UNIT 6 LAPLACE TRANSFROM METHOD

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6.1 INTRODUCTION

In the course on elementary calculus you must have studied an operator $D = \frac{d}{dx}$, which when applied to a function $f(x)$ possessing derivative, transforms it to a function $f'(x) = g(x)$ (say). Further, through your knowledge of differential equations you could realize the importance of this operator in solving linear differential equations with constant coefficients. In this unit, we introduce another transformation that is a mapping of functions onto functions that has turned out to be very useful in solving initial value problems involving linear ordinary differential equations with constant coefficients and partial differential equation with constant coefficients with special reference to three fundamental equations of mathematical physics. An interesting feature of this new operation termed as **Laplace transform** is that it replaces a problem of differential equations into an algebraic problem. This transform was first introduced by Laplace (1749-1827), a French mathematician, in the year 1790 in his work on probability theory. It became popular when Heaviside applied it to the solutions of the ordinary differential equations representing problems in electrical engineering. It is also useful in problems where the (mechanical or electrical) driving forces have discontinuities, forces acting for a short time only or are periodic and not merely sine or cosine functions.

In Sec.6.2 we start the discussion by giving the definition of Laplace transform. We have given an integral definition which transforms a given function $f(t)$ into another function $F(s)$, of the transform parameter s and stated the sufficient conditions for the existence of the transform. Laplace transforms of some elementary functions as well as some special type of functions and various properties of Laplace transforms are also discussed here. In Sec.6.3 we have concentrated on the inverse problem i.e. knowing the transform $F(s)$ of a function $f(t)$, to find $f(t)$. We have defined inverse Laplace transform, stated its properties and illustrated them through examples in this section. Finally, in Sec.6.4 we have discussed the applications of Laplace transform to initial and boundary value problems corresponding to various physical situations. At the end of the unit, a table of Laplace transform of commonly used functions is given in the appendix for ready reference.

Objectives

After studying this unit you should be able to

- define the Laplace transform and inverse Laplace transform of a function;
- derive the Laplace transforms of elementary functions;
- derive general properties of Laplace transform and inverse Laplace transform;
- use the Laplace transforms of elementary functions and general properties to obtain the Laplace transform of a given function;

- use Laplace transform method to obtain solutions of initial and boundary value problems.

6.2 LAPLACE TRANSFORMS

We start with the definition of Laplace transform.

Definition: Let $f(t)$ be a function of t defined for $t \geq 0$. The Laplace transform of $f(t)$, denoted by $\mathcal{L}[f(t)]$, is defined by the integral operation

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\, dt. \tag{1}$$

The integral in Eqn.(1) is a function of the parameter s which may be real or complex, and is denoted by $F(s)$. Accordingly,

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\, dt = F(s). \tag{2}$$

The Laplace transform of $f(t)$ exists if the integral in Eqn.(1) exists, i.e., the integral converges for some value of s.

Remark:

(i) Symbol $\mathcal{L}$ is called the Laplace transform operator.

(ii) Generally, the transform will exist for more than one value of the parameter s.

(iii) There is one to one correspondence between $f(t)$ and $F(s)$, and the relation transform $f(t)$, a function of t into a function of another variable s.

(iv) The operation defined above, which yields $F(s)$ from a given function $f(t)$ is called **Laplace transformation**.

The definition of Laplace transform as given in Eqn.(1) involves improper integral that need not converge. For the purpose of clarity let $f(x)$ be a positive function for $x > 0$.

Keeping in view the geometrical interpretation of $\int_a^b f(x)\, dx$, we find that for

$f(x) > 0, x > 0$, Eqn.(1) represents the area under the positive curve $f(x)e^{-sx}$ and above the x-axis. If this area fails to be finite we remark that the integral

$\int_0^\infty e^{-sx} f(x)\, dx$ does not converge. Alternatively, the integral diverges. This can also

be expressed by saying that in the divergent case integrand $f(x)e^{-sx}$ fails to approach zero fast enough to yield a finite area. In order to achieve this we restrict $f(x)$ to be sufficiently damped. The following definitions help clarify the terminology of sufficiently damped.

Functions of Exponential Order

Definition: The function $f(t)$ is said to be of exponential order as $t \to \infty$, if there exist positive constants M and α and a number $T > 0$, such that

$$|f(t)| \leq M e^{\alpha t}, \; t \geq T \tag{3}$$

In order to emphasize the occurrence of α in Eqn.(3) we say that $f(t)$ is of the order of $e^{\alpha t}$ as $t \to \infty$. This can also be written as $f(t) = O(e^{\alpha t})$, $t \to \infty$. Under this restriction of $f(t)$ being $O(e^{\alpha t})$ one can write

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t) dt = \int_0^T e^{-st} f(t)\, dt + \int_T^\infty e^{-st} f(t)\, dt \tag{4}$$

The first integral on the rhs of Eqn.(4) always exists and the second integral also exists because of the inequality

$$\int_T^{\infty} \left|e^{-st} f(t)\right| dt \le M \int_T^{\infty} e^{-st} e^{\alpha t} dt = M \exp\left[-T(s-\alpha)\right] / (s-\alpha) \qquad (5)$$

For $s > \alpha$, the expression on the extreme right of Eqn.(5) tends to zero as $T \to \infty$. Consequently, we have proved the following result summed up in the form of a theorem.

Theorem 1: If the integral of $e^{-st} f(t)$ between the limits zero and T exists for every finite T and if $f(t)$ is of exponential order $f(t) = O(e^{\alpha t})$ as $t \to \infty$, the Laplace transform of $f(t)$ i.e., $\mathcal{L}[f(t)] = \int_0^{\infty} e^{-st} f(t)\, dt = F(s)$ exists for $s > \alpha$.

Let us consider the following examples.

Example 1: Show that the following functions are of exponential order.

(a) $f(t) = t$ (b) $f(t) = 7\cos t$ (c) $3e^{-4t}$

Solution: In order to prove that all the three functions are of exponential order we have to find the values of M, α and T for each of the functions

(a) $f(t) = t;\ |t| < e^t;\ (\alpha = 1,\ M = 1 \text{ and } T > 0)$

(b) $f(t) = 7\cos t,\ |7\cos t| < 7e^t; (\alpha = 1,\ M = 7,\ T > 0)$

(c) $f(t) = 3e^{-4t}\ ; |3e^{-4t}| < 3e^t\ \ (\alpha = 1,\ M = 3,\ T > 0)$

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Example 2: Show that the function e^{t^2} is not of exponential order.

Solution: In this case it is not possible to determine M, α and T as the function grows faster than e^{ct} for any positive constant c such that $t > c > 0$.

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Thus, you have seen that if the function $f(t)$ is of exponential order then the improper integral in Eqn.(1) defining the Laplace transform of $f(t)$ converges. However, there exist the following possibilities under which the integral (1) may not converge:

(i) $f(t)$ has infinite number of points of discontinuities

(ii) $f(t)$ may become unbounded near a point of discontinuity

In order to avoid the above two situations we assume $f(t)$ to be sectionally continuous or piecewise continuous function.

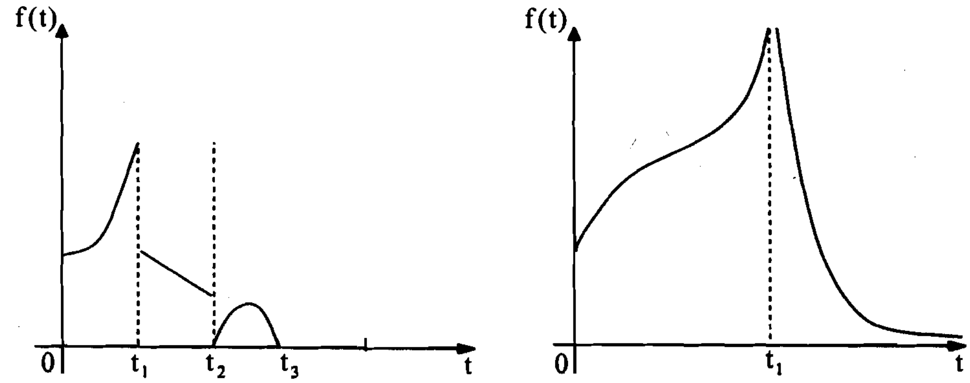
Piecewise Continuous Functions

Definition: A function $f(t)$ is said to be piecewise continuous on an interval $[a, b]$, if the interval is such that it can be subdivided into a finite number of sub-intervals such that $f(t)$ is continuous in each subinterval and $f(t)$ has finite limits as t approaches either end points of each subinterval from the interior.

For the purpose of clarity let $t_0 = 0 < t_1 < t_2 < \ldots t_n$ be the points of discontinuities. Then $f(t)$ is continuous in the open intervals

$]0, t_1[,\ ]t_1, t_2[,\ ]t_2, t_3[, \ldots]t_{k-1}, t_k[,\ ]t_k, \infty[$ and $\lim_{t \to t_i^-} f(t)$ (i.e., left handed limit), $\lim_{t \to t_i^+}$ (i.e., right handed limit) are finite at each point t_i.

Fig.1(a) gives the graph of piecewise continuous function and 1(b) gives the piecewise discontinuous function.



(a) Fig.1 (b)

The above discussion leads us to the following theorem.

Theorem 2: If $f(t)$ is piecewise continuous for $t \geq 0$ and is of exponential order for some positive constant c and for $t \geq T$ then $\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$ exists for $s > c$.

It is worth remarking here that the conditions of the above theorem for the existence of $\mathcal{L}[f(t)]$ are **sufficient** but not necessary. This means that even if these conditions are not satisfied, Laplace transform of functions exists.

For instance, the function $f(t) = t^{\frac{-1}{2}}$, which is infinite at $t = 0$ is not piecewise continuous over the domain $t \geq 0$ but $\mathcal{L}[f(t)]$ exists:

$$\mathcal{L}\left(t^{\frac{-1}{2}}\right) = \int_0^\infty e^{-st} t^{\frac{-1}{2}}\,dt$$

Substituting $t = z^2$, $dt = 2z\,dz$, when $t = 0, z = 0, t \to \infty, z \to \infty$, we get

$$\mathcal{L}\left(t^{\frac{-1}{2}}\right) = 2\int_0^\infty e^{-sz^2}\,dz$$

Putting $u = \sqrt{s}\,z$, $du = \sqrt{s}\,dz$, $z = 0$, $u = 0$, $z \to \infty$, $u \to \infty$, we get

$$\mathcal{L}\left(t^{\frac{-1}{2}}\right) = \frac{2}{\sqrt{s}}\int_0^\infty e^{-u^2}\,du = \sqrt{\frac{\pi}{s}}, \ s > o.$$

Without mentioning, we shall assume function $f(t)$ as piecewise continuous and of exponential order throughout our discussion in this unit.

Using the definition of Laplace Transform (LT) as given in Eqn.(1) we take up below the LT of some elementary functions.

6.2.1 Transforms of Some Elementary Functions

As the transforms of some functions such as exponential, trigonometric, polynomial etc. will be entering into our future discussion quite frequently, we determine the transforms of such functions, using the definition of the transform.

1. $\mathcal{L}[e^{at}]$

Using the definition of Laplace transform we can write

$$\mathcal{L}[e^{at}] = \int_0^\infty e^{-st}(e^{at})\,dt = \int_0^\infty e^{-(s-a)t}\,dt$$

For $s \le a$, it is clear that the above integral diverges. For $s > a$, the integral converges. In fact, it assumes the value $\left[\frac{-e^{-(s-a)t}}{s-a}\right]_0^\infty = 0 + \frac{1}{s-a}$.

Thus, $$\mathcal{L}(e^{at}) = \frac{1}{s-a}, \quad s > a. \tag{6}$$

For $a = 0$,. $$\mathcal{L}(1) = \left(\frac{1}{s}\right), \quad s > 0 \tag{7}$$

2. $\mathcal{L}[\cos at]$

By definition, $\mathcal{L}[\cos at] = \int_0^\infty e^{-st} \cos at \, dt$

Integrating by parts, we have

$$\int_0^\infty e^{-st} \cos at \, dt = \left[e^{-st}\left(\frac{a \sin at - s \cos at}{a^2 + s^2}\right)\right]_0^\infty = 0 + \frac{s}{(a^2 + s^2)}, \quad s > 0$$

Alternatively, through the theory of operator

$$\int e^{-st} \cos at \, dt = \frac{1}{D} e^{-st} \cos at = e^{-st} \frac{1}{D-s} \cos at = e^{-st} \frac{D+s}{D^2 - s^2} \cos at$$
$$= e^{-st} \frac{1}{-(a^2 + s^2)} \{-a \sin at + s \cos at\}$$

Thus, $$\mathcal{L}[\cos at] = \frac{s}{(s^2 + a^2)}, s > 0 \tag{8}$$

Similarly, $$\mathcal{L}[\sin at] = \left[\frac{e^{-st}(a \cos at + s \sin at)}{-(a^2 + s^2)}\right]_0^\infty = \frac{a}{(a^2 + s^2)}, s > 0. \tag{9}$$

3. $\mathcal{L}[t^n]$, $n \in I^+$

By definition, $\mathcal{L}[t^n] = \int_0^\infty e^{-st} t^n dt = I_n$ (say)

$$\therefore \quad I_n = \left(t^n \frac{e^{-st}}{(-s)}\right)_0^\infty - \int_0^\infty \frac{nt^{n-1}(e^{-st})}{(-s)} dt = 0 + \frac{n}{s}\int_0^\infty e^{-st} t^{n-1} dt, \; s > 0$$

$$\therefore \quad I_n = \frac{n}{s} I_{n-1}$$

Now, $I_{n-1} = \frac{n-1}{s} I_{n-2} = \left(\frac{n-1}{s}\right) \mathcal{L}[t^{n-2}]$

continuing, we get

$$I_n = \frac{n(n-1)(n-2)\ldots 3.2.1}{s^n} \mathcal{L}[t^0]$$

But, $\mathcal{L}[t^0] = \mathcal{L}(1) = \frac{1}{s}$

Therefore, $$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}, s > 0. \tag{10}$$

4. $\mathcal{L}[F(t)]$ where $F(t) = \begin{cases} t, & 0 < t < 3 \\ 4, & t > 3 \end{cases}$

$$\mathcal{L}[F(t)] = \int_0^\infty e^{-st} F(t) \, dt = \int_0^3 e^{-st} t \, dt + \int_3^\infty 4e^{-st} dt$$

$$= \left[\frac{te^{-st}}{(-s)}\right]_0^3 - \int_0^3 \frac{e^{-st}}{(-s)}dt + 4\left(\frac{e^{-st}}{-s}\right)_3^\infty$$

$$= \left(\frac{3e^{-3s}}{-s}\right) - 0 - \frac{1}{s^2}\left(e^{-st}\right)_0^3 - \frac{4}{s}\{0 - e^{-3s}\}.$$

$$= \frac{1}{s^2} + 4\frac{e^{-3s}}{s} - \frac{e^{-3s}}{s^2} \qquad (11)$$

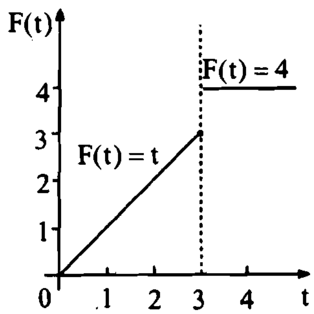


Fig.2

The graph of $F(t)$ is shown in Fig.2.

5. $\mathcal{L}$**[sinh at]**

By definition $\mathcal{L}[\sinh at] = \int_0^\infty e^{-st} \sinh at\, dt$

$$= \int_0^\infty e^{-st}\left(\frac{e^{at} - e^{-at}}{2}\right)dt$$

$$= \frac{1}{2}\left\{\int_0^\infty e^{-st}e^{at}dt - \int_0^\infty e^{-st}e^{-at}\,dt\right\}$$

$$= \frac{1}{2}\left\{\mathcal{L}[e^{at}] - \mathcal{L}[e^{-at}]\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{s-a} - \frac{1}{s+a}\right\}, \text{ if } s > a \text{ and } s > -a$$

$$= \frac{a}{s^2 - a}, \text{ if } s > |a|$$

Hence $\quad \mathcal{L}[\sinh at] = \dfrac{a}{s^2 - a^2}$, **if** $s > |a|$ (12)

Similarly, $\mathcal{L}[\cosh at] = \dfrac{s}{s^2 - a^2}$, **if** $s > |a|$ (13)

Once we know the transforms of these elementary functions, nearly all the transforms we shall need can be obtained with the use of some properties which we shall consider in the next section.

6.2.2 Properties of Laplace Transforms

In order to appreciate the usefulness of Laplace transform we develop below certain properties associated with it.

Linearity property

If $f(t)$ and $g(t)$ are functions having Laplace transforms and a, b are real arbitrary constants, then

$$\mathcal{L}\,[af(t) + bg(t)] = a\,\mathcal{L}\,[f(t)] + b\,\mathcal{L}\,[g(t)]. \qquad (14)$$

Proof: Using the definition of Laplace transform,

$$\mathcal{L}[af(t) + bg(t)] = \int_0^\infty [af(t) + bg(t)]e^{-st}dt$$

$$= a\int_0^\infty e^{-st}f(t)\,dt + b\int_0^\infty e^{-st}g(t)\,dt$$

$$= a\,\mathcal{L}[f(t)] + b\,\mathcal{L}[g(t)].$$

The above result states that the Laplace transform is a linear operator on the set of functions possessing Laplace transforms.

First Shifting Property

If $F(s)=\mathcal{L}[f(t)]$ and a is any real number then

$$\mathcal{L}[e^{at}f(t)]=F(s-a) \tag{15}$$

Proof: Using the definition of Laplace transform

$$\mathcal{L}[e^{at}f(t)]=\int_0^\infty e^{-st}e^{at}f(t)\,dt=\int_0^\infty e^{-(s-a)t}f(t)\,dt,\ s>a$$

Putting $(s-a)=r$, we get

$$\mathcal{L}[e^{at}f(t)]=\int_0^\infty e^{-rt}f(t)\,dt=F(r)=F(s-a)$$

In words this property means that the multiplication of a function $f(t)$ by e^{at} shifts its transform to $F(s-a)$. This property is also termed as **Translation theorem.**

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Second Shifting Property

If $\mathcal{L}[f(t)]=F(s)$, then

$$\mathcal{L}[u_a(t)f(t-a)]=e^{-as}F(s) \tag{16}$$

where, $u_a(t)=\begin{cases}0, & 0\le t<a\\ 1, & t>a\end{cases}$, is the unit step function.

Proof: Using the definition of unit step function we can write

$$u_a(t)f(t-a)=\begin{cases}0 & , \quad 0<t<a\\ f(t-a), & t>a\end{cases}$$

The Laplace transform of $u_a(t)f(t-a)$ can be expressed as

$$\mathcal{L}[u_a(t)f(t-a)]=\int_0^a e^{-st}(0)\,dt+\int_a^\infty e^{-st}f(t-a)\,dt$$

$$=0+\int_a^\infty e^{-st}f(t-a)\,dt$$

Putting $\sigma=t-a$, we get $d\sigma=dt$, when $t=a,\ \sigma=0$ and for $t\to\infty,\ \sigma\to\infty$ Hence

$$\mathcal{L}[u_a(t)f(t-a)]=\int_0^\infty e^{-s(\sigma+a)}f(\sigma)\,d\sigma=e^{-sa}F(s)$$

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Multiplication by Powers of t

If $\mathcal{L}[f(t)]=F(s)$, then for $n\in I^+\ge 0$

$$\mathcal{L}[t^n f(t)]=(-1)^n F^{(n)}(s)=(-1)^n\frac{d^n}{ds^n}F(s) \tag{17}$$

Proof: From the definition of Lapalce transform we have ,

$$F(s)=\int_0^\infty e^{-st}f(t)\,dt$$

Differentiating the above equation w.r.t. s, we get

$$\frac{dF}{ds}=F'(s)=\int_0^\infty(-t)e^{-st}f(t)\,dt=-\mathcal{L}[t\,f(t)]$$

$$F''(s)=\frac{d^2}{ds^2}F(s)=\frac{d^2}{ds^2}\int_0^\infty e^{-st}f(t)\,dt=\int_0^\infty(-t)^2e^{-st}f(t)\,dt$$

$$=\int_0^\infty e^{-st}(t^2f(t))\,dt=\mathcal{L}[t^2f(t)]$$

continuing the above process we obtain

$$\frac{d^n F(s)}{ds^n} = F^{(n)}(s) = (-1)^n \int_0^\infty (t)^n e^{-st} f(t)\,dt = (-1)^n \mathcal{L}[t^n f(t)]$$

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Division by t

If $\mathcal{L}[f(t)] = F(s)$ and $\frac{f(t)}{t}$ is bounded for $t > 0$, then

$$\mathcal{L}\left[\frac{1}{t} f(t)\right] = \int_s^\infty F(s)\,ds \qquad (18)$$

Proof: From the definition of Laplace transform

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$\therefore \quad \int_s^\infty F(s)\,ds = \int_s^\infty \int_0^\infty e^{-st} f(t)\,dt\,ds$$

Interchanging the order of integration in the above equation, we get

$$\int_s^\infty F(s)\,ds = \int_0^\infty \left(\int_s^\infty e^{-st}\,ds \right) f(t)\,dt$$

Performing integration w.r.t s of bracketed term on the r.h.s. of the above equation, we get

$$\int_s^\infty F(s)\,ds = \int_0^\infty \left(\frac{e^{-st}}{-t} \right)_s^\infty f(t)\,dt = \int_0^\infty \left(0 + e^{-st} \frac{f(t)}{t} \right) dt$$

Thus, from the definition of Laplace Transforms

$$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\,ds$$

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Transform of Derivative

This property is summed up in the form of the following theorem:
Theorem 3: If $f(t)$ satisfies the following conditions:

(i) it is continuous and possesses a piecewise continuous derivative

(ii) both $f(t)$ and $f'(t)$ are of exponential order as $t \to \infty$, then

$$\mathcal{L}\,[f'(t)] = s\,\mathcal{L}\,[f(t)] - f(0) \qquad (19)$$

Proof: Using the definition of Laplace transform we can write

$$\mathcal{L}[f'(t)] = \int_0^\infty e^{-st} f'(t)\,dt$$

Performing integration by parts, we get

$$= \lim_{\beta\to\infty} \left\{ \left(e^{-st} f(t)\right)_0^\beta + s \int_0^\beta e^{-st} f(t)\,dt \right\}$$

$$= \lim_{\beta\to\infty} \left\{e^{-s\beta} f(\beta)\right\} - f(0) + s\,\mathcal{L}[f(t)]$$

Now, $\lim_{\beta\to\infty} \left| e^{-s\beta} f(\beta) \right| \le \lim_{\beta\to\infty} \left| e^{-s\beta} \right| \left| f(\beta) \right|$

Since $f(\beta)$ is of exponential order as $\beta \to \infty$,

$$\lim_{\beta\to\infty} \left| e^{-s\beta} \right| \left| f(\beta) \right| < \lim_{\beta\to\infty} e^{-s\beta} M e^{\alpha\beta} = \lim_{\beta\to\infty} M e^{-(s-\alpha)\beta} \qquad \text{[ref Eqn.(3)]}$$

If $s > \alpha$, $\lim_{\beta\to\infty} M e^{-(s-\alpha)\beta} \to 0$

Consequently, $\lim_{\beta\to\infty} e^{-s\beta} f(\beta) \to 0$

Hence,

$$\mathcal{L}[f'(t)] = s\mathcal{L}[f(t)] - f(0)$$

Eqn.(19) is termed as the fundamental operation property of Laplace transform that can be easily generalized for Laplace transform of nth derivative

$$\mathcal{L}[f''(t)] = s\mathcal{L}[f'(t)] - f'(0) \qquad \text{[using Eqn.(19)]}$$
$$= s\{s\mathcal{L}[f(t)] - f(0)\} - f'(0)$$
$$= s^2\mathcal{L}[f(t)] - sf(0) - f'(0) \qquad (20)$$

continuing this process we can write

$$\mathcal{L}[f^n(t)] = s^n\mathcal{L}[f(t)] - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0) \qquad (21)$$

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Integral Property

If $\mathcal{L}[f(t)] = F(s)$, then

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}F(s). \qquad (22)$$

Proof: Introduce another function $g(t)$ as defined below:

$$g(t) = \int_0^t f(\tau)d\tau, \quad g(0) = 0$$

Differentiating above equation w.r.t. 't', we get

$$g'(t) = f(t)$$

On taking Laplace transform of this equation and using the result of Theorem 3, we get

$$\mathcal{L}[g'(t)] = s\mathcal{L}[g(t)] - g(0) = s\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} - 0 = F(s)$$

Thus,

$$\mathcal{L}\int_0^t f(\tau)\,d\tau = \frac{F(s)}{s}.$$

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Convolution Theorem

Definition: If functions $f(x)$ and $g(x)$ are piecewise continuous on $[0,\infty[$, then the convolution of f and g denoted by $(f * g)$ is given by the integral

$$(f * g)(t) = \int_0^t f(x)\, g(t-x)\,dx \qquad (23)$$

Convolution satisfies the following properties

1. Commutativity : $f * g = g * f$
2. Associativity : $f * (g * h) = (f * g) * h$
3. Distributivity: $f * (g + h) = (f * g) + (f * h)$

Theorem 4: Let $f(t)$ and $g(t)$ be piecewise continuous functions for $t > 0$ and of exponential order. Further assume that $F(s) = \mathcal{L}[f(t)]$ and $G(s) = \mathcal{L}[g(t)]$. Then,

$$\mathcal{L}[f(t) * g(t)] = \mathcal{L}[[f(t)]\,\mathcal{L}[g(t)] = F(s)\,G(s) \qquad (24)$$

Proof of Convolution Theorem

Using the definition of the Laplace transform, the product of the Laplace transform of two functions $f(t)$ and $g(t)$ can be expressed as

$$F(s)\,G(s) = \int_0^\infty e^{-sz} f(z)\,dz \int_0^\infty e^{-sx} g(x)\,dx$$

$$= \int_0^\infty f(z)\,dz \int_0^\infty e^{-s(x+z)} g(x)\,dx$$

Put $x + z = t$ so that $dx = dt, x = 0,\ t = z,\ x \to \infty,\ t \to \infty$.

$$\therefore\ F(s)\ G(s) = \int_0^\infty f(z)\ dz \int_z^\infty e^{-st} g(t-z)\ dt \qquad (25)$$

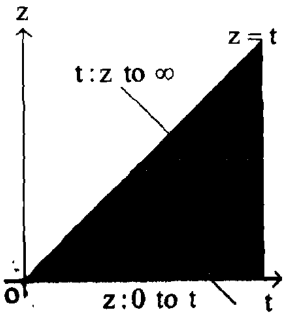

Fig.3

The region of integration of integral (25) is shown in Fig.3. In the tz plane we are integrating over the shaded region in Fig.3. Since f and g are piecewise continuous on $[0, \infty[$ and of exponential order, it is possible to interchange the order of integration.

On interchanging the order of integration in Eqn.(25), we get

$$F(s)G(s) = \int_0^\infty \int_0^t e^{-st} f(z) g(t-z)\,dz\,dt$$

$$= \int_0^\infty e^{-st} \left[\int_0^t f(z)\ g(t-z)\ dz \right] dt$$

$$= \int_0^\infty e^{-st}\ (f * g)\ (t)\,dt = \mathcal{L}(f * g)\ (t).$$

Hence result (24) is proved.

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We now illustrate the above properties through the following examples.

Example 3: Find the Laplace transform of $8t\,e^{3t}$.

Solution: Using the linearity property of Laplace transform

$$\mathcal{L}[8t\,e^{3t}] = 8\,\mathcal{L}[te^{3t}]$$

Further, using the shifting property i.e., $\mathcal{L}[e^{at} f(t)] = F(s-a)$, we can write

$$\mathcal{L}[e^{3t}\ t] = \frac{1}{(s-3)^2} \qquad \left[\text{as } \mathcal{L}(t) = \frac{1}{s^2}\right]$$

Thus, $\mathcal{L}[8t\,e^{3t}] = \dfrac{8}{(s-3)^2}$.

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Example 4: Find the Laplace transform of $e^{2at} \cos 3wt$.

Solution: Keeping in view the Laplace transform of $\cos \alpha t$ we can easily write,

$$\mathcal{L}[\cos 3wt] = \frac{s}{s^2 + 9w^2}.$$

Therefore, $\mathcal{L}[e^{2at} \cos 3wt] = \dfrac{(s-2a)}{(s-2a)^2 + 9w^2}$. [using result (15)]

***

Example 5: Find $\mathcal{L}[e^{-3t}\ t^3]$.

Solution: We know that $L\ [t^3] = \dfrac{3!}{s^4}$

$\therefore\ \mathcal{L}[e^{-3t} t^3] = \dfrac{3!}{(s+3)^4}$. [using result (15)]

***

Example 6: Find $\mathcal{L}[t^2 \cos nt]$.

Solution: $\mathcal{L}[\cos nt] = \dfrac{s}{s^2+n^2}$; $\mathcal{L}[t^2 \cos nt] = (-1)^2 \dfrac{d^2}{ds^2}\mathcal{L}[\cos nt]$ [using result (17)]

$$= (-1)^2 \frac{d^2}{ds^2}\frac{s}{(s^2+n^2)} = \frac{d}{ds}\left\{\frac{(s^2+n^2).1 - s.2s}{(s^2+n^2)^2}\right\} = \frac{d}{ds}\left\{\frac{-s^2+n^2}{(s^2+n^2)^2}\right\}$$

$$= \frac{(s^2+n^2)^2(-2s) - (n^2-s^2)2(s^2+n^2).2s}{(s^2+n^2)^4} = \frac{2s(s^2-3n^2)}{(s^2+n^2)^3}.$$

***

Example 7: Find $\mathcal{L}\left(\dfrac{\sinh 2t}{t}\right)$

Solution: Here $f(t) = \sinh 2t$. Therefore, $\mathcal{L}[f(t)] = F(s) = \dfrac{2}{s^2-4}$ [using result (12)]

$$\therefore \quad \mathcal{L}\left[\frac{\sinh 2t}{t}\right] = \int_s^\infty \frac{2}{(u^2-4)}du = \frac{1}{2}\int_s^\infty\left(\frac{1}{u-2} - \frac{1}{u+2}\right)du \qquad \text{[using result (18)]}$$

$$= \frac{1}{2}\left[\ln(u-2) - \ln(u+2)\right]_s^\infty = \frac{1}{2}\ln\left|\frac{s+2}{s-2}\right|.$$

***

Example 8: Find $\mathcal{L}\left[u_{\frac{\pi}{2}}(t)\cos t + u_{\frac{5\pi}{2}}(t)\left(t - \frac{5\pi}{2}\right)\right]$

Solution: We know that $\mathcal{L}[u_a(t) f(t-a)] = e^{-as}F(s)$. [using result (16)]
We express $\cos t$ in terms of $\sin(t-a)$ as follows:

$$\cos t = \sin\left(\frac{\pi}{2} - t\right) = -\sin\left(t - \frac{\pi}{2}\right)$$

$$\therefore u_{\frac{\pi}{2}}(t)\cos t = -u_{\frac{\pi}{2}}(t)\sin\left(t - \frac{\pi}{2}\right) = \begin{cases} 0 \quad , & t < \frac{\pi}{2} \\ -\sin t, & t > \frac{\pi}{2} \end{cases}$$

Similarly, $u_{\frac{5\pi}{2}}(t)\left(t - \dfrac{5\pi}{2}\right) = \begin{cases} 0, & t < \frac{5\pi}{2} \\ t, & t > \frac{5\pi}{2} \end{cases}$

$$\therefore \mathcal{L}[f(t)] = \mathcal{L}\left[u_{\frac{\pi}{2}}(t)\cos t + u_{\frac{5\pi}{2}}(t)\left(t - \frac{5\pi}{2}\right)\right]$$

$$= -\mathcal{L}\left[u_{\frac{\pi}{2}}(t)\sin\left(t - \frac{\pi}{2}\right)\right] + \mathcal{L}u_{\frac{5\pi}{2}}(t)\left[t - \frac{5\pi}{2}\right]$$

$$= -e^{-\frac{\pi}{2}s}\frac{1}{(s^2+1)} + e^{-\frac{5\pi}{2}s}\frac{1}{s^2}. \qquad \left[\because \mathcal{L}[\sin t] = \frac{1}{s^2+1} \text{ and } \mathcal{L}[t] = \frac{1}{s^2}\right]$$

***

Example 9: Find $\mathcal{L}[t^2u(t-4)]$

Solution: The first step in solving such problems is to express the given function in terms of unit step function.
Thus, $t^2u(t-4)$ can be written as

$$t^2u(t-4) = \{(t-4)^2 + 8(t-4) + 16\}u(t-4)$$

Hence, $\mathcal{L}[t^2u(t-4)] = \mathcal{L}[(t-4)^2u(t-4)] + 8\mathcal{L}[(t-4)u(t-4)] + 16\mathcal{L}[1.u(t-4)]$

Using the result $\mathcal{L}[u_a(t)f(t-a)] = e^{-as}F(s)$, we get

$$\mathcal{L}[t^2u(t-4)] = e^{-4s}\left\{\frac{2}{s^3}+\frac{8}{s^2}+\frac{16}{s}\right\}.$$

***

Example 10: Find $\mathcal{L}[e^{-2t}u_\pi(t)]$, where $u_\pi(t)=\begin{cases}0, & t<\pi \\ 1, & t>\pi\end{cases}$

Solution: Since $u_\pi(t)=u(t-\pi)$, therefore $\mathcal{L}[e^{-2t}u(t-\pi)]$ can be expressed as

$$\mathcal{L}[e^{-2t}u(t-\pi)] = e^{-\pi s}\,\mathcal{L}[e^{-2(t+\pi)}] = e^{-\pi s}e^{-2\pi}\mathcal{L}[e^{-2t}]$$

It may be mentioned that in writing the above result we have used the following property

$$\mathcal{L}[f(t)u(t-a)] = e^{-as}\,\mathcal{L}[f(t+a)]$$

which can be proved from the definition of LT and unit step function.

Thus, $\mathcal{L}[e^{-2t}u_\pi(t)] = e^{-(\pi s+2\pi)}\dfrac{1}{s+2} = \dfrac{e^{-(s+2)\pi}}{(s+2)}$.

***

Example 11: Evaluate $\mathcal{L}\left[\int_0^t e^{-2z}\cos(t-z)\,dz\right]$

Solution: Using convolution theorem

$$\begin{aligned}\mathcal{L}\left[\int_0^t e^{-2z}\cos(t-z)\,dz\right] &= F(s)G(s)\\ &= \mathcal{L}[f(t)]\mathcal{L}[g(t)]\\ &= \mathcal{L}[e^{-2t}]\,\mathcal{L}[\cos t] = \frac{1}{s+2}\cdot\frac{s}{s^2+1}\end{aligned}$$

***

You may now try the following exercises.

E1) Find the Laplace transforms of

a) t^2e^{-2t}

b) $t^2e^t\sin 4t$

c) $(\sin 2t)/t$

d) $\cosh^3 2t$

E2) Express the following functions $f(t)$ in terms of unit step functions and find their Laplace transforms

a) $f(t)=\begin{cases}\sin 2t, & 2\pi<t<4\pi \\ 0, & \text{otherwise}\end{cases}$

b) $f(t)=\begin{cases}(t-1), & 1<t<2 \\ (3-t), & 2<t<3\end{cases}$

E3) Find $\mathcal{L}\{(t^2-t+2)u(t-2)\}$

E4) Evaluate $\mathcal{L}\left[\int_0^t (t-z)^2\sin z\,dz\right]$

6.2.3 Transforms of Periodic, Error and Dirac-Delta Functions

Periodic Function

Let f(t) be periodic with period ω. Then

$$f(t+\omega)=f(t),$$

$\therefore \quad f(t+2\omega) = f(t+\omega+\omega) = f(t+\omega) = f(t)$

Consequently,

$$f(t+n\omega) = f(t)$$

Now, $\mathscr{L}[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$

$$= \int_0^\omega e^{-st} f(t)\,dt + \int_\omega^{2\omega} e^{-st} f(t)\,dt + \cdots + \int_{n\omega}^{(n+1)\omega} e^{-st} f(t)\,dt + \cdots \qquad (26)$$

On substituting $t = z + \omega$ in the second integral in Eqn.(26) and $t = z + 2\omega$ in the third integral and so on, that is, putting $t = z + n\omega$ in the $(n+1)$ th integral, we get

$$\mathscr{L}[f(t)] = \int_0^\omega e^{-st} f(t)\,dt + \int_0^\omega e^{-s(z+\omega)} f(z+\omega)\,dz + \int_0^\omega e^{-s(z+2\omega)} f(z+2\omega)\,dz$$

$$+ \cdots + \int_0^\omega e^{-s(z+n\omega)} f(z+n\omega)\,dz + \cdots$$

Keeping in view the fact that $f(t)$ is periodic with period ω, we can write

$$\mathscr{L}[f(t)] = \int_0^\omega e^{-sz} f(z)\,dz + e^{-s\omega}\int_0^\omega e^{-sz} f(z)\,dz + e^{-2s\omega}\int_0^\omega e^{-sz} f(z)\,dz + \cdots$$

$$\cdots + e^{-ns\omega}\int_0^\omega f(z)\, e^{-sz}\,dz + \cdots$$

$$= \{1 + e^{-s\omega} + e^{-2s\omega} + \cdots + e^{-ns\omega} + \ldots\}\int_0^\omega e^{-sz} f(z)\,dz$$

The terms in the curly bracket in the above equation constitute a G.P. with common ratio $e^{-s\omega}$, whose sum to infinity is given by $1/(1-e^{-s\omega})$.

$$\therefore \mathscr{L}\,[\mathbf{f(t)}] = \frac{\int_0^\omega \mathbf{e^{-st} f(t)\,dt}}{(\mathbf{1 - e^{-s\omega}})} \qquad (27)$$

— □ —

Error Function

Error function occurs quite frequently in statistical applications. The error function written as $\operatorname{erf}(t)$ is defined by the equation

$$\operatorname{erf}(t) = \frac{2}{\sqrt{\pi}}\int_0^t e^{-z^2}\,dz. \qquad (28)$$

When $t \to \infty$, $\operatorname{erf}(\infty) = \frac{2}{\sqrt{\pi}}\,\frac{\Gamma\left(\frac{1}{2}\right)}{2} = \frac{2}{\sqrt{\pi}}.\frac{\sqrt{\pi}}{2} = 1$.

The complementary error function written as $\operatorname{erfc}(t)$ is given by

$$\operatorname{erfc}(t) = 1 - \operatorname{erf}(t) = \frac{2}{\sqrt{\pi}}\int_t^\infty e^{-z^2}\,dz \qquad (29)$$

Now $\mathscr{L}[\operatorname{erf}(t)] = \mathscr{L}\left[\frac{2}{\sqrt{\pi}}\int_0^t e^{-z^2}\,dz\right] = \frac{2}{\sqrt{\pi}}\,\frac{1}{s}\,\mathscr{L}\left[e^{-z^2}\right]$ [using result (22)]

$$= \frac{2}{\sqrt{\pi}}\,\frac{1}{s}\int_0^\infty e^{-sz} e^{-z^2}\,dz = \frac{2}{\sqrt{\pi}}\,\frac{1}{s}\int_0^\infty e^{-\left[(z+s/2)^2 - \frac{s^2}{4}\right]}\,dz$$

$$=\frac{2}{\sqrt{\pi}}\frac{1}{s}e^{\frac{s^2}{4}}\int_0^\infty e^{-(z+s/2)^2}dz$$

$$=\frac{2}{\sqrt{\pi}}\frac{1}{s}e^{\frac{s^2}{4}}\int_{\frac{s}{2}}^\infty e^{-x^2}dx \qquad \text{[putting } z+s/2=x\text{]}$$

$$=\frac{1}{s}e^{\frac{s^2}{4}}\left[\frac{2}{\sqrt{\pi}}\int_{s/2}^\infty e^{-x^2}dx\right]=\frac{1}{s}e^{\frac{s^2}{4}}\operatorname{erfc}\left(\frac{s}{2}\right)$$

Hence, $\mathcal{L}\,[\mathbf{erf\,(t)}]=\frac{1}{s}e^{\frac{s^2}{4}}\,\mathbf{erfc}\left(\frac{s}{2}\right)$ (30)

Similarly by definition

$$\operatorname{erf}\left(\sqrt{t}\right)=\frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}}e^{-z^2}dz$$

$$=\frac{2}{\sqrt{\pi}}\int_0^t e^{-x}\left(\frac{1}{2\sqrt{x}}\right)dx \qquad \text{[putting } z^2=x\text{]}$$

$$=\frac{1}{\sqrt{\pi}}\int_0^t e^{-x}x^{-1/2}\,dx$$

$$\therefore \quad \mathcal{L}\left[\operatorname{erf}\left(\sqrt{t}\right)\right]=\frac{1}{\sqrt{\pi}}\mathcal{L}\left[\int_0^t e^{-x}x^{-1/2}\,dx\right]$$

$$=\frac{1}{\sqrt{\pi}}\frac{1}{s}\mathcal{L}\left[e^{-t}t^{-1/2}\right]$$

$$=\frac{1}{\sqrt{\pi}}\frac{1}{s}\mathcal{L}\left[t^{-1/2}\right]_{s\to s+1}$$

$$=\frac{1}{\sqrt{\pi}}\frac{1}{s}\frac{\sqrt{\pi}}{(s+1)^{1/2}}=\frac{1}{s\sqrt{s+1}}$$

Hence, $\mathcal{L}\left[\mathbf{erf}\left(\sqrt{t}\right)\right]=\frac{1}{s\sqrt{s+1}}$ (31)

— □ —

Dirac-Delta Function

Impulse Function and Heaviside Step Function

In quite a number of engineering disciplines and physical sciences we come across the word impulse. As per definition it is regarded as a force of a very large magnitude acting for a very short time. This product of a very large force and very small time is termed as an impulse function which is also known as Dirac-delta function. In a strict sense it is not a function but is known as generalized function. The unit impulse function is defined as

$$\delta(t-a)=\begin{cases}\infty, & t=a\\ 0, & t\neq a\end{cases} \quad \text{such that} \int_0^\infty \delta(t-a)\,dt=1 \tag{32}$$

or, $$\delta(t-a)=\lim_{\epsilon\to 0}F_\epsilon(t),\ \text{where } F_\epsilon(t)=\begin{cases}0\ , & t<a\\ \frac{1}{\epsilon}\ , & a\le t\le a+\epsilon\\ 0\ , & t>a+\epsilon\end{cases} \tag{33}$$

and $$\int_0^\infty f(t)\,\delta(t-a)\,dt=f(a) \tag{34}$$

where $f(t)$ is continuous and integrable in $[0, \infty[$.

Observe that the function is discontinuous at $t = a$. For many a practical application we represent delta function as follows:

$$\delta(t) = \lim_{\epsilon \to 0} \begin{cases} \frac{1}{\epsilon}, & 0 < t < \epsilon \\ 0, & t > \epsilon \end{cases} \tag{35}$$

Using the above definition, the Laplace transform of the delta function can be expressed as

$$\mathcal{L}[(\delta(t)] = \int_0^\infty e^{-st}\delta(t)\,dt = \lim_{\epsilon \to 0} \int_0^\epsilon \frac{1}{\epsilon} e^{-st}dt = \lim_{\epsilon \to 0}\left\{\frac{-1}{\epsilon s}(e^{-s\epsilon} - 1)\right\}$$

Using the expansion for exponential function, we get

$$\mathcal{L}[\delta(t)] = \lim_{\epsilon \to 0} \frac{-1}{\epsilon s}\left(1 - s\epsilon + \frac{\epsilon^2 s^2}{2!} - \frac{\epsilon^3 s^3}{3!} + \cdots - 1\right) = \lim_{\epsilon \to 0}\left(1 - \frac{\epsilon s}{2!} + \cdots\right) = 1$$

$$\therefore \quad \mathcal{L}\,[\delta(t)] = 1 \tag{36}$$

Result (34) which is called the filtering property of Dirac function can be proved by considering the integral

$$\int_0^\infty f(t) F_\epsilon(t)\,dt = \frac{1}{\epsilon}\int_a^{a+\epsilon} f(t)\,dt.$$

Using the mean value theorem of integral calculus, we obtain

$$\frac{1}{\epsilon}\int_a^{a+\epsilon} f(t)\,dt = \frac{1}{\epsilon} f(t_0) \int_a^{a+\epsilon} dt = f(t_0), \quad a < t_0 < a + \epsilon$$

$$\therefore \quad \int_0^\infty f(t)\, F_\epsilon(t)\,dt = f(t_0).$$

Taking the limit $\epsilon \to 0$, we obtain

$$\int_0^\infty \mathbf{f(t)\delta(t-a)\,dt = f(a)} \qquad \left[\begin{array}{l}\text{using (33) and} \\ f(t_0) \to f(a)\end{array}\right]$$

Another function that is discontinuous and has Laplace transform and is quite useful in application is, Heaviside step function or unit step function defined as:

$$H(t) = \begin{cases} 0, & \text{for } t < 0 \\ 1, & \text{for } t \geq 0 \end{cases}$$

using result (16) it can be easily shown that

$$\mathcal{L}\,[\mathbf{H(t)}] = \frac{\mathbf{1}}{\mathbf{s}} \text{ and } \mathcal{L}\,[\mathbf{H(t-a)}] = \frac{\mathbf{e^{-as}}}{\mathbf{s}} \tag{37}$$

— □ —

We now illustrate the application of these functions through examples. For ready reference, we have given in Appendix, at the end of the unit, the Laplace transforms of some of the commonly used functions.

Example 12: Find the Laplace transform of the function

$$f(t) = \begin{cases} \sin \omega t, & 0 < t < \frac{\pi}{\omega} \\ 0, & \frac{\pi}{\omega} < t < \frac{2\pi}{\omega} \end{cases}$$

(The above function is known as **Half wave rectifier**)

Solution: Now given $f(t)$ is periodic function with periodic $T = \frac{2\pi}{\omega}$.

$$\therefore \mathcal{L}[f(t)] = \frac{1}{(1-e^{-sT})}\int_0^T e^{-st} f(t)\,dt \qquad \text{[using Eqn,(27)]}$$

$$= \frac{1}{(1-e^{-\frac{2\pi}{\omega}s})}\left[\int_0^{\pi/\omega} e^{-st}\sin\omega t\,dt + \int_{\pi/\omega}^{2\pi/\omega} e^{-st}(0)\,dt\right]$$

$$= \frac{1}{(1-e^{\frac{-2\pi}{\omega}s})}\left\{-e^{-st}\left(\frac{\omega\cos\omega t + s\sin\omega t}{s^2+\omega^2}\right)\right\}_0^{\pi/\omega}$$

$$= \frac{\left[e^{\frac{-\pi s}{\omega}}\omega + \omega\right]}{\left(1-e^{\frac{-2\pi s}{\omega}}\right)(s^2+\omega^2)}$$

$$= \frac{\omega}{(1-e^{-\frac{\pi s}{\omega}})\,(s^2+\omega^2)}$$

***

Example 13: Show that $\int_0^\infty e^{-t}\operatorname{erf}\left(\sqrt{t}\right)dt = \frac{1}{\sqrt{2}}$.

Solution: We know that $\mathcal{L}\left[\operatorname{erf}\left(\sqrt{t}\right)\right] = \frac{1}{s\sqrt{s+1}}$

$$\therefore \int_0^\infty e^{-st}\operatorname{erf}\left(\sqrt{t}\right)dt = \frac{1}{s\sqrt{s+1}}$$

Putting $s = 1$, we get

$$\int_0^\infty e^{-t}\operatorname{erf}\left(\sqrt{t}\right)dt = \frac{1}{\sqrt{2}}$$

***

Example 14: Find $\mathcal{L}\,[t \operatorname{erf}(2\sqrt{t})]$

Solution: $\mathcal{L}\,[t\operatorname{erf}(2\sqrt{t})] = \mathcal{L}\,[t\operatorname{erf}(\sqrt{4t})]$

Now $\quad \mathcal{L}\left[\operatorname{erf}\left(\sqrt{t}\right)\right] = \frac{1}{s\sqrt{s+1}}$

$$\therefore \mathcal{L}\left[\operatorname{erf}\left(\sqrt{4t}\right)\right] = \frac{1}{4}\,\frac{1}{\frac{s}{4}\sqrt{\frac{s}{4}+1}} = \frac{2}{s\sqrt{s+4}} \qquad \left[\because \mathcal{L}\left[f(at)\right] = \frac{1}{a}F\left(\frac{s}{a}\right)\right]$$

$$\therefore \mathcal{L}\left[t\operatorname{erf}\left(\sqrt{4t}\right)\right] = (-1)\frac{d}{ds}\left(\frac{2}{s\sqrt{s+4}}\right)$$

$$= \frac{2}{s^2(s+4)}\left[\frac{2(s+4)+s}{2\sqrt{s+4}}\right] = \frac{3s+8}{s^2(s+4)^{3/2}}.$$

***

Example 15: Express the following functions in terms of Heaviside unit step functions and hence find their Laplace transform

a) $f(t) = \begin{cases} t^2, & 0<t<1 \\ 4t, & t>1 \end{cases}$ $\qquad$ b) $f(t) = \begin{cases} 0, & 0<t<\pi/2 \\ \sin t, & \pi/2<t<3\pi/2 \\ 0, & t>3\pi/2 \end{cases}$

Solution: a) $f(t)$ in terms of Heaviside unit step function can be expressed as

$$f(t) = t^2[H(t) - H(t-1) + 4t\,[H(t-1)]$$
$$= t^2H(t) + (4t - t^2)\,H(t-1)$$

$$\therefore \quad \mathcal{L}\,[f(t)] = \mathcal{L}\,[t^2H(t)] + \mathcal{L}\,[(4t - t^2)H(t-1)]$$
$$= \mathcal{L}\,[t^2] - \mathcal{L}\,[\{(t-1)^2 - 2(t-1) - 3\}\,H(t-1)]$$
$$= \mathcal{L}\,[t^2] - \mathcal{L}\,[(t-1)^2H(t-1)] + 2\,\mathcal{L}\,[(t-1)H(t-1)] + 3\mathcal{L}\,[H(t-1)]$$
$$= \frac{2}{s^3} - e^{-s}\left[\frac{2}{s^3} - \frac{2}{s^2} - \frac{3}{s}\right]$$

b) $f(t)$ in terms of Heaviside unit step function is

$$f(t) = \sin t[H(t - \pi/2) - H(t - 3\pi/2)]$$
$$\therefore \mathcal{L}\,[f(t)] = \mathcal{L}\,[\sin t\,H(t - \pi/2)] - \mathcal{L}\,[\sin t\,H(t - 3\pi/2)]$$
$$= \mathcal{L}\,\cos\,[(t - \pi/2)H(t - \pi/2)] + \mathcal{L}\,[\cos(t - 3\pi/2)H(t - 3\pi/2)]$$
$$= e^{-\frac{\pi}{2}s}\frac{s}{s^2+1} + e^{-\frac{3\pi}{2}s}\frac{s}{s^2+1}$$
$$= \left(e^{-\frac{\pi}{2}s} + e^{-\frac{3\pi}{2}s}\right)\frac{s}{s^2+1}.$$

***

You may now try the following exercises.

E5) Find the Laplace transform of the following functions

a) $f(t) = \left\{\frac{kt}{T},\ 0 < t < T,\ f(t+T) = f(t)\right.$ b) $f(t) = e^{-t^2/4}$

E6) Show that $\int_0^{\infty} t e^{-t^2}\,\mathrm{erf}\,(t)\,dt = \frac{1}{2\sqrt{2}}$.

E7) Find the Laplace transform of half wave rectified sine wave defined as

$$f(t) = \begin{cases} \sin\omega t, & 0 < t < \pi/\omega \\ 0, & \pi/\omega < t < 2\pi/\omega \end{cases}.$$

E8) For $F_\epsilon(t)$ defined by Eqn.(33), find

a) $\mathcal{L}\,[F_\epsilon(t)]$ b) $\mathcal{L}\,[\delta(t-a)]$

E9) Express the following functions in terms of Heaviside unit step functions and hence find their Laplace transforms

a) $f(t) = \begin{cases} \sin 2t, & 0 < t < \pi \\ 0, & t > \pi \end{cases}$ b) $f(t) = \begin{cases} \cos t, & 0 < t < \pi \\ \sin t, & t > \pi \end{cases}$

So far using the integral definition of the Laplace transforms of a function $f(t)$, we determine another function $F(s)$, that is, a function of the transform parameter s. We denoted this symbolically by $\mathcal{L}[f(t)] = F(s)$. However, from application point of view this will not be very useful unless we obtain inverse transform $f(t)$ of a given function $F(s)$ which we shall do in the next section.

6.3 INVERSE LAPLACE TRANSFORMS

In this section, we shall consider the inverse problem of finding $f(t)$ for a given $F(s)$ i.e., given a function $F(s)$, to find a function $f(t)$ of which $F(s)$ is Laplace transform. Consider the following definition.

Definition: If the Laplace transform of $f(t)$ is $F(s)$, then $f(t)$ is called the **inverse Laplace transform** of $F(s)$ and we write

$$\mathcal{L}^{-1}[F(s)] = f(t) \tag{38}$$

where $\mathcal{L}^{-1}$ is called the **inverse Laplace transform** operator.

Inverse Transform of Elementary Functions

Just as we obtained in Sec.6.2 the Laplace transform of some elementary functions we give below in Table-1 the inverse Laplace transform of these functions

Table-1

S.No.	F(s)	$f(t) = \mathcal{L}^{-1}[F(s)]$
1.	$\frac{1}{s}$	1
2.	$\frac{1}{s-a}$	e^{at}
3.	$\frac{1}{s^2+a^2}$	$\frac{\sin at}{a}$
4.	$\frac{s}{s^2+a^2}$	$\cos at$
5.	$\frac{1}{s^2-a^2}$	$\frac{\sinh at}{a}$
6.	$\frac{s}{s^2-a^2}$	$\cosh at$
7.	$\frac{1}{s^{n+1}}$, n is a positive integer	$\frac{t^n}{n!}$

Analogous to the properties proved in Sec.6.2 for Laplace transform, we state the following properties of inverse transform.

Properties of Inverse Transform

Linearity Property

If a and b are any constants and $F(s)$ and $G(s)$ are the Laplace transform of $f(t)$ and $g(t)$ respectively, then

$$\begin{aligned}\mathcal{L}^{-1}[a\,F(s) + b\,G(s)] &= a\,\mathcal{L}^{-1}[F(s)] + b\,\mathcal{L}^{-1}[G(s)] \\ &= a\,f(t) + b\,g(t)\end{aligned} \tag{39}$$

— □ —

First Shifting Property

If $\mathcal{L}^{-1}[F(s)] = f(t)$, then

$$\mathcal{L}^{-1}[F(s+a)] = e^{-at}f(t) \tag{40}$$

— □ —

Second Shifting Property

If $\mathcal{L}^{-1}[F(s)] = f(t)$, then

$$\mathcal{L}^{-1}[e^{-as}F(s)] = \begin{cases} f(t-a), & t > a \\ 0, & t < a \end{cases} = u_a(t)f(t-a) \tag{41}$$

— □ —

Transform of Derivatives

If $\mathcal{L}^{-1}[F(s)] = f(t)$, then

$$\mathcal{L}^{-1}\left[\frac{d}{ds}F(s)\right] = -t\,f(t) \tag{42}$$

and the generalization to higher order derivatives is

$$\mathcal{L}^{-1}\left[\frac{d^n}{ds^n} F(s)\right] = (-1)^n t^n f(t) \qquad (43)$$

— □ —

Transform of Integrals

If $\mathcal{L}^{-1}[F(s)] = f(t)$, then

$$\mathcal{L}^{-1}\left[\int_s^{\infty} F(s)\, ds\right] = \frac{f(t)}{t} \qquad (44)$$

— □ —

Multiplication by Powers of s

If $\mathcal{L}^{-1}[F(s)] = f(t)$ and $f(0) = 0$, then

$$\mathcal{L}^{-1}[s\, F(s)] = f'(t), \text{ if } f(0) = 0 \qquad (45)$$

— □ —

Division by s

If $\mathcal{L}^{-1}[F(s)] = f(t)$, then

$$\mathcal{L}^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t f(t)\, dt \qquad (46)$$

— □ —

Convolution Theorem

If the function $H(s)$ can be expressed as the product of two functions $F(s)$ and $G(s)$ whose inverse, $f(t)$ and $g(t)$ respectively, are known, then the inverse of the product $H(s) = F(s)G(s)$ can be obtained by using convolution theorem.

Theorem 5: If $\mathcal{L}^{-1}[F(s)] = f(t), \mathcal{L}^{-1}[G(s)] = g(t)$ and $H(s) = F(s)\, G(s)$, then

$$\mathcal{L}^{-1}[H(s)] = \mathcal{L}^{-1}[F(s)G(s)] = \int_0^t f(u)\, g(t-u)\, du = f(t) * g(t)$$

Proof: Since $\mathcal{L}[f(t) * g(t)] = \mathcal{L}\left[\int_0^t f(u)\, g(t-u)\, du\right]$

$$= F(s)G(s) = H(s)$$ [using result (24)]

Hence,

$$\mathcal{L}^{-1}[H(s)] = \mathcal{L}^{-1}[F(s)G(s)] = \int_0^t f(u)\, g(t-u)\, du = f(t) * g(t) \qquad (47)$$

Inversion by Contour Integration

In the application of Laplace transform methods to physical situations we come across certain functions whose inverse cannot be written through the procedure outlined above. For such situations we need to develop a direct procedure for inverting the Laplace transform. Keeping this in view Bromwich 1916, utilizing the tools of the theory of functions of a complex variable established the following result, summed up in the form of a theorem:

Theorem 6: The inverse of a Laplace transform can be expressed as the contour integral

$$\mathcal{L}^{-1}[F(s)] = f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\, e^{ts} ds$$

where $F(s)$ is the Laplace transform of $f(t)$ given by

$$F(s) = \int_0^{\infty} f(t)\, e^{-st} dt$$

and c , known as Laplace convergence abscissa , must be greater than the real part of any of the singularity of F(s).

Alternative statement of Theorem 6:

The inverse Laplace transform $\mathcal{L}^{-1}[F(s)] = f(t)$ is given by the sum of the residues of $e^{ts}F(s)$ at the poles of $F(s)$.

Proof: Keeping in view the fact that $f(t) = 0$, for $t < 0$ and the conditions that $f(t)$ satisfies for all t, we can write down the complex Fourier representation of $e^{-cx}f(x)$ as

$$f(x)e^{-cx} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iwx}\left[\int_{0}^{\infty} e^{-iwt}e^{-ct}f(t)\,dt\right]dw \tag{48}$$

where c is so chosen that $\int_0^{\infty} e^{-ct}\,|f(t)|\,dt$ exists. Eqn.(48) will become more clear to you after you study complex Fourier representation of a function in Unit 7.

Multiplying Eqn.(48) throughout by e^{cx} and adjusting this factor inside the first integral on the r.h.s, we get

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iwx+cx}\left\{\int_{0}^{\infty} e^{-(c+iw)t} f(t)\,dt\right\}dw \tag{49}$$

Putting $s = c + iw$, where s is the new variable of integration, Eqn.(49) can be expressed as

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{sx}\left[\int_{0}^{\infty} e^{-st}f(t)\,dt\right]ds \tag{50}$$

In writing Eqn.(50) we have used $ds = i dw$, when $w = -\infty$, $s = c - i\infty$ and $s = c + i\infty$ for $w \to \infty$.
Hence

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)e^{sx}\,ds \tag{51}$$

$$\therefore f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)e^{st}\,ds = \text{ sum of the residues of } e^{st}F(s) \text{ at the poles of } F(s). \tag{52}$$

— □ —

We now illustrate the use of properties discussed above through the following examples.

Example 16: Obtain the inverse Laplace transform of the following functions

a) $\dfrac{2s-5}{4s^2+25}$ b) $\dfrac{s\cos\alpha + \omega\sin\alpha}{s^2+\omega^2}$

Solution: a) $\mathcal{L}^{-1}\left[\dfrac{2s-5}{4s^2+25}\right] = \dfrac{1}{4}\mathcal{L}^{-1}\left[\dfrac{2s-5}{s^2+25/4}\right] = \dfrac{1}{4}\mathcal{L}^{-1}\left[\dfrac{2s-5}{s^2+(5/2)^2}\right]$

$$= \frac{1}{2}\mathcal{L}^{-1}\left[\frac{s}{s^2+(5/2)^2}\right] - \frac{5}{4}\mathcal{L}^{-1}\left[\frac{1}{s^2+(5/2)^2}\right]$$

$$= \frac{1}{2}\cos\frac{5}{2}t - \frac{5}{4}\frac{2}{5}\sin\frac{5}{2}t = \frac{1}{2}\left(\cos\frac{5t}{2} - \sin\frac{5t}{2}\right)$$

b) $$\mathcal{L}^{-1}\left[\frac{s\cos\alpha+\omega\sin\alpha}{s^2+\omega^2}\right]=\cos\alpha\;\mathcal{L}^{-1}\left[\frac{s}{s^2+\omega^2}\right]+\sin\alpha\,\mathcal{L}^{-1}\left[\frac{\omega}{s^2+\omega^2}\right]$$
$$=\cos\alpha\cos\omega t+\sin\alpha\sin\omega t$$
$$=\cos(\omega t-\alpha)$$

***

Example 17: Find $\mathcal{L}^{-1}\left[\ln\left(\frac{s+b}{s+a}\right)\right]$

Solution: Let $\mathcal{L}^{-1}\left[\ln\left(\frac{s+b}{s+a}\right)\right]=f(t)$

$\therefore\quad \mathcal{L}^{-1}[\ln(s+b)-\ln(s+a)]=f(t)$

$\therefore\quad \mathcal{L}^{-1}\left[\frac{d}{ds}\{\ln(s+b)-\ln(s+a)\}\right]=-t\,f(t)$

$\therefore\quad \mathcal{L}^{-1}\left[\frac{1}{s+b}-\frac{1}{s+a}\right]=-t\,f(t)$

$\therefore\quad e^{-bt}-e^{-at}=-t\,f(t)$

or, $f(t)=\dfrac{e^{-at}-e^{-bt}}{t}$

***

Example 18: Find the inverse Laplace transforms of the following functions

a) $\dfrac{s}{(s^2+a^2)^2}$ b) $\dfrac{1}{s(s^2+4)}$ c) $\dfrac{1}{(s+1)(s^2+1)}$

Solution: a) Let $\mathcal{L}^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]=f(t)$

$\therefore\quad \mathcal{L}^{-1}\left[\int_s^\infty\frac{s}{(s^2+a^2)^2}ds\right]=\frac{f(t)}{t}$ [using result (44)]

or, $\frac{1}{2}\mathcal{L}^{-1}\left[\left\{-\frac{1}{s^2+a^2}\right\}_s^\infty\right]=\frac{f(t)}{t}$

or, $\frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s^2+a^2}\right]=\frac{f(t)}{t}$

or, $\frac{1}{2}\frac{\sin at}{a}=\frac{f(t)}{t}$ or, $f(t)=\frac{t\sin at}{2a}$

Hence $\mathcal{L}^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]=\frac{t\sin at}{2a}$

b) $\mathcal{L}^{-1}\left[\frac{1}{s^2+4}\right]=\frac{\sin 2t}{2}$

$\therefore\quad \mathcal{L}^{-1}\left[\frac{1}{s}\cdot\frac{1}{s^2+4}\right]=\int_0^t\frac{\sin 2t}{2}dt=\frac{1}{2}\left[\frac{-\cos 2t}{2}\right]_0^t$ [by result (46)]

$$=\frac{1-\cos 2t}{4}=\frac{1}{2}\sin^2 t\,.$$

c) We can write $\dfrac{1}{(s+1)(s^2+1)}=\dfrac{1}{(s+1)}\cdot\dfrac{1}{(s^2+1)}$

Let $F(s)=\dfrac{1}{s+1}$ and $G(s)=\dfrac{1}{s^2+1}$

$$\therefore \quad f(t)=\mathcal{L}^{-1}\left[\frac{1}{s+1}\right]=e^{-t} \text{ and } g(t)=\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right]=\sin t$$

Hence, by convolution theorem, we have

$$\mathcal{L}^{-1}\left[\frac{1}{(s+1)(s^2+1)}\right]=e^{-t} * \sin t$$

$$=\int_0^t e^{-u}\sin(t-u)\,du$$

$$=\int_t^0 e^{-t+x}\sin x\,(-dx) \qquad \text{[putting } t-u=x\text{]}$$

$$=e^{-t}\int_0^t e^x \sin x\,dx=e^{-t}\left[\frac{e^x}{2}(\sin x-\cos x)\right]_0^t$$

$$=\frac{1}{2}(\sin t-\cos t)+\frac{1}{2}e^{-t}.$$

***

In cases where $F(s)$ is a rational algebraic fraction, it is often convenient to find inverse Laplace transform by expressing $F(s)$ in terms of partial fractions. We illustrate it through an example.

Example 19: Find the inverse Laplace transform of

a) $\dfrac{11s^2-2s+5}{(s-2)\,(2s-1)\,(s+1)}$ b) $\dfrac{21s-33}{(s+1)\,(s-2)^3}$

Solution: a) $$\mathcal{L}^{-1}\left[\frac{11s^2-2s+5}{(s-2)\,(2s-1)\,(s+1)}\right]=\mathcal{L}^{-1}\left[\frac{5}{s-2}+\frac{-3}{2s-1}+\frac{2}{s+1}\right] \qquad \text{[using partial fractions]}$$

$$=5\mathcal{L}^{-1}\left[\frac{1}{s-2}\right]-\frac{3}{2}\mathcal{L}^{-1}\left[\frac{1}{s-1/2}\right]+2\mathcal{L}^{-1}\left[\frac{1}{s+1}\right]$$

$$=5e^{2t}-\frac{3}{2}e^{t/2}+2e^{-t}.$$

b) $$\mathcal{L}^{-1}\left[\frac{21s-33}{(s+1)\,(s-2)^3}\right]=\mathcal{L}^{-1}\left[\frac{2}{s+1}+\frac{-2}{s-2}+\frac{6}{(s-2)^2}+\frac{3}{(s-2)^3}\right]$$

$$=2\mathcal{L}^{-1}\left[\frac{1}{s+1}\right]-2\mathcal{L}^{-1}\left[\frac{1}{s-2}\right]+6\mathcal{L}^{-1}\left[\frac{1}{(s-2)^2}\right]+3\mathcal{L}^{-1}\left[\frac{1}{(s-2)^2}\right]$$

$$=2e^{-t}-2e^{2t}+e^{2t}t+3e^{2t}\frac{t^2}{2}. \qquad \text{[using result (40) and formula (2) in Appendix]}$$

***

Example 20: Evaluate $\mathcal{L}^{-1}\left[\dfrac{1}{(s+1)\,(s^2+1)}\right]$ using contour integration.

Solution: Poles of $F(s)=\dfrac{1}{(s+1)\,(s^2+1)}$ are obtained by equating to zero, the terms in the denominator of $F(s)$, that is $s+1=0,\ s^2+1=0$. Thus, $s=-1,\pm i$ are the simple poles. Residue of $e^{ts}F(s)$ at the pole $s=-1$ is given by

$$\lim_{s\to -1}\frac{e^{ts}}{(s^2+1)}=\frac{e^{-t}}{2}$$

Residue of $e^{st}F(s)$ at $s = i$, is given by

$$\lim_{s\to i}\frac{(s-i)e^{st}}{(s+1)(s^2+1)} = \lim_{s\to i}\frac{e^{st}}{(s+1)(s+i)} = \frac{-e^{it}(1+i)}{4}$$

Residue at $s = -i$, is given by $\lim\limits_{s\to -i}\dfrac{e^{st}}{(s+1)(s-i)} = \dfrac{ie^{-it}}{4}(1+i)$

$$\therefore \quad \mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s^2+1)}\right\} = \frac{e^{-t}}{2} - \frac{e^{it}(1+i)}{4} + \frac{i(1+i)e^{-it}}{4} \qquad \text{[sum of the residues]}$$

$$= \frac{e^{-t}}{2} - \frac{1}{2}\left(\frac{e^{it}+e^{-it}}{2}\right) - \frac{i}{2}\left(\frac{e^{it}-e^{-it}}{2}\right)$$

$$= \frac{e^{-t}}{2} - \frac{1}{2}\cos t + \frac{1}{2}\sin t$$

***

The above result could have been easily arrived at through the other approach of resolving into partial fraction and using the properties of inverse LT. However, there are functions whose Laplace inverse is either not obtainable or is difficult to obtain via usual approach. In such cases residual method is the only alternative as illustrated below.

Example 21: Determine $\mathcal{L}^{-1}\left[\dfrac{1}{s\ \sinh as}\right]$, a is a real constant.

Solution: Using Bromwich integral

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{ts}}{s\sinh as}\,ds$$

where c is greater than the real part of any of the singularities in $\dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty}\dfrac{1}{s\sinh as}e^{ts}\,ds$. Now the Taylor's expansion of sinh as can be written as

$$\sinh as = as + \frac{(as)^3}{3!} + \frac{(as)^5}{5!} + \cdots\cdots$$

$$= as\left\{1 + \frac{(as)^2}{3!} + \frac{(as)^4}{5!} + \cdots\right\}$$

We find that $s = 0$ is a second order pole. Further, using the definition of hyperbolic function $\sinh\theta = -i\sin i\theta$, we find that there are infinite number of simple poles located where $\sin ias = 0$, that is at $s_n = \pm\dfrac{n\pi i}{a}$, $n \in I^+$, We may choose the contour as shown below:

In order to find the residue of $e^{ts}F(s)$, at the second order pole at $s = 0$, we use the standard formula to get the residue

$$= \frac{1}{1!}\lim_{s\to 0}\frac{d}{ds}\left\{\frac{(s-0)^2e^{ts}}{s\sinh as}\right\} = \lim_{s\to 0}\frac{d}{ds}\left(\frac{se^{ts}}{\sinh as}\right)$$

$$= \lim_{s\to 0}\left\{\frac{e^{ts}}{\sinh as} + \frac{st\,e^{ts}}{\sinh as} - \frac{e^{ts}s}{\sin h^2 as}a\cosh as\right\}$$

$$= \lim_{s\to 0}e^{ts}\left\{\frac{\sinh as + st\sinh as - as\cosh as}{\sin h^2 as}\right\}, \text{ this is of the form } \left[\frac{0}{0}\right]$$

Using L ' Hospital rule we get

$$= \lim_{s\to 0} e^{ts} \lim_{s\to 0}\left\{\frac{a\cosh as + t(\sinh as + sa\cosh as) - a\cosh as - a^2 s\sinh as}{2a\sinh as\cosh as}\right\}$$

$$= \lim_{s\to 0}\frac{t}{2a}\frac{1}{\cosh as} + \lim_{s\to 0}\frac{t}{2}\left(\frac{s}{\sinh as}\right) - .\lim_{s\to 0}\frac{a^2 s}{2a\cosh as}$$

$$= \frac{t}{2a} + \lim_{s\to 0}\left(-\frac{s}{\sinh as}\right)\frac{t}{2} - 0$$

$$= \frac{t}{2a} + \frac{t}{2a} + 0 \qquad \left(\because \lim_{s\to 0}\frac{s}{\sinh as} = \lim_{s\to 0}\frac{1}{a\cosh as} = \frac{1}{a}\right)$$

$$= \frac{t}{a}$$

The residue at simple poles $s_n = \pm\frac{n\pi i}{a}$, is given by

$$\lim_{s\to s_n}\frac{(s-s_n)e^{ts}}{s\sinh as} = \lim_{s\to s_n}\frac{e^{ts}}{(\sinh as + sa\cosh as)} = \frac{e^{\pm\frac{n\pi i}{a}}}{(-1)^n(\pm n\pi i)}$$

$$[\sinh as = -i\sin ias, \text{at } s = \frac{n\pi i}{a}, \sinh as = 0,\ \cosh as = \cos ias = \cos i\frac{n\pi i}{a} = \cos n\pi(-1)^n]$$

Hence,

$$f(t) = \frac{t}{a} + \sum_{n=1}^{\infty}\frac{(-1)^n\exp\left(\frac{n\pi it}{a}\right)}{n\pi i} - \sum_{n=1}^{\infty}\frac{(-1)^n\exp\left(\frac{-n\pi it}{a}\right)}{n\pi i}. \qquad \text{[sum of the residues]}$$

***

You may now try some exercises.

E10) Find the Laplace inverse of the following functions

a) $\frac{3s}{s^2+2}$

b) $\left[\frac{s}{s^2+4} - \frac{1}{3s^2}\right]$

c) $[2/(s^4-1)]$

d) $\left[\frac{s-1}{(s-2)(s+2)}\right]$

e) $[2/(s^4+2s^2]$

E11) Find $\mathcal{L}^{-1}s/[(s+a)^2+b^2]$.

E12) If $a>0$, show that

$\mathcal{L}^{-1}[F(as)] = \frac{1}{a}f\left(\frac{t}{a}\right)$; $\mathcal{L}[f(t)] = F(s)$

E13) Find the inverse Laplace transform of each of the following functions

a) $\cot^{-1}s$ b) $\frac{2s+1}{(s^2+s+1)^2}$ c) $\frac{1}{s^3(s^2+1)}$

d) $\frac{1}{s(s^2+a^2)}$ e) $\frac{2s^2-1}{(s^2+1)(s^2+4)}$

6.4 APPLICATIONS OF LAPLACE TRANSFORM TO INITIAL AND BOUNDARY VALUE PROBLEMS

The Laplace transform is very useful in solving differential equations and corresponding initial and boundary value problems. The solution of differential equations involving functions of an impulsive type can also be solved by the use of Laplace transform in a very efficient manner. The general process of solution consists of three main steps:

1. The given differential equation is transformed into a simple algebraic equation.
2. The algebraic equation is solved by pure algebraic manipulations.
3. The solution of the algebraic equation is then transformed back to obtain the solution of the given differential equation.

In this way the Laplace transform method reduces the problem of solving differential equation to an algebraic problem.

We shall now illustrate the method through the following application.

Initial Value Problems with Constant Coefficients

The advantage of using Laplace transform method for solving initial value problems is that it solves IVPs directly without first finding general solution (complete integral) and then evaluating the arbitrary constants using initial conditions.

We illustrate the method for both homogeneous and non-homogeneous IVPs through the following examples.

Example 22: Solve the initial value problem

$$\frac{d^2y(t)}{dt^2} + y(t) = 0 \ , \ \ y(0) = 0 \ , \ \ y'(0) = 1$$

Solution: On taking the Laplace transform of the given differential equation and using Eqn.(20), we get

$$s^2[\mathscr{L}\, y(t)] - s\,y(0) - y'(0) + \mathscr{L}[y(t)] = 0$$

using the initial conditions, we get

$$s^2 Y(s) - 0 - 1 + Y(s) = 0 \Rightarrow Y(s) = \frac{1}{1+s^2}$$

On taking Laplace inverse of $Y(s)$, we get

$$y(t) = \sin t \, .$$

***

Example 23: Find the solution of the IVP

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = 12e^{-2t} \ , \ \ y(0) = 2 \ , \ y'(0) = 6 \, .$$

Solution: Taking the Laplace transform of both sides of the given equation, we get

$$\mathscr{L}\left[\frac{d^2y}{dt^2}\right] - 3\mathscr{L}\left[\frac{dy}{dt}\right] + 2\mathscr{L}\,[y(t)] = 12\mathscr{L}\,[e^{-2t}]$$

if $\mathscr{L}\,[y(t)] = Y(s)$, then we get

$$s^2Y(s) - sy(0) - y'(0) - 3\{sY(s) - y(0)\} + 2Y(s) = \frac{12}{s+2} \qquad \text{[using result (20)]}$$

Substituting the given conditions $y(0) = 2$, $y'(0) = 6$, we get

$$\{s^2Y(s) - s(2) - 6\} - 3\{s\,Y(s) - 2\} + 2Y(s) = \frac{12}{s+2}$$

or, $(s^2 - 3s + 2)\ Y(s) - 2s = \dfrac{12}{s+2}$

$$\therefore \quad (s^2 - 3s + 2)\, Y(s) = \frac{2s^2 + 4s + 12}{s + 2}$$

$$\therefore \quad Y(s) = \frac{2s^2 + 4s + 12}{(s^2 - 3s + 2)\,(s + 2)} = \frac{2s^2 + 4s + 12}{(s - 1)\,(s - 2)(s + 2)}$$

Using partial fractions, we can express $Y(s)$ in the form

$$Y(s) = \frac{-6}{s - 1} + \frac{7}{s - 2} + \frac{1}{s + 2}$$

Taking the inverse Laplace transform of both sides of the above equation, we get

$$y(t) = -6e^{t} + 7e^{2t} + e^{-2t}$$

as the required solution.

***

We now give an interesting application of Laplace transform method in combination with convolution theorem to solve **non-homogeneous IVP with zero initial conditions.** The method given here serves as an alternative approach for IVP's of second order ordinary differential equations with constant coefficients which are being solved by the method of variation of parameters and turn out to be quite difficult. For this purpose, we first state and prove the following theorem:

Theorem 7: If $y = f(x)$ is the solution of the constant coefficient homogeneous initial value problem

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0, \;\; y(0) = 0, \;\; y'(0) = 1 \qquad (53)$$

then the solution of the non-homogeneous initial value problem

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = g(x), \;\; y(0) = 0, \;\; y'(0) = 0 \qquad (54)$$

is the convolution of f and g:

$$y(x) = (f * g)\,x = \int_0^x f(z)\; g(x - z)\; dz \qquad (55)$$

Proof: On taking the Laplace transform of Eqn.(53), we get on using $Y(s) = \mathcal{L}[y(x)]$

$$s^2 Y(s) - sy(0) - y'(0) + a[sY(s) - y(0)] + bY(s) = 0\,.$$

or, $\qquad Y(s) = \dfrac{1}{s^2 + as + b}$

$$\therefore y(x) = \mathcal{L}^{-1}\left[\frac{1}{s^2 + as + b}\right] = f(x) \;\; \text{(assumed)}.$$

On taking Laplace transform of Eqn.(54), we get

$$s^2 Y(s) - sy(0) - y'(0) + a[sY(s) - (0)] + bY(s) = G(s)$$

Using the zero initial conditions we get

$$Y(s) = \frac{G(s)}{(s^2 + as + b)}$$

Taking inverse of this equation

$$y(x) = \mathcal{L}^{-1}\left[\frac{G(s)}{(s^2 + as + b)}\right] \qquad (56)$$

Using convolution theorem, we get

$$y(x) = \int_0^x f(z)\; g(x - z)\; dz \qquad (57)$$

We illustrate this method through the following example.

Example 24: Solve the non-homogeneous IVP

$$\frac{d^2y}{dx^2} - y = \frac{1}{e^x + e^{-x}}, \quad y(0) = 0, \quad y'(0) = 1 \tag{58}$$

Solution: The solution of the homogeneous IVP is given by

$$y(x) = A\cosh x + B\sinh x \tag{59}$$

where A and B determined by $y(0) = 0, y'(0) = 1$ are given as

$$A = 0, \quad B = 1$$

$$\therefore \quad y(x) = \sinh x = f(x) \tag{60}$$

Using the result of Theorem-7, the solution of Eqn.(58) is given by the convolution of f and g i.e.

$$(f * g)x = \int_0^x \frac{\sinh(x - z)}{e^z + e^{-z}} dz$$

$$= \int_0^x \frac{\sin hx \cosh z - \cosh x \sinh z}{2\cosh z} dz$$

$$= \frac{1}{2}\int_0^x (\sinh x - \cosh x \tanh z) dz$$

$$= \frac{1}{2}\left[(\sinh x)x - \cosh x \ln \cosh x\right].$$

which is the required solution of Eqn.(58).

***

You may now try the following exercises.

E14) Using Laplace transform technique find the solutions of the following initial value problems:

a) $y'' + 25y = 10\cos 5t, \quad y(0) = 2, \quad y(0) = 0$

b) $y'' - 4y' + 4y = 64\sin 2t, \quad y(0) = 0, \quad y'(0) = 1$

c) $y'' + 3y' + 2y = e^t \sin t, \quad y(0) = y'(0) = 0$

d) $y' + y = e^{2t} + t, \quad y(0) = -1$

e) $y' + 4y = 1 - \sin t, \quad y'(0) = -1$

f) $2y'' + 5y' + 2y = e^{-2t}, \quad y(0) = 1, \quad y'(0) = 1$

g) $y'' + 2y' + 5y = e^{-x}\sin x, \quad y(0) = 0, \quad y'(0) = 1$

E15) A resistance R in a series with inductance L is connected with emf $E(t)$. The current t is given by $L\frac{di}{dt} + Ri = E(t)$.

If the switch is connected at $t = 0$ and disconnected at $t = a$, find the current i in terms of t

E16) Assume that an electrical circuit is governed by the initial value problem

$$\frac{d^2q}{dt^2} - \frac{dq}{dt} - 6q = E(t)$$

where $E(t) = \begin{cases} t, & 0 \le t < 1 \\ 1, & 1 \le t \end{cases}$

subject to the initial conditions $q(0) = 0$ and $q'(0) = 0$. Find the charge $q(t)$.

E17) Find the solution of the initial value problem

$$y'' + 9y = 9\, u(t-3), \quad y(0) = y'(0) = 0$$

where $u(t-3)$ is the unit step function.

We shall now show how the initial value problems involving system of linear differential equations with constant coefficients can be handled using Laplace transform method.

System of Linear Differential Equations

As pointed out earlier, the advantage of the method under consideration over the classical methods is two fold: (i) initial values are automatically satisfied and (ii) no special technique is to be devised for determining particular solution in case of repeated roots of the auxiliary equation, which in matrix form corresponds to the fundamental matrix having repeated eigenvalues. This is illustrated through the following examples.

Example 25: Use the Laplace transform method to solve the system of equations

$$y''(x) - y(x) + 5z'(x) = x, \tag{61}$$

$$z''(x) - 4z(x) - 2y'(x) = -2 \tag{62}$$

under the initial conditions.

$$y(0) = 0,\ y'(0) = 0;\ z(0) = 0,\ z'(0) = 0. \tag{63}$$

Solution: Let $Y(s)$ and $Z(s)$ denote respectively the Laplace transforms of $y(x)$ and $z(x)$. On applying LT operator to the system of Eqns.(61) – (62), we get

$$\mathcal{L}[y''(x)] - \mathcal{L}[y(x)] + 5\mathcal{L}[z'(x)] = \mathcal{L}[(x)]$$

$$\mathcal{L}[z''(x)] - 4\mathcal{L}[z(x)] - 2\mathcal{L}[y'(x)] = -\mathcal{L}[(2)]$$

Using formulas (31) and (32) of Appendix, we get

$$s^2 Y(s) - s\,y(0) - y'(0) - Y(s) + 5\{sZ(s) - z(0)\} = \frac{1}{s^2} \tag{64}$$

$$s^2 Z(s) - sz(0) - z'(0) - 4Z(s) - 2\{sY(s) - y(0)\} = \frac{-2}{s} \tag{65}$$

Using the initial conditions (63) in Eqns.(64) – (65), we get

$$(s^2 - 1)Y(s) + 5s\,Z(s) = \frac{1}{s^2}$$

$$(s^2 - 4)\,Z(s) - 2s\,Y(s) = \frac{-2}{s}$$

On solving the above equations for $Y(s)$ and $Z(s)$, we get

$$Y(s) = \frac{11s^2 - 4}{s^2\,(s^2 + 1)\,(s^2 + 4)} \tag{66}$$

$$Z(s) = \frac{-2s^2 + 4}{s(s^2 + 1)\,(s^2 + 4)} \tag{67}$$

On resolving into partial fractions, Eqns.(66) – (67) can be expressed as

$$Y(s) = -\frac{1}{s^2} + \frac{5}{s^2 + 1} - \frac{4}{s^2 + 4} \tag{68}$$

$$Z(s) = \frac{1}{s} - \frac{2s}{s^2 + 1} + \frac{s}{s^2 + 4} \tag{69}$$

On taking Laplace inverse of Eqns.(68) – (69), we get

$$y(x) = -x + 5\sin x - 2\sin 2x$$

$$z(x) = 1 - 2\cos x + \cos 2x$$

as the required solution of given system of equations.

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Example 26: Use the Laplace transform method to solve the system of equations

$$x'(t) = x - 2y(t), \tag{70}$$

$$y'(t) = 5x - y, \tag{71}$$

Subject to the initial conditions

$$x(0) = -1,\ y(0) = 2 \tag{72}$$

Solution: On taking LT of Eqns.(70) – (71), we get

$$sX(s) - x(0) = X(s) - 2Y(s) \tag{73}$$

$$sY(s) - y(0) = 5X(s) - Y(s) \tag{74}$$

On combining Eqns.(72) – (74), we can write

$$(s-1)X(s) = -1 - 2Y(s)$$
$$(s+1)Y(s) = 5X(s) + 2$$

On solving the above equations for $X(s)$ and $Y(s)$, we get

$$X(s) = \frac{(-s-5)}{(s^2+9)}$$
$$Y(s) = \frac{(2s-7)}{(s^2+9)}$$

or $X(s) = -\dfrac{s}{s^2+9} - \dfrac{5}{s^2+9}$, $Y(s) = \dfrac{2s}{s^2+9} - \dfrac{7}{s^2+9}$

On taking inverse of the above equation, we get, respectively

$$x(t) = -\cos 3t - \frac{5}{3}\sin 3t,\ y(t) = 2\cos 3t - \frac{7}{3}\sin 3t$$

***

You may now try the following exercises.

E18) If $X_1(s) = \mathcal{L}[x_1(t)],\ X_2(s) = \mathcal{L}[x_2(t)]$, using the Laplace transform convert the IVP

$$\begin{bmatrix} 2D & D-1 \\ D & D \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ t^2 \end{bmatrix}, \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

into the algebraic system

$$\begin{bmatrix} 2s & s-1 \\ s & s \end{bmatrix}\begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} 2+\dfrac{1}{s^2} \\ 1+\dfrac{2}{s^2} \end{bmatrix}$$

Hence find $X_1(s)$ and $X_2(s)$. Show that $X_1(s) + X_2(s) = \dfrac{1}{s} + \dfrac{2}{s^4}$.

Further, find $x_1(t)$ and $x_2(t)$.

E19) Use the LT method to solve the given system of equations

$$2x'(t) + 2x(t) + y'(t) - y(t) = 3t$$
$$x'(t) + x(t) + y'(t) + y(t) = 1\ ;\ x(0) = 1\ ,\ y(0) = 3\ .$$

E20) Use the LT method to solve the IVP

$$x'(t) + 2y(t) = 0\ ,$$
$$x''(t) + 2y'(t) + 2y(t) = 2e^t$$
$$x(0) = 1\ ,\ x'(0) = 0\ ,\ y(0) = 0$$

E21) Use the LT method to solve the given system of equations

$$x''(t) + 2x(t) - y'(t) - y(t) = 6e^{3t}$$
$$2x'(t) - 3x(t) + y'(t) - 3y(t) = 6e^{3t}\ ,\ x(0) = 3\ ,\ y(0) = 0\ .$$

An equation that contains a dependent variable under integral sign is termed as an **integral equation**. An equation that contains the derivative of the dependent variable and the dependent variable under an integral sign is called an integro-differential equation. Laplace transform technique is an important tool for solving integral and integro-differential equations which we shall take up now.

We shall illustrate the method of solving integral equations through examples.

Example 27: Solve the integral equation

$$\int_0^x e^{x-t}\phi(t)\,dt = x \tag{75}$$

In the above equation ϕ is the dependent variable and x is the independent variable. Further it is of convolution type where the kernel e^{x-t} is dependent solely on the difference $(x-t)$ of the arguments.

Solution: If $\mathcal{L}[\phi(t)] = \Phi(s)$ then taking Laplace transform of Eqn.(75), we get

$$\Phi(s)\frac{1}{(s-1)} = \frac{1}{s^2}$$

or, $$\Phi(s) = \frac{s-1}{s^2} = \frac{1}{s} - \frac{1}{s^2}$$

$$\therefore \quad \mathcal{L}^{-1}[\Phi(s)] = \phi(x) = 1 - x$$

***

Example 28: Solve $\int_0^t \cos(x-t)\phi(t)\,dt = \sin x$

Solution: Taking LT of the given equation we get

$$\Phi(s)\frac{s}{s^2+1} = \frac{1}{s^2+1} \Rightarrow \Phi(s) = \frac{1}{s}$$

Taking Laplace inverse of $\Phi(s)$ we get

$$\phi(x) = 1$$

***

Example 29: Solve the following integral equation using convolution theorem

$$y(x) = x + 6\int_0^x y(z)\,e^{(x-z)}\,dz$$

Solution: On taking Laplace transform of the above equation and using convolution theorem, we get

$$Y(s) = \frac{1}{s^2} + 6\frac{Y(s)}{s-1} \Rightarrow Y(s)\left\{1 - \frac{6}{s-1}\right\} = \frac{1}{s^2}$$

or, $$Y(s) = \frac{(s-1)}{(s-7)s^2} = \frac{6}{49(s-7)} - \frac{6}{49s} + \frac{1}{7s^2}$$

On taking Laplace inverse of $Y(s)$, we obtain

$$y(x) = \frac{6}{49}e^{7x} - \frac{6}{49} + \frac{7}{49}x = \left(-6 + 7x + 6e^{7x}\right)/49$$

***

You may now try the following exercises.

E22) Determine the function $y(t)$ satisfying the integral equation

$$y(t) + \int_0^t y(x)\,(t-x)\,dx = t.$$

E23) Find the solution of the equation $y' + 3y + 2\int_0^x y\,dx = f(x)$

Where $f(x) = 2u(x-1) - 2u(x-2)$.

E24) Solve the following equation

$$y' + 4y + 4\int_0^t y(\tau)d\tau = u_1(t),\ y(0) = 3.$$

E25) Solve the following integral equation using convolution theorem

$$y(x) = \cos x + e^{-x}\int_0^x y(z)\,e^{z}\,dz$$

So far we have seen that the application of Laplace transform technique for solving IVP's associated with single/system of ODEs reduces them to single/system of algebraic equations in the transformed variable. Further, the resulting system can be easily solved for the transformed variable and its inverse obtained. The question that arises in this context is- What happens when the same technique is carried over to solve initial boundary value problems of PDEs? We shall now give an answer to this question.

Initial Boundary Value Problems

Suppose we have a pde in two independent variables say x and t where 't' stands for time variable. When we apply LT w.r.t. 't' the given pde in the dependent variable $u(x,t)$ reduces to an ODE in the transformed variable $\bar{u}(x,s)$ which can be solved under the transformed boundary conditions. Finally, to obtain the original dependent variable $u(x,t)$, we search for inverse of $\bar{u}(x,s)$ either through the table given in Appendix or, through the use of any of the methods discussed earlier or, through the use of complex inversion theorem. Below we take up few examples to illustrate the method.

Example 30: Using Laplace transform technique solve the initial boundary value problem

$$\frac{\partial u}{\partial x}+x\frac{\partial u}{\partial t}=0\ ,\ u(x,0)=1\ ,\ u(0,t)=t \tag{76}$$

Solution: Taking Laplace transform with respect to 't' of Eqn.(76) and on using $\bar{u}(x,s)=\int_0^\infty e^{-st}u(x,t)\,dt$, we get

$$\frac{d\bar{u}}{dx}+x(s\bar{u}-1)=0\ ,\ \bar{u}(0,s)=\frac{1}{s^2} \tag{77}$$

On solving linear ODE (77), we get

$$\bar{u}e^{\frac{sx^2}{2}}=\frac{e^{\frac{sx^2}{2}}}{s}+C\text{, where C is constant.}$$

At $x=0$, $\bar{u}(0,s)=\frac{1}{s^2}$. Therefore $C=\frac{1}{s^2}-\frac{1}{s}$

$$\text{Hence, }\bar{u}(x,s)=\frac{1}{s}+e^{\frac{-sx^2}{2}}\left(\frac{1}{s^2}-\frac{1}{s}\right) \tag{78}$$

On taking inverse of Eqn.(78), we get

$$u(x,t)=1+\left[\left(t-\frac{x^2}{2}\right)-1\right]H\left(t-\frac{x^2}{2}\right) \tag{79}$$

In writing Eqn.(79) we have used formula (15) given in appendix i.e.,

$$f(t-a)H(t-a)=e^{-as}F(s)\ ,\quad a>0.$$

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Example 31: Solve the following initial boundary value problem

$$\frac{\partial u}{\partial x}+6x\frac{\partial u}{\partial t}=12x\ ,\ u(x,0)=2\ ,\ u(0,t)=2 \tag{80}$$

Solution: Proceeding as above, we get for the transformed variable $\bar{u}$,

$$\frac{d\bar{u}}{dx}+6x(s\bar{u}-2)=\frac{12x}{s}\ ;\ \bar{u}(0,t)=\frac{2}{s}$$

or, $$\frac{d\bar{u}}{dx}+6xs\bar{u}=12x\left(\frac{1}{s}+1\right)$$

which is a linear ODE whose solution is given by

$$\bar{u}\,e^{3x^2 s} = 2\left(\frac{1+s}{s^2}\right)e^{\frac{3x^2}{s}} + C.$$

When $x = 0, \bar{u} = \frac{2}{s}$. Hence $C = \frac{2}{s} - \frac{2(1+s)}{s^2} = \frac{-2}{s^2}$

$$\therefore \quad \bar{u} = \frac{2(1+s)}{s^2} - \frac{2}{s^2}e^{-3x^2 s}$$

Taking Laplace inverse of $\bar{u}$, we get

$$u(x,t) = 2t + 2 - 2(t - 3x^2)\,H(t - 3x^2) \tag{81}$$

***

Example 32: Using Laplace transform method solve the vibration problem

$$u_{tt} = a^2 u_{xx}\ ,\ 0 < x < \ell\ ,\ t > 0 \tag{82}$$

$$u(x,0) = 0\ ,\ u_t(x,0) = \sin\left(\frac{\pi x}{\ell}\right)\ ,\ 0 < x < \ell \tag{83}$$

$$u(0,t) = 0\ ,\ u(\ell,t) = 0\ ,\ t > 0 \tag{84}$$

Solution: On taking Laplace transform w.r.t. 't' of Eqn.(82), we get

$$s^2\bar{u}(x,s) - su(x,0) - u_t(x,0) = a^2\frac{d^2\bar{u}}{dx^2} \tag{85}$$

In writing Eqn.(85) we have used formula (31) given in the Appendix and

$$\mathcal{L}\,[u(x,t)] = \int_0^\infty e^{-st}u(x,t)\,dt = \bar{u}(x,s).$$

Laplace transform of Eqn.(84) yields

$$\bar{u}(0,s) = 0,\ \bar{u}(\ell,s) = 0$$

Using the conditions given in Eqn.(83), Eqn.(85) reduces to

$$\frac{d^2\bar{u}}{dx^2} - \frac{s^2}{a^2}\bar{u} = -\frac{\sin\left(\frac{\pi x}{\ell}\right)}{a^2} \tag{86}$$

Solving Eqn.(86), we get $\bar{u}(x,s) = A(s)\cosh\frac{s}{a}x + B(s)\sinh\frac{s}{a}x$ as the complementary solution.

Particular solution of Eqn.(86) is

$$\bar{u} = \frac{-1}{a^2}\,\frac{1}{D^2 - \frac{s^2}{a^2}}\sin\left(\frac{\pi x}{\ell}\right) = \frac{\sin\left(\frac{\pi x}{\ell}\right)}{-a^2\left(\frac{-\pi^2}{\ell^2} - \frac{s^2}{a^2}\right)} = \frac{\sin\left(\frac{\pi x}{\ell}\right)}{a^2\left(\frac{\pi^2}{a^2} + \frac{s^2}{a^2}\right)}$$

Hence $\bar{u}(x,s) = A(s)\cosh x + B(s)\sinh x + \dfrac{\sin mx}{(a^2m^2 + s^2)}$, $m = \dfrac{\pi}{\ell}$

Given $\bar{u}(0,s) = 0 \Rightarrow A = 0$,

$\bar{u}(\ell,s) = 0 \Rightarrow B(s)\sinh\ell + \dfrac{\sin m\ell}{(a^2m^2 + s^2)} = 0$

This yields $B(s) = 0$ as $m = (\pi/\ell) \Rightarrow \sin m\ell = \sin\pi = 0$ and $\sinh\ell \neq 0$. Thus

$$\bar{u}(x,s) = \frac{\sin mx}{(a^2m^2 + s^2)}$$

$$\therefore \quad \mathcal{L}^{-1}[\bar{u}(x,s)] = u(x,t) = \sin mx\ \mathcal{L}^{-1}\left[\frac{1}{a^2m^2 + s^2}\right]$$

Now poles of $F(s) = \dfrac{1}{a^2m^2 + s^2}$ are simple poles, given by $s = \pm\, iam$

Residue at $s = iam = \dfrac{e^{iamt}}{2iam}$.

Residue at $s = -iam = \dfrac{e^{-iamt}}{-2iam}$

$$\therefore \quad \mathcal{L}^{-1}\left[\frac{1}{a^2m^2 + s^2}\right] = \text{sum of residues } = \frac{\sin amt}{am}$$

$$\therefore \quad u(x,t) = \frac{1}{am}\sin mx \sin mat = \frac{\ell}{a\pi}\sin\frac{\pi}{\ell}x\sin\frac{\pi at}{\ell}.$$

***

Example 33: A semi-infinite bar $0 \le x < \infty$ is initially at temperature zero. At time $t = 0$, a constant temperature T_0 is applied and maintained at the face $x = 0$. Find the temperature at any point of the solid and any time $t > 0$.

Solution: Let $T(x,t)$ be the temperature at any point x and at any time t. The differential equation governing the flow of heat in the given solid is given by

$$\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2}, \quad x > 0, \ t > 0$$

Thus mathematically we have to solve the following initial boundary value problem

$$T_t = kT_{xx}, \quad x > 0, \ t > 0 \tag{87}$$

where $T(x,0) = 0$, $T(0,t) = T_0$, $T(x,t)$ is finite as $x \to \infty$. (88)

Taking the Laplace transform of Eqn.(87), we get

$$\int_0^\infty \frac{\partial T}{\partial t}e^{-st}dt = k\int_0^\infty \frac{\partial^2 T}{\partial x^2}e^{-st}\,dt \tag{89}$$

using formula (31) given in Appendix, we get

$$s\overline{T}(x,s) - T(x,0) = k\frac{d^2\overline{T}}{dx^2}; \quad \overline{T}(x,s) = \int_0^\infty e^{-st}T(x,t)\,dt$$

or, $$\frac{d^2\overline{T}}{dx^2} = \frac{s}{k}\overline{T} \qquad [\because T(x,0) = 0]$$

Solving the above equation for $\overline{T}$, we get

$$\overline{T}(x,s) = C_1 e^{\sqrt{s/k}\,x} + C_2 e^{-\sqrt{s/k}\,x} \tag{90}$$

Using the condition of boundedness of T as $x \to \infty$, we get

$C_1 = 0$

Thus, $\overline{T}(x,s) = c_2 e^{-\sqrt{s/k}\,x}$

On taking Laplace transform of Eqn.(88), we get

$\overline{T} = \dfrac{T_0}{s}$ at $x = 0$

Therefore, Eqn.(90) yields $C_2 = (T_0/s)$ and hence

$$\overline{T}(x,s) = \frac{T_0}{s}\,e^{-\sqrt{s/k}\,x}$$

On taking Laplace inverse of $\overline{T}$, we get

$$T(x,t) = T_0\left(1 - \text{erf}\frac{x}{2\sqrt{kt}}\right) = T_0 \ \text{erfc}\ x^* \tag{91}$$

where $x^* = x/(2\sqrt{kt})$.

In writing Eqn.(91) we have used the formula

$$\mathcal{L}^{-1}\left[e^{-b\sqrt{s}}/s\right] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\frac{e^{-b\sqrt{s}}}{s}ds = 1 - \text{erf}\left\{\frac{b}{2\sqrt{x}}\right\}$$

or, using Appendix, we have

$$\mathcal{L}^{-1}\left[\exp\left(-\sqrt{s/k}\right)x\right]=\frac{1}{2}(k\pi)^{\frac{-1}{2}}\, xt^{\frac{-1}{2}}\exp(-x^2/4kt)$$

Using the convolution property of the Laplace transform

$$T(x,t)=\frac{x}{2(k\pi)^{1/2}}\int_0^t T_0(t-u)\frac{\exp(-x^2/4ku)}{u^{3/2}}du$$

If $T_0(t)=T_0$ (constant), putting $p=\frac{1}{2}x\,(ku)^{\frac{-1}{2}}$, we get

$$T(x,t)=2T_0\pi^{\frac{-1}{2}}\int_{\frac{x}{2\sqrt{kt}}}^{\infty}e^{-p^2}dp$$

$$=T_0\ \operatorname{erfc}\left(\frac{1}{2}\frac{x}{\sqrt{kt}}\right).$$

$$=T_0\ \operatorname{erfc}\left(\frac{x}{2\sqrt{kt}}\right)$$

***

You may now try the following exercises.

E26) A semi-infinite string is initially at rest in a position coinciding with the positive half of the x-axis. At $t=0$, the left hand end of the string begins to move along the y-axis in a manner described by $y(0,t)=f(t)$ where $f(t)$ is a known function. Find the displacement $y(x,t)$ of the string at any point at any subsequent time.

E27) A semi-infinite cable of negligible leakage and inductance is initially dead. At $t=0$, an arbitrary signal voltage $E(t)$ is suddenly applied at the sending end. Find the potential $v(x,t)$ at any point on the cable at any subsequent time.

E28) A semi-infinite string is supported from below so that it lies motionless on the x-axis. At time $t=0$, the support is removed and gravity is permitted to act on the string. If the end $x=0$, is fixed at the origin, find the displacement of the string.

We now end this unit by giving a summary of what we have covered in it.

6.5 SUMMARY

In this unit we have covered the following:

1. Definition of Laplace transform of the function $f(t)$ denoted by

$$\mathcal{L}\,[f(t)]=F(s)=\int_0^{\infty}e^{-st}f(t)\,dt.$$

2. Functions of exponential order and piecewise continuous functions.
3. Laplace transform of some elementary functions like e^{at}, $\sin at, \cos at, \sinh at, \cosh at$ etc.
4. If $\mathcal{L}\,[f(t)]=F(s)$ and $\mathcal{L}[g(t)]=G(s)$ then
   - $\mathcal{L}\,[a\,f(t)+b\,g(t)]=a\,\mathcal{L}\,[f(t)]+b\,\mathcal{L}[g(t)]=a\,F(s)+b\,G(s)$ where a and b are two arbitrary scalars.
   - $\mathcal{L}\,[e^{at}f(t)]=F(s-a)$ where a is any arbitrary scalar.
   - $\mathcal{L}\,\{u_a(t)f(t-a)\}=e^{-as}$, where $u_a(t)=\begin{cases}0, & 0\le t\le a\\ 1, & t>a\end{cases}$ is the unit step function.

- $\mathcal{L}\,[t^n\,f(t)] = (-1)^n\,F^n(s) = (-1)^n\,\dfrac{d^n}{ds^n}F(s)$ for $n \in I^+$.
- $\mathcal{L}\left(\dfrac{1}{t}f(t)\right) = \int_0^\infty F(s)\,ds$, if $\dfrac{f(t)}{t}$ is bounded for $t > 0$.

5. If $f(t)$ is continuous, possesses piecewise continuous derivative and both $f(t)$ and $f'(t)$ are of exponential order then
   - $\mathcal{L}\,[f'(t)] = s\,\mathcal{L}\,[f(t)] - f(0)$
   - $\mathcal{L}\,[f''(t)] = s^2\,\mathcal{L}\,[f(t)] - s\,f(0) - f'(0)$
   - $\mathcal{L}\,[f^n(t)] = s^n\,\mathcal{L}\,[f(t)] - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - s\,f^{n-2}(0) - f^{n-1}(0)$
6. $\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \{F(s)/s\}$.
7. $\mathcal{L}\,\{f(t) * g(t)\} = \mathcal{L}[f(t)]\,\mathcal{L}\,[g(t)] = F(s)G(s)$
   where $*$ denotes the convolution of two functions f and g.
8. If $f(t)$ is periodic with period ω, then
   $$\mathcal{L}\,[f(t)] = \frac{\int_0^\infty e^{-st}f(t)\,dt}{(1-e^{-s\omega})}.$$
9. $\mathcal{L}\,[\mathrm{erf}\,(t)] = \mathcal{L}\left[\dfrac{2}{\sqrt{\pi}}\int_0^t e^{z^2}\,dz\right] = \dfrac{1}{s}e^{(s^2/4)}\,\mathrm{erfc}\left(\dfrac{s}{2}\right)$
   - $\mathcal{L}\,\mathrm{erf}\left(\sqrt{t}\right) = 1/(s\sqrt{s+1})$.
10. $\mathcal{L}\,[\delta(t)] = 1$ where $\delta(t-a) = \begin{cases}\infty\,, & t = a\\ 0\,, & t \neq a\end{cases}$ with $\int_0^\infty \delta(t-a)\,dt = 1$
11. For $H(t) = \begin{cases}0\,, & \text{for } t < 0\\ 1\,, & \text{for } t \geq 0\end{cases}$; unit step function.
    - $\mathcal{L}\,H(t) = \dfrac{1}{s}$
    - $\mathcal{L}\,H(t-a) = \left(e^{-as}/s\right)$
12. If $\mathcal{L}\,[f(t)] = F(s) = \int_0^\infty e^{-st}f(t)\,dt$ then $f(t) = \mathcal{L}^{-1}F(s)$ is known as the Laplace inverse of $F(s)$.
13. If $\mathcal{L}^{-1}\,F(s) = f(t)$ and $\mathcal{L}^{-1}\,G(s) = g(t)$ and a, b are constants then
    - $\mathcal{L}^{-1}\,[a\,F(s) + b\,G(s)] = a\,\mathcal{L}^{-1}\,(F(s)) + b\,\mathcal{L}^{-1}\,(G(s))$
      $= a\,f(t) + b\,g(t)$.
    - $\mathcal{L}^{-1}F(s+a) = e^{-at}f(t)$
    - $\mathcal{L}^{-1}\,\{e^{-as}F(s)\} = \begin{cases}f(t-a), & t > a\\ 0\,, & t < a\end{cases}$
    - $\mathcal{L}^{-1}\left[\dfrac{d^n}{ds^n}F(s)\right] = -t\,f(t)$
    - $\mathcal{L}^{-1}\left[\dfrac{d^n}{ds^n}F(s)\right] = (-1)^n\,t^n\,f(t)$

- $\mathcal{L}^{-1}\left[\int_0^\infty F(s)\right]ds = \frac{f(t)}{t}$
- $\mathcal{L}^{-1}[sF(s)] = f'(t)$, if $f(0) = 0$
- $\mathcal{L}^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t f(t)\,dt$.

14. $\mathcal{L}^{-1}\{F(s)G(s)\} = \int_0^t f(u)g(t-u) = f(t) * g(t)$

15. $\mathcal{L}^{-1}[F(s)] = f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{ts}F(s)\,ds =$ sum of the residues of $e^{ts}F(s)$ at the poles of $F(s)$.

16. If $F(x,s)$ be the $\mathcal{L}[f(x,t)]$ with respect to the variable t then
    - $\mathcal{L}\left(\frac{\partial f}{\partial t}\right) = s\mathcal{L}[f] - f(0)$
    - $\mathcal{L}\left(\frac{\partial^2 f}{\partial t^2}\right) = s^2\mathcal{L}[f] - sf(0) - f'(0)$
    - $\mathcal{L}\left(\frac{\partial f}{\partial x}\right) = \frac{d}{dx}\mathcal{L}[f] = \frac{dF}{dx}$
    - $\mathcal{L}\left(\frac{\partial^2 f}{\partial x^2}\right) = \frac{d^2F}{dx^2}$.

17. To solve initial boundary value problem using Laplace transform technique
    i) transform the pde into an ode
    ii) transform all boundary and initial conditions to the transformed variable
    iii) solve the ode for the transformed variable under transformed conditions
    iv) find the inverse Laplace transform of the transformed variable either using table of Laplace transform or the complex inversion integral theorem

6.6 SOLUTIONS/ANSWERS

E1) a) $\frac{6}{(s+2)^2}$

b) $\frac{24(s^2-2s-4)}{(s^2-2s+17)^3}$; **Hint** $\mathcal{L}[e^t \sin 4t] = \frac{4}{(s-1)^2+16}$.

c) $\cot^{-1}\frac{s}{2}$. **Hint** $\mathcal{L}\left[\frac{\sin 2t}{t}\right] = \int_s^\infty \frac{2}{s^2+4}ds$

d) $\frac{s(s^2-28)}{(s^2-4)(s^2-36)}$

E2) a) $f(t) = \sin 2t\{u(t-2\pi) - u(t-4\pi)\}$

$$\mathcal{L}[f(t)] = \mathcal{L}[\sin 2t\,u(t-2\pi)] - \mathcal{L}[\sin 2t\,u(t-4\pi)]$$
$$= e^{-2\pi s}\mathcal{L}[\sin 2(t+2\pi)] - e^{-4\pi s}\mathcal{L}[\sin 2(t+4\pi)]$$
$$= e^{-2\pi s}\mathcal{L}[\sin 2t] - e^{-4\pi s}\mathcal{L}[\sin 2t]$$
$$= (e^{-2\pi s} - e^{-4\pi s})\frac{2}{s^2+4}.$$

b) $f(t) = (t-1)\{u(t-1) - u(t-2)\} + (3-t)\{u(t-2) - u(t-3)\}$

$$= (t-1)u(t-1) - 2(t-2)u(t-2) + (t-3)u(t-3)$$

$$\mathcal{L}[f(t)] = \mathcal{L}[(t-1)u(t-1)] - 2\mathcal{L}[(t-2)u(t-2)] + \mathcal{L}[(t-3)u(t-3)]$$

$$= e^{-s}\frac{1}{s^2} - 2e^{-2s}\frac{1}{s^2} - e^{-3s}\frac{1}{s^2}.$$

E3) $\mathcal{L}[(t^2 - t + 2)u(t-2)] = \mathcal{L}[(t-2)^2 u(t-2) + 3\mathcal{L}[(t-2)u(t-2)] + 4\mathcal{L}[u(t-2)]$

$$= e^{-2s}\left(\frac{2}{s^3} + \frac{3}{s^2} + \frac{4}{s}\right)$$

E4) Here $f = \sin t$, $g = t^2$

$$\therefore \mathcal{L}\{(f * g)(t)\} = F(s)\,G(s) = \frac{1}{s^2+1}\frac{2!}{s^3} = \frac{2}{s^3(s^2+1)}$$

E5) a) Using result (27) periodic function, $\mathcal{L}(kt/T)$ can be written as

$$\mathcal{L}(kt/T) = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}\frac{kt}{T}dt$$

Evaluating the integral using integration by parts, we get

$$= \frac{-ke^{-sT}}{s(1-e^{-sT})} + \frac{k}{Ts^2}$$

b) As per definition

$$\mathcal{L}[e^{-t^2/4}] = \int_0^\infty e^{-st}.e^{-t^2/4}\,dt$$

writing $st + \frac{t^2}{4} = \left(s + \frac{t}{2}\right)^2 - s^2$

Putting $s + \frac{t}{2} = z$, $dt = 2dz$, when $t = 0$, $z = s$, $t \to \infty$, $z \to \infty$

$$\therefore \mathcal{L}[e^{-t^2/4}] = 2\int_s^\infty e^{s^2}e^{-z^2}\,dz = e^{s^2}\sqrt{\pi}\,\operatorname{erfc} s \quad \text{[using (29)].}$$

E6) Putting $t^2 = y$ the given integral reduces to

$$\int_0^\infty e^{-y}\operatorname{erf}\left(\sqrt{y}\right)\frac{dy}{2} = \frac{1}{2}\int_0^\infty e^{-t}\operatorname{erf}\left(\sqrt{t}\right)dt$$

$$= \frac{1}{2}\int_0^\infty e^{-st}\operatorname{erf}\left(\sqrt{t}\right)dt = \frac{1}{2}\mathcal{L}\left[\operatorname{erf}\left(\sqrt{t}\right)\right], \qquad \text{where } s = 1$$

$$= \frac{1}{2}\frac{1}{s\sqrt{s+1}} = \frac{1}{2\sqrt{2}} \qquad (\because s = 1)$$

E7) Here period $T = \frac{2\pi}{\omega}$. We have

$$L[f(t)] = \int_0^T \frac{e^{-st}f(t)\,dt}{1-e^{-st}} = \frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\int_0^{\frac{2\pi}{\omega}} e^{-st}f(t)\,dt$$

$$= \frac{1}{1-e^{-2\pi s/\omega}}\left\{\int_0^{\pi/\omega} e^{-st}\sin\omega t + \int_{\pi/\omega}^{2\pi/\omega} e^{-st}.0\,dt\right\}$$

$$= \frac{1}{1-e^{-2\pi s/\omega}} \left\{ \frac{e^{-st}}{s^2+\omega^2} (-s \sin \omega t - \omega \cos \omega t \right\}_0^{\pi/\omega}$$

$$= \frac{\omega}{(s^2+\omega^2)\,(1-e^{-\pi s/\omega})}.$$

E8) a) $\mathcal{L}[F_\epsilon(t)] = \int_a^{a+\epsilon} e^{-st} \frac{1}{\epsilon} dt = \frac{1}{\epsilon}\left[\frac{e^{-st}}{-s}\right]_a^{a+\epsilon} = \frac{e^{-as}}{s}\left[\frac{1-e^{-s\epsilon}}{\epsilon}\right].$

b) $\mathcal{L}[\delta(t-a)] = \lim_{\epsilon\to 0} \mathcal{L}[F_\epsilon(t)] = \lim_{\epsilon\to 0}\left[\frac{e^{-as}}{s}\left(\frac{1-e^{-s\epsilon}}{\epsilon}\right)\right]$

$$= \frac{e-as}{s} \lim_{\epsilon\to 0}\left(\frac{se^{-s\epsilon}}{1}\right) = e^{-as}.$$

E9) a) $(1-e^{-\pi s})\frac{2}{s^2+4}$ b) $\frac{1}{s^2+1}\left[s^2+(s-1)\,e^{-\pi s}\right].$

E10) a) $3\cos(\sqrt{2})\,t$ b) $\cot 2t - \frac{t}{3}$ c) $\sinh t - \sin t$

d) $\cosh 2t - \frac{\sinh 2t}{2}$ e) $t - \frac{\sin\sqrt{2}t}{\sqrt{2}}$

E11) $e^{-at}\left(\frac{a\cos bt - a\sin bt}{b}\right)$

E12) We know that

$$\mathcal{L}[f(ct)] = \frac{1}{c} F\left(\frac{s}{c}\right)$$

$$\therefore \quad \mathcal{L}^{-1}\left[\frac{1}{c}F\left(\frac{s}{c}\right)\right] = f(ct)$$

$$\therefore \quad \mathcal{L}^{-1}[a\;F(as)] = f\left(\frac{t}{a}\right) \qquad \left[\text{putting } \frac{1}{c} = a\right]$$

$$\text{or, } \mathcal{L}^{1}[F(as)] = \frac{1}{a} f\left(\frac{t}{a}\right).$$

E13) a) Let $\mathcal{L}^{-1}\left[\cot^{-1} s\right] = f(t)$

$$\therefore \quad \mathcal{L}^{-1}\left[\frac{d}{ds}\cot^{-1} s\right] = -t\,f(t)$$

$$\text{or, } \mathcal{L}^{-1}\left[\frac{-1}{s^2+1}\right] = -\sin t = -t\;f(t)$$

b) Using result (44) obtain

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+s+1}\right] = \frac{f(t)}{t}, \text{ where } f(t) = \mathcal{L}^{-1}\left[\frac{2s+1}{(s^2+s+1)^2}\right]$$

$$\therefore \quad \mathcal{L}^{-1}\left[\frac{1}{(s+1/2)^2+\left(\sqrt{3}/2\right)^2}\right] = \frac{f(t)}{t}$$

$$\text{or, } e^{-t/2}\,\mathcal{L}^{-1}\left[\frac{1}{s^2+\left(\sqrt{3}/2\right)^2}\right] = \frac{f(t)}{t}$$

$$\text{or, } \frac{2t}{\sqrt{3}} e^{-t/2} \sin\frac{\sqrt{3}}{2}\,t = f(t).$$

c) $\frac{t^2}{2} + \cos t - 1$

d) Let $F(s) = \frac{1}{s}$ and $G(s) = \frac{1}{s^2 + a^2}$

then $f(t) = 1$ and $g(t) = \frac{\sin at}{a}$

$$\therefore \quad \mathcal{L}^{-1}\left[\frac{1}{s\,(s^2 + a^2)}\right] = \int_0^t 1 \frac{\sin au}{a}\, du \qquad \text{[using convolution theorem]}$$

$$= \frac{1 - \cos at}{a^2}.$$

e) $-\sin t + 3\frac{\sin 2t}{2}$. [use partial fractions]

E14) a) $y'' + 25y = 10\cos 5t$, $y(0) = 2$, $y'(0) = 0$

On taking LT of the given DE and using the initial conditions, we get

$$Y(s) = \frac{2s}{s^2 + 25} + \frac{10s}{(s^2 + 25)^2}$$

$$y(t) = \mathcal{L}^{-1}Y(s) = 2\cos 5t + \mathcal{L}^{-1}\frac{d}{ds}\left(\frac{-5}{s^2 + 25}\right)$$

$$= 2\cos 5t + t\sin 5t.$$

b) $y'' - 4y' + 4y = 64\sin 2t$, $y(0) = 0$, $y'(0) = 1$

$$Y(s) = \frac{1}{(s-2)^2} + \frac{128}{(s^2 + 4)\,(s-2)^2}$$

$$y(t) = \mathcal{L}^{-1}Y(s) = \mathcal{L}^{-1}\left\{\frac{-8}{s-2} + \frac{17}{(s-2)^2} + \frac{8s}{s^2 + 4}\right\}$$

$$= -8e^{2t} + 17t\,e^{2t} + 8\cos 2t.$$

c) $$Y(s) = \frac{1}{(s^2 + 3s + 2)} \cdot \frac{1}{(s^2 - 2s + 2)}$$

$$= \frac{A}{s+1} + \frac{B}{(s+2)} + \frac{Cs + D}{s^2 - 2s + 2}$$

comparing the coefficients we get

$$A = \frac{1}{5},\ B = -\frac{1}{10},\ C = -\frac{1}{10},\ D = \frac{1}{5}$$

$$\therefore\ y(t) = \mathcal{L}^{-1}Y(s) = \frac{1}{5}e^{-t} - \frac{1}{10}e^{-2t} - \frac{1}{10}\mathcal{L}^{-1}\frac{s}{(s-1)^2 + 1} + \frac{1}{5}\mathcal{L}^{-1}\frac{1}{(s-1)^2 + 1}$$

$$\therefore\ y(t) = \frac{1}{5}e^{-t} - \frac{1}{10}e^{-2t} - \frac{1}{10}e^{t}\cos t + \frac{1}{5}\sin t.$$

d) On taking Laplace transform and using the initial conditions we get

$$(s+1)Y(s) = -1 + \frac{1}{s^2} + \frac{1}{(s-2)}$$

Thus, $$Y(s) = -\frac{1}{(s+1)} + \frac{1}{(s+1)s^2} + \frac{1}{(s+1)(s-2)}$$

On resolving into partial fraction we get

$$Y(s) = -\frac{1}{s+1} + \frac{1}{3}\frac{1}{(s-2)} - \frac{1}{3}\frac{1}{(s+1)} + \frac{1}{(s+1)} - \frac{1}{s} + \frac{1}{s^2}$$

$$y(t) = \mathcal{L}^{-1}Y(s) = -\frac{1}{3}\mathcal{L}^{-1}\frac{1}{s+3} - \mathcal{L}^{-1}\frac{1}{s} + \frac{1}{3}\mathcal{L}^{-1}\frac{1}{(s-2)}$$

$$= -\frac{1}{3}e^{-3t} - 1 + \frac{1}{3}e^{2t}.$$

e) On taking LT of the given DE we get

$$sY(s) - y(0) + 4Y(s) = \frac{1}{s} - \frac{1}{s^2+1}\ ;\ y(0) = -1$$

$$\therefore\ Y(s) = -\frac{1}{s+4} + \frac{1}{s(s+4)} - \frac{1}{(s+4)(s^2+1)}$$

$$= -\frac{1}{(s+4)} + \frac{1}{4s} - \frac{1}{4(s+4)} - \left(\frac{1}{17(s+4)} + \frac{4}{17(s^2+1)} - \frac{s}{17(s^2+1)}\right)$$

$$= -\frac{89}{68}\frac{1}{(s+4)} + \frac{1}{4s} + \frac{4}{17(s^2+1)} - \frac{1}{17}\frac{s}{s^2+1}$$

$$\therefore\ y(t) = \mathcal{L}^{-1}Y(s) = \frac{1}{4} - \frac{89}{68}e^{-4t} + \frac{4}{17}\sin t - \frac{1}{17}\cos t.$$

f) $2y'' + 5y' + 2y = e^{-2t}, y(0) = 1,\ y'(0) = 1$

On taking LT and using the initial conditions, we get

$$Y(s) = \frac{1}{(s+2)(2s^2+5s+2)} + \frac{2s+7}{2s^2+5s+2} = \frac{2s^2+11s+15}{(2s+1)\,(s+2)^2}$$

On resolving into partial fractions, we get

$$Y(s) = \frac{4}{9(2s+1)} - \frac{11}{9(s+2)} - \frac{1}{3(s+2)^2} = \frac{4}{9}\frac{1}{2\left(s+\frac{1}{2}\right)} - \frac{11}{9(s+2)} - \frac{1}{3(s+2)^2}$$

$$\mathcal{L}^{-1}Y(s) = y(t) = \frac{4}{18}e^{-\frac{t}{2}} - \frac{11}{9}e^{-2t} - \frac{1}{3}te^{-2t}$$

g) $y'' + 2y' + 5y = e^{-x}\sin x,\ y(0) = 0,\ y'(0) = 1$

On taking LT and using initial conditions we get

$$Y(s) = \frac{s^2+2s+3}{(s^2+2s+2)\,(s^2+2s+5)} = \frac{z+3}{(z+2)(z+5)}\ ;\ z = 2s + s^2$$

$$= \frac{2}{3}\frac{1}{z+5} + \frac{1}{3}\frac{1}{z+2}$$

$$y(x) = \mathcal{L}^{-1}Y(s) = \frac{1}{3}\mathcal{L}^{-1}\frac{2}{(s+1)^2+2^2} + \frac{1}{3}\mathcal{L}^{-1}\frac{1}{(s+1)^2+1^2}$$

$$= \frac{1}{3}e^{-x}\sin 2x + \frac{1}{3}e^{-x}\sin x.$$

E15) $L\frac{di}{dt} + Ri = E(t) = \begin{cases} E, & 0 < t < a \\ 0, & t > a \end{cases},\ i(0) = 0$

On taking LT and using the initial conditions, we get

$$(Ls+R)I(s) = \int_0^\infty E(t)e^{-st}dt = \int_0^a Ee^{-st}dt + \int_a^\infty 0.\,e^{-st}dt$$

$$I(s) = \frac{E}{s(Ls+R)}\left(1 - e^{-as}\right)$$

$$\therefore\ i(t) = \mathcal{L}^{-1}\frac{E}{s(Ls+R)} - \mathcal{L}^{-1}\frac{Ee^{-as}}{s(Ls+R)}$$

$$= \mathcal{L}^{-1}\frac{E}{R}\left(\frac{1}{s} - \frac{1}{s+R/L}\right) - \mathcal{L}^{-1}\frac{E}{R}\left(\frac{1}{s} - \frac{1}{s+R/L}\right)e^{-as}$$

$$i(t) = \frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right) - \frac{E}{R}\left(1 - e^{-\frac{R}{L}(t-a)}\right)u(t-a)$$

On simplification, we get

$$i(t) = \frac{E}{R}\left(e^{\frac{Ra}{L}} - 1\right)e^{-\frac{R}{L}t}.$$

E16) $E(t)$ can be expressed as

$E(t) = t + u(t-1)\,(1-t) = t - (t-1)u(t-1)$

On taking Laplace transform of $q'' - q' - 6q = E(t)$ and using the initial conditions $q(0) = 0,\ q'(0) = 0$, we get

$$s^2Q(s) - s\,q(0) - q'(0) - [sQ(s) - q(0)] - 6Q(s) = \frac{1}{s^2} - \frac{e^{-s}}{s^2}$$

$$(s^2 - s - 6)Q(s) = \frac{1}{s^2} - \frac{e^{-s}}{s^2}$$

$$Q(s) = \frac{1}{(s-3)\,(s+2)s^2} - \frac{e^{-s}}{(s-3)\,(s+2)s^2}$$

Now $\dfrac{1}{(s-3)\,(s+2)s^2} = \dfrac{1}{36s} - \dfrac{1}{6s^2} + \dfrac{1}{45}\dfrac{1}{(s-3)} - \dfrac{1}{20}\dfrac{1}{(s+2)}$

Thus $\mathcal{L}^{-1}Q(s) = q(t)$ is given by

$$q(t) = \frac{1}{36} - \frac{t}{6} + \frac{e^{3t}}{45} - \frac{1}{20}e^{-2t} - \left\{\frac{1}{36} - \frac{(t-1)}{6} + \frac{e^{3(t-1)}}{45} - \frac{e^{-2(t-1)}}{20}\right\}u(t-1).$$

E17) On taking LT of $y'' + 9y = 9u(t-3)$ and using the initial conditions $y(0) = y'(0) = 0$, we get

$Y(s) = 9e^{-3s}/(s^2+9)s$. Thus $y(t) = \mathcal{L}^{-1}(Y(s))$.

From table of LT given in Appendix we know that

$$\mathcal{L}^{-1}\frac{3}{s^2+9} = \sin 3t.$$

Also, from the result $\mathcal{L}\displaystyle\int_0^t f(t)\,dt = \frac{1}{s}F(s)$ one can write

$$3\mathcal{L}^{-1}\frac{3}{s(s^2+9)} = 3\int_0^t \sin 3t\,dt = 1 - \cos 3t$$

Hence, $y(t) = \mathcal{L}^{-1}Y(s) = \mathcal{L}^{-1}\dfrac{9e^{-3s}}{s(s^2+9)} = \{1 - \cos 3(t-3)\}u(t-3)$.

E18) $X_2(s) = \dfrac{4-s}{s^3(s+1)} = \dfrac{4}{s^3(s+1)} - \dfrac{1}{s^2(s+1)}$, $X_1(s)\dfrac{1}{s} + \dfrac{2}{s^4} - \dfrac{4-s}{s^3(s+1)}$

$x_2(t) = 5 - 5t^4 + 2^2 - 5e^{-t}$, $x_1(t) = 1 + \dfrac{t^3}{3} - (5 - 5t^4 + 2t^2 - 5e^{-t})$

E19) $x(t) = t + 3e^{-t} - 2e^{-3t}$; $y(t) = 1 - t + 2e^{-3t}$

E20) $x(t) = 3 - 2e^t$, $y(t) = e^t$

E21) $x(t) = (1+2t)e^t + 2e^{3t}$; $y(t) = (1-t)e^t - e^{3t}$

E22) On taking Laplace transform of the given equation we get

$$\mathcal{L}\,[y(t)] + \mathcal{L}\left[\int_0^t y(x)\,(t-z)\,dx\right] = \mathcal{L}\,(t)$$

Let $Y(s) = \mathcal{L}\,[y(t)]$. Using convolution theorem above equation can be written in the form

$$Y(s) + \mathcal{L}[y(t)]\,.\,\mathcal{L}\,[t] = \mathcal{L}[t]$$

or, $\quad Y(s) + Y(s)\dfrac{1}{s^2} = \dfrac{1}{s^2}$

$\therefore \quad \dfrac{(1+s^2)}{s^2}\,Y(s) = \dfrac{1}{s^2}$

or, $\quad Y(s) = \dfrac{1}{1+s^2}$

On taking inverse LT of the above equation, we get

$$y(t) = \sin t\,.$$

E23) $\mathcal{L}(y) = \dfrac{s}{(s+1)(s+2)} + \dfrac{2e^{-s}}{(s+1)(s+2)} - \dfrac{2e^{-3s}}{(s+1)(s+2)}$

$$y = (2e^{-2x} - e^{-x}) + 2(e^{-(x-1)} - e^{-2(x-1)})u(x-1) - 2(e^{-(x-2)} - e^{-2(x-2)})u(t-2)$$

E24) $(s\,\mathcal{L}(y) - 3) + 4\,\mathcal{L}(y) + \dfrac{4}{s}\mathcal{L}(y) = \dfrac{e^{-s}}{s}$

Then $\left\{s + 4 + \dfrac{4}{s}\right\}\,Y(s) = \dfrac{e^{-s}}{s} + 3 \qquad [\mathcal{L}\,y[(t)] = Y(s)]$

or, $Y(s) = \dfrac{3s}{(s+2)^2} + \dfrac{1}{(s+2)^2}e^{-s}$

$$= \frac{3(s+2-2)}{(s+2)^2} + \frac{1}{(s+2)^2}e^{-s}$$

$$= (3e^{-2t} - 6t\,e^{-2t})\,u_0(t) + (t-1)\,e^{-2(t-1)}\,u_1(t)$$

E25) $y(x) = \cos x + e^{-x}\displaystyle\int_0^x y(z)\,e^z\,dz$

$$= \cos x + \int_0^x y(z)\,e^{-(x-z)}\,dz$$

On taking Laplace transform of the above equation and using convolution theorem, we get $Y(s) = \dfrac{s}{s^2+1} + Y(s)\dfrac{1}{s+1}$

or $\left(1 - \dfrac{1}{s+1}\right)Y(s) = \dfrac{s}{s^2+1}$

$$Y(s) = -\frac{(s+1)}{(s^2+1)} = \frac{s}{s^2+1} + \frac{1}{s^2+1}$$

Taking Laplace inverse of the above equation, we get

$$y(x) = \cos x + \sin x\,.$$

E26) The initial boundary-value problem to be solved is

$$\frac{\partial^2 y}{\partial t^2} = a^2\,\frac{\partial^2 y}{\partial x^2},$$

where, $y(0,\,t) = f(t)$, $y(x,0) = 0$, $\left(\dfrac{\partial y}{\partial t}\right)_{t=0} = 0$ and $y(x,t)$ is bounded as $x \to \infty$.

On taking LT of given equation w.r.t t, we get

$$s^2Y(x,s) - sy(x,0) - \left(\frac{\partial y}{\partial t}\right)_{t=0} = a^2\frac{d^2Y(x,s)}{dx^2}$$

where, $Y(x,s) = \int_0^\infty e^{-st}y(x,t)\,dt$

On using the initial conditions, we get

$$\frac{d^2Y}{dx^2} - \left(\frac{s^2}{a^2}\right)Y = 0$$

whose solution is given by

$$Y(x,s) = A(s)e^{\frac{s}{a}x} + B(s)e^{-\frac{s}{a}x}$$

Since $y(x,t)$ is bounded i.e. when $x \to \infty$, Y is finite, therefore $A(s) = 0$.

Hence, $Y(x,s) = B(s)e^{-(s/a)x}$.

Since at $x = 0$, $y(0,t) = f(t)$ therefore $Y(0,s) = F(s) = B(s)$.

Thus, we get

$$Y(x,s) = F(s)e^{-\frac{s}{a}x}$$

$$\therefore \quad \mathcal{L}^{-1}Y(x,s) = y(x,t) = \mathcal{L}^{-1}F(s)e^{-\frac{s}{a}x} = f\left(t-\frac{x}{a}\right)u\left(t-\frac{x}{a}\right).$$

E27) For the case under consideration the telegraph equation is

$$\frac{\partial^2 v}{\partial x^2} = LC\frac{\partial^2 v}{\partial t^2} + (RC+GL)\frac{\partial v}{\partial t} + RGv$$

where L, C, R, G have their usual meanings. For the case $L = 0, G = 0$, equation reduces to

$$\frac{\partial^2 v}{\partial x^2} = a^2\frac{\partial v}{\partial t};\ a^2 = RC$$

Thus, we have to solve the above equation under the conditions

$$v(0,t) = E(t),\ v(x,0) = 0,\ v(x,t) \text{ is bounded as } x \to \infty$$

On taking LT of the above equation w.r.to t, and using condition $v(x,0) = 0$, we get

$$\frac{d^2V}{dx^2} = a^2sV$$

where $V(x,s) = \int_0^\infty e^{-st}v(x,t)\,dt$

Thus, $V(x,s) = A(s)e^{a\sqrt{s}\,x} + B(s)e^{-a\sqrt{s}\,x}$

Since $v(x,t)$ is bounded as $x \to \infty$, we get $A(s) = 0$. Hence

$$V(x,s) = B(s)e^{-a\sqrt{s}\,x}$$

At $x = 0$, $v(0,t) = E(t)$ $\therefore$ $\mathcal{L}[v(x,t)] = V(0,s) = \mathcal{L}[E(t)]$

Hence, $B(s) = \mathcal{L}[E(t)]$. Consequently,

$$V(x,s) = \mathcal{L}[E(t)].e^{-a\sqrt{s}\,x}.$$

From the table of LT, we know $\mathcal{L}\left\{\frac{b e^{-b^2/4t}}{2\sqrt{\pi}\,t^{3/2}}\right\} = \exp\left(-b\sqrt{s}\right)$

On taking Laplace inverse of $V(x,s)$, we get

$$v(x,t) = \mathcal{L}^{-1}\,e^{-a\sqrt{s}\,x}\,\overline{E}(s) \text{ where } \overline{E}(s) = \int_0^\infty e^{-st}\,E(t)\,dt$$

$$\therefore\ v(x,t)=\frac{ax}{2\sqrt{\pi}}\int_0^{\infty}E(t-\lambda)\frac{e^{-\frac{a^2x^2}{4\lambda}}}{\lambda^{3/2}}d\lambda$$

For the case where $E(t)=u(t)$ (a unit step function), $u(t-\lambda)=1$ for $\lambda<t$ and is zero for $\lambda>t$. Hence

$$v(x,t)=\frac{ax}{2\sqrt{\pi}}\int_0^{\infty}\frac{e^{-\frac{a^2x^2}{4\lambda}}}{\lambda^{3/2}}d\lambda.$$

E28) For this case the initial boundary value problem is formulated as follows:

$$\frac{\partial^2 y}{\partial t^2}=a^2\frac{\partial^2 y}{\partial x^2}+g\ ,\ x>0,\ t>0$$

$y(x,t)=0\ ,\ t>0;\ y(x,t)$ is bounded as $x\to\infty$.

$y(x,0)=0\ ,\ x>0;\left.\frac{\partial y}{\partial t}\right)_{t=0}=0,\ x>0$, where $g=-9.81$

On applying LT w.r. to t and using above conditions, we get

$$\left[s^2Y(x,s)-s\left(\frac{\partial y}{\partial t}\right)_{t=0}-y(x,0)\right]=a^2\frac{d^2Y(x,s)}{dx^2}+\frac{g}{s}$$

Thus, $Y''-\frac{s^2}{a^2}Y=-\frac{g}{a^2s}$

Keeping in view the condition of boundedness in

$Y(x,s)=A(s)e^{\frac{s}{a}x}+b(s)e^{-\frac{s}{a}x}+\frac{g}{s^3}$, we get

$Y(x,s)=B(s)e^{-\frac{s}{a}x}+\frac{g}{s^3}$

At $x=0,\ y(0,t)=0=Y(0,s)$

$\therefore\ B(s)=-\frac{g}{s^3}.$

Hence $Y(x,s)=\frac{g}{s^3}\left(1-e^{-\frac{s}{a}x}\right).$

On taking Laplace inverse of the above equation, we get

$$y(x,t)=\mathcal{L}^{-1}\frac{g}{s^3}-g\mathcal{L}^{-1}\frac{1}{s^3}e^{-\frac{s}{a}x}$$

$$=\frac{gt^2}{2}-\frac{g}{2}\left(t-\frac{x}{a}\right)u\left(t-\frac{x}{a}\right)$$

where $u(t)$ is the unit step function.

—x—

Appendix

Table of Laplace Transforms

	$f(t)$	$F(s) = \mathcal{L} f(t) = \int_0^\infty e^{-st} f(t) dt$
1.	1	$\frac{1}{s}$
2.	$\frac{t^n}{n!}$	$\frac{1}{s^{n+1}}, s > 0$
3.	$\frac{t^n}{\Gamma(x+1)}$	$\frac{1}{s^{x+1}}, x > -1$
4.	$(\pi t)^{-\frac{1}{2}}$	$s^{-\frac{1}{2}}$
5.	e^{-at}	$\frac{1}{s+a}$
6.	$e^{at} f(t)$	$F(s-a)$
7.	$\frac{t^n e^{-at}}{n!}$	$\frac{1}{(s+a)^{n+1}}$
8.	$\sin kt$	$k/(s^2+k^2)$
9.	$\cos kt$	$s/(s^2+k^2)$
10.	$\sinh kt$	$k/(s^2-k^2)$
11.	$\cosh kt$	$s/(s^2-k^2)$
12.	$t \sin kt$	$\frac{2ks}{(s^2+k^2)^2}$
13.	$\frac{1}{a}\exp\left(-\frac{bt}{a}\right) f\left(\frac{t}{a}\right)$	$F(as+b)$
14.	$u(t-c) = \begin{cases} 0, & 0 \le t < c \\ 1, & t \ge c \end{cases}$	$\frac{1}{s} e^{-cs}, c > 0$
15.	$f(t-c)\ u(t-c) = f(t-c)H(t-c)$	$e^{-cs} F(s), c > 0$
16.	$\int_0^t f_1(\beta) f_2(t-\beta)\ d\beta = f_1(t) * f_2(t)$	$F_1(s)F_2(s)$
17.	$(1-e^{-t})/t$	$\ln\left(1+\frac{1}{s}\right)$
18.	$\frac{2\sinh kt}{t}$	$\ln[(s+k)/(s-k)]$
19.	$\frac{\sin kt}{t}$	$\arctan(k/s)$
20.	$\delta(t-a)$	e^{-as}
21.	$\frac{1}{2a^3}(\sin at - at\cos at)$	$\frac{1}{(s^2+a^2)^2}$
22.	$\frac{1}{2a^3}(\sin at + at\cos at)$	$\frac{s^2}{(s^2+a^2)^2}$
23.	$f(at)$	$\frac{1}{a} F(s/a), a > 0$

24.	$\int_0^t f(t)dt$	$\frac{1}{s}F(s),\ s>0$
25.	$\lim_{t\to 0} f(t)$	$\lim_{s\to\infty} s\,F(s)$
26.	$t\,f(t)$	$-\frac{d}{ds}F(s)$
27.	$t^n f(t)$	$(-1)^n \frac{d^n}{ds^n}F(s)$
28.	$\frac{1}{t}f(t)$	$\int_s^\infty F(s)\,ds$
29.	$f(t)\ u(t-a)$	$e^{-as}\,\mathcal{L}\,[f(t+a)]$
30.	$\frac{df}{dt}$	$sF(s)-f(0),\ s>0$
31.	$\frac{d^2f}{dt^2}$	$s^2F(s)-s\,f(0)-f'(0),\ s>0$
32.	$\frac{d^3f}{dt^3}$	$s^3F(s)-s^2f(0)-s\,f'(0)-f''(0),\ s>0$
33.	$f(t)=f(t+T)$ periodic	$\int_0^T \frac{e^{-st}f(t)\,dt}{1-e^{-sT}}$
34.	erf (t)	$\frac{1}{s}e^{s^2/4}\ \text{erfc}\left(\frac{s}{2}\right)$
35.	erf $\left(\sqrt{t}\right)$	$\frac{1}{s\sqrt{s+1}}$

Note that same table can be used for writing Laplace inverse transform as $\mathcal{L}^{-1}\,[F(s)]=f(t)$, i.e., entries in the first column are the inverse Laplace transform of the entries in the second column.

UNIT 7 FOURIER TRANSFORM METHOD

7.1 INTRODUCTION

You know from your knowledge of Real Analysis course that **Fourier series** are powerful tools in treating various problems involving periodic functions (ref. MMT-004). Fourier series are used to represent a function f defined on a finite interval such as $[-L, L]$ or $[0, L]$. When f and f' are piecewise continuous on such an interval, a Fourier series represents the function on the interval and converges to a periodic extension of f outside the interval. However, in many practical problems, the functions involved are non-periodic for which Fourier series representations are not possible. A suitable representation for non-periodic functions can be obtained by considering the limiting form of Fourier series when the fundamental period is made infinite. In such cases, the Fourier series becomes a **Fourier integral**. Using symmetry, Fourier integral can conveniently be expressed as a pair of formulas the first giving the **Fourier transform** and the second giving its inverse. Like Laplace transform, the Fourier integrals and transforms which we shall be discussing in this unit, are useful in solving initial boundary value problems arising in science and engineering, for example, conduction of heat, wave propagation, theory of communication etc.

To begin with, in Sec.7.2, we have derived the Fourier integral representation of a function $f(x)$ defined over the interval $-\infty < x < \infty$. We have also given the Fourier cosine and sine integral representation of $f(x)$, when $f(x)$ is even and odd function respectively. Complex form of Fourier integral representation is also given in this section. In Sec.7.3, we have represented the Fourier integral representation of a function $f(x)$ as a pair of formulas, the first giving the Fourier transform of $f(x)$ and the second giving the inverse of that transform. We have also obtained the Fourier transforms and their inverses for a number of functions and discussed a number of useful properties admitted by the Fourier transform in this section. Applications of Fourier transforms to boundary value problems are discussed in Sec.7.4. An Appendix is attached at the end of the unit which gives a table of Fourier transforms of the commonly used functions for ready reference.

Objectives

After studying this unit you should be able to

- obtain the Fourier integral representation of a given non-periodic function over the given interval;
- obtain the Fourier transform and the inverse of that transform for a given function over the given interval;
- use various properties of Fourier transforms to obtain the Fourier transforms and their inverses in various applications;
- select the appropriate Fourier transform to be employed for the solution of a given boundary value problem;
- obtain the solution of a given boundary value problem using appropriate Fourier transform.

7.2 FOURIER INTEGRAL

We shall now derive, a means of representing certain kinds of non-periodic functions that are defined on either an infinite interval $]-\infty,\infty[$ or, semi-infinite interval $]0,\infty[$.

Let us consider $f(x)$ having the following properties.

1) $f(x)$ is piecewise continuous on every finite interval.
2) $f(x)$ is absolutely integrable on the interval $-\infty < x < \infty$, that is, $\int_{-\infty}^{\infty} |f(x)|dx$ converges.
3) at every x on the real line, $f(x)$ has left and right hand derivatives.

Consider the Fourier series representation of $f(x)$ on the interval $-L < x < L$. We have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right] \tag{1}$$

where, $a_0 = \frac{1}{L}\int_{-L}^{L} f(t)dt$, $a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi}{L}t\right)dt$

and $\quad b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi}{L}t\right)dt$

Note that $\frac{n\pi}{L}$ in Eqn.(1) are the frequencies – spatial or temporal, depending on whether x is a space variable or time.

If we let $\alpha_n = \frac{n\pi}{L}$, $\Delta\alpha = \alpha_{n+1} - \alpha_n = \frac{\pi}{L}$, then Eqn.(1) becomes

$$f(x) = \frac{1}{2\pi}\left(\int_{-L}^{L} f(t)dt\right)\Delta\alpha + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\left(\int_{-L}^{L} f(t)\cos\alpha_n t\,dt\right)\cos\alpha_n x + \left(\int_{-L}^{L} f(t)\sin\alpha_n t\,dt\right)\sin\alpha_n x\right]\Delta\alpha \tag{2}$$

We now expand the interval $-L < x < L$ by letting $L \to \infty$. Since $L \to \infty$ implies $\Delta\alpha \to 0$, the limit of Eqn.(2) has the form of a Riemann sum of a definite integral, i.e. of the form $\lim_{\Delta\alpha\to 0}\sum_{n=1}^{\infty} F(\alpha_n)\Delta\alpha$, which is suggestive of the definition of the integral $\int_0^{\infty} F(\alpha)d\alpha$. Thus if $\int_{-\infty}^{\infty} |f(x)|dx$ converges, the limit of the first term in Eqn.(2) is zero and by the definition of the integral as the limit of the sum, Eqn.(2) becomes

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\left[\left(\int_{-\infty}^{\infty} f(t)\cos\alpha t\,dt\right)\cos\alpha x + \left(\int_{-\infty}^{\infty} f(t)\sin\alpha t\,dt\right)\sin\alpha x\right]d\alpha \tag{3}$$

The result given in Eqn.(3) is called the **Fourier integral** of $f(x)$ or Fourier integral representation of $f(x)$ on the interval $-\infty < x < \infty$. You may **notice** that the basic structure of the Fourier integral is analogous to that of a Fourier series. We can sum up as follows:

The **Fourier integral** of a function $f(x)$ defined on the interval $-\infty < x < \infty$ is given by

$$f(x)=\frac{1}{\pi}\int_{0}^{\infty}[A(\alpha)\cos\alpha x+B(\alpha)\sin\alpha x]d\alpha \tag{4}$$

where, $$A(\alpha)=\int_{-\infty}^{\infty}f(x)\cos\alpha x\,dx \tag{5}$$

$$B(\alpha)=\int_{-\infty}^{\infty}f(x)\sin\alpha x\,dx \tag{6}$$

Eqn.(4) is called the **Fourier integral formula** for the function $f(x)$.
Sufficient conditions under which a Fourier integral converges to $f(x)$ are similar to, but are slightly more restrictive than, the conditions for a Fourier series. We state here without proof, the following convergence theorem.

Theorem 1: Let $f(x)$ be a piecewise continuous function on every finite interval and absolutely integrable on $-\infty<x<\infty$. Then the Fourier integral of $f(x)$ on the interval converges to $f(x)$ at a point of continuity. At a point of discontinuity, the Fourier integral will converge to the average

$$\frac{f(x+)+f(x-)}{2}$$

where $f(x+)$ and $f(x-)$ denote the limit of $f(x)$ at x from the right and left, respectively.

Cosine and Sine Integrals

When $f(x)$ is an **even function** on the interval $-\infty<x<\infty$, then the product $f(x)\cos\alpha x$ is also an even function, whereas $f(x)\sin\alpha x$ is an odd function. Using the property of definite integral, we obtain $B(\alpha)=0$ and so Eqn.(4) becomes

$$f(x)=\frac{2}{\pi}\int_{0}^{\infty}\left(\int_{0}^{\infty}f(t)\cos\alpha t\,dt\right)\cos\alpha x\,d\alpha$$

where in the above equation we have again used the property of definite integrals to write

$$\int_{-\infty}^{\infty}f(t)\cos\alpha t\,dt=2\int_{0}^{\infty}f(t)\cos\alpha t\,dt.$$

Similarly, when $f(x)$ is an **odd function** on the interval $-\infty<x<\infty$, the products $f(x)\cos\alpha x$ and $f(x)\sin\alpha x$ are odd and even functions respectively. Therefore $A(\alpha)=0$ and

$$f(x)=\frac{2}{\pi}\int_{0}^{\infty}\left(\int_{0}^{\infty}f(t)\sin\alpha t\,dt\right)\sin\alpha x\,d\alpha$$

We summarize:

The Fourier integral of an even function on the interval $]\infty,\infty[$ is the **cosine integral**

$$f(x)=\frac{2}{\pi}\int_{0}^{\infty}A(\alpha)\cos\alpha x\,d\alpha \tag{7}$$

where, $$A(\alpha)=\int_{0}^{\infty}f(x)\cos\alpha x\,dx \tag{8}$$

The Fourier integral of an odd function on the interval $]\infty,\infty[$ is the **sine integral**

$$f(x)=\frac{2}{\pi}\int_{0}^{\infty}B(\alpha)\sin\alpha x\,d\alpha \tag{9}$$

where, $$B(\alpha)=\int_0^\infty f(x)\sin\alpha x\,dx \tag{10}$$

Let us consider the following examples.

Example 1: Find the Fourier integral representation of the function

$$f(x)=\begin{cases}0, & x<0\\ 1, & 0<x<2\\ 0, & x>2\end{cases}$$

Solution: The graph of the function is shown in Fig.1. It satisfies the hypothesis of Theorem-1. Hence from Eqns.(5) and (6), we have

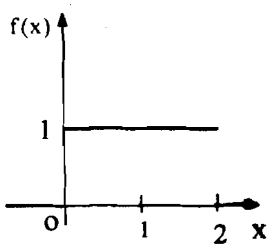

Fig.1

$$A(\alpha)=\int_{-\infty}^{\infty} f(x)\cos\alpha x\,dx$$

$$=\int_{-\infty}^{0} f(x)\cos\alpha x\,dx+\int_0^2 f(x)\cos\alpha x\,dx+\int_2^\infty f(x)\cos\alpha x\,dx$$

$$=\int_0^2\cos\alpha x\,dx=\frac{\sin 2\alpha}{\alpha}$$

$$B(\alpha)=\int_{-\infty}^{\infty} f(x)\sin\alpha x\,dx$$

$$=\int_0^2\sin\alpha x\,dx=\frac{1-\cos 2\alpha}{\alpha}$$

Substituting these coefficients in Eqn.(4) we obtain

$$f(x)=\frac{1}{\pi}\int_0^\infty\left[\left(\frac{\sin 2\alpha}{\alpha}\right)\cos\alpha x+\left(\frac{1-\cos 2\alpha}{\alpha}\right)\sin\alpha x\right]d\alpha$$

$$=\frac{1}{\pi}\int_0^\infty\frac{1}{\alpha}\left[2\sin\alpha\cos\alpha\cos\alpha x+2\sin^2\alpha\sin\alpha x\right]d\alpha$$

$$=\frac{2}{\pi}\int_0^\infty\frac{\sin\alpha}{\alpha}(\cos\alpha\cos\alpha x+\sin\alpha\sin\alpha x)\,d\alpha$$

$$=\frac{2}{\pi}\int_0^\infty\frac{\sin\alpha\cos\alpha(x-1)}{\alpha}\,d\alpha \tag{11}$$

This is the Fourier integral representation of the given function.

***

The above Fourier integral representation can be used to evaluate integrals. For example at $x=1$, it follows from Theorem-1 that integral (11) converges to $f(1)$, that is,

$$\frac{2}{\pi}\int_0^\infty\frac{\sin\alpha}{\alpha}\,d\alpha=f(1)$$

or, $$\int_0^\infty\frac{\sin\alpha}{\alpha}\,d\alpha=\frac{\pi}{2}$$

The above result is worthy of special note since it cannot be obtained in the usual manner as the integral $(\sin x)/x$ does not possess an anti-derivative that is an elementary function.

Example 2: Find the Fourier integral representation of the function

$$f(x)=\begin{cases}1, & |x|<a\\ 0, & |x|>a\end{cases}$$

Solution: The graph of the given function is shown in Fig.2. Clearly, the given function $f(x)$ is an even function. We represent $f(x)$ by the Fourier cosine integral (7). From Eqn.(8), we obtain,

$$A(\alpha)=\int_0^{\infty} f(x)\cos\alpha x\ dx$$

$$=\int_0^{a} f(x)\cos\alpha x\ dx+\int_a^{\infty} f(x)\cos\alpha x\ dx$$

$$=\int_0^{a}\cos\alpha x\ dx=\left.\frac{\sin\alpha x}{\alpha}\right|_0^a=\frac{\sin\alpha a}{\alpha}.$$

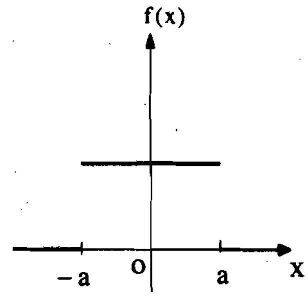

Fig.2

and thus, $$f(x)=\frac{2}{\pi}\int_0^{\infty}\frac{\sin a\alpha\cos\alpha x}{\alpha}d\alpha$$

***

Example 3: Represent $f(x)=e^{-kx}$, $x\geq 0$, k a positive constant by (a) a cosine integral (b) a sine integral.

Solution: (a) Using Eqn.(8), we obtain

$$A(\alpha)=\int_0^{\infty} e^{-kx}\cos\alpha x\, dx$$

Using integration by parts, we find

$$A(\alpha)=e^{-kx}\frac{\sin\alpha x}{\alpha}\Bigg]_0^{\infty}+\frac{k}{\alpha}\int_0^{\infty}e^{-kx}\sin\alpha x\, dx$$

$$=\frac{k}{\alpha}\left[\left(-e^{-kx}\frac{\cos\alpha x}{\alpha}\right)_0^{\infty}-\frac{k}{\alpha}\int_0^{\infty}e^{-kx}\cos\alpha x\, dx\right]=\frac{k}{\alpha^2}-\frac{k^2}{\alpha^2}A(\alpha)$$

or, $$A(\alpha)=\frac{k}{k^2+\alpha^2}$$

Therefore, the cosine integral of $f(x)$ is

$$f(x)=e^{-kx}=\frac{2k}{\pi}\int_0^{\infty}\frac{\cos\alpha x}{k^2+\alpha^2}\,d\alpha\,.$$

(b) Similarly, we have

$$B(\alpha)=\int_0^{\infty}e^{-kx}\sin\alpha x\, dx=\frac{\alpha}{k^2+\alpha^2}$$

The sine integral of $f(x)$ is then

$$f(x)=e^{-kx}=\frac{2}{\pi}\int_0^{\infty}\frac{\alpha\sin\alpha x}{k^2+\alpha^2}\,d\alpha\,.$$

***

Example 4: Find the Fourier cosine integral representation for the function

$$f(x)=\begin{cases}x^2, & 0<x<a\\ 0, & x>a\end{cases}$$

Solution: Using Eqn.(8), we obtain

$$A(\alpha) = \int_0^\infty f(x)\cos\alpha x \, dx = \int_0^a x^2 \cos\alpha x \, dx$$

$$= \left[x^2\left(\frac{\sin\alpha x}{\alpha}\right) - 2x\left(\frac{-\cos\alpha x}{\alpha^2}\right) + 2\left(\frac{-\sin\alpha x}{\alpha^3}\right)\right]_0^a \quad \left[\begin{array}{l}\text{using integration}\\ \text{by parts}\end{array}\right]$$

$$= \frac{a^2\sin\alpha a}{\alpha} + \frac{2a\cos\alpha a}{\alpha^2} - \frac{2\sin\alpha a}{\alpha^3}$$

Thus, the cosine integral representation of $f(x)$ is given by

$$f(x) = \frac{2}{\pi}\int_0^\infty A(\alpha)\cos\alpha x \, d\alpha$$

$$= \frac{2}{\pi}\int_0^\infty \left(\frac{a^2\sin\alpha a}{\alpha} + \frac{2a\cos\alpha a}{\alpha^2} - \frac{2\sin\alpha a}{\alpha^3}\right)\cos\alpha x \, d\alpha$$

$$= \frac{2}{\pi}\int_0^\infty \frac{1}{\alpha^3}\left[\left(a^2\alpha^2 - 2\right)\sin a\alpha + 2a\alpha\cos a\alpha\right] d\alpha$$

***

Example 5: Find the solution of the integral equation

$$\int_0^\infty f(x)\cos ax \, dx = e^{-a} \text{, where a is a constant.}$$

Solution: The given integral is a Fourier cosine integral representation

Let $\quad g(x) = \frac{2}{\pi}\int_0^\infty A(\alpha)\cos\alpha x \, d\alpha$, where $A(\alpha) = \int_0^\infty g(x)\cos\alpha x \, dx$

Comparing the above equation with the given equation, we get $x = a$, $\frac{\pi}{2}g(a) = e^{-a}$, $f(x) = A(x)$.

Therefore, $A(\alpha) = \int_0^\infty \frac{2}{\pi}e^{-x}\cos\alpha x \, dx = \frac{2}{\pi}\int_0^\infty e^{-x}\cos\alpha x \, dx = \frac{2}{\pi}\left(\frac{1}{1+\alpha^2}\right)$ (using Example 3 with k = 1)

Hence, $f(x) = A(x) = \frac{2}{\pi}\left(\frac{1}{1+x^2}\right)$, $\quad x > 0$.

***

Example 6: Using Fourier integral representation, show that

$$\int_0^\infty \frac{\cos\frac{\pi\alpha}{\alpha}\cos\alpha x}{1-\alpha^2} \, d\alpha = \begin{cases} \frac{\pi}{2}\cos x, & |x| \le \pi/2 \\ 0, & |x| > \pi/2 \end{cases}$$

Solution: To prove the result consider right hand side which defines the function

$$f(x) = \begin{cases} \frac{\pi}{2}\cos x, & |x| \le \pi/2 \\ 0 & |x| > \pi/2 \end{cases}$$

Here $f(x)$ is an even function of x defined in the interval $-\infty < x < \infty$ and since the integral on the left hand side involves cosine terms, it indicates that we are required to find the Fourier cosine representation of $f(x)$.

Thus from Eqn.(8), we have

$$A(\alpha)=\int_0^\infty f(x)\cos\alpha x\,dx=\int_0^{\pi/2}\frac{\pi}{2}\cos x\cos\alpha x\,dx$$

$$=\frac{\pi}{4}\int_0^{\pi/2}\left[\cos(\alpha+1)x+\cos(\alpha-1)x\right]dx \qquad \left[\because 2\cos A\cos B=\cos(A+B)+\cos(A-B)\right]$$

$$=\frac{\pi}{4}\left[\frac{\sin(\alpha+1)x}{\alpha+1}+\frac{\sin(\alpha-1)x}{\alpha-1}\right]_0^{\pi/2}$$

$$=\frac{\pi}{4}\left[\frac{\sin(\alpha+1)\pi/2}{\alpha+1}+\frac{\sin(\alpha-1)\pi/2}{\alpha-1}\right]$$

$$=\frac{\pi}{4}\left[\frac{\cos\alpha\pi/2}{\alpha+1}-\frac{\cos\alpha\pi/2}{\alpha-1}\right]=\frac{\pi}{4}\left[\frac{2\cos\alpha\pi/2}{1-\alpha^2}\right]=\frac{\pi}{2}\frac{\cos\alpha\pi/2}{1-\alpha^2}$$

Using Eqn.(7), the Fourier cosine representation of $f(x)$ is given by

$$f(x)=\frac{2}{\pi}\int_0^\infty A(\alpha)\cos\alpha x\,d\alpha=\frac{2}{\pi}\int_0^\infty\frac{\pi}{2}\frac{\cos\alpha\pi/2}{1-\alpha^2}\cos\alpha x\,d\alpha$$

$$=\int_0^\infty\frac{\cos\frac{\alpha\pi}{2}\cos\alpha x}{1-\alpha^2}\,d\alpha$$

Therefore, we have

$$\int_0^\infty\frac{\cos\alpha\pi/2\cos\alpha x}{1-\alpha^2}\,d\alpha=f(x)=\begin{cases}\frac{\pi}{2}\cos x, & |x|\le\pi/2\\ 0, & |x|>\pi/2\end{cases}$$

which is the desired result.

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Complex form of Fourier integral representation

The Fourier integral (4) also possesses an equivalent **complex form** or, **exponential form**, that is analogous to the complex form of a Fourier series. Substituting the expressions for $A(\alpha)$ and $B(\alpha)$ from Eqns.(5) and (6) into Eqn.(4), we have

$$f(x)=\frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty f(t)[\cos\alpha t\cos\alpha x+\sin\alpha t\sin\alpha x]\,dt\,d\alpha$$

$$=\frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty f(t)\cos\alpha(t-x)\,dt\,d\alpha$$

$$=\frac{1}{2\pi}\int_{-\infty}^\infty\int_{-\infty}^\infty f(t)\cos\alpha(t-x)\,dt\,d\alpha \qquad (12)$$

$$=\frac{1}{2\pi}\int_0^\infty\int_{-\infty}^\infty f(t)\left(e^{i\alpha(t-x)}+e^{-i\alpha(t-x)}\right)dt\,d\alpha.$$

$$=\frac{1}{2\pi}\int_0^\infty\int_{-\infty}^\infty f(t)e^{i\alpha(t-x)}\,dt\,d\alpha+\frac{1}{2\pi}\int_0^\infty\int_{-\infty}^\infty f(t)e^{-i\alpha(t-x)}\,dt\,d\alpha \qquad (13)$$

To combine the two terms on the right hand side of Eqn.(13), let us change the dummy integration variable from α to $-\alpha$ in the first. Thus

$$f(x)=\frac{1}{2\pi}\int_0^{-\infty}\int_{-\infty}^\infty f(t)e^{-i\alpha(t-x)}\,dt(-d\alpha)+\frac{1}{2\pi}\int_0^\infty\int_{-\infty}^\infty f(t)e^{-i\alpha(t-x)}\,dt\,d\alpha$$

$$=\frac{1}{2\pi}\int_{-\infty}^0\int_{-\infty}^\infty f(t)e^{-i\alpha(t-x)}\,dt\,d\alpha+\frac{1}{2\pi}\int_0^\infty\int_{-\infty}^\infty f(t)e^{-i\alpha(t-x)}\,dt\,d\alpha$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) e^{-i\alpha(t-x)} dt \, d\alpha$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(t) e^{-i\alpha t} dt \right] e^{i\alpha x} d\alpha \qquad (14)$$

The integral in Eqn.(14) can be expressed as

$$\mathbf{f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} C(\alpha) e^{i\alpha x} d\alpha} \qquad (15)$$

where $\mathbf{C(\alpha) = \int_{-\infty}^{\infty} f(x) e^{-i\alpha x} dx}$ (16)

Eqns.(15) and (16) give the **complex form of Fourier integral representation**. We now take up an example to illustrate its application

Example 7: Find the complex form of Fourier integral representation for the function

$$f(x) = \begin{cases} |x|, & -\pi < x < \pi \\ 0, & \text{elsewhere.} \end{cases}$$

Solution: We have

$$C(\alpha) = \int_{-\infty}^{\infty} f(x) e^{-i\alpha x} dx = \int_{-\pi}^{\pi} |x| e^{-i\alpha x} dx$$

$$= \int_{0}^{\pi} x e^{-i\alpha x} dx - \int_{-\pi}^{0} x e^{-i\alpha x} dx$$

$$= \left[\left(\frac{x}{-i\alpha} - \frac{1}{(-i\alpha)^2} \right) e^{-i\alpha x} \right]_0^{\pi} - \left[\left(\frac{x}{-i\alpha} - \frac{1}{(-i\alpha)^2} \right) e^{-i\alpha x} \right]_{-\pi}^{0}$$

$$= \left(\frac{i\pi}{\alpha} + \frac{1}{\alpha^2} \right) e^{-i\alpha\pi} - \frac{2}{\alpha^2} + \left(\frac{-i\pi}{\alpha} + \frac{1}{\alpha^2} \right) e^{i\alpha\pi}$$

$$= \frac{2}{\alpha^2} \cos \pi\alpha - \frac{2\pi}{\alpha} \sin \pi\alpha - \frac{2}{\alpha^2}$$

Therefore, $$f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{\alpha^2} [\cos \pi\alpha - \pi\alpha \sin \pi\alpha - 1] e^{i\alpha x} d\alpha$$

***

You may now try the following exercises.

E1) Find the Fourier integral representation of the following functions.

a) $f(x) = \begin{cases} 0, & x < 0 \\ x, & 0 < x < 3 \\ 0, & x > 3 \end{cases}$ b) $f(x) = \begin{cases} -(1+x), & -1 < x < 0 \\ 1 - x, & 0 \le x \le 1 \\ 0, & |x| > 1 \end{cases}$

c) $f(x) = \begin{cases} 0, & x < 0 \\ e^{-x}, & x > 0 \end{cases}$ d) $f(x) = \begin{cases} \sin x, & -2 \le x \le 0 \\ \cos x, & 0 < x \le 2 \\ 0, & |x| > 2 \end{cases}$

E2) Represent the following function by an appropriate cosine or sine integral

a) $f(x) = \begin{cases} 0, & x < -1 \\ -5, & -1 < x < 0 \\ 5, & 0 < x < 1 \\ 0, & x > 1 \end{cases}$ b) $f(x) = \begin{cases} |x|, & |x| < \pi \\ 0, & |x| > \pi \end{cases}$

c) $f(x) = \begin{cases} \sin x, & 0 \le x \le \pi \\ 0 \quad, & x > \pi \end{cases}$ d) $f(x) = \begin{cases} \sinh x, & 0 \le x \le 3 \\ 0 \quad, & x > 3 \end{cases}$

E3) Using Fourier integral representation, show that

a) $\int_0^\infty \frac{\alpha^3 \sin \alpha x}{\alpha^2 + 4} \, d\alpha = \frac{\pi}{2} e^{-x} \cos x \,, \; x > 0$

b) $\int_0^\infty \frac{1 - \cos \pi\alpha}{\alpha} \sin \alpha x \, d\alpha = \begin{cases} \frac{\pi}{2}, & 0 < x < \pi \\ 0, & x > \pi \end{cases}$

E4) Solve the following integral equation.

$$\int_0^\infty f(x) \sin \alpha x \, dx = \begin{cases} 1, & 0 < \alpha < 1 \\ 0, & \alpha > 1 \end{cases}$$

E5) a) Use Eqn.(11) to show that $\int_0^\infty \frac{\sin 2x}{x} dx = \frac{\pi}{2}$.

b) Show in general that for $k > 0$

$$\int_0^\infty \frac{\sin kx}{x} \, dx = \frac{\pi}{2}.$$

E6) For the following functions, find the complex form of the Fourier integral representation

a) $f(x) = \begin{cases} 0 \quad, & x < 0 \\ e^{-kx}, & x > 0, \, k > 0 \end{cases}$ b) $f(x) = \begin{cases} 0, & -\infty < x < -1 \\ x, & -1 < x < 0 \\ 1, & 0 < x < 1 \\ 0, & x > 1 \end{cases}$

c) $f(x) = \begin{cases} \cosh 2x, & |x| \le a \\ 0 \quad, & |x| > a \end{cases}$ d) $f(x) = \begin{cases} \sin \pi x, & -2 \le x \le 2 \\ 0 \quad, & |x| > 2 \end{cases}$

In the discussion so far, we obtained the Fourier integral representation of non-periodic functions $f(x)$ defined on $-\infty < x < \infty$ by taking the limit of the Fourier series formula as the period tends to infinity. The Fourier integral representation of a function $f(x)$, can be represented as a pair of formulas, the first giving the Fourier transform of $f(x)$ and the second giving the inverse of that transform. Like the Laplace transform, the Fourier transform, which we shall be discussing in the next section, is highly developed as a methodology.

7.3 FOURIER TRANSFORMS

Consider the Fourier integral formula

$$f(x) = \frac{1}{\pi} \int_0^\infty [A(\alpha) \cos \alpha x + B(\alpha) \sin \alpha x] \, d\alpha$$

where, $A(\alpha) = \int_{-\infty}^\infty f(x) \cos \alpha x \, dx$

$$B(\alpha) = \int_{-\infty}^\infty f(x) \sin \alpha x \, dx$$

You know that this formula possesses an equivalent complex form or an exponential form given by Eqns.(15) and (16), that is,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} C(\alpha)e^{i\alpha x}\, d\alpha$$

where, $C(\alpha) = \int_{-\infty}^{\infty} f(x)e^{-i\alpha x}\, dx$

From your knowledge of the Laplace transforms which we discussed in Unit-6, you may recall that integral transforms occur in **transform pairs** – that is, if

$$F(\alpha) = \int_{a}^{b} f(x)\ K(\alpha, x)\, dx \tag{17}$$

is an integral that transforms $f(x)$ into $F(\alpha)$, then the function $f(x)$ is recovered by means of another integral transform

$$f(x) = \int_{c}^{d} F(\alpha)\ H(\alpha, x)\ d\alpha \tag{18}$$

called the **inversion integral** or **inverse transform,** the functions K and H being the **kernels** of transforms (17) and (18), respectively.

Thus, rather than thinking of Eqn.(15) as the Fourier integral of $f(x)$ and Eqn.(16) as giving the Fourier coefficients $C(\alpha)$, we can think of Eqns.(15) and (16) as a transform pair. Eqn.(16) defines the Fourier transform $C(\alpha)$ of the given function $f(x)$ and Eqn.(15) is the inversion formula, as putting $C(\alpha)$ in it and integrating we get back $f(x)$. It is standard to use the notation $F(\alpha)$ in place of $C(\alpha)$ for the transform.

The **Fourier transform** of $f(x)$ denoted by $\mathcal{F}\,[f(x)]$ is defined as

$$\mathcal{F}\,[f(x)] = \int_{-\infty}^{\infty} f(x)e^{-i\alpha x}dx = F(\alpha) \tag{19}$$

and the **inverse Fourier transform** is defined as

$$\mathcal{F}^{-1}\,[F(\alpha)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)\ e^{i\alpha x}\ d\alpha = f(x) \tag{20}$$

Thus, the Fourier transform and inversion formula together amounts simply to the Fourier integral representation, expressed in complex exponential form. The conditions imposed on $f(x)$ are the same as in Theorem-1.

Cosine and sine transforms

When the function $f(x)$ defined in the interval $-\infty < x < \infty$ is an **even function,** then from Fourier cosine integral, we have

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty}\left(\int_{0}^{\infty} f(t)\cos\alpha t\, dt\right)\cos\alpha x\, d\alpha \tag{21}$$

If we write

$$\mathcal{F}_c[f(x)] = \int_{0}^{\infty} f(x)\cos\alpha x\, dx = F_c(\alpha)$$

then from Eqn.(21), it follows that

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty} F_c(\alpha)\cos\alpha x\, d\alpha$$

We call $F_c(\alpha)$ the Fourier cosine transform of $f(x)$, while $f(x)$ is the inverse Fourier cosine transform of $F_c(\alpha)$. Similarly, when f(x) defined in the interval $-\infty < x < \infty$ is an **odd function**, then from Fourier sine integral, we have

$$f(x) = \frac{2}{\pi}\int_0^\infty \left(\int_0^\infty f(t)\sin\alpha t\, dt \right) \sin\alpha x\, d\alpha \tag{22}$$

If we write

$$\mathcal{F}_s[f(x)] = \int_0^\infty f(x)\sin\alpha x\, dx = F_s(\alpha)$$

then from Eqn.(22), it follows that

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s(\alpha)\sin\alpha x\, d\alpha$$

We call $F_s(\alpha)$ the Fourier sine transform of $f(x)$, while $f(x)$ is the inverse Fourier sine transform of $F_s(\alpha)$.

We summarise:
The **Fourier Cosine transform** of even function $f(x)$ on the interval $-\infty < x < \infty$ is defined as

$$\mathcal{F}_c[f(x)] = F_c(\alpha) = \int_0^\infty f(x)\cos\alpha x\, dx \tag{23}$$

and the **inverse Fourier cosine transform** of $F_c(\alpha)$ is given by

$$\mathcal{F}^{-1}[F_c(\alpha)] = \frac{2}{\pi}\int_0^\infty F_c(\alpha)\cos\alpha x\, d\alpha = f(x) \tag{24}$$

The **Fourier sine transform** of odd function $f(x)$ on the interval $-\infty < x < \infty$ is defined as

$$\mathcal{F}_s[f(x)] = F_s(\alpha) = \int_0^\infty f(x)\sin\alpha x\, dx \tag{25}$$

and the **inverse Fourier sine transform** of $F_s(\alpha)$ is given by

$$\mathcal{F}^{-1}[F_s(\alpha)] = \frac{2}{\pi}\int_0^\infty F_s(\alpha)\sin\alpha x\, d\alpha = f(x) \tag{26}$$

We now illustrate the calculation of the transform $F(\alpha)$ of $f(x)$ and its inverse through the following examples.

Example 8: Find the Fourier transform of

$$f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$$

Also draw the graph of $f(x)$ and its Fourier transform for $a = 3$.

Solution: Using Eqn.(19), the Fourier transform of $f(x)$ is

$$F(\alpha) = \int_{-\infty}^{\infty} f(x)e^{-i\alpha x}dx = \int_{-a}^{a} e^{-i\alpha x}dx = \left.\frac{e^{-i\alpha x}}{-i\alpha}\right]_{-a}^{a}$$

$$= \frac{e^{i\alpha a} - e^{-i\alpha a}}{i\alpha} = \frac{2\sin\alpha a}{\alpha}, \quad \alpha \neq 0.$$

of $\alpha = 0$, we obtain $F(\alpha) = 2a$.

The graphs of $f(x)$ and $F(\alpha)$ for $a = 3$ are shown in Fig.3 and Fig.4 respectively.

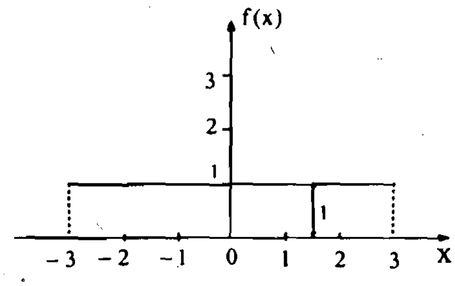

Fig.3

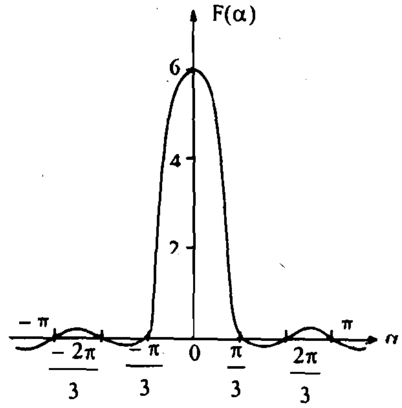

Fig.4

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Example 9: Find the Fourier transform of the function

$$f(x)=\begin{cases}0 \ , & x<0 \\ e^{-ax}, & x\geq 0 \ , \ a>0\end{cases}.$$

Solution: We have

$$\mathcal{F}[f(x)]=F(\alpha)=\int_{-\infty}^{\infty} f(x)e^{-i\alpha x}dx=\int_{0}^{\infty} e^{-ax}e^{-i\alpha x}dx$$

$$=\left[\frac{-e^{-(a+i\alpha)x}}{a+i\alpha}\right]_0^\infty=\frac{1}{a+i\alpha}-\frac{1}{a+i\alpha}\lim_{x\to\infty} e^{-(a+i\alpha)x}=\frac{1}{a+i\alpha}$$

since $\lim\limits_{x\to\infty}\left[e^{-(a+i\alpha)x}\right]=\lim\limits_{x\to\infty} e^{-ax}\left[\cos\alpha x-i\sin\alpha x\right]=0$.

Hence $\dfrac{1}{a+i\alpha}$ and $f(x)=e^{-ax}u_0(x)$ where $u_0(x)$ is the unit step function, form a transform pair.

Therefore, $\mathcal{F}[e^{-ax}u_0(x)]=\dfrac{1}{a+i\alpha}$

***

Example 10: Find the Fourier transform of e^{-ax^2}, $a>0$.

Solution: From the definition, we have

$$\mathcal{F}[e^{-ax^2}]=\int_{-\infty}^{\infty} e^{-ax^2}e^{-i\alpha x}dx=\int_{-\infty}^{\infty} e^{-a\left[x^2+\frac{i\alpha x}{a}\right]}dx$$

$$=\int_{-\infty}^{\infty} e^{-a\left[\left(x+\frac{i\alpha}{2a}\right)^2+\frac{\alpha^2}{4a^2}\right]}dx=e^{-\alpha^2/4a}\int_{-\infty}^{\infty} e^{-a\left(x+\frac{i\alpha}{2a}\right)^2}dx$$

$$=e^{\frac{-\alpha^2}{4a}}\int_{-\infty}^{\infty} e^{-t^2}\frac{dt}{\sqrt{a}} \qquad \left[\text{setting } \sqrt{a}\left(x+\frac{i\alpha}{2a}\right)=t\right]$$

$$=\frac{\sqrt{\pi}}{\sqrt{a}}e^{-\alpha^2/4a} \qquad \left[\text{since } \int_{-\infty}^{\infty} e^{-t^2}dt=2\int_{-\infty}^{\infty} e^{-t^2}dt=\sqrt{\pi}\right]$$

***

Example 11: Find the Fourier transform of

$$f(x)=\begin{cases}1-x^2, & |x|\le 1\\ 0 \quad, & |x|>1\end{cases}$$

and hence evaluate

$$\int_0^\infty\left(\frac{x\cos x-\sin x}{x^3}\right)\cos\frac{x}{2}\,dx\,.$$

Solution: The function $f(x)$ is given by

$$f(x)=\begin{cases}1-x^2, & -1\le x\le 1\\ 0 \quad, & |x|>1\end{cases}$$

This shows that $f(-x)=f(x)$ i.e. $f(x)$ is an even function in the interval $-\infty<x<\infty$. Hence by result (23), the Fourier cosine transform of $f(x)$ is

$$\mathcal{F}_c[f(x)]=\int_0^\infty f(x)\cos\alpha x\,dx=\int_0^1(1-x^2)\cos\alpha x\,dx+\int_1^\infty 0.\cos\alpha x\,dx$$

$$=\left[(1-x^2)\left(\frac{\sin\alpha x}{\alpha}\right)-(-2x)\left(\frac{-\cos\alpha x}{\alpha^2}\right)+(-2)\left(\frac{-\sin\alpha x}{\alpha^3}\right)\right]_0^1$$

$$=2\left(\frac{\sin\alpha-\alpha\cos\alpha}{\alpha^3}\right)$$

By using inverse transform [result (24)], we have

$$f(x)=\frac{2}{\pi}\int_0^\infty 2\left(\frac{\sin\alpha-\alpha\cos\alpha}{\alpha^3}\right)\cos\alpha x\,d\alpha$$

$$=\frac{4}{\pi}\int_0^\infty\left(\frac{\sin\alpha-\alpha\cos\alpha}{\alpha^3}\right)\cos\alpha x\,d\alpha \qquad (27)$$

The result (27) can be expressed as

$$\int_0^\infty\left(\frac{\sin\alpha-\alpha\cos\alpha}{\alpha^3}\right)\cos\alpha x\,d\alpha=\frac{\pi}{4}f(x)=\begin{cases}\frac{\pi}{4}(1-x^2), & |x|\le 1\\ 0 \quad, & |x|>1\end{cases}$$

Putting $x=\frac{1}{2}$, which lies in $-1\le x\le 1$, i.e., in $|x|\le 1$, we get

$$\int_0^\infty\left(\frac{\sin\alpha-\alpha\cos\alpha}{\alpha^3}\right)\cos\frac{\alpha}{2}\,d\alpha=\frac{\pi}{4}\left(1-\frac{1}{4}\right)=\frac{3\pi}{16}$$

Changing the dummy variable α by x in the above integral, we obtain

$$\int_0^\infty\left(\frac{\sin x-x\cos x}{x^3}\right)\cos\frac{x}{2}\,dx=\frac{3\pi}{16}$$

or, $$\int_0^\infty\left(\frac{x\cos x-\sin x}{x^3}\right)\cos\frac{x}{2}\,dx=\frac{-3\pi}{16}$$

which is the required result.

***

Example 12: Find the Fourier cosine transform of the function

$$f(x)=\begin{cases}\cos x, & 0<x<a\\ 0 \quad, & x>a\end{cases}$$

Solution: Using the definition, we have

$$\mathcal{F}_c[f(x)]=\int_0^\infty f(x)\cos\alpha x\,dx=\int_0^a\cos x\cos\alpha x\,dx+\int_a^\infty 0.\cos\alpha x\,dx$$

$$= \frac{1}{2}\int_0^a [\cos(\alpha+1)x + \cos(\alpha-1)x]\,dx$$

$$= \frac{1}{2}\left[\frac{\sin(\alpha+1)x}{\alpha+1} + \frac{\sin(\alpha-1)x}{\alpha-1}\right]_0^a$$

$$= \frac{1}{2}\left[\frac{\sin(\alpha+1)a}{\alpha+1} + \frac{\sin(\alpha-1)a}{\alpha-1}\right]$$

***

Example 13: Using inverse sine transform, find $f(x)$ if

$$F_s(\alpha) = \frac{1}{\alpha}e^{-a\alpha}, \quad a > 0.$$

Solution: Using result (26), inverse sine transform of $F_s(\alpha)$ is given by

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s(\alpha)\sin\alpha x\, d\alpha = \frac{2}{\pi}\int_0^\infty \frac{1}{\alpha}e^{-a\alpha}\sin\alpha x\, d\alpha = \frac{2}{\pi}I(x) \text{ (say)}$$

where, $I(x) = \int_0^\infty \frac{e^{-a\alpha}}{\alpha}\sin\alpha x\, d\alpha$.

Now, $I'(x) = \int_0^\infty \frac{\partial}{\partial x}\frac{e^{-a\alpha}}{\alpha}\sin\alpha x\, d\alpha$ [Using the rule of differentiation under the integral sign]

$$= \int_0^\infty e^{-a\alpha}\cos\alpha x\, d\alpha$$

$$= \left[\frac{e^{-a\alpha}}{a^2+x^2}(-a\cos\alpha x + x\sin\alpha x)\right]_0^\infty$$

$$= \frac{a}{a^2+x^2}.$$

Integrating, we have

$$I(x) = \int \frac{a}{a^2+x^2}dx + A = \tan^{-1}\frac{x}{a} + A$$

Putting $x = 0$, we get $I(x)]_{x=0} = A$

But we have

$$I(x)]_{x=0} = 0$$

therefore, $A = 0$ and $I(x) = \tan^{-1}\frac{x}{a}$.

Hence, we obtain

$$f(x) = \frac{2}{\pi}I(x) = \frac{2}{\pi}\tan^{-1}\frac{x}{a}.$$

***

Example 14: Find the function $f(x)$, whose Fourier cosine transform is $\frac{\sin a\alpha}{\alpha}$.

Solution: Given that $F_c(\alpha) = \frac{\sin a\alpha}{\alpha}$ and we are required to find $f(x)$. Using result (24), we get

$$f(x) = \frac{2}{\pi}\int_0^\infty F_c(\alpha)\cos\alpha x\, d\alpha = \frac{2}{\pi}\int_0^\infty \frac{\sin a\alpha}{\alpha}\cos\alpha x\, d\alpha$$

$$= \frac{1}{\pi}\int_0^\infty \frac{\sin(a+x)\alpha + \sin(a-x)\alpha}{\alpha}d\alpha$$

$$= \frac{1}{\pi}\left[\int_0^\infty \sin\frac{(a+x)\alpha}{\alpha}\,d\alpha + \int_0^\infty \sin\frac{(a-x)\alpha}{\alpha}\,d\alpha\right]$$

Using the result $\int_0^\infty \frac{\sin ax}{x}\,dx = \begin{cases} \pi/2\,, & a>0 \\ -\pi/2, & a<0 \end{cases}$, we obtain

$$f(x) = \begin{cases} \frac{1}{\pi}\left[\frac{\pi}{2}+\frac{\pi}{2}\right], & a+x>0 \text{ and } a-x>0 \\ 0 \quad , & a+x<0 \text{ and } a-x<0 \end{cases}$$

$$= \begin{cases} 1\,, & 0<x<a \\ 0\,, & x>a \end{cases}$$

***

Example 15: Solve the following integral equation

$$\int_0^\infty f(x)\sin\alpha x\,dx = \begin{cases} 1\,, & 0\le\alpha<1 \\ 2\,, & 1\le\alpha<2 \\ 0\,, & \alpha\ge 2 \end{cases}$$

Solution: Since $\sin\alpha x$ is present in the given integral, we use result (25) and write

$$F_s(\alpha) = \int_0^\infty f(x)\sin\alpha x\,dx = \begin{cases} 1\,, & 0\le\alpha<1 \\ 2\,, & 1\le\alpha<2 \\ 0\,, & \alpha\ge 2 \end{cases}$$

Now to find $f(x)$ we use result (26) and obtain the inverse Fourier sine transform. Thus we have,

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s(\alpha)\sin\alpha x\,d\alpha$$

$$= \frac{2}{\pi}\left[\int_0^1 1.\sin\alpha x\,d\alpha + \int_1^2 2.\sin\alpha x\,d\alpha + \int_2^\infty 0.\sin\alpha x\,d\alpha\right]$$

$$= \frac{2}{\pi}\left[\left(\frac{-\cos\alpha x}{x}\right)_0^1 + 2\left(\frac{-\cos\alpha x}{x}\right)_1^2\right]$$

$$= \frac{2}{\pi}\left[\left(\frac{1-\cos x}{x}\right) + 2\left(\frac{\cos x-\cos 2x}{x}\right)\right]$$

$$= \frac{2}{\pi}\left(\frac{1+\cos x-2\cos 2x}{x}\right)$$

which is the required result.

***

As with the Laplace transform, for convenience, we have given in the Appendix at the end of the unit, a table giving the Fourier transforms and their inverses of commonly used functions.

You may now try the following exercises.

E7) Find the Fourier transform of the following functions defined on $]-\infty,\infty[$

a) $f(x) = \begin{cases} a\,, & -\ell<x<0,\ a>0 \\ 0\,, & \text{otherwise} \end{cases}$ b) $f(x) = \begin{cases} a\,, & -\ell<x<\ell\,,\ a>0 \\ 0\,, & \text{otherwise} \end{cases}$

c) $f(x) = \begin{cases} a\,, & -\ell<x<0 \\ b\,, & 0<x<\ell \quad a>0,\ b>0 \\ 0\,, & \text{otherwise} \end{cases}$

E8) Find the Fourier transform of the function $f(x) = e^{-a|x|}$, $-\infty < x < \infty$, $a > 0$. Write the inverse transform also.

E9) Find the Fourier sine transform of the function $f(x) = \dfrac{e^{-ax}}{x}$ and hence evaluate

$$\int_0^\infty \tan^{-1}\frac{x}{a}\sin x\,dx.$$

E10) Find the Fourier cosine transform of the function $f(x) = e^{-x}$ and hence show that

$$\int_0^\infty \frac{\cos mx}{1+x^2}dx = \frac{\pi}{2}e^{-m}.$$

E11) Solve the following integral equations

a) $$\int_0^\infty f(x)\sin\alpha x\,dx = \begin{cases} 1-\alpha, & 0 \le \alpha \le 1 \\ 0, & \alpha > 1 \end{cases}$$

b) $$\int_0^\infty f(x)\cos\alpha x\,dx = e^{-\alpha}, \quad \alpha > 0$$

E12) Solve the integral equation $\displaystyle\int_0^\infty f(x)\cos\alpha x\,dx = \begin{cases} 1-\alpha, & 0 \le \alpha \le 1 \\ 0, & \alpha > 1 \end{cases}$ and hence

show that $\displaystyle\int_0^\infty \frac{\sin^2 z\,dz}{z^2} = \frac{\pi}{2}$.

E13) Find the Fourier cosine transform of the following function

$$f(x) = \begin{cases} x, & 0 < x < \frac{1}{2} \\ 1-x, & \frac{1}{2} < x < 1 \\ 0, & x > 1 \end{cases}$$

E14) Using inverse Fourier cosine transform, find $f(x)$, if

$$F_c(\alpha) = \begin{cases} \sqrt{\frac{2}{\pi}}\left(a - \frac{\alpha}{2}\right), & \alpha \le 2a \\ 0, & \alpha > 2a \end{cases}.$$

E15) Find the function $f(x)$, satisfying the integral equation

$$\int_0^\infty f(x)\sin\alpha x\,dx = \frac{\alpha}{\alpha^2 + k^2}.$$

The Fourier transform admits a number of useful properties which we shall discuss now. Our discussion will parallel our analogous discussion for the Laplace transform, given in Unit 6.

7.3.1 Properties of Fourier Transforms

We will assume without reiteration conditions given in Theorem-1.

Linearity property: If $\mathcal{F}[f(x)] = F(\alpha)$ and $\mathcal{F}[g(x)] = G(\alpha)$ and a and b are any scalars then

$$\mathcal{F}[a f(x) + b g(x)] = a\mathcal{F}[f(x)] + b\mathcal{F}[g(x)] = a F(\alpha) + b G(\alpha) \tag{28}$$

and $$\mathcal{F}^{-1}[a F(\alpha) + b G(\alpha)] = a\mathcal{F}^{-1}[F(\alpha)] + b\mathcal{F}^{-1}[G(\alpha)] \tag{29}$$

where, $\mathcal{F}^{-1}[F(\alpha)]$ is $f(x)$ and $\mathcal{F}^{-1}[G(\alpha)]$ is $g(x)$

Proof: By definition of Fourier transform, we have

$$\mathcal{F}[f(x)] = \int_{-\infty}^{\infty} f(x)e^{-i\alpha x}dx \text{ and } \mathcal{F}[g(x)] = \int_{-\infty}^{\infty} g(x)e^{-i\alpha x}dx$$

$$\therefore \mathcal{F}[a f(x) + b g(x)] = \int_{-\infty}^{\infty} [a f(x) + b g(x)]e^{-i\alpha x}dx$$

$$= a \int_{-\infty}^{\infty} f(x)e^{-i\alpha x}dx + b \int_{-\infty}^{\infty} g(x)e^{-i\alpha x}dx$$

$$= a\,\mathcal{F}[f(x)] + b\,\mathcal{F}[g(x)]$$

$$= a F(\alpha) + bG(\alpha)$$

— □ —

Shifting property: If $\mathcal{F}[f(x)] = F(\alpha)$ and a is any real number then

$$\mathcal{F}[f(x-a)] = e^{-i\alpha a}F(\alpha) \tag{30}$$

Proof: We have from the definition

$$\mathcal{F}[f(x-a)] = \int_{-\infty}^{\infty} f(x-a)e^{-i\alpha x}dx$$

Putting $x - a = t$ or, $x = t + a$ and $dx = dt$, we have

$$= \int_{-\infty}^{\infty} f(t)e^{-i\alpha(t+a)}dt = e^{-i\alpha a}\int_{-\infty}^{\infty} f(t)e^{-i\alpha t}dt$$

$$= e^{-i\alpha a} F(\alpha)$$

Thus,

$$\mathcal{F}^{-1}[e^{-ia\alpha}F(\alpha)] = f(x-a) \tag{31}$$

— □ —

Modulation Theorem: If $\mathcal{F}[f(x)] = F(\alpha)$ and a is any real number, then

$$\mathcal{F}[f(x) \cos a x] = \frac{1}{2}[F(\alpha + a) + F(\alpha - a)] \tag{32}$$

and

$$\mathcal{F}[f(x) \sin ax] = \frac{1}{2}[F(\alpha + a) - F(\alpha - a)] \tag{33}$$

Proof: We have

$$\mathcal{F}[f(x)\cos ax] = \int_{-\infty}^{\infty} f(x)\cos ax\, e^{-i\alpha x}dx = \int_{-\infty}^{\infty} f(x)\left(\frac{e^{iax} + e^{-iax}}{2}\right)e^{-i\alpha x}dx$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} f(x)e^{-i(\alpha+a)x}dx + \int_{-\infty}^{\infty} f(x)e^{-i(\alpha-a)x}dx\right]$$

$$= \frac{1}{2}[F(\alpha + a) + F(\alpha - a)]$$

Similarly, result (33) can be proved.

— □ —

Note: If $\mathcal{F}_s[f(x)] = F_s(\alpha)$ and $\mathcal{F}_c[f(x)] = F_c(\alpha)$ and a is any real number, then

$$\mathcal{F}_s[f(x)\cos ax] = \frac{1}{2}[F_s(\alpha + a) + F_s(\alpha - a)] \tag{34}$$

$$\mathcal{F}_s[f(x)\sin ax] = \frac{1}{2}[F_c(\alpha - a) - F_c(\alpha + a)] \tag{35}$$

$$\mathcal{F}_c[f(x)\cos ax]=\frac{1}{2}[F_c(\alpha+a)+F_c(\alpha-a)] \tag{36}$$

$$\mathcal{F}_c[f(x)\sin ax]=\frac{1}{2}[F_s(\alpha+a)-F_s(\alpha-a)] \tag{37}$$

Convolution Theorem

Definition: The convolution of the functions $f(x)$ and $g(x)$ is denoted by $f(x)*g(x)$ and is defined by

$$f(x)*g(x)=\int_{-\infty}^{\infty} f(t)g(x-t)\,dt \tag{38}$$

Theorem 2: The Fourier transform of the convolution of $f(x)$ and $g(x)$ is equal to the product of the Fourier transforms of $f(x)$ and $g(x)$. In symbols,

$$\mathcal{F}[f(x)*g(x)]=\mathcal{F}[f(x)]\,\mathcal{F}[g(x)]=F(\alpha)G(\alpha) \tag{39}$$

Proof: We have

$$\mathcal{F}[f(x)*g(x)]=\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t)g(x-t)\,dt\right]e^{-i\alpha x}dx$$

$$=\int_{-\infty}^{\infty} f(t)\left[\int_{-\infty}^{\infty} g(x-t)e^{-i\alpha x}dx\right]dt \qquad \text{[changing the order of integration]}$$

$$=\int_{-\infty}^{\infty} f(t)\left[\int_{-\infty}^{\infty} g(u)e^{-i\alpha(t+u)}du\right]dt \qquad \begin{bmatrix}\text{Putting } x-t=u\\ \therefore\ dx=du\end{bmatrix}$$

$$=\left[\int_{-\infty}^{\infty} f(t)e^{-i\alpha t}dt\right]\left[\int_{-\infty}^{\infty} g(u)e^{-i\alpha u}du\right]$$

$$=\left[\int_{-\infty}^{\infty} f(x)e^{-i\alpha x}dx\right]\left[\int_{-\infty}^{\infty} g(x)e^{-i\alpha x}dx\right]$$

$$=\mathcal{F}[f(x)]\,\mathcal{F}[g(x)]=F(\alpha)G(\alpha)$$

The inverse transform is given by

$$\mathcal{F}^{-1}[F(\alpha)G(\alpha)]=(f*g)(x) \tag{40}$$

— □ —

Remark: The following important properties of convolution can be proved easily

(i) $f(x)*g(x)=g(x)*f(x)$

(ii) $f(x)*[g(x)*h(x)]=[f(x)*g(x)]*h(x)$

(iii) $f(x)*[g(x)+h(x)]=f(x)*g(x)+f(x)*h(x)$

i.e., the convolution obeys the commutative, associative and distributive laws of algebra.

Fourier transform of n[th] derivative: If $f(x), f'(x),\ldots, f^{(n-1)}(x)$ all tend to zero as $x\to\pm\infty$ and $\int_{-\infty}^{\infty}|f^{(j)}(x)|\,dx$ converges for each $j=0,1,\ldots,n$, then

$$\mathcal{F}[f^{(n)}(x)]=(i\alpha)^n F(\alpha),\quad n=0,1,2,\ldots \tag{41}$$

Proof: Let us prove result (41) by induction. First observe that (41) holds for $n=0$, by definition. We need to establish that if it holds for $n=k$, then it also holds for $n=k+1$. Integrating by parts, we get

$$\mathcal{F}[f^{(k+1)}(x)]=\int_{-\infty}^{\infty} f^{(k+1)}(x)e^{-i\alpha x}dx$$

$$= f^{(k)}(x)e^{-i\alpha x}\Big|_{-\infty}^{\infty} + i\alpha \int_{-\infty}^{\infty} f^{(k)}(x)e^{-i\alpha x}dx$$

$$= f^{(k)}(x)e^{-i\alpha x}\Big|_{-\infty}^{\infty} + i\alpha[(i\alpha)^k F(\alpha)] \qquad \left[\begin{array}{l}\because \text{result (41) holds}\\ \text{for } n = k\end{array}\right]$$

$$= f^{(k)}(x)e^{-i\alpha x}\Big|_{-\infty}^{\infty} + (i\alpha)^{k+1}F(\alpha) \tag{42}$$

Now $|f^{(k)}(x)e^{-i\alpha x}| = |f^{(k)}(x)||e^{-i\alpha x}| = |f^{(k)}(x)|$, and since $f^{(k)}(x) \to 0$ as $x \to \pm\infty$, by assumption, the first term on r.h.s. of Eqn.(42) vanishes. Thus, $\mathcal{F}[f^{(k+1)}(x)] = (i\alpha)^{k+1}F(\alpha)$ and result (41) holds for $n = k+1$, which completes our proof.

— □ —

Differentiation with respect to frequency α: If $\mathcal{F}[f(x)] = F(\alpha)$ then

$$\mathcal{F}[x^n f(x)] = i^n F^{(n)}(\alpha) \tag{43}$$

Proof: From the definition, we have

$$F'(\alpha) = \frac{d}{d\alpha}\int_{-\infty}^{\infty} f(x)e^{-i\alpha x}dx = \int_{-\infty}^{\infty}\frac{d}{d\alpha}[f(x)e^{-i\alpha x}]dx \qquad \left[\begin{array}{l}\text{Differentiation under}\\ \text{the integral sign.}\end{array}\right]$$

$$= -i\int_{-\infty}^{\infty}[x f(x)]e^{-i\alpha x}dx = -i\mathcal{F}[x f(x)]$$

$$F''(\alpha) = -i\frac{d}{d\alpha}\int_{-\infty}^{\infty} x f(x)e^{-i\alpha x}dx = -i\int_{-\infty}^{\infty}\frac{d}{d\alpha}[x f(x)e^{-i\alpha x}]dx$$

$$= (-i)^2\int_{-\infty}^{\infty} x^2 f(x)e^{-i\alpha x}dx = (-i)^2\mathcal{F}[x^2 f(x)]$$

Using induction result (43) can be proved easily. From result (43), we have

$$\mathcal{F}^{-1}[F^{(n)}(\alpha)] = (-i)^n x^n f(x) \tag{44}$$

— □ —

Fourier transform of an integral: If $\mathcal{F}[f(x)] = F(\alpha)$ and $f(x) \to 0$ as $x \to \infty$ and $F(\alpha)$ satisfies $F(0) = 0$, then

$$\mathcal{F}\left[\int_{-\infty}^{x} f(\xi)\,d\xi\right] = \frac{1}{i\alpha}F(\alpha). \tag{45}$$

— □ —

Fourier transform of the Dirac-delta function: In Unit 6, we have shown that the Laplace transform of the Dirac-delta function is one, i.e. $\mathcal{L}[\delta(x)] = 1$. We also proved there the filtering property of Dirac-delta functions as

$$\int_{0}^{\infty} f(x)\delta(x-a)dx = f(a) \tag{46}$$

Similar results can be proved for the Fourier transform of the Dirac-delta function.
We define the delta function as

$$\delta(x) = \lim_{k\to 0}\frac{1}{k}[H(x) - H(x-k)]$$

where $H(x)$ is the Heaviside step function or unit step function.
We have,

$$H(x)-H(x-k)=\begin{cases}0, & x<0\\ 1, & 0\le x<k\\ 0, & x\ge k\end{cases}$$

Then, $\mathcal{F}[\delta(x)]=\lim\limits_{k\to 0}\left\{\frac{1}{k}\mathcal{F}[H(x)-H(x-k)]\right\}=\lim\limits_{k\to 0}\left\{\frac{1}{k}\int\limits_0^k e^{-i\alpha x}dx\right\}$

$$=\lim_{k\to 0}\frac{1}{k}\left[\frac{1-e^{-i\alpha x}}{i\alpha}\right]=1$$

Hence,
$$\mathcal{F}[\delta(x)]=1 \tag{47}$$
Therefore, Laplace transform and Fourier transform of the Dirac-delta function are both equal to 1.
Hence,
$$\mathcal{F}^{-1}[1]=\delta(x) \tag{48}$$
The filtering property given by Eqn.(46) gets modified as
$$\int\limits_{-\infty}^{\infty}f(x)\delta(x-a)dx=f(a) \tag{49}$$
Further, using the shifting property given by Eqn.(30), we obtain
$$\mathcal{F}[\delta(x-a)]=e^{-ia\alpha}F(\alpha)=e^{-ia\alpha} \tag{50}$$

— □ —

Fourier sine and cosine transform of derivatives: Let $f(x)$ and $f'(x)$ be continuous on the interval $[0,\infty[$ and $f''(x)$ is piecewise continuous on every finite interval. If all the three functions are absolutely integrable on $[0,\infty[$, then
$$\mathcal{F}_c[f''(x)]=-\alpha^2\mathcal{F}_c[f(x)]-f'(0) \quad \textbf{and} \quad \mathcal{F}_s[f''(x)]=-\alpha^2\mathcal{F}_s[f(x)]+\alpha f(0) \tag{51}$$

Proof: From the definition we have
$$\mathcal{F}_c[f''(x)]=\int\limits_0^{\infty}f''(x)\cos\alpha x\,dx$$
$$=[f'(x)\cos\alpha x+\alpha f(x)\sin\alpha x]_0^{\infty}-\alpha^2\int\limits_0^{\infty}f(x)\cos\alpha x\,dx$$
$$=-\alpha^2\mathcal{F}_c[f(x)]-f'(0)=-\alpha^2F_c(\alpha)-f'(0) \tag{52}$$
Similarly,
$$\mathcal{F}_s[f''(x)]=\int\limits_0^{\infty}f''(x)\sin\alpha x\,dx$$
$$=[f'(x)\sin\alpha x-\alpha f(x)\cos\alpha x]_0^{\infty}-\alpha^2\int\limits_0^{\infty}f(x)\sin\alpha x\,dx$$
$$=-\alpha^2\mathcal{F}_s[f(x)]+\alpha f(0)=-\alpha^2F_s(\alpha)+\alpha f(0) \tag{53}$$
In the same way it can be verified that
$$\mathcal{F}_c[f'(x)]=\alpha F_s(\alpha)-f(0) \tag{54}$$
and
$$\mathcal{F}_s[f'(x)]=-\alpha F_c(\alpha) \tag{55}$$

— □ —

Fourier Transform of partial derivatives: Let $F(\alpha,t)$ be the Fourier transform of the function $u(x,t)$ with respect to variable x, then
$$\mathcal{F}[u(x,t)]=F(\alpha,t)=\int\limits_{-\infty}^{\infty}u(x,t)e^{-i\alpha x}dx \tag{56}$$

(i) **Fourier transform of $\frac{\partial u}{\partial x}$**: Suppose $u \to 0$ as $x \to \pm\infty$, then we have

$$\mathcal{F}\left[\frac{\partial u}{\partial x}\right] = \int_{-\infty}^{\infty} \frac{\partial u}{\partial x} e^{-i\alpha x}\, dx$$

$$= e^{-i\alpha x} u \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (-i\alpha) e^{-i\alpha x} u\, dx$$

$$= 0 + i\alpha \int_{-\infty}^{\infty} u(x,t) e^{-i\alpha x}\, dx$$

$$= i\alpha\, \mathcal{F}[u(x,t)] = i\alpha\, F(\alpha, t)$$

Hence,
$$\mathcal{F}\left[\frac{\partial u}{\partial x}\right] = i\alpha\, \mathcal{F}\{u(x,t)\} = i\alpha\, F(\alpha, t) \tag{57}$$

(ii) **Fourier transform of $\frac{\partial^2 u}{\partial x^2}$**: If u and $\frac{\partial u}{\partial x} \to 0$ as $x \to \pm\infty$, then integrating by parts as in (i) above, we have

$$\mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}\right] = -\alpha^2\, \mathcal{F}[u(x,t)] = -\alpha^2 F(\alpha, t) \tag{58}$$

In general, we have

$$\mathcal{F}\left[\frac{\partial^n u}{\partial x^n}\right] = (i\alpha)^n\, \mathcal{F}[u(x,t)] = (i\alpha)^n F(\alpha, t) \tag{59}$$

where, $u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \cdots, \frac{\partial^{n-1} u}{\partial x^{n-1}} \to 0$ as $x \to \pm\infty$.

(iii) **Fourier transform of $\frac{\partial u}{\partial t}$**: By definition, Fourier transform of $\frac{\partial u}{\partial t}$ with respect to x is given by

$$\mathcal{F}\left[\frac{\partial u}{\partial t}\right] = \int_{-\infty}^{\infty} \frac{\partial u}{\partial t} e^{-i\alpha x}\, dx = \frac{d}{dt} \int_{-\infty}^{\infty} u(x,t) e^{-i\alpha x}\, dx$$

$$= \frac{d}{dt} \mathcal{F}[u(x,t)] = \frac{d}{dt} F(\alpha, t)$$

Hence,
$$\mathcal{F}\left[\frac{\partial u}{\partial t}\right] = \frac{d}{dt} \mathcal{F}[u(x,t)] = \frac{d}{dt} F(\alpha, t) \tag{60}$$

Note: that here we have written the total derivative, since $\mathcal{F}[u(x,t)]$ depends only on t and not on x.

In the same way it can be established that if $F_s(\alpha, t)$ and $F_c(\alpha, t)$ be the Fourier sine and cosine transform of $u(x,t)$ with respect to x, then for u and $\frac{\partial u}{\partial x} \to 0$ as $x \to \infty$, we have

$$\mathcal{F}_s\left[\frac{\partial^2 u}{\partial x^2}\right] = \alpha[u(x,t)]_{x=0} - \alpha^2 F_s(\alpha, t) \tag{61}$$

and
$$\mathcal{F}_c\left[\frac{\partial^2 u}{\partial x^2}\right] = -\left[\frac{\partial u}{\partial x}\right]_{x=0} - \alpha^2 F_c(\alpha, t) \tag{62}$$

We are leaving the proofs of the above results for you to do them yourself. Before illustrating the properties discussed above we make the following remark.

Remark: You may **notice** here that most of the properties (28), (29), (30), (31), (38), (39), (40), (41) and (47) correspond to the analogous properties of the Laplace transform. Properties (28) and (29) are identical to the corresponding properties of the Laplace transform. The derivative property (41) is similar to the property

$$\mathcal{L}[f^n(x)] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0) \tag{63}$$

but does not include any boundary values analogous to the initial conditions $f(0), \ldots, f^{(n-1)}(0)$ in Eqn.(63), because the boundaries are at $x = \pm\infty$, and it is assumed that $f(x), f'(x), \ldots, f^{(n-1)}(x)$ all tend to zero as $x \to \pm\infty$. The convolution properties (39) and (40) are identical to the corresponding ones for the Laplace transform, but keep in mind that the Fourier and Laplace convolutions are not quite the same. In contrast with the Fourier convolution (38), the Laplace convolution was defined as

$$(f * g)(x) = \int_0^x f(t-\tau) g(\tau)\, d\tau. \tag{64}$$

That is, the integration limits are different. Let us now consider the following examples where we have used the properties of Fourier transforms and the table given in the Appendix to solve the problems.

Example 16: Find $\mathcal{F}^{-1}\left[\dfrac{e^{4i\alpha}}{3+i\alpha}\right]$

Solution: From Example 9, we have $\mathcal{F}[e^{-3x} u_0(x)] = \dfrac{1}{3+i\alpha}$

or, $$\mathcal{F}^{-1}\left[\frac{1}{3+i\alpha}\right] = e^{-3x} u_0(x) = f(x)$$

Using shift property (31), we get

$$\mathcal{F}^{-1}\left[\frac{e^{-(-4i\alpha)}}{3+i\alpha}\right] = f(x-(-4)) = e^{-3(x+4)} u_{-4}(x) = \begin{cases} 0, & x < -4 \\ e^{-3(x+4)}, & x \ge -4 \end{cases}$$

***

Example 17: Given $f(x) = 4e^{-|x|} - 5e^{-3|x+2|}$ evaluate its transform $F(\alpha)$.

Solution: Using the linearity property (28), we have

$$\mathcal{F}[4e^{-|x|} - 5e^{-3|x+2|}] = 4\mathcal{F}\{e^{-|x|}\} - 5\mathcal{F}\{e^{-3|x+2|}\}$$

Entry 4 in Appendix gives

$$\mathcal{F}\{e^{-|x|}\} = \frac{2}{\alpha^2+1}$$ [also see E8)]

and entry 11 (with $a = 3, b = 6$) gives

$$\mathcal{F}\{e^{-3|x+2|}\} = \mathcal{F}\{e^{-|3x+6|}\}$$

$$= \frac{1}{3} e^{i6\alpha/3} \mathcal{F}\{e^{-|x|}\}\Big|_{\alpha \to \alpha/3}$$

$$= \frac{1}{3} e^{2i\alpha} \left(\frac{2}{\alpha^2+1}\right)_{\alpha \to \alpha/3}$$

$$= \frac{1}{3} e^{2i\alpha} \frac{2}{(\alpha/3)^2+1}$$

Thus, we have

$$F(\alpha) = 4\frac{2}{\alpha^2+1} - 5\left(\frac{2}{3}\right)\frac{e^{2i\alpha}}{(\alpha/3)^2+1} = \frac{8}{\alpha^2+1} - \frac{30e^{2i\alpha}}{\alpha^2+9}.$$

***

Example 18: Find $\mathcal{F}[f(x)]$, where $f(x) = x\,e^{-4x^2}$

Solution: We have

$$\mathcal{F}\{e^{-4x^2}\} = \frac{\sqrt{\pi}}{2} e^{-\alpha^2/16}$$ (see Example 10)

Entry (17) with $n = 1$ gives

$$F(\alpha) = i\frac{d}{d\alpha}\left(\frac{\sqrt{\pi}}{2} e^{-\alpha^2/16}\right) = -i\frac{\sqrt{\pi}}{16}\alpha e^{-\alpha^2/16}$$

***

Example 19: Find $\mathcal{F}^{-1}[F(\alpha)] = e^{-2|\alpha|}$.

Solution: Entry 1 with $a = 2$, gives

$$\mathcal{F}^{-1}\left\{\frac{\pi}{2} e^{-2|\alpha|}\right\} = \frac{1}{x^2+4}$$

Using the linearity property (28) with $a = \frac{\pi}{2}$ and $b = 0$, the above equation can be written as

$$\frac{\pi}{2}\mathcal{F}^{-1}\{e^{-2|\alpha|}\} = \frac{1}{x^2+4}$$

or, $$\mathcal{F}^{-1}\{e^{-2|\alpha|}\} = \frac{2}{\pi}\frac{1}{x^2+4}$$

is the desired inverse function.

***

Example 20: Evaluate $\mathcal{F}^{-1}\left\{\frac{1}{\alpha^2+4\alpha+13}\right\}$.

Solution: Completing the square, we write

$\alpha^2 + 4\alpha = (\alpha^2 + 4\alpha + 4) - 4 = (\alpha+2)^2 - 4$

$\therefore$ given problem reduces to

$$\mathcal{F}^{-1}\left\{\frac{1}{(\alpha+2)^2+9}\right\}$$

Entry 4, with $a = 3$ and linearity gives

$$\mathcal{F}^{-1}\left\{\frac{1}{\alpha^2+9}\right\} = \frac{1}{6}e^{-3|x|}$$

Now using entry 12 with $a = 1$ and $b = 2$, we have the desired result

$$\mathcal{F}^{-1}\left\{\frac{1}{(\alpha+2)^2+9}\right\} = e^{-i2x}\left(\frac{1}{6}e^{-3|x|}\right)$$

Alternatively, we could have used partial fractions, as follows

$$\frac{1}{\alpha^2+4\alpha+13} = \frac{1}{(\alpha+2-2i)\,(\alpha+2+3i)}$$

$$= \frac{1}{6i}\frac{1}{\alpha+2-3i} - \frac{1}{6i}\frac{1}{\alpha+2+3i}$$

$$= \frac{1}{6}\frac{1}{i\alpha+(3+2i)} - \frac{1}{6}\frac{1}{i\alpha+(-3+2i)}$$

$$= \frac{1}{6}\frac{1}{i\alpha(3+2i)} + \frac{1}{6}\frac{1}{-i\alpha+(3-2i)}$$

Now using the linearity property with entries 2 and 3 of the Appendix, we get for unit step function $H(x)$.

$$\mathcal{F}^{-1}\left\{\frac{1}{\alpha^2+4\alpha+13}\right\}=\frac{1}{6}H(x)e^{-(3+2i)x}+\frac{1}{6}H(-x)e^{(3-2i)x} \qquad \text{(also see Example 9)}$$

$$=\frac{1}{6}e^{-2ix}[H(x)e^{-3x}+H(-x)e^{3x}]$$

$$=\frac{1}{6}e^{-2ix}\,e^{-3|x|}$$

***

Example 21: Find $\mathcal{F}^{-1}\left[\dfrac{1}{\alpha^2+1}\right]$.

Solution: We have

$$\frac{1}{\alpha^2+1}=\frac{1}{(\alpha+i)(\alpha-i)}=\frac{1}{(1-i\alpha)(1+i\alpha)}$$

Using entries 2 and 3, respectively

$$\mathcal{F}^{-1}\left\{\frac{1}{1-i\alpha}\right\}=H(-x)e^{x}$$

$$\mathcal{F}^{-1}\left\{\frac{1}{1+i\alpha}\right\}=H(x)e^{-x}$$

Then the Fourier convolution formula (40) gives

$$\mathcal{F}^{-1}\left\{\frac{1}{\alpha^2+1}\right\}=\left[H(-x)e^{x}\right]*\left[H(x)e^{-x}\right]$$

$$=\int_{-\infty}^{\infty}H(-x+\tau)e^{x-\tau}H(\tau)e^{-\tau}d\tau$$

$$=\begin{cases} e^{x}\displaystyle\int_{x}^{\infty}e^{-2\tau}d\tau\,, & x>0 \qquad [H(\tau-x)H(\tau) \text{ is } H(\tau-x), \text{if } x>0] \\ e^{x}\displaystyle\int_{0}^{\infty}e^{-2\tau}d\tau\,, & x<0 \qquad [H(\tau-x)\,H(\tau) \text{ is } H(\tau), \text{if } x<0] \end{cases}$$

$$=\begin{cases} \dfrac{1}{2}e^{-x}\,, & x>0 \\ \dfrac{1}{2}e^{x}\,, & x<0 \end{cases}$$

$$=\frac{1}{2}e^{-|x|}$$

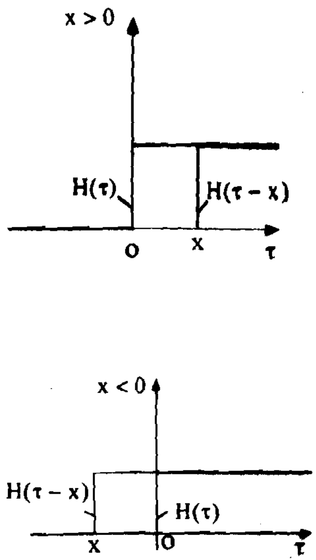

Fig.5 $H(\tau-x)$ **and** $H(\tau)$

Graphs of $H(\tau-x)$ and $H(\tau)$ both for $x>0$ and $x<0$ are shown in Fig.5.

***

Example 22: Find the solution of the differential equation

$$y'-4y=H(x)e^{-4x}\,, \quad -\infty<x<\infty$$

Using Fourier transform, where $H(x)=u_0(x)$ is the unit step function.

Solution: Applying the Fourier transform to the given differential equation, we obtain

$$\mathcal{F}[y']-4\mathcal{F}[y]=\mathcal{F}[H(x)e^{-4x}]$$

or, $\quad i\alpha\,F(\alpha)-4F(\alpha)=\dfrac{1}{4+i\alpha}$ $\qquad$ [using entry 2] or, $F(\alpha) = -\dfrac{1}{(4+i\alpha)(4-i\alpha)} = -\dfrac{1}{16+\alpha^2}$

where, $\mathcal{F}[y(x)] = F(\alpha)$

Therefore, $y(x) = \mathcal{F}^{-1}\left[-\dfrac{1}{16+\alpha^2}\right] = -\dfrac{1}{8}e^{-4|x|}$ [using entry 4]

The solution can be also be written as

$$y(x) = \begin{cases} -\dfrac{1}{8}e^{4x}, & x<0 \\ -\dfrac{1}{8}e^{-4x}, & x>0 \end{cases}$$

***

Example 23: Using the formulas for derivatives find the Fourier cosine and sine transform of

$$f(x) = e^{-ax}, \quad x \geq 0, \; a>0.$$

Solution: Given $f(x) = e^{-ax}$, we have $f'(x) = -ae^{-ax}$ and $f''(x) = a^2e^{-ax}$. If $\mathcal{F}_c[f(x)] = F_c(\alpha)$, then we have

$$\mathcal{F}_c[f''(x)] = \mathcal{F}_c[a^2e^{-ax}] = a^2\,\mathcal{F}_c[e^{-ax}] = a^2F_c(\alpha)$$

also, $\mathcal{F}_c[f''(x)] = -\alpha^2F_c(\alpha) - f'(0) = -\alpha^2F_c(\alpha) + a$ [using Eqn.(51)]

Therefore, $a^2F_c(\alpha) = -\alpha^2F_c(\alpha) + a$, or $F_c(\alpha) = \dfrac{a}{a^2+\alpha^2}$

Similarly, we have

$$\mathcal{F}_s[f''(x)] = a^2\,\mathcal{F}_s[e^{-at}] = a^2F_s(\alpha)$$

and $\mathcal{F}_s[f''(x)] = -\alpha^2F_s(\alpha) + \alpha f(0) = -\alpha^2F_s(\alpha) + \alpha$ [using Eqn.(51)]

therefore, $a^2F_s(\alpha) = -\alpha^2F_s(\alpha) + \alpha$, or, $F_s(\alpha) = \dfrac{\alpha}{a^2+\alpha^2}$

***

You may now try the following exercises.

E16) Prove the results given by Eqns.(61) and (62).

E17) Using the table given in the Appendix, or otherwise, evaluate the following

a) $\mathcal{F}\{4x^2e^{-3|x|}\}$ b) $\mathcal{F}\left\{\dfrac{\cos 3x}{x^2+2}\right\}$

c) $\mathcal{F}\left\{\dfrac{3}{2x^2+1} - 5e^{-|x|}\right\}$ d) $\mathcal{F}^{-1}\left\{\dfrac{4\sin\alpha}{\alpha} - \dfrac{1}{\sqrt{|\alpha|}}\right\}$

e) $\mathcal{F}^{-1}\left\{\dfrac{9}{2\alpha+i}\right\}$ f) $\mathcal{F}^{-1}\{e^{-|\alpha|}\cos\alpha\}$

g) $\mathcal{F}^{-1}\left\{\dfrac{1}{\alpha^2+i\alpha+2}\right\}$.

E18) Given $F(\alpha)$ find the inverse function $f(x)$, where $\mathcal{F}[f(x)] = F(\alpha)$.

a) $e^{-a|\alpha|}$, $a>0$ b) $H(\alpha+a) - H(\alpha-a)$ $(a>0)$.

E19) Using the convolution theorem find the inverse of the following functions

a) $\dfrac{1}{(i\alpha+k)^2}$, $k>0$ b) $\dfrac{1}{(i\alpha+k)^3}$, $k>0$.

E20) Using Fourier transform, find the solution of the following differential equations

a) $y' + 3y = H(x)e^{-2x}$, $-\infty < x < \infty$

b) $y'' + 3y' + 2y = \delta(x-3)$.

In many applications, we have to deal with problems defined on finite intervals. In such cases we define the finite Fourier sine and cosine transforms.

7.3.2 Finite Fourier Transforms

The finite Fourier sine and cosine transforms are obtained from the Fourier sine and cosine series.

Finite Fourier Cosine Transform: You know that if a function $f(x)$ is defined in the interval $0 \le x \le L$ and is piecewise continuous then half range cosine series of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} \tag{65}$$

where, $$a_0 = \frac{2}{L}\int_0^L f(x)\,dx \text{ and } a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx \tag{66}$$

The **finite Fourier cosine transform** of $f(x)$ is defined as

$$\mathbf{F_c(n) = \int_0^L f(x)\cos\frac{n\pi x}{L}dx} \tag{67}$$

where n is a non-negative integer.

The function $f(x)$ is then called the **inverse finite Fourier cosine transform** of $F_c(n)$ and is given by

$$\mathbf{f(x) = \frac{1}{L}F_c(0) + \frac{2}{L}\sum_{n=1}^{\infty} F_c(n)\cos\frac{n\pi x}{L}} \tag{68}$$

Note: From Eqn.(66), we have

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx = \frac{2}{L}F_c(n) \tag{69}$$

and $$a_0 = \frac{2}{L}\int_0^L f(x)dx = \frac{2}{L}F_c(0) \text{ or, } \frac{a_0}{2} = \frac{1}{L}F_c(0) \tag{70}$$

on substituting Eqns.(69) and (70) in Eqn.(65) we obtain result(68) for inverse finite Fourier cosine transform.

Finite Fourier Sine Transform: If a piecewise continuous function $f(x)$ is defined in the interval $0 \le x \le L$, then half range sine series is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \tag{71}$$

where, $$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx. \tag{72}$$

The **finite Fourier sine transform** is defined as

$$\mathbf{F_s(n) = \int_0^L f(x)\sin\frac{n\pi x}{L}dx} \tag{73}$$

where n is a non-negative integer.

The function $f(x)$ is then called the **inverse finite Fourier sine transform** of $F_s(n)$ and is given by

$$f(x)=\frac{2}{L}\sum_{n=1}^{\infty} F_s(n)\sin\frac{n\pi x}{L} \tag{74}$$

From Eqn.(72), we have

$$b_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx=\frac{2}{L}F_s(n) \tag{75}$$

and on substituting b_n from Eqn.(75) in Eqn.(71), result (74) is obtained.

Remark: The finite Fourier cosine and sine transforms can also be denoted by $C_n(f)$ and $S_n(f)$ respectively.

Formula for the second derivative: We assume that $f(x)$ and $f'(x)$ are continuous and $f''(x)$ is piecewise continuous on $[0,L]$. Then

$$C_n[f''(x)]=-\frac{n^2\pi^2}{L^2}F_c(n)-f'(0)+(-1)^n f'(L),\ n=1,2,\ldots.$$

Proof: Using the definition, we have

$$C_n[f''(x)]=\int_0^L f''(x)\cos\frac{n\pi x}{L}dx$$

$$=\left[f'(x)\cos\frac{n\pi x}{L}+\frac{n\pi}{L}f(x)\sin\frac{n\pi x}{L}\right]_0^L-\frac{n^2\pi^2}{L^2}\int_0^L f(x)\cos\frac{n\pi x}{L}dx$$

$$\therefore\quad C_n[f''(x)]=-\frac{n^2\pi^2}{L^2}F_c(n)-f'(0)+(-1)^n f'(L) \tag{76}$$

Similarly, using the definition of finite Fourier sine transform it can be proved that

$$S_n[f''(x)]=-\frac{n^2\pi^2}{L^2}F_s(n)+\frac{n\pi}{L}f(0)-\frac{n\pi}{L}(-1)^n f(L) \tag{77}$$

We now illustrate the application of finite Fourier transform through examples.

Example 24: Find the finite Fourier sine and cosine transforms of the following function

$$f(x)=\begin{cases}kx & ,\quad 0\le x\le \pi/2\\ k(\pi-x), & \pi/2\le x\le\pi\end{cases}$$

Solution: Using the result (67), we have

$$F_c(n)=\int_0^L f(x)\cos\frac{n\pi x}{L}dx=\int_0^\pi f(x)\cos nx\,dx \qquad (\because L=\pi)$$

$$=\int_0^{\pi/2} kx\cos nx\,dx+\int_{\pi/2}^{\pi} k(\pi-x)\cos nx\,dx$$

$$=k\left[x\left(\frac{\sin nx}{n}\right)-\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi/2}+k\left[(\pi-x)\left(\frac{\sin nx}{n}\right)-(-1)\left(\frac{\cos nx}{n^2}\right)\right]_{\pi/2}^{\pi}$$

$$=k\left[\left(\frac{\pi}{2}\frac{\sin n\pi/2}{n}+\frac{\cos n\pi/2}{n^2}-\frac{1}{n^2}\right)+\left(-\frac{\cos n\pi}{n^2}-\frac{\pi}{2}\frac{\sin n\pi/2}{n}+\frac{\cos n\pi/2}{n^2}\right)\right]$$

$$=k\left(2\frac{\cos n\pi/2}{n^2}+\frac{(-1)^n}{n^2}-\frac{1}{n^2}\right)$$

at $n=0$, $F_c(n)$ becomes indeterminate, hence we evaluate $F_c(0)$ separately thus

$$F_c(0)=\int_0^L f(x)\,dx=\int_0^{\pi/2} kx\,dx+\int_{\pi/2}^{\pi} k(\pi-x)\,dx$$

$$=k\left[\frac{x^2}{2}\right]_0^{\pi/2}+k\left[\pi x-\frac{x^2}{2}\right]_{\pi/2}^{\pi}$$

$$=k\left(\frac{\pi^2}{8}+\frac{\pi^2}{8}\right)=\frac{k\pi^2}{4}$$

Hence the finite Fourier cosine transform is

$$F_c(n)=k\left[2\frac{\cos n\pi/2}{n^2}+\frac{(-1)^n}{n^2}-\frac{1}{n^2}\right] \text{ and } F_c(0)=\frac{k\pi^2}{4}$$

For finding finite Fourier sine transform we use result (73) and obtain

$$F_s(n)=\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx=\int_0^{\pi} f(x)\sin nx\,dx$$

$$=\int_0^{\pi/2} kx\sin nx\,dx+\int_{\pi/2}^{\pi} k(\pi-x)\sin nx\,dx$$

$$=k\left[x\left(\frac{-\cos nx}{n}\right)+\frac{\sin nx}{n^2}\right]_0^{\pi/2}+k\left[(\pi-x)\left(\frac{-\cos nx}{n}\right)-\left(\frac{\sin nx}{n^2}\right)\right]_{\pi/2}^{\pi}$$

$$=k\left[\frac{-\pi}{2}\frac{\cos n\pi/2}{n}+\frac{\sin n\pi/2}{n^2}+\frac{\pi}{2}\frac{\cos n\pi/2}{n}+\frac{\sin n\pi/2}{n^2}\right]$$

$$=\frac{2k\sin n\pi/2}{n^2}$$

Hence, the finite Fourier sine transform is

$$F_s(n)=\frac{2k\sin n\pi/2}{n^2}.$$

***

Example 25: If $f(x)=\sin kx$, where $0\le x\le\pi$ and k is a positive integer, then show that

$$F_s(n)=0,\text{ if } n\ne k \text{ and } F_s(n)=\frac{\pi}{2},\text{ if } n=k.$$

Solution: Using result (73), we have

$$F_s(n)=\int_0^{\pi}\sin kx\sin nx\,dx$$

$$=\frac{1}{2}\int_0^{\pi}[\cos(n-k)x-\cos(n+k)x]dx$$

$$=\frac{1}{2}\left[\frac{\sin(n-k)x}{n-k}-\frac{\sin(n+k)x}{n+k}\right]_0^{\pi}$$

$$=0,\text{ if } n\ne k \qquad \text{[as } (n-k),(n+k) \text{ are integers]}$$

When $n=k$, the first term of the above expression becomes indeterminate. We evaluate $F_s(n)$ directly by putting $n=k$. Thus,

$$F_s(n)=\int_0^{\pi}\sin kx\sin kx\,dx=\int_0^{\pi}\sin^2 kx\,dx$$

$$=\int_0^{\pi}\left(\frac{1-\cos 2kx}{2}\right)dx=\frac{1}{2}\left[x-\frac{\sin 2kx}{2k}\right]_0^{\pi}$$

$$=\pi/2.$$

***

Example 26: Find $f(x)$, if $F_c(n) = \dfrac{\cos 2n\pi/3}{(2n+1)^2}$ where, $0 \le x \le 1$.

Solution: Since finite Fourier cosine transform is given, we use result (68) to find the inverse. Thus

$$f(x) = \frac{1}{L}F_c(0) + \frac{2}{L}\sum_{n=1}^{\infty} F_c(n)\cos\frac{n\pi x}{L}$$

$$= \frac{1}{1}(1) + \frac{2}{1}\sum_{n=1}^{\infty} \frac{\cos 2n\pi/3}{(2n+1)^2}\cos\frac{n\pi x}{1} \qquad [\because L = 1 \text{ and } F_c(0) = 1]$$

$$= 1 + 2\sum_{n=1}^{\infty} \frac{\cos 2n\pi/3}{(2n+1)^2}\cos n\pi x.$$

***

You may now try the following exercises.

E21) Find the finite Fourier sine and cosine transforms of the following functions:

a) $f(x) = \ell x - x^2, \quad 0 \le x \le \ell$

b) $f(x) = \begin{cases} \pi x &, \ 0 \le x \le 1 \\ \pi(2-x), & 1 \le x \le 2 \end{cases}$

c) $f(x) = \begin{cases} \pi/3 &, \quad 0 \le x \le \pi/3 \\ 0 &, \quad \pi/3 \le x \le 2\pi/3 \\ -\pi/3 &, \quad 2\pi/3 \le x \le \pi \end{cases}$

d) $f(x) = \left(1 - \dfrac{x}{\pi}\right)^2, \ 0 \le x \le \pi.$

E22) Find $f(x)$, if

a) $F_s(n) = \dfrac{2\ell}{n\pi}\sin^2\dfrac{n\pi}{4}, \quad 0 \le x \le \ell.$

b) $F_c(n) = \dfrac{c\ell}{n\pi}\sin\dfrac{n\pi a}{\ell}$ and $F_c(0) = ac$, where $0 \le x \le \ell$.

c) $F_s(n) = \dfrac{1}{n^2}\sin\dfrac{n\pi}{3}, \quad 0 \le x \le \pi.$

Fourier transform method has become a powerful tool in diverse fields of science and engineering. It can provide a means of solving equations that describe dynamic responses to electricity, heat or light and can also identify the regular contributions to a fluctuating signal, useful in making sense of observations in astronomy, medicine and chemistry. Fourier transforms become indispensable in the numerical calculations needed to design electrical circuits, to analyse mechanical vibrations, and to study wave propagation. They are widely used to solve problems involving semi-infinite or totally infinite range of the variables. In the next section, we deal with some applications of Fourier transforms to diffusion, wave and Laplace equations.

7.4 APPLICATIONS OF FOURIER TRANSFORMS TO BOUNDARY VALUE PROBLEMS

Fourier transforms are of great importance in solving boundary value problems and form a basis of powerful method for the solution of partial differential equations (pde). For solving boundary value problems, we take Fourier transform of a given pde using given boundary and initial conditions. The required solution is then obtained by taking corresponding inverse transform. The choice of particular transform to be employed for the solution of an equation depends on the boundary conditions of the problem.

We use the following notations for different types of Fourier transform and corresponding inverse transform

For the interval $-\infty < x < \infty$ **:** We use infinite Fourier transform

$$\mathcal{F}[u(x,t)] = \bar{u}(\alpha,t) = \int_{-\infty}^{\infty} u(x,t)e^{-i\alpha x}\,dx \tag{78}$$

and corresponding inverse transform is given by

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{u}(\alpha,t)e^{i\alpha x}\,d\alpha \tag{79}$$

For the interval $0 < x < \infty$ **:** We use Fourier sine or cosine transforms depending upon the boundary conditions of the problem.

Fourier sine transform:

$$\mathcal{F}_s[u(x,t)] = \bar{u}_s(\alpha,t) = \int_0^{\infty} u(x,t)\sin\alpha x\,dx \tag{80}$$

and the inverse sine transform is

$$u(x,t) = \frac{2}{\pi}\int_0^{\infty} \bar{u}_s(\alpha,t)\sin\alpha x\,d\alpha \tag{81}$$

Fourier cosine transform:

$$\mathcal{F}_s[u(x,t)] = \bar{u}_c(\alpha,t) = \int_0^{\infty} u(x,t)\cos\alpha x\,dx \tag{82}$$

and the inverse cosine transform is

$$u(x,t) = \frac{2}{\pi}\int_0^{\infty} \bar{u}_c(\alpha,t)\cos\alpha x\,d\alpha \tag{83}$$

For the interval $0 < x < L$ **:** We use finite Fourier sine or cosine transforms depending upon the boundary conditions of the problem.

Finite Fourier sine transform:

$$\mathcal{F}_s[u(x,t) = \bar{u}_s(n,t) = \int_0^{L} u(x,t)\sin\frac{n\pi x}{L}\,dx \tag{84}$$

and inverse finite Fourier sine transform is

$$u(x,t) = \frac{2}{L}\sum_{n=1}^{\infty} \bar{u}_s(n,t)\sin\frac{n\pi x}{L} \tag{85}$$

Finite Fourier cosine transform:

$$\mathcal{F}_c[u(x,t) = \bar{u}_c(n,t) = \int_0^{L} u(x,t)\cos\frac{n\pi x}{L}\,dx \tag{86}$$

and inverse finite Fourier cosine transform is

$$u(x,t) = \frac{1}{L}\bar{u}_c(0,t) + \frac{2}{L}\sum_{n=1}^{\infty} \bar{u}_c(n,t)\cos\frac{n\pi x}{L}\,dx \tag{87}$$

The choice of Fourier transform:

Depending on boundary conditions, we select the Fourier transform to be employed for the solution of pdes.

- **If the interval is** $-\infty < x < \infty$ and if the boundary conditions are $u \to 0$ and $\frac{\partial u}{\partial x} \to 0$ as $x \to \pm\infty$ we use infinite Fourier transform.
- **If the interval is** $0 < x < \infty$ and if the boundary conditions are

(i) u and $\frac{\partial u}{\partial x} \to 0$ as $x \to \infty$ and $u(x, t) = 0$ or, $f(t)$ at $x = 0, \forall t$, we use Fourier sine transform

(ii) u and $\frac{\partial u}{\partial x} \to 0$ as $x \to \infty$ and $\frac{\partial u}{\partial x} = 0$ or $f(t)$ at $x = 0, \forall t$, we use Fourier cosine transform.

Note: For the interval $0 < x < \infty$, we always assume u and $\frac{\partial u}{\partial x} \to 0$ as $x \to \infty$, even if it is not given in the problem.

- **If the interval is $0 < x < L$** and if the boundary conditions are
  (i) $u(0,t) = u(L,t) = 0 \; \forall t$, we use finite Fourier sine transform
  (ii) $\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0 \; \forall t$, we use finite Fourier cosine transform.

We now take up illustrations on solutions of boundary value problems.

Diffusion Equation

Let us consider the problem of flow of heat in an infinite medium $-\infty < x < \infty$, when the initial temperature distribution $f(x)$ is known and no heat sources are present.

Example 27: Use Fourier transforms to solve the boundary value problem

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0$$

subject to the conditions

(i) $u, \frac{\partial u}{\partial x} \to 0$ as $x \to \pm\infty$

(ii) $u(x,0) = f(x)$.

Solution: Since the interval is $-\infty < x < \infty$, we use infinite Fourier transform. Taking the infinite Fourier transform with respect to x of both sides of the given pde, we have

$$\mathcal{F}\left[\frac{\partial u}{\partial t}\right] = k\,\mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}\right] \tag{88}$$

Let $\bar{u}(\alpha, t)$ be the infinite Fourier transform of $u(x,t)$, we then have from result (78)

$$\bar{u}(\alpha, t) = \int_{-\infty}^{\infty} u(x,t)e^{-i\alpha x}\,dx$$

Now, $\mathcal{F}\left[\frac{\partial u}{\partial t}\right] = \int_{-\infty}^{\infty} \frac{\partial u}{\partial t} e^{-i\alpha x}\,dx = \frac{d}{dt}\int_{-\infty}^{\infty} u(x,t)e^{-i\alpha x}\,dx$

$$\therefore \mathcal{F}\left[\frac{\partial u}{\partial t}\right] = \frac{d\bar{u}}{dt} \qquad \text{[by result (60)]} \tag{89}$$

Similarly, $\mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}\right] = \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2} e^{-i\alpha x}\,dx = -\alpha^2\bar{u}(\alpha, t)$ [by result (58)] (90)

Substituting from Eqns.(89) and (90) in Eqn.(88), we get

$$\frac{d\bar{u}}{dt} = -k\alpha^2\bar{u}$$

The solution of this ordinary differential equation is

$$\bar{u}(\alpha, t) = A\,e^{-k\alpha^2 t} \tag{91}$$

To find the arbitrary constant A, we take infinite Fourier transform of initial condition (ii), then

$$\bar{u}(\alpha,0)=\int_{-\infty}^{\infty}u(x,0)e^{-i\alpha x}\,dx=\int_{-\infty}^{\infty}f(x)e^{-i\alpha x}\,dx$$

$$\therefore \bar{u}(\alpha,0)=F(\alpha) \qquad (92)$$

Now if we put $t=0$ in Eqn.(91) and use result (92), we obtain

$$\bar{u}(\alpha,0)=A=F(\alpha)$$

Thus,

$$\bar{u}(\alpha,t)=F(\alpha)e^{-k\alpha^2 t}$$

Taking the inverse Fourier transform of the above equation, we have

$$u(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\alpha)e^{-k\alpha^2 t}e^{i\alpha x}\,d\alpha$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}f(u)e^{-i\alpha u}\,du\right)e^{-k\alpha^2 t}e^{i\alpha x}\,d\alpha$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(u)e^{-k\alpha^2 t}e^{i\alpha(x-u)}\,du\,d\alpha$$

which is the required solution. This can be further simplified as

$$u(x,t)=\frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty}f(u)e^{-k\alpha^2 t}\cos\alpha(x-u)\,du\,d\alpha$$

$$=\frac{1}{\pi}\int_{-\infty}^{\infty}f(u)\,du\int_{0}^{\infty}e^{-k\alpha^2 t}\cos\alpha(x-u)\,d\alpha$$

Using result $\int_0^{\infty}e^{-a\alpha^2}\cos 2b\alpha\,d\alpha=\frac{1}{2}\sqrt{\frac{\pi}{a}}\,e^{b^2/a}$ with $a=kt$ and $b=\frac{x-u}{2}$, we have

$$u(x,t)=\frac{1}{2\sqrt{k\pi t}}\int_{-\infty}^{\infty}f(u)e^{\frac{(x-u)^2}{4kt}}\,du$$

$$=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}f(x+2\sqrt{kt}\,v)e^{-v^2}\,dv \qquad \left\{\text{putting } v=\frac{u-x}{2\sqrt{kt}}\right\}$$

***

Remark: The conditions $\frac{\partial u}{\partial x}\to 0$ as $x\to\pm\infty$ represents that temperature $u(x,t)$ must be bounded. This condition can also be expressed as $|u(x,t)|<M$.

We now take up an example on heat-flow in semi-infinite bar.

Example 28: Determine the distribution of temperature in the semi-infinite medium $x\geq 0$, when the end $x=0$ is maintained at zero temperature and the initial distribution of temperature is $f(x)$.

Solution: Let $u(x,t)$ be the temperature at any point x at any time t, we have to solve the heat flow equation

$$\frac{\partial u}{\partial t}=c^2\frac{\partial^2 u}{\partial x^2}, \quad 0<x<\infty \ , t>0$$

subject to the initial condition

$$u(x,0)=f(x), \quad x>0$$

and the boundary condition

$$u(0,t)=0, \quad t>0$$

u and $\dfrac{\partial u}{\partial x} \to 0$ as $x \to \infty \ \forall \ t$.

Since the interval is $0 < x < \infty$ and $u(x,t)$ at $x = 0$ is given, we use Fourier sine transform given by Eqn.(80).

Taking Fourier sine transform with respect to x of both sides of the given partial differential equation, we have

$$\mathcal{F}_s\left[\frac{\partial u}{\partial t}\right] = c^2 \ \mathcal{F}_s\left[\frac{\partial^2 u}{\partial x^2}\right] \tag{93}$$

Now $\mathcal{F}_s\left[\dfrac{\partial u}{\partial t}\right] = \displaystyle\int_0^\infty \frac{\partial u}{\partial t} \sin \alpha x \, dx = \frac{d}{dt}\int_0^\infty u(x,t) \sin \alpha x \, dx$

$$= \frac{d\bar{u}_s(x,t)}{dt} \tag{94}$$

and $\quad \mathcal{F}_s\left[\dfrac{\partial^2 u}{\partial x^2}\right] = \displaystyle\int_0^\infty \frac{\partial^2 u}{\partial x^2} \sin \alpha x \, dx$

$$= \alpha[u(x,t)]_{x=0} - \alpha^2 \bar{u}_s(\alpha,t) \qquad \text{(using result (61))}$$

$$= -\alpha^2 \bar{u}_s(\alpha,t) \tag{95}$$

Substituting from Eqns.(94) and (95) in Eqn.(93), we have

$$\frac{d\bar{u}_s}{dt} = -c^2\alpha^2\bar{u}_s$$

Solution of this equation is

$$\bar{u}_s(\alpha,t) = A\, e^{-c^2\alpha^2 t} \tag{96}$$

Now we take Fourier sine transform of initial condition, i.e.

$$\mathcal{F}_s[u(x,0)] = u_s(\alpha,0) = \int_0^\infty u(x,0) \sin \alpha x \, dx$$

$$= \int_0^\infty f(x) \sin \alpha x \, dx$$

$$= F_s(\alpha) \tag{97}$$

Putting $t = 0$ in Eqn.(96) and using Eqn.(97) we obtain

$$\bar{u}_s(\alpha,0) = A = F_s(\alpha)$$

The solution (96) thus reduces to

$$\bar{u}_s(\alpha,t) = F_s(\alpha) e^{-c^2\alpha^2 t}$$

and its inverse Fourier sine transform yields,

$$u(x,t) = \frac{2}{\pi}\int_0^\infty F_s(\alpha) e^{-c^2\alpha^2 t} \sin \alpha x \, d\alpha$$

$$= \frac{2}{\pi}\int_0^\infty \int_0^\infty f(u) e^{-c^2\alpha^2 t} \sin \alpha u \sin \alpha x \, du \, d\alpha$$

which is the required solution.

Using the identity $\sin \alpha x \sin \alpha u = \frac{1}{2}[\cos\alpha(u - x) - \cos\alpha(u + x)]$ and the result

$\displaystyle\int_0^\infty e^{-a\alpha^2} \cos b\alpha \, d\alpha = \frac{1}{2}\sqrt{\frac{\pi}{a}}\, e^{b^2/4a}$, we can write the above solution(on substituting

$\dfrac{u - x}{2c\sqrt{t}} = v$ in the first integral and $\dfrac{u + x}{2c\sqrt{t}} = v$ in the second integral) as

$$u(x,t)=\frac{1}{\sqrt{\pi}}\left[\int_{\frac{-x}{2c\sqrt{t}}}^{\infty} e^{-v^2} f(x+2c\sqrt{t}\,v)\,dv-\int_{\frac{x}{2c\sqrt{t}}}^{\infty} e^{-v^2} f(2c\sqrt{t}\,v-x)\,dv\right]$$

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Wave Equation

Wave motion that occurs in nature, viz sound waves, surface waves, transverse waves of an infinite string, and of mechanical systems are governed by the wave equation. We now illustrate the application of Fourier transform to wave equation.

Example 29: Find the solution of the wave equation

$$\frac{\partial^2 y}{\partial t^2}=c^2\frac{\partial^2 y}{\partial x^2}$$

subject to the conditions

(i) $y(0,t)=y(\pi,t)=0$

(ii) $y(x,0)=f(x)$

(iii) $\left.\frac{\partial y}{\partial t}\right)_{t=0}=g(x)$

Solution: Since the boundary conditions are $y(0,t)=y(\pi,t)=0$, we take finite Fourier sine transform with respect to x of both sides of the given equation. We get

$$\mathcal{F}_s\left[\frac{\partial^2 y}{\partial t^2}\right]=c^2\ \mathcal{F}_s\left[\frac{\partial^2 y}{\partial x^2}\right]$$

Using formula (84) with $L=\pi$, we get

$$\bar{y}_s(n,t)=\int_0^{\pi} y(x,t)\sin nx\,dx$$

$$\therefore \int_0^{\pi}\frac{\partial^2 y}{\partial t^2}\sin nx\,dx=c^2\int_0^{\pi}\frac{\partial^2 y}{\partial x^2}\sin nx\,dx$$

or, $$\frac{d^2}{dt^2}\int_0^{\pi} y(x,t)\sin nx\,dx=c^2\left[\left(\frac{\partial y}{\partial x}\sin nx\right)_0^{\pi}-n\int_0^{\pi}\frac{\partial y}{\partial x}\cos nx\,dx\right]$$

$$=c^2\left[0-n\left\{(y\cos nx)_0^{\pi}+n\int_0^{\pi} y(x,t)\sin nxdx\right\}\right]$$

$$=-c^2n^2\bar{y}_s$$

or, $$\frac{d^2\bar{y}_s}{dt^2}=-c^2n^2\bar{y}_s \qquad (98)$$

Solution of Eqn.(98) is

$$\bar{y}_s(n,t)=A(n)\cos cnt+B(n)\sin cnt \qquad (99)$$

Now taking the finite Fourier sine transform of conditions (ii) and (iii), we get

$$\bar{y}_s(n,0)=\int_0^{\pi} f(x)\sin nx\,dx=F_s[f(n)] \qquad (100)$$

$$\left.\frac{d\bar{y}_s}{dt}\right)_{t=0}=\int_0^{\pi} g(x)\sin nx\,dx=F_s[g(n)] \qquad (101)$$

From Eqn.(99), we have

$$\frac{d\bar{y}_s}{dt}=-A(n)(cn\sin cnt)+B(n)(cn\cos cnt) \qquad (102)$$

Putting $t=0$ in Eqn.(99) and (102) and using Eqns.(100) and (101), we obtain

$$\bar{y}_s(n,0) = A(n) = F_s[f(n)]$$

$$\frac{d\bar{y}_s(n,0)}{dt} = cnB(n) = F_s[g(n)]$$

Hence, on substituting the above values of $A(n)$ and $B(n)$ in Eqn.(99), we obtain

$$\bar{y}_s(n,t) = F_s[f(n)]\cos cnt + \frac{1}{cn}F_s[g(n)]\sin cnt$$

Taking inverse Fourier transform, we have

$$y(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty}[F_s[f(n)]\cos cnt + \frac{1}{cn}F_s[g(n)]\sin cnt]\sin nx$$

using results (100) and (101), we have

$$y(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\left\{\int_0^{\pi} f(u)\sin nu\,du\right\}\cos cnt + \frac{1}{cn}\left\{\int_0^{\pi} g(u)\sin nu\,du\right\}\sin cnt\right]\sin nx$$

which is the required solution.

***

Laplace Equation

Steady flow of current in solid conductors, the velocity potential of inviscid irrotational fluids etc are governed by Laplace equation. We shall now consider a few related examples.

Example 30: Find the steady state temperature distribution $u(x,y)$ in a long square bar of side π with one face maintained at constant temperature u_0 and the other faces at zero temperature.

Solution: Mathematically, the given problem is described by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < \pi, \quad 0 < y < \pi$$

subject to the boundary conditions

$$u(0,y) = u(\pi,y) = 0$$

$$u(x,0) = u(x,\pi) = u_0$$

Taking the finite Fourier sine transform with respect to the variable x, we get

$$\int_0^{\pi}\frac{\partial^2 u}{\partial x^2}\sin nx\,dx + \int_0^{\pi}\frac{\partial^2 u}{\partial y^2}\sin nx\,dx = 0$$

which after using the conditions $u(0,y) = u(\pi,y) = 0$, yields

$$\frac{d^2\bar{u}_s}{dy^2} - n^2\bar{u}_s = 0 \tag{103}$$

General solution of Eqn.(103) is given by

$$\bar{u}_s(n,y) = A(n)\cosh ny + B(n)\sinh ny \tag{104}$$

Taking the finite Fourier sine transform of the second set of boundary conditions, we have

$$\bar{u}_s(n,0) = 0$$

and $$\bar{u}_s(n,\pi) = \int_0^{\pi} u_0\sin nx\,dx = u_0\left(\frac{1-\cos n\pi}{n}\right)$$

Using these in Eqn.(104), we get

$$A = 0, B = u_0\frac{(1-\cos n\pi)}{n\sinh n\pi}.$$

Hence, $$\bar{u}_s(n,\pi) = \frac{u_0}{\sinh n\pi}\left(\frac{1-\cos n\pi}{n}\right)\sinh ny$$

Finally, taking the inverse finite Fourier sine transform, we obtain

$$u(x,y)=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{u_0}{\sinh n\pi}\left(\frac{1-\cos n\pi}{n}\right)\sinh ny\sin nx$$

Thus the required temperature distribution is

$$u(x,y)=\begin{cases}\dfrac{4u_0}{\pi}\displaystyle\sum_{k=0}^{\infty}\dfrac{\sinh(2k+1)y\sin(2k+1)x}{(2k+1)\sinh(2k+1)\pi}, & \text{for n odd}\\ 0, & \text{for n even}\end{cases}$$

***

Example 31: Find the bounded harmonic function $V(x,y)$ in the semi-infinite strip $x>0,\ 0<y<1$ that satisfies the boundary conditions

$$v_x(0,y)=0\ ,v_y(x,0)=0\ ,\ v(x,1)=f(x)$$

where (a) $f(x)=\begin{cases}1, & 0<x<1\\ 0, & x>1\end{cases}$ (b) $f(x)=e^{-x}\ ,\ x>0$.

Solution: Here we have to solve the two dimensional Laplace's equations

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0\ ,\ \ x>0\ \ 0<y<1$$

subject to conditions

(i) $\dfrac{\partial v}{\partial x}=0$ for $x=0$, for all y

(ii) $\dfrac{\partial v}{\partial y}=0$ for $y=0$, for all x

(iii) $v(x,1)=f(x)$

Taking Fourier cosine transform with respect to x of both the sides of the given equation, we have

$$\mathcal{F}_c\left[\frac{\partial^2 v}{\partial x^2}\right]+\mathcal{F}_c\left[\frac{\partial^2 v}{\partial y^2}\right]=0$$

Let $v_c(\alpha,y)=\mathcal{F}_c[v(x,y)]=\int_0^{\infty}v(x,y)\cos\alpha x\,dx$.

Hence we obtain

$$-\alpha^2 v_c(\alpha,y)-\frac{\partial v}{\partial x}(0,y)+\frac{d^2 v_c}{dy}(\alpha,y)=0$$

or, $$\frac{d^2 v_c}{dy}-\alpha^2 v_c=0$$

whose solution is given by

$$v_c(\alpha,y)=C_1\cosh\alpha y+C_2\sinh\alpha y \tag{105}$$

$$\therefore\ \frac{\partial v_c}{\partial y}=\alpha C_1\sinh\alpha y+\alpha C_2\cosh\alpha y \tag{106}$$

To find C_1 and C_2 we take Fourier cosine transform of given conditions (ii) and (iii)

when $y=0$ $$\frac{\partial v_c}{\partial y}=\int_0^{\infty}\frac{\partial v}{\partial y}\cos\alpha x\,dx=0 \tag{107}$$

From Eqn.(106), putting $y=0$ and using Eqn.(107), we obtain $C_2=0$.

$$\therefore\ v_c(\alpha,y)=C_1\cosh\alpha y \tag{108}$$

To obtain C_1 we take Fourier cosine transform of condition (iii), we get

$$v_c(\alpha,1)=\int_0^{\infty}v(x,1)\cos\alpha x\,dx=\int_0^{\infty}f(x)\cos\alpha x\,dx$$

Therefore, from Eqn.(108), we have

$$C_1 \cosh \alpha = \int_0^\infty f(x) \cos \alpha x \, dx$$

(a) when $f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & x > 1 \end{cases}$

$$C_1 \cosh \alpha = \int_0^1 1.\cos \alpha x \, dx + \int_1^\infty 0.\cos \alpha x \, dx$$

$$= \left.\frac{\sin \alpha x}{\alpha}\right]_0^1 = \frac{\sin \alpha}{\alpha}$$

$$\therefore \quad C_1 = \frac{\sin \alpha}{\alpha \cosh \alpha}$$

Hence $v_c(\alpha, y) = \dfrac{\sin \alpha \cosh \alpha y}{\alpha \cosh \alpha}$

Taking inverse Fourier cosine transform we obtain

$$v(x, y) = \frac{2}{\pi} \int_0^\infty \frac{\sin \alpha \cosh \alpha y}{\alpha \cosh \alpha} \cos \alpha x \, d\alpha \qquad (109)$$

(b) when $f(x) = e^{-x}, \quad x > 0$

$$C_1 \cosh \alpha = \int_0^\infty e^{-x} \cos \alpha x \, dx = \frac{1}{1+\alpha^2}$$

$$\therefore \qquad C_1 = \frac{1}{(1+\alpha^2)\cosh \alpha}$$

and $\qquad v_c(\alpha, y) = \dfrac{\cosh \alpha x}{(1+\alpha^2)\cosh \alpha}$

Taking inverse Fourier cosine transform, we obtain

$$v(x, y) = \frac{2}{\pi} \int_0^\infty \frac{\cosh \alpha x}{(1+\alpha^2)\cosh \alpha} \cos \alpha x \, d\alpha \qquad (110)$$

***

Remark: You may be wondering whether the integrals like the one in Eqn.(109) or (110) can be evaluated. In most cases the answer is no. One way of finding a specific value, say, of the temperature $v(1,1)$ in Eqn.(110) is to resort to the numerical integration.

You may now try the following exercises.

E23) The initial temperature along the length of an infinite bar is given by

$$u(x, 0) = \begin{cases} 2, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

If the temperature $u(x, t)$ satisfies the equation.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0$$

Find the temperature at any point of the bar at any time t.

E24) Use Fourier transform to solve the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \infty, \quad t > 0$$

subject to the following conditions

(i) $u(0, t) = 0, \; t > 0$

(ii) $u(x,0)=\begin{cases}1, & 0<x<1\\ 0, & x>1\end{cases}$

(iii) $u(x,t)$ is bounded.

E25) Use finite Fourier transform to solve

$$\frac{\partial u}{\partial t}=k\frac{\partial^2 u}{\partial x^2}\ ,\ 0<x<4\ ,\ t>0$$

subject to the conditions

(i) $u(x,0)=2x\ ,\ 0<x<4$

(ii) $u(0,t)=u(4,t)=0\ ,\ \ 0<x<4\ ,\ \ t>0$.

E26) Show that the solution of Laplace equation $\nabla^2 v=0$ for v inside the semi-infinite strip $x>0$, $0<y<b$, such that $v=f(x)$ when $y=0, 0<x<\infty$; $v=0$ when $y=b, 0<x<\infty$ and $v=0$ when $x=0$, $0<y<b$ is given by

$$v(x,y)=\frac{2}{\pi}\int_0^\infty\int_0^\infty f(u)\frac{\sinh(b-y)\alpha}{\sinh b\alpha}\sin\alpha x\sin\alpha u\,du\,d\alpha\,.$$

Before we end the discussion in this unit we would like to make the following remark.

Remark: We have seen that the Fourier transform and the corresponding inversion formula are actually just a restatement of the Fourier integral representation of a non-periodic function defined on $-\infty<x<\infty$. We have also seen that the Fourier transform methodology is analogous, and in some aspects identical, to the Laplace transform methodology discussed in Unit 6. Given that similarity, the important question arises, as to which of the two transform methods to use in a given differential equation application. The general guideline is as follows:

- The **Laplace transform** is tailored to **initial-value problems** on a **semi-infinite interval** $0<t<\infty$.
- The **Fourier transform** is tailored to **boundary-value problems** on an infinite interval $-\infty<x<\infty$.

The independent variables are usually t (time) and x (linear dimension) but not always. Further, there does exist Laplace transform defined on an infinite interval, the Laplace transform can sometimes be used for boundary-value problems on finite intervals, and the Fourier transform can be adapted to boundary-value problems on a semi-infinite interval $0<x<\infty$ or, a finite interval $0<x<L$. But the guideline given above provides the general rule of thumb for selecting one transform or the other.

We now end this unit by giving a summary of what we have covered in it.

7.5 SUMMARY

In this unit we have covered the following points.

1. The Fourier integral

$$f(x)=\frac{1}{\pi}\int_0^\infty[A(\alpha)\cos\alpha x+B(\alpha)\sin\alpha x]\,d\alpha$$

where, $\quad A(\alpha)=\int_{-\infty}^{\infty}f(x)\cos\alpha x\,dx$

$$B(\alpha)=\int_{-\infty}^{\infty}f(x)\sin\alpha x\,dx$$

is a way of representing a non-periodic function that is defined on the infinite interval $]-\infty,\infty[$.

2. When the function $f(x)$ is an even function, expression for $f(x)$ in 1) above becomes a **cosine integral** and when $f(x)$ is odd it reduces to a **sine integral.**

3. The **complex form** of **Fourier integral representation** of a function $f(x)$ defined over the interval $]-\infty,\infty[$ is given by

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} C(\alpha)e^{i\alpha x}\,d\alpha$$

where, $C(\alpha)=\int_{-\infty}^{\infty} f(x)e^{-i\alpha x}\,dx$.

4. Integral transforms occur in **transform** pairs: the transform itself and the inversion integral.
Some of the transforms and their inverses are as follows:

- **Fourier transform:** $\mathcal{F}[f(x)]=\int_{-\infty}^{\infty} f(x)e^{-i\alpha x}dx=F(\alpha)$
- **Inverse Fourier transform:** $\mathcal{F}^{-1}\{F(\alpha)\}=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)e^{i\alpha x}d\alpha=f(x)$
- **Fourier sine transform:** $\mathcal{F}_s[f(x)]=\int_{0}^{\infty} f(x)\sin\alpha x\,dx=F_s(\alpha)$
- **Inverse Fourier sine transform:** $\mathcal{F}_s^{-1}[F_s(\alpha)]=\frac{2}{\pi}\int_{0}^{\infty} F_s(\alpha)\sin\alpha x\,d\alpha=f(x)$
- **Fourier cosine transform:** $\mathcal{F}_c[f(x)]=\int_{0}^{\infty} f(x)\cos\alpha x\,dx=F_c(\alpha)$
- **Inverse Fourier cosine transform:**

  $\mathcal{F}_c^{-1}[F_c(\alpha)]=\frac{2}{\pi}\int_{0}^{\infty} F_c(\alpha)\cos\alpha x\,d\alpha=f(x)$

5. If $\mathcal{F}[f(x)]=F(\alpha)$ and $\mathcal{F}[g(x)]=G(\alpha)$ and a and b are arbitrary scalars then

- $\mathcal{F}[\mathbf{a\,f(x)+b\,g(x)}]=\mathbf{a}\,\mathcal{F}[\mathbf{f(x)}]+\mathbf{b}\,\mathcal{F}[\mathbf{g(x)}]=\mathbf{a\,F(\alpha)+b\,G(\alpha)}$
- $\mathcal{F}[\mathbf{f(x-a)}]=\mathbf{e^{-i\alpha a}\,F(\alpha)}$
- $\mathcal{F}[\mathbf{f(x)\cos ax}]=\mathbf{\frac{1}{2}[F(\alpha+a)+F(\alpha-a)]}$
- $\mathcal{F}[\mathbf{f(x)\sin ax}]=\mathbf{\frac{i}{2}[F(\alpha+a)-F(\alpha-a)]}$
- $\mathcal{F}[\mathbf{f(x)*g(x)}]=\mathcal{F}[\mathbf{f(x)}]\,\mathcal{F}[\mathbf{g(x)}]=\mathbf{F(\alpha)G(\alpha)}$

  where, $f(x)*g(x)=\int_{-\infty}^{\infty} f(t)g(x-t)\,dt$
- $\mathcal{F}[\mathbf{f^{(n)}(x)}]=\mathbf{(i\alpha)^n\,F(\alpha)},\ \mathbf{n=0,1,2,\ldots,}$
- $\mathcal{F}[\mathbf{x^n f(x)}]=\mathbf{i^n F^{(n)}(\alpha)}$
- $\mathcal{F}\left[\int_{-\infty}^{x} \mathbf{f(\xi)d\xi}\right]=\mathbf{\frac{1}{i\alpha}F(\alpha)}$
- $\mathcal{F}[\mathbf{\delta(x)}]=\mathbf{1}$

where $\delta(x) = \lim\limits_{k\to 0}[H(x) - H(x-k)]$, $H(x)$ is the unit step function

- $\mathcal{F}\,[\delta(x-a)] = e^{-ia\alpha}\ \mathcal{F}[\delta(x)] = e^{-ia\alpha}$
- $\mathcal{F}_c\,[f''(x)] = -\alpha^2\ \mathcal{F}_c\,[f(x)] - f'(0)$
- $\mathcal{F}_s\,[f''(x)] = -\alpha^2\ \mathcal{F}_s[f(x)] + \alpha f(0)$
- $\mathcal{F}_c\,[f'(x) = \alpha F_s(\alpha) - f(0)$ where, $F_s(\alpha) = \mathcal{F}_s\,[f(x)]$
- $\mathcal{F}_s[f'(x)] = -\alpha F_c(\alpha)$ where, $F_c(\alpha) = \mathcal{F}_c[f(x)]$

6. If $F(\alpha, t)$ be the Fourier transform of the function $u(x,t)$ with respect to the variable x, then

- $\mathcal{F}\left[\frac{\partial u}{\partial x}\right] = i\alpha\mathcal{F}[u(x,t)] = i\alpha F(\alpha\, t)$
- $\mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}\right] = -\alpha^2\mathcal{F}[u(x,t)] = -\alpha^2\, F(\alpha, t)$, u and $\frac{\partial u}{\partial x} \to 0$ as $x \to \pm\infty$.
- $\mathcal{F}\left[\frac{\partial u}{\partial t}\right] = \frac{d}{dt}\mathcal{F}[u(x,t)] = \frac{dF(\alpha,t)}{dt}$

7. To solve a boundary value problem by an integral transform we :

(i) transform the partial differential equation into an ordinary differential equation

(ii) transform all boundary or initial conditions that depend on the transformed variable

(iii) solve the ordinary differential equation subject to the conditions found in step (ii)

(iv) find the inverse transform of the function found in step (iii). This is the solution of the original boundary-value problem.

7.6 SOLUTIONS/ANSWERS

E1) a) $f(x) = \frac{1}{\pi}\int_0^\infty [A(\alpha)\cos\alpha x + B(\alpha)\sin\alpha x]\,d\alpha$

Where $A(\alpha) = (3\alpha\sin 3\alpha + \cos 3\alpha - 1)/\alpha^2$

$B(\alpha) = (\sin 3\alpha - 3\alpha\cos 3\alpha)/\alpha^2$

b) $A(\alpha) = 0,\ B(\alpha) = 2[\alpha - \sin\alpha]/\alpha^2,$

$$f(x) = \frac{1}{\pi}\int_0^\infty [A(\alpha)\cos\alpha x + B(\alpha)\sin\alpha x]\,d\alpha$$

c) $f(x) = \frac{1}{\pi}\int_0^\infty \frac{\cos\alpha x + \alpha\sin\alpha x}{1+\alpha^2}\,d\alpha$

d) $f(x) = \frac{1}{\pi}\int_0^\infty [A(\alpha)\cos\alpha x + B(\alpha)\sin\alpha x]\,d\alpha$

$$A(\alpha) = \frac{1}{2}\left[\frac{2}{\alpha^2 - 1} + \frac{1}{\alpha+1}\{\cos 2(\alpha+1) + \sin 2(\alpha+1)\} + \frac{1}{\alpha - 1}\{\sin 2(\alpha-1) - \cos 2(\alpha-1)\}\right]$$

$$B(\alpha) = \frac{1}{2}\left[\frac{2\alpha}{\alpha^2 - 1} - \frac{1}{\alpha+1}\{\sin 2(\alpha+1) + \cos 2(\alpha+1)\}\right.$$

$$+\frac{1}{\alpha-1}\{\sin 2(\alpha-1)-\cos 2(\alpha-1)\}\Bigg]$$

E2) a) $f(x)=\dfrac{10}{\pi}\displaystyle\int_0^\infty \frac{\cos\alpha x+\alpha\sin\alpha x}{1+\alpha^2}\,d\alpha$

b) $f(x)=\dfrac{2}{\pi}\displaystyle\int_0^\infty \frac{(\pi\alpha\sin\pi\alpha+\cos\pi\alpha-1)\cos\alpha x}{\alpha^2}\,d\alpha$

c) $f(x)=\dfrac{2}{\pi}\displaystyle\int_0^\infty -2\frac{[1+\cos\alpha\pi]}{\alpha^2-1}\cos\alpha x\,d\alpha$

d) $f(x)=\dfrac{2}{\pi}\displaystyle\int_0^\infty \frac{2[\sin 3\alpha\cosh 3-\alpha\cos 3\alpha\sinh 3]}{1+\alpha^2}\sin\alpha x\,d\alpha$.

E3) a) Consider the right hand side, which defines the function

$$f(x)=\frac{\pi}{2}e^{-x}\cos x,\quad x>0$$

Since the integral on the left hand side is from 0 to ∞ and involves $\sin\alpha x$, we take Fourier sine representation of $f(x)$

We have, $B(\alpha)=\displaystyle\int_0^\infty \frac{\pi}{2}e^{-u}\cos u\sin\alpha u\,du$

$$=\frac{\pi}{4}\int_0^\infty e^{-u}[\sin(\alpha+1)u+\sin(\alpha-1)u]\,du$$

Integrating by parts, we obtain

$$B(\alpha)=\frac{\pi}{4}\left[\frac{\alpha+1}{\alpha^2+2\alpha+2}+\frac{\alpha-1}{\alpha^2-2\alpha+2}\right]$$

$$=\frac{\pi}{4}\left[\frac{2\alpha^3}{(\alpha^2+2)^2-4\alpha^2}\right]=\frac{\pi}{2}\left[\frac{\alpha^3}{\alpha^4+4}\right]$$

Using Eqn.(9), we get

$$f(x)=\frac{2}{\pi}\int_0^\infty \frac{\pi}{2}\frac{\alpha^3}{\alpha^4+4}\sin\alpha x\,d\alpha=\int_0^\infty \frac{\alpha^3\sin\alpha x}{\alpha^4+4}\,d\alpha$$

or, $\displaystyle\int_0^\infty \frac{\alpha^3\sin\alpha x}{\alpha^4+4}\,d\alpha=f(x)=\frac{\pi}{2}e^{-x}\cos x$

b) Consider the function $f(x)=\begin{cases}\dfrac{\pi}{2}, & 0<x<\pi\\ 0, & x>\pi\end{cases}$, and take the Fourier sine representation of $f(x)$.

Thus, $B(\alpha)=\displaystyle\int_0^\infty f(u)\sin\alpha u\,du=\int_0^\pi \frac{\pi}{2}\sin\alpha u\,du+\int_\pi^\infty 0.\sin\alpha u\,du$

$$=\frac{\pi}{2}\left[\frac{1-\cos\alpha\pi}{\alpha}\right]$$

Using Eqn.(9), we get

$$f(x)=\frac{2}{\pi}\int_0^\infty \frac{\pi}{2}\left[\frac{1-\cos\alpha\pi}{\alpha}\right]\sin\alpha x\,d\alpha$$

$$= \int_0^\infty \frac{1-\cos \alpha\pi}{\alpha} \sin \alpha x \, d\alpha .$$

E4) Compare with Fourier sine integral representation

Let $g(x) = \frac{2}{\pi}\int_0^\infty B(\alpha)\sin \alpha x \, d\alpha$ where $B(\alpha) = \int_0^\infty g(x)\sin \alpha x \, d\alpha$

Set $x = \alpha$; $\pi g(\alpha) = 1$ for $0 < \alpha < 1$ and 0 for $\alpha > 1$

$B(x) = f(x)$; $f(x) = 2(1-\cos x)/(\pi x)$, $x > 0$, $0 < \alpha < 1$.

E5) a) Let $x = 2$ in Eqn.(11). Use a trigonometric identity and replace α by x.

b) Make the change of variable $2x = kt$.

E6) $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} C(\alpha)e^{i\alpha x} d\alpha$ where $C(\alpha) = \int_{-\infty}^{\infty} f(x)e^{-i\alpha x} dx$

a) $C(\alpha) = \frac{k - i\alpha}{k^2 + \alpha^2}$ b) $C(\alpha) = \frac{1}{\alpha}(i + 2\sin\alpha) + \frac{1}{\alpha^2}(1 - e^{i\alpha})$

c) $C(\alpha) = \frac{4}{4+\alpha^2}\cos a\alpha \sinh 2a + \frac{2\alpha}{4+\alpha^2}\sin a\alpha \cosh 2a$

d) $C(\alpha) = \frac{2\pi i}{\pi^2 - \alpha^2}\sin 2\alpha$

E7) a) $\mathcal{F}[f(x)] = \int_{-\infty}^{\infty} f(x)e^{-i\alpha x}dx = a\int_{-\ell}^{0} e^{-i\alpha x} = \frac{-a}{i\alpha}\left[e^{-i\alpha x}\right]_{-\ell}^{0}$

$$= \frac{-a}{i\alpha}\left[1 - e^{i\alpha\ell}\right] = \frac{ia}{\alpha}\left[1 - e^{i\alpha\ell}\right]$$

b) $\mathcal{F}[f(x)] = a\int_{-\ell}^{\ell} e^{-i\alpha x}dx = \frac{-a}{i\alpha}\left[e^{-i\alpha x}\right]_{-\ell}^{\ell} = \frac{2a}{\alpha}\sin(\alpha\ell)$

c) $\mathcal{F}[f(x)] = a\int_{-\ell}^{0} e^{-i\alpha x}dx + b\int_{0}^{\ell} e^{-i\alpha x}dx$

$$= \frac{1}{i\alpha}\left[(b-a) + a\,e^{i\alpha\ell} - b\,e^{-i\alpha\ell}\right]$$

E8) We have $f(x) = \begin{cases} e^{ax}, & x < 0 \\ e^{-ax}, & x > 0 \end{cases}$

Therefore, $\mathcal{F}[f(x)] = \int_{-\infty}^{0} e^{ax}e^{-i\alpha x}dx + \int_{0}^{\infty} e^{-ax}e^{-i\alpha x}dx$

$$= \left[\frac{e^{(a-i\alpha)x}}{a - i\alpha}\right]_{-\infty}^{0} + \left[\frac{e^{-(a+i\alpha)x}}{-(a+i\alpha)}\right]_{0}^{\infty}$$

$$= \frac{1}{a - i\alpha} + \frac{1}{a + i\alpha} = \frac{2a}{a^2 + \alpha^2}$$

The inverse transform is given by $\mathcal{F}^{-1}\left[\frac{2a}{a^2+\alpha^2}\right] = e^{-a|x|}$.

E9) Taking the Fourier sine transform of the given function, we obtain

$$F_s(\alpha) = \int_0^\infty f(x)\sin\alpha x\, dx = \int_0^\infty \frac{e^{-ax}}{x}\sin \alpha x\, dx = I(\alpha)\ (\text{say})$$

$$\frac{dI}{d\alpha} = \int_0^\infty \frac{\partial}{\partial\alpha}\frac{e^{-ax}}{x}\sin\alpha x\,dx = \int_0^\infty e^{-ax}\cos\alpha x\,dx$$

$$= \left[\frac{e^{-ax}}{a^2+\alpha^2}(-a\cos\alpha x + \alpha\sin\alpha x)\right]_0^\infty = \frac{a}{\alpha^2+a^2}$$

$$\therefore\ I(\alpha) = \int \frac{a}{\alpha^2+a^2}\,d\alpha + A = \tan^{-1}\frac{\alpha}{a} + A$$

$$I(\alpha)\Big]_{\alpha=0} = A = \int_0^\infty \frac{e^{-ax}}{x}\sin\alpha x\,dx\Bigg]_{\alpha=0} = 0$$

$$\therefore A = 0 \text{ and } I(\alpha) = \tan^{-1}\frac{\alpha}{a}$$

Inverse sine transform of $F_s(\alpha)$ is written as

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s(\alpha)\sin\alpha x\,d\alpha = \frac{2}{\pi}\int_0^\infty \tan^{-1}\frac{\alpha}{a}\sin\alpha x\,d\alpha$$

$$\therefore\ \int_0^\infty \tan^{-1}\frac{\alpha}{a}\sin\alpha x\,d\alpha = \frac{\pi}{2}f(x) = \frac{\pi}{2}\frac{e^{-ax}}{x}$$

Putting $x = 1$ in the above equation, we obtain

$$\int_0^\infty \tan^{-1}\frac{\alpha}{a}\sin\alpha\,d\alpha = \frac{\pi}{2}e^{-a}$$

$\therefore\ \displaystyle\int_0^\infty \tan^{-1}\frac{x}{a}\sin x\ dx = \frac{\pi}{2}e^{-a}$ is the required solution.

E10) Taking Fourier cosine transform of $f(x) = e^{-x}$, we obtain

$$F_c(\alpha) = \int_0^\infty e^{-x}\cos\alpha x\,dx = \frac{1}{1+\alpha^2}$$

Inverse Fourier cosine transform of $F_c(\alpha)$ gives

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{1}{1+\alpha^2}\cos\alpha x\,d\alpha$$

$$\therefore \int_0^\infty \frac{\cos\alpha x}{1+\alpha^2}\,d\alpha = \frac{\pi}{2}f(x) = \frac{\pi}{2}e^{-x}$$

Putting $x = m$ in above equation, we have

$$\int_0^\infty \frac{\cos\alpha m}{1+\alpha^2}\,d\alpha = \frac{\pi}{2}e^{-m}$$

Hence, $\displaystyle\int_0^\infty \frac{\cos mx}{1+x^2}\,dx = \frac{\pi}{2}e^{-m}$

E11) a) The given integral is the Fourier sine transform of $f(x)$. Its inverse Fourier transform is obtained as

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s(\alpha)\ \sin\alpha x\,d\alpha = \frac{2}{\pi}\int_0^1 (1-\alpha)\sin\alpha x\,d\alpha$$

$= \dfrac{2}{\pi}\left(\dfrac{x-\sin x}{x^2}\right)$, is the required solution.

b) $f(x) = \dfrac{2}{\pi}\left(\dfrac{1}{1+x^2}\right)$.

E12) We have $F_c(\alpha) = \int_0^\infty f(u)\cos\alpha u\, du = \begin{cases} 1-\alpha, & 0 \le \alpha \le 1 \\ 0, & \alpha \ge 1 \end{cases}$

The inverse Fourier cosine transform of $F_c(\alpha)$ is given by

$$f(x) = \frac{2}{\pi}\int_0^\infty F_c(\alpha)\cos\alpha x\, d\alpha = \frac{2}{\pi}\left[\int_0^1 (1-\alpha)\cos\alpha x\, d\alpha + \int_1^\infty 0.\cos\alpha x\, d\alpha\right]$$

$$= \frac{2}{\pi}\left[(1-\alpha)\left(\frac{\sin\alpha x}{x}\right) - (-1)\left(\frac{-\cos\alpha x}{x^2}\right)\right]_0^1$$

$$= \frac{2}{\pi}\left(\frac{1-\cos x}{x^2}\right)$$

$\therefore$ we have $F_c(\alpha) = \frac{2}{\pi}\int_0^\infty \left(\frac{1-\cos u}{u^2}\right)\cos\alpha u\, du = \frac{2}{\pi}\int_0^\infty \frac{2\sin^2 u/2}{u^2}\cos\alpha u\, du$

$\therefore$ $F_c(\alpha)\big]_{\alpha=0} = \frac{2}{\pi}\int_0^\infty \frac{2\sin^2 u/2}{u^2}\, du = 1$

or, $\frac{2}{\pi}\int_0^\infty \frac{2\sin^2 z}{(2z)^2}\, 2dz = 1$ (Putting $u = 2z$)

or, $\int_0^\infty \frac{\sin^2 z}{z^2}\, dz = \frac{\pi}{2}$.

E13) $\mathcal{F}_c[f(x)] = F_c(\alpha) = \frac{-\cos\alpha + 2\cos\alpha/2 - 1}{\alpha^2}$.

E14) $f(x) = \frac{2\sin^2 ax}{\pi x^2}$.

E15) $f(x) = e^{-kx}$, $x > 0$.

E16) $\mathcal{F}_s\left[\frac{\partial^2 u}{\partial x^2}\right] = \int_0^\infty \frac{\partial^2 u}{\partial x^2}\sin\alpha x\, dx$

$$= \sin\alpha x\frac{\partial u}{\partial x}\bigg]_0^\infty - \int_0^\infty (\alpha\cos\alpha x)\frac{\partial u}{\partial x}\, dx$$

$$= 0 - \alpha\int_0^\infty \frac{\partial u}{\partial x}\cos\alpha x\, dx \qquad \left\{\because \frac{\partial u}{\partial x} \to 0 \text{ as } x \to \infty\right\}$$

$$= \alpha[u(x,t)]_{x=0} - \alpha^2\int_0^\infty u\sin\alpha x\, dx \qquad \{\because u \to 0 \text{ as } x \to \infty\}$$

$$= \alpha[u(x,t)]_{x=0} - \alpha^2\mathcal{F}_s[u(x,t)]$$

$$= \alpha[u(x,t)]_{x=0} - \alpha^2 F_s(\alpha, t)$$

which proves the result (61). Result (62) can be proved similarly.

E17) a) $\mathcal{F}[4x^2 e^{-3|x|}] = 4i^2\frac{d^2}{d\alpha^2}\mathcal{F}\{e^{-3|x|}\}$ (entry 17)

$$= -4\frac{d^2}{d\alpha^2}\left(\frac{6}{\alpha^2+9}\right) \qquad \text{(entry 4)}$$

$$= 48\left[\frac{1}{(\alpha^2+9)^2} - \frac{4\alpha^2}{(\alpha^2+9)^3}\right]$$

b) With $c = 3, a = 1$, $f(x) = \frac{1}{x^2+2}$ in entry 13 and $F(\alpha) = \frac{\pi}{\sqrt{2}} e^{-\sqrt{2}|\alpha|}$, we have

$$\mathcal{F}\left\{\frac{\cos 3x}{x^2+2}\right\} = \frac{1}{2}[F(\alpha-3) + F(\alpha+3)] = \frac{\pi}{2\sqrt{2}}\left(e^{-\sqrt{2}|\alpha-3|} + e^{-\sqrt{2}|\alpha+3|}\right).$$

c) $$\mathcal{F}\left\{\frac{3}{2x^2+1} - 5e^{-|x|}\right\} = \frac{3}{2}\mathcal{F}\left\{\frac{1}{x^2+1/2}\right\} - 5\mathcal{F}\left\{e^{-|x|}\right\}$$

$$= \frac{3}{2}\pi\sqrt{2}\, e^{-|\alpha|/\sqrt{2}} - 5\frac{2}{\alpha^2+1} \qquad \text{(entries 1, 4)}$$

$$= \frac{3\pi}{\sqrt{2}} e^{-|\alpha|/\sqrt{2}} - \frac{10}{\alpha^2+1}$$

d) $$\mathcal{F}^{-1}\left\{\frac{4\sin\alpha}{\alpha} - \frac{1}{\sqrt{|\alpha|}}\right\} = 2\mathcal{F}^{-1}\left\{\frac{2\sin\alpha}{\alpha}\right\} - \frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}\left\{\sqrt{\frac{2\pi}{|\alpha|}}\right\}$$

$$= 2[H(x+1) - H(x-1)] - \frac{1}{\sqrt{2\pi|x|}} \qquad \text{(entries 9, 7)}$$

e) $$\mathcal{F}^{-1}\left\{\frac{9}{2\alpha+i}\right\} = \frac{-9i}{2}\mathcal{F}^{-1}\left\{\frac{1}{1/2 - i\alpha}\right\} = \frac{-9i}{2}H(-x)e^{x/2} \qquad \text{(entry 3)}$$

f) $$\mathcal{F}^{-1}\left\{e^{-|\alpha|}\cos\alpha\right\} = \frac{1}{2\pi}\mathcal{F}^{-1}\left\{2(\pi e^{-|\alpha|})\cos\alpha\right\}$$

$$= \frac{1}{2\pi}\left[\frac{1}{(x+1)^2+1} + \frac{1}{(x-1)^2+1}\right] \qquad \text{(entries 15, 1)}$$

g) $$\mathcal{F}^{-1}\left\{\frac{1}{\alpha^2+i\alpha+2}\right\} = \mathcal{F}^{-1}\left\{\frac{1}{3i}\left(\frac{1}{\alpha-i} - \frac{1}{\alpha+2i}\right)\right\} = \mathcal{F}^{-1}\left\{\frac{1}{3}\left(\frac{1}{1+i\alpha} - \frac{1}{-2+i\alpha}\right)\right\}$$

$$= \frac{1}{3}\mathcal{F}^{-1}\left\{\frac{1}{1+i\alpha} + \frac{1}{2-i\alpha}\right\} = \frac{1}{3}\left[H(x)e^{-x} + H(-x)e^{2x}\right]$$

(entries 2, 3)

E18) a) $f(x) = \frac{a}{\pi(a^2+x^2)}$ b) $f(x) = \frac{\sin ax}{\pi x}$.

E19) a) $$\mathcal{F}^{-1}\left[\frac{1}{(i\alpha+k)^2}\right] = \mathcal{F}^{-1}\left[\frac{1}{(i\alpha+k)(i\alpha+k)}\right] = \left[e^{-kx}H(x)\right] * \left[e^{-kx}H(x)\right]$$

$$= \int_{-\infty}^{\infty} e^{-k\tau}H(\tau)e^{-k(x-\tau)}H(x-\tau)\,d\tau$$

$$= e^{-kx}\int_{-\infty}^{\infty} H(\tau)H(x-\tau)\,d\tau = x\,e^{-kx}H(x)$$

since $H(\tau)H(x-\tau) = 0$ for $\tau < 0$ and $\tau > x$, and 1 for $0 < \tau < x$.

b) $x^2 e^{-kx} H(x)/2$ [use the result of a) above]

E20) a) $F(\alpha) = \frac{1}{[(2+i\alpha)(3+i\alpha)]}$, $y(x) = e^{-2x}(1-e^{-x})H(x)$

b) $F(\alpha) = \frac{e^{-3i\alpha}}{[(2+i\alpha)(1+i\alpha)]}$, $y(x) = \left[e^{-(x-3)} - e^{-2(x-3)}\right]H(x-3)$.

E21) a) We have $F_c(n) = \int_0^L f(x)\dfrac{\cos n\pi x}{L}dx = \int_0^{\ell} (\ell x - x^2)\dfrac{\cos n\pi x}{\ell}dx$

$\because L = \ell$ and $f(x) = \ell x - x^2$

$$= \left[(\ell x - x^2)\left(\frac{\ell}{n\pi}\frac{\sin n\pi x}{L}\right) - (\ell - 2x)\left(\frac{-\ell^2}{n^2\pi^2}\frac{\cos n\pi x}{\ell}\right) + (-2)\left(\frac{-\ell^3}{n^3\pi^3}\frac{\sin n\pi x}{\ell}\right)\right]_0^{\ell}$$

$$= \frac{-\ell^3}{n^2\pi^2}\cos n\pi \frac{-\ell^3}{n^2\pi^2} = \frac{-\ell^3}{n^2\pi^2}(\cos n\pi + \ell)$$

at $n = 0$, $F_c(n)$ becomes indeterminate, hence we evaluate $F_c(0)$ separately,

$$F_c(0) = \int_0^{\ell} (\ell x - x^2)dx = \frac{\ell^3}{6}$$

Hence finite Fourier cosine transform of $f(x)$ is

$$F_c(n) = \frac{-\ell^3}{n^2\pi^2}(\cos n\pi + \ell) \text{ and } F_c(0) = \frac{\ell^3}{6}.$$

Now $F_s(n) = \int_0^{\ell} (\ell x - x^2)\sin\dfrac{n\pi x}{\ell}dx$ $\quad [\because L = \ell$ and $f(x) = \ell x - x^2]$

$$= \left[(\ell x - x^2)\left(\frac{-\ell}{n\pi}\cos\frac{n\pi x}{\ell}\right) - (\ell - 2x)\left(\frac{-\ell^2}{n^2\pi^2}\sin\frac{n\pi x}{\ell}\right) + (-2)\left(\frac{\ell^3}{n^3\pi^3}\cos\frac{n\pi x}{\ell}\right)\right]_0^{\ell}$$

$= \dfrac{2\ell^3}{n^3\pi^3}(\ell - \cos n\pi)$ is the finite Fourier sine transform of $f(x)$

b) $F_c(n) = \dfrac{4}{n^2\pi}\left(2\cos\dfrac{n\pi}{2} - \cos n\pi - 1\right)$, $F_c(0) = \pi$

$F_s(n) = \dfrac{8\pi}{n^2\pi^2}\sin\dfrac{n\pi}{2}$.

c) $F_c(n) = \dfrac{\pi}{3}\sin\dfrac{n\pi}{3}\left(1 + 2\cos\dfrac{n\pi}{3}\right)$, $F_c(0) = 0$

$F_s(n) = \dfrac{\pi}{3n}\left(\cos\dfrac{n\pi}{3} - 1 - \cos n\pi + \cos\dfrac{2n\pi}{3}\right)$

d) $F_c(n) = \dfrac{2}{\pi n^2}$, $F_c(0) = \dfrac{\pi}{3}$

E22) a) $f(x) = \dfrac{4}{\pi}\sum_{n=1}^{\infty} \dfrac{1}{n}\sin^2\dfrac{n\pi}{4}\sin\dfrac{n\pi x}{\ell}$

b) $f(x) = \dfrac{ac}{\ell} + \dfrac{2c}{\pi}\sum_{n=1}^{\infty} \dfrac{1}{n}\sin\dfrac{n\pi a}{\ell}\cos\dfrac{n\pi x}{\ell}$

c) $f(x) = \dfrac{2}{\pi}\sum_{n=1}^{\infty} \dfrac{1}{n^2}\sin\dfrac{n\pi}{3}\sin nx$

E23) Taking Fourier transform of both the sides of given pde, we obtain

$$\frac{d\bar{u}}{dt} = -\alpha^2 \bar{u}$$

whose solution is given by

$$\bar{u}(\alpha, t) = A e^{-\alpha^2 t}, \text{ A being an arbitrary constant} \qquad (111)$$

Taking Fourier transform of initial condition, we obtain

$$\bar{u}(\alpha, 0) = \int_{-\infty}^{\infty} u(x,0) e^{-i\alpha x} dx = \int_{-1}^{1} 2.\, e^{-i\alpha x} dx = \frac{4}{\alpha} \sin\alpha \qquad (112)$$

Substituting $t = 0$ in Eqn.(111) and using Eqn.(112), we get

$$\bar{u}(\alpha, 0) = A = \frac{4}{\alpha} \sin\alpha$$

$$\therefore \quad \bar{u}(\alpha, t) = \frac{4}{\alpha} \sin\alpha\, e^{-\alpha^2 t}$$

Taking inverse Fourier transform of $\bar{u}(\alpha, t)$, we obtain

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{4}{\alpha} \sin\alpha e^{-\alpha^2 t} e^{i\alpha x} d\alpha$$

$$= \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\sin\alpha}{\alpha} e^{-\alpha^2 t} (\cos\alpha x + i \sin\alpha x) d\alpha$$

$$= \frac{4}{\pi} \int_{0}^{\infty} \frac{e^{-\alpha^2 t}}{\alpha} \sin\alpha \cos\alpha x\, d\alpha$$

(second integral vanishes as its integrand is an odd function)

$$= \frac{2}{\pi} \int_{0}^{\infty} e^{-\alpha^2 t} \left[\frac{\sin\,(1+x)\,\alpha + \sin\,(1-x)\,\alpha}{\alpha} \right] d\alpha$$

which is the required solution.

E24) Taking Fourier sine transform of both sides of the given pde, we obtain

$$\frac{d\bar{u}_s}{dt} + \alpha^2 \bar{u}_s = 0$$

whose solution is given by

$$\bar{u}_s(\alpha, t) = A e^{-\alpha^2 t}, \text{ A being constant} \qquad (113)$$

Now $\bar{u}_s(\alpha, 0) = \int_0^\infty u(x,0) \sin\alpha x\, dx = \int_0^1 \sin\alpha x\, dx + \int_1^\infty 0.\sin\alpha x\, dx$

$$= \frac{1 - \cos\alpha}{\alpha} \qquad (114)$$

Putting $t = 0$ in Eqn.(113) and comparing with Eqn.(114), we obtain

$$A = \frac{1 - \cos\alpha}{\alpha}$$

$$\therefore \quad \bar{u}_s(\alpha, t) = \frac{1 - \cos\alpha}{\alpha} e^{-\alpha^2 t}$$

and $u(x,t) = \frac{2}{\pi} \int_0^\infty \frac{1 - \cos\alpha}{\alpha} e^{-\alpha^2 t} \sin\alpha x\, d\alpha$, is the required solution.

E25) Taking finite Fourier sine transform of given pde with $L = 4$, we obtain

$$\int_0^4 \frac{\partial u}{\partial t} \sin\frac{n\pi x}{4} dx = \frac{d}{dt} \int_0^4 u(x,t) \sin\frac{n\pi x}{4} dx = \frac{d\bar{u}_s}{dt}$$

$$\int_0^4 \frac{\partial^2 u}{\partial x^2} \sin\frac{n\pi x}{4} dx = \left[\left(\frac{\partial u}{\partial x} \sin\frac{n\pi x}{4} \right)_0^4 - \int_0^4 \frac{\partial u}{\partial x} \frac{n\pi}{4} \cos\frac{n\pi x}{4} dx \right]$$

$$= \frac{-n\pi}{4}\left[\left(u\cos\frac{n\pi x}{4}\right)_0^4 - \int_0^4 u\left(\frac{-n\pi}{4}\sin\frac{n\pi x}{4}\right)dx\right]$$

$$= \frac{-n^2\pi^2}{16}\int_0^4 u(x,t)\sin\frac{n\pi x}{4}dx = \frac{-n^2\pi^2}{16}\bar{u}_s$$

$$\therefore \quad \frac{d\bar{u}_s}{dt} = \frac{-kn^2\pi^2}{16}\bar{u}_s$$

The solution of this ode is written as

$$\bar{u}_s(n,t) = Ae^{\frac{-kn^2\pi^2}{16}t} \tag{115}$$

$$\text{Now } \bar{u}_s(n,0) = \int_0^4 2x\sin\frac{n\pi x}{4}dx = \frac{-32}{n\pi}\cos n\pi \tag{116}$$

Substituting $t = 0$ in Eqn.(115) and using Eqn.(116), we obtain

$$A = \frac{-32}{n\pi}\cos n\pi$$

$$\therefore \quad \bar{u}_s(n,t) = \frac{-32}{n\pi}\cos n\pi\, e^{\frac{-kn^2\pi^2}{16}t}$$

Taking inverse finite sine transform of $\bar{u}_s(n,t)$, we obtain

$$u(x,t) = \frac{2}{4}\sum_{n=1}^{\infty}\bar{u}_s(n,t)\sin\frac{n\pi x}{4}$$

$$= \frac{16}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{\frac{-kn^2\pi^2}{16}t}\sin\frac{n\pi x}{4}$$

E26) Here we have to solve two-dimensional Laplace equation

$$\nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0, \quad x > 0, \; 0 < y < b$$

subject to the conditions

(i) $v(x,0) = f(x)$, $x > 0$

(ii) $v(x,b) = 0$, $x > 0$

(iii) $v(0,y) = 0$, $0 < y < b$

Taking Fourier sine transform of the given pde, we obtain

$$\frac{d^2 v_s}{dy^2}(\alpha,y) - \alpha^2 v_s(\alpha,y) = 0$$

General solution of the above equation is ,

$$v_s(\alpha,y) = C_1\cosh\alpha y + C_2\sinh\alpha y, \; C_1, C_2 \text{ being constants} \tag{117}$$

To find C_1 and C_2, take Fourier sine transform of conditions (i) and (ii)

$$\text{when } y = 0, \qquad v_s(\alpha,0) = \int_0^\infty v(u,0)\sin\alpha u\,du = \int_0^\infty f(u)\sin\alpha u\,du \tag{118}$$

$$\text{And when } y = b,\; v_s(\alpha,b) = \int_0^\infty v(u,b)\sin\alpha u\,du = 0 \tag{119}$$

Putting $y = 0$ in Eqn.(117) and using Eqn.(118), we obtain

$$C_1 = \int_0^\infty f(u)\sin\alpha u\,du$$

Putting $y = b$ in Eqn.(117) and using Eqn.(119), we obtain

$$C_2 = -C_1 \frac{\cosh \alpha b}{\sinh \alpha b} = -\frac{\cosh \alpha b}{\sinh \alpha b} \int_0^\infty f(u) \sin \alpha u \, du$$

$$\therefore \; v_s(\alpha, y) = \cosh \alpha y \int_0^\infty f(u) \sin \alpha u \; du - \frac{\cosh \alpha b}{\sinh \alpha b} \sinh \alpha y \int_0^\infty f(u) \sin \alpha u \; du$$

$$= \int_0^\infty f(u) \sin \alpha u \frac{\sinh \, (b-y) \, \alpha}{\sinh \alpha b} \, du$$

Taking inverse Fourier sine transform of $v_s(\alpha, y)$, we obtain

$$v(x, y) = \frac{2}{\pi} \int_0^\infty \int_0^\infty f(u) \sin \alpha u \frac{\sinh (b-y) \alpha}{\sinh \alpha b} \sin \alpha x \, du \, d\alpha \, ,$$

as the required solution.

—x—

Appendix

Table of Fourier Transforms

	$f(x)$	$F(\alpha) = \int_{-\infty}^{\infty} f(x)e^{-i\alpha x}\,dx$
1.	$\frac{1}{x^2+a^2}\ (a>0)$	$\frac{\pi}{a}e^{-a\lvert\alpha\rvert}$
2.	$H(x)e^{-ax}\ (\operatorname{Re} a>0)$	$\frac{1}{a+i\alpha}$
3.	$H(-x)e^{ax}\ (\operatorname{Re} a>0)$	$\frac{1}{a-i\alpha}$
4.	$e^{-a\lvert x\rvert}\ (a>0)$	$\frac{2a}{\alpha^2+a^2}$
5.	e^{-x^2}	$\sqrt{\pi}\,e^{-\alpha/4}$
6.	$\frac{2}{2a\sqrt{\pi}}e^{-x^2/(2a)^2}\ (a>0)$	$e^{-a^2\alpha^2}$
7.	$\frac{1}{\sqrt{\lvert x\rvert}}$	$\sqrt{\frac{2\pi}{\lvert\alpha\rvert}}$
8.	$e^{-a\lvert x\rvert/\sqrt{2}}\sin\left(\frac{a}{\sqrt{2}}\lvert x\rvert+\frac{\pi}{4}\right)\ (a>0)$	$\frac{2a^3}{\alpha^4+a^4}$
9.	$H(x+a)-H(x-a)$	$\frac{2\sin\alpha a}{\alpha}$
10.	$\delta(x-a)$	$e^{-i\alpha a}$
11.	$f(ax+b)\ (a>0)$	$\frac{1}{a}e^{ib\alpha/a}F\left(\frac{\alpha}{a}\right)$
12.	$\frac{1}{a}e^{-i(bx/a)}f\left(\frac{x}{a}\right)\ (a>0,\ b \text{ real})$	$F(a\alpha+b)$
13.	$f(ax)\cos cx\ (a>0,\ c \text{ real})$	$\frac{1}{2ai}\left[F\left(\frac{\alpha-c}{a}\right)+F\left(\frac{\alpha+c}{a}\right)\right]$
14.	$f(ax)\sin cx\ (a>0,\ c \text{ real})$	$\frac{1}{2ai}\left[F\left(\frac{\alpha-c}{a}\right)-F\left(\frac{\alpha+c}{a}\right)\right]$
15.	$f(x+c)+f(x-c)\ (c \text{ real})$	$2F(\alpha)\cos\alpha c$
16.	$f(x+c)-f(x-c)\ (c \text{ real})$	$2iF(\alpha)\sin\alpha c$
17.	$x^n f(x)\ (n=1,2,\ldots)$	$\frac{i^n d^n F(\alpha)}{d\alpha^n}$