belongs to the 1st row and 3rd column; in general, a_{ij} refers to the element of j^{th} row and i^{th} column. If the matrix has total number of elements = mn, arranged in m rows and n column, it is said to be of the order of m by n, which is often written as m x n matrix.

11.3.2 Types of Matrix

Rectangular Matrix

A matrix of the order m by n is also called as 'rectangular matrix'. And if m=n, then the matrix is said to be a '*square matrix*' of order 'n'.

Column Matrix

If the matrix consists of only one column as $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$, then it is called a '*column matrix*' or '*column vector*'.

Null Matrix

If all the elements of a matrix be zero, then it is called a 'null matrix' or 'zero matrix'. The null matrix of order m x n is denoted by O_{mxn}.

e.g., $0_{2x3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

If the elements of a matrix be not all zero, then it is called a '*non-zero matrix*'

The negative of a matrix is $[-aij]_{mxn}$ and is denoted by (-A).

Thus, if $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}$

then $-A = \begin{bmatrix} -a_1 & -b_1 & -c_1 \\ -a_2 & -b_2 & -c_2 \end{bmatrix}$

The '*transpose of a matrix*' A is the matrix obtained from Δ by changing its rows into columns and columns into rows. It is denoted by A' or A^T. Hence if A be of order (m x n), then A^T is of order (n x m).

Thus, the transpose of $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}$ is $\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{bmatrix}$

Note that, $\left[A^T\right]^T = A$

If the number of columns be equal its number of rows, then the matrix is called a '*square matrix*'. There are some special forms of square matrices, viz.,

Diagonal Matrix

If all off-diagonal elements of a square matrix are zero, that matrix is called diagonal matrix. Thus

$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} = (a_{ij})_{nxn}$ is a diagonal matrix if and only if

$$a_{ij} = 0 \text{ for } i \neq j \text{ and } a_{ij} \neq 0 \text{ for at least one } i = j$$

For example, $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is a diagonal matrix.

Scalar Matrix

If the diagonal elements of a diagonal matrix be all equal, then the matrix is called a 'scalar matrix'.

e.g., $\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$ is a scalar matrix [a]

Identity Matrix

If the diagonal elements of a diagonal matrix are all equal to 1, it is defined as a 'unit matrix' or 'identity matrix'. A unit matrix of order 'n' is denoted by I_n or by [I] or simply by I. Thus,

$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} = (a_{ij})_{nxn}$ is an identity matrix if and only if

$$a_{ij} = 0 \text{ for } i \neq j \text{ and } a_{ij} = 1 \text{ for } i = j$$

Example: $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is an identity matrix of order 3.

Note that $I = I'$

11.3.3 Matrix Operations

11.3.3.1 Equality of Matrices

Two matrices A and B are said to be '*conformable for equality*' if they were of same order.

Two matrices of same order are said to be '*equal*', if and only if, the corresponding elements of the two are equal.

Example 16: $\begin{bmatrix} 1^2 & 2^2 \\ 3^2 & 4^2 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 9 & 16 \end{bmatrix}$

If the two matrices are not of the same order, then they are '*not conformable for equality*'.

11.3.3.2 Addition and Subtraction of Matrices

If the matrices A and B are of the same order (m x n), then they can be added or subtracted.

If $A=[a_{ij}]$ and $B=[b_{ij}]$, then $A+B=[a_{ij}+b_{ij}]$ and $A-B=[a_{ij}-b_{ij}]$

In other words, corresponding elements are added or subtracted. The new matrix will be of the same order (m x n).

Example: $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 2 \\ 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1+0 & 2+1 & 3+2 \\ 3+2 & 4+3 & 5+4 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 5 \\ 5 & 7 & 9 \end{bmatrix}$

Some Properties:

i) $A + B = B + A$ (so, matrix addition is commutative)

ii) $(A + B) + C = A + (B + C)$ where C is the third matrix (so, matrix addition is associative).

Note: $[A + B]^T = A^T + B^T$

iii) $A + 0 = 0 + A = A$, thus zero matrix plays the same role in matrix algebra as zero in ordinary algebra.

Some Results:

a) $A + (-A) = (-A) + A = 0$

b) $A + A = 2A$

c) $k(A + B) = kA + kB$

d) $(k_1 + k_2)A = k_1A + k_2A$

11.3.3.3 Matrix Multiplication

If the number of columns of a matrix A be equal to the number of rows of another matrix B, then the matrices A and B are said to be 'conformable for the product' AB and the product AB is said to be defined.

If $A = [a_{ij}]_{m \times n}$ and $B = [b_{jk}]_{n \times p}$,

then $AB = [c_{ik}]_{m \times p}$ where $c_{ik} = \sum_{j=1}^{n} a_{ij}\, b_{jk}$

Thus, $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 3 \\ 5 & 2 \\ 9 & 7 \end{bmatrix} = \begin{bmatrix} 1+10+27 & 3+4+21 \\ 0+5+18 & 0+2+14 \end{bmatrix} = \begin{bmatrix} 38 & 28 \\ 23 & 16 \end{bmatrix}$

Some Properties:

i) In general $AB \neq BA$ [so, matrix multiplication is not communicative in general.]

ii) For conformable matrices (AB) C=A (BC) [so, matrix multiplication is associative.]

iii) (B+C) D= BD+CD [so, matrix multiplication is distributive with respect to addition]

iv) $(AB)^{T} = B^{T}.A^{T}$

11.3.3.4 Adjoint and Reciprocal Matrices

If $A = [a_{ij}]_{n \times n}$ be a square matrix, then the transpose of the matrix $[A_{ij}]_{nxn}$ whose elements are the cofactors of the corresponding element in $|A|$ is called the '<u>adjoint</u>' or '<u>adjugate</u>' matrix of A and is denoted by adj.A It is equal to $[A_{ji}]_{n \times n}$. If $A = [a_{ij}]_{n \times n}$ be a non-singular square matrix, the matrix $\left[\frac{Aji}{|A|}\right]_{n x m}$ or, $\frac{\text{adj.A}}{|A|}$ is called the '<u>reciprocal matrix</u>' of A.

Thus, if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$\text{adj.A} = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$ [where A_{ij} is the cofactor of a_{ij} in $|A|$]

and the reciprocal of A is $\begin{bmatrix} \frac{A_{11}}{|A|} & \frac{A_{21}}{|A|} & \frac{A_{31}}{|A|} \\ \frac{A_{12}}{|A|} & \frac{A_{22}}{|A|} & \frac{A_{32}}{|A|} \\ \frac{A_{13}}{|A|} & \frac{A_{23}}{|A|} & \frac{A_{33}}{|A|} \end{bmatrix}$

11.3.3.5 Trace of a Matrix

The trace of a square matrix is the sum of its diagonal elements.

Thus, in the matrix $\begin{bmatrix} 1 & 1 & 2 \\ 2 & 0 & 1 \\ 3 & 3 & -1 \end{bmatrix}$ the trace (tr) is = 1 + 0 – 1=0

In general, trace $tr(A) = a_{11} + a_{22} + \ldots + a_{nn} = \sum_{i=1}^{n} a_{ii}$

Properties: i) $tr(A+B) = tr(A) + tr(B)$

ii) $tr(A') = tr(A)$

iii) $tr(AB) = tr(BA)$ and

iv) $tr(ABC) = tr(BCA) = tr(CAB)$.

11.3.3.6 Sub Matrices and Minors

Any matrix obtained by omitting some rows and columns of a given m x n matrix A is called 'sub matrix' of A.

Thus, $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ is a sub matrix of $\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix}$

The determinant of the square matrix of order 'r', obtained from a given m x n matrix A by omitting (m – r) rows and (n – r) columns, is called a 'minor' of A of order r'.

Thus, $\begin{bmatrix} a & h \\ h & b \end{bmatrix}$ is a minor of $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ of order 2

11.3.3.7 Transpose of Sum or Difference of Matrices

The transpose of a sum or difference of matrices is equal to the sum or difference of the transpose of the matrices. Thus,

$$(A_{mxn} \pm B_{mxn} \pm C_{mxn})' = A'_{mxn} \pm B'_{mxn} \pm C'_{mxn}$$

Example: $A = \begin{bmatrix} 2 & -1 \\ 0 & -1 \end{bmatrix}$ $B = \begin{bmatrix} 6 & 8 \\ 10 & -12 \end{bmatrix}$ $C = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$ then

$$[A + B + C]' = \begin{bmatrix} -4 & -10 \\ -10 & 11 \end{bmatrix}' = \begin{bmatrix} -4 & -10 \\ -10 & 11 \end{bmatrix}$$ or alternatively

$$A' - B' + C' = \begin{bmatrix} 2 & 0 \\ -1 & -1 \end{bmatrix} - \begin{bmatrix} 6 & 10 \\ 8 & -12 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -4 & -10 \\ -10 & 11 \end{bmatrix}$$

11.3.3.8 Transpose of a Product of Matrices

The transpose of a product of matrices is equal to the product of the transpose of the matrices in inverse sequence; thus

$$\left[A_{mxn}\ B_{nxp}\ C_{pxq}\right]' = C'_{qxp}\ B'_{pxn}\ A'_{nxm}$$

Example: $A = \begin{bmatrix} 3 & 0 \\ -4 & -1 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 5 & -7 \\ 0 & -1 & 8 \end{bmatrix}$ $C = \begin{bmatrix} 6 \\ -1 \\ 0 \end{bmatrix}$ then

$$ABC = \begin{bmatrix} 39 \\ -53 \end{bmatrix} \quad [ABC]' = [39 \quad -53] = C'B'A'$$

$$= \begin{bmatrix} 6 & -1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 5 & -1 \\ -7 & 8 \end{bmatrix} \begin{bmatrix} 3 & -4 \\ 0 & -1 \end{bmatrix}$$

11.3.4 Matrix Inversion

A square matrix $A_{n\ x\ n}$ with $|A| \neq 0$ is called a non-singular matrix. $A_{n\ x\ n}$ is called singular matrix if $|A| = 0$.

The inverse of a non-singular matrix A is given by $A^{-1} = \frac{1}{|A|}.\text{Adj.A}$

Example 17: $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 3 & 2 & -1 \end{bmatrix}$ $\therefore |A| = 1(1-6) - 1(-2-9) + 1(4+3)$

$$= -5 + 11 + 7 = 13 \text{ i.e., } |A| \neq 0$$

So A is a non-singular matrix, its inverse exists.

Now, matrix of cofactors of $A = \begin{bmatrix} -5 & 11 & 7 \\ 3 & -4 & 1 \\ 4 & -2 & -3 \end{bmatrix}$

$$\therefore \text{Adj.A} = \begin{bmatrix} -5 & 3 & 4 \\ 11 & -4 & 1 \\ 7 & 1 & -3 \end{bmatrix} \quad \text{and} \quad A^{-1} = \frac{1}{13} \begin{bmatrix} -5 & 3 & 4 \\ 11 & -4 & -1 \\ 7 & 1 & -3 \end{bmatrix}$$

Some Properties:

i) The inverse of a matrix, if exists, is unique.

ii) The necessary and sufficient condition that the inverse of a square matrix A exist, is that A is non-singular.

iii) $[AB]^{-1} = B^{-1}.A^{-1}$ (this is known as '*law of reversal of inverse*')

iv) $(A^{-1}) = A$

v) $(A^{-1})^{T} = (A^{T})^{-1}$

and $(A^{-1})^{-1\ T}.\ A^{T} = I = A^{T}\ (A^{-1})^{T}$ (operations of reciprocating and transposing are commutative)

Check Your Progress 3

1) Obtain the inverse of $A = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{vmatrix}$

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2) Prove that, if $A = \begin{bmatrix} 2 & -3 & 0 \\ 3 & 1 & -2 \\ -1 & 0 & -4 \end{bmatrix}$ then $A^{-1} = -\frac{1}{50}\begin{bmatrix} -4 & -12 & 6 \\ 14 & -8 & 4 \\ 1 & 3 & 11 \end{bmatrix}$

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3) Given a matrix, $\begin{bmatrix} 7 & -1 & 0 \\ 1 & 1 & 4 \\ 13 & -3 & -4 \end{bmatrix}$, what would you conclude about its rank?

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11.3.5 Some Other Types of Matrices

11.3.5.1 Orthogonal Matrix

A square matrix A is said to be orthogonal, if its transpose A^T produces a unit matrix when multiplied by A, i.e., $A^T . A = I$.

Some Properties:

i) Orthogonal matrices are non-singular,

i.e., if $A^T.A = I$ then $|A \neq 0|$

ii) Unit matrices are orthogonal, i.e.,

$I^T.I = I.I = I$

iii) The value of the determinant of an orthogonal matrix is either 1 or (-1). If A is an orthogonal matrix, then $A^T A = I$

$$\therefore |A^T.A| = |A^T|.|A| = |A|^2 = |I|$$

$$\therefore |A| = \pm 1$$

iv) The product of two orthogonal matrices is orthogonal.

v) The transpose of an orthogonal matrix is orthogonal.

vi) The inverse of an orthogonal matrix is orthogonal.

11.3.5.2 Symmetric and Skew-Symmetric Matrices

A square matrix A is said to be 'symmetric' if its transpose be equal to it, i.e., if $A^T=A$. Hence, the matrix $[a_{ij}]$ is symmetric, if $a_{ij} = a_{ji}$ for all i and j. For example,

$$\begin{bmatrix} a & b & c \\ b & e & d \\ c & d & f \end{bmatrix}$$ is a symmetric matrix.

A square matrix A is said to be 'skew-symmetric', if its transpose is equal to its negative. i.e., if $A^T = -A$. Hence, the matrix $[a_{ij}]$ is skew-symmetric, if $a_{ij} = -a_{ji}$ for all i and j. The diagonal elements of skew – symmetric matrix are obviously zero. For example,

$$\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$$ is a skew – symmetric matrix.

Some Properties:

i) The product of a matrix and its transpose is a symmetric matrix

Now, $[A\,A^T]^T = [A^T]^T.\,[A]^T = A.\,[A]^T$

Hence, $A.A^T$ is symmetric.

ii) The product of two symmetric matrices of the same order is symmetric if the product by commutative.

If A and B be two symmetric matrices of the same order and AB = BA, then $[AB]^T = B^T.A^T = B.A = AB$.

Hence, AB is symmetric.

iii) If A is a square matrix, then $(A + A^T)$ is symmetric and $(A - A^T)$ is skew-symmetric.

$(A + A^T)^T = A^T + (A^T)^T = A^T + A$

Thus, $(A + A^T)$ is symmetric.

Similarly, $(A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T)$

Hence, $(A - A^T)$ is skew-symmetric.

iv) Every square matrix can be expressed as a sum of a symmetric matrix and a skew symmetric matrix uniquely.

Let A be a square matrix, then,

$$A = \frac{1}{2}\left(A + A^T\right) + \frac{1}{2}\left(A - A^T\right)$$

11.3.5.3 Idempotent Matrix

A symmetric matrix that reproduces itself when multiplied by itself is termed as an idempotent matrix.

That is, A will be termed as idempotent matrix if AA = A.

11.3.6 Partitioned Matrices

It is often convenient to partition a matrix into sub matrices. These sub matrices can then be treated as scalars in performing operations on the original matrix. Partitioning of a matrix is indicated by horizontal and vertical lines between rows and columns.

The mxn matrix A may be partitioned as follows $A = (A_1 \vdots A_2)$

If they are partitioned conformably, partitioned matrices can be added, subtracted or multiplied. If an mxn matrix A is partitioned $A = (A_1 \vdots A_2)$, where A_1 is mxn_1, A_2 is mxn_2 and $n_1+n_2=n$, and mxn matrix B is partitioned $B = (B_1 \vdots B_2)$, where B_1 is mxn_1, B_2 is mxn_2 and $n_1+n_2=n$, then $A \pm B = [A_1 \pm B_1 \vdots A_2 \pm B_2]$

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Similarly, if $A=\begin{bmatrix} A_1 \\ \cdots \\ A_2 \end{bmatrix}$, where A is mxn, A_1 is $m_1$xn, A_2 is $m_2$xn and $m_1+m_2=m$, and $B=\begin{bmatrix} B_1 \\ \cdots \\ B_2 \end{bmatrix}$, where B is mxn, B_1 is $m_1$xn, B_2 is $m_2$xn and $m_1+m_2=m$ then $A \pm B=\begin{bmatrix} A_1 \pm B_1 \\ \cdots \\ A_2 \pm B_2 \end{bmatrix}$

Partitioning is frequently conceptually convenient when matrices are to be added or subtracted, but the computational advantages of matrix partitioning are associated primarily with multiplication and other more complex operations. Partitioned matrices must be partitioned conformable for multiplication. If matrix A of order mxn is partitioned $A = (A_1 \vdots A_2)$, where A_1 is mxn_1, A_2 is mxn_2 and $n_1+n_2=n$ and nxp matrix B is partitioned $B=\begin{bmatrix} B_1 \\ \cdots \\ B_2 \end{bmatrix}$,

where B_1 is $n_1$xp and B_2 is $n_2$xp then $AB=\begin{bmatrix} A_1 & \vdots & A_2 \end{bmatrix}\begin{bmatrix} B_1 \\ \cdots \\ B_2 \end{bmatrix}=A_1B_1+A_2B_2$

Another example: if $A=\begin{bmatrix} A_{11} & \vdots & A_{12} \\ \cdots & \cdots & \cdots \\ A_{21} & \vdots & A_{22} \end{bmatrix}$ where A_{11} is m_1xn_1, A_{12} is m_1xn_2, A_{21} is m_2xn_1, A_{22} is m_2xn_2 and $m_1+m_2=m$, $n_1+n_2=n$ and if

$B=\begin{bmatrix} B_{11} & \vdots & B_{12} \\ \cdots & \cdots & \cdots \\ B_{21} & \vdots & B_{22} \end{bmatrix}$, where B_{11} is n_1xp_1, B_{12} is n_1xp_2, B_{21} is n_2xp_1, B_{22} is n_2xp_2 and $n_1+n_2=n$, $p_1+p_2=p$, then A and B are partitioned conformably for multiplication and $AB=\begin{bmatrix} A_{11} & \vdots & A_{12} \\ \cdots & \cdots & \cdots \\ A_{21} & \vdots & A_{22} \end{bmatrix}\begin{bmatrix} B_{11} & \vdots & B_{12} \\ \cdots & \cdots & \cdots \\ B_{21} & \vdots & B_{22} \end{bmatrix}$

$$=\begin{bmatrix} A_{11}B_{11}+A_{12}B_{21} & \vdots & A_{11}B_{12}+A_{12}B_{22} \\ \cdots & \cdots & \cdots \\ A_{21}B_{11}+A_{22}B_{21} & \vdots & A_{21}B_{12}+A_{22}B_{22} \end{bmatrix}$$

Check Your Progress 4

1) List the Properties of orthogonal matrix.

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2) Define an idempotent matrix.

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3) If

$$A = \begin{pmatrix} 1 & 3 & 1 \\ -1 & 0 & 1 \\ 2 & -1 & 4 \\ 0 & 2 & -3 \end{pmatrix} \text{ and } B = \begin{pmatrix} 3 & -2 & 1 & 0 & -1 \\ 5 & -1 & 4 & -3 & 2 \\ 3 & -2 & 0 & 1 & -1 \end{pmatrix}.$$

$$\text{then show } AB = \begin{bmatrix} A_{11} & \vdots & A_{12} \\ \cdots & \cdots & \cdots \\ A_{21} & \vdots & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & \vdots & B_{12} \\ \cdots & \cdots & \cdots \\ B_{21} & \vdots & B_{22} \end{bmatrix}$$

$$= \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & \vdots & A_{11}B_{12} + A_{12}B_{22} \\ \cdots & \cdots & \cdots \\ A_{21}B_{11} + A_{22}B_{21} & \vdots & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}.$$

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11.3.7 Rank of a Matrix

A number 'r' is said to be the rank of a matrix A, if

i) there is at least one (r x r) sub-matrix of A whose determinant is not equal to zero; and

ii) the determinant of every (r+1) – rowed square sub-matrix of A is zero.

In other words, rank of a matrix is equal to the order of the highest order non-singular square matrix contained in A. Moreover, rank of a matrix can also be defined as the maximum number of rows (or, columns) in a matrix.

For example, the rank of matrix $\begin{bmatrix} 5 & 2 \\ -2 & -3 \end{bmatrix}$ is 2; because this second order matrix corresponds to the determinant $\begin{bmatrix} 5 & 2 \\ -2 & -3 \end{bmatrix}$, which is not zero, hence this given matrix contains a non-singular matrix of order 2.

Whereas, the rank of the matrix $\begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}$ is 1; the highest order determinant $\begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}$ is zero, next to this would be of the 1st order, which is non-zero.

Again, the rank of $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is zero.

Note:

i) The rank of a matrix cannot exceed the number of its rows or columns - whichever is less.

ii) The rank of a column or row matrix is at most 1, unless the matrix is a null matrix, where the rank is always zero.

iii) Rank of the transpose of a matrix is the same as the rank of that matrix.

Example 18: Determine the rank of the matrix $\begin{bmatrix} 2 & -1 & 3 & 1 & 5 \\ 3 & 2 & 0 & 2 & 1 \end{bmatrix}$. Here, picking any sub-matrix of order 2, it can easily be shown that value of the determinant is non-zero, hence rank = 2.

There is another method of determining rank

$$\begin{bmatrix} 2 & -1 & 3 & 1 & 5 \\ 3 & 2 & 0 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 & 1 & 5 \\ -1 & 4 & -6 & 2 & 1 \end{bmatrix}, \text{ by, } \begin{matrix} c_1' \to c_1 - 2c_4 \\ c_2' \to c_2 - c_4 \\ c_3' \to c_3 - 3c_4 \end{matrix}$$

$$\sim \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ -1 & 4 & -6 & 2 & -9 \end{bmatrix}, \text{ by, } c_5' \to c_5 - 5c_4$$

$$\sim \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & -9 \end{bmatrix}, \text{ by, } \begin{matrix} c_1'' \to 6c_1' - c_3' \\ c_2'' \to 6c_2' - 4c_3' \\ c_3'' \to 9c_3' - 6c_5' \end{matrix}$$

$$\sim \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \text{ by, } R_2' \Rightarrow R_2 - 2R_1$$

$$\sim [\,0 \;\; I_2\,], \text{ by, } R_2'' \Rightarrow \frac{1}{9} R_2'$$

So, rank of the matrix is 2

Check Your Progress 5

1) Show that rank of A=2 and rank of B=2, where

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & -2 & 3 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}$$

but rank of (A+B)=3.

Show further that, rank of (AB) = 1, while rank of (BA) = 2.

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2) How would you determine linear dependence of matrix?

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3) Define the rank of matrix in terms of linear in dependence.

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11.3.8 Linear Dependence (Independence) and Rank of a Matrix

The vector of rows (columns) of a matrix are said to be linearly dependent if and only if some linear combination of them is a null vector or null row (column). The rows (columns), which are not linearly dependent, are said to linearly independent.

Consider the matrix:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

A linear combination of the rows 1, 2 and 3 are obtained by multiplying the 1st, 2nd and 3rd rows by any constants k_1, k_2 and k_3 and adding them, where at least one constant is not zero. Thus, the new row will be

$[k_1a_{11}+k_2a_{21}+k_3a_{31} \quad k_1a_{12}+k_2a_{22}+k_3a_{32} \quad k_1a_{13}+k_2a_{23}+k_3a_{33}]$

$= [c_1 \quad c_2 \quad c_3]$ (say)

The rows are linearly dependent if all the three elements c_1, c_2 and c_3 are equal to zero for some values of k_1, k_2, k_3. Of course, k_1, k_2, k_3 should all not be equal to zero, at least one of these must be non-zero.

With regard to the concept of linear dependence (independence), rank of a matrix can also be defined as follows:

The rank of matrix A (denoted as $\rho(A)$,), is the maximum number of linearly independent rows (or columns) in A.

11.4 LET US SUM UP

Through this unit we have learnt how by using matrix and determinants method an equation system can be written in a compact way. Moreover, we studied the way of testing the existence of a solution by evaluating matrices and determinants and finding the solutions (if exist) of n equation system.

11.5 KEY WORDS

Cofactor: The 'cofactor' of any element in a determinant is its coefficient in the expansion of the determinant. It is therefore equal to the corresponding minor with a proper sign. The sign of the cofactor of an element in the i^{th} row and j^{th} column is $(-1)^{i+j}$.

Cramer's Rule: In particular, let the system of equations be

$$a_i x + b_i y + c_i z = k_i \quad (i = 1,2,3) \ \ldots\ldots\ldots\ldots\ldots (1)$$

$$\text{And } \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$$

$$\text{Then } x\Delta = \begin{vmatrix} xa_1 & b_1 & c_1 \\ xa_2 & b_2 & c_2 \\ xa_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1x + b_1y + c_1z & b_1 & c_1 \\ a_2x + b_2y + c_2z & b_2 & c_2 \\ a_3x + b_3y + c_3z & b_3 & c_3 \end{vmatrix} \quad [c_i' = c_1 + yc_2 + zc_3]$$

$$= \begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix} \quad [\text{from}(1)] = \Delta_1 \text{ (say)}$$

$$\therefore \quad x = \frac{\Delta_1}{\Delta}$$

Similarly, $y = \frac{\Delta_2}{\Delta}$ and $z = \frac{\Delta_3}{\Delta}$ where Δ_2 and Δ_3 are the determinants obtained from Δ substituting the columns (k_1, k_2, k_3) for the second and third column respectively? Thus, by Cramer's rule variables of an equation system can be obtained.

Determinants: If the linear equation $a_1x + b_1 = 0$ and $a_2x + b_2 = 0$ have the same solution, then $\frac{a_1}{a_2} = \frac{b_1}{b_2}$, i.e., $\frac{a_1}{b_1} = \frac{a_2}{b_2}$, *or*, $a_1b_2 - a_2b_1 = 0$. The expression $(a_1b_2 - a_2b_1)$ is called a determinant and is denoted by the symbol $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$.

Inverse of a Matrix: A square matrix an $_{x\ n}$ with $|A| \neq 0$ is called a non-singular matrix. The inverse of a non-singular matrix A is given by $A^{-1} = \frac{1}{|A|}.\text{Adj.A}$.

Linear Dependence: The vector of rows (columns) of a matrix are said to be linearly dependent, if and only if, some linear combination of them is a null vector or null row (column).

Matrix: A '**matrix**' is defined as a rectangular array of elements arranged in rows and columns and **'matrix algebra'** basically provides methods for dealing with sets of simultaneous relationships involving many variables.

Minor: The 'minor' of any element in a determinant is another determinant of lower order that can be obtained by deleting the row and column containing that element. So in a 3rd order determinant minor of any element will be a 2nd order determinant.

Rank: Rank is the maximum number of independent rows (or, columns) in a matrix.

Trace: The trace of a square matrix is the sum of its diagonal elements.

11.6 SOME USEFUL BOOKS

Chiang Alpha C. (1984), *Fundamental Methods of Mathematical Economics*, Third edition, McGraw-Hill Book Company.

Henderson J.M. & Quandt R.E, *Microeconomic Theory: A Mathematical Approach*, Third edition, McGraw-Hill Book Company.

Chakravorty, J.G & P.R. Ghosh (1996), *Higher Algebra Including Modern Algebra*, Twelfth Edition, U.N. Dhur & Sons Pvt. Ltd. Calcutta.

11.7 ANSWER OR HINT TO CHECK YOUR PROGRESS

Check Your Progress 1

1) -9

2) Do yourself.

3) $\left[\text{Hint}: \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c \end{vmatrix}\right]\left[\text{Hint}: \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c \end{vmatrix}\right]$

$$= \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix} \quad \begin{vmatrix} c_2^1 = c_2 - c_1 \\ c_3^1 = c_3 - c_1 \end{vmatrix}$$

$$= 1\begin{vmatrix} b-a & c-a \\ b^2-a^2 & c^2-a^2 \end{vmatrix} \text{ [Expanding by rows]}$$

$= (a-b)(b-c)(c-a)$]

4) [Hint: given determinant $= \begin{vmatrix} (b+c)-(b+c) & a-(2c+a) & a-(2b+a) \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$

$(R_1' = R_2 + R_3 - R_1)$

$$= -2\begin{vmatrix} 0 & c & b \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = -2\begin{vmatrix} 0 & c & b \\ b & a & 0 \\ c & 0 & a \end{vmatrix} \begin{pmatrix} R_2'' = R_2 - R_1' \\ R_3'' = R_3 - R_1' \end{pmatrix}$$

$$= +2\begin{vmatrix} 0 & b & c \\ b & 0 & a \\ c & a & 0 \end{vmatrix} = 4abc \text{ (after evaluating)}$$

Check Your Progress 2

1) Do yourself read Section 11.2.

2) [Hint: $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 2 & 9 \end{vmatrix} = -20 \neq 0 \ \therefore \Delta_1 = \begin{vmatrix} 6 & 2 & 3 \\ 7 & 4 & 1 \\ 14 & 2 & 9 \end{vmatrix} = -20$

$\Delta_2 = \begin{vmatrix} 1 & 6 & 3 \\ 2 & 7 & 1 \\ 3 & 14 & 9 \end{vmatrix} = -20 \qquad \Delta_3 = \begin{vmatrix} 1 & 2 & 6 \\ 2 & 4 & 7 \\ 3 & 2 & 14 \end{vmatrix} = -20$

$\therefore x = \frac{\Delta_1}{\Delta} = 1, \quad y = \frac{\Delta_2}{\Delta} = 1, \quad z = \frac{\Delta_3}{\Delta} = 1$]

Check Your Progress 3

1) [Hint: $|A| = 1(36-25) - 2(12-5) + 3(5-3) = 11-14+6 = 3 \neq 0$

$\therefore \text{Ad } A = \begin{bmatrix} 11 & -9 & 1 \\ -7 & 9 & -2 \\ 2 & -3 & 1 \end{bmatrix} \ \therefore A^{-1} = \frac{\text{Adj. } A}{|A|} = \frac{1}{3}\begin{bmatrix} 11 & -9 & 1 \\ -7 & 9 & -2 \\ 2 & -3 & 1 \end{bmatrix}$

2) Follow Example 17

3) Rank is less than 3

Check Your Progress 4

1) See 11.2.5.1

2) See 11.2.5.3

3) First partition the matrices A and B then, separately determine $[A_{11}B_{11} + A_{12}B_{21}]$, $[A_{11}B_{12} + A_{12}B_{22}]$, $[A_{21}B_{11} + A_{22}B_{21}]$ and $[A_{21}B_{12} + A_{22}B_{22}]$

Check Your Progress 5

1) Follow Example 18

2) See 11.2.7

3) See 11.2.7

11.8 EXERCISES

1) Use Cramer's rule to solve the following national income model:

$Y = C + I + G$ And $C = a + bY$

[Hint: Find Y and C, consider I, G, a, b, as constants]

Answer: $Y = \frac{I+G+a}{1-b}, \quad C = \frac{a+b(I+G)}{1-b}$

2) Consider the following national income determination model:

$Y = C + I + G, \quad c = a + b(Y - T), \qquad T = d + tY$

Where Y (national income), C (consumption expenditure) and T (tax collection) are endogenous variables, I (investment) and G (government expenditure) are exogenous variables; t is the income tax rate.

Solve for the endogenous variables, using Cramer's rule.

$$\left[Ans: y = \frac{bd - (I + G + a)}{b(1 = t) - 1}, \quad c = \frac{-a + bd - b(1-t)(I+G)}{b(1-t) - 1}, \quad T = \frac{-d + bd - at - (I + G)t}{b(1-t) - 1} \right]$$

3) The following is an example of the Keynesian model with the money.

Behavioral equations are

$C = 0.8Y$, $I = 102\text{-}0.2r$,

$M^d = 0.25Y - 2.5r$, $M^s = 100$

Equilibrium conditions are $Y = C + I$ and $M^d = M^s$

ii) Write down the equations of the IS and LM curves.

iii) Evaluate the equilibrium value of Y and r using Cramer's rule.

[Ans: $Y = 500$, $r = 10$]

4) Solve the equations:
$x + y + z = 1, x + 2y + 3z = 6,$ and $x + 3y + 4z = 6.$

5) A set of linear equations in which the right-hand side consisting of constants has zeros, is called a set of homogeneous equations.

Show that $ax + by + cz = 0, bx + cy + az = 0, cx + cy + bz = 0$

with $D = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

has the trivial solution $(x = y = z = 0)$ if $D \neq 0$ and has a non-trivial solution

if $a^3 + b^3 + c^3 - 3abc = 0.$

Again, find the solution treating x to be constant.

6) Find the inverse of the matrix $\begin{bmatrix} 2 & -3 & 4 \\ 1 & 0 & 1 \\ 0 & -1 & 4 \end{bmatrix}$ and hence obtain the solution of the equations

$2x_1 - 3x_2 + 4x_3 = -4,\ x_1 + x_3 = 0,\ -x_2 + 4x_3 = 2.$

7) Consider the national income model given by

$$Y = C + I + G$$
$$c = a + b(Y - T) \qquad (a > 0,\ 0 < b < 1)$$
$$T = d + tY \qquad (d > 0,\ 0 < t < 1)$$

Solve this model for the variables Y, C and T by matrix inversion.

8) If $U = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$, $V = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $X = \begin{bmatrix} 4 & 2 & -1 \\ -1 & 3 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ and $Y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

then find a. UV; b. VU+X; c. XY

Use partitioned matrices to check the results.

9) If A is a non-zero (n x 1) column matrix, while B is a non-zero matrix, then show that P(BA) is unity and P(AB) is n.

10) Show that rank of the transpose of the matrix $\begin{bmatrix} 4 & 1 & 2 \\ -3 & 2 & 4 \\ 8 & -1 & -2 \end{bmatrix}$ is the same as that the original matrix.

UNIT 12 INPUT-OUTPUT ANALYSIS

Structure

12.0 Objectives

12.1 Introduction

12.2 Main Features of Input-Output Analysis
- 12.2.1 Assumptions
- 12.2.2 Input-Output Table
- 12.2.3 Technological Coefficient Matrix

12.3 Closed and Open Input-Output Models

12.4 Coefficient Matrix and Open Model

12.5 Solution to Open Model
- 12.5.1 Determination of Gross Output
- 12.5.2 The Hawkins-Simon Conditions
- 12.5.3 Consumption Possibility Locus
- 12.5.4 Determination of Equilibrium Prices

12.6 Linear Programming Formulation

12.7 Coefficient Matrix and Closed Model

12.8 Dynamic Input-Output Model

12.9 Limitations of Input-Output Analysis

12.10 Let Us Sum Up

12.11 Key Words

12.12 Some Useful Books

12.13 Answer or Hints to Check Your Progress

12.14 Exercises

12.0 OBJECTIVES

After studying this unit, you will be able to answer the following:

- with what proportions one sector of the economy are related to other sectors;
- how solution is attained in a framework of several variables production problems related to input-output;.
- how the system of equations input-output can be represented graphically by the linear programming; and
- how the system of input-output can be solved in a dynamic framework.

12.1 INTRODUCTION

The concept of Input-Output (I-O) analysis originated by the eighteenth century French economists Quesnay in the name of 'Tabelau Economique'. In its modern form, the I-O analysis has been developed by the American economist Wassily W. Leontief in his famous work 'Structure of the American Economy' in the year 1951.

I-O analysis is a method of analysing how an industry undertakes production by using the output of other industries in the economy and how the output of the given industries used up in other industries or sectors. Since various industries are interdependent, i.e., the output of one industry is an input for the others, their mutual relationship ultimately must lead to equilibrium between supply and demand in the economy consisting of 'n' industries. I-O analysis is also known as the inter-industry analysis as it explains the interdependence and interrelationship among various industries. Since the analysis is concerned with the relations among inputs, use of a commodity in the production by an industry, it is known as the input-output analysis. In other words, the I-O analysis explains the interdependence of inputs and outputs of various industries in the economy.

12.2 MAIN FEATURES OF INPUT-OUTPUT ANALYSIS

I-O analysis was an attempt made by Prof. Leontief to take account of '*general equilibrium*' phenomena in the '*empirical*' analysis of '*production*'. These three italicized elements are the main features of I-O analysis. First, the I-O analysis deals almost exclusively with production. The problem is essentially technological. Given the quantities of available resources and the state of technology, the analysis is concerned with the use of various inputs by the industries and outputs derived from them.

The second distinctive feature of I-O analysis is its devotion to empirical investigation. This is primarily what distinguishes it from the work of Walras and later general equilibrium theorists. I-O employs a model, which is more simplified and also narrower in the sense that it seeks to encompass fewer phenomena than does the usual general equilibrium theory. Its narrowness lies in its exclusive emphasis on the production side of the economy.

The third distinctive feature is its emphasis of general equilibrium phenomena where everything depends on everything else. Thus in the two-industry model coal is an input for steel industry and steel is an input for coal industry, though both are the output of respective industries. According to I-O analysis, it is not possible to find some industries as being in the 'earlier' stages of production and some other industries as being in the 'later' stages. For the production of coal, steel is needed; whereas, for the production of steel, coal is required. No one can say whether the coal industry or the steel industry is earlier or later in the hierarchy of production.

The basic problem, then, is to see what can be left over for final consumption and how much of each output will be used up in the course of the productive activities which must be undertaken to obtain these net outputs. Solution of this problem can be used in predicting future production requirements if usable

demand estimates can somehow be obtained. Particularly, it can be used for economic planning including problems of economic development in 'backward areas' as well as problem of military mobilisation. A more modest purpose that it has already successfully begun to serve is the provision of a very illuminating detailed structure for national income accounting.

12.2.1 Assumptions

As it was stated earlier, the intransigence of empirical materials and the computational problems have forced on I-O analysis to adopt a number of simplifying assumptions even more extreme than those usually employed in our theoretical models.

The economy can be meaningfully divided into finite number of sectors (industries) on the basis of the following assumptions:

i) Each industry produces only one homogeneous output. No two products are produced jointly; but if at all there is such a case, then it is assumed that they (products) are produced in fixed proportions.

ii) Each producing sector satisfies the properties of linear homogeneous production function. In other words, production of each sector is subject to constant returns to scale so that k-fold change in every input will result in an exactly k-fold change in the output.

iii) A far stronger assumption is that each industry uses a fixed input ration for the production of its output; in other words, input requirement per unit of output in each sector remains fixed and constant. The level of output in each sector (industry) uniquely determines the quantity of each input, which is purchased.

iv) The final demand for the commodities is given from outside the system. The total amount of the primary factor is also given. These two are called open end of the system and for this, the model is called '*open model*'. In contrast in the '*closed model*' all the variables are determined within the system.

v) The model is static in the sense that all variables in it refer to the same period of time. The static model is also known as flow model.

Note on assumption (iii)

The fixed proportion assumption is far more restrictive and is a special case of an assumption of constant returns to scale (CRS). CRS is perfectly consistent with the substitution of one factor for another. A liner homogeneous production function (CRS) permits both labor-intensive and capital-intensive processes. The firm whose production function exhibits constant returns can, if it wishes say, to have 100 workers for every $1000 invested in machinery, or it may use machines which require only 10 workers per $1000 machine investment. A linear homogeneous production function requires only that if the firm decides to triple the scale of either of these types of operations, the result will be a tripling of output. Whereas Leontief fixed proportion premise requires that a manufacturing process, which is labor-intensive, offers no option of a capital-intensive alternative. If 53 men per $1000 of investment are required at any

level of operation, it is assumed that the same ratio will be required no matter how much the size of the firm expands or contracts. Whether this assumption is relatively innocuous or does considerable violence to the input-output results is still under dispute. But the premise is certainly never absolutely true, even in those cases where chemistry and engineering dictate fixed proportions between some ingredients and outputs.

12.2.2 Input-Output Table

I-O table shows the disposition of the total products and total inputs among the different industries. Let us assume that an economy consists of 4 producing sectors only; and that the production of each sector is being used as the input in all the sectors and used for final consumption. Suppose,

i) X_1, X_2, X_3 and X_4 are total outputs of 4 sectors;

ii) F_1, F_2, F_3 and F_4 are the amounts of final demand, consumption, capital formation and exports for output of these sectors.

iii) X_{ij} be the amount of output of the i^{th} Industry used up as an intermediary input by the j^{th} industry (i,j = 1,2,3,4)

iv) L represents the given amount of primary factor (here, labor) and L_i is the amount of primary factor used in the i^{th} Industry.

Then the following table represents the I-O table for the simplified economy.

Producing sector number	Total output of the sector	Input requirements of producing sectors				Requirements for final uses
		X_1	X_2	X_3	X_4	
1	X_1	X_{12}	X_{12}	X_{13}	X_{14}	F_1
2	X_2	X_{21}	X_{22}	X_{23}	X_{24}	F_2
3	X_3	X_{31}	X_{32}	X_{33}	X_{34}	F_3
4	X_4	X_{41}	X_{42}	X_{43}	X_{44}	F_4
Primary input (labour)	Total primary input = L	L_1	L_2	L_3	L_4	.

Here it should be remembered that all the items in the above table are flows, i.e. physical units per year. Since the entries in any row are all measured in the same physical units, it makes good sense to add across the rows. The 'total output' column gives the overall input of labor and output of each commodity. On the other hand, items in the same column are not measured in the same units, so that it would not be correct to add down the columns. But each column, thought as a whole (i.e., as a vector) does have a meaning. The third column describes the input or cost structure of the first industry: X_1 units of output of the first industry was produced with the use of X_{11} units of first good, X_{21} units of second good, X_{31} units of third good, X_{41} units of fourth good and L_1 units of labor. Similar meanings will follow for other columns, i.e., Columns 4, 5 and 6. The 'final demand' or 'requirement of final uses' i e., Column (7)

shows the commodity breakdown of what is available for consumption and government expenditure. It is, for convenience, assumed that labor is not consumed directly. Now suppose that each unit of output of each industry has a price of $1 (say) and each unit of labor receives a wage rate of $1. Then each entry of the above table can be expressed in terms of money value (rather than being a physical unit). It is then possible to add down the columns. The sum of each column gives the total cost of the corresponding industry. Thus, revenue of industry 1 is X_1 units ($= X_{11} + X_{12} + X_{13} + X_{14} + F_1$) and cost of that industry is ($X_{11} + X_{21} + X_{31} + X_{41} + L_1$) units. It is true for other industries also. Clearly, these prices are long-run competitive equilibrium prices. (In the long-run, competitive equilibrium price is equal to average cost and there is neither profit nor loss).

Those items in the above table that show the sales of the four industries to themselves and to each other might be described as 'non-GNP' items – because these transactions are intermediate transactions, which are not considered in the national income accounting. The 'final demand' column represents the output side of GNP, as final transactions are included in GNP accounting. The labor row represents the factor cost side of GNP. The inter-industrial sales have no welfare significance at all. Social benefits come from final consumption and social cost comes from use of labor. The economy can be viewed as a machine that used up labor and produces final consumption.

The above table represents a particular technology, a particular combination of inputs to get the outputs. If any element, along any row, changes, then other elements have to change accordingly so as to maintain the same total output of that industry. A particular technology is thus characterized by a column vector. Changing any elements of this vector results in a new vector and therefore represents a new technology.

From the above table, the production unction for the four industries can be written as follows: Para $X_1 = f_1 (X_{11}, X_{21}, X_{31}, X_{41}, L_1)$;

$X_2 = f_2 (X_{12}, X_{22}, X_{32}, X_{42}, L_2)$;

$X_3 = f_3 (X_{13}, X_{23}, X_{33}, X_{43}, L_3)$; and

$X_4 = f_4 (X_{14}, X_{24}, X_{34}, X_{44}, L_4)$.

In general terms, if there are 'n' number of producing sectors, then the production function of sector 'n' will be represented by $X_n = f_n (X_{1n}, X_{2n}, X_{3n}, X_{4n}, L_n)$. Further, we can always add across the rows, which, basically gives us the equality between the demand and the supply of each product. In general terms,

$X_1 = X_{11} + X_{12} + \ldots + X_{1n} + F_1$;

$X_2 = X_{21} + X_{22} + \ldots + X_{2n} + F_2$;

.....;

$X_n = X_{n1} + X_{n2} + \ldots + X_{nn} + F_n$; and

$L = L_1 + L_2 + \ldots + L_n$

That is $X_i = \sum_{j=1}^{n} X_{ij} + F_i$ and $L = \sum_{i=1}^{n} L_i$.

Here, X_i = total output of the i^{th} sector, X_{ij} = output of i^{th} sector used as input in j^{th} sector, and F_i = final demand for i^{th} sector.

The above identity states that all the output of a particular sector could be utilised as an input in one of the producing sectors of the economy and/or as a final demand. Basically, therefore, I-O analysis is nothing more than finding the solution of these simultaneous equations.

12.2.3 Technological Coefficient Matrix

From the assumption of fixed input requirements, we see that in order to produce 1 unit of j^{th} commodity, the input used of i^{th} commodity must be a fixed amount, which we denote by a_{ij}; thus $a_{ij} = \frac{X_{ij}}{X_j}$. If X_j represents the total output of the j^{th} commodity (or j^{th} producing sector) the input requirements of the i^{th} commodity will be equal to $a_{ij}X_j$ or $X_{ij} = a_{ij}X_j$.

So the production function can be written as follows:

X_1 = min. $(X_{11}/a_{11}, X_{21}/a_{21},\ldots, L_1/a_{L1})$; ....; X_4 = min.$(X_{14}/a_{14}, X_{24}/a_{24},\ldots, L_4/a_{L4})$.

In other words, this means that X_i (i =1, 2, 3, 4) will be equal to the minimum of the five ratios within the bracket.

It can be seen that if each of the X_{ij} is multiplied by a constant, the corresponding X_j is multiplied by the same constant so that we have a constant returns to scale. If any a_{ij} is zero, we can ether omit the corresponding term from the right hand side or we can think of X_{ij}/a_{ij} as infinity in which case it will certainly never be the smallest ratio.

An alternative way of writing the production function is to note that since X_1 equals the smallest of X_{11}/a_{11}, X_{21}/a_{21}, X_{31}/a_{31}, X_{41}/a_{41}, L_1/a_{L1}, it must be less than or equal to all five of the ratios, i.e., $X_{11}/a_{11} \geq X_1$, i.e., $X_{11} \geq a_{11}X_1$. Similarly, $X_{21} \geq a_{21}X_1$...and $L_1 \geq a_{L1}X_1$.

In general, $X_{ij} \geq a_{ij}X_j$ and $L_j \geq l_jX_j$.

It should be noted that equality would hold at least once in each row. In fact, if none of the commodities concerned is free good, the equality will hold everywhere. Assuming that no goods are free, we can consider all the above equalities. Then putting the values of X_{ij} in equation (1), (2),..., we get,

$X_1 = a_{11}X_1 + a_{12}X_2 + \ldots + a_{1n}X_n + F_1$;

$X_2 = a_{21}X_1 + a_{22}X_2 + \ldots + a_{2n}X_n + F_2$;

.....;

$X_n = a_{n1}X_1 + a_{n2}X_2 + \ldots + a_{nn}X_n + F_n$; and

$L = l_1X_1 + l_2X_2 + l_3X_3 + l_4X_4$.

In general, $X_i = \sum_{j=1}^{n} a_{ij}X_j + F_j$, (i=1, 2, …, n) and $L = \sum_{i=1}^{n} l_i X_i$.

The equation may be put in matrix notations:

$$\begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} + \begin{pmatrix} F_1 \\ \vdots \\ F_n \end{pmatrix}$$

$X = AX + F$ and, $L = \sum_{i=1}^{n} l_i X_i$.

Check Your Progress 1

1) What is an input coefficient matrix?

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2) What is an open input-output model?

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3) The input coefficient matrix is given as

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 0.2 & 0.3 & 0.2 \\ 0.4 & 0.1 & 0.2 \\ 0.1 & 0.3 & 0.2 \end{pmatrix}$$

Get the technology matrix.

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12.3 CLOSED AND OPEN INPUT-OUTPUT MODELS

In the above example, besides n industries, our model contains exogenous sector of final demand, which supplies primary input factors (labor services-which are not produced by these n industries) and consumes the output of n producing industries (not as input). Such an I-O model is known as '*open model*'. It includes exogenous sectors in terms of 'final demand bill' along with the endogenous sectors in terms of n producing sectors. I-O model, which has endogenous final demand vector, is known as '*closed model*'.

12.4 COEFFICIENT MATRIX AND OPEN MODEL

Our open model in matrix notation is given by, $X = AX + F$, where A is the input coefficient matrix, F is the final demand vector and X is the total output matrix.

The input coefficient matrix or 'technology matrix' represented by $[a_{ij}]$ is of great importance. Each element must be non-negative, i.e., we rule out the possibility of negative inputs. But to maintain complete interdependence among the industries each element of $[a_{ij}]$ matrix must be positive and no element can exceed unity, i.e., we rule out the possibility of negative outputs. Each column of this matrix specifies the input requirements for the production of 1 unit of a particular commodity; thus, sum of the elements in each column must be less than unity. Symbolically, this fact may be stated as:

$\sum_{i=1}^{n} a_{ij} < 1$, (j = 1, 2,..., n), and each a_{ij} is non-negative, i.e., either zero or greater than zero. The cost of the primary inputs (which is also termed as '*value added*') needed in producing a unit of j^{th} commodity would be $(1 - \sum_{i=1}^{n} a_{ij})$. [Note here that a_{ij}'s are in value terms]. If this were not true, it would mean that the total value of intermediate products used by an industry exceeded the value of its input. This in turn would mean that the value added by that industry was negative. Now, this is not impossible, but, if we assume that the wage bill cannot be negative, it means that the industry must be making losses (indeed, losses greater in absolute value than its wage bill). An industry in which value added is negative is not covering variable costs (intermediate inputs plus the wage bill), and we know from the elementary micro theory that in such a case, losses will be reduced by closing down. Thus, we do not want to describe such an industry in our technology matrix at all. Given the assumption of CRS, we've described the technology by a constant coefficient matrix. We should notice that we have also built in the assumption that there are no externalities. An externality in production would exists if, for example, a factory discharged waste into a river so that a factory further downstream had to use the resources to clean the water before it could use it. In this case, the resource requirement of the later factory would not depend solely on its output but would, also depend on the activity of the former.

12.5 SOLUTION TO OPEN MODEL

Let us consider an economy with n industries. If producing sector 1 is to produce an output just sufficient to meet the input requirements of the n industries as well as the final demand of the exogenous sector, its output level X_1 must satisfy the following equations:

$X_1 = a_{11}X_1 + a_{12}X_2 + \ldots + a_{1n}X_n + F_1$, or,

$(1-a_{11})X_1 - a_{12}X_2 - \ldots - a_{1n}X_n = F_1$

For the entire set of n industries, the correct output levels, therefore, can be symbolized by the following set of n linear equations:

$(1-a_{11})X_1 - a_{12}X_2 - \ldots - a_{1n}X_n = F_1$

$-a_{21}X_1 + (1-a_{22})X_2 - \ldots - a_{1n}X_n = F_2$

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$-a_{n1}X_1 - a_{n2}X_2 - \ldots\ldots + (1-a_{nn})X_n = F_n$

In the matrix notation this may be written as:

$$\begin{pmatrix} 1-a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & -a_{22} & \cdots & -a_{2n} \\ \cdots & \cdots & \ddots & \cdots \\ -a_{n1} & -a_{n2} & \cdots & 1-a_{nn} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{pmatrix}$$

$[I-A]\ X = F$, i.e. $X = [I-A]^{-1}\ F$

Here A is the given matrix of input coefficients, while X and F are the vectors of output and final demand of each producing sector. If $|I-A| \neq 0$, then $[I-A]^{-1}$ exists, we can then estimate for either of the two matrices X and F by assuming one of them to be given exogenously. It is to be observed that, the assumptions made in I-O analysis go a long way in making the problem simplified. For example, with the assumption of linear homogenous function, it is possible to write a linear equation of each producing sector, which then can be easily transformed into matrix notation. On the other hand, as long as the input coefficients remain fixed (as assumed), the matrix A will not change or [I-A] will not change. Therefore, in finding the solution of $X = [I-A]^{-1}\ F$, only one matrix inverse needs to be performed even if we are to consider thousand of different final demand vectors according to alternative development targets. Hence, such an assumption of fixed technical coefficient has meant considerable savings in computational effort.

12.5.1 Determination of Gross Output

Suppose, X stands for the column matrix $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$, F stands for the column matrix $\begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$, and a stands for the matrix of a_{ij} coefficients $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. Then from $X = [I-A]^{-1}\ F$ we can get the value of each element of X. Before this, we need to evaluate $[I-A]^{-1}$.

$$I - A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 1-a_{11} & -a_{12} \\ -a_{21} & 1-a_{22} \end{pmatrix}, \text{ matrix of co-factors} = \begin{pmatrix} 1-a_{22} & a_{21} \\ a_{12} & 1-a_{11} \end{pmatrix};$$

Transpose of this matrix or adjoint matrix of [I-A] $= \begin{pmatrix} 1-a_{22} & a_{12} \\ a_{21} & 1-a_{11} \end{pmatrix}$. Now let D = | I-A | = determinant of [I-A] matrix. Assume that $D \neq 0$, i.e., [I-A] is non-singular. Then the inverse matrix can be written as $[I-A]^{-1} = \begin{pmatrix} \frac{1-a_{22}}{D} & \frac{a_{12}}{D} \\ \frac{a_{21}}{D} & \frac{1-a_{11}}{D} \end{pmatrix} =$

$$\frac{1}{D}\begin{pmatrix} 1-a_{22} & a_{12} \\ a_{21} & 1-a_{11} \end{pmatrix}$$

Therefore, $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \frac{1}{D}\begin{pmatrix} 1-a_{22} & a_{12} \\ a_{21} & 1-a_{11} \end{pmatrix}\begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$

Or, $X_1 = \frac{(1-a_{22})F_1 + a_{12}F_2}{D}$, $X_2 = \frac{a_{21}F_1 + (1-a_{22})F_2}{D}$ where, $D = \begin{vmatrix} 1-a_{11} & -a_{12} \\ -a_{21} & 1-a_{22} \end{vmatrix} = (1\text{-}a_{11})$ $(1\text{-}a_{22}) - a_{12}\, a_{21}$.

12.5.2 The Hawkins-Simon Conditions

Often I-O solution may give output expressed by negative numbers. If our solution gives negative outputs, it means that more than one unit of a product is used up in the production of every one unit of that product; it is definitely an unrealistic situation. Such a system is not a viable. Hawkins-Simon condition guards against such eventualities. Our basic equation is $X = [I-A]^{-1} F$, in order that this does not give negative numbers as a solution, the matrix [I-A], which in fact is:

$$\begin{pmatrix} 1-a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & -a_{22} & \cdots & -a_{2n} \\ \vdots & \cdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & 1-a_{nn} \end{pmatrix}$$ should be such that

i) the determinant of the matrix must always be positive, and

ii) the diagonal elements: $(1\text{-}a_{11})$, $(1\text{-}a_{22})$,..., $(1\text{-}a_{nn})$ should all be positive or, in other words, elements, a_{11}, a_{22},...,a_{nn} should all be less than one. Thus, one unit of output of any sector should use not more than one unit of its own output. These are called Hawkins-Simon conditions. Further, the first condition, that implies $D > 0$, implies that (for 2 industry case) $\begin{vmatrix} 1-a_{11} & -a_{12} \\ -a_{21} & 1-a_{22} \end{vmatrix} > 0$, or, $(1\text{-}a_{11})\,(1\text{-}a_{22}) - a_{12}\, a_{21} > 0$. This condition implies that the direct and indirect requirement of any commodity to produce one unit of that commodity must also be less than one. On the other hand, the interpretation is always that all subgroups of commodities should be 'self sustaining', directly and indirectly.

Example 1: Suppose $[I-A] = \begin{bmatrix} 0.2 & -0.2 \\ -0.9 & 0.3 \end{bmatrix}$.

Then from [I-A] = $\begin{bmatrix} 0.2 & -0.2 \\ -0.9 & 0.3 \end{bmatrix}$ we can get the value of the determinant [I-A] = (-) 0.12, which is less than zero. As the Hawkins-Simon conditions are not satisfied, no solution will be possible in this case.

Check Your Progress 2

1) Describe the features of a closed input-output model.

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2) Discuss the importance of Hawkins-Simon conditions in an input-output model.

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3) Suppose [A] = $\begin{bmatrix} 0.8 & 0.2 \\ 0.9 & 0.7 \end{bmatrix}$, then check whether any solution will be possible for the system or not.

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12.5.3 Consumption Possibility Locus

So long we assumed that the final demands F_1 and F_2 are given and our main problem was to determine the gross outputs of the two industries to support the final demands and also to know whether these gross outputs can be produced with the given labor force or not. Consider now a different type of problem. Let us suppose that the total labor supply is fully employed while F_1 and F_2 are not given. Let us now consider different combinations of F_1 and F_2 which can be obtained from the full employment of the given labor force. The different combinations of F_1 and F_2 which can be obtained from the full employment of the given labor force is known as consumption possibility locus. Let us see how we can derive this consumption possibility locus. Consider, for simplicity, 2 industry cases. Then equations represent the level of outputs can be written as

$X_1 = a_{11}X_1 + a_{12}X_2 + F_1$ and $X_2 = a_{21}X_1 + a_{22}X_2 + F_2$. Therefore, $X_1 = \frac{(1-a_{22})F_1 + a_{12}F_2}{D}$, $X_2 = \frac{a_{21}F_1 + (1-a_{11})F_2}{D}$.

Let us denote the coefficients of F_1 and F_2 in these solutions by A_{ij}, i.e., let, $A_{11} = \frac{1-a_{11}}{D}$, $A_{12} = \frac{a_{12}}{D}$, $A_{21} = \frac{a_{21}}{D}$, and $A_{22} = \frac{1-a_{22}}{D}$. Then the solution can be written as: $X_1 = A_{11} F_1 + A_{12} F_2$; and $X_2 = A_{21} F_1 + A_{22} F_2$. These A_{ij} coefficients have economic meanings: A_{ij} is the direct and indirect input requirement of the i^{th} commodity to support one unit of final demand of the j^{th} commodity. Note that, $A_{ij} = \frac{\delta X_i}{\delta F_j}$ ($\forall$ i, j = 1, 2). The derivative $\frac{\delta X_i}{\delta F_j}$ represents the change in X_i (i=1, 2) if F_j (j=1, 2) is changed by one unit. Thus, if F_j (j=1, 2) is to be changed by 1 unit, we require A_{1j} units of X_1 and A_{2j} units of X_2. Therefore, if we like to get F_1 units of the first commodity and F_2 units of the second commodity, then to meet these final demand the total direct and indirect requirement of the first commodity is equal to $A_{11}F_1 + A_{12}F_2$. Therefore, the equation $X_1 = A_{11}F_1 + A_{12}F_2$ represents the equality between the supply and demand in the market for the first commodity. Similar analysis will be followed for the second commodity. In the same way the equation $L = l_1X_1 + l_2X_2$ can be interpreted as the supply-demand equation for the labor market. Leontief's model can thus be regarded as a general equilibrium system of three markets (two commodity markets + one labor market).

Let us put the values of X_1 and X_2 as obtained from the solution of the equations:

$X_1 = a_{11}X_1 + a_{12}X_2 + F_1$; $X_2 = a_{21}X_1 + a_{22}X_2 + F_2$; and $L = l_1X_1 + l_2X_2$.

Therefore, $L = l_1 (A_{11} F_1 + A_{12} F_2) + l_2 (A_{21} F_1 + A_{22} F_2)$. [Substituting the values of X_1 and X_2]

Or, $L = A_{L1}F_1 + A_{L2}F_2$, where $A_{L1} = l_1A_{11} + l_2 A_{21}$; $A_{L2} = l_1A_{12} + l_2 A_{22} F_2$.

The coefficients A_{L1} and A_{L2} can be given economic interpretations: A_{Li} (i=1, 2) represents the direct and the indirect requirement of labor to support 1 unit of final demand of the i^{th} commodity. Alternatively, A_{Li} (i=1, 2) is the total labor cost of 1 unit of final demand of the i^{th} commodity. A_{Li} coefficients are derived from a_{ij} coefficients, which are technologically given. Hence A_{Li}'s are also technologically given. Since A_{Li}'s are given, the equation $L = A_{L1}F_1 + A_{L2}F_2$ is the equation of a straight line. This is the equation of the '*consumption possibility locus*' (CPL). This equation gives us different combinations of F_1 and F_2, which can be obtained from the full employment of the given labor force. The equation can be written as $A_{L2}F_2 = - A_{L1}F_1 + L$; or, $F_2 = - (A_{L1}/A_{L2}) F_1 + L/A_{L2}$

If we plot the above equation on the (F_1, F_2) plane then, we will get a downward sloping straight line with absolute slope (A_{L1}/A_{L2}).

A consumption possibility locus can be regarded as a social transformation curve. If the society desires to consume only F_i (i=1, 2), it can consume an amount equal to L/A_{Li}. Because the CPL is a straight line, substitution of F_1 and F_2 is possible and it takes place at a constant cost. The rate at which F_2 is substituted for F_1 is called the marginal rate of substitution (MRS). The MRS is

equal to the absolute value of the slope of the CPL, i.e., (A_{L1}/A_{L2}). This means that giving up of 1 unit of F_1 the society can get (A_{L1}/A_{L2}) units of F_2. This straight-line transformation curve reflects not only the linearity of the technology but also the presence of only one primary factor and the absence of joint production.

We know that under the assumption of perfect competition, the MRS between any two commodities will be equal to the price ratio. Let p_i (i=1, 2) represents the price of the i^{th} commodity. Then we get $(p_1/p_2) = (A_{L1}/A_{L2})$. This represents that the price ratio is equal to the ratio of labor costs of the two commodities. This is nothing but an expression of the Ricardian labor theory of value which states that the value of a commodity determined by the amount of labor embodied in that commodity. It also shows that the relative price is determined by technological coefficients and is independent of demand. In other words, whatever may be the composition of the final demand, the price ratio (p_1/p_2) remains the same.

Check Your Progress 3

1) In a two-industry input-output model we have the following information:

$[I-A]^{-1} = \frac{1}{0.84}\begin{bmatrix} 0.9 & 0.4 \\ 0.3 & 0.8 \end{bmatrix}$, $l_1 = 0.2$, $l_2 = 0.3$ and $L = 10$. Find the equation of the consumption possibility locus.

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12.5.4 Determination of Equilibrium Prices

Let the prices of commodities 1, 2, 3,... be p_1, p_2, p_3,...respectively, and the price of the primary factor inputs be *w* (here, primary factor is labor; so w represents wage rate), then the technology matrix or transaction matrix in quantity may be converted into that in value terms. The problem can be posed as follows:

Sector	1	2	Final demand
1	$a_{11}X_1p_1$	$a_{12}X_2p_1$	F_1p_1
2	$a_{21}X_1p_2$	$a_{22}X_2p_2$	F_2p_2
Primary Input	$l_1X_1\,w$	$l_2X_2\,w$	
Total Costs	$a_{11}X_1p_1 + a_{21}X_1p_2 + l_1X_1\,w$	$a_{12}X_2p_1 + a_{22}X_2p_2 + l_2X_2\,w$	

With pure competition and free entry, profit in each industry must be zero, i.e., revenues equal costs. Hence, for the first industry receipts are (output x price) and cost is $a_{11}X_1p_1 + a_{21}X_1p_2 + l_1X_1 w$. Same is true for the second industry.

Hence, for equilibrium; $p_1X_1 = a_{11}X_1p_1 + a_{21}X_1p_2 + l_1X_1w$; and $p_2X_2 = a_{12}X_2p_1 + a_{22}X_2p_2 + l_2X_2w$, which simplify to $p_1 = a_{11}p_1 + a_{21}p_2 + l_1w$; and $p_2X_2 = a_{12}p_1 + a_{22}p_2 + l_2 w$, which can be put in matrix form as under:

$\begin{bmatrix} 1-a_{11} & -a_{21} \\ -a_{12} & 1-a_{22} \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} l_1\omega \\ l_2\omega \end{bmatrix}$. Notice that the set of coefficients here are transposed, this matrix is transposed of [I-A].

Therefore $\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 1-a_{11} & -a_{21} \\ -a_{12} & 1-a_{22} \end{bmatrix}^{-1}\begin{bmatrix} l_1\omega \\ l_2\omega \end{bmatrix} = [\text{I-A}]^{-1}\begin{bmatrix} l_1\omega \\ l_2\omega \end{bmatrix}$.

Therefore, $p_1 = \frac{1}{D}(A_{11}l_1 + A_{12}l_2)w$, and $p_2 = \frac{1}{D}(A_{21}l_1 + A_{22}l_2)\ w$, where A_{11}, A_{12}, etc., are the cofactors of the matrix [I-A] as in the preceding cases.

Check Your Progress 4

1) Consider the following technology matrix

	Steel	*Coal*	*Final Demand*
Steel	0.4	0.1	50
Coal	0.7	0.6	100
Labor	5	2	

Determine the equilibrium prices, if the wage rate is Rs. 10 per man day.

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12.6 LINEAR PROGRAMMING FORMULATION

Leontief's I-O model (in case of 2 industries) can be interpreted as a simplified linear programming model. Suppose that the final demand F_1 and F_2 are given. The society likes to get these final demands. Now if X_1 is the total output of the first industry, then out of this $a_{11}X_1$ will be used up in the first industry itself and $a_{12}X_2$ will be used up in the second industry. The amount left is, therefore, $(X_1\text{-}a_{11}X_1 - a_{12}X_2)$ and it must be equal to the final consumption F_1 if this final consumption is to be achieved. From this relation we get one constraint $(1\text{-}a_{11})$ $X_1 - a_{12}X_2 \geq F_1$, which will be utilized in the linear programming formulation. Similarly, for the second industry we can get another constraint $-a_{21}X_1 + (1\text{-}a_{22})\ X_2 \geq F_2$. The gross output levels X_1 and X_2 cannot be negative so that $X_1 \geq 0$ and $X_2 \geq 0$ are the non-negativity conditions. Consider now the objective

function. Here our problem is that the final demands F_1 and F_2 are specified. To get these specified final demands we have to choose gross outputs X_1 and X_2 in such a manner that the total labor cost is minimized. The labor cost (in money terms) of X_1 and X_2 is equal to $l_1X_1w + l_2X_2w$. This is the objective function to be minimised.

Thus, in the Leontief model we get a typical linear programming problem, which can be stated as follows:

Minimise $W = l_1X_1w + l_2X_2w$

Subject to $(1-a_{11})\ X_1 - a_{12}X_2 \geq F_1$

$$-a_{21}X_1 + (1-a_{22})\ X_2 \geq F_2$$

$$X_1 \geq 0 \text{ and } X_2 \geq 0.$$

By following the rules regarding the primal problem and the dual problem, the dual of the above primal problem can be written as follows:

Maximise $R = p_1F_1 + p_2F_2$

Subject to $(1-a_{11})\ p_1 - a_{21}p_2\ p \leq l_1w$

$$-a_{12}p_1 + (1-a_{22})\ p_2 \leq l_2w$$

$$p_1 \geq 0, \text{ and } p_2 \geq 0.$$

Here it is to be noted that minimum value of the objective function in the primal problem is equal to the maximum value of the objective function in the dual problem. This means that total value of net output is just imputed as wages to the scarce factor, labour.

12.7 COEFFICIENT MATRIX AND CLOSED MODEL

We shall now examine whether we will be able to estimate F or X if the model is closed one. If the exogenous sector (final demand bill) of the open model is absorbed into the system of endogenous sectors, the model would turn into a closed one. In such a model final demand bill and primary inputs will not appear any more; rather in their place, we shall have the input requirements and output of this newly conceived industry, the 'household industry' producing the primary input labor. Final demand sector would now be considered as one of the endogenous sectors. As such now we shall have (n+1) industries in place of n industries and all producing for the sake of satisfying the requirements. This will mean, for example, that households will consume each commodity in fixed proportion to the labour services they supply.

Let us assume that, there are four industries only – including the new one (of final demand) designated b subscript 0(zero).We shall, therefore, have the following set of equations:

$X_0 = a_{00}X_0 + a_{01}X_1 + a_{02}X_2 + a_{03}X_3$; $X_1 = a_{10}X_0 + a_{11}X_1 + a_{12}X_2 + a_{13}X_3$;

$X_2 = a_{20}X_0 + a_{21}X_1 + a_{22}X_2 + a_{23}X_3$; and $X_3 = a_{30}X_0 + a_{31}X_1 + a_{32}X_2 + a_{33}X_3$.

This gives us a homogenous set of equation system

$$\begin{pmatrix} 1-a_{00} & -a_{01} & -a_{02} & -a_{03} \\ -a_{10} & 1-a_{11} & -a_{12} & -a_{13} \\ -a_{20} & -a_{21} & 1-a_{22} & -a_{23} \\ -a_{30} & -a_{31} & -a_{32} & 1-a_{33} \end{pmatrix} \begin{pmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Since the four rows of the input coefficient matrix happen to be linearly dependent, |I-A| will turn out to be zero. Hence the solution is indeterminate.

This means that in a closed model no output-mix of each sector exists. We can at most determine the output levels of endogenous sectors in proportion to one another, but cannot fix their absolute levels unless additional information is made available exogenously.

12.8 DYNAMIC INPUT-OUTPUT MODEL

The Leontief I-O model has appeared in a number of modified forms. We have discussed above how the model can be changed to an open or close one by including or ignoring the exogenous sectors. Here we shall discuss a dynamic form of I-O model in which specific account is taken of inter-relationship of the current and past outputs.

In our static open model, we assumed that current output can be used for current consumption and/or as current input in production of some other outputs. But in a dynamic model it is assumed that the current output can also be used (along with these two purposes) as an addition to industry's inventory stock and fixed capital formation.

The balance equation will, therefore, be

$$X_i = X_{i1} + X_{i2} + \ldots + X_{in} + \Delta \sum_{j=1}^{n} S_{ij} + F_i$$

$$(X_i)_t = (X_{i1})_t + (X_{i2})_t + \ldots + (X_{in})_t + \Delta\left(\sum_{j=1}^{n} S_{ij}\right)_t + (F_i)_t$$

Here X_i is the total output of the i^{th} (endogenous) producing sector in period t, which is used for three purposes:

i) as an input for production in economy's n industries, i.e., $(X_{i1})_t + (X_{i2})_t + \ldots + (X_{in})_t$

ii) as remaining final demand: (Fi) t (which now includes purchases by government sector, household and foreign sector).

iii) as an addition to the stock of n industries: $\Delta\left(\sum_{j=1}^{n} S_{ij}\right)_t$. If depreciation is ignored, the net addition to stock of i^{th} output in the n producing sectors will be equal to the difference between the current year's and last year's accumulated capital.

$$\left[\Delta\left(\sum_{j=1}^{n} S_{ij}\right)_t\right]_{net} = \left(\sum_{j=1}^{n} S_{ij}\right)_t - \left(\sum_{j=1}^{n} S_{ij}\right)_{t-1}.$$

Given the fixed relation between capital and output: $b_{ij} = \frac{S_{ij}}{X_j}$,

so that $b_{ij}(X_j)_t = (S_{ij})_t$ and $b_{ij}(X_j)_{t-1} = (S_{ij})_{t-1}$

Therefore, the balance equation of dynamic model turns out to be:

$$(X_i)_t = \sum_{j=1}^{n} a_{ij}(X_j)_t + (F_i)_t + \sum_{j=1}^{n} b_{ij}\left[(X_j)_t - (X_j)_{t-1}\right]$$

This is a basic equation of dynamic model representing a set of first order linear difference equations with constant coefficients: a_{ij}'s and b_{ij}'s.

Another way of formulating dynamic model is as follows:

Let the investment demand by sector j be $I_{jt} = b_j (X_{j,t} - X_{j,t-1})$, where b_j is the acceleration coefficient. In matrix notation $I = b'(X_t - X_{t-1})$. In the continuous case the above relation will be $I = b'\left(\frac{dX}{dt}\right)$

Now in order to transform these aggregate figures of capital formation (C.F) by sector of use to sector of origin or to demand for sectoral outputs, a capital formation matrix has to be defined. Let it be, $B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix}$

where b_{ji}=(demand for j^{th} sector's output by i^{th} sector for C.F)/(C.F in sector i)

Hence the vector of investment by origin is $= Bb'(X_t - X_{t-1})$ or, $Bb'\frac{dX}{dt}$ (in continuous case).

Hence the Leontief consistency conditions are

$X_t = AX_t + F_t + Bb'(X_t - X_{t-1})$ [in discrete case]

$X = AX + F + Bb'\frac{dX}{dt}$ [in continuous case]

Thus, the system may be described in first order difference or differential equations. However, in a dynamic (model) situation it is better to use the inequalities to account for over production and excess capacity. Planning for long-term cannot be reduced to simple matter of solution of a system of simultaneous equations as in the input-output case.

12.9 LIMITATIONS OF INPUT-OUTPUT ANALYSIS

i) Errors in forecasting final demand will have grave consequence.

ii) Current relative prices of inputs may not be same as the ones implied in the table.

iii) The assumption of linear homogenous production function may not be valid. The technical coefficients will not remain constant even if input price ratios are held constant in such circumstances.

iv) The constant coefficient formulation also ignores the possibility of industry outputs reaching capacity, changing prices and input proportions in the table.

v) The assumption of constant technical coefficients goes counter to the possibility of substitutions of inputs and factors.

vi) Sectoral division is, for practical purposes, limited. Such a sectorisation is not good enough for many forecasting purposes.

vii) Sectorisation (grouping of commodities in sectors) is often arbitrary. The intra-sectoral heterogeneity with respect to technologies, efficiency and demand is not invariant over time.

viii) Input-output model building is highly costly in terms of time and money.

ix) Regional input-output analysis involves many more assumptions and difficulties in construction of such tables.

12.10 LET US SUM UP

This unit tells us about the interrelationship among different industries in the market. It also shows the way of determining output and price of the product for each industry, which is the most important thing for this kind of inter linkage among the industries. Relation between input-output analysis and linear programming problem reflects the important linkage of two real life applications of matrix algebra.

12.11 KEY WORDS

Closed and Open Input-Output Model: The I-O model that considers 'final demand bill' as exogenous factor is said to be as open I-O model and in closed I-O model 'final demand bill' is considered as endogenous factor.

Consumption Possibility Locus: The different combinations of F_1 and F_2 (final demands) which can be obtained from the full employment of the given labor force is known as consumption possibility locus.

Dynamic Input-Output Model: In dynamic I-O model it is assumed that the current output can also be used (along with current consumption and/or current input in production) as an addition to industry's inventory stock and fixed capital formation.

Hawkins-Simon Conditions: It basically states that more than on unit of a product cannot be used up in the production of every unit of that product. If A is the technological coefficient matrix then, according to Hawkins-Simon

condition, determinant of [I-A] must be positive and all principal minors of [I-A] must also be positive.

Technological Coefficient Matrix: The matrix $[a_{ij}]$, which basically represents input requirement from ith industry to produce one unit output of jth industry, is known as technological coefficient matrix.

12.12 SOME USEFUL BOOKS

Chiang Alpha C. (1984), *Fundamental Methods of Mathematical Economics*, Third edition, McGraw-Hill Book Company.

Archibald, G.C & R.G. Lapse (1976), *Introduction to Mathematical Economics: Methods and Applications*, Harper & Row, New York.

Dorman, Samuelson & Sallow, *Linear Programming and Economic Analysis*, McGraw-Hill Book Company, New York, 1958.

Mukherji, B. & V. Pandit, *Mathematical Methods for Economic Analysis*, Allied Publishers Pvt. Ltd., New Delhi, 1982.

12.13 ANSWER OR HINTS TO CHECK YOUR PROGRESS

Check Your Progress 1

1) See Section 12.2 and answer.

2) See Section 12.2 and answer.

3) $$A=\begin{pmatrix}(1-a_{11}) & -a_{12} & -a_{13}\\ -a_{21} & (1-a_{22}) & -a_{23}\\ -a_{31} & -a_{32} & (1-a_{33})\end{pmatrix}=\begin{pmatrix}0.8 & -0.3 & -0.2\\ -0.4 & 0.9 & -0.2\\ -0.1 & -0.3 & -0.8\end{pmatrix}$$

Check Your Progress 2

1) See Section 12.3 and answer

2) See Sub-section 12.5.2 and answer

3) Follow Example 1.

Check Your Progress 3

1) Let F_1 and F_2 be the final demands for the output of the two industries X and Y respectively. Then, $X_1 = 0.9X_1 + 0.4X_2 + F_1$; $X_2 = 0.3X_1 + 0.8X_2 + F_2$; solving these equations for X_1 and X_2 with the help of Cramer's rule we get $X = [I\text{-}A]^{-1}F$, i.e. $X_1 = \frac{1}{0.84}[0.9F_1 + 0.4F_2]$ and $X_2 = \frac{1}{0.84}[0.3F_1 + 0.8F_2]$

Putting these values of X_1 and X_2 in the labor supply equation we get and $10 = 0.2X_1 + 0.3X_2$

i.e., $10 = (0.2)\frac{1}{0.84}[0.9F_1 + 0.4F_2] + (0.3)\frac{1}{0.84}[0.3F_1 + 0.8F_2]$

$8.4 = 0.27F_1 + 0.32F_2$. This is the required equation of the consumption possibility locus.

Check Your Progress 4

1) Here $I-A = \begin{bmatrix} 0.6 & -0.1 \\ -0.7 & 0.4 \end{bmatrix}$, $\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 0.6 & -0.1 \\ -0.7 & 0.4 \end{bmatrix}^{-1} \begin{bmatrix} 5x10 \\ 2x10 \end{bmatrix} = \begin{bmatrix} 200 \\ 100 \end{bmatrix}$.

Therefore, price of steel = Rs. 200 per ton and price of coal = Rs. 100 per ton.

12.14 EXERCISES

1) The input coefficient matrix is of an open input-output system is given as $A = \begin{pmatrix} 0.2 & 0.3 & 0.2 \\ 0.4 & 0.1 & 0.2 \\ 0.1 & 0.3 & 0.2 \end{pmatrix}$. If the final demand vector in thousand rupees happens to be $d = \begin{bmatrix} 10 \\ 5 \\ 6 \end{bmatrix}$, solve the system for output production.

[Hints: Primary input requirement is Rs. 21,000.]

2) Given the input-output matrix and final demand vector as

$A = \begin{pmatrix} 0.05 & 0.25 & 0.34 \\ 0.33 & 0.10 & 0.12 \\ 0.19 & 0.38 & 0 \end{pmatrix}$ and $d = \begin{bmatrix} 1800 \\ 200 \\ 900 \end{bmatrix}$, find the solution for output levels of three industries .

[Answer $\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} = \begin{bmatrix} 3091 \\ 1638 \\ 2114 \end{bmatrix}$]

3) Consider the following inter-industry transactions table:

Industry	1	2	Final Consumption	Total
1	500	1,600	400	2,500
2	1,750	1,600	4,650	8,000
Labor	250	4,800	...	5,050
Total	2,500	8,000	5,050	15,500

Construct technology coefficient matrix showing direct requirements. Does a solution exist for this system?

[Hint: Determine $a_{ij} = X_{ij} / X_j$, then construct $[A] = [a_{ij}]$. Then evaluate $|I-A|$.]

Ans: Find the demand vector D in the equation $AX + D = X$ consistent with the output vector $X = \begin{bmatrix} 25 \\ 21 \\ 18 \end{bmatrix}$ and the coefficient matrix $A = \begin{bmatrix} 0.2 & 0.3 & 0.2 \\ 0.4 & 0.1 & 0.2 \\ 0.1 & 0.3 & 0.2 \end{bmatrix}$. Test whether the Hawkins-Simon conditions for the viability of the system are satisfied.

4) Given $A = \begin{bmatrix} 0.1 & 0.3 & 0.1 \\ 0 & 0.2 & 0.2 \\ 0 & 0 & 0.3 \end{bmatrix}$ and final demands are F_1, F_2 and F_3. Find the output levels consistent with the model. What will be the output levels if $F_1 = 20$, $F_2 = 0$ and $F_3 = 100$?

[Answer: $X_1 = 1.11F_1 + 0.42F_2 + 0.28F_3$, $X_2 = 1.25F_2 + 0.36F_3$, and $X_3 = 1.43F_3$]

5) In the above example if final demand changes by 10, 10, 10, then what will be the change in sector outputs?

[Hint: $X = [I-A]^{-1} F$, so, $\Delta X = [I-A]^{-1} \Delta F$]

6) A three sector input-output matrix [I-A] is given as: $\begin{bmatrix} 1 & -0.5 & 0 \\ -0.2 & 1 & -0.5 \\ -0.4 & 0 & 1 \end{bmatrix}$ with labour coefficients (per unit of output) as 0.4, 0.7, 1.2, if the household demand for the outputs of the 3 sectors is 1000, 5000 and 4000, determine the level of output and employment.

If the wage rate is Rs. 10 per labor day, find the equilibrium prices.

7) Determine the consumption possibility locus, given the total available labor supply = 1000 units and the technology matrix is given by $A = \begin{bmatrix} 0.4 & 0.5 \\ 0.4 & 0.3 \end{bmatrix}$ and $L = \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix}$.

UNIT 13 LINEAR PROGRAMMING

Structure

13.0 Objectives

13.1 Introduction

13.2 Linear Programming: Basic Concept

13.3 Formulation: Structure and Variables of Linear Programming

13.4 Graphic Solution

13.5 Simplex Method

13.6 Duality of Linear Programming

13.7 Economic Importance of Duality

13.8 Duality Theorems

13.9 Zero-sum Games and Linear Programming

13.9.1 Basic Concept

13.9.2 Relationship between Game Theory and Linear Programming

13.10 Let Us Sum Up

13.11 Key Words

13.12 Some Useful Books

13.13 Answer or Hints to Check Your Progress

13.14 Exercises

13.0 OBJECTIVES

The objectives of this unit is to:

- enable you to grasp the basic idea of linear programming principles;
- explain the difference between different processes of solution and their application;
- introduction to the game theory and basic methods of solution of a game; and
- enable to get the idea about how these mathematical tools are applied in our basic problems.

13.1 INTRODUCTION

Linear programming (LP) is a technique used for deriving optimum use of limited resources. Specifically, it deals with maximising a linear function of

variables subject to linear constraints. Applications range from economic planning and environmental management to the diet problem.

13.2 LINEAR PROGRAMMING: BASIC CONCEPT

LP deals with a class of programming problems where both the objective function to be optimised is linear and all relations among the variables corresponding to resources are linear.

Remember that optimisation problems consist of two basic features, viz.,

- An objective function that you want to minimise or maximise.
- The objective function describes the behavior of the measure of effectiveness and captures the relationship between that measure and those variables that cause it to change. System variables can be categorised as decision variables and parameters. A decision variable is directly controlled by the decision-maker. Thus constraints are the relations between decision variables and the parameters.

Thus, any LP problem will have an objective function and a set of constraints. In most cases, constraints come from the nature of problem you work with. For example, if you want to achieve the desirable objective (maximisation or minimisation) you do so keeping in view some constraints (i.e., the difficulties, restrictions).

Check the following conditions while formulating a LP problem:

- The objective function must be linear. That is, check if all variables have power of 1 and they are added or subtracted (not divided or multiplied)
- The objective must be either maximisation or minimisation of a linear function. The objective must represent the goal of the decision-maker.
- The constraints must also be linear. Moreover, the constraint must be of the following forms ($\leq$, $\geq$, or =, that is, the LP-constraints are always *closed*) giving the LP in standard format:

Max c^T subject to (s.t.) $A x = b$

for $x \geq 0$

Feasible and Optimal Solutions: A solution value for decision variables, where all of the constraints are satisfied, is called a feasible solution. Most solution algorithms proceed by first finding a feasible solution, then seeking to improve upon it, and finally changing the decision variables to move from one feasible solution to another feasible solution. This process is repeated until the objective function has reached its maximum or minimum. This result is called an optimal solution. The basic goal of the optimisation process is to find values of the variables that minimise or maximise the objective function while satisfying the constraints. This result is called an optimal solution.

Check Your Progress 1

1) What is a linear programming problem?

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2) Define the following terms:

Constraint

Feasible solution

Optimal solution

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3) What conditions you have to keep in mind while formulating a linear programming problem?

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13.3 FORMULATION: STRUCTURE AND VARIABLES OF LINEAR PROGRAMMING

We can translate a real world problem in a format of LP. Consider the following example:

Example 1: Diet problem

A doctor chooses the least cost diet that will meet the nutritional requirement of her patients. The patients require 4, 8, 9 units of nutrition A, B, C per week. There are four varieties of food available. The nutritional contents of food per kg are given in the following table.

		food			
		1	2	3	4
	A	1	2	1	4
nutrient	*B*	1	3	0	2
	C	4	2	6	1

If the food cost Rs. 5, 7, 7, 9 per kg respectively, find an optimal weekly diet for the patients.

If we use x_j, units of food j, then we must solve the following liner programming problem.

minimise $5x_1 + 7x_2 + 7x_3 + 9x_4$

subject to $1x_1 + 2x_2 + 1x_3 + 4x_4 \geq 4$

$1x_1 + 3x_2 + 2x_4 \geq 8$

$4x_1 + 2x_2 + 6x_3 + 1x_4 \geq 9$

$x_1, x_2, x_3, x_4 \geq 0$

We can write this in matrix form as

$$\min_x c^T x \quad \text{subject to } \mathbf{Ax} \geq \mathbf{b}$$

$$\mathbf{x} \geq \mathbf{0}$$

Please note that

- minimizing $\mathbf{c}^T\mathbf{x}$ is the same as maximizing $-\mathbf{c}^T\mathbf{x}$.
- $\mathbf{Ax} \geq \mathbf{b} \Leftrightarrow -\mathbf{Ax} \leq -\mathbf{b}$

Therefore, we write the problem

- $\max_x -c^T x$ subject to $-\mathbf{Ax} \leq -\mathbf{b}$

$$\mathbf{x} \geq \mathbf{0}$$

Example 2: A manufacturer has two types of machines. Her production requires that she must have at least three A type of machines and one B type of machine. The cost of production in type A machine is Rs.1000 while that in type B is Rs.1200 for. The floor areas taken up by each of these machines are $4m^2$ and $5m^2$ respectively. The total cost of production must not exceed Rs.15000/- and the available floor space is $40m^2$.

Let x, y be the number of type A and type B machines. Then the equalities will be $1000x + 1200y \leq 15000$ for the constraint as regards money.

Such a formulation implies $5x + 6y \leq 75$ .....(1)

For space constraint we have $4x + 5y \leq 40$ .....(2)

Again for A type of machine $x \geq 3$ .....(3)

And for B type of machine $x \geq 1$ .....(4)

Now, suppose the weekly profit from the output as Rs.120 for each type A and Rs.100 for each type B machine. Our problem is to find the combination of machine use giving maximum profit. If the total profit $p = 120x + 100y$, then the problems is to find x and y which will maximise the objective function. $p = 120x + 100y$ subject to constraints (1), (2), (3) and (4) above. Here x and y are decision variables.

13.4 GRAPHIC SOLUTION

Graphic method becomes handy in the obtaining solution from simple LP problems. Just follow the sequence:

Step 1: Plot the constraints on the graph paper and find the feasible region.

Step 2: Find the coordinates of that verifies (corner points) of the feasible region.

Step 3: Find the values of the objective function of these corner points.

Step 4: Select the corner point at which the value of the objective function is maximum (or minimum as in minimisation problem). This corner point will be the required solution of the given L.P.P.

Example: maximise $z = 6x_1 + 7x_2$

Subject to $2x_1 + 3x_2 \leq 12$

$2x_1 + x_2 \leq 8$

$(x_1\ x_2) \geq 0.$

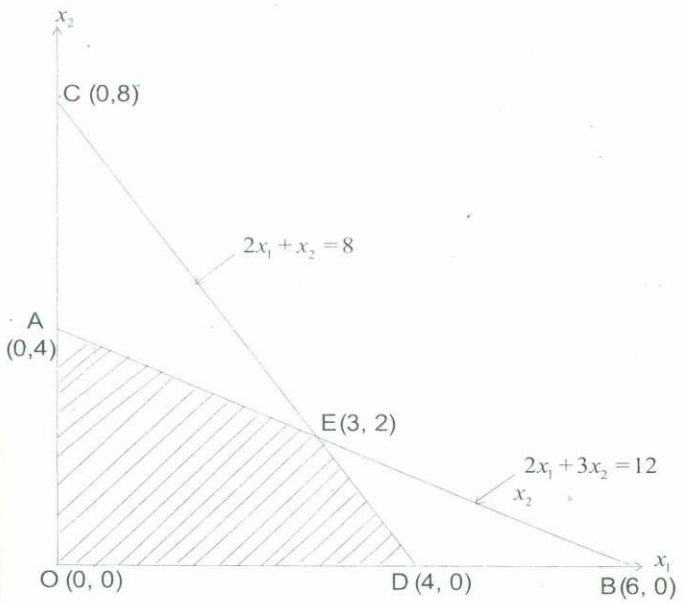


Fig. 13.1

The shaded region OAED is the set of points that are feasible under all the constraints it is called the feasible region. The task now is to find which combination of x_1 and x_2 in OAED for which the value of z is maximised.

The task is added by a theorem: If there exists an optimum solution to the problem, it will be found among the combinations of x_1 and x_2 values represented by the vertices (or the extreme corners) of the feasible solution polygon.

Corner Point	Coordination (x_1 x_2)	Value of z (=$6x_1 + 7x_2$)
O	(0, 0)	6.(0) + 7.(0) = 0
D	(4, 0)	6.(4) + 7.(0) = 24
E	(3, 2)	6.(3) + 7.(2) = 32
A	(0, 4)	6.(0) + 7.(4) = 28

The value of z is maximum at E whose coordinates are (3, 2). Hence, the required solution is $x_1 = 3$, $x_2 = 2$ and the corresponding maximum value of z =32.

The same process is followed for a minimisation process but only then we have to consider the minimum value of z.

Extra Variables

Before looking into further solution options of a LP, it will be useful to understand the nature of variables that enter into its formulation. For that purpose let us take a simple example: A textile firm produces two products I and II in its plant consisting of three departments dyeing, stitching and packaging. The machines in each department can be used for 8 hours a day. The production process can be described as follows:

- product I is stitched first and then packaged. One ton of product I requires $\frac{1}{2}$ hour of stitching and $\frac{1}{3}$ hour of packaging,
- product II is first dyed and than packaged. One ton of product II requires 1 hour of dying and $\frac{2}{3}$ hours of packaging;
- products I and II are sold at prices Rs. 80 and Rs. 60 per ton respectively after the deduction of variable costs incurred;
- the gross profit per ton of products I and II is given as Rs. 40 and Rs. 30.

Our interest in the problem is to find out the output combination that would maximise the total profit of the firm.

LP Formulation:

Let x_1 and x_2 be the number of tons to be produced of products I and II. The problem, therefore, is:

maximise $\pi = 40x_1 + 30x_2$ (profit in Rs.)

subject to $x_1 \leq 16$ (stitching constraint)

$x_2 \leq 8$ (dyeing constraint)

$x_1 + 2x_2 \leq 24$ (packaging constraint)

with $x_1, x_2 \geq 0$

We can rewrite the problems as:

maximise $\pi = 40x_1 + 30x_2 + 0s_1 + 0s_2 + 0s_3$

subject to $x_1 + s_1 = 16$

$+ x_2 + s_2 = 8$

$x_1 + 2x_2 + s_3 = 24$

$x_1, x_2, s_1, s_2, s_3 \geq 0$

Thus, there are altogether five variables. From these s_i (for I=1, 2, 3) are called ***slack variables***. Note that the reformulated problem is equivalent to the original formulation except that the constraints are of the form

$$\mathbf{Ax} = \mathbf{b}$$

$$\mathbf{x} = \mathbf{0}$$

and we write the matrix equation as

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} 16 \\ 8 \\ 24 \end{pmatrix}$$

When some component of $\boldsymbol{x}$ is of unrestricted sign, may be $s_3 \geq 0$ is omitted, then s_3 is called ***free variable.*** You can eliminate a free variable by setting

$s_3 = u_3, v_3$ where $u_3, v_3 \geq 0$.

Two different standard forms of LP:

Observe that any LP can be converted to a problem in the form

$\min_x c^T x$ subject to $\mathbf{Ax} = \mathbf{b}$

$$\mathbf{x} \leq \mathbf{0}$$

This is because:

- Minimizing $\mathbf{c}^T\mathbf{x}$ is equivalent to maximizing $-\mathbf{c}^T\mathbf{x}$,
- Inequalities in the constraints can be converted to equalities by adding by slack variables,
- Free variables can be eliminated as above.

Similarly, any LP can be converted into a problem in the from

$$\min_x c^T x \quad \text{subject to } \mathbf{Ax} \leq \mathbf{b}$$

$$\mathbf{x} \geq \mathbf{0}$$

since

$$\mathbf{Ax} = \mathbf{b} \Leftrightarrow \begin{Bmatrix} Ax = b \\ -Ax = -b \end{Bmatrix}$$

The advantage of such a feature is that you can concentrate analysing any given forms of LP programming problem inter-changeably.

Assumptions of Linear Programming

Divisibility: Decision variables in a linear programming model are allowed to have any values, including non-integer values that satisfy the constraints. Since each decision variable represents the level of some activity, it is being assumed that the activities can be run even at fractional levels.

Proportionality: The contribution of each activity to the value of the objective function is proportional to the level of the activity x_j. Similarly, the contribution of each activity to the left-hand side of each constraint is proportional to the level of the activity x_j.

Additivity: Every function in a linear, programming model is the sum of the individual contributions of the respective activities.

Certainty: The value assigned to each parameter of a linear programming model is assumed to be a known constant.

Check Your Progress 2

1) Discuss the assumptions made in a linear programming problem.

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2) What are the slack and free variables in a linear programming problem?

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3) Formulate a linear programme taking an example of diet problem. Draw the graph of the problem for finding its solution.

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13.5 SIMPLEX METHOD

The simplex method is based on the property that the optimum solution of a L.P. problem if exists can always be found in one of the basic feasible solutions.

Example: Maximise $z = x + y$, subject to $x + y \leq 5$

$$x + 3y \leq 12,\ x \geq 0,\ y \geq 0.$$

Step 1: Convert the inequalities to equalities by addition of non-negative slack variables.

Let s_1 and s_2 be the slack variables, which convert the inequalities into equations. Then

$x + y + s_1 = 5$ and

$3x + 2y + s_2 = 10$ where $s_1 \geq 0$ and $s_2 \geq 0$.

The problem can then be written as follows:

maximise $z = 5x + 6y + 0s_1 + 0s_2$

subject to $x + y + s_1 + 0s_2 = 5$

$$2x + 3y + 0s_1 + s_2 = 12$$

$$x \geq 0,\ y \geq 0,\ s_1 \geq 0,\ s_2 \geq 0.$$

Step 2: Put the problem in a simplex tableau.

The simplex table is given as,

			5	6	0	0	
C	Basic variable	Values of the basic variable	x	y	s_1	s_2	Ratio
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
0	s_1	5	1	1	1	0	5/1 =5
0	s_2	12	2	3	0	1	12/3=4
	z_j	0	0	0	0	0	
	$C_j - z_j$	-	5	6	0	0	

The 1st column denoted by C_j, is known as the objective column, which represents the coefficients of the objective function of the basic variables listed in Column(2). The 2nd column shows the basic in the solution. The 3rd column represents the values of basic variables listed in Column (2).

Columns (4), (5), (6) and (7) represent the 4 variables x, y, s_1 and s_2 respectively. The row shows the coefficients of the respective variables in the objective function. This row is known as the objective row. The Column (8) known as the ratio column. The elements in z_j row are obtained by multiplying the elements of that column by the compounding elements of the objective column (C_j column) and then added up. Similarly, other elements of the z_j row all zero since all the elements in the C_j column are zero. The row $C_j - z_j$ is known as the net evaluation row or index row.

Step 3: (a) Calculate the net evaluation of $(C_j - z_j)$. To get an element in the net evaluation row under any column multiply the entries in that column by the corresponding entities in the objective column (C_j) and all them up. Next, subtract this sum from the element in the objective row listed at the top of the table.

(b) After examining that the net evaluation row of all the elements are zero or negative, the optimum solution is reached. But if any positive element is present it indicates that a better program can be formulated.

(c) Revise the program.

1) Find the pivot column: The column under which falls the largest positive element of the net evolution row is pivot column.

2) Find the pivot row and the pivot number: Divide the elements of the constant column by corresponding non-negative elements of the pivot column to form replacement ratio. The row in which the replacement ratio is the smallest is the pivot row. The number lying at the intersection of the pivot row and the pivot column is the pivot number.

3) Transform the pivot row: Divide all the elements of the pivot row (starting from the constant column) by the pivot number. The resulting numbers will form the corresponding row of the next table.

4) The non-pivot rows are transformed by using the rule,

New Number =

$$\text{Old Number} = \frac{\text{Corresponding number in the pivot row} \times \text{Corresponding number in the pivot column}}{\text{Pivot number}}$$

5) With the results of (3) and (4) form a new table representing a new basic solution. In the new table the variable of the pivot row of previous table will be replaced by the variable of the column of the previous table.

Then steps 3 and 4 are repeated until an optimal column is reached.

Illustration: maximise $z = 3x_1 + 7x_2 + 6x_3$

subject to $2x_1 + 2x_2 + 2x_3 \leq 8$

$x_1 + x_2 \leq 3$

$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$

Next step is to convert it into equalities by two slack variables s_1 and s_2.

Maximise $z = 3x_1 + 7x_2 + 6x_3 + 0s_1 + 0s_2$

Subject to $2x_1 + 2x_2 + 2x_3 + s_1 + 0s_2 = 8$

$x_1 + x_2 + 0x_3 + 0s_1 + s_2 = 3$

$(x_1, x_2, x_3, s_1, s_2) \geq 0.$

Simplex Table

				3	7	6	0	0	
	C_j	Basic variable	Values of the basic variable	x_1	x_2	x_3	s_1	S_2	Ratio
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
Table 1	0	s_1	8	2	2	2	1	0	8/2 =4
	0	s_2	3	1	1	0	0	1	3/1=3
		z_j	0	0	0	0	0	0	
		C_j,-z_j	-	3	7	6	0	0	
Table 2	0	s_1	2	0	0	2	1	-2	2/2=1
	7	x_2	3	1	1	0	0	1	
		z_j,	21	7	7	0	0	7	
		C_j-z_j	-	-4	0	0	0	-7	
Table 3	6	x_3	1	0	0	1	½	-1	
	7	x_2	3	1	1	0	0	1	
		zj,	27	7	7	6	3	1	
		C_j-z_j	-	-4	0	0	-3	-1	

In the 1st program we include only the slack variables $s_1 = 8$, $s_2 = 3$, $x_1 = 0$, $x_2 = 0$ and $x_3 = 0$. In this solution, we find that the value of the objective function is zero. Further, from the C_j-z_j row it is found that there are positive elements. Hence, a better program can be formulated. The highest positive element in the net evaluation row is 7, which lies in the x_2 column. Then the x_2 column is the pivot column and in the next program x_2 has to be included as one of the basic variables. Now dividing the elements of the constant column by the non-negative elements of the pivot column, we get the replacement ratios of column (9). The lowest ratio appears to be in the second column (s_2) column, which is now the pivot column. Hence in the next program s_2 will be replaced by x_2. The pivot number is 1, which lies at the intersection of the pivot row and pivot column. The 2nd program is formulated in Table 2, which is given below Table 1. In Table 2 the basic variables are s_1 and s_2. The 2nd row (x_2 row) of Table 2 has been obtained by dividing the elements of the 2nd row of Table 1 by the pivot number, 1. The elements of the first now (s_1 row) of Table 2 have been obtained by following the rule for the transformation of the non-pivot row. The calculations are shown below:

$$8-\frac{3\times 2}{1}=2 \qquad 2-\frac{1\times 2}{1}=0$$

$$2-\frac{1\times 2}{1}=0 \qquad 2-\frac{0\times 2}{1}=0$$

$$1-\frac{0\times 2}{1}=1 \qquad 0-\frac{1\times 2}{1}=-2$$

Thus the 2nd table represents a basic solution where $x_1 = 0$, $x_2 = 3$, $x_3 = 0$, $s_1 = 2$, $s_2 = 0$ and where the value of the objective function is 21. Inspecting the net evaluation row (C_j-z_j) it is found that one element remains positive. The highest positive number in the net evaluation row lies in the x_3 column, which is now the pivot column. The variable x_3 will have to be introduced as a basic variable in the next programme. The smallest replacement ratio in Table 2 falls in the s_1 row. Therefore, this row is the pivot row and in the new program s_1 will be replace by x_3. The s_1 row in Table 2 is the pivot row and 2 is the pivot number.

The 3rd program is shown in Table 3, which is placed immediately below Table 2. By diving all the elements of the s_1 row of Table 2 by 2, the pivot number we get the corresponding elements of the first row (x_3 row) in Table 3. The elements of the second row (x_2 row) of Table 3 have been obtained from the corresponding elements of the 2nd row (x_2 row) of Table 2 by using the rule for transformation of the non-pivot row(s). These calculations are shown below:

$$3-\frac{2\times 0}{2}=3 \qquad 1-\frac{0\times 0}{2}=1$$

$$1-\frac{0\times 0}{2}=1 \qquad 0-\frac{2\times 0}{2}=0$$

$$0-\frac{1\times 0}{2}=0 \qquad 1-\frac{(-2)\times 0}{2}=1$$

In the third the basic solution is given by $x_1 = 0$, $x_2 = 3$, $x_3 = 1$, $s_1 = 0$, $s_2 = 0$. In the net evaluation row of the table it is found that all the elements are either zero or negative. This means that the optimum program has been attained and

there is no scope for further improvement. Hence, the required optimum solution is $x_1 = 0$, $x_2 = 3$ and $x_3 = 1$. The corresponding value of $z = 27$.

Check Your Progress 3

1) Using simplex method solve the problem:

Maximise $\pi = 6x_1 + 2x_2 + 5x_3$

subject to $\begin{pmatrix} 2 & 3 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \leq \begin{pmatrix} 10 \\ 8 \\ 19 \end{pmatrix}$ and

$x_1, x_2, x_3 \geq 0$

1) ans. $(\bar{\pi}, x_1, x_2, x_3, s_1, s_2, s_3) = (43, 4, 0, 2, 0, 0, 5)$

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2) Maximise $z = 4x + 8y + 2k$

subject to $\frac{1}{2}x + 2y + 4k \geq 4$

$x + y - 2k \geq 6$

$x \geq 0, y \geq 0, k \geq 0$ and minimise the same for z.

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13.6 DUALITY OF LINEAR PROGRAMMING

Every L.P. problem is intimately related another called its "dual". For purposes of identification, the original problem is called the primal problem. The relationship between the primal problem and its dual can be summarised as follows:

1) The dual has as many variables as there are constraints in the original problem.

2) The dual has as many constrains as there are variables in the original problem.

3) The dual of a maximisation problem is a minimisation problem and vice versa.

4) The coefficients of the objective function of the original problem appear as the constant terms of the constraints of the dual and the constant terms of the original constraints are the coefficients of the objective function of the dual.

5) The coefficients of a single variable in the original constraints become the coefficients of a single constraint in the dual. Stated visually, each column of coefficients in the constraints of the original problem becomes a row of coefficients in the dual.

6) The sense of the inequalities, the dual is the reverse of the inequalities in the original problem, except that the inequalities restricting the variables to be non-negative have the same in the primal and the dual.

Example: Suppose the primal problem is

$$\text{maximise } z = c_1 x_1 + c_2 x_2$$

$$\text{subject to } a_{11} x_1 + a_{12} x_2 \leq b_1$$

$$a_{21} x_1 + a_{22} x_2 \leq b_2$$

$$a_{31} x_1 + a_{32} x_2 \leq b_3$$

$$x_1 \geq 0, x_2 \geq 0$$

We have to form the dual of this problem by applying the 6 rules described above. Since the original problem has 3 constraints the dual problem will have 3 variables. Let y_1, y_2 and y_3 be the dual variables. Again, since the original problem has two variables, the dual problem will have 2 constraints. Since the original problem is a maximisation problem, the constants b_1, b_2 and b_3 will appear as the coefficients in the objective function of the dual and the constants c_1 and c_2 will appear as the constant terms in the right hand side of the constraints of the dual. Further, the first column of coefficients $\begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix}$ in the constraints will be the first row of coefficients in the constraints of the dual. Similarly, the second column of coefficients $\begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix}$ will be the second row of coefficients in the constraints of the dual. Again, since the constraints in the original problem all the "less than equal to" type, the constraints in the dual problem will be "greater than equal to" type.

The dual problem will then be as follows:

$$\text{maximise } w = b_1 y_1 + b_2 y_2 + b_3 y_3$$

$$\text{subject to } a_{11} y_1 + a_{21} y_2 + a_{31} y_3 \geq c_1$$

$$a_{21}y_1 + a_{22}y_2 + a_{32}y_3 \geq c_2$$

$$y_1 \geq 0, y_2 \geq 0, y_3 \geq 0.$$

13.7 ECONOMIC IMPORTANCE OF DUALITY

The importance of the dual problem lies in the fact that the dual variables can be given economic interpretation in all situations. Suppose a company produces two products in x_1 and x_2 quantity respectively. The profits per unit of these are c_1 and c_2 respectively. Suppose also that the company uses two factors to produce these two products. Let b_1 and b_2 be the fixed quantities of the two factors and let a_{ij} be the amount of the i^{th} factor required to produce one unit of the j^{th} product (where i, j = 1, 2). Let z represent total profit subject to the condition that no more than 100% of the two given factors used. This is a typical linear programming maximisation problem:

maximise $z = c_1x_1 + c_2x_2$

subject to $a_{11}x_1 + a_{21}x_2 \geq b_1$

$$a_{21}x_1 + a_{22}x_2 \geq b_2$$

$$x_1 \geq 0, x_2 \geq 0.$$

Consider now the economic interpretation of the dual problem if y_1 and y_2 are the dual variables then the dual of the above problem is

maximise $w = b_1y_1 + b_2y_2$

subject to $a_{11}y_1 + a_{21}y_2 \geq c_1$

$$a_{21}y_1 + a_{22}y_2 \geq c_2$$

$$y_1 \geq 0, y_2 \geq 0.$$

The dual variables y_1 and y_2 here can be interpreted as imputed costs or **shadow prices** of the two factors. Then the total imputed cost of the two factors is equal to $b_1y_1 + b_2y_2$. Its dual problem represents minimisation of the total cost. Consider now the first inequality. In production one unit of the first product, the first inequality states that the unit cost should be at least as much as the unit profitability of the product. Similar is the interpretation of the second inequality. Thus, while the original problem is concerned with allocation, its dual is a valuation problem. It should be noted that if in the dual problem the unit cost is greater than the unit profitability for any product, then the unit profitability for the corresponding output will be zero. Thus, if in the first inequality $a_{11}y_1 + a_{21}y_2 > c_1$, then it implies that $x_2 = 0$. Similarly if $a_{12}y_1 + a_{22}y_2 > c_2$ then it implies that $x_2 = 0$. Alternatively, if in the original problem any inequality is of less than type then the corresponding valuation of the factor will be zero. In other words, if any factor is not fully utilised then its shadow price (or valuation) is zero.

Example: Original problem:

maximise $z = 4x + 6y$

subject to $\frac{1}{2}x + y \leq 4$

$2x + y \leq 8$

$4x - 2y \leq 2$

$x \geq 0, y \geq 0.$

Dual problem:

minimise $w = 4u + 8v + 2w.$

subject to $\frac{1}{2}u + 2v + 4w \geq 4$

$u + v - 2w \geq 6$

$u \geq 0, v \geq 0, w \geq 0.$

Last Table for the solution of the original problem

			4	6	0	0	0
C_j	Basic variable	Values of the basic variable	x	y	s_1	s_2	s_3
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
6	y	3	0	1	4/5	0	-1/10
0	s_2	1	0	0	-8/5	1	-3/10
4	x	2	1	0	2/5	0	1/5
	z_j	26	4	6	32/5	0	1/5
	$C_j - z_j$	-	0	0	-32/5	0	-1/5

Last Table for the solution of the dual problem

			4	8	2	0	0	M	M
C_j	Basic variable	Values of the basic variable	u	v	w	s_1	s_2	A_1	A_2
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
2	w	1/5	0	3/10	1	-1/5	1/10	1/5	-1/10
4	u	32/5	1	16/10	0	-2/5	-4/5	2/5	4/5
	z_j	26	4	7	2	-2	-3	2	3
	$C_j - z_j$	-	0	1	0	2	3	M-2	M-3

Here also it is seen that the value of the objective function is the same in both the problems. In the original problem the elements in the net evaluation row under columns s_1, s_2 and s_3 are -32/5, 0 and -1/5. Hence, the solution of the three dual variables will be $u = 32/5$, $v = 0$, and $w = -1/5$.

The knowledge of the dual is important for two main reasons.

1) The dual variables have economic interpretations. The values of the dual variables may be useful in taking managerial decisions.

2) The solution of a L.P. problem may be easier through the dual than through the primal problem.

Check Your Progress 4

1) A primal problem is given as

minimise $C = x_1 + 4x_2$

subject to $\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \geq \begin{pmatrix} 8 \\ 12 \end{pmatrix}$

$x_1, x_2 \geq 0$.

Find its solution.

(a) Write its dual programme and solve the problem.

(b) Show that solution to objective functions of primal and dual are the same.

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2) Construct the dual of the following L.P. problem and solve the primal and the dual.

Maximise $z = 3x_1 + 4x_2$

subject to $x_1 + x_2 \leq 12$

$2x_1 + 3x_2 \leq 21$

$x_1 \leq 8$

$x_2 \leq 6$

$x_1, x_2 \geq 0.$

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13.8 DUALITY THEOREMS

We have not covered technical themes underlying the solution process of LP problems. Therefore, it will be difficult for us to work on the proofs of the duality theorems. However, it will be important to document the main theorems as the properties of duality evolve new algorithms for solving linear programming problems. We state the following theorems without proof.

Theorem 1: The dual of the dual is the primal

Theorem 2: If x be any feasible solution to the primal problem and v be any feasible solution to the dual problem, then

$$Cx \leq b'v$$

Theorem 3: If x^* be a feasible solution to the primal problem and v^* be the feasible solution to the dual problem such that

$Cx^* = b'v^*$

Then both x^* are optional solution to the respective problems.

Theorem 4: A feasible solution x^* to the primal problem is optimal if an only if these exists a feasible solution v^* to the dual problem such that $Cx^* = b^1v^*$

This theorem is referred to as Fundamental Duality Theorem.

Theorem 5: If the primal has an unbounded objective function, then the dual has no feasible solution.

Theorem 6: If the dual problem has no feasible solution and the primal problem has a feasible solution, then the primal objective function is unbounded.

13.9 ZERO-SUM GAMES AND LINEAR PROGRAMMING

13.9.1 Basic Concept

We will present the linear programming formulation of zero-sum games in this section. Before doing that, however, it will be useful to touch upon some of the basic concepts of game theory. As you will be exposed to most of the important features of game theory in MEC-001, we restrict the present discussion to two-player zero-sum games only.

In a competitive situation, the final outcome of any action depends primarily on the combinations of strategies selected by the participants. The decision-making processes of these participants (players) are captured by game theory in a formal and abstract way. For that purpose, a game is conceived as a set of values and regulations of play. When there are two players, the gain of one is a loss to another. These types of games are, therefore, called two-person 'zero-sum games'.

A two-person zero-sum game is characterised by

- the strategies of Player I;
- the strategies of Player II; and
- the pay-off matrix.

Note that each player knows her own strategies, other player's strategies, and the payoff matrix before the game begins. The strategies for each player may not be the same, and the payoff matrix $A = a_{ij}$ demonstrates the amount Player II gives to Player I under each combination of strategies. For example, consider Payoff Matrix 1.

Payoff Matrix 1

		II plays j		
		1	*2*	*3*
I plays i	*1*	1	2	4
	2	1	0	5
	3	0	1	-1

The matrix has two-players each having three options $(1,2,3)$. In the game, if Player I makes move i and Player II responds with j, then Player I wins (and Player II loses) a_{ij}.

Consider the example of children's game of "Matching Pennies" to understand the underlying idea. In this game, the two players agree that one will be "even" and the other will be "odd." Each one then shows a penny. The pennies are shown simultaneously, and each player may show either a head or a tail. If both show the same side, then "even" wins the penny from "odd;" or if they show different sides, "odd" wins the penny from "even".

The payoff matrix for the game "Matching Pennies" is given below (see Payoff Matrix 2).If we add up the payoffs in each cell, we find 1-1=0. This is a "zero-sum game."

Payoff Matrix 2

		Odd	
		Head	Tail
Even	Head	1,-1	-1,1
	Tail	-1,1	1,-1

Strategies

A strategy is the course of action taken by the players. An optimum strategy maximises and ensures a payoff to one player while minimising the loss of the other. Therefore, we need to find out the best strategy the player has to adopt.

In a game, first option available to a player is to eliminate the **dominated strategy**. Such a move helps rule out the inferior strategies until only one choice remains. We call a strategy is **dominated** by another if the latter is

always at least as good or sometimes better regardless of what the opponent does.

In the Payoff Matrix 1 above, strategy I_3 is dominated by that of I_1 regardless what Player II does. Elimination of strategy I_3 leads to Payoff Matrix 3.

Payoff Matrix 3

		II plays *j*		
		1	*2*	*3*
I plays *i*	*1*	1	2	4
	2	1	0	5

Since both players are assumed to be rational, Player II can obtain this reduced payoff matrix and find that her strategy II_3 is dominated by both II_1 and II_2. Elimination of strategy II_3 results in Payoff Matrix 4.

Payoff Matrix 4

		II plays *j*	
		1	*2*
I plays *i*	*1*	1	2
	2	1	0

An examination of Payoff Matrix 4 shows that for Player I, strategy I_2 becomes dominated by that of I_1. After the elimination of I_2, we can see that for Player II, the strategy II_2 is dominated by II_1. Thus, both players would select their strategy 1 as a result.

Another strategy is based on **minimax** or **maximin** criterion. Here, the game problem is viewed in such a manner that each player tries to determine her optimal strategy and selects that offers a payoff, which can never be worsened by the choice of her opponent. Thus, a player attempts to minimise her maximum losses given the move of the other. She lists the worst possible outcomes of all her potential strategies and chooses the one, which corresponds to the best of these worst outcomes. Such a strategy is called minimax criterion. On the other hand, her rival would select the strategy whose minimum payoff is the largest (maximin strategy).

Example: Solve the following game whose payoff matrix is given by

		B			
		I	II	III	IV
	I	-5	3	1	20
A	II	5	5	4	6
	III	-4	-2	0	-5

Let A be the maximising player and B the minimising player. Put squares around the row minimums and circles around the column maximums. It is seen that the element 4 of the 2nd row and 3rd column has been marked by both a square and a circle. See that $\underline{u} = Max\{-5, 4, -5\} = 4$ and $\overline{u} = Min\{5, 5, 4, 20\}$. Such a position is the **saddle point** of the payoff matrix and the strategies II (for A) and III (for B) provides **equilibrium solution** of the game.

Using this result, we can define the **value of a game.** It is determined if the maximum guaranteed gain to the maximising player or the minimum possible loss to the minimising player is reached when best strategies are used by both. Thus, 4 is the value of the game as $\underline{u} = 4 = \overline{u}$

Remember that such a value is optimal as neither player finds it beneficial to alter her strategy and in this state the game is said to be stable.

Example 2: Show that whatever many be the value of a, the game with the following payoff matrix is strictly determinable.

		B	
		I	II
A	I	3	7
	II	-3	a

We first ignore a. The payoff matrix then yields the following row minimums and column maximums.

		B		
		I	II	Row Min
A	I	3	7	3
	II	-3	a	-3
Col Max		3	7	

Thus, we have $\underline{u}$=3 and $\overline{u}$=3.

Therefore, the game is strictly determinable and u=3 and the best strategies for A and B are (I, I).

Strategies can be **pure** or **mixed**. In a pure strategy is the decision-making is such that one particular course of action is selected. On the other hand, in a mixed strategy a player decides, in advance to choose her course of action with some definite probability distribution. Thus, we can assume that that Player A selects i with probability p_i; $i = 1, \ldots\ldots, n$ while Player B moves j with probability q_j, $j = 1, \ldots\ldots, m$. Player A's expected payoff if Player

B playes j is $\sum_{i=1}^{n} a_{ij} p_i$.

In fact, we can assume that Player A maximises her minimal expected payoff,

while Player B minimises her maximal expected loss to reach the solution. Such a payoff happens to be the saddle point of a game.

Let us return to the game of matching pennies and assume that the payers can exercise the option of pure strategy, head and tail. Alternatively, a player may randomise her strategy by offering either a head or a tail, at random, with specific probabilities. There are infinitely many mixed strategies corresponding to the infinitely many ways probabilities can be assigned to the two pure strategies.

Going by the maxmin criterion discussed above, we would like to conclude that the matching penny game does not have a unique solution. The minimum payoff for each of the two strategies is the same: (-1). That is to say, we do not have a saddle point in this game.

However; the game can have more than two strategies. In addition to the two obvious strategies, head and tail, a player can "randomise" her strategy by offering either a head or a tail, at random, with specific probabilities ((i.e., play with mixed strategy). Then the game of matching pennies has a solution and it is to offer heads or tails at random with probabilities 0.5 each way. We can reason it out: if odd offers heads with any probability greater than 0.5, then even can do better by offering heads with probability 1. On the other hand, if odd offers heads with any probability less than 0.5 , then even can do better by offering tails with probability 1. The only way odd can win is to choose a randomised strategy with probability 0.5. See that 0.5 probability maximises the minimum payoff over all pure *or mixed* strategies. Even can reason the same way (reversing heads and tails) and come to the same conclusion. Consequently, both players choose 0.5.

13.9.2 Relationship between Game Theory and Linear Programming

Every finite two-person zero-sum game can be expressed as a linear programme problem and conversely every linear programme can be represented on a two person zero sum game.

Method

Consider a game problem with payoff matrix $[a_{ij}]_{m \times n}$

The maximizing player A has m strategies which she has chosen with probabilities. $x_1, x_2 \ldots . x_m$ respectively such that

$x_1 + x_2 + \ldots + x_m = 1$ where $x_1, x_2, \ldots x_m \geq 0$ .....(1)

The expected gain to A which the minimizing player B has chosen is j^{th} course of action out of her n courses is $\sum_{i=1}^{m} a_{ij} x_i$

If $\min_j \left(\sum_{i=1}^{m} a_{ij} x_i \right) = \vartheta$, then A expects to get at least ϑ. A will choose her strategies with such probabilities as she has at least a guaranteed gain of ϑ.

Now

$$\sum_{i=1}^{m} a_{ij} x_i \geq \vartheta \text{ for all j} \qquad \text{.....(2)}$$

as ϑ is the minimum of all expected gains and the value of the game is clearly the maximum value of ϑ, if it exists, Thus, x_1, x_2 ...., x_m are to be so determined as to satisfy (1) and maximise ϑ.

To put this problem in a standard L.P format, we divide (1) and (2) by ϑ, which is positive. If ϑ be not positive, which will be indicated by the presence of some negative elements in the payoff matrix, then we add to all the elements of payoff matrix a sufficiently large positive quantity e such that ϑ becomes positive. This operation does not change the optimal solution but only increases the value of the game by e.

Now dividing (1) and (2) by ϑ (>0) and setting $\frac{x_i}{\vartheta} = X_i$ (i = 1, 2, ... m), we write the problem as

$$a_{ij} X_1 + a_{2j} X_2 + \ldots\ldots\ldots + a_{mj} X_m \geq 1, j = 1, 2, \ldots, n$$

$$\text{and } X_1 + X_2 + \ldots\ldots\ldots + X_m = \frac{1}{\vartheta}$$

$$\text{with } X_1, X_2, \ldots\ldots\ldots X_m \geq 0$$

The maximisation of ϑ is equivalent to the minimisation of $\frac{1}{\vartheta}$. Hence we can state the problem for A as

$$\text{minimise } \frac{1}{\vartheta} = X_1 + X_2 + \ldots\ldots\ldots + X_m$$

$$\text{subject to } a_{ij} X_1 + a_{2j} X_2 + \ldots\ldots\ldots a_{mj} X_m \geq 1 \qquad \text{.....(3)}$$

$$\text{and } x_i \geq 0 \qquad \begin{matrix} j = 1, 2, \ldots\ldots, n. \\ i = 1, 2, \ldots, m. \end{matrix}$$

This is an L.P. Problem written from the point of view of A

Considering the problem from the point of view of B, who will expect to minimise all the expected loss for her, we can arrive in a similar manner at the following L.P. problem.

$$\text{Maximise } \frac{1}{\vartheta} = Y_1 + Y_2 + \ldots\ldots\ldots + Y_m$$

$$\text{Subject to } a_{i1} Y_1 + a_{i2} Y_2 + \ldots\ldots\ldots + a_{jn} Y_n \leq 1 \qquad \text{.....(4)}$$

$$\text{and } Y_j = \frac{y_j}{\vartheta} \qquad \begin{matrix} i = 1, 2, \ldots\ldots, m \\ j = 1, 2, \ldots\ldots, n \end{matrix}$$

where Y_j's are the mixed strategies of B.

Having found X_i and Y_j, we can find x_i and y_j from the relation $x_i = \vartheta X_i$ and $y_j = \vartheta Y_j$.

Finally, the value of the original game is obtained by subtracting e, if needed.

It can be seen here that the problems (3) and (4) as stated above are dual to one another. Thus, the optimal solution of one problem will afford an optimal solution for the other. Players B's problem can be solved by regular simplex method while that of A can be solved by the method of duality.

Example: Suppose we have the following payoff matrix.

		Player B		
		B_1	**B_2**	**B_3**
Player A	A_1	1	−1	3
	A_2	3	5	−3
	A_3	6	2	−2

Here some of the elements of the payoff matrix are negative. The largest minimum element is -3. Hence to make all elements positive, let us add a constant $c \geq 3$ to all the elements. Let $c = 4$. Then the payoff matrix becomes.

		B_1	**B_2**	**B_3**
$[a]_{m \times n}$	A_1	5	3	7
	A_2	7	9	1
	A_3	10	6	2

Now let the mixed strategies of the two player be

$$S_A = \begin{bmatrix} A_1 & A_2 & A_3 \\ p_1 & p_2 & p_3 \end{bmatrix} \text{ where } p_1 + p_2 + p_3 = 1$$

and $S_B = \begin{bmatrix} B_1 & B_2 & B_3 \\ q_1 & q_2 & q_3 \end{bmatrix}$ where $q_1 + q_2 + q_3 = 1$

The expected gain of A when the plays S_A against the pure strategy B_1 is given by $q_1 = 5p_1 + 7p_2 + 10p_3$

Similarly, $q_2 = 3p_1 + 9p_2 + 6p_3$

$q_3 = 7p_1 + p_2 + 2p_3$

In the same way the expected losses of B when she plays S_B against the pure strategies A_1, A_2, ... and A_3 are given by

$l_1 = 5q_1 + 3q_2 + 7q_3$

$l_2 = 7q_1 + 9q_2 + q_3$

$l_3 = 10q_1 + 6q_2 + 2q_3$

Player A is interested in determining p_i's that will maximise her minimum expected gain u = min(q_1, q_2, q_3) while player B is interested in determining q_j that will minimise her maximum expected loss v = max (l_1, l_2, l_3). Let us define new variables p_i' and q_j' such that

$$p_i' = \frac{p_i}{u}, i = 1,2,3$$

and $q_j' = \frac{q_j}{v}, j = 1,2,3$

The problem of player A then becomes maximise u = minimise $\frac{1}{u}$ = minimise

$p_0 = p_1' + p_2' + p_3'$ subject to constraints

$$5p_1' + 7p_2' + 10p_3' \geq 1$$
$$3p_1' + 9p_2' + 6p_3' \geq 1$$
$$7p_1' + p_2' + 2p_3' \geq 1$$
$$p_1' \geq 0, p_2' \geq 0, p_3' \geq 0$$

The problem of player B is to minimise v = maximise $\frac{1}{9}$ = maximise

$q_0 = q_1' + q_2' + q_3'$ subject to constraints

$$5q_1' + 3q_2' + 7q_3' \leq 1$$
$$7q_1' + 9q_2' + q_3' \leq 1$$
$$10q_1' + 6q_2' + 2q_3' \leq 1$$
$$q_1' \geq 0, q_2' \geq 0, q_3' \geq 0$$

We can solve either of these two LP problems by using the simplex method. The solution of one gives also the solution of the other.

Let us solve the problem of player B. If, we introduce three stack variables, q_4', q_5' and q_6' respectively and solve the problem in simplex method,

we get

$$q_1' = 0, q_2' = \frac{1}{10}, q_3' = \frac{1}{10} \text{ and } q_0 = \sum q_i' = \frac{1}{5}$$

Hence, $v = \frac{1}{q_0} = 5$. Therefore,

$q_1 = q_1'\vartheta = 0.(5) = 0; q_2 = q_2'\vartheta = \frac{1}{10}.5 = \frac{1}{2}$

$q_3 = q_3'\vartheta = \frac{1}{10}.5 = \frac{1}{2}$

The value of the game $= \frac{1}{q_0} - c = 5 - 4 = 1$.

The optimum strategy for B is $S_B = \begin{bmatrix} B_1 & B_2 & B_3 \\ 0 & 1/2 & 1/2 \end{bmatrix}$

Again from the net evaluation row of the last table we can get the values of p_1', p_2', p_3'. From this we can get the values of p_1, p_2 and p_3. It can be seen that $p_1 = 2/3$. $p_2 = 1/3$ and $p_3 = 0$. Then the optimum strategy for A is

$S_A = \begin{bmatrix} A_1 & A_2 & A_3 \\ 2/3 & 1/3 & 0 \end{bmatrix}$.

Check Your Progress 5

1) Use linear programming to solve the game problem whose payoff matrix is given below:

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{ccc} \mathbf{B_1} & \mathbf{B_2} & \mathbf{B_3} \end{array} \\ \begin{bmatrix} 1 & -1 & -2 \\ -1 & 1 & 1 \\ 2 & -1 & 3 \end{bmatrix} \end{array}$$

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2) Solve the following game problem and determine the value of the game in each case.

a)
$$\begin{array}{c} \\ A_1 \\ A_2 \end{array} \begin{array}{c} \begin{array}{cc} \mathbf{B_1} & \mathbf{B_2} \end{array} \\ \begin{bmatrix} -4 & 6 \\ 2 & -3 \end{bmatrix} \end{array}$$

b)
$$\begin{array}{c} \\ A_1 \\ A_2 \end{array} \begin{array}{c} \begin{array}{cc} \mathbf{B_1} & \mathbf{B_2} \end{array} \\ \begin{bmatrix} 3 & -3 \\ -2 & 3 \end{bmatrix} \end{array}$$

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13.10 LET US SUM UP

This unit presents the basic principles of linear programming the application of linear programming in solving different events of mathematical and economical importance. Both the processes of maximisation and minimisation are analysed by the help of game theory and linear programming principles. These important mathematical tools can be used for determining the proper course of action in making any suitable decision. The different conditions and situations are explained in detail.

13.11 KEY WORDS

Basic Feasible Solutions: These solutions are basic as well as feasible.

Basic Solution: Any set of values of the variables in which the number of non-zero valued variables is equal to the number of constraints is called a Basic Solution.

Constraints: The linear inequalities or the side condition.

Dual Problem: Associated with every linear programming there is a linear programming problem. Which is called its dual problem.

Feasible Solution: A set of values of decision variables, which satisfies the set of constraints and the non-negativity restrictions.

Linear Programming: Linear programming is the analysis of problems in which linear function of a number of variables is to be maximised (or minimised) when those variables are subject to a number of restraints in the form of linear inequalities.

Maximin or Minimax Principle: If a player lists the worst possible outcomes of all his potential strategies then he will choose that strategy to be the most suitable for him which corresponds to the best of these worst outcomes.

Mixed Strategy: Decision making rule in which a player decides in advance, to chose his courses of action with some definite probability distribution.

Objective Function: The function to be maximised or minimised.

Primal: The original L.P.P. is called the primal problem.

Pure Strategy: Pure Strategy is a decision making rule in which one particular cause of action is selected.

Saddle Point: A saddle point in payoff matrix is that position in the matrix where the maximum of row minima coincide with the minimum of column maxima.

13.12 SOME USEFUL BOOKS

F. S. Hillier / G. J. Lieberman (2001), *Introduction to Operations Research*, McGraw - Hill ,7th Edition.

A. M. Glicksman (2001), *An Introduction to Linear Programming and the Theory of Games,* Dover Publ., Mineola, NY.

Jiun-Yu Yu (2005), *Linear Programming*, www.stats.ox.ac.uk (portions of this note is used in the present unit).

13.13 ANSWER OR HINTS TO CHECK YOUR PROGRESS

Check Your Progress 1

1) Do yourself after reading Section 13.2

2) Do yourself after reading Section 13.2

3) Do yourself after reading Section 13.2

Check Your Progress 2

1) Do yourself after reading Section 13.4

2) Do yourself after reading Section 13.4

3) Do yourself after reading Section 13.4

Check Your Progress 3

1) $(\bar{\pi}, x_1, x_2, x_3, s_1, s_2, s_3) = (43, 4, 0, 2, 0, 0, 5)$

2) Do Yourself after reading Section 13.6

Check Your Progress 4

1) Do Yourself after reading Section 13.7

2) Do Yourself after reading Section 13.8

Check Your Progress 5

1) Do Yourself after reading Section 13.9

2) Do Yourself after reading Sub-Section 13.9.2

13.14 EXERCISES

1) In a diet problem with quantities of two food items purchased each day are given as x_1 and x_2. Assume that only three nutrients needed to be taken are calcium, protein and vitamin A. You need to find out the combination of the two food items that will satisfy the daily nutrient requirement with least cost. If you are given the following information, solve the problem graphically.

Minimise $C = 0.6x_1 + x_2$

Subject to $10x_1 + 4x_2 \geq 20$ [calcium constraint]

$5x_1 + 5x_2 \geq 20$ [protein constraint]

$2x_1 + 6x_2 \geq 12$ [vitamin A constraint]

$$x_1, x_2 \geq 0$$

2) Solve the following problem:

maximise $10x_1 + 10x_2 + 20x_3 + 20x_4$

subject to

$$12x_1 + 8x_2 + 6x_3 + 4x_4 \leq 210$$
$$3x_1 + 6x_2 + 12x_3 + 24x_4 \leq 210$$
$$x_1, x_2, x_3, x_4 \geq 0$$

Ans.: $[x] = [1\ 0\ 0\ 1\ 5\ 0]$

3) Apply the maximin and minimax principle to solve the games whose payoff are given bellow:

		B			
		B_1	B_2	B_3	B_4
	A_1	-5	2	1	6
A	A_2	5	6	4	8
	A_3	4	0	1	-3

Ans.: $(A_1\ B_3)$, $v = 4$

4) For what value of k the game with the following payoff matrix is strictly determinable.

	B		
	K	7	3
A	-2	K	-8
	-3	4	K

Ans.: $-2 \leq K \leq 3$]

NOTES

MPDD/IGNOU/P.O.7K/November 2018 (Reprint)

ISBN : 978-93-86375-15-5