UNIT 14 REINSURANCE

Structure

14.0 OBJECTIVES

After going through this unit you will be able to:

- understand the underlying idea behind reinsurance;
- identify the forms of reinsurance operation;
- estimate reinsurance claims and losses; and
- analyse reinsurance pricing provisions contained in different models.

14.1 INTRODUCTION

Reinsurance is a means by which an insurance company shares the risk of loss with another insurance company. It is not difficult to see that the demand for reinsurance emerges as a part of the responsibility of the insurance company to manage a portfolio of risks for the benefit of its policy holders. Reinsurance allows the insurance (ceding) company to assume individual risks greater than its size would otherwise allow. Moreover it protects the cedent against catastrophic losses.

The major functions performed by reinsurance are:

Risk Transfer: The insurance company can absorb larger losses and would reduce the amount of capital needed to provide coverage.

Surplus Relief: Reinsurance effectively reduces the amount of net liability and helps build increase company's surplus.

Arbitrage: An insurance company may be motivated to go for arbitrage by way of purchasing reinsurance coverage at a lower rate than the expected cost of the underlying risk. Viewed in the above context, the demand for reinsurance brings forth a case of more general issue concerning how one has designed contract so as to allow for efficient allocation, pricing and management of risk.

Remember that the traditional approach to risk management is typically built upon an analysis of specific type operating or financial risk. If you do not remember, operating risks include uninsurable business risk such as loss of competitive position, product substitution and obsolescence as well as risk for which insurance is often available (cases such as property-liability risks or workplace risks that can adversely effect employee life and health). Financial risks, on the other hand, include credit risk as well as risks resulting from market fluctuations in the commodity prices, currencies and financial assets. Insurance and derivatives are commonly used to control the cost of 'insurable' operating and financial risks. In case of the class of uninsurable business risks that are usually difficult to quantify and manage, one has to resort to alternative contract designs and organizational structures.

In contrast to the traditional approach, there has emerged an integrated approach to risk management in recent years. This approach is basically adopted by brining together the insurance and financial markets along with the increasing frequency and severity of natural and man-made catastrophes.

Once we accept that risk is costly, our primary concern is to find out how reinsurance can be used to reduce the cost of risk for insurance companies. We will have to address this question by analysing the economic and financial conditions that are necessary in order for reinsurance to add value to the insurance enterprise.

Discussion in the following on the section of reinsurance pricing draws heavily from the work of Kreps (1998). The equations and their sequencing have been adopted without change. Keeping in view the overall level of exposition of the themes in the present course we have selected the basic concepts of the above work. However, it will be useful to see the entire work once you have understood the arguments employed in analysing the reinsurance pricing.

14.2 TYPES AND FORMS OF REINSURANCE

For the purposes of this unit, it will be assumed that the reinsurance considered is of two forms, viz., proportional and non-proportional

14.2.1 Proportional Reinsurance

Under a proportional reinsurance arrangement, the original insurance company and the reinsurer share the cost of all claims for each risk. The direct writer must pay a premium to effect this reinsurance. This premium may, or may not, be calculated on the same basis as the premiums charged by the insurer.

Proportional reinsurance operates in two forms, quota shares and surplus. With quota share reinsurance, the proportions are the same for all risks. On the other hand, with surplus reinsurance, the proportions can vary from one risk to another.

14.2.2 Non-Proportional Reinsurance

Under a non-proportional reinsurance, the direct writer pays a fixed premium to the reinsurer. The reinsurer will be required to make payments where part of the claim amount falls in a particular reinsurance layer. The layer will be defined by a lower limit, and an upper limit (or "infinity" if the cover is unlimited).

Two forms of non-proportional reinsurance are:

Individual excess of loss (XOL) reinsurance: In this the reinsurer is required to make a payment when the claim amount for any individual claim exceeds a specified excess point or retention.

Stop loss reinsurance: With this form the reinsurer makes payments when the total claim amount for a specified group of policies exceeds a specified amount (usually expressed as a percentage of the gross premium). For example, the reinsurer might agree to pay 95% of the excess when the total claim amount for all car insurance policies exceeds 102% of the total gross premium, with no upper limit.

14.3 REINSURANCE ARRANGEMENT

Let us start with some helpful terms to be used in the discussion that follows:

1) Net Claim

The actual claim amount that the direct insurer has to pay after allowing for payments under the reinsurance arrangements is called the net claim amount.

2) Net Premium

The actual premium that the direct writer keeps after making any payments for reinsurance is called the net premium.

3) Gross Claim/Premium

The original amounts without adjustment for reinsurance are referred to as the gross claim amount and the gross premium.

14.3.1 Excess of Loss Reinsurance –The Insured

In excess of loss reinsurance, the insurer pays any claim in full up to an amount M, the retention level; an amount above M is borne by the reinsurer. Symbolically, we can write

$$Y = X \text{ if } X \leq M$$

$$Y = M \text{ if } X > M \text{ i.e., } Y = \min(X,M)$$

where X denotes the claim amount and Y denotes the amount paid by the insurer.

The reinsurer pays the amount $Z = Y - Y$, i.e., $Z = \max(X - M,0)$.

when a layer between M and 2M is reinsured, the amount paid by the insurer is

$$Y = X \text{ if } X \leq M$$

$$Y = M \text{ if } M < X \leq 2M$$

$$Y = X - M \text{ if } X > 2M$$

The insurer's liability is affected in two obvious ways by reinsurance:

1) the mean amount paid is reduced;

2) the variance of the amount paid is reduced.

Both the mean and the reduction in the mean of the amount paid by the insurer under excess of loss reinsurance can be obtained.

Observe that the mean amount paid by the insurer without reinsurance is

$$E(X) = \int_0^\infty xf(x)dx \text{ where } f(x) \text{ is the PDF of the claim amount X.}$$

With a retention level of M the mean becomes

$$E(Y) = \int_0^M x f(x) dx + M \, P(X > M)$$

The main difficulty in dealing with excess of loss reinsurance is that the integral is incomplete.

With excess of loss reinsurance there is a method that converts the incomplete integral to a complete integral.

$$E(Y)=\int_0^\infty xf(x)dx-\int_M^\infty xf(x)dx+M\int_M^\infty f(x)\,dx$$
$$=E(X)-\int_M^\infty (x-M)f(x)dx$$
$$=E(X)-\int_0^\infty zf(z+M)dz \text{ by setting } z=x-M.$$

If $X\sim Pareto(\alpha,\lambda)$, and the retention level is M, then

$$E(Y)=\frac{\lambda}{\alpha-1}-\int_0^\infty z\frac{\alpha\lambda^\alpha}{(\lambda+z+M)^{\alpha+1}}dz$$
$$=\frac{\lambda}{\alpha-1}-\left(\frac{\lambda}{\lambda+M}\right)^\alpha\int_0^\infty z\frac{\alpha(\lambda+\alpha)^\alpha}{(\lambda+z+M)^{\alpha+1}}dz$$
$$=\frac{\lambda}{\alpha-1}-\left(\frac{\lambda}{\lambda+M}\right)^\alpha\frac{\lambda+M}{\alpha-1}$$

For cases where this integral is not easy to calculate, the formula has limited use.

If $X\sim Gamma(\alpha,\lambda)$, and the retention level is M, then

$$E(Y)=\frac{\alpha}{\lambda}-\int_0^\infty z\frac{1}{\Gamma(\alpha)}\lambda^\alpha(z+M)^{\alpha-1}e^{-\lambda(z+M)}dz$$

This is not going to be easy to evaluate either.

14.3.2 Excess of Loss Reinsurance-The Reinsurer

The reinsurer may have a record only of claims that are greater than M. If a claim is for less than M, then the reinsurer may not even know a claim has occurred. The reinsurer thus has the problem of estimating the underlying claims distribution when only those claims greater than M are observed.

The statistical terminology is to say that the reinsurer observes claims from a truncated distribution.

In this case, the values observed by the reinsurer relate to a conditional distribution, since the numbers are conditional on the original claim amount exceeding the retention limit.

We write the random variable for the amount paid by the reinsurer as

$$Z=X-M\mid X>M.$$

Suppose that the underlying claim amounts have $PDF-f(x)$ and CDF – $F(x)$. Suppose that the reinsurer is only informed of claims greater than the retention

M and has a record of $z = x - M$. What is the PDF g(z) of the amount, z, paid by the reinsurer?

The argument goes as follows:

$$P(Z < z) = P(X < z + M \mid X > M)$$
$$= \int_M^{z+M} \frac{f(x)}{1 - F(M)} dx$$
$$= \frac{F(z+M) - F(M)}{1 - F(M)}$$

the PDF of the reinsurer's claims is

$$g(z) = \frac{f(z+M)}{1 - F(M)} \quad z > 0.$$

Note that the original PDF applied to the gross amount z + M, divided by the probability that the claim exceeds M.

It is also possible that the reinsurer is aware of all the claims that are made under the original policies. In this case the distribution of the reinsurer's outgo may include those claims on which it does not in fact pay anything.

In such a case we are looking at the unconditional distribution of the reinsurer's outgo Z, where:

$$Z = 0 \text{ if } X < M$$

$$Z = X - M \text{ if } X \geq M$$

The calculation of the mean and variance of the reinsurer's outgo in this case will be similar to the corresponding calculations for the insurer.

If we use W to denote the amount payable by the reinsurer on claims in which it is involved, i.e.,

$$W = Z \mid Z > M$$
$$= X - M \mid X > M,$$

then

$$g(w) = \frac{fz(w)}{1 - Fz(0)}$$
$$E(W) = \frac{E(Z)}{P(Z > 0)} = \frac{E(Z)}{P(X > M)}$$

14.3.3 Proportional Reinsurance

In proportional reinsurance the insurer pays a fixed proportion of the claim, whatever the size of the claim. Using the same notation as above, the proportional reinsurance arrangement can be written as follows:

If the claim is for an amount X, then the company will pay Y where

$$Y = \alpha X \qquad 0 < \alpha < 1$$

The parameter α is known as the retention level.

As the amount paid by the insurer on a claim X is $Y = \alpha X$ and the amount paid by the reinsurer is $Z = (1-\alpha)X$, the distribution of both of these amounts can be found by a simple change of variable.

$$f_Y(y) = \frac{1}{\alpha} f_X\left(\frac{y}{\alpha}\right)$$
$$f_Z(z) = \frac{1}{1-\alpha} f_X\left(\frac{z}{1-\alpha}\right)$$

If X represents the size of an individual claim, the retention level α is 0.8, then the amount paid by the insurer is $Y = 0.8X$, and
$E(Y) = 0.8\ E(X)$, $Var(Y) = 0.64\ Var(X)$. The amount paid by the reinsurer is $Z = 0.2X$, and $E(Z) = 0.2E(X)$, $Var(Z) = 0.04\ Var(X)$.

14.4 REINSURANCE CALCULATION

There are a number of useful integral formulae that simplify reinsurance calculations when working with particular distributions. Two useful distributions for this purpose could be lognormal and normal distributions.

14.4.1 Inflation

The examples so far have assumed that claim distribution remain constant over time. In fact claims are likely to increase because of inflation, at least in the longer term. Thus, we need to adjust out claim distributions to allow for inflation.

In this section we will look at how claims in inflation affect reinsurance arrangements.

Suppose that the claims X are inflated by a factor of k but the retention M remains fixed. The amount claimed is kX, and the amount paid by the insurer, Y, is

$$Y = kX \text{ if } kX \leq M$$
$$Y = M \text{ if } kX > M$$

The mean amount paid by the insurer is

$$E(Y) = \int_0^{M/k} kxf(x)\,dx + M\ \mathrm{P}(X > M/k)$$

which can be written as

$$\begin{aligned} E(Y) &= \int_0^{\infty} kxf(x)\,dx - \int_{M/k}^{\infty} kxf(x)\,dx + M\int_{M/k}^{\infty} f(x)\,dx \\ &= kE(X) - k\int_{M/k}^{\infty}(x - M/k)f(x)\,dx \\ &= k\left(E(X) - \int_0^{\infty} yf(y + M/k)dy\right) \text{ where } \left(y \equiv x - \frac{M}{k}\right) \end{aligned}$$

One important general point that can be made is that the new mean claim amount is not k times the mean claim amount without inflation.

Of course if the retention limit increases by a factor k as well, both mean claim amounts for the insurer and reinsurer will increase by the same factor.

When examining the details of a reinsurance arrangements in real life it is very important to check whether the retention limits are fixed or linked to an agreed inflation index.

14.5 REINSURANCE ESTIMATION

Consider the problem of estimation in the presence of excess of loss reinsurance. Suppose that the claims record shows only the net claims are paid by the insurer.

A typical claims record might be

$$x_1,\ x_2,\ M\,x_3,\ M\,x_4,\ x_5,\ \ldots$$

and an estimate of the underlying gross claims distribution is required.

The statistical terminology for a sample of this form is censored. In general, a censored sample occurs when some values are recorded exactly and the remaining values are known only to exceed a particular value.

The method of moments is not available since even the mean claim amount cannot be computed.

It may be possible to use the method of percentiles without alteration; this would happen if the retention level M is high and only the higher sample percentiles were affected by the (few) reinsurance claims.

Maximum likelihood can be applied to censored samples.

The likelihood is made up of two parts. The complete likelihood is

$$L(\underline{\theta}) = \prod_{1}^{n} f(x_i;\underline{\theta}) \times \left[1 - F(M;\underline{\theta})\right]^m$$

Policy Excess

Insurance policies with an excess are common in motor insurance and many other kinds of property and accident insurance.

Under this kind of policy, the insured agrèes to carry the full burden of the loss up to a limit, L, called excess. If the loss is an amount X, greater than L, then the policyholder will claim only $X - L$. If Y is the amount actually paid by the insurer, then

$$Y = 0 \text{ if } X \leq L$$

$$Y = X - L \text{ if } X > L.$$

Clearly, the premium due on any policy with excess will be less than that on a policy without an excess.

The position of the insurer for a policy with excess is exactly the same as that of the reinsurer under excess of loss reinsurance. The position of the policyholder as far as losses are concerned is exactly the same as that of an insurer with an excess of loss reinsurance contract.

In practice, expenses form a significant part of the insurance cost. So the presence of an excess might not affect the premium as much as might be expected.

Check Your Progress 1

1) What are the main functions of reinsurance?

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2) Explain the meaning of proportional and non-proportional reinsurance.

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3) What is excess of loss insurance from the view point of an insured?

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4) Why do you need to take account of inflation while calculating reinsurance?

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5) What technique do you want to use for an estimation of reinsurance in the presence of excess of loss?

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14.6 REINSURANCE PRICING

We may think that reinsurance pricing is market-driven. However, to appreciate its basis some description of the economic origin of reinsurance risk load needs to be given. Let us look at the return from the investment to be made when there is insurance risk. Here we have to take decision considering a combination of the reinsurance contract and financial techniques comes to play. The inputs are the loss distributions and reinsurer's underwriting constraints.

Although the forces of market will determine the equilibrium price to be charged for contracts, both insurer and reinsurer have to use an economic pricing model to help decide whether to write the contract. Correspondingly the combination of a reinsurance arrangement and suitable financial techniques will become a basis for making investment choice.

Let us start with the assumption that investment criteria is based on the form of a target mean return and risk measure thereon.

When the reinsurer accepts a contract, she arranges the availability of sufficient liquefiable assets to cover possible losses up to some safety level. These assets arise from premium and assets allocated from surplus, both of which are invested in appropriate financial instruments. So the reinsurer attempts to have at least as favourable return and risk over the period of the contract as possible when setting the target investment.

Such a behaviour looks at risk load as an opportunity cost and represents it as a (partially offset) cost of liquidity.

The loss safety level is essentially a measure of reinsurer constraint in the insurance business. The more weightage attached to it the less probable it is that the safety level will be exceeded. However, higher safety levels will typically result in more expensive contracts.

To understand the idea consider the following example (due to Kreps, 1998) Consider a person who decides to build a house in snow country. The question is, how strong to build the roof for snow load? If it is a cabin for only a few years, perhaps building to survive the 10 year storm will be enough. If it is meant for the grandchildren, perhaps the 200 year storm is more appropriate. It is, of course, more expensive to build it stronger. In any case some level is chosen depending on the builder's criteria.

Thus, risk load involves the cost of capital and measures of investment return and risk in particular. If capital were freely available, insurance, much less reinsurance, would be unnecessary since a firm would borrow to overcome difficulties. We will consider here standard deviation (or variance) measure of investment risk although there are more sophisticated strictly downside measures, such as a semi-variance or the average value of the (negative) excess of return. In the cases where very large losses may generate negative results, such a downside risk measure may be more desirable.

We consider two types of simple financial techniques: First, risk-free instrument such as government securities. This is called a swap in finance literature. Cost associated with it is the loss of investment income. There is a gain in terms of reduced risk, however.

This technique offers a simple formula. But often it turns out to create a higher risk load. Hence it is more expensive to the cedent.

The second technique is buying "put" options. You may recall that in put option there is the right to sell the underlying target investment at a predetermined strike price at maturity (European options). Here the strike price will be equal to investment in risk-free securities. So the reinsurer can buy the right to sell the target investment at a return not less than the risk-free rate.

The Black-Scholes formula is used to price the option. The distribution of investment returns underlying this formula is assumed for the reinsurer's target investment. The cost of these options will contribute to the risk load, but this is partly offset because the options both increase the return and decrease the variance of the target investment.

14.6.1 Single Payment at One Year

We follow Kreps (1998)'s formulation and let the principal determinants of interest to be

s = the dollar safety level associated with the loss distribution.

L = the amount of the loss.

μ_L = the mean value of the loss.

σ_L = the standard deviation of the loss.

r_f = the risk-free rate.

y = the yield rate of the target investment.

σ_y = the standard deviation of the investment yield rate.

P = the premium net to the reinsurer after expenses.

Quantities derived from the above are

A = the assets allocated by the reinsurer.

F = the funds initially invested: premium and assets less option cost, if applicable.

R = the risk load in the premium: the premium less the discounted expected loss.

In all cases the premium comprises the risk load the expected loss discounted at the risk-free rate. Note that the premium above does not include any reinsurer expenses. Then for a single payment at one year,

$$p = R + \frac{\mu_L}{1+r_f} \qquad (14.6.1)$$

The constraints are: (1) the investment result from F as input must be at least s, and (2) the standard deviation of the overall result must be no larger than σ_y.

The cash flow relations are stochastic. However, it is possible to obtain explicit formulae for the mean and variances involved and get explicit forms for the risk load. Let us work to obtain mean and variance.

14.6.1.1 Swap Case

Let the reinsurer have an inflow of P and an outflow of

$$F = (P + A) \tag{14.6.2}$$

Since the investment is in risk-free securities, at the end of the year she has an inflow of $(1+r_f)F$ and an outflow of the loss L. The internal rate of return (IRR) on these cash flows is defined by the stochastic relation

$$(1+\text{IRR})\text{A} = (1+r_f)F - L \tag{14.6.3}$$

where L and IRR are stochastic variables. Take the mean value of this equation and the mean value of the IRR to be the yield rate y to get,

$$(1+y)A = (1+r_f)F - \mu_L \tag{14.6.4}$$

which may be expressed (See Appendix 3)as

$$R = \frac{(y-r_f)}{(1+r_f)}A \tag{14.6.5}$$

We need another equation to solve the system with two constraints, a loss safety constraint and an investment variance constraint. In general, it is by making the asset base large enough, we can make the fractional variability as small as desired and the funds available as large as desired. Hence there is always a solution. Both constraints may be placing lower limits on the allocated assets.

Safety constraint: funds available at the year end has to be greater than or equal to the safety level. So

$$(1+r_f)F \geq s. \tag{14.6.6}$$

Combining Eqs. (14.6.4) and (14.6.6) to eliminate F, we have

$$A \geq \frac{(s-\mu_L)}{1+y} \tag{14.6.7}$$

From Eqs. (14.6.5) and (14.6.7) the risk load at the equality is

$$R = \frac{(y-r_f)(s-\mu_L)}{(1+r_f)(1+y)} \tag{14.6.8}$$

From Equation (14.6.1) premium before expenses is

$$P = R + \frac{\mu_L}{1+r_f} \tag{14.6.9}$$

which gives the result for the safety constraint.

Variance Constraint:

Since there is no variability in the investment return because it is risk-free, the standard deviation of the IRR is

$$A\sigma_{IRR} = \sigma_L \tag{14.6.10}$$

from Equation (14.6.3).

Investment Constraint: The IRR should have variance less than or equal to that of the target investment, so that

$$A \geq \sigma_L / \sigma_y \tag{14.6.11}$$

and from equation (14.6.5), we get

$$R = \frac{(y - r_f)}{(1 + r_f)}(\sigma_L / \sigma_y). \tag{14.6.12}$$

Given the values for the loss distribution and the target investment the latter is likely to be the more stringent constraint will hold when

$$(s - \mu_L)/\sigma_L < (1 + y)/\sigma_y. \tag{14.6.13}$$

14.6.1.2 Option Case

Consider the case where the reinsurer receives the premium at time zero, but keeps the initial assets invested in the target investment. She also buys an option to sell the target investment at the end of the year for the value that the risk-free technique would have achieved. By doing so she has obtained an instrument that eliminates a portion of the investment return distribution which lies below the risk-free rate. This will have two effects: increasing the mean return from the investment and decreasing its standard deviation.

Let

r = the rate (cost per dollar of investment protected) of a put option.

I = investment return

i = mean investment return (see Appendix 2).

The value of r depends upon the underlying investment parameter σ, which is determined by y and σ_y, (see Appendix 1). For small values of the ratio of σ_y, to $(1 + y)$, it is approximately true that

$$\sigma = \frac{\sigma_y}{(1 + y)} \tag{14.6.14}$$

and

$$r = \frac{1}{\sqrt{2\pi}}\sigma\left(1 - \frac{\sigma^2}{24}\right). \tag{14.6.15}$$

While at time zero the reinsurer has an inflow of P and an outflow of $(P + A)$, the funds available for investment have decreased by the cost of the option. so, Eq. (14.6.2) indicates

$$F = P + A - rF \tag{14.6.16}$$

or
$$F = \frac{(P + A)}{(1 + r)}. \tag{14.6.17}$$

We are considering investment in risky securities (hedging at the bottom end so that these to do not drop below the risk-free rate). At the end of the year, the reinsurer has an inflow of $(1 + I)F$ and an outflow of the loss, L. The internal rate of return on these cash flows is defined by the stochastic relation

$$(1 + IRR)A = (1 + I)F - L. \tag{14.6.18}$$

Since we must have mean value of IRR = target yield rate,

$$(1 + y)A = (1 + i)F - \mu_L. \tag{14.6.19}$$

While it is not easy to simplify this we have two unknowns R and A at our disposal and this is one equation relating them. The other equation will come from constraints, the one which is more restrictive.

Take the loss safety constraint on the funds available:

$$(1 + r_f)F \geq S .$$

Note that the actual funds available are likely to be larger than this, since r_f, represents the minimum value of the realisable investment return, due the option.

Combining Eqs. (14.6.6) and (14.6.19) to eliminate F, the allocated assets are obtained such that

$$\mathrm{A} \geq \frac{1}{1 + y}\left\{\frac{(1 + i)}{(1 + r_f)}s - \mu_L\right\}. \tag{14.6.20}$$

This is larger than in the swap case since $i > y > r_f$.

The risk load at equality is given by

$$\mathrm{R} = \frac{1}{(1 + r_f)}\left[s\{(1 + y)(1 + r) - (1 + i)\} - \mu_L(y - r_f)\right]. \tag{14.6.21}$$

Using the above two formulae and putting $i = r_f$ and $r = 0$, the results of the previous section can be obtained.

For formulating the investment variance constraint it is necessary to obtain the correlation between the loss and the investment return. The linkage by inflation suggests that there may be a negative correlation-if inflation rises, (claims costs rise and bond values fall). We make a simplifying assumption that the correlation is zero. The standard deviation of the investment return is derived in Appendix 2 and written as σ_i. When the variance of the IRR is required to be that of the target investment, we get

$$\left(A\sigma_y\right)^2 = \left(F\sigma_i\right)^2 + \left(\sigma_L\right)^2 . \qquad (14.6.22)$$

On substituting the value of the initial fund F from the equation for the mean, we have

$$-aA^2 + 2bA + c = 0 \qquad (14.6.23)$$

with

$$a = \sigma_y^{\,2}\left(1+i\right)^2 - \sigma_i^{\,2}\left(1+y\right)^2 \qquad (14.6.24)$$

$$b = L\left(1+y\right)\sigma_i^{\,2} \qquad (14.6.25)$$

$$c = L^2\sigma_i^{\,2} + \sigma_L^{\,2}\left(1+i\right)^2 \qquad (14.6.26)$$

The coefficients, a, b, and c are positive: last two due to their explicit construction and the first because the option decreases the variance and increases the mean of the investment return compared to the target values.

Thus,

$$A = \frac{b + \sqrt{b^2 + ac}}{a} \qquad (14.6.27)$$

and

$$R = A\frac{(1+r)(1+y)-(1+i)}{1+i} + L\left[\frac{1+r}{1+i} - \frac{1}{1+r_f}\right]. \qquad (14.6.28)$$

In the limit as $\sigma_i \to 0$ the solution for A is same as the ratio of standard deviations; with $i = r_f$ and $r = 0$ the risk load is the same as earlier form found in case of swap.

14.6.1.3 High Excess Layer and Minimum Premium

The above formulae can be applied to the case of a high excess layer or any similar finite rare event cover. A non-zero rate on line (ratio of premium to limit) is predicted even for cases where the loss probability goes to zero.

Consider the loss distribution to be binomial. The probability, p, is hitting the layer, and if it hits there is a total loss. The safety level, S, is the limit (total amount payable) of the layer.

The mean loss μ_L is ps and the variance of the loss is $p(1-p)s^2$. As p gets smaller, corresponding to higher and higher layers, in both the swap and option cases the variance constraint yields A and R to be zero. However, the safety constraint in both cases is linear in p with a non-zero intercept. In the option case, the rate on line (ROL) in the limit $p \to 0$ is

$$ROL = \frac{(1+y)(1+r)-(1+i)}{(1+r_f)(1+y)}. \tag{14.6.29}$$

We get this when in Eq. (14.6.21) $L = 0$ and recalling that ROL is the ratio of R to S.

See that the swap version may be obtained by letting $r = 0$ and $i = r_f$, which results in

$$ROL = \frac{1}{1+r_f} - \frac{1}{1+y} = \frac{y-f}{(1+r_f)(1+y)}. \tag{14.6.30}$$

Such a from suggests that the minimum ROL is of the order of the real target return i.e., the excess of the return over the risk-free rate.

14.6.2 Single Payment at Arbitrary Time

If we went to express the returns in the preceding discussion returns in terms of the equivalent annualized returns, the results hold after the following replacements are made:

$$(1+i) \to (1+i)^t \tag{14.6.31}$$

$$(1+y) \to (1+y)^t \tag{14.6.32}$$

$$(1+r_f) \to (1+r_f)^t \tag{14.6.33}$$

The forms for the option rate and the standard deviations given in Appendix 2 contain the time dependence.

14.6.3 Multiple Payments

In case of possible multiple loss payments, the same formulation in case single payment can be used. However, it needs a more complex formulation. In the single payment case, simultaneously enforcing the rate of return (through the mean value of the stochastic equation) and the safety constraint yielded an easy solution. However, in multiple payments, the variance constraint cannot be derived easily and it can be evaluated only for any given level of allocated assets.

Thus general procedure to follow is: (1) express the fundamental stochastic process on a spreadsheet. It is now more than a simple equation because of the interaction of the fund levels at different times, but it is still easily expressed; (2) define the safety levels; (3) use those to define what funds are needed and what options need to be bought; (4) find the risk load corresponding to the target return for the indicated safety constraint by putting all the stochastic variables at their mean values; (5) simulate to see if the variance constraint is satisfied; (6) if it is not, then add "excess capital" and simulate again; (7) repeat step until both constraints are satisfied.

Check Your Progress 2

1) Which measures of risk, you would like to use while taking decision on investment in reinsurance?

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2) What is the difference between swap and option?

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3) What do mean by single payment at one year in reinsurance pricing?

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4) Distinguish between single payment and multi-payment cases in reinsurance pricing.

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14.7 LET US SUM UP

In this unit we have discussed the functions and types of reinsurance along the pricing options in it. We defined reinsurance as a means by which an insurance company shares risk of loss with another insurance company. Its functions are concerned with risk transfer, surplus relief and arbitrage. Reinsurance could be proportional and non-proportional. The proportional reinsurance is also known as quota share reinsurance. The reinsurance here takes a stated percent share of each policy written by the insurer and presents the premium and losses in that same proportion. The non–proportional (or excess of loss) reinsurance helps only if the loss suffered by the insured exceeds a certain amount (retention). Excess of loss reinsurance can be of two forms – per risk or per occurrence (catastrophe). In per risk case, the insurance policy limits are exposed within the reinsurance limits. On the other hand, in catastrophe excess loss, the insurance policy limits must be less than the reinsurance retention.

While calculating reinsurance premium, it helps to consider distribution such as log normal and normal. Presence of inflation may have to be considered in reinsurance provisions. An estimation of reinsurance claims with provision of excess loss is considered. The censored sample methods with some value recorded exactly and remaining known only to exceed a particular value has been pointed out.

In the last part of the discussion, we have presented the pricing options for reinsurance. The major focus of discussion in respect of pricing is the presence of investment return and risk. It was pointed out that in single payment case IRR was used because it was unequivocally defined. In multiple payment case the IRR might not even be definable. In order to consider further value of the cash flow, it was found to be necessary to set up some description of the investment policy on the released funds.

In the single payment at one year framework we have considered there cases, viz., option and high excess layer and minimum premiums, and discussed the required changes to be introduced in multiple payments.

14.8 KEY WORDS

Censored Sample: Case where the variable of interest is only observable under certain conditions.

Exposure: Possibility of loss.

Loss Development: The difference between the estimated amount of loss as initially reported to the reinsurer and the amount of an evaluation at a late amount paid in final settlements(s).

Loss Loading: A factor applied to pure loss cost or losses to produce a reinsurance rate or premium.

Loss Ratio: The ratio of incurred losses and loss-adjustment expenses to net premiums earned. This ratio measures the company's underlying profitability, or loss experience, on its total book of business.

Losses Incurred (Pure Losses): Net paid losses during the current year plus the change in loss reserves since the prior year-end.

Net Loss: The amount of loss sustained by an insurer after giving effect to all applicable reinsurance, salvage, and subrogation recoveries.

Net Retention: The amount of insurance which a ceding company keeps for its own account and does not reinsure in any way.

Reinsurance: In effect, insurance that an insurance company buys for its own protection. The risk of loss is spread, so a disproportionately large loss under a single policy doesn't fall on one company. Reinsurance enables an insurance company to expand its capacity; stabilise its underwriting results; finance its expanding volume; secure catastrophe protection against shock losses; withdraw from a line of business or a geographical area within a specified time period.

Reinsurance Ceded: The unit of insurance transferred to a reinsurer by a ceding company.

Reinsurance Premium: The consideration paid by the ceding company to the reinsurer for the reinsurance afforded by the reinsurer.

Reinsurance: Assumption by one insurance company of all or part of a risk underwritten by another insurer.

Reinsurer: The insurer, which assumes all or a part of the insurance or reinsurance written by another insurer.

Retention: Net retention plus any amounts of other cedes all of the reinsurance it has assumed to another reinsurer.

Stop Loss: Any provision in a policy designed to cut off an insurer's losses at a given point.

Stop Loss Reinsurance: A form of excess of loss reinsurance which indemnifies the ceding company against the amount by which the ceding company's losses incurred during a specific period (usually 12 months) exceed either (1) a predetermined dollar amount or (2) a percentage of the company's subject premiums (loss ratio) for the specific period.

Swaps: The simultaneous buying, selling or exchange of one security for another among investors to change maturities in a bond portfolio.

Truncated Sample: Case where whole observations are missing.

14.9 SOME USEFUL BOOKS

Deykin, C. D. Pentiken, T., Pesonan, M. (1994), *Practical Risk Theory for Actuaries,* Chapman and Hall, London

M. Goovaerts & D. Vyncke (2004), *Encyclopedia of Actuarial Science, Wiley,* Vol III, 1403-1404

Kreps, Rodney (1998), Investment-Equivalent Reinsurance pricing, proceedings of the Casualty Actuarial Society, 85, PP. 101-131 (see Internet)

Richard, A. Brealey and Stewart. C. Myers (1996), *Principles of Corporate Finance,* 5th edition, Irwin-McGraw-Hill

Rolski, T., Schmidli, H., Schmidt, V., Teugels, J. (1998), Stochastic Processes for Insurance and Finance. J. Wiley & Sons, Chichester

14.10 ANSWER OR HINTS TO CHECK YOUR PROGRESS

Check Your Progress 1

1) See Section 14.1 and answer

2) See Section 14.2 and answer

3) See Section 14.3 and answer

4) See Section 14.4 and answer

5) See Section 14.5 and answer

Check Your Progress 2

1) See Section 14.6 and answer

2) See Sub - section 14.6.1.1 and 14.6.1.2 and answer

3) See Sub - section 14.6.1 and answer

4) See Sub – section 14.6.1 and 14.6.1.3 and answer

APPENDICES

(Taken from Kreps, Rodney)

Appendix 1

The form of the Black-Scholes formula for the price of a European call option on a security is

$$\text{call price} = \Phi(\Delta_1)P_o - \Phi(\Delta_2)PV(E)$$

where $P(E)$ = present value of the exercise price discounted at the risk-free rate,

P_o = price of the security at time zero,

$$\Delta_1 = \frac{In\left(\dfrac{p_o}{PV(E)}\right)}{\sigma\sqrt{t}} + \frac{\sigma\sqrt{t}}{2}$$

$$\Delta_2 = \Delta_1 - \sigma\sqrt{t}$$

where σ is a parameter of the distribution of the underlying security and is $\Phi(x)$ the cumulative distribution function for the normal distribution, that is

$$\Phi(x) = \int_{-x}^{x} \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz.$$

This function is available in at least one standard spreadsheet program.

The option is the right to buy the underlying security at the exercise price at the time t.

The logarithm of the value of the security is assumed to follow a normal distribution with parameters μt and $\sigma\sqrt{t}$ for the mean and standard deviation, respectively. Given the expected annual yield rate y and its standard deviation σ_y,

$$\sigma^2 = In\left\{1 + \left(\frac{\sigma_y}{1+y}\right)^2\right\}$$

and,

$$\mu = \text{In}(1+y) - \frac{\sigma^2}{2}$$

The price for a put option, which is actually what is of interest here, is given by put-call parity as

$$\text{put price} = \text{call price} + PV(E) - P_0,$$

In the case of interest here $PV(E) = P_0$ since we want the exercise price to be the growth at the risk-free rate. Hence the put price is equal to the call price, and for either option the

$$\text{option cost} = P_0\Phi\left(\frac{\sigma\sqrt{t}}{2}\right) - P_0\Phi\left(-\frac{\sigma\sqrt{t}}{2}\right)$$

so the

$$\text{option rate} = \Phi\left(\frac{\sigma\sqrt{t}}{2}\right) - \Phi\left(-\frac{\sigma\sqrt{t}}{2}\right)$$

or,

$$\text{Option rate} = \sqrt{\frac{2}{\pi}}\int_0^{\sigma\sqrt{t}/2} e^{-z^2/2}\,dz.$$

The exponential may be expanded to first order in a Taylor series to get the

approximation quoted, which is actually rather good for the order of magnitude of numbers used here.

As stated in appendix 1, the probability density function for the investment value (which is 1 + return) is lognormal with parameters μt and $\sigma\sqrt{t}$. That is,

$$f(x)=\frac{1}{\sigma x\sqrt{2\pi t}}\exp\left\{\frac{-(In(x)-\mu)^{2}}{\sigma^{2}t}\right\}.$$

The investment hedged with the option to time t has the characteristics (f is the risk-free rate)

$$\begin{aligned}\text{investment} &= x \text{ for } x \geq (1+f)^{t}\\ &= (1+f)^{t} \text{ for } x<(1+f)^{t}\end{aligned}$$

What is needed are the moments of the investment; in particular its mean and standard deviation.

Define

$$\begin{aligned}F_{n} &= \int_{0}^{(1+f)^{t}} x^{n} f(x)dx\\ &= \Phi\left(\varsigma-n\sigma\sqrt{t}\right)\exp\left\{n\mu t+n^{2}\sigma^{2}t/2\right\}\end{aligned}$$

where

$$\xi=\sqrt{t}\left\{\frac{\text{In}(1+f)-\mu}{\sigma}\right\}.$$

In general,

$$\begin{aligned}\text{moment}(n) &= \int_{0}^{\alpha}\text{investment}^{n} f(x)dx\\ &= (1+f)^{nt}\int_{0}^{(1+f)^{t}} f(x)dx+\int_{(1+f)^{t}}^{x} x^{n} f(x)dx\end{aligned}$$

Using the results for F_{n} above, the moment of order n of the investment is

$$\begin{aligned}\text{moment}(n) &= (1+f)^{nt} F_{o}+\exp\left(n\mu t+n^{2}\sigma^{2}t/2\right)-F_{n}\\ &= (1+f)^{nt}\,\Phi(\varsigma)+\exp\left(n\mu t+n^{2}\sigma^{2}t/2\right)\left[1-\Phi\left(\varsigma-n\sigma\sqrt{t}\right)\right]\end{aligned}$$

The mean value is just moment (1) and the variance of the investment is (moment (2) - moment$(1)^{2}$ }.

Appendix 3

Derivation of Eq. (14.6): Substitute for μ_L and F in Eq. (14.6):

Eq. (14.6) may be solved for μ_L as

(A.1) $$\mu_L = \left(1+r_f\right)\left(p-R\right)$$

Substitute F from Eq. (14.6) and L from Eq. (A.l) into Eq. (14.6):

$$\left(1+y\right)A = \left(1+r_f\right)\left(\mathrm{P}+A\right)-\left(1+R_f\right)\left(\mathrm{P}-r\right)$$

(4) $$=\left(1+r_f\right)A+\left(1+r_f\right)R$$

Solving for R gives Eq. (14.6).

Derivation of Eq. (14.6):

Eq. (14.6) can be written

$$\left(1+r\right)F = p+A = A+\frac{\mu_L}{\left(1+r_L\right)}+R$$

from Eq. (14.6). Rearranging to solve for R, and subsequently using Eq. (6) for F and Eq. (14.6) for A,

$$R = \left(1+r\right)F - A - \frac{\mu_L}{\left(1+r_f\right)}$$

$$=\left(1+r\right)\frac{S}{1+r_f}-\frac{1}{1+y}\left(\frac{1+i}{1+r_f}S-\mu_L\right)-\frac{\mu_L}{1+r_f}$$

$$=\frac{S}{1+r_f}\left[\left(1+r\right)-\frac{1+i}{1+y}\right]+\mu_L\left[\frac{1}{1+y}-\frac{1}{1+r_f}\right]$$

$$=\frac{1}{\left(1+r_f\right)\left(1+y\right)}\left[S\left\{\left(1+r_f\right)\left(1+r_f\right)-\left(1+i\right)\right\}-\mu_L\left\{y-r_f\right\}\right]$$

Derivation of Eqs. (14.6)-(14.6):

Eq. (14.6) can be written

$$F=\frac{\left(1+y\right)A+\mu_L}{1+i}$$

Substituting for F in Eq. (14.6) gives

$$A^2\sigma^2 = \left[(1+y)^2 A^2 + 2A\mu_L(1+y) + \mu_L^2\right]\frac{\sigma_i^2}{(1+i)^2} + \sigma_L^2$$

Multiplying through by the denominator and collecting terms,

$$0 = A^2\left[(1+y)^2\sigma_i^2 - \sigma_y^2(1+i)^2\right] + 2A\mu_L(1+y)\sigma_i^2 + \mu_L^2\sigma_i^2 + \sigma_L^2(1+i)^2$$

This is Eqs. (14.6)-(14.6). If there is a correlation ρ_{iL} between investment and loss. then this equation becomes

$$0 = A^2\left[(1+y)^2\sigma_i^2 - \sigma_y^2(1+i)^2\right] + 2A(1+y)\sigma_i\left[\mu_L\sigma_i + \sigma_{iL}(1+i)\right]$$
$$+\mu_L^2\sigma_i^2 + \sigma_L^2(1+i)^2 + 2\mu_L\sigma_i\sigma_L(1+i)\rho_{iL}$$

Derivation of Ea. (14.6):

By substituting for F from Eq. (14.6) into Eq. (14.6)

$$(1+y)A = (1+i)\frac{p+A}{1+r} - \mu_L.$$

Multiplying through by the denominator and using Eq. (14.6) for P,

$$A(1+y)(1+r) = (1+i)\left(R + \frac{\mu_L}{1+r_f} + A\right) - \mu_L(1+r)$$

Rearranging terms,

$$(1+r) = A\left[(1+y(1+r) - (1+i)\right] + \mu_L\left[(1+r) - \frac{1+i}{1+r_f}\right]$$

Eq. (14.6) for R results immediately.

UNIT 15 EXTREME VALUE

Structure

15.0 OBJECTIVES

After going through this unit, you will be able to:

- model insurance when extreme levels (maximum or minimum) values are present; and
- apply tools of extreme value estimation.

15.1 INTRODUCTION

There are unusual or rare events such as earthquakes, hurricanes and even stock market crashes, which seem to follow no rule of occurrence. Insurance companies have to deal with such external event (catastrophe insurance, for example). It is essential for an insurance company to set its premiums at levels high enough to cover losses from the rare but large claims. For example, a property and casualty insurance company that must be able to withstand a loss proportional to the magnitude of the loss caused by Tsunami of 2004 or Hurricane Andrew in 1992. Note that the occurrence of such events is infrequent but the magnitude of loss is enormous. Extreme value theory of statistics deals with such extreme deviations from the median of probability distributions. In general, a method used is to assess the type of probability distribution generated by the process. You cannot use the usual rule of normal distribution where the distribution of observations fit into the well-known bell-shaped curve. Essentially, one is now looking for extremes, which are in the tails of distributions. These are flatter (more heavy) than the normal events.

Some examples in this category could be,

- estimation of the probability of extreme insurance claims and typical sizes of such claims in order to compute the premium volume that covers future losses.
- modelling of extreme individual and simultaneous price movements for financial assets in order to build an accurate risk model for a financial portfolio consisting of several risky assets.

Thus, important feature of extreme value theory (EVT) is to provide solutions to problems associated with extreme risks, which are extremely rare. In some of the sections below, we have followed the notations and adopted the discussion of Bensalah (2000).

15.2 KEY COMPONENTS OF EVT

EVT draws from the limiting distribution of sample extreme (maxima or minima). To appreciate its underlying idea, take $X = (X_1, X_2, \ldots X_n)$ as *a* sequence of independent identically distributed observations with distribution function F. Let the sample maximum be denoted by $M_n = \max\{X_1, \ldots X_n\}$. Under certain assumption – subexponential distributions – the tail of the maximum determines the tail of the sum as $n \to \infty$. More generally, the generalised extreme value distribution (GEV) given by $H_\xi(x)$ describes the limit distributions of suitably normalised maxima. We can obtain a standard GEV, for example, by writing $\frac{X-\mu}{\sigma}$ for a random variable X and consequently get a distribution function that is specified as

$$H_{\xi,}(x) = \begin{cases} \exp\left[-\left(1+\xi\dfrac{X-\mu}{\sigma}\right)^{-1/\xi}\right], & \text{if } \xi \neq 0, 1+\xi\dfrac{x-\mu}{\sigma} > 0 \\ \exp\left[-\exp\left(-\dfrac{x-\mu}{\sigma}\right)\right], & \text{if } \xi = 0 \end{cases}$$

where μ, σ and ξ are the location, scale and shape parameters, respectively. Note that μ converges to a scalar whereas σ is a tendency. The remaining parameter ξ is called the tail index. It indications the thickness of the tail of distribution, so that the larger the index, the thicker the tail.

15.3 SOME THEORETICAL RESULTS OF EVT

There are two significant results of EVT. First, the asymptotic distribution of a series of maxima (minima) modelled under certain conditions (i.e., the distribution of the standardised maximum of the series) converges to the Gumbel, Frechet, or Weibull distribution. These three distributions give the GEV distribution we have noted above. The second result is the distribution of excess over a given threshold, where we are interested in modelling the behavior of the excess loss once a high threshold (loss) is reached. This result is used to estimate the very high quantiles (0.999 and higher). EVT shows that the limiting distribution is a generalised Pareto distribution (GPD). An often used method is called Peaks-Over-Threshold (POT) method. Here the maxima (or minima) is assumed to follow a GEV distribution. For example, if we were concerned with monthly peaks in interest rates, we could fit a GEV distribution to the monthly maxima. However, excesses over a given high threshold, follow a GPD. If we are interested in the distribution of insurance claims over some high threshold, such excesses would be best modelled by a GPD. Alternatively, if we were concerned about the occurrence times of the losses over some threshold and the excess distribution, we would fit a POT model. In a POT model, the number of events during a given time follows a Poisson process and the exceedances over a given high threshold follow a GPD. The number of events being Poisson, the interarrival times (time between events) are exponentially distributed. Consequently, by fitting a POT model we can estimate the average time between events of a given magnitude (threshold), and the distribution of the excess over the threshold.

15.3.1 GEV Distribution

Classical extreme value theory was formatted by Fisher and Tippett (1928). The Fisher-Tippet theorem specifies the form of the limit distribution for centered and normalised maxima. To see its construction suppose $\{X_n\}$ is a sequence of independent and identically distributed (iid) random variables and M_n is the $\max(X_1,\ldots,X_n)$. If there exist constants $c_n > 0$ and a real number $d_n \in R$, then $(M_n - d_n)/c_n$ yields a centered and normalised maximum. When $(M_n - d_n)/c_n \xrightarrow{d} H$ (that is, converges in distribution to H), for some non-degenerate distribution function H, then H belongs to one of the three families of extreme value distribution functions: Fréchet, Weibull, and Gumbel. We return to key elements of EVT given above and let $\{X_1\ldots,X_n\}$ be a sequence of iid random variables. Then the maximum $M_n = Max(X_1\ldots,X_n)$ converges in law (weakly) to the following distribution:

$$H_{(\xi,\mu,\sigma)}(x) = \begin{cases} \exp\left(-\left[1+\xi(X-\mu)/\sigma\right]^{-1/\xi}\right) & if\,\xi \neq 0. \\ \exp\left(-e^{-(X-\mu)/\sigma}\right) & if\,\xi = 0 \end{cases}$$

where $1+\xi(X-\mu)/\sigma>0$

When the index is equal to zero, the distribution H corresponds to a Gumbel type. When the index is negative, it corresponds to a Weibull; when the index is positive, it corresponds to a Frechet distribution. That is, these distributions corresponded to different values of ξ. They are,

Gumbel distribution $\wedge, \xi=0$

Frlchet distribution $\phi_\alpha, \xi=\alpha^{-1}>0$

Weibull distribution $\Psi_{\alpha,} \xi=-\alpha^{-1}<0$

The Frechet distribution corresponds to fat-tailed distributions and has been found to be the most appropriate for fat-tailed financial data. Since the asymptotic distribution of the maximum always belongs to one of these three distributions, irrespective of the original distribution we should note the importance of this result. The asymptotic distribution of the maximum can be estimated without making any assumptions about the nature of the original distribution of the observations.

15.3.2 Procedure for Tail Estimation

You may note that the purpose of tail estimation procedures is to estimate the values of X outside the range of existing data. To do this, we have to employ both extreme events and exceedances of a specified level. The standard approach assumes that the tail of the population follows the selected family of distribution as seen above and we will discuss more in the following.

15.3.2.1 Maximum Domain of Attraction (MDA)

The extreme value distributions introduced above represent the limit laws for the normalised maxima of *iid* random variables. We have to consider the conditions on a distribution function F that will give the normalised maxima M_n that converges to H. In other words, we need to choose constants $c_n>0$ and d_n such that $(M_n-d_n)/c_n \xrightarrow{d} H$. If this condition is satisfied, then we say the distribution function F belongs to the maximum domain of attraction of H.

Definition

Let us consider the random variable X (or the distribution function F of X) that belongs to the maximum domain of attraction of the extreme value distribution H. If for some constants $c_n>0$ and $d_n \in R$, we have $(M_n-d_n)/c_n \xrightarrow{d} H$, then we write $X \in MDA(H)$.

Two examples illustrate the concept of the MDA and the procedure to find the appropriate constants.

Example 1.

Consider a sample $\{X_1, X_2, \ldots, X_n\}$, from the logistic distribution $F(x)=1/\left(1+e^{-x}\right)$. First, find the constants c_n and d_n. That helps achieve normalised distribution. Second, we need to find the limiting distribution of

$(M_n - d_n)/c_n$. Third, find the "centering" constants $\{d_n\}$ close to $E[M_n]$ (i.e., the expected value of M_n). When all these steps are completed, we have $F(X_1),\dots,F(x_n)$ as a random sample from the uniform distribution over (0,1) and $F(x_n)$ is the largest of a sample size n from the uniform distribution over (0,1). This gives $E[F(M_n)] = n/(n+1)$ and we write,

$$F(E[M_n]) = \left(1+\exp\{-E[M_n]\}\right)^{-1} = 1-\left(1+\exp\{E[M_n]\}\right)^{-1}$$

and

$$E[F(M_n)] = \frac{n}{n+1} = 1-(n+1)^{-1}.$$

If $F(E[M_n]) \approx E[F(M_n)]$, then $n \approx \exp\{E[M_n]\}$ or

$E[M_n] \approx \log n$. That is to say, a reasonable choice for the centering constants $\{d_n\}$ is the sequence $\{\log_n\}$. To find the limit of the distribution we proceed as follows:

$$\begin{aligned}
\lim_{n\to\infty} p\left[\frac{M_n - d_n}{c_n} \le y\right] &= \lim_{n\to\infty} P\left[\frac{M_n - \log n}{c_n} \le y\right] \\
&= \lim_{n\to\infty} P\left[M_n \le c_n y + \log n\right] \\
&= \lim_{n\to\infty}\left[F(c_n y + \log n)\right]^n \\
&= \lim_{n\to\infty}\left(1+e^{-c_n y - \log n}\right)^{-n} \\
&= \lim_{n\to\infty}\left(1+(1/n)e^{-c_n y}\right)^{-n} \\
&= \exp\{-e^{-y}\} \quad \text{for } c_n \equiv 1.
\end{aligned}$$

Consequently, when we choose $\{d_n\} = \{\log n\}$ and $\{c_n\} = 1$, the limiting distribution of $(M_n - d_n)/c_n$ is $\exp\{-e^{-y}\}$, which is the Gumbel distribution.

Example 2.

Suppose we have a sample from the exponential distribution, $F(x) = 1 - e^{x/\beta}$, for $x \in [0,\infty)$. Then as in case of example above, $E[F(M_n)] = n/(n+1) = 1 - 1/(n+1) \Rightarrow$

$$F(E[M_n]) = 1 - \exp\{-E[M_n]/\beta\} \approx E[F(M_n)].$$

Then, $\frac{1}{n+1} \approx \exp\{-E[M_n/\beta]\}$, or $E[M_n] \approx \beta\log(n+1) \approx \beta\log n$.

Thus, we can write use $d_n = \beta \log n$.

$$\lim_{n\to\infty} P\left[\frac{M_n - d_n}{c_n} \le y\right] = \lim_{n\to\infty} P\left[M_n - \beta\log n \le c_n y\right]$$

$$= \lim_{n\to\infty}\left[F\left(c_n y + \beta\log n\right)\right]^n$$

$$= \lim_{n\to\infty}\left(1 + e^{-(1/\beta)c_n y - \log n}\right)^n$$

$$= \lim_{n\to\infty}\left(1 - \left(1/n\right)e^{-(1/\beta)c_n y}\right)^n$$

$$= \exp\left\{-e^{-y}\right\} \textit{for } c_n \equiv \beta.$$

Thus, when we choose $\{d_n\} = \{\beta\log n\}$ and $c_n = \beta$, distribution at limit of $(M_n - d_n)/c_n$ is a Gumbel distribution.

The lesson you draw from the above examples is that MDA of Gumble distribution can be obtained with the normalised maxima from the logistic distribution and the exponential distribution.

15.3.3 Excess beyond a Threshold

Let X be a random variable with a distribution F and a threshold given X_F, for U fixes $< x_F$. F_u is the distribution of excesses of X over the threshold U

$$F_u(x) = P\left(X - u \le x \mid X > u\right),\ x \ge 0$$

If we estimate the threshold with help of VaR, the conditional distribution F_u can be approximated by a generalised Pareto distribution (GDP).

We can write:

$$F_u(x) \approx G_{\xi,}\beta_{(u)}(x), u \to \infty,\ \ x \ge 0$$

where:

$$G_{\xi,\beta_{(u)}}(x) = \begin{pmatrix} 1 - \left(1 + \dfrac{\xi x}{\beta}\right)^{-1/\xi} \\ 1 - e^{-x/\beta} \end{pmatrix} \textit{if} \Big|_{\xi=0}^{\xi\neq 0}$$

Let us elaborate a little more of the above formulation.

The generalised Pareto distribution (GPD) – the limt distribution of excesses $Y = \max\{X - u, 0\}$ over sufficiently high thresholds u – offers a good approximation of the tail of F for some fixed ξ and β which depend upon u.

Thus the distribution of Y may be thought of as the conditional distribution of X given $X > u$.

The GDP with shape parameter ξ and scale parameter β is specified as

$$G_{\xi,\beta}(y)=\begin{cases}1-\left(1+\xi\dfrac{y}{\beta}\right)^{-1/\xi} & if\,\xi\neq 0\\ 1-\exp\left(-\dfrac{y}{\beta}\right) & if\,\xi=0\end{cases}$$

$$\text{where } y\in\begin{cases}[0,\infty] & if\,\xi\geq 0\\ \left[0,-\dfrac{\beta}{\xi}\right] & if\,\xi<0\end{cases}$$

and the sign of the shape parameter ξ determines its tail behaviour and thus the tail behaviour of the original distribution.

If $\xi>0$, then tail of the distribution function F of X decays like a power function $X^{-1/\xi}$. In that case, F belongs to a family of heavy-tailed distributions that includes, among others, the Pareto, log-Gamma, Cauchy and t-distributions.

If $\xi=0$, then tail of F decreases exponentially, and belongs to a class of medium-tailed distributions which include the normal, exponential, Gamma and log-normal distributions.

If $\xi<0$, the underlying distribution F is characterised by a finite right endpoint, which class of short-tailed distributions includes the uniform and beta distributions.

The mean excess function (expectation) of the GDP is given by

$$e(u)=E(X-u\mid X>u)=\frac{\beta+\xi_u}{1-\xi}\text{, where }\beta=\sigma+\xi(u-\mu)$$

and $\max_n y_n$ follows a GEV distribution with parameters ξ,μ,σ.

Distributions of the type {H} are used to model the behavior of the maximum of a series. The distributions {G} of the second result model excess beyond a given threshold, where this threshold is supposed to be sufficiently large to satisfy the condition $u\to\infty$.

The GEV describes the limit distributions of normalised maxima. The GPD describes the limit distribution of scaled excesses over high thresholds. While modelling external events, often we are concerned with excesses over some high value. For example, in a catastrophe bond structure that has a $1 billion attachment point, we need to ascertain if the loss claims are greater than $1 billion. To estimate the claims, let us work with the following definition of the excess distribution function.

Definition

Let X be a random variable with distribution function F that has right endpoint x_F. If we have fixed $u<x_F$, then $F_u(x)=P(X-u\leq x\mid X>u)$, $x\geq 0$, is the excess distribution function of the random variable X over the threshold u and the function $e(u)=E[X-u\mid X>u]$ gives the mean excess function of X.

The above definition says that for any high threshold u, we have the distribution of the amounts over u, and $e(u)$ gives the average excess over u.

Thus, when our objective is to find an appropriate value of high threshold, the mean excess function provides a necessary tool. Particularly when the mean excess function of a GPD is linear, we can select a threshold from the region where the empirical mean-excess function is roughly linear.

Danish Fire Insurance Claims

Empirical Data and Mean Excess plot used by ABS Research (2000) have been used to demonstrate the use of the mean excess function. In its Chart 3, the mean-excess plot is shown which is roughly linear in the range up to about 10-15, indicating a threshold of 10 is appropriate (the claims are in millions of Danish Kroner, 1985 prices). On the basis of such result, it has concluded that this threshold could be used to model the excess distribution.

When the number of exceedances of a high threshold is roughly a Poisson process, it is useful to model a peaks-over-threshold (POT) model. Therefore, the study suggested the following:

- The number of exceedances of a high threshold follows a Poisson process.
- Excesses over high thresholds can be modelled by a GPD.
- Plotting the mean-excess function is useful in determining an appropriate value of the high threshold.
- The distribution of the maximum of a Poisson number of *iid* excesses over a high threshold is a GEV.

Let us discuss the POT model

Peaks-over-thresholds (POT) models have been used by hydrologists for many years in modelling such events as flood levels and dam heights. The basic steps for formulation of the POT model are:

- The occurrence times of the excesses of an iid (or stationary) sequence over a high threshold follow a Poisson process;
- the corresponding excesses are independent and have a GPD; and,
- the excesses and the occurrence times of the excesses are independent of each other.

The POT model can be used to estimate the excess distribution with respect to a threshold level u, and to estimate the tail shape of the original distribution. It should be remembered that the threshold setting of the POT model is data dependent. The model defines a two- dimensional (Y_n, N_u) space-time point process on, $X_n \geq u, n = 1 \ldots N_u$. Here Y_n and N_u are independent random variables such that Y_n ~ GDP (ξ, b) and the number of excesses N_u follows a Possion process with intensity λ representing the average number of exceedances over the time interval used for sampling process and given by

$$\lambda = \left(1 + \xi \frac{X - \mu}{r}\right)^{-\frac{1}{\xi}} \text{ for } X \geq u .$$

The threshold u is usually chosen using mean excess plots.

While assessing the POT models, it will be useful to consider an example such as catastrophe bonds. These instruments become handy because occurrence times of excesses are important. The fitted model helps determine the encounter probabilities of extreme events. Moreover, you can determine the non-encounter probabilities of such events as well. As the return period is the expected time to an event of a given threshold from some starting point and the occurrence times are independent, the return period is the average time to a given event. Such a feature help in making statements such as "a loss of X is expected to occur every four years on average". The non-encounter probability of an event is indicated by likelihood that such an event will not occur in a given period, for example, one year. Thus, the model offers an ideal method for simulation experiments and stress scenarios.

To appreciate the kind of prediction made through POT model, we may see the study of Parisi and Lund (2000). They modelled the Atlantic basin hurricanes making landfall in the United States from 1935 to 1998 and then simulating from the fitted GPD estimate the return periods and non-encounter probabilities for storms of various intensities. Three findings are:

A storm with a wind speed of 240 km/h (150 mph) is expected to make landfall about once every 10.71 ± 0.104 years; the non-encounter probability is 0.912 ± 0.0028. That is, there is about a 91% chance that no such storm will make land-fall in any given year. Thus, the complement is that there is about a 9% chance that this event will occur in any given year. In other words, such an event occurs about once every 11 years on average but there is a 9% chance it could occur in any year. As year-to-year storm counts and intensities are independent, even if a 240 km/h storm makes landfall in one year, there is still a 9% chance it could occur the following year.

In another similar analysis, Parisi and Herlihy (1999) modelled catastrophic losses to the global reinsurance industry. Their paper predicts the return periods and non-encounter probabilities for various levels of global catastrophic losses. An important result to note is, losses of $1 billion or more are expected to occur about once a year.

Note, however, that the application of EVT involves a number of considerations. It is important at the outset of data analysis to determine whether the series has the fat tail needed to apply the EVT results. Also, the parameter estimates of the limit distributions H and G depend on the number of extreme observations used. The choice of a threshold should be large enough to satisfy the conditions to permit its application (u tends to infinity), while at the same time leaving sufficient observations for the estimation. Different methods of making this choice will be examined in the following discussion. Finally, until now, it was assumed that the extreme observations are iid. The choice of the method for extracting maxima can be crucial in making this assumption viable. However, there are some extensions to the theory for estimating the various parameters for dependent observations; see Embrechts, Kluppelberg and Mikosch (1997).

Check Your Progress 1

1) Why the need for reinsurance arises?

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2) What is generalised extreme value distribution?

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3) Name the three distributions to which extreme of a series converges to.

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4) What is maximum domain of attraction?

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5) Write the meaning of the excess beyond a threshold.

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15.4 STEPS FOR APPLYING EVT

Sections 15.4 to 15.7 are developed borrowing the discussion from Bensalah (2000).

15.4.1 Exploratory Data Analysis

The first step is to explore the data; we use here Q-Q plots and the distribution of mean excess.

15.4.1.1 Q-Q Plots

Essentially we start by studying a histogram of the data. In most of the VaR methods, the approximation by a normal distribution remains a basic assumption. However, as most financial series are fat-tailed, the graph of the quantiles makes it possible to assess the goodness of fit of the series to the parametric model. We consider the model below.

Let $X_1...,X_n$ be a succession of random variables iid and $X_{n,n} < ... < X_{1,n}$, the order d statistics, F_n being d, the empirical distribution. Note that $F_n(X_k, n) = (n-k+1)/n$ and F, the estimated parametric distribution of the data.

To construct the Q - Q plot the graph of quantiles is to be drawn. It is defined by the set of the points,

$$\left\{X_{k,n}, F^{-1}\left(\frac{n-k+1}{n}\right), k = 1,...,n\right\}$$

In order to judge that the parametric model fits the data well, we should draw the graph in a linear form. In practice, we need to compare such a graph through various estimated models and select the best. The rule is, more linear the Q-Q plots, the more appropriate the model in terms of goodness of fit. Additionally, the Q-Q plots can help to detect out layers.

15.4.1.2 Mean Excess Function

Let X be a random variable and given threshold x_F. If we take,

$$e(u) = E(X - u \mid X > u), \quad 0 \le u < x_F$$

then *e(.)* is called the mean excess function. In the above equation *e(u)* is the mean excess over the threshold *u*. To see its specific form, let us take *X* to follow an exponential distribution with a parameter λ. Then the above function is equal to $e(u) = \lambda^{-1}$ for any $u > 0$. For the GPD, we have

$$e(u) = \frac{\beta + \xi u}{1 - \xi}, \ \beta + \xi u > 0$$

As given in Bensalah (2000), the mean excess function for a fat-tailed series is located between the constant mean excess – function of an exponential distribution $e(u) = \lambda^{-1}$ and the GPD, which is linear and tends towards infinity for high thresholds as *u* tends towards infinity.

In order to establish the behaviour of the tail, we need to examine the graph. If $X_1, \ldots, X_n$ is iid and with F_n the corresponding empirical distribution with $\Delta_n(u) = \{i, i = 1, \ldots, n, X_i > u\}$, then

$$e_n(u) = \frac{1}{card(\Delta_n(u))} \sum_{i \in \Delta_n(u)} (X_i - u), \; u \geq 0$$

where card refers to the number of points in the set $\Delta_n(u)$.

In this formulation, the set $\{e_n(X_{k,n}), \; k = 1, \,,, n\}$ gives the mean excess graph.

When the function $e(u)$ that tends towards infinity for high-threshold u (linear shape with positive slope) we have a fat-tailed distribution.

15.4.2 Sampling the Maxima and Some Data Problems

There are two approaches which can be considered in building the series of maxima or minima. First, divide the series into non-overlapping blocks of the same length and choose the maximum from each block. The assumption of extreme observations are iid is viable in such a case.

Intuitive idea in this can be formed by looking at the financial data. These data contain periods of high volatility followed by periods of low volatility (clustering). Sampling the maxima using this technique reduces such a feature when we increase the size of the block. However, the risk of losing extreme observations within the same block is still present and this makes the choice of the size of the block problematic.

Second, choose a given threshold (high enough) and consider the extreme observations which exceed this threshold. As regards the choice of the threshold, there is a trade-off between variance and bias. When the number of observations is increased for the series of maxima leading to a lower threshold, some observations from the centre of the distribution are introduced in the series. Consequently, the index of tail is more precise due to less variance but it is biased. On the other hand, when a high threshold chosen, it reduces the bias but makes the estimator more volatile due to fewer observations.

15.4.3 Choice of the Threshold

15.4.3.1 Graph of Mean Excess

From the preceding discussion we know that it is possible to provide a graphical tool to choose the threshold. The mean excess function for the GPD is linear (tends towards infinity) and for a high threshold, the excess over a threshold for a given series converges to a GPD. Therefore, it is possible to choose the threshold where an approximation by the GPD is reasonable and the area has a linear shape on the graph.

15.4.3.2 Hill Graph

Hill graph provides another tool for selecting the threshold. If $X_1 > \ldots > X_n$ are the ordered statistics of random variables iid, then Hill estimator of the tail index ξ using $k+1$ ordered statistics is defined by:

$$H_{k,n} = \frac{1}{k}\sum_{i=1}^{k} \ln\left(\frac{Hi}{H_{k+1}}\right) = \hat{\xi}$$

In the equation, the Hill graph is defined by the set of points

$$\left\{\left(k, H_{k,n}^{-1}\right),\ 1 \le k \le n-1\right\}$$

we can select the threshold u from this graph for the stable areas of the tail index. However, this method applies well for a GPD or close to GPD type distribution. The Hill estimator is the maximum likelihood estimator for a GPD and since the extreme distribution converges to a GPD over a high threshold u, we may justify its use.

15.5 PARAMETERS AND QUANTILES ESTIMATION

Usually, the parameters of the extreme distribution are estimated taking different assumptions. First, we can assume that the extreme observations follow exactly the GEV distribution. Second, and possibly more realistically, we can assume that the observations are roughly distributed like the GEV distribution. More specifically, the distribution of the observations can be assumed to belong to the maximum domain of attraction (MDA) of H_{ξ}. Lastly, the parameters and quantiles can be estimated for the distribution of excess over a threshold.

15.5.1 Estimate of the Distribution of Extremes

Parametric Methods: We may assume that the extreme observations follow exactly the GEV distribution, so that the maximum likelihood estimation (MLE) can be used. As there is no closed form for the parameters, numerical methods may be used to provide good estimates. The p-quantile method can be used by defining it as $\hat{x}_p = H_{\xi,\hat{\mu},\hat{\sigma}}^{-1}(p)$. Then,

$$In(p) = \left(-\left[1+\hat{\xi}\left(\widehat{X}_p - \hat{\mu}\right)/\hat{\sigma}\right]^{-1/\hat{\xi}}\right.$$

$$\left(-In(p)\right) = \left[1+\hat{\xi}\left(\widehat{X}_p - \hat{\mu}\right)/\hat{\sigma}\right]^{-1/\hat{\xi}}$$

$$\widehat{X}_p = \hat{\mu} + \frac{\hat{\sigma}}{\hat{\xi}}\left[\left(-In(p)\right)^{-\xi} - 1\right]$$

$$p = \exp\left(-\left[1+\hat{\xi}\left(\widehat{X}_p - \hat{\mu}\right)/\hat{\sigma}\right]^{-1/\hat{\xi}}\right.$$

MLE has the advantage of simultaneous estimation of the three parameters, and it applies well to the series of maxima per block. Especially, in the case $\xi > -1/2$, MLE is said to yield good estimates. As the majority of financial series have a positive tail index $\xi > 0$, it becomes a good tool for estimation in these data.

Semi-parametric Methods: The assumption that the extreme observations converge or are distributed exactly as H_{ξ} is very strong. When we relax this assumption, we may find the observation to be distributed like the GEV. Then F, the distribution of the observations, belongs to the MDA of H_ξ. Thus, the parameters and quantities estimation differ from that of the *Parametric methods*.

Let $X_1 ..., X_n$ be random variables that are iid, with a distribution $F \in$ MDA$\left(H_\xi\right)$. This is equivalent to $\lim_{n\to\infty} \bar{F}\left(\sigma_n x + \mu_n\right) = -In\left(H_\xi\left(x\right)\right)$

In this case (> 0):

$$\bar{F}\left(x\right) = x^{-1/\xi} L\left(x\right),\ x > 0$$

with L a slow variation function, or more exactly,

$$\forall \lambda > 0,\ \lim_{x\to\infty} \frac{L\left(\lambda x\right)}{L\left(x\right)} = 1$$

Therefore, for a high u such as $u = \sigma_n x + \mu_n$, we have

$$\bar{F}\left(u\right) = \left[1 + \hat{\xi}\left(u - \hat{\mu}_n\right) / \hat{\sigma}_n\right]^{-1/\hat{\xi}}$$

The p-quantile is defined as $\hat{x}_p = H^{-1}_{\xi,\hat{\mu}_n,\hat{\sigma}_n}\left(p\right)$ then,

$$\hat{X}_p = \hat{\mu}_n + \frac{\hat{\sigma}_n}{\hat{\xi}}\left[\left(n\left(1-p\right)\right)^{-\xi} - 1\right]$$

Usually we are more interested in very high quantiles beyond the series or, in other words, out-of-sample estimation. For this purpose, let $u = X_k$, where u is a very high threshold, *k/n* its associated probability (probability from the empirical distribution of a series of length N and u is the K order statistic $X_1 > .. > X_n$), and p is the associated probability of the p-quantile x_p; then,

$$\bar{F}\left(\mathrm{X}_k\right) = \mathrm{X}_{k,n}{}^{-1/\xi} L\left(X_{k,n}\right) = \frac{k}{n} \qquad (1)$$

$$\bar{F}\left(\hat{\mathrm{X}}_p\right) = \hat{X}_p^{-1/\xi} L\left(\hat{X}\right) = 1 - p \qquad (2)$$

Dividing (2) by (1), and with $x_p > X_k$, we have $L\left(\hat{X}_p\right) / L\left(X_{k,n}\right) \approx 1$; then,

$$\hat{X}_p = \left(\frac{n}{k}\left(1-p\right)\right)^{-\hat{\xi}} X_{k,n}$$

The tail index ξ is chosen from a stable area on the Hill graph.

15.5.2 Fitting Excesses over a Threshold

The GPD estimation involves two steps:

1) The choice of the threshold *u*. The mean excess graph can be used where *u* is chosen such that *e(x)* is approximately linear for $x > u\left(e\left(u\right)\right.$ is linear for a GPD).

2) The parameter estimations for ξ and β can be estimated MLE.

When the distribution of excesses over a threshold is estimated, an approximation of the unknown original distribution (that generates the extreme observations) and an estimation of the p-quantile from it can be used to estimate the extreme VaR.

Thus, let u be a threshold, $X_1 \ldots, X_n$ the random variables exceeding this threshold (following a distribution $F \in MDA\left(H_{\xi}\right)$), and $Y_1 \ldots, Y_n$ the series of exceedances $\left(Y_i = X_i - u\right)$. The distribution of excesses beyond u is given by:

$$F_u(y) = P(X - u \leq y \mid X > u) = P(Y \geq y \mid X > u), y \geq 0$$

and the distribution, F, of the extreme observations, X_i, is given by:

$$\bar{F}(u+y) = P(X \geq u+y) = P\left((X \geq u+y \mid X > u).P(X > u)\right)$$

$$F(u+y) = P(X - u \geq y \mid X > u).\,P(X > u)$$

$$F(u+y) = \bar{F}_u(y).\,\bar{F}_u(u)$$

Such a result makes it possible to estimate the tail of the original distribution, by separately estimating F and F_u. For a high threshold u:

$$\left(\bar{F}_u(y)\right) \approx \bar{G}\hat{\xi}, \hat{\beta}_{(u)}(y)$$

F(u) can be estimated from the empirical distribution of the observations:

$$\left(\hat{F}(u)\right) = \frac{1}{n}\sum_{i=1}^{n} I_{\{X_i > u\}} = \frac{N_u}{n}$$

and

$$\left(\bar{F}(u+y)\right) = \frac{N_u}{n}\left(1 + \hat{\xi}\frac{y}{\beta}\right)^{-1/\hat{\xi}}$$

The estimation of the p-quantile for a given threshold, u, using this distribution is straightforward:

$$\hat{X}_p = u + \frac{\hat{\beta}}{\hat{\xi}}\left[\left(\frac{n}{N}(1-p)\right)^{-\hat{\xi}} - 1\right]$$

15.6 EXTREME VaR

The following discussion is adopted from Bensalah (2000). By definition, VaR is the p-quantile of the distribution of the log change in price. EVT makes it possible to model the empirical distribution of the extreme observations. Extreme VaR is defined as the p-quantile estimated from the extreme distribution. Various estimators are available, depending on the estimation method and the assumptions.

VaR is estimated assuming that the extreme observations follow exactly the GEV distribution (see parametric methods):

$$VaR_{extrene} = \hat{\mu} + \frac{\hat{\sigma}}{\hat{\xi}}\left[\left(-In(p)\right)^{-\xi} - 1\right]$$

It is estimated assuming that the observations follow approximately the GEV distribution (see semi-parametric methods).

VaR in sample:

$$VaR_{extreme\ (in-sample)} = \hat{\mu}_n + \frac{\hat{\sigma}_n}{\hat{\xi}}\left[\left(n(1-p)\right)^{-\xi} - 1\right]$$

and VaR out-of-sample:

$$VaR_{extreme\ (out-of-sample)} = \left(\frac{n}{k}(1-p)\right)^{-\xi} X_{k,n}$$

The approximation of excesses over a threshold by a GPD leads to the following estimator:

$$VaR_{extreme\ GPD} = u + \frac{\hat{\beta}}{\hat{\xi}}\left[\left(\frac{n}{N}(1-p)\right)^{-\xi} - 1\right]$$

In practice estimation using a GPD is repeated several times to have a graph of quantiles. Various results might be presented: quantiles for a stable area of the graph and a quantile corresponding to the pessimistic version (peak). This method is called peak over threshold (POT) method.

15.7 LIMITATIONS OF THE EVT

The following discussion is due to Bensalah (2000). As stated earlier, the approaches described here relate to aggregate positions. Most of the articles in the literature use the same approach. The application of the EVT results in a multivariate case faces a basic problem: There is no standard definition of order in a vectorial space with dimensions greater than 1 and thus it is difficult to define the extreme observations for n-dimension vectors $(n > 1)$.

To solve this problem, it is proposed that estimation of the extreme marginal distribution for each asset (use the maxima for the short positions w_i and the minima for the long positions). Attempt is made for solving for the extreme p-quantiles VaR_i computing the correlations r_{ij} between the series of maxima and minima, and calculating the extreme VaR of a portfolio of N assets:

$$Va\text{R}_{extreme} = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\rho_{ij}.w_i.w_j.VaR_i.\ VaR_j}$$

Unfortunately, the joint distribution of the extreme marginal distributions is not necessarily the distribution of the extremes for the aggregate position. In other words, extreme movements of the log change in prices for the different assets do not necessarily result in extreme movements for the whole portfolio. This will depend on the composition of the portfolio (position on each instrument) and on the relations (dependencies or correlations) between the various assets.

Check Your Progress 2

1) What is Q-Q Plot?

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2) How do you formulate mean excess function?

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3) What is Quantiles' estimation?

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4) What methods will you employ to estimate distribution of extremes?

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15.8 LET US SUM UP

In this unit we have discussed modelling of extreme event. Our concern in the field of insurance is when maximum or minimum level of some value rather than the expected for average case is being examined. The basic features of extreme value theory (EVT) are included by describing the Fisher – Tippet theorem. It specifies the form of the limit distribution for centred and normanised maxima. The generalised extreme value distribution (GEV) is stated to be given by $H_{\xi}(x)$. It describes the limit distribution of suitably normalised minima. Three standard distributions that correspond to different values of ξ are Gumbel, Frichet and Weibull to which H converges. When the conditions under which a distribution function that implies the normalised maxima converges of H are satisfied, we say that the distribution function has belonged to the maximum domain of attraction of H.

Another distribution that plays an important role in modelling external events is the generalised Pareto distribution (GPD). Whereas GEV describes the limit distributions of normalised maxima, the GPD describes the limit distribution of scaled excesses over length thresholds.

Peaks-Over–Threshold (POT) models are introduced to modelling extremes when the occurrence times of the excesses are important.

Steps required in applying EVT include Q–Q plots and distribution of mean excess. While the Q–Q plots help identity the distribution through its histogram, the mean excess function establishes the form of the distribution of mean excess. While building the series of maxima or minima, the series could be divided into non-overlapping blocks or one may choose a given threshold (through the graph of mean excess or Hill graph). To estimate the parameters of extreme distribution, parametric methods like maximum likelihood estimation or p-quantile can be used.

With EVT techniques it is possible to capture the behaviour of extreme observations. Also, the loss over a very large threshold can be estimated. While these results apply well to the univariate case, the multivariate one seems to define the limits of this theory. Indeed, the joint distribution of the marginal extreme distributions is not necessarily an extreme distribution. However, the joint distribution can be used to estimate the probability associated with an extreme scenario, but incorporating this information in the market risk framework remains an open question. The majority of the results were presented assuming that the observations are independent and identically distributed. The sampling of maxima can make this assumption viable, but it is also possible to estimate an index of dependency to incorporate it in the calculation of the extreme VaR.

15.9 KEY WORDS

Degenerate Distribution: Probability distribution of a discrete random variable that assigns all of the probability, i.e., probability 1, to a single number, a single point, or otherwise to just one outcome of a random experiment.

Fat Tail: A phenomenon of approximately normal distribution. Fat tail introduces additional risk.

Heavy-Tailed Distribution: Probability distribution with infinite variance. The simplest heavy-tailed distribution is the Pareto distribution.

Logistic Distribution: A continuous probability distribution. This distribution has longer tails than the normal distribution and a higher kurtosis of 1.2 (compared with 0 for the normal distribution).

Quantiles: A set of 'cut points' that divide a sample of data into groups containing (as far as possible) equal numbers of observations.

15.10 SOME USEFUL BOOKS

Bensalah, Younes (2000), Steps in Applying Extreme Value Theory to Finance: A Review, Bank of Canada Working Paper 2000-20, Bank of Canada, Ottawa (see Internet)

Embrechts, P., C. Klüppelberg and T. Mikosch (1977), Modelling Extremal Events for Insurance and Finance, Berlin Springer

Parisi, Fancis (2000), Extreme Value Theory and Standard & Poor's Ratings, Structured Finance Special Report, New York

Parisi, F. and Lund, R. (2000), Seasonality and Return Periods of Land falling Atlantic Basin

Hurricanes, Australian and New Zealand Journal of Statistics, 42, 271-282

Parisi, F. and Herilihy, H.M. (1999), Modelling Catastrophe Reinsurance Risk: Implications for the CAT Bond Market

Embrechts, P., C. Kluppelber, and T. Mikosch, (1997), Modelling Extremal Events for Insurance and Finance. Berlin: Springer

McNeil, A.J. (1999), "Exteme Value Theory for Risk Managers." Risk special volume, to be published. ETH preprint (www.math.ethz.ch/~mcneil/pub_list.html).

15.11 ANSWER OR HINTS TO CHECK YOUR PROGESS

Check Your Progress 1

1) See Section 15.1 and answer.
2) See Section 15.2 and answer.
3) See Section 15.3 and answer.
4) See Sub-section 15.3.2.1 and answer.
5) See Sub-section 15.3.3 and answer.

Check Your Progress 2

1) See Sub-section 15.4.1.1 and answer.
2) See Sub-section 15.4.1.2 and answer.
3) See Section 15.5 and answer.
4) See the discussion on parametric method at Sub-section 15.5.1 and Section 15.6 and answer.

UNIT 16 STOCHASTIC CONTROL AND INSURANCE

Structure

16.0 OBJECTIVES

After going through this unit, you will be able to:

- identify insurance risk;
- pick up possible control variables including stochastic control;
- evaluate optimal investment for insurers; and
- decide on optimal reinsurance.

16.1 INTRODUCTION

Our main objective here is to consider optimization problems faced by an insurer. We shall focus on problems with infinite planning horizon. As the themes are to be seen over time, essentially, we need to work with objective functions that have infinite time ruin (survival) probability. While reading these lines immediately you can anticipate that we will be discussing some of the concepts such as compound Poisson process, renewal process, exponential as well as Pareto claim function and geometric Brownian process. Thus you need to brush up your lessons on these before reading the present unit.

Dealing with stochastic control, undoubtedly, takes us to a higher analytical level in insurance problem. There are a good deal of models already available. Since we propose this discussion to be an introduction to such studies, let us follow the framework of our discussion to a work of Christain Hipp entitled, **Stochastic Control with Application in Insurance** (see Hipp, 2004). The material presented in the following borrows freely from this work. For further reference materials, you may see the original article, which lists the recent studies in the area.

16.2 INSURANCE RISK

16.2.1 The Lundberg Risk Model

The randomness of claim sizes and claim occurrence times reflect a risk process. The Lundberg model for the risk process uses a compound Poisson process for the claims of such risk process. It is given as

$$R(t) = s + ct - S(t),$$

$$S(t) = X_1 + \ldots + X_{N(t)},$$

with a homogeneous Poisson process $N(t)$ with constant intensity λ and independent claim sizes $X_1, X_2, \ldots$ with distribution Q (the claim size distribution) which are independent of $N(t), t \geq 0$. The initial surplus is s, and c is the constant premium intensity. This process is generated by independent random variables $X_1, X_2, \ldots$, $W_1, W_2, \ldots$ with $X_i \sim Q$, $W_i \sim Exp(\lambda)$, where W_i is the interarrival time between claim X_{i-1} and X_i if $i \geq 2$, and W_i is the waiting time until the first claim. Then $N(t)$ can be written as

$$N(t) = \max\{k : W_1 + \ldots + W_k \leq t\}.$$

Claim X_i occurs at time $T_i = W_1 + \ldots + W_i$, $i \geq 1$. The process $R(t)$ has independent stationary increments. In particular it is Markov with respect to the natural filtration F_t generated by $R(t)$, in the following sense: for any set A in the sigma-field generated by $R(u), u \geq t$, the conditional probability

$P\{A \mid F_t\}$ depends on $R(t)$ alone,

$$P\{A \mid F_t\} = P\{A \mid R(t)\}.$$

This is the standard model for nonlife insurance, simple enough to calculate probabilities of interest. It is a useful formulation as it models the two major reasons for big losses: frequent claims and large claims separately.

16.2.2 Sparre-Andersen and Markov-Modulated Models

In insurance, other risk models discussed are the Sparre-Andersen model and the Markov-modulated risk process. In both classes of models, the claims X_i are iid independent of the claims arrival process $N(t)$. In the Sparre-Andersen model, it is developed as a renewal process

$$N(t) = \max\{k : W_1 + \ldots + W_k \leq t\}$$

with iid positive random variables W_i which are independent of the sequence of claim sizes X_i. In this model the parameters are s, c, Q, R where R is the distribution of the interarrival times W_i.

The Markov-modulated risk model takes a continuous time homogeneous Markov process $M(t)$ on the state space $\{1, \ldots I\}$, and with intensities $0 \leq \lambda_1 < \lambda_2 < \ldots < \lambda_I$ and uses the process $\lambda(t) = \lambda_{M(t)}$ as stochastic intensity of an inhomogeneous Poisson process $N(t)$. Here the parameters are $s, c, Q, \lambda_1, \ldots, \lambda_I, b_{ij}, i, j = 1, \ldots, I$, where b_{ij} are the transition intensities of the Markov process $M_{(t)}$. Also in this model, $R(t)$ is not Markovian, while the process $(R(t), \lambda(t))$ is a Markov process.

By extending the Lundberg risk process we can include constant interest, in which the reserve earns interest at a constant rate r. In this process, the jumps are the same as in the Lundberg process, and between claims the process evolves with the dynamics

$$R'(t) = c + rR(t)$$

16.2.3 Ruin Probability

A classical risk measure is the infinite time ruin probability,

$$\psi(s) = P\{R(t) < 0 \text{ for some } t \geq 0\}$$

which equals one as long as $c \leq \lambda E[X_1]$ without safety loading. When there is safety loading, we have

$c > \lambda \;\; \mu$ with

$R(t) \to \infty$. The ruin probability satisfies the following first order integro-differential equation

$$0 = \lambda E[\psi(s - X) - \psi(s)] + c\;\psi'(s), s \geq 0 \qquad (16.1)$$

For exponential claim sizes with density

$$f(x) = \theta \exp(-\theta x), x > 0$$

the ruin probability equals

$$\psi(s)=\frac{\lambda\mu}{c}\exp(-Rs), \quad (16.2$$

where $\mu=1/\theta$ is the mean claim size, and $R=(c-\lambda\mu)/(c\mu)$ is the adjustment coefficient of the problem which is the positive solution r of the Lundberg equation

$$\lambda+rc=\lambda E\left[\exp(rX)\right]. \quad (16.3)$$

You can derive the equation (16.2) from (16.1). Consider the survival probability $\delta(s)=1-\psi(s)$, for which $\delta(s)=0$ for $s<0$ and

$$0=\lambda(g(s)-\delta(s))+c\delta'(s), s\geq 0 \quad (16.4)$$

where

$$g(s)=g(s)=E\left[\delta(s-X)\right]=\int_{\theta}^{\delta}\delta(s-x)\theta e^{-\theta x}dx=\int_{\theta}^{\delta}\delta(x)\theta_e^{-\theta(s-x)}dx$$

It is easy to see that on the set $\{s\geq 0\}$, $g(s)$ satisfies the differential equation

$$g'(s)=\theta m(\delta(s)=g(s)),$$

Therefore, on the set $\{s\geq 0\}$, the function $\delta(s)$ has a continuous second derivative $\delta''(s)$ for which

$$0=\lambda(g'(s)-\delta'(s))+c\,\delta''(s)$$

$$=\lambda\theta(\delta(s)-g(s))-\lambda\delta'(s)+c\delta''(s)$$

$$c\theta\delta'(s)-\lambda\delta'(s)+c\delta''(s).$$

The general solution of the above linear differential equation with constant coefficients can be obtained from

$$\delta(s)=C_1+C_2\exp(-Rs)$$

since $z=0$ and $z=-R$ are the solutions to the characteristic equation

$$0=(c\theta-\lambda)z+cz^2.$$

Using $\delta(s)\rightarrow 1$ for $s\rightarrow\infty$ we obtain $C_1=1$. From (16.4) at the point s = 0 we obtain $\lambda\delta(0)=c\delta'(0)$ or $(1+C_2)=-cRC_2$, or

$$-c_2=\frac{\lambda}{cR+\lambda}=\frac{\lambda\mu}{c}.$$

In the Markov modulated environment, the ruin probability $\psi(s,i)$ depends on the initial surplus and the initial value of the process $\lambda(t):\lambda(0)=\lambda_i$. The functions $\psi(s,i)$ satisfy the following interacting system of first order integrodifferential equations:

$$0=\lambda_i E\left[\psi(s-X,i)-\psi(s,i)\right]+c\psi_s(s,i)+\sum_{j=1}^{I}b_{ij}\psi(s,j), s\geq 0, i=1,\ldots,I.$$

The ruin probability $\psi(s)=\psi(s,0)$ can be derived from a function $\psi(s,t)$, where s is the initial surplus and t is the current time since the last claim. If the waiting times W_i have a continuous density *f(x)*, then the function $\psi(s,t)$ satisfies the following integro-differential equation:

$$0=\frac{f(t)}{1-F(t)}E[\psi(s-X,t)-\psi(s,+)]+c\psi_s(s,t)+\psi_t(s,t), s\geq 0,\ t\geq 0(s-x,t)$$

In case of Lundberg risk process with constant interest, the ruin probability $\psi(s)$ satisfies the integro-differential equation

$$0=\lambda E[\psi(s-X)-\psi(s)]+(c+rs)\psi'(s),\ s\geq 0\ .$$

As we will see below, these integro-differential equations are derived with the infinitesimal generators of the underlying risk processes. In the remainder of this section we shall restrict ourselves to Lundberg risk processes.

16.2.4 Asymptotics for Ruin Probabilities

Equation (16.2) shows the behaviour of ruin probabilities for small claim sizes for which the adjustment coefficient exists,

$$\psi(s)\sim C\exp(-Rs)$$

with $C>0$, and where $a(s)\sim b(s)$ means $a(s)/b(s)\to 1$. For example this relation holds, if $r_0=\sup\{r:E[\exp(rX)]<\infty\}>0$

and $\lim_{r\to r_0} E[\exp(rX)]=\infty$.

For large claims with heavy tailed distributions (for which $r_0=0$) the behaviour is totally different: for Pareto claims with density $f(x)=ax^{-(a+1)},x>1,a>1$, we have $\psi(s)\ \sim Cs^{-(a-1)},s\to\infty$,

with a positive constant C. For heavy tailed claim size distributions Q, we have

$$\psi(s)\sim C\int_s^{\infty}Q(x,\infty)dx\ .$$

If you increase the initial surplus, the ruin probability reduces to half. In the exponential claims case, the increased initial surplus is $s_1=s+\log(2)/R$ while in the Pareto claims case $s_1\sim 2^{\frac{1}{(a-1)}}s$.

Check Your Progress 1

1) What is Lundberg risk model?

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2) Discuss the features of infinite time ruin probability.

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3) What do you mean by asymptotics for ruin probabilities?

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16.3 CONTORL VARIABLES

Let us work with a Lundberg risk process with parameters c, λ, and Q. To see how an insurance company manages the risk in portfolio with claim in this model, we will consider the possible options in reinsurance, investment, volume control (through setting of premia), portfolio selection (through combination of the given risk portfolio with other risks which can be written), and the combination of all these actions. Since we consider the problem overtime, the risk management requires decisions making which adjust with time. We will treat actions involving only one control variable and try to find the optimal (with respect to some given objective) dynamic strategy.

16.3.1 Optimal Investment

Let the insurer has the option of investing in a risky asset or a riskless asset (say, in a bank account) with interest r. Allow the insurer with wealth $R(t)$ to invest at each point in time t an amount $A(t)$ into the risky asset, and what is left is on the bank account earning (costing) interest r if $R(t)-A(t)>0\left(\text{if } R(t)-A(t)<0\right)$. For simplicity, let us take the classical Samuelson model ie., logarithmic Brownian motion for the dynamics of the asset prices $Z(t)$:

$$dZ(t)\,aZ(t)\,dt + dZ(t)\,dW(t), Z(0) = z_0,$$

where $W_1(t)$ is a standard Wiener process. If $\theta(t)=A(t)/Z(t)$ is the number of shares held at time t, then the wealth distribution of the insurer has the following dynamics:

$$dR(t) = rR(t)\,dt + cdt - dS(t) + \theta(t)\,dZ(t) - r\theta(t)\,Z(t)\,dt, R(0) = s,$$

or

$$dR(t) = rR(t)\,dt + cdt - dS(t) + A(t)\left((a-r)\,dt + bdW(t)\right), R(0) = s.$$

To simplify the setup and the notation we shall restrict ourselves to the case $r = 0$.

Let us have all possible trading strategies $\theta(t)$ which – as stochastic processes – are (F_t) –predictable, where (F_t) is the filtration generated by the two processes $Z(t)$ and $S(t)$, $t \geq 0$. For selection $\theta(t)$, we may use the knowledge of all stock prices and claims before time t and not the knowledge at time t which might be the size of a claim happening at time t. There is no budget constraint such as $\theta(t)Z(t) \leq R(t)$, one can borrow an arbitrary amount of money and invest it into the risky asset. We will also neglect transaction costs and allow for shares of any size.

16.3.2 Optimal Proportional Reinsurance

Recall that in a proportional reinsurance contract each individual claim of size X is divided between first insurer and reinsurer according to a proportionality factor a. In this form of insurance the insurer pays aX, whereas reinsurer pays $(1-a)X$. Moreover, the insurer pays a reinsurance premium $h(a)$ to the reinsurer. We allow a continuous adjustment of the proportionality factor, such that $a(t)$ is (F_t) predictable. Under the strategy $a(t)$ the risk process of the first insurer is given by

$$R(t) = s + ct - \int_0^t h(a(v))dv - \sum_{i=1}^{N(t)} a(T_i)X_i, t \geq 0.$$

We follow the premium rule of

$h(a) = a\rho\lambda E[X]$, the expectation principle

with $\rho > 1$. If $c \geq \rho\lambda E[X]$ and the first insurer wants to minimize her risk then she would choose $a(t) = 0$ and pass on all the risk to the reinsurer. As such a situation is not interesting we will assume that reinsurance is expensive:

$c < \rho\lambda\ E[X]$.

Optimal Unlimited *XL* Reinsurance

As we know, in excess of loss (XL) reinsurance each claim of size X is divided between the first insurer and the reinsurer according to a priority $0 \leq b \leq \infty$: the insurer pays $\min(X, b)$, and the reinsurer pays $(X-b)^{+} - \max\{X-b, 0\}$. In this form, the insurer pays a reinsurance premium $h(b)$ to the reinsurer. In order to put the problem in a dynamic set up we allow a continuous adjustment of the proportionality factor so that $b(t)$ is (F_t)-predictable. Under the strategy $b(t)$ the risk process of the first insurer is given by

$$R(t) = s + ct - \int_0^t h(b(v))dv - \sum_{i=1}^{N(t)} \min\{b(T_i), X_i\}, t \geq 0.$$

As in case of proportional reinsurance, we will go for the expectation principle for premium, i.e.,

$$h(b) = \rho\lambda E\left[(X-b)^{+}\right]$$

with $\rho > 1$. Further we assume that reinsurance is expensive, so that $c < \rho\lambda\, E[X]$. Other premium principles would be the variance principle

$$h(b) = \lambda E\left[(X-b)^{+}\right] + \beta\lambda E\left[\left((X-b)^{+}\right)^{2}\right].$$

which puts more weight to the tail of the distribution of the claim size, or the standard deviation principle

$$h(b) = \lambda E\left[(X-b)^{+}\right] + \beta\sqrt{\lambda E\left[\left((X-b)+\right)^{2}\right]}.$$

Note that $c < h(0)$ so that reinsurance is expensive.

16.3.4 Optimal XL-Reinsurance

When XL-reinsurance contracts are limited by some constant $0 \le L \le \infty$, which leads to the following division of a claim of size X, the reinsurer pays $\min\left\{(X-b)^{+}, L\right\}$, and the first insurer pays what is left:

$g(X,b,L) = \min\{X,b\} + (X-b-L)^{+}$. For this the insurer pays a reinsurance premium $h(b,L)$. Under a dynamic XL-reinsurance contract with strategy $(b(t), L(t))$ the insurer has the following risk process:

$$R(t) = s + ct - \int_0^t h(b(v))L(v)\,dv - \sum_{i=1}^{N(t)} g\left(b(T_i), L(T_i)\right).$$

Reinsurance is expensive if $c < h(0,\infty)$. Possible premium schemes are the expectation principle

$$h(b,L) = \rho\lambda E\left[\min\left\{(X-b)^{+} L\right\}\right]$$

the variance principle

$$h(b,L) = \lambda E\left[\min\left\{(X-b)^{+}, L\right\}\right] + \beta\lambda E\left[\min\left\{(X-b)^{+}, L\right\}^{2}\right]$$

or the standard deviation principle.

To minimize her risk, an insurer will choose $L(t) = \infty$ if she can afford it. So $L(t) < \infty$ will be a reasonable choice for her only if the tail of the claim size distribution matters for the reinsurance premium, as is the case in the variance or in the standard deviation principle.

16.3.5 Optimal Premium Control

It is reasonable to assume that an insurer can control the volume of her business by setting the premium c. The higher the premium rate c, the smaller the number of contracts in her portfolio, which in turn will decrease the claims intensity λ. If we take a non-increasing function $\lambda(c)$, with $c(t)$ is the instantaneous premium rate charged, then $\lambda(c(t))$ will be the instantaneous intensity of the claims process. A realistic model has $\lambda(\infty)=0$, and in order to get a non trivial risk minimization problem, one has to change the framework a bit since otherwise the insurer would reduce her risk to zero by the choice of an infinite premium rate. One possibility is the introduction of cost of capital, i.e., for the initial surplus an interest rate ρ has to be paid continuously.

16.3.6 Optimal New Business

Consider an insurer controlling the risk in one given portfolio by writing an appropriate proportion of business in a second independent portfolio. If $R(t)$ and $R_1(t)$ are the two independent insurance portfolios, which are both modelled as Lundberg risk processes with parameters λ, c, Q and λ_1, c_1, Q_1, respectively, and if $b(t)$ is the proportion written at time t in portfolio $R_1(t)$ then the wealth position of the insurer comprises premium income up to time t, that will be equal to

$$ct + \int_0^t \lambda_1 b(u)\,du.$$

Let us look at the claims to the paid by the insurer. The claims paid up to time t are $S(t) = X_1 + \ldots + X_{N(t)}$ for the first portfolio with of distribution compound Poisson and parameters λt and Q. In case of the second portfolio, claims paid $S(t)$ have with instantaneous claims intensity $\lambda_1 b(t)$ and claim size distribution Q_1. For practical applications one has to assume that $b(t) \geq 0$ (no short selling of insurance business) and $b(t) \leq 1$ (the maximum possible volume written is $R_1(t)$).

Check Your Progress 2

1) How do you identify the control variables in Lundberg risk process?

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2) Describe the model formulated by Samuelson for investment in risky assets.

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3) How would you formulate the problem of portfolio allocation in optimal new business?

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16.4 STOCHASTIC CONTROL

To see the working of the stochastic control consider Merton's formulation of optimal investment and consumption problem. Recall that in its simplest form an investor has initial wealth r_0. She may dynamically consume a part of her wealth and invest dynamically another part of it in a risky asset with price process modelled as logarithmic Brownian motion. So, we have

$$dX(t) = X(t)\big(adt + bdW(t)\big), X(0) = x_0.$$

The remaining portion of the wealth is put on a bank account earning interest at a constant rate r. If $A(t)$ is the amount invested and $c(t)$ the rate of consumption at time t, then the wealth $R(t)$ has the dynamics

$$dR(t) = A(t)\big(adt + bdW(t)\big) + \big(R(t) - A(t)\big)rdt - c(t)dt, R(0) = r_0$$

Our objective is to find the optimal strategy $\big(A(t), c(t)\big)$ which maximises expected accumulated utility of consumption

$$E\left[\int_0^\tau e^{-\rho t} u\big(c(t)\big)dt\right]$$

where $\tau = \inf\{t : R(t) < 0\}$ is the ruin time for the investor, $\rho > 0$ is a subjective interest rate (appreciation rate), and $u(x) = x^\gamma, \gamma < 1$, is a special utility function. Here we maximise the risky asset by the two control variables $A(t)$ and $c(t)$.

16.4.1 Objective Functions

In order to define the optimization problem, we need to specify the planning horizon and the quantity, which should be maximised. To do this, we consider finite horizon and infinite horizon. In case of the finite horizon, optimization is done over a finite interval $[0, T]$, with a general objective function that has two components: the running cost and the final cost. If $\sigma(t)$ is the strategy with values in an action space Σ, then the general objective function to be maximised is

$$E\left[\int_0^{\tau\wedge T} u(R(t),\sigma(t),t)\,dt + U(R(T),\sigma(T))\right],$$

where τ is some stopping time (such as ruin time or first entry time into a certain region).

In the case of infinite horizon, a general objective function considered running costs and terminal costs and we write

$$E\left[\int_0^{\tau} u(R(t),\sigma(t),t)\,dt + U(R(\tau),\sigma(\tau),\tau)\right],$$

where τ is an unlimited stopping time and $U(R(\tau),\sigma(\tau),\tau)$ could be bankruptcy cost when τ is time of ruin. For the above mentioned infinite horizon Merton problem, we have

$u(r,\sigma,t) = \exp(-\rho t)(c(t))^{\gamma}$, $\gamma < 1$ with $U = 0$.

If $c(t)$ is taken to be a dividend rate, then the quantity

$$E\left[\int_0^{\tau} \exp(-\rho t)c(t)^{\gamma}\,dt + U(R(\tau))\right]$$

yields the value of the company where $U(s)$ is the cost of default when the final capital is s.

To focus on the present problem our concern is with ruin (survival) probabilities, i.e., running cost is zero, and $U = 1$ if $\tau = \infty$, $U = 0$ elsewhere.

16.4.2 Infinitesimal Generators

Infinitesimal generators L are defined for Markov processes $R(t)$ and for (sufficiently smooth) functions $f(s)$ on the state space taking

$$L_t f(s) = \lim_{h\to\infty}\frac{1}{h}E\left[f(R(t+h)-f(t)) \mid R(t)=s\right],$$

where the function $f(s)$ is restricted to the domain D of L for which this limit exists. Taking the Markov process to be time homogeneous, we can have the infinitesimal generator that does not depend on t and D can be linear, with L_t a linear operator. Note that we cannot specify the domains of the generator. However, it should be clear in each case that it contains the set of all functions $f(s)$ having bounded derivatives of all orders. If the process are stationary, than we have

1) $R(t) = a + bt : Lf(s) = cf'(s);$

2) $dR(t) = a(R(t))dt + b(R(t))dW(t) : Lf(s) = a(s)f'(s) + \frac{1}{2}b^2(s)f''(s);$

3) $R(t) = s + ct - S(t)$, the Lundberg risk process:
   $Lf(s) = \lambda E\left[f(s-X) - f(s)\right] + cf'(s);$

4) $R(t)$ the Lundberg risk process with constant interest r :
   $Lf(s) = \lambda E\left[f(s-X) - f(s)\right] + (c+rs)f'(s);$

5) $(R(t), M(t))$ from the Markov- modulated risk process:

$$Lf(s,i) = \lambda_i E\left[f(s-X,i)\right] + cf_s(s,i) + \sum_{j=1}^{J} b_{ij} f(s,j)$$

6) $(R(t), T(t))$ from the Sparre-Andersen model:

$$Lf(s,t) = \frac{f(t)}{1-F(t)} E\left[f(s-X,t) - f(s,t)\right] + cf_s(s,t) + f_t(s,t).$$

In the following we will need the infinitesimal generator for a controlled risk process, where the control strategy is constant. In case of optimal investment with a constant amount A invested in a risky asset, for example, the total position $R(t)$ of the insurer has the dynamics

$$dR(t) = cdt - dS(t) + A\left(adt + ndW(t)\right).$$

The infinitesimal generator for the process $(R(t), X(t))$ (which is Markov) will be

$$Lf(s,x) = \lambda E\left[f(s-X,x) - f(s,x)\right] + cf_s(s,x) + Aaf_s(s,x) + \frac{1}{2} A^2 b^2 f_{ss}(s,x).$$

See that the infinitesimal generator is independent of x (for logarithmic Brownian motion, the initial value has no influence on the return of an investment). This helps us write

$$Lf(s) = \lambda E\left[f(s-X) - f(s)\right] + cf'(s) + Aaf'(s) + \frac{1}{2} A^2 b^2 f''(s).$$

Consequently, for proportional reinsurance with a constant proportion the risk process of the insurer is

$$R(t) = s + \left(c - h(a)\right)t - a\sum_{i=1}^{N(t)} X_i,$$

and the corresponding infinitesimal generator is

$$Lf(s) = \lambda E\left[f(s-aX) - f(s)\right] + \left(c - h(a)\right) f'(s).$$

In case of unlimited XL-reinsurance with constant probability $b \in [0,\infty]$ the generator is

$$Lf(s) = \lambda E\left[f(s - X \wedge b) - f(s)\right] + \left(c - h(b)\right) f'(s),$$

and for the general reinsurance contract $g(X,a)$ with reinsurance premium $h(a)$ and fixed decision vector a the generator is

$$Lf(s) = E\left[f\left(s - g(X,a)\right) - f(s)\right] + \left(c - h(a)\right) f'(s).$$

The integro-differential equations for the ruin probability (s) are all of the form

$$L\psi(s) = 0, s \geq 0,$$

where L is the infinitesimal generator of the underlying risk process.

Check Your Progress 3

1) Discuss the meaning of stochastic control taking logarithmic Brownian motion.

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2) How do you formulate the objective functions in a model for stochastic control?

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3) What do you mean by infinitesimal generators?

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16.5 LET US SUM UP

In this unit we have discussed the optimization decision making in insurance given the presence of stochastic processes. The Lundberg risk model is used to settle the claims under a compound Poisson process. As an alternative to this, we have presented Sparre – Andersen and Markov – modulated models. Both these highlight the claim arrivals processes where claims are iids. The ruin probabilities derived from these models can be obtained using integro-differential equations. It is seen that such a formulation requires infinitesimal generators of the underlying risk processes. Taking investment in risky assets as a control variable, we have formulated the optimization problems for reinsurance, new business and premium settings.

16.6 KEY WORDS

Asymptotic Analysis: A method of classifying limiting behaviour by concentrating on some *trend.* It can be brooadly expressed in terms of equvalece relations.

Geometric Brownian Motion (GBM) (or Exponential Brownian Motion): A continuous-time stochastic process in which the logarithm of the randomly varying quantity follows a Brownian motion, or, a Wiener process. It is used in option pricing because a quantity that follows a GBM may take any value strictly greater than zero, and only the fractional changes of the random variate are significant. This is precisely the nature of a stock price.

Infinitesimal Transformation: A limiting form of small transformation. To smoothen a function it can be used as a differential operator just as is done with Taylor's theorem.

Log-Normal Distribution: The probability distribution of a random variable whose logrithm is normally distributed. If X is a random variable with a normal distribution, then $\exp(X)$ has a log-normal distribution.

Pareto Distribution: If X is a random variable with a Pareto distribution, then the probability that X is greater than some number x is given by

$$\Pr(X > x) = \left(\frac{x}{x_m}\right)^{-k}$$

for all $x \geq x_m$, where x_m is the (necessarily positive) minimum possible value of X, and k is a positive parameter.

16.7 SOME USEFUL BOOKS

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Rolski, T. Schmidli, H. Schmidt, V. Teugels, J. (1998), Stochastic Processes for Insurance and Finance, Wiley Series in Probability and Statistics

16.8 ANSWER OR HINTS TO CHECK YOUR PROGESS

Check Your Progress 1

1) See Sub-section 16.2.1 and answer.

2) See Sub-section 16.2.3 and answer.

3) See Sub-section 16.2.4 and answer.

Check Your Progress 2

1) See Section 16.3 and answer.

2) See Sub-section 16.3.1 and answer.

3) See Sub-section 16.3.6 and answer.

Check Your Progress 3

1) See section 16.4 and answer.
2) See sub-section 16.4.1 and answer.
3) See sub-section 16.4.2 and answer.