UNIT 17 RESERVING TECHNIQUES

Structure

17.0 OBJECTIVES

After going through this unit, you will be able to:

- appreciate the provision of reserve in insurance;
- identify the reserve-types;
- see the principles underlying reserving methods;
- select the deterministic and stochastic reserve techniques from insurance data; and
- understand the reserving models.

17.1 INTRODUCTION

The reserves required at any time are the resources needed to meet the costs, as they arise, of all claims not finally settled at that time. The insurer must be able to quantify this liability if it is to assess its financial position correctly, both for statutory and for internal resource planning purposes.

Insurance companies hold reserves because the timing of premiums receipt and claims payment does not coincide. When there is a delay between the claim event and the claim settlement dates, the insurer must have reserves in respect of those claims still to be settled to help maintain business credibility.

Therefore, the primary reasons to hold reserves are: (i) premiums have been received but the cover relating to part of the premiums has not yet been completed; (ii) claims are due but not yet have been paid, or claims that have been reported and not yet reported; (iii) other expenses incurred by the insurance company that have yet to be paid; (iv) the nature of business, as in case of a general insurance company, requires to hold technical reserves; and (v) contingent reserves are to be held to provide a buffer against adverse developments of claims and to smoothen the emergence of profit.

There are various reasons that require estimation of the amount to be held in reserve. The main reasons include: assessment of solvency and profitability, business planning (such as the negotiation of contact commutation, assistance in sale and purchase negotiations), rating, claim management, as well as for tax purposes and advice on portfolio reinsurance.

The following material for your study is prepared taking inputs from the work of Pantzopoulou (2003) and Jamenez-Huerta, D.(2005). The tables with numerical data are taken from these studies without acknowledging the source. Our thanks to these authors. We have not covered the projections of reserves in this unit. If you are interested for a reference on such aspects the first study will be quite useful.

17.2 CATEGORIES OF RESERVES

Usually the reserves held fall into the following categories:

Unexpired or unearned exposure

- Unearned premium reserve (UPR)
- Deferred acquisition costs (DAC)
- Additional unexpired risk reserve (AURR)

Contingent reserves

- Catastrophe reserves
- Claims equalisation reserves

Earned exposure

- Notified outstanding claims, which can be subdivided into:
  - notified (open) claims
  - reopened claims
- Incurred but not reported claims (IBNR)
- Incurred but not enough reported (IBNER) on existing notified claims

Unexpired or Unearned Exposure Reserve

Usually the accounting period for reporting purposes does not coincide with the end of the policy period. Such a feature creates a liability for unexpired cover and the insurer is to provide coverage beyond the accounting date. Unearned premium reserves relate mostly to business accounted for one-year and will be substantial for policies which have a duration of greater than one year.

There are occasions when no unexpired cover exists at the valuation date. For example, a company providing holiday insurance cover, where the contract duration is generally short, and having a financial year outside the holiday period.

There is a distinction between 'unearned premium' and 'unexpired risk' reserve. Essentially, the reserve for unexpired risk could exceed the UPR if the premium basis upon which the original business was written proves to be inadequate. In such an event, the UPR is increased to a level consistent with the true risk profile of the business through an 'additional unexpired risk reserve' (AURR). So,

$$\text{UPR} = \text{URR} + \text{AURR}.$$

If the UPR is likely to exceed the insurer's liabilities on the unexpired cover, then URR = UPR. Note that due to accounting practices, any insurance profits implicit in the UPR are not usually recognised until the premium has been earned. Therefore, to establish the UPR, one can simply apply proportion:

$$\text{UPR} = \frac{\text{period of unexpired cover} \times \text{original premium}}{\text{duration of original policy}}$$

The problem with such an approach, however, is that the unexpired cover may not occur uniformly neither during in case of the calendar year nor throughout the policy period. For example, in case of crop insurance worst claims period is during bad weather seasons, particularly coinciding with harvest time.

When the risk of the policy over the term is be uniform, and other items paid for by the premium are not be uniformly spread over the policy, a more generalised formula for UPR is adhered to. The UPR in respect of an individual policy is expressed as:

$UPR = P(n) * F(n) * (1-k)$ where, n is the date of policy inception;

P(n) = premium received in respect of a policy written at time n ;

F(n) = proportion of the cover not exposed for this policy at the d accounting date;

k = proportion of the gross premium lost through acquisition expenses that has been treated as earned (that is, a revenue account expense) when the policy was written.

Note that, $DAC = P(n)^* \; F(n)^*(1-k)$.

It is not a common practice to calculate UPR on an individual basis. Generally, the premiums received are aggregated within similar classes of business by month or quarter. Then UPR is calculated on the assumption that the premiums on average were received half way through the period. For example, take a business to be monitored on a monthly basis with a 12-month policy duration and with intensity of uniform risk spread through out the policy period. Such an approach gives rise to the '24ths' method like,

$$F(n) = \frac{2n-1}{24},$$

where n is the month in the accounting year of the inception of cover (n =1,2,....12). Similarly, if premiums are assumed to be received uniformly over each quarter of a year, we will have a '8ths' method.

For the purpose of operational convenience, the UPR is usually set up net of reinsurance since it is not always possible to identify the particular reinsurance premium or unexpired reinsurance cover that corresponds with either an individual policy or a group of policies.

When consideration is given to the other elements of the premium, such as profit and contingent loading, along with unexpired risk reserve, an additional amount is added to reserve amount. The additional amount for unexpired risks can be expressed as:

$$P(n)_* \; F(n)^*(1-k)_* \; (L-1),$$

where $L(>1)$ is the claim plus claim expense ratio, excluding acquisition costs.

17.2.1 Contingent Reserves

Contingent reserves are set up as a precautionary measure to provide additional funds. Such reserves are generally known as catastrophe reserves and claims equalisation reserves.

Catastrophe reserves are basically held for catastrophe events, as well as a reserve for any adverse experience.

Claims equalisation reserves are set up to smooth profits. They are often used to try and dampen the effects of the business cycle or volatile claims experience.

17.2.2 Earned Exposure Reserve

Companies need to reserve for such claims that have occurred and have not yet been paid. There are delays in the claim recognition process which necessitates reserves for outstanding claims. The basic categories of reserve for this purpose are:

- reserves in respect of advised losses,
- reserves in respect of losses that have been incurred but not yet reported (IBNR).

Amounts in respect of advised losses can be further broken down into:

- specific case reserves;
- reserves for reopened claims;
- additional reserves in respect of case reserves believed to be inadequate (reserves for claims 'incurred but not enough reported', or IBNER).

IBNR and IBNER require a statistical treatment based on expected trends. There are several statistical methods available that can be used to estimate IBNR claims separately. Such methods include the **delay table** method and the **projection method**; the choice of method will depend mainly on the length of the tail of the class of business and the relative size of this particular class of business to the company.

The case reserve normally reflects the expected ultimate settlement value of an individual claim against the insurance company, such as expenses paid to external parties involved in the settlement. The basic process of arriving at a case reserve include: claim notification, file establishment, reserve estimation, loss adjustment, claim settlement and salvage. The following discussion gives a general overview of the claims reserving methodology before getting into the methods of reserving in detail.

Check Your Progress 1

1) What do you understand by

   i) UPR ii) IBNR and iii) IBNER?

   ....

   ....

   ....

   ....

2) Distinguish between unearned premium and unexpired risk reserve.

   ....

....

....

3) How would you establish UPR?

....

....

....

....

17.3 BASIC PRINCIPLES OF RESERVING METHODS

In general, the claims reserving process consists of two parts. The first part is the analysis of the data available, and the fitting of suitable models. The second part consists of using the results of the modelling process to project future claims experience.

17.3.1 Run-off Triangles

(In the following discussion we have taken the presentation sequence and numerical example from the lecture note of D. Jimenez-Huerta and thank the author)

A claims settlement process follows the pattern:

Claim event occurred → Claim reported → Claim payment(s) made → Claim file closed.

There are certain insurance contracts in which claim payments are settled only after some considerable time after a claim is reported. For example, such a requirement may come after the end of the origin year. That is, the year in which the policy was written (started). In other words, in a future claim year.

However, since claims are attributed to the origin year, the insurer needs enough reserves to cover future payments from claim events that have been

- reported but not yet settled (agreed);
- incurred but not yet reported (IBNR).

In such type of situation, the insurer does not know the exact total claim amount from the policies written in each origin year. We will discuss some of the ways of estimating such amounts. There are two types of approaches that are used, viz., deterministic and stochastic. We start with the deterministic methods where an estimate is obtained from claims data (for a portfolio) taking into account the year that policies were written and the year they were settled. As we will see these data are summarised in a run-off triangle.

Three methods of carrying out the estimation are

- Basic chain-ladder method

- Average cost per claim method
- Bornheutter-Ferguson method.

Note that in each of these methods, the calculations can be adjusted such that they allow for inflation.

17.3.2 Chain Ladder Method

Let us start in the some terminology that will be used.

Origin year (or accident year): year when policy is written in order to cover an event.

Development year (or delay year): number of years until payment from a claim is made.

Presentation of claims data (run-off triangle)

We present claim payments per year in a run-off triangle or delay triangle as in the following (fictitious) example.

Example

Let us take the following example, (the amounts considered here are in thousand rupees).

Table 3.1

		Development year			
		0	1	2	3
	1997	300	250	100	65
Origin	1998	400	350	160	
Year	1999	500	360		
	2000	700			

The 'diagonals' in the ↗ direction give payments for a particular year.

Alternatively, we can present cumulative claim payments in each row as in Table 3.2 for the claim payments in given in Table 3.1.

Table 3.2

		Development year			
		0	1	2	3
	1997	300	550	650	715
Origin	1998	400	750	910	
Year	1999	500	860		
	2000	700			

In this example our objective to estimate the entries in the lower right triangle of Table 3.1, which represent the future payments from policies written over the origin years in the table, i.e., project the future development of the run-off triangle.

Projections using development factors

Assume that payments will emerge in a similar way in each origin year. However, the observed proportionate increases in cumulative payments will not be same as that of the origin year. For that see Table 3.3 for the data in Table 3.2.

Table 3.3

		From year – year		
		0-1	1-2	2-3
	1997	1.83	1.18	1.10
Origin	1998	1.88	1.21	
year	1999	1.72		
	2000	-		

For example, for 1997 origin year, cumulative payment to development year 1999 is 1.18 times that for 1998 (development years 1 to 2).

The increases vary within each column of Table 3.3, so what single value should we use to represent the increase in each column (i.e., as we change from one development year to the next)?

We could use the largest (the most conservative), or the mean (a disadvantage is that the mean doesn't take into account that the years with more claims provide more information).

So we could use a weighted mean with weights as cumulative claims so far.

Let us go from development year 0 to 1, where the numerator is really the sum of the cumulative claims for the next development year from Table 3.2.

The quantity of development factor for the cumulative payments from year 0 to year 1 from Table 3.2 are:

Table 3.4

From year – year		
0 – 1	1 – 2	2 – 3
$\frac{2160}{1200}$ =1.80	$\frac{1560}{1300}$ =1.20	$\frac{715}{650}$ =1.10

Note that the development factors are calculated for cumulative payments rather than the original yearly payments. The underlying logic is greater stability of the cumulative payements in comparison with yearly payments.

For origin year 1999, we obtain cumulative claims for years 2 and 3 as follows:

Development year	
2	3
860×1.20 =1032	1032×1.10 =1135.2

Same method can be applied to origin year 1998 as well. Table 3.5 summarises the projections of cumulative payments. We can then calculate the projected payments per development year; these are given in Table 3.6

Table 3.5

		Development year			
		0	1	2	3
	1997				
Origin	1998				1001.0
year	1999			1032	1135.2
	2000		1260	1512	1663.2

Table 3.6

		Development year			
		0	1	2	3
	1997				
Origin	1998				91.0
year	1999			172	103.2
	2000		560	252	151.2

Model Checking

To check whether the method of projection is reasonable, we start with the actual claims in development year 0 and project them using the development factors. Table 3.7 shows the results for the cumulative claims in Table 3.2 using the development factors calculated for these data in Table 3.4.

Table 3.7

Development year:		0	1	2	3
Factor:			1.80	1.20	1.10
	1997	300	540	648	712.8
Origin	1998	400	720	864	
year	1999	500	900		
	2000	700			

Table 3.8 shows the resulting estimated claim payments assuming the cumulative claims shown in Table 3.7 and compares these with the actual claim payments given in Table 3.1.

Table 3.8

			Development Year			
			0	1	2	3
Origin year	1997	Actual		250	100	65
		Estimate	300	240	108	64.8
		Error		10	-8	0.2
	1998	Actual		350	160	
		Estimate	400	320	144	
		Error		30	16	
	1999	Actual		360		
		Estimate	500	400		
		Error		-40		

However, we still need to decide whether an 'error' is too large – we could look at 'relative error' but a better approach is to use statistical modelling and to look at residuals.

17.3.3 Average Cost Per Claim Method

This method also takes into account the number of claims made and estimates of the number of future claims. Then the average cost per claim per development year can be obtained for the claims fully processed and estimates obtained for the future years.

For the data presented in Table 3.1, suppose the claim amounts were from the number of claims shown in Table 3.9.

Table 3.9

		Development year			
		0	1	2	3
	1997	300	150	13	5
Origin	1998	380	200	20	
year	1999	450	215		
	2000	650			

At this stage you can derive the average costs per claim. Refer to Tables 3. 1 and 3.9. Calculate the average cost figures to be placed as diagonal elements in the triangle 13{=(65/5)}; 7.879{=(100+160)/(13+20)}; and 1.699

{=(250+350+360)/150+200+215)}.

Estimates of the future number of claims in the bottom part of the run-off triangle in Table 3.9 can be obtained using the basic chain ladder method. The development factors based on the cumulative number of claims (not given here; you do this as has been done in Table 3.2) are:

Table 3.10

From year – year		
0 – 1	1 – 2	2 – 3
$\frac{1695}{1130}$ =1.500	$\frac{1063}{1030}$ =1.032	$\frac{468}{463}$ =1.011

Hence, after estimating the future cumulative numbers of claims, estimates of the future numbers of claims are shown in Table 3.11. Estimates of the future claim amounts, obtained by multiplying the entries in Table 3.11 by the average claim amounts for each development year derived above (1.699,7.879 and 13.0), are shown in Table 3.12.

Table 3.11

		Development year			
		0	1	2	3
	1997				
Origin	1998				6.48
year	1999			21.31	7.41
	2000		325	31.24	10.86

Table 3.12

		Development year			
		0	1	2	3
	1997				
Origin	1998				84.24
year	1999			167.90	96.33

From Table 3.12, an estimate of the reserve needed to meet outstanding claims from these policies is 1287.99 i.e.,

{(84.24)+(167.9+96.33)+(552.21+246.13+141.18)}.

17.3.4 Bornheutter-Ferguson Method

A simple estimate of reserve for each origin year could be made by multiplying the premiums received for that year by the expected loss ratio, where the loss ratio for a given year is the ratio of ultimate claim losses to the total premium received.

In this method we need to know the loss ratio expected for each origin year. The Bornheutter-Ferguson method combines the prior expectation of losses provided by simple loss ratio estimates with the actual rate of emergence of claims.

While working with it, we require the initial expected ultimate ratio for each origin year, the total premium for each origin year and the expected development pattern of claims.

Let us use the claims experience given in Table 3.1 and assume an initial loss ratio of 80% along with the total premiums (in 000's) in the second column of Table 3.14. The claims data in Table 3.1 provide the expected development pattern of claims through the development factors (see Table 3.13). We will need the proportion of ultimate claims paid off by the end of development year j, denoted earlier by r_j. The calculation of these is shown in Table 3.13, where the ultimate year(see next section for its meaning) is year 3. The final row in Table 3.13 gives the r_j's (left for you as an exercise).

Table 3.13

Development period	0 – 1	1 – 2	2 – 3
Development factor	1.80	1.20	1.10
Development period	**0 – ultimate**	**1 – ultimate**	**2 – ultimate**
Development factor	2.376	1.32	1.10
Inverse	0.4209	0.7576	0.9091

Table 3.14

Origin year	Total Premium	Naïve estimate of ultimate claims	Estimated proportion outstanding, $1 - r_j$	Estimated reserve
1997	1300	1040		
1998	1400	1120	0.0909	101.82
1999	1600	1280	0.2424	310.30
2000	1800	1440	0.5791	833.94

From the sum of the entries in the last column of Table 3.1, an estimate of the reserve needed to meet outstanding claims from these policies is 1246.06.

Adjusting for Inflation

Consider to adjust for inflation we have neglected so far. Let the claims inflation be at the same annual rate for all claims within a calendar year of payment (i.e., on a $\nearrow$ diagonal of the run-off triangle).

Reformatting the data in Table 3.1 with the following annual claims inflation rates (%) we get,

1997/1998	1998/1999	1999/2000
4.0	3.5	3.0

When adjusted with the claim figures given in Table 3.1 using these rates, i.e., by converting the amounts to 2000 figures, we get,

Table 3.15

		Development year			
		0	1	2	3
	1997	332.61	266.51	103	65
Origin	1998	426.42	360.05	160	
Year	1999	515	360		
	2000	700			

The above table records, for the 1997 origin year, 65 is at the 2000 value, 103 is the original 100 multiplied by the 1999/2000 inflation rate of 1.03, and similarly 266.51 is $250 \times 1.03 \times 1.035$ and 332.61 is $300 \times 1.03 \times 1.035 \times 1.04$. Use the same process for the other origin years.

The last diagonal in Table 3.1 is at 2000 prices, the penultimate diagonal is at 1999 prices and so needs to be multiplied by 1.03, and so on. Then proceed as before. If we have estimates of future inflation rates, then the projected amounts in the lower triangle can be adjusted for inflation too.

Check Your Progress 2

1) Write the meaning of Run-off triangles and development factors.

....

....

....

2) What is difference between chain ladder method and average cost per claim method?

....

....

....

....

....

....

3) What is Bornleutter – Ferguson method?

....

....

....

....

17.4 RUN-OFF TECHNIQUES

To project the future claims run-off techniques are used. The format used in this technique contains the claim data (payments, numbers, case estimates, etc.) according to the year in which the claim arose and the year in which the payment was made. The difference between the payment year and the accident year is called the development year. Other time periods, particularly for short-tail classes, can also be put into this format. However, instead of grouping by year of accident, we may group according to year when the claim was reported, or even the year of underwriting.

The methods of reserves are grouped into (1) statistical reserves and (2) case reserves. Statistical reserves are calculated for groups of policies all at once usually based on the past development pattern of similar business written in previous years. Case reserves are calculated for each policy on an individual basis.

Case estimates are mainly used when there are (i) few estimates left, (ii) when there are large claims and (iii) assumptions for with statistical methods are invalid. Remember that case estimates do not calculate any reserve for IBNR claims.

Example

Consider the following example in order to illustrate the chain ladder method. The data are taken by Pantzopoulou from Ackman et al (1983).

Table 4.1 Incremental Paid Claims

Year of Origin	Development Period					
	1	2	3	4	5	6
1991	52 546	28 729	9186	7816	4885	3102
1992	62 285	36 210	11 601	8250	5336	
1993	72 173	41 126	11 041	8543		
1994	86 135	41 224	11 050			
1995	97 068	53 408				
1996	128 982					

The first stage is to turn the incremental data into cumulative data. The cumulative paid claims are shown below in Table 4.2.

Table 4.2 Cumulative Paid Claims

Year of Origin	Development Period					
	1	2	3	4	5	6
1991	52 546	81 275	90 461	98 277	103 162	106 264
1992	62 285	98 495	110 096	118 346	123 682	
1993	72 173	113 299	124 340	132 883		
1994	86 135	127 359	138 409			
1995	97 068	150 476				
1996	128 982					

At this point it is essential to study the data to see if there have been any disturbances in the patterns of claims development.

Plotting the data to get a visual indication of the development pattern will enable us to see the consistency of the development for each year of origin and to identify any unusual features.

It is also useful to calculate the development ratios between successive cumulative payments, within the year of origin. The development ratio is obtained by dividing one years' figure by the previous years' figure. The results are shown below in Table 4.3.

Table 4.3 Ratio of Cumulative Payments in Successive

Year of Origin	Development Period				
	1-2	2-3	3-4	4-5	5-6
1991	1.547	1.113	1.086	1.050	1.030
1992	1.581	1.118	1.075	1.045	
1993	1.570	1.097	1.069		
1994	1.479	1.087			
1995	1.550				
1996					

The best-case scenario is that the claim development ratios in each column are very similar to each other and do not depend on the year of origin. For this set of data, a variance can be observed for the development ratios in each development period.

However, the development ratios can still be considered stable, despite the observed variance.

The next stage now, having assumed stable development ratios, is to calculate weighted development factors.

The volume weighted development factors are calculated as:

$$\text{development factor } (j \to j+1) = \frac{\sum_{i=1}^{n-j} c_{i,j+1}}{\sum_{i=1}^{n-j} c_{i,j}},$$

The calculations for the weighted average succession ratios are shown below in Table 4.4.

Table 4.4 The Weighted Average Succession Ratios

	Development Period				
	1-2	2-3	3-4	4-5	5-6
Ratio	$\frac{570\,904}{370\,207}$ = 1.542	$\frac{463\,306}{420\,428}$ = 1.102	$\frac{349\,506}{324\,897}$ = 1.076	$\frac{226\,844}{216\,623}$ = 1.047	$\frac{106\,264}{103\,162}$ = 1.030

The calculated average development factors are then used to complete the development triangle of year-to-year ratios as shown in Table 4.5.

Table 4.5 Ratio of Cumulative Payments in Successive Development Periods

We complete here the Table 4.3. See Table 4.5

	Development Period				
Year of Origin	1-2	2-3	3-4	4-5	5-6
1991	1.547	1.113	1.086	1.050	1.030
1992	1.581	1.118	1.075	1.045	**1.030**
1993	1.570	1.097	1.069	**1.047**	**1.030**
1994	1.479	1.087	**1.076**	**1.047**	**1.030**
1995	1.550	**1.102**	**1.076**	**1.047**	**1.030**
1996	**1.542**	**1.102**	**1.076**	**1.047**	**1.030**

Table 4.5 leads to the completed cumulative period claims triangle, shown in Table 4.6.

Table 4.6 Cumulative Paid Claims

Year of Origin	Development Period					
	1	2	3	4	5	6
1991	52 546	81 275	90 461	98 277	103 162	106 264
1992	62 285	98 495	110 096	118 346	123 682	**127 401**
1993	72 172	113 299	124 340	132 883	**139 153**	**143 337**
1994	86 135	127 359	138 409	**148 893**	**155 998**	**160 606**
1995	97 068	150 476	**165 823**	**178 383**	**186 799**	**192 416**
1996	128 982	**198 906**	**219 192**	**235 794**	**246 920**	**254 344**

The estimated reserve is found by taking the difference between column 6, the estimated ultimate claim value, for years of origin 1992 to 1996, and the current cumulative paid claims. The results are shown in Table 4.7. Note that the 1991 year of origin is assumed to be fully developed.

Table 4.7 Reserve Estimates on Chain-Ladder Basis

Year of Origin	Estimated Ultimate Claims	Payments to Date	Reserve
1992	127 401	123 682	3719
1993	143 337	132 883	10 454
1994	160 606	138 409	22 197
1995	192 416	150 476	41 940
1996	254 344	128 982	125 362
Total	**878 104**	**674 432**	**203 672**

Additionally, the year-to-year factors can be multiplied together to give factors that take claims from a given level of maturity to ultimate. The inverse of these factors represents the proportion of ultimate that is expected at each delay period. Table 4.8 shows the calculations.

Table 4.8 Calculation of Development Year Factor R_j

Development Period	1-2	2-3	3-4	4-5	5-6
Development Factors	1.542	1.102	1.076	1.047	1.030
Development Period	1 – ultimate	2 – ultimate	3 – ultimate	4 – ultimate	5 – ultimate
Cumulative Factors	1.972	1.279	1.160	1.079	1.030
Inverse (R_j)	0.057	0.782	0.862	0.927	0.971

From the base model of the chain-ladder,

$$C_{ij} = S_i R_j \text{ and } S_i = \frac{C_{ij}}{R_j}$$

S_i is usually estimated using the latest calendar period information, that is, in *this* case,

S_{1991} =106 264/1=106 264

S_{1992} =123 682/0.971=127 376

S_{1993} =132 883/0.927=143 347

S_{1994} =138 309/0.862=160 567

S_{1995} = 150 476/0.782=192 425

S_{1996} = 128 982/0.507=254 402

Table 4.9 gives estimates of ultimate claims using the above Rj at each point in the triangle. The differences between the S_i values and the ones given in the table are due to rounding the factor R_j to three decimal places in the above calculations.

Note that, the estimates of S_i should be constant if the trends within the data are stable, regardless of the delay period at which the estimate is made.

Table 4.9 Estimated Ultimate Claims at Each Development Period

	Development Period					
Year of Origin	1	2	3	5	5	6
1991	103 617	103 928	104 969	106 009	106 264	106 264
1992	122 822	125 947	127 753	127 656	127 401	
1993	142 320	144 877	144 281	143 337		
1994	169 853	162 856	160 606			
1995	191 412	192 416				
1996	254 344					

This simple analysis provides initial insights into trends within the data that might be outside the scope of the model. It further provides a crude indication of the stability with which the ultimate claims would have been estimated in the past, and, hence, the reserving error that could be inherent in using this method.

Estimates of the incremental future payments can also be obtained using this model, by simply taking the differences of the successive values of Table 4.6. The results are shown in Table 4.10.

Table 4.10 Incremental Paid Claims

Year of Origin	Development Period					
	1	2	3	4	5	6
1991	52 546	28 729	9186	7816	4885	3102
1992	62 285	36 210	11 601	8250	5336	**3719**
1993	72 173	41 126	11 041	8543	**6270**	**4184**
1994	86 135	41 224	11 050	**10 484**	**7025**	**4688**
1995	97 068	53 408	**15347**	**12 560**	**8417**	**5617**
1996	128 982	**69 924**	**20 286**	**16 602**	**11 126**	**7425**

Projected future payments by calendar period can be obtained from the diagonals of the incremental cash flow triangle, shown in Table 4.11.

Table 4.11 Estimated Future Payments by Calendar Year

Year of Origin	Calendar Year of Payment				
	1997	1998	1999	2000	2001
1991					
1992	3719				
1993	6270	4184			
1994	10 484	7025	4688		
1995	15 347	12 560	8417	5617	
1996	69 924	20 286	16 602	11 126	7425
Total	**105 744**	**44 055**	**29 707**	**16 743**	**7425**

At this point, having obtained estimates of future payments, estimates of reserves on a discounted basis can also be calculated. When discounting claims, a view is taken on the returns achievable on the invested funds that match the technical liabilities. In practice, most companies choose not to discount outstanding payment liabilities, as this provides a useful, implicit margin against claims deteriorating significantly.

Generally, insurance regulators permit discounting provided that all future costs have been recognised; that the selected discount rate is conservative and that there has been full disclosure in the company's accounts.

The basic chain-ladder is usually used to estimate the numbers of claims incurred or to be finalised, for use in payment per claim analyses and projections. It can also be used with claim payments and incurred costs. However, extreme care is required in the interpretation of the results of the analysis, in the cases where there have been large payments or large changes in estimates. Nevertheless, the method is useful as a check on other, more sophisticated methods, because it has fewer estimated parameters and therefore is subject to less estimation error. For small and highly variable portfolios, the problems of parameter estimation may make most other methods unusable.

There are many variations on the basic chain-ladder method but all have the same objective: to extract from the loss development triangle a pattern for the claims run-off that can, possibly with some additional manual adjustment, be used to extrapolate the less mature years of account.

17.5 ALTERNATIVE: METHOD OF RATIOS

An alternative variation on the chain-ladder method is to use the ratios of the payments in succeeding development years instead of the ratios of the cumulative payments.

When this approach is used, there is some loss of stability, particularly at the higher development years, where the numbers of open claims are smaller and the payments are more variable. However, more information is available, since all of the payment data for earlier origin years can be used. For later development periods, the chain -ladder ratios are based on a very small number of data points. Stability may be achieved well by using the extra data points.

Additionally, in order to keep the form of the valuation basis the same as for the chain-ladder method, the ratios of payments in succeeding years can be converted into cumulative ratios. The method of ratios can be applied to the original or inflation-adjusted payments, which we have given in the last scheme. Lastly, the outstanding claim estimates may be discounted or undiscounted as for the chain-ladder method.

17.6 STOCHASTIC CLAIMS RESERVING

The chain-ladder technique we have discussed above is a deterministic method for predicting claim amounts.

Stochastic models make assumptions about the expected value of future payments and also model the variation in those. By making assumptions about the random component of a model, these models allow the validity of the assumptions to be tested statistically, and produce estimates not only of the expected value of the future payments, but also of the variation about that expected value.

A disadvantage of a stochastic reserving method is that it models an immensely complex series of events with a few parameters, and as a result it is open to criticism that its assumptions are far too simple and hence unrealistic.

Overall, the primary advantage of stochastic reserving models is the availability of measures of precision of reserve estimates, and in this respect, attention is focused on the root mean squared error of prediction in the following.

The following discussion illustrates stochastic models which reproduce reserve estimates given by the chain-ladder technique.

17.6.1 Stochastic Chain-Ladder Models

While considering claims reserving in a stochastic context, we will assume that the data consist of a triangle of incremental claims.

Therefore, we assume that we have the following set of incremental claims data:

$$\{C_{ij} : i = 1, K, n; j = 1, K, n - i + 1\} \text{ where}$$

i refers to the row, and could indicate accident year or underwriting year;

j refers to the column, and indicates the delay, assumed to be measured in years.

Then the cumulative claims are defined by:

$$D_{ij} = \sum_{k=1}^{j} C_{ik},$$

and the development factors are denoted by $\{\lambda_j : j = 2, k, n\}$. The development factors are estimated by the chain-ladder technique such that we have

$$\hat{\lambda}_j = \frac{\sum_{i=1}^{n-j+1} D_{ij}}{\sum_{i=1}^{n-j+1} D_{i,j-1}}.$$

The development factors are then applied to the latest cumulative claims in each row $(D_{i,n-i+1})$ to produce forecasts of future values of cumulative claims:

$$\hat{D}_{i,n-i+2} = D_{i,n-i+1}\hat{\lambda}_{n-i+2}$$

In the chain-ladder technique forecasts of claims are the ultimate claims only. The term, 'Ultimate' refers to the latest delay year so far observed and the does not include any tail factors. In the outcome, in order to be able to assess whether extra reserves should be held for prudence, over and above the predicted values. The measure of variability which is used is the prediction error, defined as the standard deviation of the distribution of possible reserve outcomes.

To obtain the prediction error, we formulate an underlying statistical model making assumptions about the data. The predicted values should be the same as those of the chain-ladder technique to provide a stochastic model which is analogous to the chain-ladder technique. For that purpose we have to specify the distributions of the data or specify the first two moments.

Note that the basic objective of all the models is to produce the same reserve estimates as the chain-ladder technique.

17.6.2 Over-Dispersed Poisson Model

Under the over-dispersed Poisson distribution the variance is proportional to the mean. In claims reserving, the over-dispersed Poisson model assumes that the C_{ij} are distributed as independent over-dispersed Poisson random incremental claim variables, with mean and variance

$$E\left[C_{ij}\right] = m_{ij} = x_i y_j \text{ and } Var\left[C_{ij}\right] = \phi x_i y_j$$

where,

$$\sum_{k=1}^{n} y_k = 1$$

x_i is the expected ultimate claim and y_j is the proportion of ultimate claim to emerge in each development year; ϕ is a parameter which is unknown and estimated from the data.

y_j appears in the variance, and constrained to be positive. As a result, the sum of incremental claims in column j must also be positive.

In this formulation, the mean has a multiplicative structure, by being a product of a row effect and a column effect. However, re-parameterisation of the model is often applied for estimation purposes, so that the mean has a linear form. A log link function is used in the generalised linear models, so that

$$\log(m_{ij}) = c + \alpha_i + \beta_j$$

Note that, $\log(m_{ij})$ is known as the linear predictor, denoted by η_{ij}. This predictor structure is still a chain-ladder type, in the sense that there is a parameter for each row i, and a

To obtain the well behaved estimates constraints are applied to the sets of parameters, for example,

$$\alpha_i = \beta_1 = 0$$

17.6.3 Negative Binomial Model

The negative binomial model is derived from the over-dispersed Poisson model using different parameterisation. It is expressed as a model for either incremental or cumulative claim C_{ij}, incremental claims, have an over-dispersed negative binomial distribution, where

$$\text{mean} = (\lambda_j - 1)D_{i,j-1} \text{ and variance} = \phi\lambda_j(\lambda_j - 1)D_{i,j-1}.$$

λ_j is analogous to the standard chain-ladder development factor; ϕ is an unknown dispersion parameter in the variance, making the distribution 'over-dispersed'.

Note that $D_{ij} = D_{i,j-1} + C_{ij}$, assuming that $D_{i,j-1}$ is known. This implies that the model can also be written in terms of the cumulative claims, where D_{ij} has an over dispersed negative binomial distribution, with

$$\text{mean} = \lambda_j D_{i,j-1} \text{ and variance} = \phi\lambda_j(\lambda_j - 1)D_{i,j-1}.$$

17.6.4 Normal Approximation to Negative Binomial Model

The normal model as an approximation to the negative binomial model has the advantage that it can produce estimates for a wide range of data sets, and is less affected by the presence of negatives.

Using a normal approximation for the distribution of incremental claims, C_{ij} is approximately normally distributed, with

$$\text{mean } D_{i,j-1}(\lambda_j - 1) \text{ and variance } \phi_j D_{i,j-1},$$

or equivalently D_{ij} is approximately normally distributed with

$$\text{mean } \lambda_j D_{i,j-1} \text{ and variance} = \phi_j D_{i,j-1}.$$

This holds where $D_{i,j-1}$ is known, and a recursive approach is required to estimate the variance where cumulative claims are forecasted.

The negative binomial model, as well as the normal approximation model are recursive models where the likelihood is written down by taking the data in a certain order.

17.6.5 Mack's Model

The model of Mack (1993) takes a similar recursive approach as the negative binomial model and the normal model.

Limited assumptions are made as to the distribution of the underlying data. Only the first two moments are specified. The mean and variance of D_{ij} are

$\lambda_j D_{i,j-1}$ *and* $\sigma_j^2 D_{i,j-1}$ respectively.

Mack produced estimators of the unknown parameters λ_j and σ_j^2. Making further limited assumptions, Mack also provided formulae for the prediction errors of predicted payments and reserve estimates.

Mack considers the model to be distribution-free, since the full distribution of the underlying data is not specified. This simplifies the model but it limits analysis of the distribution of outstanding reserves to the first two moments only. If the results are used in a dynamic financial analysis exercise (see next unit), where the distribution of outstanding reserves might be simulated, further with assumptions are to be made.

Lastly, note that the mean and variance of D_{ij} under the Mack's model is similar to the mean and variance of D_{ij} in the normal approximation to the negative binomial model, with the unknown scale parameters ϕ_j of the normal approximation being replaced by σ_j^2 in Mack's.

Check Your Progress 3

1) Differentiate between deterministic and stochastic chain reserving techniques.

....

....

....

....

....

2) How deterministic chain-ladder model is different from stochastic chain-ladder model?

....

....

....

....

....

3) Discuss how negative binomial model can be presented as a normal model.

....

....

....

....

....

17.7 LET US SUM UP

In this unit we have discussed the reserving concept in insurance and techniques used for that purpose. It is seen that reserve is required to meet the costs that arise from the unsettled claims. As the timing of premium receipt and claims payment do not coincide, such a provision helps maintain business credibility. Main categories of reserves held by companies are unexpired or unearned exposure, contingent reserves and earned exposure. The basic principle of reserve methods are related to the pattern of claim settlement process.

Run-off triangles or delay triangles are used to prepare the claims data for insurance contracts when claim payments are settled only after a considerable time. Chain ladder method, average cost per claim method and Bernheutter-Forguson methods are used for carrying out reserve estimation.

The chain ladder method estimates the number future claims on the basis of available data on claims already made. In average cost per claim method, the average cost per claim per development year is obtained to fully process the claims and obtains the estimates for the future years. Lastly, in Bornheutter-Forguson method, a simple estimate of reserve for each origin year is made by multiplying the premiums received for that year by the expected loss ratio. The loss ratio is derived by dividing the ultimate claim losses with the total premium received. These three methods are deterministic. Stochastic claim reserving is based on the expected values of future payments. The stochastic chain-ladder models are built accordingly. There are three other models, viz., over-dispersed Poisson model, negative binomial model and Mack's model, which are developed on the basis of probability distribution on claim settlement process.

17.8 KEY WORDS

Chain-Ladder: A method of claim reserve during the development period of insurance.

Claim Ladder Statistics: A method of showing loss development statistics.

Contingent Reserves: Reserves set up as a precautionary measure to provide additional funds.

Development Year: Number of years until payment from a claim is made.

Loss Development: Refers to estimating the amount of loss that has not yet happened.

Loss Ratio: The ratio of incurred losses and loss-adjustment expenses to net premiums earned. This ratio measures the company's underlying profitability, or loss experience, on its total book of business.

Loss Reserve: The estimated liability, as it would appear in an insurer's financial statement, for unpaid insurance claims or losses that have occurred as of a given evaluation date. Usually includes losses incurred but not reported (IBNR), losses due but not yet paid, and amounts not yet due. For individual claims, the loss reserve is the estimate of what will ultimately be paid out on that claim.

Origin Year: Year when policy is written to cover an event.

Reserve: An amount representing actual or potential liabilities kept by an insurer to cover debts to policyholders. A reserve is usually treated as a liability.

Run-off Triangle: A claim payment format given in a tabular form and read from the diagonal bottom left to top right in each payment year.

Stop Loss: Any provision in a policy designed to cut off an insurer's losses at a given point.

Surplus: The amount by which assets exceed liabilities.

17.9 SOME USEFUL BOOKS

Pantzopoulou, E. (2003), The Concept of Reserving and Reserving Methodologies in General insurance, a dissertation submitted for Ph. D. in M. Sc. Actuarial Management, Dept. of Actuarial Science and Statistics, city University, London (available in Internet).

Booth, P., Chadburn, R., Cooper, D., Haberman, S., James, D. (1996), Modern *Actuarial Theory and Practice, Chapman & Hall/CRC, London*

Daykin, C.D., Pentiken, T., Pesonen, M. (1994), Practical Risk Theory for Actuaries, Chapman & Hall, London

Jamenet-Huerta, D. (2005), Reserving Methods: Run-off Triangles, Lecture Notes (posted in Internet).

Taylor, G.C. (2000), Loss Reserving: An Actuarial Perspective, Kluwer Academic Publishers, London

17. 10 ANSWER OR HINTS TO CHECK YOUR PROGRESS

Check Your Progress 1

1) See Section 17.2 and answer
2) See Section 17.2 and answer

3) See Section 17.2 and answer

Check Your Progress 2

1) See Sub-section 17.3.1 and answer
2) See Sub-section 17.3.2 and 17.3.3 and answer
3) See Sub-section 17.3.4 and answer

Check Your Progress 3

1) See Section 17.5 and 17.6 and answer
2) See Sub-section 17.3.2 and 17.6.1 and answer
3) See Sub-section 17.6.3 and 17.6.4 and answer

UNIT 18 DYNAMIC FINANCIAL ANALYSIS

Structure

18.0 OBJECTIVES

After going through this unit, you will be able to

- understand the scope of dynamic financial analysis (DFA) in insurance industry; and
- apply the DFA technique to life and non-life insurance problems.

18.1 INTRODUCTION

Insurance companies, especially those dealing with non-life insurance, often witness pricing cycles accompanied by volatile insurance profits and increasing catastrophe losses. Under such circumstances, shareholders is, therefore, necessary to evaluate the economic factors behind such cycles and to identify their nature and interrelationships.

The discussion that follows has three major components, viz.,

i) What is DFA,

ii) Stochastically modelled variables and

iii) Corporate model.

The following discussion is based on the work entitled, **Introduction to Dynamic Financial Analysis** (see Kaufmann, et al (2001)). To facilitate learner's exposition to formulation of the technical ideas, we have followed the sequence and notation of the above work borrowing freely. Keeping in view the introductory requirement of the theme for our course we propose to use the important portions of the work to help understand the basic formulations of DFA. However, once we understand the basic concepts, if will be useful to go through the entire work.

18.2 WHAT IS DFA

There are two primary techniques available to analyse financial effects of different business strategies viz., scenario testing and stochastic simulation or dynamic financial analysis (DFA).

The scenario testing predicts the business results only under deterministic scenario and when risk associated with a specific case can be quantified. The other technique overcomes such a flaw. It has the advantage of generating a number of scenarios stochastically by allowing for the full probability distributions of important variables such as surplus, written premiums or loss ratios.

18.2.1 ALM in Life Insurance

There exist many traditional Asset-Liability Management (ALM) approaches in life insurance which consider the liabilities as more or less deterministic due to their low variability. This approach, however, cannot be extended to nonlife insurance where we are faced with much more volatile liability cash flows. In nonlife insurance both the date of occurrence and the size of claims are uncertain. Moreover, claim costs in it are information sensitive in contrast to life insurance where these are expressed in nominal terms. Thus, it would be better to deal with nonlife insurance liabilities and assets on stochastic simulations.

18.2.2 Objectives of DFA

DFA is a part of the financial management of a firm. Therefore, we analyse the management of profitability and financial stability (risk control function of DFA) through its results. While the first task aims at maximising shareholder valuation of the firm, the second serves maintaining customer value. Within these two parameters DFA tries to explain strategic management decisions such as asset allocation, capital allocation, performance measurement, market strategies, business mix, pricing decisions and product design.

The list above suggests that DFA can be used to answer broader issues such as, the primary beneficiaries (shareholder and management) objectives of an individual company.

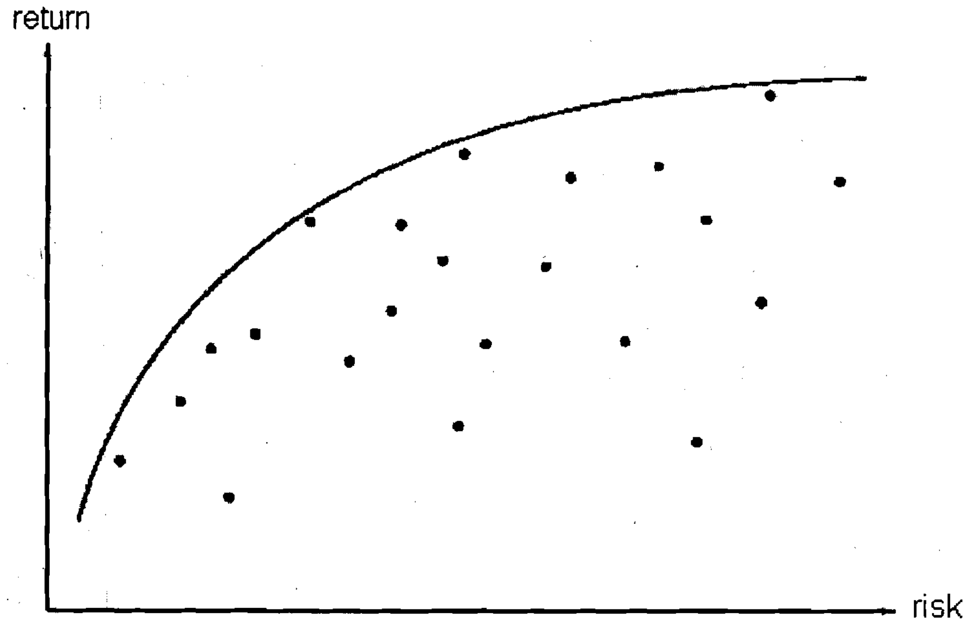

Fig. 18.1: Efficient frontier
(Source: Kaufmann et al (2001))

18.2.3 Fixing the Time Period

It is necessary to start with a time horizon that has to be fixed for analysing the business strategies. On the one hand, we would like to model over a long time period to see the long-term effects of a chosen strategy. Particularly, effects concerning long-tail business are realised after a lapse of some years and can hardly be recognised in the first few years. On the other hand, simulated values become more unreliable when the projection period is longer, due to accumulation of process and parameter risk over time. Therefore projection period of five to ten years seems to be a reasonable choice. Usually the time period is split into yearly, quarterly or monthly sub-periods.

18.2.4 Analysing DFA and Efficient Frontiers

The most common framework used in DFA is the efficient frontier concept, which is used in modern portfolio theory. Thus, we have to choose a return measure (e.g., expected surplus) and a risk measure (e.g., expected policyholder deficit). Then the measured risk and return of each strategy can be plotted as shown in Figure 18.1. The figure gives each strategy representing one spot in the risk-return intersection. A strategy is called efficient if there is no other one with lower risk at the same level of return, or higher return at the same level of risk. For each level of risk there is a maximal return that cannot be exceeded, giving rise to an efficient frontier.

Thus the efficient frontier serves as a tool to compare different strategies through the variables risk and return.

Although efficient frontiers are a good means of communicating the results of DFA, there are criticisms against these. A typical efficient frontier uses risk measures that mix together systematic risk(non-diversifiable by shareholders) and non-systematic risk, which blurs the shareholder value perspective. Moreover, efficient frontiers might give misleading insight if they are used to address investment decisions once the concept of systematic risk has been plugged into the equation.

18.2.5 Solvency Testing

DFA is used as solvency testing tool where the financial position of the company is evaluated from the perspective of the customers. It is used to quantify in probabilistic terms whether the company will be able to meet its commitments in the future. In the process, it determines the necessary amount of capital necessary for a given the level of risk the company is exposed to.

18.2.6 Structure of a DFA Model

It can be seen from Figure 18.2 that (i) the **stochastic scenario generator** produces realisations of random variables representing the most important drivers of business results. A realisation of a random variable in the course of simulation corresponds to fixing a scenario; (ii) **data source** consists of company specific inputs (e.g., mean severity of losses per line of business and per accident year), assumptions regarding model parameters (e.g., long-term mean rate in a mean reverting interest rate model), and strategic assumptions (e.g., investment strategy); and (iii) the **output** provided by the DFA model, can then be analysed to improve the strategy, i.e., to make new strategic assumptions.

Check Your Progress 1

1) What is ALM approach to life insurance? Why this approach cannot be relied upon for nonlife insurance?

....

....

....

....

....

2) What is DFA?

....

....

....

....

3) What do mean by efficient frontiers in DFA?

....

....

....

....

....

4) Explain the concept of solvency testing.

....

....

....

....

18.3 STOCHASTICALLY MODELLED VARIABLES

While building a stochastic model, it is necessary is to identify the key random variables affecting asset and liability cash flows.

Specifically, the risks affecting the financial position of a life/nonlife insurer can be categorised in various ways. For example, we must have groups like pure asset, pure liability and asset/liability risks. A DFA model should at least address the following risks:

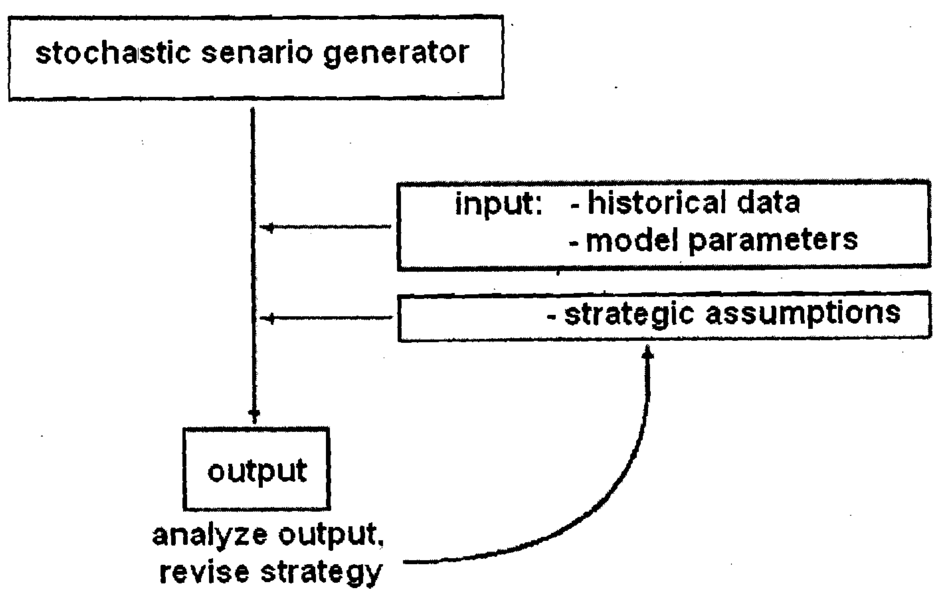

Fig. 18.2: Main structure of a DFA model.

(Source: Kaufmann et al (2001))

- risk of inadequate premiums,
- interest rate risk,
- risk of insufficient reserves,
- investment risk associated with volatile investment returns and capital gains and

We would also like to include credit risk related to reinsurer default, currency risk and exchange rate risks. A critical part of a DFA model is the interdependencies between different risk categories. Particularly, we have to take into account risks associated with the asset side and those belonging to liabilities.

18.3.1 Evaluation of Variables

Looking at the list of variables given above it will become apparent that these belong to asset and liability categories of the financial statement of a company. While evaluating the variables we may follow such a categorisation. It is necessary to be able to tackle the problem of evaluating interest rate risk. Since nonlife insurance companies are often exposed to interest rate behaviour due to large investments in fixed income assets, it will be useful to assume that interest rates are strongly correlated with inflation. Consequently, the inflationary influences of the future changes in claim size and claim frequency must be included. As there is a correlation between interest rates and stock returns, investment returns are affected. The model should explicitly recognise such a feature.

Let us turn to the liability side. Four sources of randomness, viz., *non-catastrophe losses, catastrophe losses, underwriting cycles, and payment patterns* may have to be considered from it. We may simulate catastrophes separately due to their different statistical behaviour from non-catastrophe losses. Such a separation helps in having more homogeneous data for non-catastrophe losses, which makes fitting the data by well-known (right skewed) distributions easier; in case of reinsurance programme evaluation of external events with severity of claims have also to be treated separately.

Underwriting Cycles: In case of non-life insurance, underwritings reflect market and macroeconomic conditions, which affect the business results. Therefore, it would be useful to include these in a DFA model.

Losses influence insurance firms by their size and piecewise payment over time. Such features increase the uncertainties of the claims process by introducing the time value of money and future inflation considerations. Consequently, it is necessary to model claim frequency and severity keeping in view the uncertainties involved in the settlement process. In order to allow for reserving risk we may use stochastic payment patterns as a means of estimating loss reserves on a gross and on a net basis.

Note that for each of the DFA components considered, there are numerous alternatives evaluation methods, which may be more appropriate in particular situations. We have provided a model framework here that serves as a suggested reference point that can be adjusted or improved on case-by-case basis.

18.3.2 Short-Term Interest Rate

A specified choice of an interest rate model is not available. A list of some general features and identified by Kaufmann et al (2001) is given below:

- Volatility of yields varies at different maturities
- Interest rates are mean-reverting.
- Rates at different maturities are positively correlated.
- Interest rates should not be allowed to become negative.
- The volatility of interest rates should be proportional to the level of the rate.

In addition to these characteristics, the authors specify the practical issues that may be addressed are:

- *flexible* enough to cover most situations arising in practice,
- *simple* enough that one can compute answers in reasonable time,
- *well-specified,* in that required inputs can be observed or estimated,
- *realistic,* in that the model will not include unsubstantiated elements.

Needless to point out that an interest rate model meeting all the above criteria does not exist. Therefore, studies often rely on the one-factor Cox – Ingersoll –Ross (CIR) model. CIR is one of the models where the instantaneous rate is modelled as a special case of an Ornstein– Uhlenbeck process. Thus,

$$dr = k(\theta - r)dt + \sigma r^{\gamma} dZ \qquad (2.1)$$

By setting $\gamma = 0.5$, we arrive at CIR also known as the square root process

$$dr_t = a(b - r_t)dt + s\sqrt{r_t}dZ_t \qquad (2.2)$$

where r_t = instantaneous short-term interest rate,

b = long-term mean,

a = constant that determines the speed of reversion of the interest rate toward its long-run mean b,

s = volatility of the interest rate process,

(Z_t) = standard Brownian motion.

CIR is a mean-reverting process where the short rate stays positive almost surely.

For simulating the short rate dynamics over the projection period we need to discretise the mean reverting model (2.2). This is done in (2.3) as

$$r_t = r_{t-1} + a(b - r_{t-1}) + s\sqrt{r_{t-1}}Z_t, \qquad (2.3)$$

where

r_t = the instantaneous short-term interest rate at the beginning of year t,

$$Z_t \sim N(0,1), Z_1, Z_2, \ldots\ \text{i.i.d.}$$

a, b, s same as in (2.2).

Although the short rate process does not have negative values, in its discrete version, the last equation's probability is not zero. To meet such a condition Equation (2.3) is changed to

$$r_t = r_{t-1} + a(b - r_{t-1}) + s\sqrt{r_{t-1}^{+}}Z_t.. \qquad (2.4)$$

A generalisation of CIR is given by the following equation, where setting $g = .05$ yields again CIR:

$$r_t = r_{t-1} + a(b - r_{t-1}) + s(r_{t-1}^{+})^{g} Z_t. \qquad (2.5)$$

This general version is flexible in determining the degree of dependence between conditional volatility of interest rate changes and the level of interest rates.

It is important to remember the following points while calibrating the above model;

i) choice of valuing g is not unique.

ii) the parameters, a, b, s and g must be determined so as to ensure that modelled spot rates (based on the instantaneous rate) correspond to empirical term structures derived from traded financial instruments;

iii) building models of in inflation based on historical data may be a feasible approach. However, it is not certain if the future evolution in action will follow historical patterns. Therefore, besides historical data, economic reasoning and actuarial judgment of future development may have to be considered.

18.3.3 Term Structure

Based on equation (2.2) we can calculate the prices $F(t,T,(r_t))$ at time t of zero-coupon bonds which pay 1 monetary unit at time of maturity t +T. Thus,

$$F(t,T,(r_t)) = \mathrm{E}_Q\left[e^{-\int_0^T r_{t+s}ds} \mid \gamma_t\right] = e^{\log A_T - r_t B_T}, \qquad (2.6)$$

where

$$A_T = \left(\frac{2Ge^{(a+G)T/2}}{(a+G)(e^{GT-1})+2G}\right)^{2ab/s^2},$$

$$B_T = \frac{2(e^{GT}-1)}{(a+G)(e^{GT}-1)+2G},$$

$$G = \sqrt{a^2 + 2s^2}$$

The continuously compounded spot rates $R_{t,T}$ at time t derived from equation (2.6) determine the modelled term structure of zero-coupon yields at time t:

$$R_{t,T} = -\frac{\log F\left(t,T,(r_t)\right)}{T} = \frac{r_t B_T - \log A_T}{T}, \qquad (2.7)$$

where T is the time to maturity.

18.3.4 General Inflation

While modelling loss payments, inflation has to be taken into account and we may use the (annualised) short-term interest rate r_t for that purpose. By using a linear regression model on the short-term interest rate we can have

$$i_t = a^I + b^I r_t + \sigma^I \varepsilon_t^I \ , \qquad (2.8)$$

where

$$\varepsilon_t^I \sim N(0,1), \varepsilon_1^I, \varepsilon_2^I, \ldots \ i.i.d.,$$

a^I, b^I, σ^I : parameters that can be estimated by regression, based on historical data; and index I indicates general inflation.

18.3.5 Change by Line of Business

Lines of business are affected differently by macro economic decision-making. Costs of claims for specific lines of business are strongly affected by legislative and court decisions, e.g., product liability. This gives rise to inflation (due to superimposed exogenous intervention).

To model the change in loss frequency δ_t^F i.e., the ratio of number of losses divided by number of written exposure units, the change in loss severity δ_t^X, and the combination of both of these, δ_t^P, the following formulae can be used:

$$\delta_t^F = \max\left(a^F + b^F i_t + \sigma^F \varepsilon_t^F, -1\right), \qquad (2.9)$$

$$\delta_t^X = \max\left(a^X + b^X i_t + \sigma^X \varepsilon_t^X, -1\right), \qquad (2.10)$$

$$\delta_t^P = \left(1+\delta_t^F\right)\left(1+\delta_t^X\right) - 1, \qquad (2.11)$$

where

$$\varepsilon_t^F \sim N(0,1), \varepsilon_1^F, \varepsilon_2^F, \ldots \ i.i.d.,$$

$$\varepsilon_t^X \sim N(0,1), \varepsilon_1^X, \varepsilon_2^X, \ldots \text{ i.i.d.}, \quad \varepsilon_{t_1}^F, \varepsilon_{t_2}^X \text{independent } \forall t_1, t_2,$$

$a^F, b^F, \sigma^F, a^X, b^X, \sigma^X$: parameters that can be estimated by regression, based on historical data.

In the above variable δ_t^P represents changes in loss trends triggered by changes in inflation rates. δ_t^P is applied to premium rates as will be explained below. Its construction through (2.11) ensures correlation of aggregate loss amounts and premium levels that can be attributed to dynamics of inflation.

It is necessary to impose restriction of setting δ_t^X and δ_t^X to at least -1 so that number of losses and loss severities have negative values.

Loss frequency changes due to general inflation. When inflation is high, policyholders tend to report more claims in certain lines of business.

The corresponding cumulative changes $\delta_t^{F,c}$ and $\delta_t^{X,c}$ are calculated by

$$\delta_t^{F,c} = \prod_{s=t_0+1}^{t} \left(1+\delta_s^X\right), \qquad (2.12)$$

$$\delta_t^{F,c} = \prod_{s=t_0+1}^{t} \left(1+\delta_s^X\right) \qquad (2.13)$$

where $t_0 + 1 =$ first year to be modelled.

18.3.6 Stock Returns

For modelling stocks we consider either stock prices or stock returns. Let us take into account the insight gained from Capital Asset Pricing Model (CAPM).

In order to apply CAPM we need to model the return of a portfolio. Assuming a significant correlation between stock and bond prices and taking into account multi-periodicity of a DFA model, we can have the following linear model for the stock market return in projection year *t* conditional on the one-year spot rate $R_{t,1}$ at time *t*.

$$\mathbb{E}\left[r_t^M \mid R_{t,1}\right] = a^M + b^M \left(e^{R_{t,1}} - 1\right), \qquad (2.14)$$

where

$e^{R_{t,1}} - 1 =$ risk-free return, see *(2.7)*,

a^M, b^M = parameters that can be estimated by regression,

based on historical data and economic reasoning.

When modelling sub-periods, usually of length one year is often used by studies. Note that you don't confuse r_t^M with the instantaneous short-term interest rate r_t in CIR. Moreover a negative value of b^M means that increasing interest rates signal falling of expected stock prices.

When such considerations are included, you can apply the CAPM formula to get the conditional expected return on an arbitrary stock S:

$$\mathbb{E}\left[r_t^S \mid R_{t,1}\right] = \left(e^{R_{t,1}} - 1\right) + \beta_t^s \left(\mathbb{E}\left[r_t^M \mid R_{t,1}\right] - \left(e^{R_{t,1}} - 1\right)\right), \qquad (2.15)$$

where $e^{R_{t,1}} - 1$ = risk-free return,

r_t^M = return on the market portfolio,

$\beta_t^s = \beta -$ coefficient of stock S

$$= \frac{Cov\left(r_t^S, r_t^M\right)}{Var\left(r_t^M\right)}.$$

Remember that a geometric Brownian motion for the stock price can be projected as a lognormal distribution for $1 + r_t^S$:

$$1 + r_t^S \sim \text{lognormal}\left(\mu_t, \sigma^2\right), r_1^S, r_2^S, \ldots \text{ independent}, \quad (2.16)$$

with μ_t chosen to yield

$$m_t = e^{\mu_t + \sigma^2/2},$$

where

$$m_t = 1 + \mathbb{E}\left[r_t^s \mid R_{t,1}\right], \text{ see (2.15)},$$

σ^2 = estimated variance of logarithmic historical stock returns .

Again, it may be noted that stock returns can be modelled differently than we have done above.

18.3.7 Non-Catastrophe Losses

In non-catastrophe case, loss amounts depend on the age of insurance contracts. The aging phenomenon describes the fact that the loss ratio – *i.e.*, the ratio of (estimated) total loss divided by earned premiums – decreases when the age of policy increases. For this reason Kaufmann et al (2001), divide the insurance business into three classes, viz.,

- new business (superscript 0),
- renewal business – first renewal (superscript 1), and
- renewal business – second and subsequent renewals (superscript 2).

There are two main stochastic factors that affect total claim amount. These are: number of losses and severity of losses. The choice of a specific claim number and claim size distribution depends on the line of business and is the result of fitting distributions to empirical data requiring prior adjustments of historical loss data. In the following we will use non-catastrophe losses by referring to a negative binomial (claim number) and a gamma (claim size) distribution.

The loss numbers N_t^j and mean loss severities $X_t^j = \frac{1}{N_t^j}\sum_{i=1}^{N_t^j} X_t^j(i)$ for period t and renewal category j mean values $\mu^{F,j}, \mu^{X,j}$ and standard deviations $\sigma^{F,j}, \sigma^{X,j}$ of historical data can be used for loss frequencies and mean loss severities. We also have to take into account inflation and written exposure units. Because loss frequencies behave more stable than loss numbers, estimation of loss frequencies instead of relying on estimates of loss numbers can be resorted to.

We consider the negative binomial distribution with mean $m_t^{N,j}$ and variance $v_t^{N,j}$ where variables m and v represent mean and variance of different factors for the distribution of the number of N_t^j. These factors can be referred by attaching a superscript (N,X,Y,...) to m or v:

$$N_t^j \sim NB(a,p), j = 0,1,2, \qquad (2.17)$$

$N_1^j, N_2^j, \ldots$ independent,

with a and p chosen to yield

$$m_t^{N,j} = \mathbb{E}\left[N_t^j\right] = \frac{a(1-p)}{p}, \qquad (2.18)$$

where $$v_i^{N,j} = \text{var}\left(N_i^j\right) = \frac{a(1-p)}{p^2}$$

$$m_i^{N,j} = w_i^j \mu^{F,j} \delta_i^{F,c},$$

$$v_i^{N,j}, = \left(w_i^j \sigma^{F,j} \delta_i^{F,c}\right)^2,$$

w_i^j = written exposure units; introduced in more detail and modelled in (3.3),

$\mu^{F,j}$ = estimated frequency, based on historical data,

$\sigma^{F,j}$ = estimated standard deviation of frequency, based on historical data,

$\delta_i^{F,c}$ = cumulative change in loss frequency, see (2.12).

Next, we take up claim size distribution for high frequency, low severity losses.

$$X_i^j \sim Gamma(\alpha,\theta), j = 0,1,2, \qquad (2.19)$$

$X_{1,}^j X_{2,...}^j$ independent,

with α and θ chosen to yield

$$m_i^{X,j} = E\left[X_i^j\right] = \alpha\theta,$$

$$v_i^{X,j} = Var\left(X_i^j\right) = \alpha\theta^2,$$

where

$$m_i^{X,j} = \mu^{X,j} \delta_i^{X,c},$$

$$v_i^{X,j} = \left(\sigma^{X,j} \delta_i^{X,c}\right)^2 / \delta_i^{F,c},$$

$\mu^{X,j}$ = estimated mean severity, based on historical data,

$\sigma^{X,j}$ = estimated standard deviation, based on historical data,

$\delta_i^{X,c}$ = cumulative change in loss severity, see (2.13),

$\delta_i^{F,c}$ = cumulative change in loss frequency, see (2.12).

When the number of losses is multiplied with the mean severity, the total (non-catastrophic) loss amount in respect of a certain line of business: $\sum_{j=0}^{2} N_i^j X_i^j$ is obtained.

18.3.8 Catastrophes

Consider losses triggered by catastrophic events like windstorm, flood, hurricane and earthquake. Such events, are modelled through negative binomial, Poisson, or binomial distribution with mean m^M and variance v^M. We assume that there are no trends in the number of catastrophes:

$M_t \sim NB,\ Pois,\ Bin, \ldots$(mean m^M, variance v^M),

$$M_1, M_2, \ldots i.i.d., \qquad (2.20)$$

where

m^M = estimated number of catastrophes, based on historical data,

v^M = estimated variance, based on historical data.

The total (economic) loss (i.e., not only the part the insurance company in consideration has to pay) for each catastrophic event $i \in \{1,\ldots, M_t\}$ can be simulated by taking probability distribution, GPD (generalised Pareto distribution) $G_{\xi,\beta}$. We have seen GPD in Unit 15 where $G_{\xi,\beta}$ was considered as the limit distribution of scaled excesses over high thresholds. In the following equation Y_t^i describes the total economic loss caused by catastrophic event $i \in \{1,\ldots, M_t\}$ in projection period t.

$Y_{t,i} \sim$ *lognormal, Pareto, GPD, …*(mean m_t^Y, variance v_t^Y),

$$Y_{t,1}, Y_{t,2}, \ldots\ i.i.d., \qquad (2.21)$$

Y_{t_1,i_1}, Y_{t_2,j_2} independent, $\forall (t_1, i_1) \neq (t_2, i_2)$

where

$$m_t^Y = \mu^Y \delta_t^{\bar{X},c},$$

$$v_t^Y = \left(\sigma^Y \delta_t^{X,c}\right)^2$$

μ^Y = estimated loss severity, based on historical data,

σ^Y = [illegible]ted standard deviation, based on historical data,

$\delta_t^{X,c}$ = cumulative change in loss severity, see (2.13).

When Y_t' is generated, we split it into pieces reflecting the loss portions of different lines of business:

$$Y_{t,i}^{k} = a_{t,i}^{k} Y_{t,i}, \qquad (2.22)$$

where

k = line of business,

l = total number of lines considered,

$\forall i \in \{1,\ldots,M_t\} : \left(a_{t,i}^{1} \cdots a_{t,i}^{l}\right) \in \left\{x \in [0,1]^l, \|x\|_1 = 1\right\} \subset \mathbb{R}^l$ *is a* random convex combination, whose probability distribution within the $(l-1)$ dimensional test objects can be specified arbitrarily.

18.3.9 Underwriting Cycles

Some of the important factors behind underwriting cycles as documented by Kaufmann at (2001) are:

- time lag effect of the pricing procedure,
- trends, cycles and short-term variations of claims,
- fluctuations in interest rate and market values of assets.

Short-term interest rates are the main factors affecting all other variables in the model. For this we have to look at the premium cycles by including competitive strategies. To see this, take a homogeneous Markov chain model (in discrete time) and assign one of the following states to each line of business for each projection year:

i) weak competition,

ii) average competition,

iii) strong competition.

Let state 1 be weak competition and the insurance company demands high premiums since it knows that its market share cannot be increased. Let state 3 be strong competition and the insurance company accepts low premiums so as to keep its current market share. If we assume a stable claim environment, high premiums are equivalent to high profit margin over pure premium, and low premiums equal low profit margin. When change occures from one state to another, premiums undergo change commensurately.

Consider the transition probabilities $p_{ij}, i, j \in \{1, 2, 3\}$, which denote the probability of changing from state i to state j from one year to the next and are assumed to be equal for each projection year. That means, the Markov chain is homogeneous. p_{ij} form a matrix T such that

$$T = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}.$$

The set of transition probabilities $p_{ij}, i, j \in \{1, 2, 3\}$ above depicts more than one possibilities. However, it is possible to model the p_{ij}s depending on current market conditions applicable to each line of business separately. If the company writes l lines of business this will imply 3^l states of the world. Since the business cycles of different lines of business are strongly correlated, only few of the 3^l states are attainable. As a result, we have to model $L \ll 3^l$ states, where the transition probabilities $p_{ij,} i, j \in \{1,...,L\}$ remain constant over time. It is not unreasonable to that some of these are zero, because there may exist some states that cannot be attained directly from certain other states. Thus, taking the attainable states, L, the matrix T can be assigned dimension $L \times L$:

$$T = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1L} \\ p_{21} & p_{22} & \cdots & p_{2L} \\ \vdots & \vdots & \ddots & \vdots \\ p_{L1} & p_{L2} & \cdots & p_{LL} \end{pmatrix}.$$

To fix the transition probabilities p_{ij} in any of the above mentioned cases, each state i has to be treated separately and probabilities have to be assigned to the variables $p_{i1},..., p_{iL}$ such that $\sum_{j=1}^{L} p_{ij} = 1 \ \forall_i$. Then the stationary probability distribution π has to be considered to which the chosen probability distribution will converge. It is extremely difficult to choose an estimate of appropriate transition probability and one has to rely both on historical data and experience based knowledge.

18.3.10 Payment Patterns

The model deals with uncertainties of the claim settlement process, i.e., to pay at the random time. It is not difficult to see that a whole loss portfolio belonging to a specific line of business exists. We need to consider its aggregate yearly loss payments in different calendar years (or development periods). The incremental payment of aggregate losses stemming from one and the same accident year forms a payment pattern. To form a broad idea on technique related these payments you may see the portion dealing with loss triangle of Unit 17. If we consider yearly loss payments pertaining to a specific accident year t then the i^{th} development year refers to calendar year

$t+i$. In the following we will denote accident years by t_1 and development years by t_2.

Here we need to distinguish two cases: First, we have to explain the modelling of outstanding loss payments pertaining to previous accident years followed by a description to model loss payments in respect of future accident years. For that purpose we use a chain-ladder procedure.

Since a lognormal distribution usually provides a good fit to historical loss development factors we may use it for outstanding loss payment.

Reserve Estimation

For each accident year t_1 we have to estimate the ultimate claim amount in each development year t_2 through:

$$\hat{Z}^{ult}_{t_1,t_2} = \prod_{t=t_2+1}^{\tau}\left(1+e^{\mu_t}\right)\sum_{t=0}^{t_2} Z_{t_1,t}, \qquad (2.30)$$

where

μ_t = estimated logarithmic loss development factor for development year t, based on historical data,

$Z_{t_1,t}$ =simulated losses for accident year t_1, to be paid in development year t.

Check Your Progress 2

1) Point out the main structure of a DFA model.

....

....

....

....

....

2) What are the practical issues you may consider for selecting the interest rate while building a DFA model?.

....

....

....

....

....

3) What is CIR model?

....

....

....

....

....

4) How would you apply CAPM in DFA?

....

....

....

....

18.4 CORPORATE MODEL

As pointed out earlier, DFA facilitates and helps arrive management decisions. Let us consider one of the decisions, that is, maximisation of shareholder's value in the following discussion. That is, we assume that the stackholders of the company rely on financial reports in making decisions while valuing the company.

Consider economic surplus U_t, which is computed as the difference between the market value of assets and the market value of liabilities (derived by discounting loss reserves and unearned premium reserves) of a company. The amount of available surplus reflects the financial strength of an insurance company and it in turn serves as a measure for shareholder value.

To see the change in the cash flows write the following equation:

$$\Delta U_t = P_t + (I_t - I_{t-1}) + (C_t - C_{t-1}) - Z_t - E_t - (R_t - R_{t-1}) - T_t, \quad (3.1)$$

where

P_t = earned premiums,

I_t = market value of assets (including realised capital gains in year t),

C_t = equity capital,

Z_t = losses paid in calendar year t,

E_t =expenses,

R_t = (discounted) loss reserves,

T_t = taxes.

Before discussing (3.1), it may be useful to remember that a company becomes insolvent for $U < 0$. In (3.1) C_t - C_{t-1} gives the net additions to capital from issuance of new equity capital etc.

To derive the earned premium from written premiums, take variables such as:

each line of business, written premiums P_t^j for renewal class j change in loss trends, the position of the underwriting cycle and number of written exposures. Putting together in a functional form written premium $\tilde{P}_t^j$ is derived as

$$\tilde{P}_t^j = \left(1+\delta_t^P\right)\left(1+c_{m_{t-1},m_t}\right)\frac{w_t^j}{w_{t-1}^j}\tilde{P}_{t-1}^j,\ j=0,1,2, \qquad (3.2)$$

where

δ_t^P = change in loss trends, see (2.11),

m_t = market condition in year t,

$c_{A,B}$ = constant that describes how premiums develop when changing from market condition A to B; $c_{A,B}$ can be estimated from historical data,

w_t^0 = written exposure units for new business,

w_t^1 = written exposure units for renewal business, first renewal,

w_t^2 = written exposure units for renewal business, second and subsequent renewals.

Studies have concluded that the written premiums given by equation (3.2) would come close to be adequate if the realisations of all random variables referring to projection year t $\left(\delta_t^P, c_{m_{t-1},m_t}, w_t^j\right)$ were known in advance and assuming adequacy of current premiums $\tilde{P}_{t_0}^j$. There is a problem however. Premiums to be charged in year t have to be determined prior to the beginning of year t. As a solution, random variables in (3.2) have to be replaced by estimations in order to model written premiums P_t^j, which would be charged in projection year t.

$$P_t^j = \left(1+\hat{\delta}_t^P\right)\left(1+\hat{c}_{m_{t-1},m_t}\right)\frac{\hat{w}_t^j}{w_{t-1}^j}\tilde{P}_{t-1}^j,\ j=0,1,2, \qquad (3.3)$$

where estimates can be obtained via their expected values:

$$\hat{\delta}_t^P = \left[1+a^X+b^X\left(a^I+b^I\left(ab+(1-a)r_{t-1}\right)\right)\right]\left[1+a^F+b^F\left(a^I+b^I\left(ab+(1-a)r_{t-1}\right)\right)\right]-1.$$

For these variable see (2.11), (2.10), (2.9), (2.8) and (2.4).

$$\hat{c}_{mt-1,mt} = \sum_{m=1}^{l(k)} p_{mt-1,m}\ c_{mt-1,m},$$

$l(k)$ = number of states for line of business k,

$p_{m_{t-1},m}$ = transition probability,

To get $\hat{w}_t$, estimate the equation

$$w_t^j = \left(a^j + b^j w_{t-1}^j + \varepsilon_t^j\right)^+, j = 0,1,2,$$

where

$$\varepsilon_t^j \sim N\left(0,\left(\sigma^j\right)^2\right),\ \varepsilon_1^j, \varepsilon_2^j, \ldots \text{ i.i.d.},$$

a^j, b^j, σ^j = parameters that can be estimated based on historical data. Thus,

$$\hat{w}_t^j = a^j + b^j w_{t-1}^j$$

Equation, (3.3) gives the expected value of random variable representing actually written premiums. Note that the time index t = t_0 refers to the year prior to the first projection year. By combining (3.2) and (3.3) the initial values $\tilde{P}_{t_0}^j$ can be calculated via $P_{t_0}^j$:

$$\tilde{P}_{t_0}^j = \frac{1+\delta_{t_0}^P}{1+\hat{\delta}_{t_0}^P} \frac{1+c_{m_{t_0-1},m_{t_0}}}{1+\hat{c}_{m_{t_0-1},m_{t_0}}} \frac{w_{t_0}^j}{\hat{w}_{t_0}^j} P_{t_0}^j, \ j = 0,1,2, \tag{3.4}$$

where

$P_{t_0}^j$ =t written premiums charged for the last year and still valid just before the start of the first projection year.

Taking written premiums $P_t^j(k)$ from (3.3) where k denotes line of business, the total earned premiums of all lines and renewal classes will be:

$$P_t = \sum_{k=1}^{l} \sum_{j=0}^{2} a_t^j(k)\ P_t^j(k) + \left(1 - a_{t-1}^j(k)\right) P_{t-1}^j(k), \tag{3.5}$$

where

$a_t^j(k)$ = percentage of premiums earned in year written, estimated on the basis of historical data.

Check Your Progress 3

1) Which economic surplus is used in modelling DFA?

....

....

....

....

....

2) List the cash flows that determine the change in surplus in financial statements.

....

....

....

....

....

18.5 LET US SUM UP

In this unit we have discussed DFA as a tool of financial management in insurance. It helps in decision-making when variable characterised by random occurrence are present through an evaluation of their interrelationships. DFA scores over the deterministic models on this count.

If a DFA model is built taking a time period of about 5 to 10 years, its results may be able to capture the movements of the variables better.

Being a part of financial management, DFA's framework is developed on an efficiency frontier comprising risk and return. Its structure consists of stochastic scenario generator, data sources and output. The stochastically modelled variables are selected using the available data sources to evaluate the financial position of a firm, business line selection strategies and economic surplus generation potentials.

18.6 KEY WORDS

Capital Asset Pricing Model (CAPM): CAPM is used theoretically to relate securities to the market as a whole, and practically, as the discount rate in discounted cash flow calculations to establish the fair value of an investment

Efficient Frontier: Every possible asset combination can be plotted in risk-return space, and the collection of all such possible portfolios defines a region in this space. The line along the upper edge of this region is known as the efficient frontier.

Exposure: Measure of vulnerability to loss, usually expressed in dollars or units.

Liability: Legally enforceable obligation. The term is most commonly used in a pecuniary sense.

Ornstein-Uhlenbeck Process: A stochastic process used as a theoretical model for Brownian motion.

Solvency: Having sufficient assets-capital, surplus, reserves-and being able to satisfy financial requirements-investments, annual reports examinations-to be eligible to transact insurance business and meet liabilities.

Underwriting: The process of selecting risks for insurance and classifying them according to their degrees of insurability so that the appropriate rates may be assigned. The process also includes rejection of those risks that do not qualify.

Zero Coupon Bond: A bond that sells at a huge discount and pays no interest.

[See, Units 2 – 4 (Block 1) for more key words]

18.7 SOME USEFUL BOOKS

Björk T. (1996) Interest Rate Theory, In *Financial Mathematics* (ed. W. Runggaldier), Lecture Notes in Mathematics 1656, 53-122, Springer, Berlin

Cox J. C., Ingersoll J. E. and Ross S. A. (1985), A Theory of the Term Structure of Interest Rates, *Econometrica* 53, 385-407

Embrechts P., Kl ü ppelberg C. and Mikosch T. (1997), *Modelling Extremal Events for Insurance and Finance,* Springer, Berlin

Kaufmann, R., Andeas Gadmeraved Ralf kalf(2001), Introduction to Dynamic Financial Analysis, Asian Bulletin, vlo.31,No.1, pp.213-249 (see Internet)

Lamberton D. and Lapeyre B. (1996), *Introduction to Stochastic Calculus Applied to Finance,* Chapman & Hall, London

18.8 ANSWER OR HINTS TO CHECK YOUR PROGRESS

Check Your Progress 1

1) See Sub-section 18.2.1 and answer.
2) See Section 18.2 and answer.
3) See Sub-section 18.2.4 and answer.
4) See Sub-section 18.2.5 and answer.

Check Your Progress 2

1) See Sub-section 18.2and answer.
2) See Sub-section 18.3.2 and answer.
3) See Sub-section 18.3.2 and answer.
4) See Sub-section 18.3.6 and answer.

Check Your Progress 3

1) See Section 18.4 and answer.
2) See Section 18.4 and answer.