UNIT 8 LIFE INSURANCE

Structure

8.0 OBJECTIVES

After going through this unit, you will be able to:

- understand the basic cancepts used in life insurance;
- appreciate the underlying principles of premium; and
- analyse the cost and benefit estimation in life insurance with the probabilistic approach.

8.1 INTRODUCTION

Life insurance operates on the basis of probabilitic estimation of survival and mortality of the insured. Consequently the fixation of premiums and making provisions for meeting the costs of contract benefits (claims) have to be decided taking into account the their probability distributions. The following discussion introduces the basic concepts of life insurance and highlights the tools used for insurance operations.

8.2 HOW LIFE INSURANCE WORKS

Life insurance is a form insurance that envisages three parties, viz., the insurer, the insured, and the owner of the policy (policyholder). Although the owner and the insured are often the same person at times such a view does not hold good. For example, if Rita buys a policy on her own life, she is both the owner and the insured. But if Adam, her husband, buys a policy for her, he is the owner and she is the insured.

In an insurance transaction there is always a beneficiary. The beneficiary is the person who receives the policy proceeds upon the death of the insured. The face amount of the policy is normally the amount paid when the policy matures, i.e., when the insured dies or reaches a specified age.

It is important to note that the insurer (the life insurance company) calculates the policy prices with an intent to recover claims to be paid and administrative costs, and to make a profit. To operate the business, it receives the premiums from the policy owner and invests such receipts to create a pool of resources from which to pay claims, and to finance the insurance company's operations.

8.3 TYPES OF LIFE INSURANCE

Life insurance may be divided into two basic classes – temporary and permanent.

Temporary: In case of temporary insurance, there is definite time period which is mentioned when the contract is signed. In the case of annual renewable term (ART), however, this practice is not followed as the coverage is provided for one year.

Term: Term life insurance provides for life insurance coverage for a specified term of years for a specified premium. The policy does not accumulate cash value. In this policy the premium buys protection only in the event of death.

Permanent: Permanent life insurance remains in force until the policy matures (pays out), and when the owner fails to pay the premium on time the policy expires.The owner can access the money in the cash value by withdrawing money, borrowing the cash value, or surrendering the policy and receiving the surrender value.

There are three basic types of permanent insurance, viz., **whole life, universal life**, and **endowment**.

- **Whole life insurance** provides for a level of premium, and a cash value table included in the policy guaranteed by the company. It's features include guaranteed death benefits, guaranteed cash values, fixed and known annual premiums. Mortality and expense charges do not reduce the cash value shown in the policy.
- **Universal life insurance** is intended to provide permanent insurance coverage with greater flexibility in premium payment and the potential for a higher internal rate of return.
- **Endowments life insurance** policies have provision for the cash value build up inside the policy that equals the death benefit (face amount) at a certain age. The age at which it commences is called the endowment age.

In terms of annual premiums endowments are more expensive than either whole life or universal life.

Accidental death

Accidental death is a limited life insurance. It is designed to cover the insured when death occurred due to an accident. In the contract accidents means an injury, but not deaths due to health problems or suicide. Because such policies cover accidents only, these are less expensive than other life insurances.

Related Life Insurance Products

Riders are modifications to the insurance policy added at the time of signing the contaract. Such provisions change the basic policy to include some feature desired by the policy owner. A common rider is accidental death, which used to be commonly referred to as "double indemnity". It pays twice the amount of face value or the policy if death results from accidental causes.

Joint life insurance is either a term or permanent policy that insures two or more lives with the proceeds payable on the first death.

Survivorship life is a whole life policy which insures two lives with the proceeds payable on the second (later) death.

Single premium whole life is a policy with only one premium that is payable at the time the policy is issued.

Modified whole life is a policy that charges smaller premiums for a specified period of time after which the premiums increase for the remainder of the policy.

Group life insurance is term insurance that covers a group of people, usually employees of an organisation or members of a union or association. Individual proof of insurability is not normally a consideration in the underwriting. Instead the underwriter takes into account the size and turnover of the group. An important feature of Group life insurance is the provision that a member exiting the group has the right to buy individual insurance coverage.

Prepaid insurance policies are like that of whole life policies. However, these are available at any age and are usually offered to older applicants. This is intended in a major way to to cover funeral expenses when the insured person dies.

8.4 PREMIUM PRINCIPLES

Rule through which an insurance risk is assigned a premium is called premium principle. In case of life insurance we focus on the premium that accounts for the monetary payout by the insurer for insurable losses plus fair risk loading. Note that the main idea behind the pricing of life insurance is the concept of net single premium. It gives the actuarial present value at the time of issuing insurance contract keeping in view the product's total future costs of benefit. Actuaries, on the basis of a list of desirable properties, put forth three methods to develop these principles.

8.4.1 Properties of Premium Principles

(Adopted from the Encyclopedia of Actuarial Science, John Wile & Sons Ltd, 2004)

Notations: Let X denote the set of nonnegative random variables on the probability space (Ω, F, P); this space gives the collection of insurance-loss random variable—also called insurance risks. Let X, Y, Z etc. denote elements of X. Let H denote the premium principle, or function, from X to the set of (extended) nonnegative real numbers.

1) *Independence:* $H[X]$ depends only on the (de)cumulative distribution function of X, namely, S_X such that $S_X(t) = P\{\omega \in \Omega : X(\omega) > t\}$. That is, the premium of X depends only on the tail probabilities of X.

   According to this property, the premium depends only on the monetary loss of the insurable event.

2) *Risk loading:* $H[X] \geq EX$ for all $X \in \chi$.

   Loading for risk is desirable because there has to a rule to charge at least the expected payout of the risk X, namely, EX, in exchange for insuring the risk. Otherwise, the insurer will lose money on average.

3) *No unjustified risk loading:* If a risk $X \in \chi$ is identically equal to a constant $c \geq 0$ (almost everywhere), then $H[X] = c$.

4) *Maximal loss (or no rip-off):* $H[X] \leq \sup[X]$ for all $X \in \chi$

5) *Transaction equivariance: (or transaction invariance):* $H[X + a] = H[X] + a$ for all $X \in \chi$ and all $a \geq 0$.

   If we increase a risk X by a fixed amount a, then Property 5 states that the premium for $X + a$ should be the premium for X increased by that fixed amount a.

6) *Scale equivariance (or scale invariance):*

   $H[bX] = bH[H]$ for all and all $X \in \chi$ and all $b \geq 0$.

   Properties 5 and 6 imply Property 3 as long as there exists risk Y such that $H[Y] < \infty$. See that, if $X \equiv c$, then
   $H[X] = H[c] = H[0 + c] = H[0] + c = 0 + c$. We have $H[0] = 0$ because
   $H[0] = H[0Y] = 0H[Y] = 0$. Scale equivariance is also known as homogeneity of degree one. It means the premium for doubling a risk is twice the premium of the single risk.

7) *Additivity:* $H[X + Y] = H[X] + H[Y]$ for all $X\ Y \in \chi$

8) *Subadditivity:* $H[X + Y] \leq H[X] + H[Y]$ for all $X, Y \in \chi$.

9) *Superadditivity:-* $H[X + Y] \geq H[X] + [Y]$ for all $X, Y \in \chi$.

10) *Additivity for independent risks:* $H[X+Y]=H[X]+H[Y]$ for all $X,Y \in \chi$

11) *Additivity for comonotonic risks:* $H[X+Y]=H[X]+H[Y]$ for all $X,Y \in \chi$ such that X and Y are comonotonic.

Next, we consider property of premium rules that require that they preserve common **orderings of risks**

12) *Monotonicity:* If $X(\omega) \le Y(\omega)$ all $\omega \in \Omega$, then $H[X] \le H[Y]$.

13) *Preserves first stochastic dominance* (FSD) *ordering:* If $S_X(t) \le S_Y$ for all $t \ge 0$, then $H[X] \le H[Y]$.

14) *Preserves stop-loss* (SL) *ordering:* If $E[X-d]_+ \le E[Y-d]_+$ for all $d \gtrsim 0$, then $H[X] \le H[Y]$.

Finally, we present a technical property that is useful in characterizing certain premium principles.

15) *Continuity:* Let $X \in \chi$; then, $\lim_{a \to 0+} H[\max(X-a,0)] = H[X]$, and $\lim_{a \to 0+} H[\min(X,a)] = H[X]$.

8.4.2 Important Premium Principles

Net Premium Principle:	The net premium from the above is the special case H = E, i.e., $H(\mathrm{X}) = \mathrm{E}(\mathrm{X})$.
Expected Value Principle:	Take $H(\mathrm{X}) = (1+\lambda)E(\mathrm{X})$ which suggests a safety loading proportional to $E(\mathrm{X})$.
Variance Principle:	If $H(\mathrm{X}) = E(\mathrm{X}) + \alpha Var(\mathrm{X})$, then a safety loading is proportional to $Var(\mathrm{X})$.
Standard Deviation Principle:	If $H(\mathrm{X}) = E(\mathrm{X}) + \beta\sqrt{Var(\mathrm{X})}$ then a safety loading is proportional to the standard deviation of X.
Percentile Principle:	If $H(\mathrm{X}) = \min\{A \mid \mathrm{P}(\mathrm{X} > \mathrm{A}) \le \delta\}$, then the minimal premium is bound by the probability of given level loss. The level is not exceeded but the premium is not minimal in this property.
Exponential Principle:	$H(S) = \dfrac{\log(E(\exp\{a\mathrm{X}\}))}{a}$.

Esscher premium Principle: $H[X]=\left(\dfrac{E(Xe^{Z})}{E(e^{Z})}\right)$ for some random variable Z.

Proportional Hazards Premium Principle: $H(X)=\int_{0}^{\infty}[S_X(t)]^{c}dt$

for some $0<c<1, H[X]$ solves the equation.

Principle of Equivalent Utility: $u(\omega)=E[u(\omega-X+H)]$ where u is an increasing concave utility wealth held by an insurer and ω is her initial wealth.

8.5 SURVIVAL DISTRIBUTIONS

An insurance policy covers two different types of risk. In some of insurances, such as life insurance, often the claim is made at the time specified by the policy. In other types such as auto or casualty the claims vary in both the time and amount.

Determination of the length of the future life of the insured is the central problem for a life insurance company. The future lifetime of a newborn is usually represented by a random variable X. For simplicity, assume that the distribution function of X is absolutely continuous. The **survival function** of X, denoted by $s(x)$ is defined by the formula, $s(x)=P[X>x]=P[X\geq x]$ and we always assume that $s(0)=1$.

The simplest way to model the survival function is to resort to DeMoivre Law. It is assumed that $s(x)=1-\dfrac{x}{\omega}$ for $0<x<\omega$ where ω is the **limiting age** by which all have died and X has the uniform distribution on the interval $(0,\omega)$.

Life insurance is usually issued on a person who has already attained a certain age x. For notational convenience denote such a **life aged** x by (x), and denote the future lifetime of a life aged x by $T(x)$. Note that $T(x)$ is a random variable indexed by x, the age. Thus, $T(x)$ represents the *remaining life* for an individual currently aged x. The survival function for (x) is $p[T(x)>t]$.

We say, following the actuarial notation, ${}_tp_x=P[T(x)>t]$, i.e., the function $({}_tP_x)$ is the probability of surviving to t. Now we have ${}_tq_x=P[T(x)\geq t]$ to *represent* probability of dying before t. When $t=1$, the prefix is omitted to be written as P_x and q_x respectively.

Conditional Probability of Survival

If an individual is currently aged x, then the probability of surviving to age n is defined by:

$$({}_nP_x) = \prod_{t=0}^{n-1} (1 - q_{x+t})$$

Remaining Lifetime: An additional note

The remaining lifetime random variable T_x has a probability density function (PDF) denoted and defined by: $F_x(t) = \int_0^t f_x(s)ds$ when the random variable $f_x(t)$ is continuous, or $F_x(n) = \sum_{i=1}^{n} \Pr[T_x = x_i]$, if the random variable T_x is discrete.

Here x_i denotes the ages at which people are "allowed" to die. For example, $x_1 = 70$, $x_2 = 85$ and $x_3 = 95$. In this case, $T_x = \{70, 85, 90\}$ and the probability mass function (PMF) is used instead of the PDF. A possible example would be:

T_{60}	$\Pr[T_{60} = x_1]$
70	8/12
85	3/12
95	1/12

The expected value of the remaining lifetime, which is a random variable (r.v.) would be:

$$E[T_{60}] = \frac{8}{12} * 10 + \frac{3}{12} * 25 + \frac{1}{12} * 35 = 15.833 \text{ years}$$

Note also that:

$$\Pr[T_{60} \leq 10] = 8/12$$
$$\Pr[T_{60} \leq 25] = 8/12 + 3/12 = 11/12$$
$$\Pr[T_{60} \leq 35] = 8/12 + 3/12 + 1/12 = 1$$

For all positive real x and t, $s(x) - s(x+t)$ gives a life which dies between ages x and $x+t$, and $\left(\frac{s(x) - s(x+t)}{s}\right)$ represent the fraction of those alive at age x who fail before $x+t$. An evident representation is $s(x) = \int_x^{\infty} f(t)dt$, where $f(t) \equiv -s(t)$ is called the failure density. Thus, we can as well have

$$ {}_t p_x = P\left[T(x) > t\right] = p\left[X > x + t \mid X > x\right] = \frac{s(x+t)}{s(x)} \qquad (1) $$

Correspondingly,

$$ {}_t q_x = 1 - {}_t p_x = P\left(T \le x + t \mid T \ge x\right) = \left[\frac{s(x) - s(x+t)}{s(x)}\right] = 1 - \frac{s(x+t)}{s(x)} \qquad (2) $$

The density of $T(x)$ is given by

$$ f_{T(x)}(t) = \frac{-s'(x+1)}{s(x)} = \frac{f_X(x+t)}{1 - F_X(x)} \qquad (3) $$

representing the rate of death of (x) at time t. We can define the moments of remaining life $T(x)$. Thus,

$$ E\left[T(x)\right] = \int_0^\infty t f_{T(x)}(t) \mid t = -\int_0^\infty \frac{t\, s'(x+t)}{s(x)} dt $$

$$ = \int_0^\infty \frac{s(x+t)}{s(x)} dt = \int_0^\infty tP_x \; dt. \qquad (4) $$

This expectation is called **complete expectation of life** and denoted by e_x^0. Using the same procedure, it can be shown that $E\left[T(x)^2\right] = 2\int_0^\infty t \;{}_t p_x \; dt$.

Force of Mortality: We consider the quantity

$$ \mu_x = \frac{f_X(x)}{1 - F_X(x)} = -\frac{s'(x)}{s(x)} \qquad (5) $$

which intuitively gives instantaneous rate of death of (x) and at times used as **hazard rate.** Thus we can pose question like –

If $$ ({}_t p_x) = e^{-\int_x^{x+t} \lambda(s) ds} \qquad (6) $$

where the curve $\lambda(s) \geqslant 0$ for all $s \geqslant 0$, then we can take $\lambda(s)$ as the instantaneous rate of death at age s. When $s = 0$, then $({}_s p_x) \to 1$, and if s $\to \infty$, then that $({}_s p_x) \to 0$. Thus, $\int_x^{x+\infty} \lambda(s) ds \to \infty$. Note that by putting $u = s - x$, we can re-write equation $({}_t p_x) = e^{-\int_x^{x+t} \lambda(s) ds}$ above as

$$ ({}_t p_x) = e^{-\int_0^t \lambda(x+u) du} \qquad (7) $$

Integrating the curve $\lambda(s)$ from a lower bound of x to an upper bound $x + t$ is mathematically equivalent to integrating the curve starting at $\lambda(x+s)$ from a lower bound of 0 to an upper bound of t.

From $({}_t p_x)$ above when you take derivatives of both sides you get

$$\frac{\partial}{\partial t}({}_tp_x) = -({}_tp_x)\lambda(x+t)$$

Therefore, $f_x(t) = (1 - F_x(t))\lambda(x+t)$ (8)

Ordinary Differential Equation Presentation

Putting in terms of Ordinary Differential Equation (ODE) for the function $F_x(t)$ displayed above, we can represent the IFM as:

$$\lambda(x+t) = \frac{f_x(t)}{1-F_x(t)}, t \geq 0, \quad (9)$$

In (9) above, $F_x(t) \to 1$ as $t \to \infty$ (everyone dies eventually) and therefore $\lambda(t) \to \infty$ as $t \to \infty$, unless $f_x(t) \to 0$ in the numerator "faster".

Further, the relationship implied by equation (9) leads to:

$$F_x(t) = 1 - \frac{f_x(t)}{\lambda(x+t)} \quad (9.1)$$

which implies $f_x(t) = ({}_tp_x)\lambda(x+t)$. (9.2)

On the basis of the above discussion mortality laws can be written in two different ways:

- We start with a CDF $F_x(t) = 1 - ({}_tp_x)$ and then take the derivative to create the PDF $f_x(t)$ and using equation (9) we get instantaneous forced mortality (IFM) $\lambda(x)$.
- Alternatively, we start with the IFM and build the CDF $F_x(t) = 1 - ({}_tp_x)$ using equation (6) and then take derivatives to arrive at the PDF $f_x(t)$.

Question: Using some of the qualitative features we would expect from the IFM curve, can we use any functional form for $f_x(t)$ and $F_x(t)$ or are there some natural restrictions on the remaining lifetime random variable?

Example: The normal distribution we are all familiar with. In this case the CDF is:

$$N(m,b,t) = \int_{-\infty}^{t} \frac{1}{b\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{z-m}{b}\right)^2} dz \quad (10)$$

Exponential Law of mortality

Assume that the (IFM) curve satisfies $\lambda(x+t) = \lambda$, which is a constant across all ages and times. In such a case, let us "build" $F_x(t)$ and $f_x(t)$ functions using (6).

We will have: $({}_tP_x) = e^{-\int_x^{x+t} \lambda ds} = e^{-\lambda t}$ (11)

The integral in the exponent collapses, i.e., it can be solved to produce a linear function λt. In this case the current age x does not really impact the probability of survival since all that matters is the magnitude of the IFM, λ.

From the above discussion, let us look at following results:

$$F_x(t) = 1 - e^{-\lambda t},$$
$$f_x(t) = \lambda e^{-\lambda t}. \tag{12}$$

The expected remaining lifetime (ERL) is:

$$E[T_x] = \int_0^\infty t\lambda e^{-\lambda t} dt = \frac{1}{\lambda} \tag{13}$$

Gompertz-Makeham Law of Mortality

The Gompertz-Makeham (GM) law of mortality is derived using the IFM curve $\lambda(x)$. In GM's case, it is:

$$\lambda(x) = \lambda + \frac{1}{b} e^{(x-m)/b}, t \geq 0 \tag{14}$$

Note that the instantaneous force of mortality is a constant λ along with a time-dependent exponential curve. This curve increases with age and goes to infinity as $t \to \infty$.

Relation between age and GM-IFM curve

When the individual is, $x = m$ years old, GM-IFM curve is $\lambda(m) = \lambda + 1/b$; $(x, < m)$ i.e., younger, GM-IFM curve is $\lambda(x) < \lambda + 1/b$; $(x > m)$ i.e., older, GM-IFM curve is $\lambda(x) > \lambda + 1/b$. Thus, $x = m$ is a special age point on the IFM curve.

There is an alternative representation for the GM law:

$$\lambda(t) = h + h_1 e^{h_2 t}, t \geq 0.$$

to see that these functional forms are equivalent, take $h = \lambda, h_2 = 1/b$ and $h_1 = \exp\{-m/b\}/b$. Or move from the (λ, m, b) specification to the (h, h_1, h_2) specification by the transformation $b = 1/h_2$ and $m = -In[h_1/h_2]/h_1$.

If we recall from equation (6), the conditional probability of survival under the GM-IFM curve is equal to:

$$({}_t p_x) = e^{-\int_x^{x+t}\left(\lambda + \frac{1}{b}e^{(s-m)/b}\right)ds}$$
$$= e^{-\lambda t + b(\lambda(x) - \lambda)(1 - e^{t/b})},$$

and $\quad F_x(t) = 1 - ({}_t p_x)$.

By taking derivatives of $F_x(t)$ with respect to t, the probability density function (PDF) of the remaining lifetime random variable can be derived, which is $f_x(t) = F'_x(t)$.

Using equation (9.1), we derive

$$f_x(t) = e^{-\lambda t + b(\lambda(x) - \lambda)(1 - e^{t/b})} \times \left(\lambda + \frac{1}{b} e^{(x-m)/b}\right),$$

which is the $\left({}_{t}p_{x}\right)$ of the Gompertz-Makeham law multiplied by the IFM curve.

The Expected Remaining Lifetime (ERL) under the Gompertz-Makeham law of mortality is:

$$E[T_x] = \int_0^\infty e^{-\lambda t + b(\lambda(x) - \lambda)\left(1 - e^{t/b}\right)} dt$$

$$= \frac{b\Gamma\left(-\lambda b, b(\lambda_x - \lambda)\right)}{e^{(m-x)\lambda + b(\lambda - \lambda_x)}}, \text{ where,}$$

$\Gamma(a,c) = \int_c^\infty e^{-t} t^{(a-1)} dt$. We get an incomplete Gamma function.

Curtate Future Lifetime

The curtate future lifetime of (x), is denoted by $K(x)$. It is defined by the relation $K(x) = [T(x)]$. Note that in this formulation *[t]* is the greatest integer function and $K(x)$ is a discrete random variable with density

$$P[K(x) = k] = P[k \geq T(x) < k + 1].$$

The curtate lifetime, $K(x)$, represents the number of complete future years lived (x).

Check Your Progress 1

1) Define the terms: whole life insurance and endowment insurance.

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2) List the important Premium Principles.

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3) Write the survival function and explain its meaning.

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4) How do you differentiate between mortality and instantaneous force mortality?

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5) What is the difference between exponential law and Gompertz law of mortality?

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8.6 LIFE TABLE

In the following discussion we have adopted a substantial portion of the lecture note of J.R. Carey (1999) to discuss the life table and acknowledge the work of the author.

We will use the following terms and concepts while discussing to life table techniques.

- **Cohort**: The basic unit of the life table defined as a group of same-aged persons (birth cohort) who experience the same significant event in a particular time period (e.g. marriage of graduation cohort).
- **Life expectancy:** The average number of years remaining to a person of specified age
- **Life span:** The maximal number of years for a given species.
- **Cross sectional survey:** Data gathered from multiple groups over a short period of time.

A life table is a detailed description of the age-specific mortality, survival and expectation of life a population. It helps answer questions such as: What is the life expectancy of the average newborn? How many years remains for the average 75 year old woman in the country? Over what age groups do most death occur? What is the probability of dying in the next year given that you are aged 18? What is this probability at age 85? What fraction of all 2000 newborn will live to age 85? What proportion will live from live from age 50 to age 90?

There are two general forms of the life table. The first is the cohort life table, which includes the mortality experience of a particular cohort from the moment of birth through consecutive ages until none remains in the original cohort. The second basic form is the current life table, which is cross-sectional. This table assumes a hypothetical cohort subject throughout its lifetime to the age-specific mortality rates prevailing for the actual population over a specified period and is used to construct a synthetic cohort. Both the forms of the life table may be either single decrement or multiple decrements. The first of these lumps all forms of death into one and the second disaggregates death by cause.

Note the following sample calculations that help use a life table. Remember that these samples may not be obvious when you overlook the probability theory. With your understanding of discrete probability theory it will be possible to introduce some new ideas that becomes handy in understanding the table. We start with the following notations:

q_x: the probability that someone aged exactly x will die before $(x+1)^{th}$ birthday

p_x: the probability of surviving from age x to age $(x+1)$ and

$p_x = 1 - q_x$

l_x: number of people who survive to age x. Note that this is based on starting point of l_0 lives, typically, 100,000. Thus,

$l_{x+1} = l_x.(1-q_x) = l_x.p_x$

$\frac{l_x+1}{l_x} = p_x$

d_x : number of people who die aged x

So $d_x = l_x - l_{x+1}$

${}_tP_x$: the probability that someone aged exactly x will survive for t more years, i.e., live up to at least age $x+t$ years, i.e.,

${}_tP_x = \frac{l_{x+t}}{l_x}$

Life Table Functions

The life table is organised in seven columns starting with an age x column from 0 through the age of the oldest person ever to live around 120 year.

Table 1:Hypothetical Life table showing the main Functions

Age x (1)	Living at age x N_X (2)	Cohort survival to age x l_X (3)	Survival from x to $x+1$, p_x (4)	Mortality in interval x to $x+1$ q_x (5)	Death in interval x to $x+1$ dx (6)	Expectation of life at age x e_X (7)
0	100,000	1.000	.900	.100	.100	2.40
1	90,000	.900	.556	.444	.400	1.61
2	50,000	.500	.800	.200	.100	1.50
3	40,000	.400	.250	.750	.300	.75
4	10,000	.100	.000	1.000	.100	.50
5	0	--	--	--	--	--

l_X = Proportion of all newborn surviving to age X

P_X = Proportion of individuals alive at age X that survive to $x+1$

q_X = Proportion of individuals alive at age X that die prior to $x+1$

d_X = Proportion of all newborn that die in the interval x to $x+1$

e_X = Expected lifetime remaining to the average individual age X

Column 1. The first column of a life table contains all age classes denoted X and ranges from 0 (newborn) through oldest possible age, ω

Column 2. This column gives the number of the original cohort alive X and is denoted N_X. The initial number is typically 100,000 and is known as the life table radix.

Column 3. This column contains cohort survival, l_X, defined as the fraction of the original cohort surviving to age X. The general formula is given as $l_x = \frac{N_x}{N_0}$.

For example: $l_0 = \frac{N_0}{N_0} = \frac{100000}{100000} = 1$

and $l_3 = \frac{N_3}{N_0} = \frac{40,000}{100,000} = 0.4$

Column 4. The parameter defined in this column is known as period survival, P_X, defined as the fraction of individuals alive at age X that survive to age $X+1$. The general formula is given as

$$P_X = \frac{l_{x+1}}{l_x}$$

For example, the period of survival rates for ages $X=0$ and $X=3$ from Table 1 are

$$p_0 = \frac{l_1}{l_0} = \frac{0.90}{1.00} = 0.90 \text{ and}$$

$$p_3 = \frac{l_4}{l_3} = \frac{0.10}{0.25} = 0.25$$

Thus, the probability of surviving from age 0 to 1 is 0.90 and from age 3 to 4 is 0.25.

Column 5. This column contains the compliment of period survival and is known as period mortality, q_x, defined as the fraction of individuals alive at age X that die prior to age $X+1$. The general formula is

$$q_x = 1 - \frac{l_{x+1}}{l_x} \text{ or, } q_X = 1 - p_x$$

For example, the period of mortality rates for ages $x=0$ and $x=3$ (see Table 1)

$$q_0 = 1 - p_0 = 1.00 - 0.90 = 0.10 \text{ and}$$

$$q_3 = 1 - p_3 = 1.00 - 0.25 = 0.75$$

Thus, the fraction of individuals that are alive at age 0 but die prior to age 1 is 0.10 and fractions that are alive at age 3 but die prior to age 4 is 0.75

Column 6. The parameter contained in this column is the fraction of the original cohort that dies in the interval x to $x+1$ and is denoted d_X. It is the frequency distribution of deaths and is given by the formula

$$d_x = l_x - l_{x+1}$$

For example, the value of d_X for ages $x = 0$ and $x = 3$ from Table 1 are

$$d_0 = l_0 - l_1 = 1.00 - 0.90 = 0.10 \text{ and}$$

$$d_3 = l_3 - l_4 = 0.14 - 0.10 = 0.30$$

Thus, the fraction of the original cohort that dies in the interval 0 to 1 is 0.10 and the fraction that dies in the interval 3 to 4 is 0.30.

Column 7. This column contains the parameter expectation of life, e_X, which is defined as, the average number of years (days, weeks) remaining to individual age X.

The general Formula is given by

$$e_x = \frac{1}{2} + \frac{l_{x+1} + l_{x+2} + l_{x+3} + \ldots\ldots + l_w}{l_x}$$

For example, the value of dx for ages $x = 0$ and $x = 3$ from Table 1 are

$$e_0 = \frac{1}{2} + \frac{l_1 + l_2 + l_3 + \ldots\ldots + l_\omega}{l_0} = \frac{1}{2} + \frac{0.90 + 0.50 + 0.40 + 0.10}{1.00} = 2.40$$

$$e_3 = \frac{1}{2} + \frac{l_4 + l_5}{l_3} = \frac{1}{2} + \frac{0.10 + 0.00}{0.40} = 0.75$$

Thus the expectation of life at birth, e_0, is 2.4 and at age 3, e_3, is 0.75

Current Life Table

A current life table is based on the concept of a *synthetic* cohort in which the probabilities of dying from one age class to the next are based on the death rates of each of the $115+$ cohorts living at any one time. The idea is based on the notion that the probability of surviving to say, age 5, is the probability (or fraction) of newborn individual living to age 1 times the probability of age 1 individuals living to age 2 times the probability of age 2 individuals living to age 3 and so forth. Thus, the general form for reconstructing a survival curve and, in turn, all other life table parameters form period survival (or period mortality) data is

$$l_x = p_0 \times p_1 \times p_2 \times \ldots\ldots \times p_{x-1}$$

For example, suppose we wish to construct a survival curve from the death rates prevailing from 1990 to 1991 as in Table 2. The probability of an individual age 0 in 1990 surviving to age 1 in 1991 is

$$p_0 = \frac{P_0 p_1}{P_0 p_0} = \frac{92,00}{110,000} = 0.8364$$

Table 2. Number of persons in hypothetical census by age class and year.

Year	0	1	2	3
1990	110,000	121,000	110,000	--
1991	--	92,000	100,000	101,000

Similarly for the probabilities for individuals surviving from age 1 to 2 and age 2 to 3

$$p_1 = \frac{P_0 p_2}{P_0 p_0} = \frac{100,000}{121,000} = 0.8264$$

$$p_2 = \frac{P_0 p_3}{P_0 p_2} = \frac{101,000}{110,00} = .09182$$

Thus, a synthetic survival schedule can be constructed by multiplying progressively as

Age	P_X	l_X	N_X
0	0.8364	1.0000	100,000
1	0.8264	0.8364	83,640
2	0.9182	0.6912	69,120
3	--	0.6347	63,470

Note that in the above, out of 100,000 individual's age 0, 83640 will survive to age 1. This is based on the 1990 to 1991 probability of survival from 0 to 1 years. Out of these 83,640 individuals alive at age 1, 0.8264 of these or 69, 120 will survive to age 2. Out of these 69, 120 individuals alive at age 2, 0.9182 or 63, 470 will survive to age 3. In other words, the fractions that survive to age x equal the product of the fractions (or probabilities) that survive through each of the previous age classes.

Both the cohort life table and the current life table are based on assumptions that are violated in reality. The cohort life table is based on the mortality experience of the same group of individuals from birth through the age when last individual dies. Thus if we were to use only cohort life tables to characterise longevity in a country then only life tables available would be for cohorts in which all individuals have died, the earliest of which would be individuals born around 1880. First, there is the problem of obtaining accurate records of mortality rates over a century ago and second, there is the problem that mortality rates in 1880 were much higher at all age classes than are current mortality rates. Third, the centenarians of today through much more difficult times than will future centenarians. Thus, their mortality rates may be much higher than will be the future centenarians. But the current life also has problems (though many fewer). First, the current life table is cross sectional. Thus, every cohort will have a different history. And it is well known that experiences at younger ages have an impact on mortality rates at older ages. Take for example two cohorts aged 10 and 50. Suppose the cohort age 50 was severely malnourished. When they were aged 10 years assumes that when the 10-year-old cohort attains age 50, they will experience the same mortality rate as the 50-year old cohort does this year. Clearly, they have different experiences that will have a major bearing on their mortality rates at older ages.

Table 1: Life table for birth for cohort subjected to mortality rates of 1990 U.S females

Year	Age	Number Remaining alive	Survival to age x l_x	Survival From x to $x+1$ P_x	Mortality From x to $x+1$ q_x	Fraction Dying x to $x+1$ d_x	Number Dying x to $x+1$ D_x	Expectation of life e_x
1970	0	3731000	1.0000	.9914	.0086	.0086	31975	79.9
1971	1	3699025	.9914	.9993	.0007	.0007	2537	79/5
1972	2	3696488	.9908	.9996	.0004	.0004	1567	78.6
1973	3	3694921	.9903	.9997	.0003	.0003	1194	77.6
1974	4	3693727	.9900	.9997	.0003	.0002	933	76.7
1975	5	3692795	.9898	.9998	.0002	.0002	821	75.7
1976	6	3691974	.9895	.9998	.0002	.0002	746	74.7
1977	7	3691228	.9892	.9998	.0002	.0002	709	73.7
1978	8	3690519	.9890	.9998	.0002	.0002	634	72.7
1979	9	3689884	.9888	.9998	.0002	.0002	597	71.7
1980	10	3689287	.9887	.9998	.0002	.0002	597	70.7
1981	11	3688690	.9887	.9998	.0001	.0001	560	69.8
1982	12	3688131	.9885	.9998	.0002	.0002	672	68.8
1983	13	3687459	.9883	.9998	.0002	.0002	784	67.8
1984	14	3686676	.9881	.9997	.0003	.0003	1007	66.8
1985	15	3685668	.9879	.9997	.0003	.0003	1269	65.8
1986	16	3684400	.9875	.9996	.0004	.0004	1455	64.8
1987	17	3682945	.9871	.9996	.0004	.0004	1642	63.9
1988	18	3681303	.9867	.9995	.0005	.0005	1716	62.9
1989	19	3679587	.9862	.9995	.0005	.0005	1791	61.9
1990	20	3677796	.9857	.9995	.0005	.0005	1791	60.9
1991	21	3676005	.9853	.9995	.0005	.0005	1866	60.0
1992	22	3674140	.9848	.9995	.0005	.0005	1903	59.0
1993	23	3672237	.9837	.9995	.0005	.0005	1940	58.0
1994	24	3670297	.9832	.9995	.0005	.0005	1977	57.1
1995	25	3668319	.9821	.9994	.0006	.0006	2089	56.1
1996	26	3666230	.9815	.9994	.0006	.0006	2127	55.1
1997	27	3664103	.9809	.9994	.0006	.0006	2127	54.2

1998	28	3659813	.9804	.9994	.0006	.0006	2164	53.2
1999	29	3657686	.9798	.9994	.0006	.0006	2127	52.2
2000	30	3655522	.9792	.9994	.0006	.0006	2164	51.20
2001	31	3653358	.9786	.9994	.0006	.0006	2164	49.3
2002	32	3651119	.9780	.9994	.0006	.0006	2239	48.3
2003	33	3648843	.9773	.9994	.0006	.0006	2276	47.4
2004	34	364656	.9767	.9993	.0007	.0006	2388	46.4
2005	35	3643956	.9752	.9993	.0007	.0007	2500	45.4
2006	36	3641307	.9767	.9993	.0007	.0007	2649	44.5
2007	37	3638359	.9760	.9992	.0008	.0008	2947	43.5
2008	38	3634964	.9752	.9991	.0009	.0009	3395	42.5
2009	39	3631084	.9743	.9989	.0011	.0010	3880	41.6
2010	40	3626569	.9732	.9988	.0012	.0012	4515	41.6
2011	41	3621421	.9720	.9986	.0014	.0014	5149	40.6
2012	42	3615637	.9706	.9984	.0016	.0015	5783	39.7
2013	43	3609257	.9691	.9982	.0018	.0017	6380	38.8
2014	44	3602281	.9674	.9981	.0019	.0019	6977	37.8
2015	45	3594669	.9655	.9979	.0021	.0020	7611	36.9
2016	46	3586349	.9635	.9977	.0023	.0022	8320	36.0
2017	47	3577208	.9612	.9975	.0025	.0025	9141	35.1
2018	48	3567134	.9588	.9972	.0028	.0027	10074	34.1
2019	49	3556091	.9561	.9969	.0031	.0030	11044	33.2
2020	50	3543890	.9531	.9966	.0034	.0033	12200	32.3
2021	51	3530533	.9499	.9962	.0038	.0036	13357	31.4
2022	52	3515945	.9463	.9959	.0041	.0039	14588	30.6
2023	53	3500088	.9424	.9955	.0045	.0042	15857	29.7
2024	54	3482889	.9381	.9951	.0049	.0046	17200	28.8
2025	55	3464308	.9335	.9947	.0053	.0050	18580	28.0
2026	56	3444161	.9285	.9942	.0058	.0054	20147	27.1
2027	57	3422222	.9231	.9936	.0064	.0059	21938	26.3
2028	58	3398195	.9172	.9930	.0070	.0064	24028	25.4
2029	59	3371854	.9108	.9922	.0078	.0071	26341	24.6
2030	60	3343088	.9037	.9915	.0085	.0077	28766	23.8
2031	61	3311748	.8960	.9906	.0094	.0084	31340	23.0
2032	62	3277795	.8876	.9997	.0103	.0091	33952	22.2
2033	63	3241194	.8785	.9888	.0112	.0098	36601	21.4
2034	64	3201870	.8687	.9879	.0121	.0105	39325	20.7
2035	65	3159635	.8582	.9868	.0132	.0113	42235	19.9
2036	66	3114378	.8469	.9857	.0143	.0121	45257	19.2

2037	67	3066210	.8347	.9845	.0155	.0129	48167	18.4
2038	68	3066621	.8218	.9835	.0165	.0136	50742	17.7
2039	69	3015469	.8082	.9823	.0177	.0143	53316	17.0
2040	70	2962153	.7939	.9811	.0189	.0150	56114	[illegible]
2041	71	2906039	.7789	.9796	.0204	.0159	59286	15.6
2042	72	2846753	.7789	.9780	.0220	.0168	62596	14.9
2043	73	2784184	.7630	.9763	.0237	.0177	66039	14.3
2044	74	2718145	.7462	.9744	.0256	.0187	69695	13.6
2045	75	2648450	.7285	.9721	.0279	.0198	73799	12.9
2046	76	2574651	.7099	.9696	.0304	.0210	78351	12.3
2047	77	2496300	.6991	.9667	.0333	.0223	83127	11.7
2048	78	2413173	.6468	.9635	.0365	.0236	88126	11.1
2049	79	2325047	.6232	.9599	.0401	.0250	93312	10.5
2050	80	2231735	.5982	.9557	.443	.0265	98909	9.9
2051	81	2132826	.5717	.9509	.0491	.0281	104692	9.3
2052	82	2028134	.5436	.9457	.0543	.0295	110139	8.8
2053	83	1917995	.5141	.9401	.0599	.0308	110139	8.2
2054	84	1803080	.4833	.9339	.0661	.0320	114915	7.7
2055	85	1683875	.4543	.9269	.0731	.0330	119205	7.2
2056	86	1560827	.4183	.9190	.0810	.0339	123048	6.8
2057	87	1434383	.3845	.9100	.0900	.0346	126444	6.3
2058	88	1305253	.3498	.8998	.1002	.0350	129130	5.9
2059	89	1174519	.3148	.8885	.1115	.0351	130724	5.5
2060	90	1043598	.2797	.8761	.1239	.0347	130921	5.1
2061	91	914319	.2451	8626	.1374	.0321	129279	4.8
2062	92	788696	.2114	.8481	.1519	.0300	125623	4.5
2063	93	668856	.1793	.8324	.1676	.0300	119840	4.2
2064	94	556777	.1492	.8159	.1841	.0275	112079	3.9
2065	95	454249	.1218	.7994	.2006	.0244	102528	3.7
2066	96	363138	.0973	.7835	.2165	.0177	91111	3.5
2067	97	284526	.0763	.7686	.2314	.0144	78612	3.3
2068	98	218674	.0586	.7546	.2314	.0114	65852	3.1
2069	99	165022	.0442	.7423	.2454	.0089	53652	3.0
2070	100	122489	.0328	.7295	.2577	.0068	42533	2.8
2071	101	89357	.0240	.7161	.22705	.0051	33131	2.7
2072	102	699[illegible]	.0172	.7015	.2839	.0038	25371	2.6
2073	103	44884	.0120	.6874	.2985	.0027	19103	2.4
2074	104	30855	.0083	.6711	.3126	.0019	14029	2.3
2075	105	20707	.0056	.6541	.3289	.0013	10148	2.2

2076	106	13544	.0036	.6391	.3459	.0009	7164	2.2
2077	107	8656	.0023	.6164	.3836	.0006	4888	2.1
2078	108	5335	.0014	6014	.3986	.0004	3321	2.0
2079	109	3209	.0009	.5814	.4186	.0002	2127	1.9
2080	110	1866	.0005	.5800	.4200	.0001	1343	1.8
2081	111	1082	.0003	.5517	.4483	.0001	784	1.6
2082	112	597	.0002	.5000	.5000	.0001	485	1.4
2083	113	298	.0001	.5000	.5000	.0000	298	1.4
2084	114	149	.0000	.5000	.5000	.0000	149	1.3
2085	115	75	.0000	.5000	.5000	.0000	75	1.0
2086	116	37	.0000	.0000	1.0000	.0000	37	0.5

Source: HDE Lecture Note by J.R.Carey (1999)

8.7 VALUING CONTINGENT PAYMENTS

In insurance the actuarial present value is a result of formulating money to be paid as well as time of such payment as random variables. If the random amount of money is denoted by A and the random time of payment by T, then the present value is AV^T and its expected value, i.e., $E\left(AV^T\right)$ is the value today of the random future payment.

In life insurance, the amount that is paid at the time of death is usually fixed by the policy and is non-random. If the force of interest is constant and known to be equal to δ, we simply write $T = T\left(X\right)$. The actuarial present value of an insurance policy when $A = 1$ at the time of death is given by E $[v^T]$ according to the contingent payment defined above. Viewed differently, the actuarial present value of the benefit is equivalent to single premium payment that an insurance company with no operating expenses and no desire for profit would charge today. For this reason the actuarial present value of a benefit is called the net single premium. Thus, the net single premium is the idealised amount you would like to pay as a lump sum (single premium) at the time of taking the policy. This will exactly compensate for the average of the future payments, which the insurer will have to make.

Thus, if the valuation rate of insurance is $r = 5\%$ and the insured person dies at time $T = 10$ years, then the discounted value of the death benefit at time zero is: $e^{-(0.05)10} = \$0.606$ per dollar of face value. In other words, an initial premium of 0.606 dollars invested at a rate of r =5% would be enough to pay the death benefit to the beneficiary. If, on the other hand, the insured person dies at time $T = 20$ years, then the discounted value of the death benefit is $e^{-(0.05)20} = \$0.368$ per dollar of the face value, which is much lower. In this case, an initial premium of 0.368 dollars is sufficient.

Taking the remaining lifetime as a random variable T_x, the stochastic discounted value (SDV) of a \$1 death benefit at a valuation rate r is: $A_x = e^{-rT_x}$.

The standard notation for the associated net single premium is given below. We present these following Veeh (2001) and will see more of these notations subsequently. In most cases it is assumed that the benefit amount is \$1, and is paid at the time of death. Note that a fixed constant force of interest is assumed and $v = 1/(1+i) = e^{-\delta}$.

Insurances Payable at the Time of Death (see Veeh 2001)

Type	Net Single Premium
n-year pure endowment A	$\bar{A}_{\overset{}{x}:\overset{1}{\overline{n}\rvert}} = {}_nE_x = E\left[v^n 1_{(n,\infty)}(T)\right]$
n-year term	$\bar{A}^{1}_{x:\overline{n}\rvert} = E\left[v^T 1_{(0,n)}(T)\right]$
whole life	$\bar{A}_x = E\left[v^T\right]$
n-year endowment	$\bar{A}_{x:\overline{n}\rvert} = E\left[v^{T\wedge n}\right]$
m-year deferred n-year term	${}_{m\mid n}\bar{A}_x = E\left[v^T 1_{(m,n+m)}(T)\right]$
whole life increasing mthly	$(I\bar{A})^{(m)}_x = E\left[v^T [Tm+1]/m\right]$
n-year term increasing annually	$(I\bar{A})^{1}_{x:\overline{n}\rvert} = E\left[v^T [T+1] 1_{(0,n)}(T)\right]$
n-year term decreasing annually	$(D\bar{A})^{1}_{x:\overline{n}\rvert} = E\left[v^T (n-[T]) 1_{(0,n)}(T)\right]$

In the table above, the bar indicates that an insurance paid at the time of death, whereas the subscripts denote the status whose death causes the insurance to be paid. These insurances are now reviewed on a case-by-case basis (see Unit 12 for more notations).

(i) **n-year Pure Endowment Insurance:** It pays the full benefit amount at the end of the n[th] year if the insured survives at least n-years. The notation for the net single premium for a benefit amount of 1 is $\bar{A}_{x:\overset{1}{\overline{n}\rvert}}$. As the payment of benefit is to be made at time n, we can write

$$ {}_nE_x = E\left[v^n 1_{(n,\infty)}(T(x))\right] = v^n P\left[T(x) \geq n\right] = v^n_n p_n $$

Pure Endowment: In this case, the net single premium happens to be the actuarial present value of a lump sum payment made at a future date. Thus, you pay one unit at the end of the year of death if death occurs within n years. Otherwise pay one unit after n years.

(ii) **n-year Term Insurance:** Under this type of contract the net single premium with a benefit of 1 is payable at the time of death for an insured (x) and is denoted $\bar{A}^{1}_{x\overline{n}\rvert}$. So, you pay one unit at the end of the year if death occurs within n years and pay nothing if the life survives n years.

(iii) **Whole Life Insurance:** In this form full benefit is paid at time (K+1) after the insured dies. The whole life benefit can be obtained by taking the limit as $n \to \infty$ in the n-year term insurance setting. The notation for the net single premium for a benefit of 1 is $\bar{A}_x$. The random discounted value at time 0 is $e^{-\delta(K+1)}$ and its expected value is $E\left[e^{-\delta(K+1)}\right]$ which happens to be the fair premium.

Example: (due to Veeh, 2001) Suppose T (x) follows an exponential distribution with mean 50. The force of interest is 5%. The net single premium for a whole life policy for (x) would be

$$E\left[1000V(x)\right] = \int_0^\infty 1000\, e^{-0.05t}\left(\frac{1}{50}\right)e^{-t/50}dt = 285.71$$

if the benefit of \$1000 is payable at the moment of death.

(iv) n**-year Endowment Insurance:** It has the provision of paying the full benefit at the time of death of the insured if this occurs before time n and for the payment of the full benefit at time n otherwise. For example, the net single premium for a 5 year pure endowment policy for (x)=(30) when interest rate = 5% is $V^5{}_5p_{30} = 0.74091$.

The insurances discussed above carry a fixed constant benefit. However, increasing whole life insurance has a provision of benefit, which increases linearly in time. Similarly, increasing and decreasing n-year term insurance has the provision for linearly increasing (decreasing) benefit over the term of the insurance.

Just as the cases of insurance payable at the time of death there are same types of policies available in which the benefit is paid at the end of the year of death. The definition and notation for these net single premiums will now be introduced. The only difference between these insurances and those already described is that these insurances depend on the distribution of the curtate life variable $K = K\,(x)$ instead of T. For the notations used see the following table due to Veeh 2001.

Insurances Payable at the Time of Death

Type	**Net Single Premium**
n-year term	$A^{\,1}_{x:\overline{n}\vert} = E\left[v^{K+1}1_{(0,n)}(K)\right]$
whole life	$A_x = E\left[v^{K+1}\right]$
n-year endowment	$A^{\,1}_{x:\overline{n}\vert} = E\left[v^{(K+1)\wedge n}\right]$
m-year deferred n-year term	${}_{m\vert n}A_x = E\left[v^{K+1}1_{(m,n+m)}(K)\right]$
whole life increasing mthly	$(IA)_x = E\left[v^{K+1}(K+1)\right]$
n-year term increasing annually	$(IA)^{1}_{x:\overline{n}\vert} = E\left[v^{K+1}(K+1)1_{[0,n)}(K)\right]$
n-year term decreasing annually	$(DA)^{1}_{x:\overline{n}\vert} = E\left[v^{k+1}(n-k)^1(0,n)^K\right]$

8.8 LIFE ANNUITIES

A Life Annuity contract is an agreement to pay a scheduled payment to the policyholder. It needs to be paid at every interval $\frac{1}{m}$ of a year while the annuitant is alive up to a maximum number of nm payments (see Unit 11 for more discussion).

See that these are contingent annuities in which payments are made for a random time interval. To appreciate the idea, suppose that in a life insurance policy, (x) pays a premium at the beginning of each year until the time of death. Thus, the premium payments represent a life annuity due for (x). Take a case in which the payment amount is 1. As the premiums are paid annually, the term of this life annuity depends on the curtate life of (x) and there will be $K(x) + 1$ payments. We can write the actuarial present value of the payments as $\ddot{a}_x = E\left[\ddot{a}_{\overline{K(x+1)}|}\right]$. Thus,

$$\ddot{a}_x = E\left[\ddot{a}_{\overline{K(x)+1}|}\right] = [\frac{1-v^{K(x)+1}}{d}] = \frac{1-A_x}{d}$$

Note that the above formulation gives the relationship between this life annuity due and the net single premium for a whole life policy.

Let us consider the actuarial present value of a life annuity immediate. Here the first payment is made at the $\frac{1}{m}$ and the last payment at time n in the case of a finite term n over which the annuitant survives. Thus,

$$E\left[a_{\overline{K(x)}|}\right] = E\left[\frac{1-v^{(K+1)}}{i}\right] = \frac{1-v^{-1}A_x}{i}.$$

When a life annuity due in which payments are made m times per year and each payment is $1/m$ the actuarial present value denoted by $\ddot{a}_x^{(m)}$. In such a case, there are ${}_mT+1$ payments. Therefore,

$$\ddot{a}_x^{(m)} = E\left[\sum_{j=1}^{(mT)}\left(\frac{1}{m}\right)V^{j/m}\right] = E\left[\frac{\left(\frac{1}{m}\right)1-V^{([mT]+1/m)}}{1-V^{\frac{1}{m}}}\right]$$

If we take $m\left(1-v^{\frac{1}{m}}\right) = d^{(m)}$, then $A_x^{(m)} + d^m\ddot{a}_x^{(m)} = 1$.

An annuity is called a temporary **life annuity,** when pension benefits are in the form of a life annuity immediate. In this arrangement the insured exercises the option of receiving a higher benefit only for a fixed number of years or unit death occurs depending on whichever comes first.

Consider a situation of annuities payable continuously. Suppose that the rate of benefit payment is constant and is 1 per unit time. Then in a time interval $(t, d+dt)$, the amount paid is dt and its present value is $e^{-\delta t}dt$. Thus, the present value of the continuously paid annuity over a period of n years would be

$$\bar{a}_{\overline{n}|} = \int_0^n e^{-\delta t} dt = \frac{1-e^{\delta n}}{\delta}$$

A life annuity payable continuously will have actuarial present value

$$\bar{a}_x = E\left[\bar{a}_{\overline{T(x)}|}\right] = E[\frac{1-e^{-\delta T(x)}}{\delta}].$$

8.9 NET PREMIUMS

Let us consider the case of an insurer selling a fully discrete whole life policy. It will be paid for by equal annual premium payments during the life of the insured. Note that fully discrete refers to the fact that the benefit is to be paid at the end of the year of death and the premiums are to be paid on a discrete basis as well. We turn to the idea behind the pricing of life insurance, that is, premium calculation. A first approximation yields the **net premium**, which is found by the **equivalence principle:** It says, the premium should be set such that the actuarial present value of the benefits paid is equal to the actuarial present value of the premiums received. On the basis of this principle, therefore, the net premium P should satisfy

$$E\left[v^{K(x)+1}\right] = PE\left[\ddot{a}_{\overline{K(x)+1}|}\right]$$

or, $\quad A_x - p\ddot{a}_x = 0$.

To find the net premium, p_{30}, for (30) if $i = 0.05$, we take the help of a life table. It can be shown in that $p_{30} = A_{30} / \ddot{a}_{30} = 0.10248/15.8561$. You many extend net premiums for fully discrete insurances.

8.9.1 Multiple Lives

There are cases of insurance in which the benefit is paid contingent on the death or survival of more than one life. Thus, we have to study the time of the benefit payment that depends on more than one life. Let us define that a **status** as an artificially constructed life form for which the notice of life and death can be related. The simplest type of status is the single life status. The **single life** status, where (x) dies exactly when (x) does. Another artificial life is the simple status. This status dies at the end of n years. The **joint life status** for the n lives $(x_1),...(x_n)$ is the status, which survives until the first member of the group dies. This status is denoted by $(x_1x_2...x_n)$. The **last survivor status**, denoted by $(\overline{x_1x_2...x_n})$ is the status, which survives until the last member of the group dies.

For the purpose of insurance, if the constituents of the status are assumed to die independently then one way to solve the problem is the treatment of single life case. Thus, a fully discrete whole life policy issued to the joint status (xy), with the net annual premium to be paid for such a policy is computed as follows:

The premium, P, must satisfy $A_{xy} = P\ddot{a}_{xy}$.

When we use the definition of the joint life status, we get

$$A_{xy} = E\left[v^{K(x)\wedge K(y)+1}\right] \text{ and } \ddot{a}_{xy} = \frac{1-A_{xy}}{d}.$$

Moreover, ${}_tp_{xy} = {}_tp_x\,{}_tp_y$ (due to independences assumption)

$$\Rightarrow A_{xy} = E\left[v^{K(xy)+1}\right] = \sum_{K=0}^{\infty}\left({}_kp_{xy} - {}_{kl}p_{xy}\right)$$

$$= \sum_{l=0}^{\infty} v^{k+1}\left({}_kp_x\,{}_kp_y - {}_{k+1}p_{xk+1}p_y\right),$$

that is, we can use life table for computation.

8.10 MULTIPLE DECREMENT MODELS

Instead of treating $\lambda(x)$ as a "rate of death" we can think of $\lambda(x)$ as an abstract rate at which individuals are "leaving" a group over time. For example, we might say that: $\lambda(x) = \lambda^1(x) + \lambda^2(x) + \lambda^3(x)$, in which case we have three causes of death, cancer $\lambda^1(x)$, strokes $\lambda^2(x)$ and heart disease $\lambda^3(x)$. We can then model the probability of dying from one cause versus the probability of dying from another.

Multiple decrement models are important in the context of the analysis of pension plans. In such a setting a person may withdraw from the workforce (a 'death') due to accident, death, or retirement. Different benefits may be payable in each of these cases. A common type of insurance in which a multiple decrement model is appropriate is in the double indemnity life policy. In such case the benefit is twice the amount of the policy if the death is accidental.

To analyze the insurance plans take random variable $J = J(x)$ which is a discrete random variable denoting the cause of decrement of the status (x). Assume that $J(x)$ takes values 1, . . . , m. For information on these take joint distribution of the random variables $T(x)$ and $J(x)$ where the joint distribution is of mixed type since $T(x)$ is assumed to be absolutely continuous while $J(x)$ is discrete.

Consider ${}_tq_x^{(j)} = P\left[0 < T(x) \le t, J(x) = j\right]$ and ${}_tp_x^{(j)} = P\left[T(x) > t,\ J(x) = j\right]$ where ${}_\infty q_x^{(j)}$ is the marginal density of J(x). The probability of death due to all causes is represented by the superscript τ. Thus,

$${}_tq_x^{(\tau)} = P\left[T(x) \le t\right] = \sum_{j=1}^{m} {}_tq_x^{(j)}$$

and a similar expression for the survival probability holds. Although ${}_tq_x^{(\tau)} + {}_tp_x^{(\tau)} = 1$ a similar equation for the individual causes of death fails unless $m = 1$. For the force of mortality from all causes

$$\mu_{x+t}^{(\tau)} = \frac{f_{T(x)}(t)}{P[T(x)>t]} \quad \text{and} \quad \mu_{x+t}^{(j)} = \frac{f_{T(x)J(x)}(t,j)}{P[T(x)>t]}.$$

While using these pieces of information, we have to be careful, however. Particularly,

${}_tp_x^{(j)} \neq \exp\left\{-\int_0^t \mu_{x+s}^{(j)} ds\right\}$ is generally true.

There are problems in the construction of a multiple decrement life table. For example, a problem arises when we take of the case of a double indemnity whole life policy. To see it, let us assume that the policy will pay an amount \$1 at the end of the year of death if death occurs due to non-accidental causes and an amount of \$2 if the death is accidental. If we denote the type of decrement as 1 and 2 respectively, the present value of the benefit is

$$v^{K(x)+1} 1_{\{1\}}(J(x)) + 2v^{K(x)+1} 1_{\{2\}}(J(x)) = J(x) v^{K(x)+1}$$

To get the net premium we take the expected value of this quantity, which can be completed if $p_x^{(j)}$ is known.

To calculate these probabilities two basic methodologies are used. If a large group of people for whom records are maintained is available, the actual survival data with the deaths in each year of age broken down by cause would also be known. The multiple decrement tables can then be easily constructed.

Example (modified from Veeh, 2001): An insurance company issues life insurance to coal miners. There are three causes of decrement (death): mining accidents, lung disease, and other causes. Experience of the company allows it construct a decrement (life) table for these three causes of decrement. However, the company now wants to enter the life insurance business for salt miners where the two causes of decrement (death) are mining accidents and other. The problem is how can the information about mining accidents for coal miners be used to get useful information about mining accidents for salt miners?

Perhaps a simple approach could be to lift the appropriate column from the coal miners life table and use it for the salt miners. However, such an approach is not appropriate, as it does not take into account the fact that there are competing risks, that is, the accident rate for coal miners is affected by the fact that some miners die from lung disease and thus are denied the opportunity to die from an accident. The death rate for each cause in the absence of competing risk is needed.

Thus, to solve the multiple decrement process some auxiliary information is introduced. Define ${}_tq_x^{\prime(j)} = 1 - {}_tp_x^{\prime(j)}$.

Such an approach helps because a person cannot simultaneously die of 2 causes. Thus, there is no competing risk instantaneously.

The probability ${}_tq_x^{\prime(j)}$ *x is the* ***net probability of decrement*** (or **absolute rate of decrement).**

Such probabilities can be used to obtain the desired entries in a multiple decrement table as follows: First ${}_tp_x^{(\tau)} = \prod_{j=1}^{m} {}_tp_x^{\prime(j)}$.

In case of a single decrement there is the assumption of uniform distribution of deaths in the year of death. In the multiple decrement context this translates into the statement that for $0 \le t \le 1$, ${}_tq_x^{\prime(j)} = tq_x^{\prime(j)}$. Another approximation that can connect single and multiple decrement tables is the life table functions $L_x = \int_0^1 l_{x+t}\,dt$

$m_x = \dfrac{l_x - l_{x+1}}{L_x}$. where m_x is called central death rate at age x.

Check Your Progress 2

1) What steps will you take to read a life table?

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2) What do you mean by valuation of contingent payment?

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3) How net premium is related to equivalent principle?

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4) Define status as a life form.

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5) Explain the meaning of multiple decrement in life insurance.

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8.11 LET US SUM UP

In this unit we have discussed the basic features of life insurance with a view to understanding it's modeling in cases related to single and multiple decrements. The presentation comprises, preliminary idea of the working of life insurance, survival distributions and single and multiple decrement models. A life insurance contract has three parties, the insurance contract may be for temporary or permanent life insurance. The permanent life insurance are of three basic types- whole life, universal life and endowment. Life insurance operates by assigning premiums to different categories of risks and follows the premium principles. An important issue with insurance company is the claims of the insured. The claim could be one time in case life insurance or time varying in case of non-life insurance. To determine the claims, the company depends a survival function of the insured. This, if an individual is presently aged x, then it probabilistically defines the survival of x. With such a for formulation the mortality rate is derived as a complement to survival rate. The mortality is modelled by taking future life (the aging process) (De Movire Law) or exponential distribution (Gompertz Makeham Law).

A life table is used to get information on age specific mortality, survival and expectation of life or a population.

8.12 KEY WORDS

Annuity: A sequence of payments of a limited duration.

Central Mortality Rate: Probability that a person whose exact age is not known but lies in between x and $x + 1$ will die within one year following the attainment of that age,

Cohort: A hypothetical group of population gradually diminished by deaths.

Complete Expectation of Life or Complete Future Life Time: The average number of years a person of given age can be expected to live under the prevailing mortality conditions. It gives the number of years of life entirely completed and includes the fraction of the year survived in the year in which death occurs which on the average is taken as ½ year. So complete future life time = curtate future life time + ½.

Curtate Future Life Time: Curtate future lifetime is the number of completed years lived by a person aged x years.

Force of Mortality: Ratio of instantaneous rate of decrease in the number of persons living at age x to the value of the number of persons living at age x.

Joint-Life Status: To exist as long as all participating lives survive.

Last-Survivor Status: It is defined to be an insurance policy remaining intact when at least one of all the participating lives survive so that it falls with the last death.

Life Annuity: A series of payments, which are made while the beneficiary (of initial age) lives.

Life Table: A conventional method of expressing the most essential facts about the age distribution of mortality in a tabular form and it gives the life history of a hypothetical population which is gradually diminished by deaths. It is a powerful tool for measuring the probability of life and death at various age factors.

Net Premium Reserve: The net premium reserve at time t is defined as the conditional expectation of the difference between the present value of the future benefit payments and the present value of the future premium payment given that the future life time is greater than the given time.

Net Premium: The expected value of the loss is zero.

Net Single Premium: The expected value of present value of the payment.

Perpetuity: A sequence of payments of a unlimited duration.

8.13 SOME USEFUL BOOKS

Carey J.R.(1999), Life Tables and Mortality, HDE Lecture Notes (see internet)

Mathias Winkel (2003), Actuarial Science University of Oxford (see Internet)

Trowbridge, C.L (1989), Fundamental concepts of Actuarial science, Actuarial Education and Research Fund (see internet)

Veeh, Jerry Alan (2001), Lecture Notes on Actuarial Mathematics (see Internet)

8.14 ANSWER OR HINTS TO CHECK YOUR PROGRESS

Check Your Progress 1

1) See Section 8.3 and answer
2) See Sub-Section 8.4.2 and answer
3) See Section 8.5 and answer
4) See Equation (5) to (9) in Section 8.5 and answer
5) See Equations (11) to (14) in Section 8.5 and answer

Check Your Progress 2

1) See section 8.6 and answer
2) See Sub-Section 8.7 and answer
3) See Section 8.9 and answer
4) See Sub-Section 8.9.1 and answer
5) See section 8.10 and answer

UNIT 9 COLLECTIVE INSURANCE AND COMPANY OPERATIONS

Structure

9.0 OBJECTIVES

After going through this unit, you will be able to:

- understand individual and collective risk models of insurance; and
- apply important techniques used in risk modelling.

9.1 INTRODUCTION

Insurance operations relies on premiums to meet the costs (claims). The claim arrivals are random and the insurance portfolios need to be designed to cover the risks. The following discussion gives the operational tools on which the insurance companies rely to take decision.

9.2 ACCOUNTING PRACTICES OF INSURANCE COMPANY

9.2.1 Insurance Company Operation

The goal of insurance modelling is to develop a probability distribution for the total amount paid in return as benefits. Such a strategy helps manage company's capital account and honour its commitments. It becomes necessary, therefore, to understand their overall approach to achieve the basic objective. For that purpose we will take up three aspects, viz.,

1) Accounting practices of an insurance company and reserve
2) Behaviour of the loss characteristics of a groups of similar polices and
3) Other methods of premium setting for a policy.

9.2.2 Net Premium Reserve

To understand the concept of net premium reserves, we need to start with some basic terms used in accounting. Suffice it will be to remember five of these:

1) Asset
2) Liability
3) Equity
4) Reserve and
5) Surplus

Broadly speaking the terms are used to mean the following:

- Asset: Tangible and intangible properties owned by the business
- Liability: Payments owed by the business
- Equity: (Assets – liabilities).
- Reserve: Insurance name of liability
- Surplus: Insurance name of equity

A major source of liabilities in case of insurance as opposed to other business is the payment in return for premiums received from the insured. Therefore an insurance company's strategy to run its business successfully always aims at measuring the liabilities. In general, it is an accepted principle for a company to have

Reserve at time t = (Actuarial present value at time t of future benefits) −(Actuarial present value at time to of future premiums).

In the book of record, the accountant assumes that the premium charged for a policy is net level premium and makes provision for reserves accordingly. For this reason, the liabilities are measured as the net level premium reserve.

To understand the idea of reserve adopted in accounting consider the following example.

Let the income of a company for the year ending in March 2006 is given as follows:

S.No	Premiums	200
1.	Investment income	100
2.	Expenses	105
3.	Claims and maturities	70
4.	Increases in reserve	--
5.	Net income	--

In addition, let the balance sheet of the company is presented as

S.No	Balance sheet as on		
		March 31, 2005	March 31, 2006
1.	Assets	2000	--
2.	Reserve	--	700
3.	Surplus	1400	--

With the information given above, it will be possible for you to fill up the missing entries (--) in the tables. We know that net income is,

total income $-$ total expenditure

So, $(200+100) - (105 + 70) = 300 - 175 = 125$

From the balance sheet data reserve at the end of 2005

$2000 - 1400 = 600$

The net increase in reserve

$= 700 - 600 = 100.$

The net income $= 125 - 100 = 25.$

In 2006, surplus $= 1400 + 25 = 1425$ and asset $= 700 + 1425 = 2125.$

In case of prospective reserve formula, the expected size of future loss at time K is considered which is based on the evaluation of future performance of the insurance portfolio. To appreciate the underlying idea, consider the example of a fully discrete whole life policy issued to (x). Remember that premium under this scheme is payable annually and is equal to net premium. To compute the reserve at time k, where k is an integer, you need to note the following: Given the underlying accounting assumption, the amount of reserve at any time t can be computed with the help of two methods, viz., ***prospective reserve formula*** and ***retrospective reserve formula***.

Prospective Reserve Formula = (APV of future benefits) – (APV of future assessments), where APV = actuarial present value. A scenario-based on stochastic prospective reserve formula is similar, except that the above formula is calculated for each scenario on stochastically generated variable.

Retrospective Reserve Formula = accumulated value of the net premiums received based on realised return – accumulated value of completed insurance payments.

Since an insurance company operates with a large number of policies at any given time, it has to manage the financial risk involved in such operations. Keeping such a possibility in view, we study the problem of risk analysis in terms of individual and collective risk models. One of our main tasks in the following is to estimate the amount of risk both in terms of its quantification and time of occurrence.

9.3 INDIVIDUAL RISK MODEL

We consider a certain portfolio of insurance policies and the total amount of claims arising from it during a given period (usually a year). The joint premium for the whole portfolio that will cover the accumulated risk connected with all policies will be determined at the beginning.

Let us assume that the portfolio consists of n insurance and the claim (or loss) made in respect of the policy K is denoted by X_k. Then the total, or aggregate, amount of claims (or total risk) is expressed in the form $S = X_1 + X_2 + ... + X_n$, where $X_1, X_2, ..., X_n$ are independent random variables and X_i = loss to the insurer on insured unit i. There are two types of models - closed and open - through which we analyse the probable losses. In case of closed model, it is assumed that the number of insured units n is known and fixed. On the other hand, the insurance system contains provision for inflow and outflow of the insured units in case if open model.

The claim amount variable X_k for each policy is usually presented as $X_k = I_k B_k$ where random variables $I_1, ..., I_n, B_1, ..., B_n$ are independent. The random variable I_k indicates whether or not k^{th} policy produced a payment. If the claim has occurred, then $I_k = 1$, if not, $I_k = 0$. We denote $q_k = P(I_k = 1)$ and $1 - q_k = P(I_k = 0)$. The random variable B_k can have an arbitrary distribution and represents the amount of the payment in respect of the K^{th} policy given that the payment was made.

.9.3.1 General Premium Formulae

In order to determine the formulae for premiums, let us assume that the expectations and variances of $B_k's$ exist and denote $\mu_k = E(B_k)$ and $\sigma^2{}_k = Var(B_k), k = 1, 2, ...,, n$.. Then

$$E(X_k) = \mu_k q_k, \tag{9.1}$$

and the mean of the total loss in the individual risk model is given by

$$E(S) \sum_{k=1}^{n} \mu_k q_k \tag{9.2}$$

The variance of X_k is

$$Var(X_k) = Var\left\{E(X_k|I_k)\right\} + E\left\{Var(X_k|I_k)\right\} \tag{9.3}$$

$$= Var\left\{I_k E(B_k)\right\} + E\left\{I_k Var(B_k)\right\}$$

$$= \left\{E(B_k)\right\}^2 Var(I_k) + Var(B_k)E(I_k)$$

$$= \mu^2{}_k q_k(1-q_k) + \sigma^2{}_k q_k.$$

Since it is assumed that X_k's are independent, the variance of S is

$$Var(S) = \sum_{k=1}^{n}\left\{\mu^2{}_k q_k(1-q_k) + \sigma^2{}_k q_k\right\}. \tag{9.4}$$

With these distributional information, premiums for the individual risk model are as follows:

Pure risk premium: $P = \sum_{k=1}^{n} \mu_k q_k,$

Premium with safety loading: $P_{SL}(\theta) = (1+\theta)\sum_{k=1}^{n} \mu_k q_k,\ \theta \geq 0,$

Premium with variance loading:

$$P_V(a) = \sum_{k=1}^{n} \mu_k q_k + a\sum_{k=1}^{n}\left\{\mu^2{}_k q_k(1-q_k) + \sigma^2{}_k q_k\right\},\ a \geq 0,$$

Premium with standard deviation loading:

$$P_{SD}(b) = \sum_{k=1}^{n} \mu_k q_k + b\sqrt{\sum_{k=1}^{n}\left\{\mu^2{}_k q_k(1-q_k) + \sigma^2{}_k q_k\right\}},\ b \geq 0.$$

If we assume that for each $k = 1, 2, \ldots, n$ the moment generating function $M_{X_k}(t)$ exists, then

$$M_{X_k}(t) = 1 - q_k + q_k M_{B_k}(t), \tag{9.5}$$

and hence $$M_S(t) = \prod_{k=1}^{n}\left\{1 - q_k + q_k M_{B_k}(t)\right\}. \tag{9.6}$$

which gives the exponential premium as

$$P_E(c) = \frac{1}{c}\sum_{k=1}^{n} \ln\left\{1 - q_k + q_k M_{B_k}(c)\right\},\ c > 0. \tag{9.7}$$

In the individual risk model, claims of an insurance company are modelled as a sum of the claims of many insured individuals. Therefore, to find the quantile premium given by

$$P_Q(\varepsilon) = F_S^{-1}(1-\varepsilon),\ \varepsilon \in (0,1) \tag{9.8}$$

the distribution of the sum of independent random variable has to be determined.

9.3.2 Premiums in the Case of the Normal Approximation

The distribution of the total claim in the individual risk model can be approximated by means of the central limit theorem. In such a case, it is sufficient to evaluate means and variances of the individual loss random variables, sum them to obtain the mean and variance of the aggregate loss of the insurer and apply the normal approximation. However, it is important to remember that the quality of the approximation depends not only on the size of the portfolio, but also on its homogeneity.

The approximation of the distribution of the total loss S in the individual risk model can be applied to find a simple expression for the quantile premium. If the distribution of S is approximated by a normal distribution with mean E (S) and variance Var (S), the quantile premium can be written as

$$P_Q(\varepsilon)=\sum_{k=1}^{n}\mu_k q_k+\Phi^{-1}(1-\varepsilon)\sqrt{\sum_{k=1}^{n}\left\{\mu^2{}_k q_k(1-q_K)+\sigma^2{}_k q_k\right\}}, \tag{9.9}$$

where $\varepsilon \in (0,1)$ and $\Phi(.)$ denote the standard normal distribution function. Moreover, in the case of this approximation, it is possible to express the exponential premium as

$$P_E(c)=\sum_{k=1}^{n}\mu_k q_k+\frac{c}{2}\sum_{k=1}^{n}\left\{\mu^2{}_k q_k(1-q_k)+\sigma^2{}_k q_k\right\},\ c>0, \tag{9.10}$$

and you may see that such a premium is equal to the one with variance loading of $d=\frac{c}{2}$.

As the distribution of S is approximately normal with the same mean and variance, premiums defined in the expected value of the aggregate claims are given by the same formulae as given above.

To derive the probabilistic properties of the random variable S, we take the help of conditioning and obtain the formula for the distribution function of a sum of independent random variables. For that purpose, look at the following example.

Le there be two independent random variables X and Y with exponential distribution and with parameter 1. By conditioning

$$\begin{aligned}P[X+Y\le t]&=\int_{-\infty}^{\infty}P[X+Y\le t=y]f_Y(y)dy\\&=\int_0^t P[X\le t-y]f_Y(y)dy\\&=\int_0^t\left(1-e^{-(t-y)}\right)f_Y(y)dy\\&=1-e^{-t}-te^{-t}\text{ for } t\ge 0\end{aligned}$$

Thus, if X and Y are independent and absolutely continuous, then

$$F_{x+y}(t)=\int_{-\infty}^{\infty}Fx(t-y)fy(y)dy$$

The integral above is called the convolution of the two distribution function and is often denoted $F_X * F_Y$

Another approach is to appeal to the Central Limit Theorem we have discussed above in the context of individual loss model premium. As you may recall, the theorem states that if $X_1, \ldots\ldots, X_n$ are independent random variables, then the distribution of $\sum_{i=1}^{n} X_i$ is approximately the normal distribution with mean $\sum_{i=1}^{n} E[X_i]$ and variance $\sum_{i=1}^{n} Var(X_i)$. Thus, you have a case where an approximation of normal distribution does not depend on the detailed nature of the original distribution but only on the first two moments.

Note the insight gained from above theorem. If an insurance company has sold insurance at the pure premium, it will not achieve break-even and also face bankruptcy due to amount of claims arrivals. Therefore, insurance companies must charge an amount greater than the pure premium. A common methodology for the company is to charge $(1+\theta)$ times the pure premium. When this scheme is followed, θ is called the *relative security* loading and the amount $\theta \times$ (pure premium) is called the *security loading.*

9.4 THE COLLECTIVE RISK MODEL

Let us consider an alternative model describing the total claim amount in a fixed period in a portfolio of insurance contracts.

Let N be the number of claims arising from policies in a given period; X_1 be the amount of the first claim, X_2 be the amount of the second claim and so on. In the collective risk model, the random sum

$$S = X_1 + X_2 + \ldots + X_n \tag{9.11}$$

represents the aggregate claims generated by the portfolio for the period under study. The number of claims N is a random variable and is associated with the frequency of claim. The individual claims X_1, X_2...are identically distributed random variable and the random variables N, X_1, X_2...are mutually independent.

9.4.1 General Premium Formulae

In order to find formulae for premiums based on the expected value of the total claim, let us assume that *E (X), E (N), Var (X)* and *Var (N)* exist. For the collective risk model, expected value of aggregate claims is the product of the expected individual claim amount and the expected number of claims,

$$E(S) = E(N)\, E(X), \tag{9.12}$$

while the variance of aggregate claims is the sum of two components where the first is attributed to the variability of individual claims amounts and the other to the variability of the number of claims:

$$Var(S) = E(N)\, Var(X) + \{E(x)\}^2 + Var(N). \tag{9.13}$$

Thus, it is easy to obtain the following premium formulae in the collective risk model:

Pure risk premium: $P = E(N)E(X)$,

Premium with safety loading: $P_{SL}(\theta) = (1+\theta)E(N)E(X)$, $\theta \geq 0$,

Premium with variance loading:

$$P_v(a)=E(N)E(X)+a\left[E(N)Var(X)+\{E(X)\}^2 Var(N)\right],\quad a\geq 0$$

Premium with standard deviation loading:

$$P_{SD}(b)=E(N)E(X)+b\sqrt{E(N)Var(X)+\{E(X)\}^2 Var(N)},\quad b\geq 0$$

If we assume that $M_N(t)$ and $M_X(t)$ exist, the moment generating function of S can be derived as

$$M_S(t)=M_N\{\ln M_X(t)\}$$

and thus the exponential premium is of the form

$$P_E(c)=\frac{\ln\left[M_N\{\ln M_X(c)\}\right]}{c},\quad c>0$$

It is often difficult to determine the distribution of the aggregate claims and this fact causes problems with calculating the quantile premium given by

$$P_Q(\varepsilon)=F_s^{-1}(1-\varepsilon),\quad \varepsilon\in(0,1)$$

Although the distribution function of S can be expressed by means of the distribution of N and the convolution of the claim amount distribution, this is too complicated in practical applications. Therefore, approximations for the distribution of the aggregate claims are usually considered.

9.4.2 Normal and Gamma Approximations

In case of individual risk model above we have employed the normal approximation as an approximation for the distribution of aggregate claims. The same technique can be extended to the collective model when the expected number of claims is large.

As was observed earlier the normal approximation simplifies the calculations. If the distribution of S can be approximated by a normal distribution with mean $E(S)$ and variance $Var(N)$, the quantile premium is given by the formula

$$P_Q(\varepsilon)=E(N)E(X)+\Phi^{-1}(1-\varepsilon)\sqrt{E(N)Var(X)+\{E(X)\}^2 Var(N)},$$

where $\varepsilon\in(0,1)$ and $\phi(.)$ denote the standard normal distribution function. It is easy to notice, that this premium is equal to the standard deviation-loaded premium with $b=\Phi^{-1}(1-\varepsilon)$.

Moreover, in the case of the normal approximation, it is possible to express the exponential premium as

$$P_E(c)=E(N)E(X)+\frac{c}{2}[N]Var(X)+\{E(X)\}^2 Var(N)\ ],\quad c>0,$$

which is the same premium as resulting from the variance principle with $a=\frac{c}{2}$.

Let us also mention that since the mean and variance in the case of the normal approximation are the same as for the distribution of S, the premiums based on the expected value are given by the general formula.

Unfortunately, the normal approximation is not usually sufficiently accurate. The disadvantage of this approximation lies in the fact that the skewness of the normal distribution is always zero, as it has a symmetric probability density function. Since the distribution of aggregate claims is often skewed, another approximation of the distribution of aggregate claims that accommodates skewness is required.

The distribution function of the translated (shifted) gamma distribution is given by

$$G^{tr}(x;a,\beta,x_0) = F(x-x_0;a,\beta),\ x,a,\beta > 0$$

where $F(x,\alpha,\beta)$ denotes the distribution function of the gamma distribution with parameters α and β

$$F(x;a,\beta) = \int_0^x \frac{\beta^a}{\Gamma(a)} t^{a-1} e^{-\beta t} dt,\ x,a,\beta > 0$$

To apply the approximation, the parameters α, β and x_0 have to be selected so that the first, second, and third central moments of S equal the corresponding items for the translated gamma distribution. This procedure leads to the following result:

$$\alpha = 4\frac{\{Var(S)\}^3}{\left(E\left[\{S-E(S)\}^3\right]\right)^2}$$

$$\beta = 2\frac{Var(S)}{E\left[\{S-E(S)\}^3\right]}$$

$$x_0 = \beta = -2\frac{\{Var(S)\}^2}{E\left[\{S-E(S)\}^3\right]}$$

In the case of the translated gamma distribution, it is impossible to give a simple analytical formula for the quantile premium. Therefore, in order to find this premium a numerical approximation must be used. However, it is worth noticing that the exponential premium can be presented as

$$P_E(c) = x_0 + \frac{\alpha}{c}\ln\left(\frac{\beta}{\beta-c}\right), c > 0$$

9.4.3 Compound Poisson Distribution

In many applications, the number of claims N is assumed to be described by the Poisson distribution with the probability function given by

$$P(N=n) = \frac{\lambda^n e^{-\lambda}}{n!}, n = 0,1,2,.....,$$

where $\lambda > 0$. With this choice of the distribution of N, the distribution of S is called a compound Poisson distribution.

The compound Poisson distribution has a number of useful properties. Formulae for the exponential premium and for the premiums based on the expectation of the aggregate claims simplify because

$$E(N) = Var(N) = \lambda \text{ and } M_N(t) = exp\{(\lambda e^t - 1)\}.$$

Moreover, for large λ, the distribution of the compound Poisson can be approximated by a normal distribution with mean $\lambda E(X)$ and variance $\lambda E(X^2)$, and the quantile premium is given by

$$P_Q(\varepsilon) = \lambda E(X) + \Phi^{-1}(1-\varepsilon)\sqrt{\lambda E(X^2)}\varepsilon \in (0,1),$$

and the exponential premium is of the form

$$P_E(c) = \lambda E(X) + \frac{c}{2}\lambda E(X^2), \quad c > 0$$

If the first three central moments of the individual claim distribution exist, the compound Poisson distribution can be approximated by the translated gamma distribution with the following parameters

$$a = 4\lambda \frac{\{E(X^2)\}^3}{\{E(X^3)\}^2},$$

$$\beta = 2\frac{E(X^2)}{E(X^3)},$$

$$x_0 = \lambda E(X) - 2\lambda \frac{\{E(X^2)\}^2}{E(X^3)}$$

With these parameters one can obtain the formula for the exponential premium.

It is worth mentioning that the compound Poisson distribution has many attractive features. Modelling for example, the combination of a number of portfolios, each of which has a compound Poisson distribution of aggregate claims, also has a compound Poisson distribution of aggregate claims. Moreover, this distribution can be used to approximate the distribution of total claims in the individual model. Although the compound Poisson distribution is normally appropriate in life insurance modelling, it sometimes does not provide an adequate fit to insurance data in other coverage.

9.4.4 Compound Negative Binomial Distribution

When the variance of the number of claims exceeds its mean, the Poisson distribution is not appropriate. In this situation, the use of the negative binomial distribution with the probability function given by

$$P(N=n)=\binom{r+n-1}{n}p^r q^n,\ n=0,1,2,.....,$$

where $r>0, 0<p<1$, and $q=1-p$, is suggested. In many cases it provides a significantly improved fit to that of the Poisson distribution. When a negative binomial distribution is selected for N, the distribution of S is called a compound negative binomial distribution.

Since for the negative binomial distribution we have

$$E(N)=\frac{rq}{p},\quad Var(N)=\frac{rq}{p^2},$$

and

$$M_N(t)=\left(\frac{p}{1-q^{e^t}}\right)^r$$

the formulae for the exponential premium and for the premiums based on the expectation of the aggregate claims simplify.

For large r, the distribution of the compound negative binomial can be approximated by a normal distribution with the mean $\frac{rq}{p}\mathrm{E}(X)$ and variance $\frac{rq}{p}\mathrm{Var}(X)+\frac{rq}{p^2}\{\mathrm{E}(X)\}^2$. In this case the quantile premium is given by

$$P_Q(\varepsilon)=\frac{rq}{p}E(X)+\Phi^{-1}(1-\varepsilon)\sqrt{\frac{rq}{p}Var(X)+\frac{rq}{p^2}\{E(X)\}^2},\quad \varepsilon\in(0,1),$$

and the exponential premium is of the form

$$P_E(c)=\frac{rq}{p}E(X)+\frac{c}{2}\left[\frac{rq}{p}Var(X)+\frac{rq}{p^2}\{E(X)\}^2\right],\ c>0$$

It is worth mentioning that the negative binomial distribution arises as a mixed Poisson variate. More precisely, various distributions for the number of claims can be generated by assuming that the Poisson parameter Λ is a random variable with probability distribution function $u(\lambda)$, $\lambda>0$, and that the conditional distribution of N, given $\Lambda=\lambda$, is Poisson with parameter λ. In such a case the distribution of S is called a compound mixed Poisson distribution. This choice might be useful. For example, when we consider a population of insured where various classes of insured within the population generate number of claims according to the Poisson distribution. But the Poisson parameters may be different for the various classes. The negative binomial distribution can be derived in this fashion when $u(\lambda)$ is the gamma probability density function.

9.4.5 Importance of Surplus at Time t

As seen above in the collective risk model, the time at which claims are made is taken into account. Here the aggregate claims up to time t *is* assumed to be given by $\sum_{k=1}^{N(t)} X_k$, where $X_1, X_2, ...$ are independent identically distributed random variables representing the sizes of the respective claims, $N(t)$ is stochastic process representing the number of claims up to time t, and N and the $X's$ are independent. It is important to note the insurer's surplus at time t, denoted by $U(t)$, which is assumed to be of the form $U(t) = u + ct - \sum_{k=1}^{N(t)} X_k$, where u is the surplus at time $t = 0$, and c represents the rate of premium income.

Special attention will have to be paid to the problem of estimating the probability that the insurance company has negative surplus at some time t since this would mean that the company is ruined. To gain familiarity with some of the ideas involved, the classical gambler's ruin problem is studied below.

Check Your Progress 1

1) Write the meaning of net premium reserve in life insurance.

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2) Differentiate between prospective reserve and reserve formulae.

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3) What is the meaning of individual risk model in life insurance?

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4) How collective risk model is different from individual risk model in life insurance?

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5) How will you compute the formula of compound Poisson distribution in life insurance premium?

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9.5 CLASSICAL RISK MODEL

The Classical Model

Suppose we model a risk process as follows: The occurrence of the claims is described by a point process and the amount of money to be paid by the company at each claim by a sequence of random variables. The company receives a certain amount of premium of cover is liability and the difference between the premium income and the (average) cost for the claims is the 'safety loading' r.

In the classical Cramer-Lundberg-model the claim sizes $Y_1, Y_2, \ldots$ are independent identically distributed (iid) positive random variables with distribution function F and finite mean $\mu = E(Y_1)$. The claims Y_k arrive at time $T_k = D_1 + \ldots + D_k$ according to a homogenous Poisson process, where the inter-occurrence times $D_1, D_2, \ldots$ are independent exponentially distributed random variables with intensity α. If the corresponding claim number is $N(t) = \sup\{k \geq 1 : T_k \leq t\}, t \geq 0$, then the capital at time t is

$$U(t) = u + ct - V(t), \qquad t \geq 0,$$

where u is initial capital, $V(t) = \sum_{n=1}^{N(t)} Y_n$ the total claim amount until time t and $c > 0$ is the premium income rate. The ruin probability in infinite time is then defined by

$$\psi(u) = P\left(\inf_{0 \leq t \leq \infty} U(t) < 0\right).$$

For the case of exponentially distributed claim sizes Y_k there exists an explicit formula for the ruin probability first established by [Lundberg 1903]:

$$\psi(u) = \frac{1}{1+r} e^{-\frac{ru}{\mu(1+r)}}, \tag{1}$$

where $r = \frac{c}{\alpha\mu} - 1$ is the safety loading, which is positive for a reasonable insurance policy. Using an empirical value for μ and α, a desired value for the ruin probability can in practice (under the assumption of independence between the claims) be approximated by choosing a suitable risk premium rate C (this procedure relies on the law of large numbers which only holds for independent events).

In the more general case of $Y_1, Y_2, \ldots$ being identically distributed according to an arbitrary (but exponentially bounded) distribution function F, there exists the famous Cramer-Lundberg approximation

$$\lim_{u\to\infty} e^{Ru}\psi(u) = C, \tag{2}$$

where C is a constant and the so-called Lundberg exponent R depends on F and is given by a functional equation.

This asymptotic result (2) is only proved for exponentially distributed D_i. However, the theorem shows below that it holds for a still wider class of distributions.

For further purposes it is convenient to rewrite the process in the form of a random walk

$$S_i = \sum_{j=1}^{i} X_j \tag{3}$$

where $X_i = Y_i - cD_i$, so the process direction is reversed (starting at $U(t) = 0$ and causing ruin, if $U(t) = u$). In the classical model, $X_1, X_2, \ldots$ is an iid sequence of random variables. Note that for a reasonable choice of the premium rate $c, E(X_i) < 0$, so the random walk has negative drift. Using this notation the ruin probability can equivalently be defined by

$$\psi(u) = P(\max(0, S_1, S_2, \ldots) \geq u) \tag{4}$$

If the distribution function F is normal with mean μ and variance σ^2, then the random walk S_n can be seen as discrete equidistant points taken from its continuous analog to the well-known Wiener process $W(t)$. But for a Wiener process $W(t)$, it is known that the ruin probability is

$$\psi(u) = P\left(\max_{t\geq 0}\left[\sigma W(t) - \mu t\right] \geq U\right) = e^{-\frac{2\mu u}{\sigma^2}},$$

so there exists a Lundberg exponent $R = 2\mu/\sigma^2$ for the continuous case. For our discrete random walk with normal distribution, the following theorem applies and hence shows the existence of a Lundberg exponent R:

Theorem 1 Let F be the common (compound) distribution function of the *iid* X_i. If F is such that there exists a constant V with

$$\int_{-\infty}^{\infty} (1 - F(u)) e^{vu} du = 1, \tag{5}$$

then the Cramer-Lundberg approximation

$$\lim_{u\to\infty} e^{Ru}\psi(u) = C$$

holds for some constant C.

9.6 STOPPING TIMES AND MARTINGALES

Let us see the problem as given in Veeh (2001)

Discrete time collective risk model: We start with the gambler's ruin problem in which a gambler enters a casino with z dollars and plays a game of chance. She wins with probability p and her loss is \$1with probability $q = 1 - p$. Suppose also that the gambler will quit playing if her fortune ever reaches $a > z$ and will be forced to quit by being ruined if her fortune reaches 0. Note that in this problem, we are interested to find the probability that the gambler is ultimately ruined.

Consider a simple case $p = q = \frac{1}{2}$. Let X_j be the amount won or lost on the j^{th} play of the game. When there is no restriction on quitting this game, the fortune of the gambler after k plays of the game is given by

$$z + \sum_{j=1}^{k} X_j$$

When the actual game is played the gambler either reaches her goal or is ruined..Let T be the game on which this occurs. So we write

$$T = \inf\left(k : z + \sum_{j=1}^{k} X_j = 0 \ or \ a \right)$$

where $t =$ random time. Note that for this event $[T \le k]$ depends only on the random variables $X_1, ...X_k$. and in order to decide at time k whether or not necessary to look into the future. Such special random times are called stopping times. Thus $X_1, X_2, ...$ are random variables and T is a nonnegative integer valued random variable with the property that for each integer k the event $[T \le k]$depends only on $X_1, ...X_k$. Then T is said to be a *stopping time* (relative to the sequence $X_1\, X_2 ...$)

The random variable $z + \sum_{j=1}^{T} X_j$ gives the gambler's fortune when she leaves the casino, which is either $a \ or \ 0$. Let$\Psi(z)$ be probability that the gambler leaves the casino with 0. Then by direct computation

$$E\left[z + \sum_{j=1}^{T} X_j\right] = a(1 - \Psi(z)).$$

The ruin probability $\Psi(z)$ can be obtained by computing the expectation in another way. Since each of the random variables X_j takes values 1 and -1 with equal probability, $E[X_j] = 0$. Hence for any integer $k, E[\sum_{j=1}^{k} X_j] = 0$. So it is at least plausible that $E[\sum_{j=1}^{k} X_j] = 0$ and $E\left[z + \sum_{j=1}^{k} X_j\right] = z$. Using the above result, we get $z = a(1 - \Psi(z))$. Thus $\Psi(z) = 1 - \frac{z}{a}$ for $0 \le z \le a$ are the ruin probabilities.

To compute the ruin probabilities we rely on stopping time T and fair game feature of gambling which gives $p = q = \frac{1}{2}$. Close to the notion of a fair game is the idea behind a martingale. To appreciate the idea you need to recall that a sequence of random variable $M_0, M_1 ....M_n$ is a martingale if

$E[M_k | M_{k-1} ....M_0] = M_{k-1}$ for all $k \geq 1$. In the context of the gambling, if M_k is the gambler's fortune after k plays of a fair game, then given M_{k-1} the expected fortune after one more play is still M_{k-1}.

Example 1.

$$M_k = z + \sum_{i=1}^{k} X_j$$

With $M_0 = Z$ is a martingale. Because $X_1 ....X_{K-1}$ give the same information as that of $M_0, M_1, M_2 ...$ we have

$$E[M_k | M_0 ....M_{k-1}] = E[M_k | X_0, ...X_{k-1}]$$
$$= t + \sum_{j=1}^{k=1} X_j + E[X_k | X_0, ...X_{k-1}] = M_{k-1}$$

as $X_0, ...X_{k-1}$ are independent variable the expectation on these is zero.

Example 2.

The sequence $M_0 = z^2$ and $M_k = (z + \sum_{j=1}^{k} X_j)^2 - k$ for $k \geq 1$ is a martingale, if knowing $M_0, ...M_{k-1}$ is the same as knowing $X_1, ..., X_{k-1}$ and $X's$ are independent. To see why, write

$$M_k = (z + \sum_{j=1}^{k} X_j + X_k)^2 - k = \left(z + \sum_{j=1}^{k-1} X_j\right)^2 + 2X_k \left(z + \sum_{j=1}^{k-1} X_j\right) + X_k^2 - k$$

On taking conditional expectation and recalling that X_K is independent of other $X's$, we get $E(X_k) = 0$, $E(X_k^2) = 0$ and above results follow.

Optional Stopping Theorem:

It states that if $\{M_k\}$ is martingale and T is a stopping time, then $E[M_T] = E[M_o]$. In the gambling context, this concept shows gambling strategy T can make a fair game biased. We can obtain the duration of gambler stay in the casino using the martingale $M_0 = z^2$ and $M_K = \left(z + \sum_{j=1}^{k} X_j\right)^2 - k$ for $k \geq 1$ along with the same stopping time T. The expected value of the random variable $M_T = \left(z + \sum_{j=1}^{T} X_j\right)^2 - T$ can be computed directly as $E[M_T] = a^2(1 - \Psi(z) - E(T)$. By the optional stopping theorem, $E[M_T] = E[M_0] = z^2$. When these two expressions are compared we get, $E[T] = a^2(1 - \Psi(z)) - z^2 = az - z^2$.

This is the expected duration of the game.

To derive the appropriate martingale, we often take the help of Wald's martingale. This formulation is also known as exponential martingale. To get this, take $X_1, X_2 .....$ as independent and identically distributed random variables to define

$$W_k = \frac{e^{t\sum_{j=1}^{k} X_j}}{E\left[e^{t\sum_{j=1}^{k} X_j}\right]}$$

Note that in the denominator above we have the moment generating function of the sum evaluated at t. For each fixed t the sequence W_k is a martingale (here $W_0 = 1$). This follows if X and Y are independent and that $E\left[e^{t(X+Y)}\right] = E\left[e^{tX}\right]E\left[e^{tY}\right]$ which is Wald's martingale (or the exponential martingale) for the X sequence. The independence assumption result in

$$E\left[e^{t\left(\sum_{j=1}^{k} X_j\right)}\right] = E\left[e^{t\left(\sum_{j=1}^{k-1} X_j\right)}\right] \times E\left[e^{tX_k}\right]$$

and $\{W_k : k \geq 0\}$ is a martingale irrespective of any fixed value of t.

A non-zero value of t can be found in many cases so that the denominator of the Wald martingale is 1. Use of this particular value of t makes application of the Optional Stopping Theorem easy.

We use the above technique for analysing the collective risk model. Our objective is to derive ruin probability $\Psi(t)$. Let the insurer has initial reserve z and that premium income is collected at the rate of c per unit time. Let X_K be the claims that are payable at time k, and the X's are independent and identically distributed random variables. Then the insurers reserve at time k is $z + ck - \sum_{j=1}^{k} X_j = z + \sum_{j=1}^{k} (c - X_j)$ if t denotes the time of ruin, we can write

$$T = \min\left\{k : z + ck - \sum_{j=1}^{k} X_j 0\right\}.$$

In the above formulation if $E[c - X_j] \leq 0$, then ruin is guaranteed since premium income in each period is not adequate to balance the average amount of claims in the period. Thus, to arrive at meaningful result let us assume that $E[c - X_j] > 0$.

If we suppose there is a number τ such that $E\left[e^{\tau(c-X_j)}\right] = 1$, we will have Wald's martingale with the denominator 1. Moreover, we will get

$$M_R = e^{t((t+(k-\sum_{j=1}^{k} X_j)}$$

which is martingale. Taking expectation of M_T and using the Optional Stopping Theorem, we have $E[M_T] = E[M_0] = e^{Tt}$. Direct computation of it is, therefore,

$$E[M_T] = E\left[e^{\tau(z+cT-\sum_{j=1}^{T} X_j)} | T < \infty\right]\Psi(z).$$

$$\Psi(z) = e^{\tau z} / E\left[e^{\tau(z+cT-\sum_{j=1}^{T} X_j)} | T < \infty\right]$$

Note that if $\tau < 0$, the denominator of above fraction is greater than 1. Hence $\Psi(z) \leq e^{\tau z}$. Consequently, the ruin probability decays exponentially as the initial reserve increases. If we put $R = -\tau$ as an adjustment coefficient, then we have $\psi(z) \leq e^{-Rz}$ and a large adjustment coefficient implies that the ruin probability declines rapidly as the initial reserve increases:

9.7 COLLECTIVE RISK MODEL AND RUIN PROBLEM

We will use the gambler's ruin problem discussed above to compute the ruin probability in collective risk model. Instead of discrete time being used so far, we will take the help of continuous time t and calculate the ruin probability. The discussion that follows takes into account development of the collective risk model using martingale framework and adopts the notations as given by Veeh (2001).

To understand technical derivation there, we have discussed these separately. Let us take the claim process as $\sum_{k=1}^{N(t)} X_k$ where $X_1 X_2$.....are independent identically distributed random variables representing the sizes of the formulation $N(t)$ is number of claims up to time t which is a stock size process. Let N and X^{s} be independent and take insurance surplus as $u(t) = u + ct - \sum_{k=1}^{N(t)} X_K$ where $u > 0$ is the surplus at time $t = 0$; $c > 0 =$ rate of premium arrival per unit of time. We denote the probability of ruin with initial surplus by u. As has been done above in case of discrete time, the Wald martingale will be used with Optional Stopping Theorem to derive the information on ruin probability.

As in discrete time setting the Wald martingale will be used together with the Optional Stopping Theorem in order to obtain information about the ruin probability. See that the denominator of the Wald martingale is $E[e^{vU(t)}]$, so first step is to find a $v \neq 0$ such that

$$E\left[e^{v(ct-\sum_{k=1}^{N(t)} X_k)}\right] = 1 \text{ irrespective of the value of } t.$$

Consider the Wald martingale as the random sum $\sum_{k=1}^{N(t)} X_k$. For each moment generating function, this sum can be easily computed by conditioning on the value of the discrete random variable $N(t)$.

$$E\left[e^{v\sum_{k=1}^{N(t)} X_k}\right] = E\left[E\left[e^{v\sum_{k=1}^{N(t)} X_k} N(t)\right]\right]$$

$$= \sum_{j=0}^{\infty} E\left[e^{v\sum_{k=1}^{N(t)} X_k} N(t) = j\right] P[N(t) = j]$$

$$= \sum_{j=0}^{\infty} E\left[e^{v\sum_{k=1}^{j} X_k}\right] P[N(t) = j]$$

$$= \sum_{j=0}^{\infty} \left(E\left[e^{vX}\right]\right)^j P[N(t) = j]$$

$$= \sum_{j=0}^{\infty} e^{j\ln(E[e^{vX}])} P[N(t) = j]$$

$$= M_{N(t)} \ln(M_X(v))).$$

- Hence there exists a $v \neq 0$ so that $E\left[e^{v(ct-\sum_{k=1}^{N(t)} X_k}\right] = 1$ if and only if $e^{vct} M_{N(t)} \ln(M_X(-v))) = 1$ for all t.

If there is number $R > 0$ so that $e^{-Rct} M_{N(t)}(\ln(M_X(R))) = 1$ for all t, R is called the adjustment coefficient. Using $-R$ as the value of v in Wald's martingale we have

$$W_t = e^{-R\left(u+ct-\sum_{k=1}^{N(t)} X_k\right)} \text{ is a martingale.}$$

Define a stopping time T_a and $T_a = \inf\{s : u + cs - \sum_{k=1}^{N(s)} X_k \leq 0 \ \ or \geq a\}$ where a is an arbitrary fixed positive number. Intuitively, T_a is stopping time. By the Optional Stopping Theorem, $E[W_{T_a}] = e^{-Ru}$. which gives

$$E[W_{Ta}] = E[e^{-R(u+cT_a-\sum_{k=1}^{N(T_a)} X_k)} \mid u + cT_a - \sum_{k=1}^{N(T_a)} X_k \leq 0] P[u + cT_a - \sum_{k=1}^{N(T_a)} X_k \leq 0]$$

$$+ E[e^{-R(u+ct-\sum_{k=1}^{N(T)} X_k)} \mid u + cT_a - \sum_{k=1}^{N(T_a)} X_K \geq a] P[u + cT_a - \sum_{k=1}^{N(T_a)} X_k \geq a]$$

Since this equation is valid for any fixed positive a, and as $R > 0$, we can set limits as $a \to \infty$. Since $\lim_{a\to\infty} P[u + cT_a - \sum_{k=1}^{N(T_a)} X_k) \leq 0 = \Psi(u)$ and $\lim_{a\to\infty} e^{-Ra} = 0$ the following result is obtained.

Theorem.

Suppose that in the collective risk model the adjustment coefficient $R > 0$ satisfies $e^{-Rct} M_{N(t)}(\ln(M_x(R))) = 1$ for all t. Let $T \inf\{s : u + cs - \sum_{k=1}^{N(s)} X_k \leq 0\}$ be the random time at which ruin occurs. Then

$$\Psi(u) = \frac{l^{Ru}}{E[e^{R(u+cT-\sum_{k=1}^{N(T)} X_k} \mid T < \infty]} \leq e^{-Ru}.$$

As in the case of discrete time model, the existence of an adjustment coefficient guarantees that the ruin probability decreases exponentially as the initial surplus u increases. To see how it works, we consider a Poisson process with constant intensity $\lambda > 0$ and suppose $W_1, W_2, \ldots$ are independent identically distributed exponential random variables with mean $\frac{1}{\lambda}$ and common density $\lambda e^{-\lambda x} 1_{(0,\infty)}(x)$. Let $W's$ be the waiting times between claims. The Poisson process can then be viewed as the number of claims that arrive up to time t. This means that $N(t) = \inf\{k : \sum_{j=1}^{k+1} W_j > t\}$ and it can be shown that for any fixed t the random variable, $N(t)$ has independent increments. Thus, whenever $t_1 < t_2 < \ldots < t_n$ are fixed real numbers, the random variables $N(t_2) - N(t_1) \ldots N(t_n) - N(t_{n-1})$ are independent. Using this, we can get

$$E[e^{vN(t)}]$$

$$= \sum_{j=0}^{\infty} e^{vj} P\left[N(t) = j\right]$$

$$= \sum_{j=0}^{\infty} h^{e^{vj}} \; e^{-\lambda t} (\lambda t)^j / j!$$

$$= e^{-\lambda t} \sum_{j=0}^{\infty} \left(e^{v} \lambda t\right)^j / j!$$

$$= e^{\lambda t} e^{v} - 1$$

From the moment generating function of $N(t)$, we derive a simple formula for the adjustment coefficient in this case. The general equation for the adjustment coefficient was earlier found to be $e^{-Rct} \mathrm{M}_{N(t)}(\ln(M_X(R))) = 1$. Taking logarithms and using the form of the moment generating function of $N(t)$ shows that the adjustment coefficient is the positive solution of the equation

$$\lambda + cR = \lambda M_X(R).$$

If we take $N(t)$ as a Poisson process, more specific results can be generated. To do this define a stopping time $T_{u\lambda} = \inf\{s : U(s) < u\}$ to be the first time that the surplus falls below its initial level and denote by $L_1 = u - U(T_u)$, the amount by which the surplus falls below its initial level. Then

$$P[T_u < \infty, L_1 \geq y] = \frac{1}{(1+\theta)E[X]} \int_y^{\infty} (1 - F_x(x))dx.$$

There are two useful consequences of the above formula. First, by taking $y = 0$, the probability that the surplus ever drops below its initial level is $1/(1+\theta)$. Second, an explicit formula for the size of the drop below the initial level is obtained as

$$P[L_1 \leq y \mid T_u < \infty] = \frac{1}{E[X]} \int_0^{y} (1 - F_X(x))dx$$

which can be evaluated in specific cases.

Consider a random variable L, which represents the maximum aggregate loss, and is defined by

$$L = \max_{t\geq 0}\left\{\sum_{k=1}^{N(t)} X_k - ct\right\}.$$

Note that $P[L \leq u] = 1 - \Psi(u)$ from which the distribution of L has discontinuity at the origin of size $1-\Psi(0) = \theta/(1+\theta)$, and is continuous otherwise. In fact a reasonably explicit formula for the moment generating function of L can be obtained.

If $N(t)$ is a Poisson process and $L = \max_{t\geq o}\left\{\sum_{k=1}^{N(t)} X_k - ct\right\}$, then

$$M_L(v) = \frac{\theta E[X]v}{1+(1+\theta)E[X]v - M_X(v)}.$$

This formula for the moment generating function of L can sometimes be used to find an explicit formula for the distribution function of L, and hence $\Psi(u) = 1 - P[L \leq u]$.

Example. Suppose X is exponential with mean 1 and that $\theta = 1$. Then substitution gives

$$M_L(t) = \frac{1}{2} + \frac{1}{2}\frac{1}{1-2t},$$

after using long division (or partial fractions). The second term is half of the moment generating function of an exponential random variable with mean $\frac{1}{2}$, while the first term is half the moment generating function of a random variable that is degenerate at zero. The ruin probability is therefore $\Psi(u) = P[L > u] = e^{-2u}/2$ for $u > 0$. Notice that $\Psi(0) = \frac{1}{2}$.

The analysis of random sums of the form $\sum_{j=1}^{N} X_j$ in which the $X's$ are independent and identically distributed has played a key role in the preceding analysis. The distribution of such a sum is called a compound distribution, with N as the compounding variable and X as the compounded variable. The conditioning method used earlier shows that the moment generating function of such a sum is $M_N(\ln M_x(t)$.

The first two moments are $E\left[\sum_{j=1}^{N} X_j\right] = E[N]E[X]$ and

$$Var\left(\sum_{j=1}^{N} X_j\right) = E[N]Var(X) + (E[X])^2 Var(N).$$

A random variable S has the compound Poisson distribution with Poisson parameter λ and mixing distribution $F(x)$, denoted $S = CP(\lambda, F)$, if S has the same distribution as $\sum_{j=1}^{N} X_j$ where $X_1, X_2, \ldots$ are independent identically distributed random variables with common distribution function F and N is random variable which is independent of the $X's$ and has a Poisson distribution with parameter λ.

Example. For each fixed t, the aggregate claims process $CP(\lambda_t, F_t)$.

Example. If S has the $CP(\lambda, F)$ distribution then the moment generating function of S is

$$M_S(v) = \exp\left\{\lambda \int_{-\infty}^{\infty} (e^{uv} - 1) dF(u)\right\}$$

This follows from the earlier general derivation of the moment generating function of a random sum.

Suppose that S has the $CP(\lambda, F)$ distribution and T has the $CP(\delta, G)$ distribution and that S and T are independent. Then $S+T$ has the

$$CP(\lambda+\delta, \frac{\lambda}{\lambda+\delta}F + \frac{\delta}{\lambda+\delta}G) \text{ distribution.}$$

This property is very useful in the insurance and results of the analysis of different policy types can be easily combined into one grand analysis of the company's prospects as a whole. A compound Poisson distribution can also be decomposed.

Example. Suppose each claim is either for \$1 or \$2, each event having probability 0.5. If the number of claims is Poisson with parameter λ then the amount of total claims, S, is compound Poisson distributed with moment generating function

$$M_S(v) = \exp\{0.5\lambda(e^{v} - 1) + 0.5\lambda(e^{2v} - 1)\}.$$

Hence S has the same distribution as $Y_1 + 2Y_2$ where Y_1 and Y_2 are independent Poisson random variable with mean $\lambda/2$. Thus the number of claims of each size are independent.

Example. The collective risk model can be used as an approximation to the individual risk model. In the individual risk a product often represents model the claims amount $B_j X_j$ in which B is a Bernoulli random variable which represent whether a claim is paid or not and X is the amount of the claim. Then

$$BX = \sum_{j=1}^{B} X'_j \approx \sum_{j=1}^{N} X'_j$$

where N has a Poisson distribution with parameter $P[B=1]$ and $X'_1, X'_2, \ldots$ are independent random variable each having the same distribution as X. Thus the distribution of BX may be approximated by the $CP(P[B=1], F_X)$ distribution.

The analysis of the aggregate loss random variable L introduced the idea of mixing. A mixture of distributions often arises when the outcome of an experiment is the result of a two-step process.

9.7.1 Collective Model of Risk Theory: A Discussion

- The following discussion is taken from Mackova, Monika (2005).

The collective risk model is a stochastic model that uses losses as random variables in certain portfolio of insurance policies.

The basic mathematical assumptions of the model are

- The number of individual claims (losses) within a certain portfolio of insurance policies during a certain period of time, denoted as N, is a non-negative, integer valued random variable, called frequency.
- The individual claims (losses) occurring during this period, denoted as X_i, for $i = 1, \ldots, N$, are positive random variables, stochastically independent, identically distributed,
- The individual claims (losses) $X_1, \ldots, X_N$ are independent also from frequency N.
- If $N = 0$ then $X_1 + X_2 + \ldots + X_N = 0$.

Then the random variable

$$S = \sum_{i=1}^{N} X_i .$$

is called compound variable and describes the aggregate claim or aggregate loss for the some fixed period $(0, T]$ under consideration (mostly $T = 1$ year) for a (collective) insurance contract.

We do not restrict assumption only to the discrete random variable X. By extending to continuous variable, we can make a rescaling. Thus, we work with $E[S] = E[E[S/N]] = E[p_1 N] = p_1 E[N],\ for\ p_1 = E[X]$, and

$$Var[S] = E[Var(S/N)] + Var(E[S/N]) = E[NVarX] + Var(p_1 N)$$

$$= E[N]VarX + p_1^2 VarN$$

That is, the expected value of aggregate claim depends on expected value of individual claim and the expected number of individual claims; a similar logic applies to the variance.

It may be useful to note that the period under consideration is mostly one year. We suppose that distribution of individual claims is not changing within this period of time. Therefore, the model is much more used for non-life insurance contract.

A frequently used distribution of X is Gamma, log-Gamma or log-normal distribution. In case of N, the distribution are assumed to be N Poisson, binomial, negative binomial.

Let us look at the notations used.

- If the probability distributions for the claims (losses) $X_i, i = 1, .., N$ are continuous with a density function (df) then f gives cumulative distribution function $(cdf)\, F$, which is given by $F(x) = \int_0^x f(u)\,du, x \geq 0$.

  The corresponding survival function (sf) is given by $\bar{F}(x) = 1 - F(x) = \int_x^{\infty} f(u)\,du, x \geq 0$.

- If the probability distribution for the claims (losses) $X_i\, i = 1,..,N$ are discrete with a probability function $\{f_j, j \in N s.t. f_j = f(j): p(X_i = j)\}$, then, a cumulative distribution function $(cdf) F$ is given by $F(x) = P(X_i \leq x)$ *for* $x \geq 0$, and 0 otherwise. The corresponding survival function (sf) is given by $\bar{F}(x) = 1 - F(x)$.
- The cdf of the aggregate claim (loss) G is given by $G(x) = P(S \leq x)$ with a density function g or probability function $\{g_k, k \in N\ s.t. g_k = g(k) = P(S = k)\}$.
- The compounding probability function of N is denoted as $\{p_n, n \in N_0 s.t. p_n = p(N = n)\}$.

Discretisation or Scaling

When the distribution function of random variable X is continuous we often use discretisation for numerical simulation (convolution, Panjer recursion, fast Fourier transform, etc.).

Let us write

$$X_\Delta = \left\lceil \frac{X}{\Delta} \right\rceil = \min\{k \in N \mid k\Delta \geq X\}$$

with $\Delta > 0$ and with probabilities

$$P(X_\Delta = k) = P\left(\left\lceil \frac{X}{\Delta} \right\rceil = k\right) = P\left(k - 1 < \frac{X}{\Delta} < k\right) = F\left(k\Delta - F\left(k-1\right)\Delta\right), k \in N$$

Then the discretised aggregate claim loss S_Δ has the probability generating function $\varphi_{S_\Delta}(s) = \varphi_N\left(\varphi_{S_\Delta}(s)\right)$, $|s| \leq 1$.

Convolution Method

To compute the aggregate claim distribution in the collective risk model we introduce recursive methods – the convolution principle.

As in the collective risk model we are looking for the compound distribution function $G(x)$ of random variable S, the distribution of aggregate claim losses, S for any discrete distribution of individual losses, r.v. X can be written as:

$$G(x) = P\left(\bigcup_{n=0}^{\infty}(S \leq x \wedge N = n)\right) = \sum_{n=0}^{\infty} P(S \leq x \wedge N = n)$$

$$= \sum_{n=0}^{\infty} P(S \leq x / N = n).P(N = n)$$

$$= \sum_{n=0}^{\infty} P(X_1 + X_2 + ... + X_n \leq x).P(N = n) = \sum_{n=0}^{\infty} P(N = n).F^{*n}(x),$$

where F^{*n} denotes the n-fold convolution of F. So,

$$F^{*n}(x) = P(S \le x) = P(X_1 + ... + X_n \le x).$$

Denote: $f^{*n}(x) = F^{*n}(x) - F^{*n}(x-1)$.

For the continuous random variable X, we use integral version of convolution.

Step 1:

In practice we often assume that the individual claims have positive integer value. So, for the discrete *r.v.* X we evaluate following expressions:

$$g_{k=0} = P(S=0) = P(N=0) = p_0$$

$$g_k = P(S=k) = \sum_{n=1}^{\infty} P(N=n).f^{*n}(k) \quad \text{k=1, 2,...}$$

If the value of each claim is greater then 1, we decrease the upper bound of the sum operator to level k:

$$g_{k=0} = P(S=0) = p(N=0) = p_0$$

$$g_k = P(S=k) = \sum_{n=1}^{k} p(N=n).f^{*n}(k) \quad \text{k=1, 2,...}$$

The computation difficulty of the modified formula is confined to only k^2.

Step 2:

Consider the discrete *r.v.* X_i, i=1,..., N, which reaches the values x_i with probability function $f_j = P(X_j = x_j)$, as probability generating function

$$\varphi(Z) = \sum_{j=1}^{\infty} P(X_j = x_j).z^{x_j} = f_1 z^{x_1} + f_2 z^{x_2} + ...$$

Then it holds for $\varphi^n(z) = (f_1 z^{x_1} + f_2 z^{x_2} + ...)^n$

If *r.v.* X_i are *i.i.d.* then the polynomial term $(f^j.z^{xj})^n$ of probability generating function implies that by the occurrence of n losses the aggregate claim value will be $(z^{xj})^n$ with the probability $(f^j)^n$. Then the probability of aggregate claims value reaches x_j which we derive in following steps:

1) exponentiate the probability generating function (with power from1 to x_j),
2) sum the exponent coefficient of term x_j, and
3) obtain following formula:

$$g_{x_j=0} = P(S=0) = P(N=0) = p_0$$

$$g_{x_j} = P(S = x_j) = \sum_{n=1}^{x_j} f_j^n \quad x_j = 1, 2, ...$$

Panjer Recursion

Panjer presented results for some specific distributions, when the claim amount is discrete and the recursive definition can be used to compute the aggregate claims distribution without the use of convolution.

Panjer Theorem

Consider the family of claim number distribution satisfying the recursion:

$$p_n = P(N=n) = (a + b/n)p_{n-1} \quad n = 1,2,\ldots \quad a,b \in R$$

where p_n denotes the probability that exactly n claim occurs in the fixed time interval. Consider the compound distribution function

$$G(x) = \sum_{n=1}^{\infty} p_n F^{*n}(x) \quad x > 0$$
$$= p_0 \quad x = 0$$

for arbitrary claim amount distribution $F(x)$, $x > 0$. Consider the continuous function (corresponding results will be given for the discrete case as well). Then the density of total claim is:

$$g_k = \sum_{n=1}^{\infty} p_n f_k^{*n} \quad k > 0$$
$$g_{k=0} = p_0 \quad k = 0$$

when $f(x)$ is the density associated with $F(x)$. Then the following recursion holds:

$$g_k = p_1 f_k + \int_0^k (a + by/k).f_y^{*}.g_{k-y}dy \quad k > 0$$

If the claim amount distribution is discrete and defined on the positive integers, the corresponding recursive definition of total claims is:

$$g_0 = P(S=0) = P(N=0) = p_0 \quad k = 0$$
$$g_k = P(S=k) = \sum_{j=1}^{k}\left(a + \frac{bj}{k}\right).f_j.g_{k-j} \quad k = 1,2,\ldots$$

The formula for computing g_k is only k. In comparison to convolution, the computation the modified convolution formula is equal to k^2. Therefore the Panjer recursion formula is much more efficient.

Members of the family of claim number distribution satisfying the recursion:

$$p_n = P(N=n) = (a + b/n)p_{n-1} \quad n = 1,2,\ldots \quad a,b \in R$$

are Poisson distribution, binomial distribution, negative binomial distribution and geometric distribution (as a special type of negative binomial).

Let us drive the recursive definition of the density of total claims:

- If random variable N is a Poisson distribution, then:

$$g_0 = e^{-\lambda}, g_k = \frac{\lambda}{k}\sum_{j=1}^{k} j.f_j.g_{k-j} \qquad k = 1,2,...$$

- If random variable N is a binomial distribution, then

$$g_0 = (1-p)^N, g_k = \frac{p}{1-p}\sum_{j=1}^{k}\left[(N+1)\frac{j}{k}-1\right].f_j.g_{k-j} \qquad k = 1,2,...$$

- If random variable N is a negative binomial distribution, then

$$g_0 = (1-p)^{\alpha}, g_k = p\sum_{j=1}^{k}\left[(\alpha-1)\frac{j}{k}+1\right].f_j.g_{k-j} \qquad k = 1,2,..$$

- If random variable N is a geometric distribution, then

$$g_0 = 1-p, g_k = p\sum_{j=1}^{k} f_j.g_{k-j} \qquad k = 1,2,...$$

Probability of Ruin and Panjer Recursion

In the following we will evaluate upper and lower bound of the probability of ruin under assumption of the classical risk model.

The following discussion illustrate the typical behavior of the insurer's surplus process:

The Surplus Process

Our interest in ruin theory is in the ruin probability for the insurer, i.e., the probability that at some point in time the claims exceed the premium income.

We formulate this by considering the following three elements:

- the initial surplus of the insurer – a fixed amount,
- the income flow of the premium – in the simplest case we consider here, aggregate premium income is a linear function of time t,
- the outflow of aggregate claims amount.

The insurer's surplus at time t is:

$$R(t) = R_0 + P(t) - S(t)$$

where $R(t)$ denotes the insurer's surplus at time t, R_0 represents the initial insurer's surplus at time $t = 0$; $P(t)$ represents insurer's premium income. Assume deterministic premium income $P(t) = P.t$, where P is a constant at time t.

Hence, we have $R(t) = R_0 + P.t - S(t)$

Let $N(t)$ be the number of claim that has occurred up to time t. Then $S(t)$ denotes the aggregate claims up to time t, $S(t) = X_1 + X_2 + ... + X_{N_t}$. Random variable X_i denotes the individual claim (loss) at the time t, $t_i = T_1 + T_2 + ... + T_i$. Individual claims amounts have a continuous distribution with distribution function $P(x)$ and the mean (μ).

Since $\{N(t):t\geq 0\}$ and $\{S(t):t\geq 0\}$ are stochastic processes, i.e., families of random variables, for fixed t, N(t) and S(t) we have exactly the set-up of a collective risk model.

Note that the process $\{S(t):t\geq 0\}$ is called the aggregate claims process. It is in fact a compound Poisson process with parameter λ. When we have fixed t, S(t) has a compound Poisson distribution.

The insurer's premium income per unit time to cover the risk is $P=(1+\theta)\lambda\mu$,

where θ is a relative security loading factor, λ is Poisson parameter and μ is the mean of individual claims. The classical risk model uses equality $\psi(R_0)=1/(1+\theta)$.

The probability of ruin is:

$$\psi(R_0)=p\left[\min R(t)<0,t\geq 0\right]$$

When the surplus become negative for the first time it does not mean that the company becomes bankrupt. The initial capital is interpreted as the capital the company is willing to risk. If ruin occurs at this point, the company has to take action so that the business becomes profitable.

We may also define the surplus process in terms of the aggregate loss process $\{L(t)\}$, where

$$L=\max_{t\geq 0}\{S(t)-Pt\}$$

In this formulation, for a fixed value of t, $L(t)$ represents the accumulated loss up to time t.

For $R_0\geq 0$ to holds,

$$\begin{aligned}1-\psi(R_0)&=P\left[R(t)\geq 0,\forall t\right]\\&=P\left[R_0+Pt-S(t)\geq 0,\forall t\right]\\&=P\left[S(t)-Pt\leq R_0,\forall t\right]\end{aligned}$$

Hence left side of the equation is equivalent to $P\left[L\leq R_0\right]$ and we get

$$1-\psi(R_0)=p\left[L\leq R_0\right]$$

The term on the left side $1-\psi(R_0)$ can be interpreted as a distribution function of random variable L.

Ruin occurs on the first occasion so that the aggregate loss exceeds R_0. Hence, the variable N denotes the number of occasion that the aggregate loss process reaches a new record value. In this context N has geometric distribution with probability function:

$$\begin{aligned}p(N=n)&=\left[1-\psi(0)\right]\left[\psi(0)\right]^n\\&=\theta\left(\frac{1}{1+\theta}\right)^{n+1}\qquad n=0,1,2,\ldots\end{aligned}$$

The distribution function of i.i.d. random variables X_i is independent of N has the distribution function:

$$F(x)=\int_0^x \frac{1-p(y)}{\mu}dy$$

Next, we can use Panjer recursion, by replacing $F(x)$ by a discrete function,

$$p[X_1+X_2+...+X_N \le x]=F^{*n},$$

$$F^{0*}=1 \, for \quad x \ge 0, \text{ otherwise } 0.$$

Hence N has geometric distribution and we get the relationship for $\psi(R_0)$:

$$1-\psi(R_0)=\sum_{n=0}^{\infty}\psi(0)^n(1-\psi(0)).F^{n*}(R_0).$$

Now define two discrete distribution which have the probability functions $l(x) \, and \, u(x)$. Taking the relevant distribution function $L(x)$ and $U(x)$ as:

$$u(x)F(x+h)-F(x) \quad x=0,h,2h,...,$$
$$l(x)=F(x)-F(x-h) \quad x=h,2h,3h,...$$

for a very small h, near zero.

From these definitions we have $L(x)\le F(x)\le U(x)$

Then $L^{n*}(x)\le F^{n*}(x)\le U^{n*}(x)$,

where L^{*n} and U^{*n} denote the n- fold convolution. Hence,

$$1-\sum_{n=0}^{\infty}\psi(0)^n(1-\psi(0))U^{n*}\le\psi(R_0)\le 1-\sum_{n=0}^{\infty}\psi(0)^n(1-\psi(0))L^{n*}.$$

Since $L(x)$ and $U(x)$ are discrete function (discretisation of $F(x)$), we apply Panjer recursion and derive the lower and the upper bounds for the ruin probability ψ, i.e., evaluate values for $\psi(jh)$, $j=0,1,...$

Check Your Progress 2

1) What is Cramer-Lundberg approximation in classical risk theory?

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2) How do you explain the individual risk in insurance with gambler's ruin problem?

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3) Why martingale is used while modelling individual risk in life insurance?

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4) What framework you will use to model the collective risk in life insurance?

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5) How convolution method is helpful in collective risk modelling?

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6) At which point of collective risk modelling you will make use of Panjer recursion.

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9.8 LET US SUM UP

In this unit we have discussed the insurance companies' operational tools for dealing with individual and collective risks. The probability distribution of claims has been developed, which help company's decision-making process to honour its commitments. The individual risk model considers a specific portfolio of insurance policies. The premium collection under such a model is discussed by introducing normal distribution. Necessary formulae for premium setting under expected value of total claims have been highlighted. This line of discussion has been extended to normal and gamma approximations as well as to the class of distributions such as compound Poisson and negative binomial. Classical gambler's ruin probability problem is discussed and risk modelling with martingales has been included. From individual risk model, the analysis is broadened for collective risk model. The collective risk model uses losses as random variables in certain portfolio of insurance polices. Under this theme, we have covered the probability distribution of aggregate claims and compound the ruin probability.

9.9 KEY WORDS

Conditioning: A measure on problem's amenability to digital computation. For example, the condition number associated with the linear equation $Ax = b$ gives a bound indicating degree of the inaccurate the solution given by x. Also, you may think of the condition number as an approximate rate at which the solution, x, will change with respect to a change in b. That is, if the condition number is large, a small error in b may cause a large error in x. On the contrary, with a small condition number, the error in x will not be much bigger than the error in b.

Convolution: A mathematical operator which takes two functions f and g and produces a third function that in a sense represents the amount of develop between f and a reversed and translated version of g. A convolution is a kind of very general moving average as one can see by taking one of the functions to be an indicator function of an interval.

Discretisation: The process of transferring continuous models and equations into discrete counterparts. This process usually carried out as a first step toward making them suitable for numerical evaluation.

Net Level Premium Reserves (or benefit reserves): A premium value of the reserve such that at time zero is it is equal to zero. The net level premium reserve is found by taking the expected value of the loss random variable.

Recursion: A method of defining functions in which the function being defined may be used within its own definition.

Scaling: A linear transformation that enlarges or diminishes objects.

9.10 SOME USEFUL BOOKS

Bowers, Gerber, Hickman, Jones Nesbit (1986), Actuarial Mathematics, The Society of Actuaries, Illinois

Mathias Winkel (2003), Actuarial Science University of Oxford (see Internet)

Mackova, Monika (2005), Selected Recursive Methods of Collective Risk Insurance – Convolution vs. Panjer Recursion, University of Vienna (see Internet)

Trowbridge, C.L (1989), Fundamental concepts of Actuarial science, Actuarial Education and Research Fund (see internet)

Veeh, Jerry Alan (2001), Lecture Notes on Actuarial Mathematics (see Internet)

9.11 ANSWER OR HINTS TO CHECK YOUR PROGRESS

Check Your Progress 1

1) See Section 9.2 and answer.
2) See Section 9.2 and answer.
3) See Section 9.3 and answer.
4) See Section 9.4 and answer.
5) See Section 9.4.3 and answer.

Check Your Progress 2

1) See Section 9.5 and answer.
2) See Section 9.6 and answer.
3) See Section 9.6 and answer.
4) See Section 9.7 and answer.
5) See Section 9.7 and answer.
6) See Section 9.7 and answer.

UNIT 10 PRICING INSURANCE

Structure

10.0 OBJECTIVES

After going through this unit, you will be able to:

- understand the basics of insurance pricing from the perspective finance-theory; and
- apply the non-arbitrage pricing principles to insurance.

10.1 INTRODUCTION

We have seen in Block 2 risks involved in financial and insurance markets can be covered jointly. However, these two markets differ in terms of their features. Whereas insurance market is a warehousing of risk, financial market is an intermediation of risk. Consequently, fixation of price in each differs. In general, insurance pricing is based on classical theory of risk while that of finance relies on non-arbitrage pricing.

Note that the insurance market operates mostly with conditions that are neither liquid nor efficient. In contrast, non-arbitrage pricing is based on efficient market conditions as well as with good deal of liquidity. Despite these differences a part of the risk in insurance can be transferred to the financial markets or even insurance risk can be traded. The following discussion is based on the work of Kull (2002). We have used the equations as well as interpretations of the work. In view of the overall introductory scope of the will be useful to see the entire discussion of the paper as your reference reading.

10.2 BASICS OF INSURANCE PRICING

We know that pricing of insurance contracts is based on the principle of equivalence. It considers the probabilities of loss events. Thus, in one period case, the premium for an insurance contract covering losses X is

$$\text{premimum} = \frac{1}{1+r_t} E^P[X] + S[X] \tag{1.1}$$

where $E^P[X]$ is the loss expectation calculated under the real probability measure P, discounted with a risk-discount rate r_t chosen according to actuarial judgement. $S[X]$ is the safety loading or risk premium. The choice of r_t and $S[X]$ is based on the overall risk related to the contract, the risk-free interest rate, the cost of capital, the expected investment return, market conditions etc.

10.3 FINANCE PRICING

In finance, the non-arbitrage pricing principle is adhered to which shifts away from the real probability measure P to equivalent martingale measures Q. Valuation of a future (stochastic) cash flow F under the equivalent martingale measure Q can be written as

$$price = \frac{1}{1+r_{rf}} E^Q[F] \tag{1.2}$$

where $E^Q[\cdot]$ denotes the expectation operator under the measure Q. See that this pricing again takes into account the equivalence principle. However, it corresponds to the equivalent martingale measure Q instead of probability. Under this changed formulation, discounting is performed with the directly observable risk-free interest rate r_{rf} and there is no modification due to a loading. A useful feature of (1.2) is that with the knowledge of equivalent martingale measure Q, the valuation of the cash flow F can be undertaken without taking recourse to subjective criteria. And a non-arbitrage pricing can be justified due to the existence of a hedge portfolio that creates an overall risk-less position. A hedge portfolio in turn relies on prevalence of liquid and efficient market.

We may note the main difference between classical risk theory and the non-arbitrage approach to pricing: the non-arbitrage approach substitutes real probabilities and knowledge for 'preference-free' probabilities that comply with the assumptions of non-arbitrage. What is common in both the pricing approaches is to find the corresponding probabilities.

In the following discussion we will focus on the procedure of insurance contracts. When this is done, we must remember that the performance of martingale based price will have to depend on financial markets. Since the insurance market is incomplete, equivalent martingale probabilities cannot be uniquely defined. Moreover, since the return distribution of insurance contract is fundamentally different from that of asset returns ('heavy tails'), we have to resolve the problem that will be encountered from martingale based pricing. Thus we cannot apply the standard financial techniques as is done by the Black-Scholes and Merton. The information theoretical maximum entropy principle will have to be relied on while choosing a particular equivalent martingale measure for pricing the insurance products.

The role of investment opportunities for insurance operations will be crucial while working for a solution. It will be important to decide between risk free investment and a risky investment of the insurance premiums received. We will show that these are connected to the Esscher measure of premium calculation principle.

10.4 INTEGRATION OF INSURANCE AND FIANCE PRICING

To bride up the gap between financial and insurance pricing, we take the help of modern finance and information theories. The concepts of non-arbitrage pricing and equivalent martingale probabilities are important. Note that non-arbitrage pricing relies on the insight that, in the absence of arbitrage opportunities, the price (or premium) of some contingent claim should match the price for a position perfectly hedging its risk. As has first been shown by Cox and Ross (1976) and Harrison and Kreps (1979), this insight can be reformulated mathematically in terms of 'equivalent martingale probabilities', i.e., in terms of a probability measure Q satisfying for a given random process X

$$E^{Q}[X_T] = X_t \qquad (2.1)$$

The Equation (2.1) shows that the best forecast of future values X_t at $T > t$ is the observed value X.

Two considerations are important when pricing contingent claims in the non-arbitrage framework: First, equivalent martingale probabilities substitute the 'real' probabilities P that can be derived from historical data. It is, again necessary to keep in mind that equivalent martingale probabilities emerge as a consequence of the non-arbitrage assumption which are 'artificial' as they do not need to correspond to any real probabilities or beliefs. Second, it is important to note that there exists a unique equivalent martingale measure for complete and efficient markets. In incomplete markets, equivalent martingale measures are not uniquely defined and we cannot hedge contingent claims fully.

In the presence of incomplete insurance markets, you can use the basic concepts of information theory. A particular equivalent martingale measure Q has to be selected from the infinite possibilities to provide a measure of the information on equivalent martingale measure Q. In the absence of information other than the non-arbitrage assumption, the rationale is to choose the distribution that embodies least additional information. In a discrete setting, we have to use the method of Information theoretical maximum Entropy Principle and formulate

$$S = -\sum_i q_i . \ln[q_i] \qquad (2.2)$$

where the q_i's are the probabilities associated with a particular equivalent martingale measure Q.

10.4.1 Non-Arbitrage Constraint

Take a simplest form of insurance such that: An insurer accepts the liability to pay for the compound loss

$$X = \sum_{j=1}^{N} L_j \tag{3.1}$$

occurring over a period [0,T] where L_j is the (random) jth claim amount during the time period and N is the random number of claims. Moreover, we assume that the claim amounts L_j are independently and identically distributed and $L_j \equiv L$ refers to the claim distribution without deductible or limit. In exchange for this liability, the insurer receives a premium b. It is assumed that there are no costs or investment returns and payments made at time T only. Then, the premium b can be interpreted as a risky asset generating one-period returns

$$R = \frac{b-X}{b} \tag{3.2}$$

where R is insurance related return. It depends on b. So we will have realisations X_i an r_i of the compound loss X and return R respectively.

As the non-arbitrage theorem is developed in different ways, Kull (2005) presents an intuitive formulation of it in a discrete setting. Let there be different states i of the world, each of which is characterised by a set of payoffs. These payoffs originate from assets, (in our case from the premium b or a risk-free bond). To formulate the non-arbitrage condition, realisations $l_{j,i}$ and n_i of the single claim amount L and the claim number N are considered. These can be translated by (3.1) and (3.2) into realisations x_i and r_i of the compound loss X and R respectively. In addition, a risk-free bond with return r_{rf} is considered. Let us take $(1+r_i)$ and $(1+r_{rf})$ as the insurance and bond related payoffs in the ith state. When the payoffs are grouped in a matrix $D \in \Re^{2.k}$ where K is the total number of states, we have

$$D = \begin{bmatrix} (1+r_{rf})(1+r_{rf}) \wedge (1+r_{rf}) \\ (1+r_1)(1+r_2) \wedge (1+r_k) \end{bmatrix} \tag{3.3}$$

Then the non-arbitrage theorem states that if there is no arbitrage, positive constants ψ_i exist such that

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} (1+r_{rf})(1+r_{rf}) \wedge (1+r_{rf}) \\ (1+r_{rf})(1+r_2) \wedge (1+r_k) \end{bmatrix} \cdot \begin{bmatrix} \psi_1 \\ \psi_2 \\ M \\ \psi_k \end{bmatrix} \tag{3.4}$$

holds. With $q_i = \psi_1 \cdot (1+r_{rf})$ relation (3.4) becomes

$$\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}1 & 1 & L & 1\\ \frac{(1+r_1)}{(1+r_{rf})} & \frac{(1+r_2)}{(1+r_{rf})} & L & \frac{(1+r_k)}{(1+r_{rf})}\end{bmatrix}\cdot\begin{bmatrix}q_1\\q_2\\M\\q_k\end{bmatrix} \quad (3.5)$$

From (3.5) it is evident that the quantities q_i can be interpreted as probabilities (see that first component q_i's sum up to one). The q_i's are interpreted as 'risk-adjusted' or 'risk-neutral' probabilities. From (3.5) it follows that under the probability measure Q we have

$$E^Q\left[\frac{1+R}{1+r_{rf}}\right]=\frac{1}{1+r_{rf}}E^Q\left[\frac{X-b}{b}+1\right]=1 \quad (3.6)$$

So under the measure $Q, (1+R)/(1+r_{rf})$ *is* a martingale and the q_i's are equivalent martingale probabilities.

Given the probabilities q_i, you can have the arbitrage-free pricing of an arbitrary insurance contract referring to the losses L and the claim number N. Thus

$$\text{non-arbitrage premium} = \frac{1}{1+r_{rf}}E^Q\left[f(L,N)\right] \quad (3.7)$$

where *f(L,N)* = loss amount to be paid.

Note that (3.7) is identical to (1.2) and the relationship shows that non-arbitrage premium value is calculated under the equivalent martingale measure Q, discounted by the risk-free interest rate r_{rf}. Since the risk adjustment is internalised by the change of the probability measure we have not added the risk premium. Thus pricing any insurance contract whose loss can be written as *f(L,N), which is* equal to the equivalent martingale probabilities q_1.

In complete and efficient markets, the non-arbitrage premium contained in (3.7) coincides with the overall risk-less position. In incomplete insurance markets, on the other hand, the situation is more complex. It can be shown that no unique and perfect hedging portfolio exists. For this reason, the equivalent martingale probabilities q_i cannot be uniquely defined. A major task before us, therefore, is to define the martingale probabilities. In other words, we most have a particular equivalent martingale measure Q chosen out of the infinitely many possibilities.

10.4.2 Equivalent Martingale Measure

The insurance market is neither liquid effective nor complete. As a result, pricing contingent claims by their replication cost is not possible. It implies that no unique equivalent martingale measure exists. There are also practical difficulties in this approach since the loss distribution of L is fundamentally different from that of asset returns. In particular, the loss distribution L may often show heavy tails, i.e., relatively high probabilities of large losses. For example, consider a straightforward application of standard financial techniques like the Black-Scholes-Merton which implicitly relies on complete markets and log-normally distributed random variables. In view of this, we need to identify the properties of non-arbitrage pricing that can be applied to insurance market.

The non-arbitrage assumption in (3.5) above gives the equivalent martingale probabilities q_i which can be reduced to a linear system of two equations and K unknowns q_i. Then we write

$$s = \frac{1}{1+r_o} D * q \tag{4.1}$$

For given $S = [1,1]^T$ and D (specified by K realisations r_i), q is not determined uniquely. On the other hand, there exist an infinite number of solutions for q_i reflecting market incompleteness.

The Equation (4.1) above is known as the 'Inverse Problem'. This problem can be dealt with a number of methods. We discuss one provided by the maximum entropy principle of information theory, which we will present below.

10.4.2.1 Determination of Equivalent Martingale Probabilities

Kull (2005) depends on the information available about the equivalent martingale probabilities q_i it is the non-arbitrage constraint as specified by (3.5) and the assumption that all realisations r_i are equally likely. Any other additional information like the volatility of the equivalent martingale probabilities q_i would be imposed on the martingale probabilities q_i without justification as the relation (3.5) is the only constraint defining properties of the equivalent martingale probabilities. A choice that is consistent along this line is the additional information a particular distribution that has to be added to the constraint (3.5) and try to minimise it. Minimising additional information is equivalent to maximise the entropy

$$S = -\sum_i q_i . \ln[q_i] \tag{4.2}$$

associated with a particular distribution $\{q_i\}$. This information on theoretical *Maximum Entropy Principle* guarantees the chosen distribution to incorporate no information other than specified by the constraint (3.5). It is equivalent to choose the distribution that is 'most unbiased' or is 'embodying least structure'.

The problem of maximising the entropy associated with the distribution q_i is a constraint maximising problem:

$$\max\ S = -\sum_i q_i . \ln[q_i] \tag{4.3}$$

under the constraints:

I. $\sum_i q_i = 1$ (normalisation) (4.4)

II. $q_i \geq 0, \forall i$ (positivity) (4.5)

III. $E^{Q}\left[\frac{1+R}{1+r_{rf}}\right]=\frac{1}{1+r_{rf}}\sum_{i} q_i(1+r_i)=1$ (non-arbitrage condition) (4.6)

For solution use Lagrangian multiplier techniques, i.e., maximise the expression

$$L=-\sum_{i} q_i.\ln[q_i]+\gamma\left(\frac{1}{1+r_{rf}}\sum_{i} q_i(1+r_i)\right)+\beta\left(\sum_{i} q_i\right) \quad (4.7)$$

where, γ and β represent the Lagrangian multipliers associated with the constraints (4.4) and (4.6). Maximising (4.7) yields

$$q_i=\frac{Exp\left[\hat{\gamma}\frac{1+r_i}{1+r_{rf}}\right]\hat{y}}{\sum_{j} Exp\left[\hat{\gamma}\frac{1+r_j}{1+r_{rf}}\right]\hat{y}} \quad (4.8)$$

where $\hat{\beta}^{-1}=\sum Exp\left[\hat{\gamma}(1+r_i)/(1+r_{rf})\right]$ from constraint (4.4). The parameter $\hat{\gamma}$ is to be determined numerically by the non-arbitrage constraint (4.6). Since the distribution (4.8) is an exponential, constraint (4.5) is automatically fulfilled.

In the above distribution (4.8) is the maximum entropy probability distribution. It has been deduced from the maximum entropy principle and the non-arbitrage constraint. By construction, the probabilities (4.8) are equivalent martingale probabilities. Their distribution is consistent with the non-arbitrage theorem and without any other subjective probability assumptions.

10.4.2.2 Information Theoretical Justification

Note that (4.8) is equivalent to a special case of the Esscher Transform

$$F(x,\gamma)=\Pr[X<x,\gamma]=\frac{\int_{-\infty}^{x} Exp[\gamma\cdot y]\cdot dF(y,t)}{\int_{-\infty}^{\infty} Exp[\gamma\cdot y]\cdot dF(y,t)} \quad (4.9)$$

where the parameter y can be chosen to be consistent with a non-arbitrage condition similar to (4.6). For discrete probabilities q_i as discussed here, the Esscher Transform becomes

$$F(r,\gamma)=\Pr[R<\gamma]=\frac{\sum_{r_i<r} Exp\left[\gamma\frac{1+r_i}{1+r_{rf}}\right]}{\sum_{r_i} Exp\left[\gamma\frac{1+r_i}{1+r_{rf}}\right]} \quad (4.10)$$

When $\gamma=\hat{\gamma}$, (4.10) is consistent with the non-arbitrage constraint. This shows that the Esscher equation is such that it is transformed to be equivalent to the maximum entropy probability distribution. In this sense, the Esscher transformation (4.10), which is essentially the well-known Esscher premium principle, has an information theoretical justification.

Check Your Progress 1

1) Write the equation of equivalent principle used in insurance premium.

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2) What is the equation for valuation of a future cash-flow under equivalent martingale measure?

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3) What is equivalent martingale probability?

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4) How do you formulate the equation of arbitrage-free pricing of an arbitrary insurance given losses are claim number?

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5) What is the meaning of maximum entropy principle?

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10.4.3 Costs and Investment Return

As in case of any other economic activity costs and investments are crucial for the profitability of insurance operations. Remember that costs enter premium calculations in a straightforward way, and this feature is less obvious for investments. Pricing of contracts and assessing reserves depends directly on the anticipated investment return. Usually, investment opportunities are taken into account by a risk-adjusted discounting factor in (1.1). We need to consider arbitrage-free pricing of insurance operations that reflects and incorporates investment possibilities.

The maximum entropy approach provides a way to account for investment returns and cost simply by modifying or adding constraints. Fixed costs c modify the return related to insurance (risky asset b) so that,

$$R = \frac{b - c - X}{b} \tag{5.1}$$

It is assumed that the insurer is investing the amount b during the period [0,T] in which no payments are made in a risky asset with (stochastic) return Y, the overall return of the insurance operation can be written as

$$R^{tot} = \frac{b - c - X}{b} + Y \tag{5.2}$$

where return R^{tot} is the over all return. It includes the diversification effects that are present due to the non (or only partially) correlated nature of the returns R and Y.

We can also calculate equivalent martingale probabilities as is discussed in Sections 10.4 and 10.5. The only differences are that the constraint (4.6) is modified to reflect the cost c and that a new non-arbitrage constraint

$$E^Q\left[\frac{1+Y}{1+r_{rf}}\right] = \frac{1}{1+r_{rf}}\sum_i q_i\left(1+y_i\right) = 1 \tag{5.3}$$

is added to capture the investment return. Linearity property ensures that $\left(1+R^{tot}\right)/\left(1+r_{rf}\right)$ is a martingale, and no modification for R^{tot} is required. Maximising the entropy as in case of (4.7) yields a martingale

$$q_i = \frac{\exp\left[\left(1+r_{rf}\right)^{-1}\left(\hat{\gamma}_1 + r_1\right) + \hat{\gamma}_2\left(1+y_i\right)\right)\right]}{\sum_i \exp\left[\left(1+r_{rf}\right)^{-1}\left(\hat{\gamma}_1\left(1+r_i\right) + \hat{\gamma}_2\left(1+y_i\right)\right)\right]} \tag{5.4}$$

where the parameters γ_1 and γ_2 are determined by the constraints (4.6) and (5.3), respectively. What is attempted here is to develop the non-arbitrage value of an insurance contract according to (3.7). In this modified formulation the discounting is done with respect to the risk-free interest rate r_{rf}, i.e., the dependence on the investment return is internalised.

You can compare (3.7) with the equivalent martingale measure (5.4) and consider the role of investment opportunities. Specifically, it yields the non-arbitrage value of the combination of insurançe and investment operation. Through this, we could present the market price of the overall risk. The non-arbitrage pricing principle (3.7) together with the equivalent martingale measure (5.4) provides a unified valuation of assets and liabilities as a result.

Look at the diversification effects due to the non (or only partially) correlated nature of the returns R and Y present in the equivalent measure of martingale

(5.4). To see this, consider that (5.4) depends on the correlation between the non-arbitrage premium of R and Y. However, it is difficult to discern the effect of correlation. See that it depends on the parameters γ_1 and γ_2 that in turn would depend on the correlation themselves. From (5.4) it can be inferred that for negative, fixed $\hat{\gamma}_1$ and $\hat{\gamma}_2$, a positive correlation between R and Y will have to be produced by putting relatively more weight on low returns. On the contrary, no correlation, positive or negative, will decrease the relative weight on low returns. Such a feature of (5.4) can be interpreted in terms of a portfolio which we will take up separately in the following.

Thus, we have seen that (5.4) can be interpreted as a generalisation of the Esscher transformation (4.10). The transformed measure Q in (5.3) depends not on a single measure P but on two measures P_1 and P_2 related to insurance and investment returns.

10.4.4 Valuation and Implied Discounting

In the preceding discussion we have seen that in comparison to real probabilities P, the equivalent martingale probabilities Q would put more weight on low returns. This feature is related to the concept of the safety loading. In fact common premium and loading principles can be obtained by suitably choosing martingale probabilities if we take the case of a compound Poisson process. Two different conditions included in the process of new formulation are as follows: First, in addition to the insurance return, the stochastic investment return is considered. Second, a discounting factor is included that is implied by the overall non-arbitrage value of the contract and a particular loading. Why? Because, in finance, the non-arbitrage value coincides with the (unique) market price in the presence of liquidity, effectiveness and completeness of the market. These conditions are not found in the current case. However, the implied discounting rate may shed some light on how the discounting factor in (6.1) has to be chosen.

To obtain the implied discounting factor, solve the equation

$$\frac{1}{1+r_i} E^r\left[f(L)\right] + S\left[f(L)\right] = \frac{1}{1+r_{rf}} E^Q\left[f(L)\right] \qquad (6.1)$$

for $(1+r_i)$ where *f(L)* stands for the loss incurred by an insurance structure. Such a formulation comes closer to the actuarial and financial view as it is consistent with the loading principle and the market value of the overall risk of insurance and investment operation as implied by non-arbitrage pricing.

An examination of (6.1) shows that it has defined the implied discounting factor $(1+r_i)$ in a unique way. While interpreting it, however, you must remember that its right hand side depends on the martingale measure Q, which is not uniquely defined since insurance market is incomplete. Moreover, Q is linked by construction to the reference premium b. Thus, the implied discounting factor $(1+r_i)$ is relative to the reference premium b as defined in (6.1) and we have

$$B = \frac{1}{1+r_1} E^P\left[X\right] + S\left[X\right]$$

Alternative definitions for b can be considered in a real life situation. A reference premium could be considered that is based on $E^{r}\left[f'(L)\right]$ and $S(f'(L))$.

Example. Simple Reinsurance Structures

In reinsurance contracts, premiums are calculated under the assumption of an arbitrage-free reinsurance market and in the context of the continuous Lundberg model. Discounted cash flows based on stochastic interest rates link the insurance and finance side. However, there is no explicit non-arbitrage condition for investments in financial assets.

We will consider two simple reinsurance structures, one of which is related to compound claim amounts X (stop-loss insurance) and the other one to single claim amounts L_i (excess of loss insurance). These two have the same underlying asset b and the one-period returns defined by (3.2) are derived probabilistically.

Stop-Loss Reinsurance

In stop-loss reinsurance, an insurer cedes the risk of being exposed to compound losses X surpassing some threshold d to a reinsurer. For simplicity, we will assume that no payments are made in the time period $[0,T]$. Our objective is to find the non-arbitrage value of reinsurance in this situation and its relation to the premium b. The incurred loss, which at time T is ceded to the reinsurer, is given by

$$f(L,N)=f(X)=\max(X-d,0) \tag{7.1}$$

where d is the deductible. On the other hand, the insurer incurs the loss

$$\min(X,d) \tag{7.2}$$

Taking the expected value of (7.1) with respect to the equivalent martingale measure Q and discounting with the risk-free rate, the non-arbitrage reinsurance premium is worked out to be

$$b_{reins}=\frac{E^{Q}\left[\max(X-d,0)\right]}{1+r_{rf}} \tag{7.3}$$

Because of the relation,

$$\max(X-d,0)=X-\min(X,d)=X-\left(d-\max(d-X,0)\right) \tag{7.4}$$

the reinsurance premium b_{reins} can also be expressed as

$$b_{reins}=\frac{(b-d)}{1+r_{rf}}+\frac{E^{Q}\left[\max(d-X,0)\right]}{1+r_{rf}} \tag{7.5}$$

In (7.5), the first term is the discounted value of a cash amount $(b-d)$ while the second term corresponds to the value of a 'put option on insured losses' with strike d. That means the value of reinsurance for the insurer equals to holding (or lending) the cash amount $(b-d)$ at the risk-free rate r_{rf} and selling a put option with strike d. Recall that the equivalent martingale measure Q expression (7.3) (or (7.5)) for the reinsurance premium b_{reins} implicitly takes into account the investment return Y. Such a result indicates the equivalent martingale measure Q depends on the insurance return related to X and the investment return Y.

From Equation (7.5), the interconnection between the insurance premium b and reinsurance premium b_{rein} is established when no arbitrage is present. The 'call-put parity' in finance following from (7.4) can also be deduced such that

$$b + E^Q\left[\max(d - X, 0)\right] = d + E^Q\left[\max(X - d, 0)\right] \tag{7.6}$$

To get the premium b and the deductible d, we return to Equation (6.6). It may be seen that these are related to the value of put and call options with equal strike (i.e., deductible) on compound losses X.

Excess-of-Loss Reinsurance

We have been considering a compound claim amount X so far. However, in insurance and reinsurance, the amount to be paid depends on single claim amounts. As an example, consider excess-of-loss reinsurance where the amount for the *ith* claim to be paid by the reinsurer is

$$\max(L_i - d, 0) \tag{7.7}$$

The total amount paid in the period $[0,T]$ is obtained by

$$f(L,N) = \sum_{i=1}^{N} \max(L_i - d, 0) \tag{7.8}$$

where, N is the (random) number of claims occurring in the period $[0,T]$. If we assume (7.8) as a payoff function $f(L,N)$ of the underlying asset b, then non-arbitrage premium can be calculated by taking the expected value of (7.9) with respect to the equivalent martingale measure Q

$$b_{reins} = \frac{E^Q\left[\sum_{i=1}^{N} \max(L_i - d, 0)\right]}{1 + r_{rf}} \tag{7.9}$$

The expectation value $E^Q[.]$ in Equation (7.9) can be written

$$b_{reins} = \frac{E^Q[N].E^Q\left[\max(L_i - d, 0)\right]}{1 + r_{rf}} \tag{7.10}$$

and, from discussion of stop-loss-reinsurance above, an equivalent expression

$$b_{reins} = \frac{\left(b - E^Q[N].d\right)}{1 + r_{rf}} + \frac{E^Q[N].E^Q\left[\max(d - L, 0)\right]}{1 + r_{rf}} \tag{7.11}$$

can be obtained. Similarly, relation (7.6) now becomes

$$E^Q\left[\max(L - d, 0)\right] = \left(\frac{b}{E^Q[N]} - d\right) + E^Q\left[\max(d - L, 0)\right] \tag{7.12}$$

The interpretation of (7.12) is similar to the one in the case of compound losses. With respect to the premium b, the deductible d and the expected value of number of claims evaluated under the equivalent martingale measure Q, Equation (7.12) interconnects the value of put and call options with equal strike (i.e., deductible) on individual losses L_i.

10.4.5 Multi-Period Contracts

The one-period setting considered so far is easily generalised to the multi-period case. Basically, the constraints (3.6) and (5.3) have to capture multi-period payoffs. In addition the discounting with respect to the risk-free interest rate r_{rf} would also account for multiple periods. For example, in the simplest case where no term structure is present, constraint (3.6) becomes

$$E^Q\left[\sum_t \frac{1+R_t}{\left(1+r_{rf}\right)^t}\right]=1$$

Check Your Progress 2

1) Interpret the equation $R=\dfrac{n-c-X}{b}$ given for arbitrage-free pricing of insurance.

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2) Write the generalisation equation of Esscher transformation.

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3) How would you find an equation for non-arbitrage value of stop-loss reinsurance?

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4) How will you formulate the problem of non-arbitrage premium of excess-of-loss reinsurance?

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5) What is the difference between single and multi-period contract equations of non-arbitrage insurance premium?

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10.5 LET US SUM UP

In this unit we have discussed the method of using the insight gained in the finance theory to insurance. It is seen that in finance the arbitrage-free pricing relies on the liquidity and efficient market conditions. However, liquidity and efficiency are not the features of the insurance market in general and thus the non-arbitrage approach to pricing of insurance contracts may be inadequate. When insurance risk is traded or transferred to the financial markets, the non-arbitrage approach may be satisfied due to hedging. Even for then, traditional insurance structures the financial risk due to investments may not be comparable to the insurance risk itself. In such cases, the price for the overall risk would reflect the price financial markets are putting on risk. This problem is examined by extending the non-arbitrage principle to the insurance side. Due to incompleteness of the insurance market, no unique equivalent martingale measure exists and the information theoretical maximum entropy principle can be applied to make a choice.

Equivalent martingale probabilities consistent with the combined non-arbitrage conditions for insurance and investment operations for a one-period horizon have been discussed. Their relations with a generalised form of the Esscher measure have been highlighted. The latter illustrates the presence of a portfolio effect.

10.6 KEY WORDS

Complete Market: The complete set of possible gambles on future states-of-the-world can be constructed with existing assets.

Compound Poisson Process: A process with rate $\lambda > 0$ and jump size distribution G in a continuous-time stochastic process $\{Y(t): t \geq 0\}$ given by

$$Y(t) = \sum_{i=1}^{N(t)} D_i$$

where, $\{N(t): t \geq 0\}$

is a Poisson process with rate λ, and $\{D_i : i \geq 1\}$ are independent and identically distributed random variables, with distribution function G, which are also independent of $\{N(t): t \geq 0\}$

Efficient Market Hypothesis: The financial markets are "efficient", or that prices on traded assets, e.g., stocks, bonds, or property, already reflect all known information and therefore are unbiased in the sense that they reflect the collective beliefs of all investors about future prospects.

Hedge: An investment made to reduce or cancel out the risk in another investment. **Hedging** strategy is used to minimise exposure to an unwanted business risk, while still allowing the business to profit from an investment activity.

Poisson distributions: A distribution of discrete random variables. Usually, a Poisson random variable is a count of the number of events that occur in a certain time interval or spatial area.

Principle of Maximum Entropy: A method for analysing the available information to determine a unique probability distribution.

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10.8 ANSWER OR HINTS TO CHECK YOUR PROGRESS

Check Your Progress 1

1) See Section 10.2 and answer.
2) See Section 10.3 and answer.
3) See Section 10.4 and answer.
4) See Sub-section 10.4.1 and answer.
5) See Sub-section 10.4.2.1 and answer.

Check Your Progress 2

1) Read Sub-section 10.4.3 and answer.
2) Read Sub-section 10.4.3 and answer.
3) See Sub-section 10.4.4 and answer.
4) See Sub-section 10.4.4 and answer.
5) Read carefully Sub-section 10.4.4.2 and answer.