UNIT 11 FINITE DIFFERENCE METHODS

Structure **Page No.**

11.1 INTRODUCTION

A number of physical processes can be described by a partial differential equation or a system of partial differential equations. For example, conduction of heat in a rod is governed by the partial differential equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad 0 \le x \le \ell, \quad t > 0$$

where $u(x, t)$ is the temperature of the rod at any instant of time and x and t are the independent variables representing the direction of the rod and time respectively. The vibrations of an elastic string is governed by the partial differential equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}, \quad 0 \le x \le \ell, \quad t > 0$$

where $u(x,t)$ is the displacement of the string at any instant of time and $\partial u/\partial t$ is the velocity of any point on the string. Under suitable initial and boundary conditions, it is possible to obtain Fourier series solution of the above problems.

However there are complex physical processes which can be described by single/system of partial differential equations in two or three space dimensions for which it is not always possible to find the closed form solutions. Often, analytical methods do not exist to solve such problems. In such situations, an alternative is to solve them numerically. In this unit we shall discuss finite difference methods of solving partial differential equations.

We have started the unit by briefly reviewing some definitions and concepts of differential equations and numerical analysis in Sec.11.2. In Sec.11.3 we have derived five-point difference method for general elliptic equation and used this method to solve Laplace and Poisson equations under Dirichlet and mixed boundary conditions. Finite difference methods for solving parabolic and hyperbolic equations are derived in Sec.11.4 and Sec.11.5 respectively, and the solutions of heat and wave equations under different types of initial and boundary conditions are obtained. Finally, in Sec.11.6 conditions for the convergence of finite difference methods are given and Von Neumann stability analysis to test the stability of difference methods is discussed.

Objectives

After studying this unit you should be able to

- derive finite difference methods to solve general elliptic equation;
- use these methods to solve Laplaoe and Poisson equation under Dirichlet or mixed boundary conditions;
- derive difference methods to solve parabolic and hyperbolic equations and solve these equations under different types of initial and boundary conditions;
- derive difference methods and use them to solve more complex problems occurring in various physical applications;
- examine the stability of the difference method used.

11.2 PRELIMINARIES

In this section we shall briefly review the definitions/concepts that are required in this unit. You must be familiar with these definitions/concepts through your knowledge of differential equations and numerical analysis which you may have studied at the undergraduate level (ref. MTE-08 and MTE-10).

Classification of partial differential equations

Consider the general quasilinear second order partial differential equation

$$A\frac{\partial^2 u}{\partial x^2}+2B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}-F\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)=0 \qquad (1)$$

in two-dimensional region R, where A, B and C are functions of x, y, u, $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$.

When A, B and C are functions of x, y only and F is a linear function of u, $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, then Eqn.(1) reduces to a linear equation. A linear partial differential equation (pde) can be written as

$$A\frac{\partial^2 u}{\partial x^2}+2B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu+G=0 \qquad (2)$$

where $A, B, \ldots, G$ are functions of x, y.

If $B^2-AC>0$, for all $(x,y)\in R$, then Eqn.(2) is called a **hyperbolic** partial differential equation. In this case, Eqn.(2) has two real characteristics.

If $B^2-AC=0$, for all $(x,y)\in R$, then Eqn.(2) is called a **parabolic** partial differential equation. In this case, Eqn.(2) has one real characteristic.

If $B^2-AC<0$, for all $(x,y)\in R$, then Eqn.(2) is called an **elliptic** partial differential equation. In this case, Eqn.(2) has no real characteristics.

The simplest examples of these equations are given below.

Hyperbolic equations

$$\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}, \text{ (wave equation in one dimension).}$$

$$\frac{\partial^2 u}{\partial t^2}=c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right), \text{ (wave equation in two dimensions).}$$

Parabolic equations

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}, \quad \text{(heat conduction equation in one dimension).}$$

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}, \text{ (heat conduction equation in two dimensions).}$$

Elliptic equation

$$\nabla^2 u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0, \text{ (Laplace equation).}$$

Example 1: Classify the partial differential equation into elliptic, hyperbolic or parabolic equation.

$$\frac{\partial^2 u}{\partial x^2}+4x\frac{\partial^2 u}{\partial x\partial y}+(1-y^2)\frac{\partial^2 u}{\partial y^2}=0.$$

Solution: Comparing the given equation with Eqn.(2), we find

$B^2-AC=4x^2-(1-y^2)=4x^2+y^2-1$. Therefore, if $4x^2+y^2-1>0$, that is, outside the ellipse $4x^2+y^2>1$, the given pde is an hyperbolic pde. On the ellipse

$4x^2 + y^2 = 1$, the equation is a parabolic pde and inside the ellipse $4x^2 + y^2 < 1$, the equation is an elliptic pde.

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The classification of the second order linear pde is very important. It governs the number and nature of conditions that must be prescribed in the problem in order that a unique solution may exist.

Parabolic and hyperbolic problems are classified as **initial boundary value problems** whereas the elliptic problems are classified as **boundary value problems (bvp).** Elliptic bvp usually holds inside a closed region (or in an open region which can be mapped conformally onto a closed region). Elliptic bvp is further classified in the following way.

(i) **Dirichlet bvp:** The pde holds in a region R, bounded by a closed curve C. On the boundary C, the unknown is prescribed, that is, $u(x, y) = f(x, y)$ on C.

(ii) **Neumann bvp:** The pde holds in a region R, bounded by a closed curve C. On the boundary C, the normal derivative is prescribed, that is,

$$\frac{\partial u}{\partial n} = g(x, y) \text{ on } C$$

where n is the unit outward normal to C.

(iii) **Mixed bvp:** A simple case is when $u(x, y)$ is prescribed on the part of the boundary and on the remaining part, $\frac{\partial u}{\partial n}$ is prescribed. However, we may have a general boundary condition like

$$a(s)\,u + b(s)\frac{\partial u}{\partial n} = h(s) \text{ on } C$$

where s is the arc length.

Let us now quickly review some of the concepts from numerical analysis.

Finite differences

Consider a function $f(x)$ defined on an interval $[a, b]$. Subdivide this interval into n equal parts, such that $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, where $x_i = x_0 + ih$, and $h = (b - a)/n$. The points x_i are called the nodal points. Then, we define the following.

Shift operator E

We define, $E\,f(x) = f(x + h)$. In general, $E^k f(x) = f(x + kh)$, where k is any real number, (integer or a fraction). For example, $E^{1/2} f(x) = f[x + (h/2)]$.

Forward difference operator Δ

We define, $\Delta f(x) = f(x + h) - f(x) = E\,f(x) - f(x) = (E - 1)f(x)$.
Hence, $\Delta = E - 1$, or $E = 1 + \Delta$.
The second order forward difference is defined as follows.

$$\begin{aligned}\Delta^2 f(x) &= \Delta[\Delta f(x)] = \Delta[f(x + h) - f(x)] = \Delta f(x + h) - \Delta f(x)\\ &= f(x + 2h) - 2f(x + h) + f(x).\end{aligned}$$

In general, higher order forward differences are defined by $\Delta^n f(x) = \Delta[\Delta^{n-1} f(x)]$. The coefficients are binomial coefficients with alternating signs.

Backward difference operater ∇

We define, $\nabla f(x) = f(x) - f(x - h) = f(x) - E^{-1} f(x) = (1 - E^{-1}) f(x)$.

Hence, $\nabla = 1 - E^{-1}$, or $E = (1 - \nabla)^{-1}$.
The second order backward difference is defined as follows.

$$\nabla^2 f(x) = \nabla[\nabla f(x)] = \nabla[f(x) - f(x-h)] = \nabla f(x) - \nabla f(x-h)$$
$$= f(x) - 2f(x-h) + f(x-2h).$$

In general, higher order backward differences are defined by $\nabla^n f(x) = \nabla[\nabla^{n-1} f(x)]$. The coefficients are binomial coefficients with alternating signs.

Central difference operator δ

We define, $\delta f(x) = f\left(x + \dfrac{h}{2}\right) - f\left(x - \dfrac{h}{2}\right) = (E^{1/2} - E^{-1/2}) f(x).$

Hence, $\delta = E^{1/2} - E^{-1/2}$.

The second order central difference is defined as follows.

$$\delta^2 f(x) = \delta\left[f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)\right] = \delta f\left(x + \frac{h}{2}\right) - \delta f\left(x - \frac{h}{2}\right)$$
$$= f(x+h) - 2f(x) + f(x-h).$$

In general, $\delta^n f(x) = \delta[\delta^{n-1} f(x)]$.

Note that the even differences contain the values of $f(x)$ at the nodal points.

Relationship of the difference operators with the derivative operator D:

We define the derivative operators as $D = \dfrac{d}{dx}$, $D^2 = \dfrac{d^2}{dx^2}$, etc.. We have

$$E f(x) = f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \cdots$$
$$= \left[1 + hD + \frac{h^2 D^2}{2!} + \cdots\right] f(x) = e^{hD} f(x).$$

Hence, $E = e^{hD}$, or $hD = \ln(E)$. Therefore, we can write

$$hD = \ln(E) = \ln(1+\Delta) = \Delta - (\Delta^2/2) + (\Delta^3/3) - \cdots$$
$$= \ln[(1-\nabla)^{-1}] = -\ln(1-\nabla) = \nabla + (\nabla^2/2) + (\nabla^3/3) + \cdots$$
$$\delta = E^{1/2} - E^{-1/2} = e^{hD/2} - e^{-hD/2} = 2\sinh(hD/2)$$

or, $\quad hD = 2\sinh^{-1}(\delta/2)$.

Therefore, some approximations to $f'(x_k)$, using the forward differences are the following:

$$f'(x_k) = \frac{1}{h}\Delta f(x_k) = \frac{1}{h}[f(x_{k+1}) - f(x_k)] \tag{3}$$

$$f'(x_k) = \frac{1}{h}\left[\Delta - \frac{1}{2}\Delta^2\right] f(x_k) = \frac{1}{2h}[2\Delta f(x_k) - \Delta^2 f(x_k)]$$
$$= \frac{1}{2h}[-3f(x_k) + 4f(x_{k+1}) - f(x_{k+2})] \tag{4}$$

Using Taylor series expansions, we can show that the error of approximation in the formulas (3) and (4) are

Formula(3): $\quad E(f, x_k) = -\dfrac{h}{2} f''(\xi), \quad x_k < \xi < x_{k+1}$

Formula(4): $\quad E(f, x_k) = \dfrac{h^2}{3} f'''(\xi), \quad x_k < \xi < x_{k+2}$.

Therefore, the method (3) is of order 1 or $O(h)$ formula and the method (4) is of order 2 or $O(h^2)$ formula.

Similarly, we can write approximations to $f'(x_k)$, using the backward differences as

$$f'(x_k) = \frac{1}{h}\nabla f(x_k) = \frac{1}{h}[f(x_k) - f(x_{k-1})] \qquad (5)$$

$$f'(x_k) = \frac{1}{h}[\nabla f(x_k) + \frac{1}{2}\nabla^2 f(x_k)] = \frac{1}{2h}[2\nabla f(x_k) + \nabla^2 f(x_k)]$$

$$= \frac{1}{2h}[3f(x_k) - 4f(x_{k-1}) + f(x_{k-2})]. \qquad (6)$$

The error of approximation in the formula (5) is of order $O(h)$ and the error in the formula (6) is of order $O(h^2)$.

Using the central differences, we have the approximation to $f'(x_k)$ as

$$f'(x_k) = \frac{1}{h}\delta f(x_k) = \frac{1}{h}\left[f\left(x_k + \frac{h}{2}\right) - f\left(x_k - \frac{h}{2}\right)\right]. \qquad (7)$$

If we want to use the mesh points, we can write the approximation as

$$f'(x_k) = \frac{1}{2h}\left[\delta f\left(x_k + \frac{h}{2}\right) + \delta f\left(x_k - \frac{h}{2}\right)\right] = \frac{1}{2h}[f(x_{k+1}) - f(x_{k-1})] \qquad (8)$$

The error of approximation is of order $O(h^2)$. Sometimes, we write this approximation as

$$f'(x_k) = \frac{1}{2h}\delta_{2x} f(x_k).$$

where δ_{2x} represents central difference with double spacing, that is, $2(h/2) = h$.

Now, we want to write approximations to the second derivative $f''(x_k)$.

We have, $(hD)^2 = [\Delta - (\Delta^2/2) + (\Delta^3/3) - \cdots]^2$ etc.

Hence, an approximation to $f''(x_k)$, using the forward differences is

$$f''(x_k) = \frac{1}{h^2}\Delta^2 f(x_k) = \frac{1}{h^2}[f(x_{k+2}) - 2f(x_{k+1}) + f(x_k)] \qquad (9)$$

An approximation to $f''(x_k)$, using the backward differences is

$$f''(x_k) = \frac{1}{h^2}\nabla^2 f(x_k) = \frac{1}{h^2}[f(x_k) - 2f(x_{k-1}) + f(x_{k-2})] \qquad (10)$$

Using the central differences, we have an approximation to $f''(x_k)$ as

$$f''(x_k) = \frac{1}{h^2}\delta^2 f(x_k) = \frac{1}{h^2}[f(x_{k+1}) - 2f(x_k) + f(x_{k-1})] \qquad (11)$$

Using Taylor series expansions, we can show that the error of approximation in the formulas (9) and (10) are of order $O(h)$ and the formula (11) is of order $O(h^2)$.

We can extend the above results to higher dimensions, that is, approximations to partial derivatives can be written. Let the dependent variable be $u(x, y, t)$, where x, y denote the space directions and t denotes the time direction. In elliptic problems, the variable t does not appear.

We superimpose on the region R of interest, a mesh of lines parallel to x and y axis. The points of intersection of these mesh lines, lying in the region R are called the nodes or mesh points.

In one-dimensional case, we define $x_j = a + jh$, $j = 0,1,2,\ldots,$ where h is the mesh size or mesh length in the x-direction and a is a suitable initial point.

In the two-dimensional case, we define

$$x_i = a + ih_1, y_j = b + jh_2, i = 0,1,2,\ldots, j = 0,1,2,\ldots$$

where h_1, h_2 are the mesh lengths in the x and y directions respectively. If we are considering an initial value problem, then we also have the mesh lines $t_n = nk, n = 0,1,2,\ldots$ where k is the mesh length in the t-direction. The mesh and nodes for a bvp are shown in Fig.1.

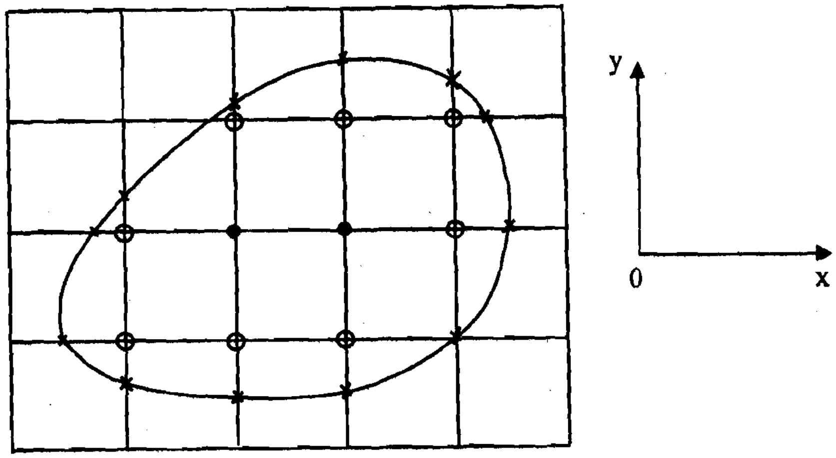


Fig.1: A typical mesh and nodes

In the direction of x-axis, we denote the forward differences by $\Delta_x, \Delta_{xx}, \ldots$, the backward differences by $\nabla_x, \nabla_{xx}, \ldots$, and the central differences by $\delta_x, \delta_{xx}, \ldots$. We shall use similar notations for differences in the y and t directions. We have

$$h\frac{\partial}{\partial x} = hD_x = \ln(1+\Delta_x) = \Delta_x - (\Delta_x^2/2) + (\Delta_x^3/3) - \cdots$$

$$= \ln[(1-\nabla_x)^{-1}] = -\ln(1-\nabla_x) = \nabla_x + (\nabla_x^2/2) + (\nabla_x^3/3) + \cdots$$

$$h^2\frac{\partial^2}{\partial x^2} = h^2 D_x^2 = [\ln(1+\Delta_x)]^2 = [\Delta_x - (\Delta_x^2/2) + (\Delta_x^3/3) - \cdots]^2$$

$$= \Delta_x^2 - \Delta_x^3 + \cdots$$

$$h^2\frac{\partial^2}{\partial x^2} = h^2 D_x^2 = [2\sinh^{-1}(\delta_x/2)]^2 = \delta_x^2 + \cdots$$

Therefore, $O(h)$ approximations to $\frac{\partial u}{\partial x}$ are given by

$$\frac{\partial u}{\partial x} = \frac{1}{h}\Delta_x u = \frac{1}{h}[u(x+h, y, t) - u(x, y, t)] \tag{12}$$

$$\frac{\partial u}{\partial x} = \frac{1}{h}\nabla_x u = \frac{1}{h}[u(x, y, t) - u(x-h, y, t)]. \tag{13}$$

Similarly, $O(h)$ approximations to $\frac{\partial^2 u}{\partial x^2}$ are given by

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{h^2}\Delta_x^2 u = \frac{1}{h^2}[u(x+2h, y, t) - 2u(x+h, y, t) + u(x, y, t)]. \tag{14}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{h^2}\nabla_x^2 u = \frac{1}{h^2}[u(x, y, t) - 2u(x-h, y, t) + u(x-2h, y, t)]. \tag{15}$$

An $O(h^2)$ approximation to $\frac{\partial^2 u}{\partial x^2}$ is given by

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{h^2}\delta_x^2 u = \frac{1}{h^2}[u(x+h, , t) - 2u(x, y, t) + u(x-h, y, t)]. \tag{16}$$

We can verify the orders using Taylor series expansions in three variables.

An $O(k)$ approximation to $\frac{\partial u}{\partial t}$ is given by

$$\frac{\partial u}{\partial t} = \frac{1}{k}\Delta_t u = \frac{1}{k}[u(x, y, t+k) - u(x, y, t)] \tag{17}$$

$$\frac{\partial u}{\partial t} = \frac{1}{k}\nabla_t u = \frac{1}{k}[u(x, y, t) - u(x, y, t-k)]. \tag{18}$$

E1) Using Taylor series, derive the orders of the formulas given in Eqns.(12) to (18).

In **difference methods,** the partial derivatives in the differential equation are replaced by suitable difference quotients, converting the differential equation to a difference equation at each nodal point. This procedure is called **discretization** of the differential equation. The difference equation is applied at each nodal point. The given data is used to modify the difference equation at the nodes near or on the boundary. The solution of this algebraic system of equations (consisting of the difference equation and the initial/boundary conditions) gives the numerical solution of the given initial or boundary value problem.

We shall now discuss difference methods for solving elliptic differential equations.

11.3 DIFFERENCE METHODS FOR ELLIPTIC EQUATIONS

Consider the linear partial differential equation

$$Lu = A\frac{\partial^2 u}{\partial x^2} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G \qquad (19)$$

defined in a region R with boundary Γ, where $A, C, \ldots, G$ are continuous functions of x and y in R and on Γ. We assume that $A > 0, C > 0$ and $F \le 0$ (so that the ellipticity condition and the weak max-min principle are satisfied). Let suitable boundary conditions (Dirichlet or mixed boundary conditions) be given.

Max-min principle: The maximum and minimum values of $u(x, y)$ occur inside the region R.

Weak max-min principle: The maximum and minimum values of $u(x, y)$ occur either inside the region R or/and on the boundary Γ.

We superimpose on R, a rectangular mesh with step lengths h_1 and h_2 in the directions x and y respectively. The nodal points are given by

$$x_i = x_0 + ih_1,\ i = 0, \pm 1, \pm 2, \ldots;\ y_j = y_0 + jh_2,\ j = 0, \pm 1, \pm 2, \ldots$$

If all the four neighbouring nodes of a mesh point are inside R, then this node is called an internal node. Otherwise, it is called a boundary node as shown in Fig.2. Here, we have represented an internal node by a solid circle and a boundary node by a hollow circle O.

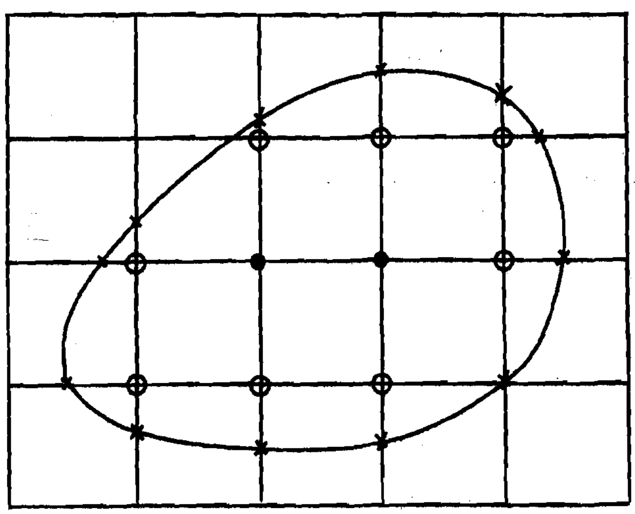


Fig.2: Internal nodes (•) and boundary nodes (O)

We denote the numerical solution at the nodal point (x_i, y_j) by $u_{i,j}$. We write the following approximations at (x_i, y_j).

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{1}{2h_1}\delta_{2x}\, u_{i,j} + O(h_1^2), \left(\frac{\partial u}{\partial y}\right)_{i,j} = \frac{1}{2h_2}\delta_{2y} u_{i,j} + O(h_2^2)$$

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} = \frac{1}{h_1^2}\delta_x^2\, u_{i,j} + O(h_1^2), \left(\frac{\partial^2 u}{\partial y^2}\right)_{i,j} = \frac{1}{h_2^2}\delta_y^2\, u_{i,j} + O(h_2^2)$$

where, $\delta_{2x} u_{i,j} = u_{i+1,j} - u_{i-1,j}, \delta_x^2 u_{i,j} = u_{i+1,j} - 2u_{i,j} + u_{i-1,j}$ etc.

At the nodal point (x_i, y_j), the approximation to the differential Eqn.(19), becomes

$$\left[\frac{1}{h_1^2} A_{i,j}\delta_x^2 + \frac{1}{h_2^2} C_{i,j}\,\delta_y^2 + \frac{1}{2h_1} D_{i,j}\delta_{2x} + \frac{1}{2h_2} E_{i,j}\delta_{2y} + h_1^2\, F_{ij}\right] u_{i,j} = G_{i,j}$$

or,
$$\frac{1}{h_1^2}\left[A_{i,j}\delta_x^2 + p^2 C_{i,j}\,\delta_y^2 + \frac{h_1}{2} D_{i,j}\delta_{2x} + \frac{ph_1}{2} E_{i,j}\delta_{2y} + F_{ij}\right] u_{i,j} = G_{i,j}$$

where, $p = \frac{h_1}{h_2}$. Writing the expressions for each term and simplifying, we get

$$\frac{1}{h_1^2}[a_{i,j} u_{i+1,j} + b_{i,j}\, u_{i-1,j} + c_{i,j}\, u_{i,j+1} + d_{i,j}\, u_{i,j-1} - e_{i,j}\, u_{i,j}] = G_{i,j} \tag{20}$$

where, $a_{i,j} = A_{i,j} + \frac{h_1}{2} D_{i,j}$; $b_{i,j} = A_{i,j} - \frac{h_1}{2} D_{i,j}$;

$c_{i,j} = p^2\, C_{i,j} + \frac{ph_1}{2} E_{i,j}$; $d_{i,j} = p^2\, C_{i,j} - \frac{ph_1}{2} E_{i,j}$;

$e_{i,j} = 2(A_{i,j} + p^2\, C_{i,j}) - h_1^2\, F_{i,j}$.

This method is called the **five-point formula** for the solution of (19). We can also write the formula as

$$a_{i,j}\, u_{i+1,j} + b_{i,j}\, u_{i-1,j} + c_{i,j}\, u_{i,j+1} + d_{i,j}\, u_{i,j-1} - e_{i,j} u_{i,j} = h_1^2\, G_{i,j}. \tag{21}$$

If the step lengths in both directions, x and y, are same that is, $h_1 = h_2 = h$, then $p = 1$ and the coefficients simplify as

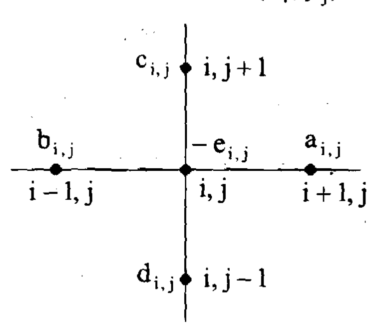
$a_{i,j} = A_{i,j} + \frac{h}{2} D_{i,j}$; $b_{i,j} = A_{i,j} - \frac{h}{2} D_{i,j}$;

$c_{i,j} = C_{i,j} + \frac{h}{2} E_{i,j}$; $d_{i,j} = C_{i,j} - \frac{h}{2} E_{i,j}$;

$$e_{i,j} = 2(A_{i,j} + C_{i,j}) - h^2\, f_{i,j}. \tag{22}$$

The difference Eqn.(20) is applied at all the internal nodal points in R. When the difference Eqn.(20) is applied at a node numbered (i, j), it uses the information at the four neighbouring points $(i + 1, j)$, $(i - 1, j)$, $(i,\ j + 1)$, and $(i, j - 1)$, which lie on the mesh lines $x = x_i$ and $y = y_j$. We define these five mesh points as the computational molecule to derive the difference equation at (i, j).

The computational molecule at (x_i, y_j) is given in Fig.3.



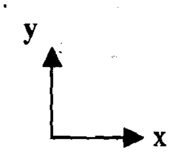


Fig.3: Computational molecule for five-point formula

The above formulas simplify when we consider a Poisson equation or a Laplace equation.

Let us now derive methods for

Poisson equation: $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = G(x, y).$

Let us first consider the corresponding homogeneous equation i.e.,

Laplace equation: $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0.$

In these cases, $A = 1, C = 1, D = 0 = E = F$. For Laplace equation $G = 0$ as well. Then the coefficients in the five point formula (20) simplify as

$$a_{i,j} = 1 = b_{i,j};\ \ c_{i,j} = p^2 = d_{i,j};\ \ e_{i,j} = 2(1 + p^2),$$

and the formula becomes

$$u_{i+1,j} + u_{i-1,j} + p^2(u_{i,j+1} + u_{i,j-1}) - 2(1 + p^2)u_{i,j} = h_1^2\ G_{i,j}. \qquad (23)$$

when $h_1 = h_2 = h$, the five point formula simplifies as

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = h^2\ G_{i,j}. \qquad (24)$$

The computational molecule is as given in Fig.3, where $a_{i,j} = 1 = b_{i,j} = c_{i,j} = d_{i,j}$ and $e_{i,j} = 4$. For the Laplace's equation $G = 0$.

Let us apply the above methods using the following regions and meshes.

(i) Square, $0 \le x \le 3,\ 0 \le y \le 3$, with $h_1 = h_2 = 1.0$. The mesh is as given in Fig.4. Let the Dirichlet boundary conditions be given, that is, on the boundary $u(x, y)$ is prescribed. There are four nodes and the above formulas can be applied as the step lengths on all four sides are equal.

(ii) Rectangle, $0 \le x \le 3.5, 0 \le y \le 3$, with $h_1 = h_2 = 1.0$. The mesh is as given in Fig.5. Let the Dirichlet boundary conditions be given. There are four nodes numbered 1, 2, 4, 5 for which the step lengths on all sides are equal and the above formulas can be applied. However, at the nodes numbered 3 and 6, the mesh lengths on the right and left are 0.5 and 1.0 respectively. Hence, the above formulas cannot be applied at these points.

(iii) Triangle, $0 \le x \le 7,\ 0 \le y \le 7,\ 0 \le x + y \le 7$ with $h_1 = h_2 = 2.0$. The mesh is as given in Fig.6. Let the Dirichlet boundary conditions be given. There are three nodes inside the triangle. The coordinates of the nodes are $1(2,2), 2(4,2), 3(2,4)$. At the node numbered 1 for which the step lengths on all sides are equal, the above formulas can be applied. The steps lengths on the right and left, above and below at the nodes numbered 2 and 3 are 1.0 and 2.0 respectively. Hence, the above formulas cannot be applied at these nodes.

Therefore, we need to derive a separate formula to take care of the situations mentioned in the illustrations (ii) and (iii).

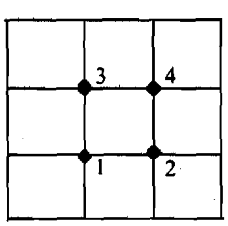


Fig.4: Square mesh

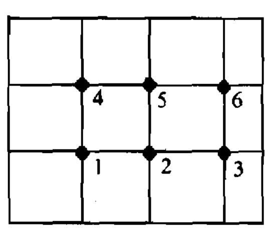


Fig.5: Rectangular mesh

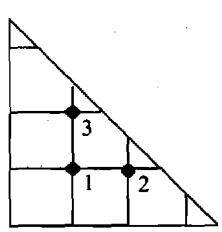


Fig.6: Triangular mesh

Let us now describe how we have drawn the above meshes.

How to draw mesh

We shall describe the method for the problem in (ii), (Fig.5). We first draw the boundary lines $x = 0,\ y = 0,\ x = 3.5$ and $y = 3$. The region inside this rectangle is the required region R. The step length is $h_1 = h_2 = 1.0$. Now, we draw the mesh lines $x = 1$, $x = 2$, $x = 3$, $y = 1$ and $y = 2$. The points of intersection of these mesh lines in R are the nodal points. Since Dirichlet boundary conditions are given, the solutions at the boundary points are known. Hence, we number the internal nodes from 1 to 6 (in an ordered way) at which the solutions are to be obtained, the coordinates of these points are $1(1,1)$, $2(2,1)$, $3(3,1)$, $4(1,2)$, $5(2,2)$ and $6(3,2)$, (see Fig.5).

Difference formula for nodes near the boundary

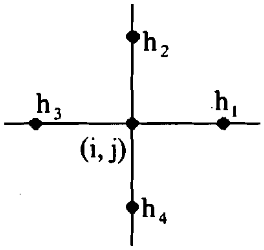


Fig.7: Non-uniform mesh

Let the step lengths on the four sides of the node (x_i, y_j) be h_1, h_2, h_3 and h_4 (see Fig.7).
We write the formula as

$$\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)_{i,j} = a_0 u(x_i, y_j) + a_1 u(x_i + h_1, y_j) + a_2 u(x_i, y_j + h_2)$$

$$+ a_3 u(x_i - h_3, y_j) + a_4 u(x_i, y_j - h_4). \qquad (25)$$

Writing the Taylor series expansions of the terms on the right hand side, we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = a_0 u + a_1\left[u + h_1 \frac{\partial u}{\partial x} + \frac{h_1^2}{2}\frac{\partial^2 u}{\partial x^2} + \cdots\right] + a_2\left[u + h_2 \frac{\partial u}{\partial y} + \frac{h_2^2}{2}\frac{\partial^2 u}{\partial y^2} + \cdots\right]$$

$$+ a_3\left[u - h_3 \frac{\partial u}{\partial x} + \frac{h_3^2}{2}\frac{\partial^2 u}{\partial x^2} - \cdots\right] + a_4\left[u - h_4 \frac{\partial u}{\partial y} + \frac{h_4^2}{2}\frac{\partial^2 u}{\partial y^2} - \cdots\right]$$

where all the terms on the right hand side are evaluated at (x_i, y_j). Comparing the coefficients, we get

$$a_0 + a_1 + a_2 + a_3 + a_4 = 0. \qquad (26)$$

$$a_1 h_1 - a_3 h_3 = 0. \qquad (27)$$

$$a_2 h_2 - a_4 h_4 = 0. \qquad (28)$$

$$a_1 h_1^2 + a_3 h_3^2 = 2. \qquad (29)$$

$$a_2 h_2^2 + a_4 h_4^2 = 2. \qquad (30)$$

Multiply Eqn.(27) by h_1 and subtract it from Eqn.(29) to eliminate a_1. We get

$$a_3 h_3^2 + a_3 h_1 h_3 = 2, \quad \text{or} \quad a_3 = \frac{2}{h_3 (h_1 + h_3)}.$$

From Eqn.(27), we get $a_1 = \dfrac{a_3 h_3}{h_1} = \dfrac{2}{h_1(h_1 + h_3)}$.

Multiply Eqn.(28) by h_2 and subtract it from Eqn.(30) to eliminate a_2. We get

$$a_4 h_4^2 + a_4 h_2 h_4 = 2, \quad \text{or} \quad a_4 = \frac{2}{h_4(h_2 + h_4)}.$$

From Eqn.(28), we get

$$a_2 = \frac{a_4 h_4}{h_2} = \frac{2}{h_2(h_2 + h_4)}.$$

From Eqn.(26), we get

$$a_0 = -(a_1 + a_2 + a_3 + a_4)$$

$$= -\left[\frac{2}{h_1(h_1 + h_3)} + \frac{2}{h_2(h_2 + h_4)} + \frac{2}{h_3(h_1 + h_3)} + \frac{2}{h_4(h_2 + h_4)}\right]$$

$$= -2\left[\frac{1}{h_1 h_3} + \frac{1}{h_2 h_4}\right].$$

Substituting the values of $a_0, a_1, \ldots, a_4$ in Eqn.(25), we obtain the required formula. When $h_1 = h_3$ and $h_2 = h_4$, we obtain formula (23) and when $h_1 = h_2 = h_3 = h_4 = h$, we obtain the formula (24).

Application of the method

We form the mesh with the given step lengths. We number the unknowns at the nodes in an ordered way, from left to right and bottom to top, or from left to right and top to bottom. We apply the method at the nodal points in the same order, that is, first equation at point1, second equation at point 2 etc. The difference equations give rise to a system of algebraic equations for the required number of unknowns. The solution of this system is obtained numerically by a direct method (Gauss elimination or LU decomposition) if the system is small and by an iterative method (Gauss-Seidel or any other suitable method) if the system is large.

We now illustrate the above methods through the following examples.

Example 2: Find the solution of $\nabla^2 u = 0$ in R subject to the boundary conditions $u(x,y) = x^2 - y^2$ on Γ, where R is the square $0 \le x \le 1, 0 \le y \le 1$, using the five point formula. Assume uniform step length $h = \frac{1}{3}$ along the axes.

Solution: The mesh is given in Fig.8. We have four mesh points 1, 2, 3, 4 whose coordinates are $(1/3,1/3),(2/3,1/3),(1/3,2/3),(2/3,2/3)$ respectively. Denote the solutions at these points as u_1, u_2, u_3, u_4 respectively. From the boundary condition $u(x,y) = x^2 - y^2$, we have the following values.

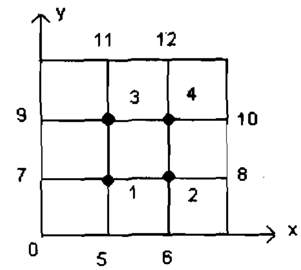


Fig.8: Mesh in Example 2

$$u_5 = u\left(\frac{1}{3},0\right) = \frac{1}{9}, \quad u_6 = u\left(\frac{2}{3},0\right) = \frac{4}{9},$$
$$u_7 = u\left(0,\frac{1}{3}\right) = -\frac{1}{9}, \quad u_8 = u\left(1,\frac{1}{3}\right) = \frac{8}{9},$$
$$u_9 = u\left(0,\frac{2}{3}\right) = -\frac{4}{9}, \quad u_{10} = u\left(1,\frac{2}{3}\right) = \frac{5}{9},$$
$$u_{11} = u\left(\frac{1}{3},1\right) = -\frac{8}{9}, \quad u_{12} = u\left(\frac{2}{3},1\right) = -\frac{5}{9}.$$

Hence, we obtain the following difference equations using formula (24).

At 1: $u_2 + u_3 + u_7 + u_5 - 4u_1 = 0$, or $-4u_1 + u_2 + u_3 = 0.$

At 2: $u_8 + u_4 + u_1 + u_6 - 4u_2 = 0$, or $u_1 - 4u_2 + u_4 = -\frac{4}{3}.$

At 3: $u_4 + u_{11} + u_9 + u_1 - 4u_3 = 0$, or $u_1 - 4u_3 + u_4 = \frac{4}{3}.$

At 4: $u_{10} + u_{12} + u_3 + u_2 - 4u_4 = 0$, or $u_2 + u_3 - 4u_4 = 0.$

We have the system of equations $\mathbf{Ax} = \mathbf{B}$, where

$$A = \begin{bmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 1 & 0 & -4 & 1 \\ 0 & 1 & 1 & -4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ -4/3 \\ 4/3 \\ 0 \end{bmatrix}$$

We can solve this system by Gauss elimination method. The solution is

$$u_1 = 0, u_2 = 1/3, u_3 = -1/3, u_4 = 0.$$

We also note that $u(x,y) = x^2 - y^2$ satisfies the differential equation $\nabla^2 u = 0$ and the given boundary conditions. Hence, $u(x,y) = x^2 - y^2$ is the exact solution of the bvp. Hence the numerical solution (u_1, u_2, u_3, u_4) obtained is same as the exact solution.

***

Example 3: Find the solution of $\nabla^2 u = x + y$ in R subject to the boundary conditions $u(x, y) = (x^3 + y^3)/6$ on Γ, where R is the triangle $0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le x + y \le 1$, using the five point formula. Assume uniform step length $h = \dfrac{1}{4}$ along the axes.

Solution: The boundaries of the triangle are $x = 0$, $y = 0$ and $x + y = 1$. The mesh is given in Fig.9. We have three mesh points 1, 2, 3 whose coordinates are $(1/4, 1/4)$, $(1/2, 1/4)$, $(1/4, 1/2)$. Denote the solutions at these points as u_1, u_2, u_3 respectively. From the boundary condition, $u(x, y) = (x^3 + y^3)/6$, we have the following values.

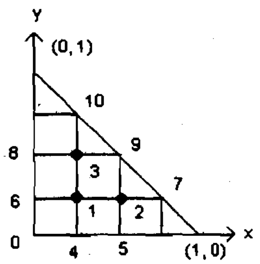


Fig.9: Mesh in Example 3

$$u_4 = u\left(\frac{1}{4}, 0\right) = \frac{1}{384}\ ,\quad u_5 = u\left(\frac{1}{2}, 0\right) = \frac{1}{48},$$

$$u_6 = u\left(0, \frac{1}{4}\right) = \frac{1}{384}\ ,\quad u_7 = u\left(\frac{3}{4}, \frac{1}{4}\right) = \frac{7}{96},$$

$$u_8 = u\left(0, \frac{1}{2}\right) = \frac{1}{48}\ ,\quad u_9 = u\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{24},$$

$$u_{10} = u\left(\frac{1}{4}, \frac{3}{4}\right) = \frac{7}{96}.$$

Hence, we obtain the following difference equations

At 1: $$u_2 + u_3 + u_6 + u_4 - 4u_1 = \frac{1}{16}\left(\frac{1}{4} + \frac{1}{4}\right)$$

or, $$-4u_1 + u_2 + u_3 = \frac{1}{32} - \frac{1}{384} - \frac{1}{384} = \frac{5}{192}.$$

At 2: $$u_7 + u_9 + u_1 + u_5 - 4u_2 = \frac{1}{16}\left(\frac{1}{2} + \frac{1}{4}\right)$$

or, $$u_1 - 4u_2 = \frac{3}{64} - \frac{7}{96} - \frac{1}{24} - \frac{1}{48} = -\frac{34}{384}.$$

At 3: $$u_9 + u_{10} + u_8 + u_1 - 4u_3 = \frac{1}{16}\left(\frac{1}{4} + \frac{1}{2}\right)$$

or, $$u_1 - 4u_3 = \frac{3}{64} - \frac{1}{24} - \frac{7}{96} - \frac{1}{48} = -\frac{34}{384}.$$

Subtracting the last two equations, we get $u_2 = u_3$. Therefore, we have the equations

$$-4u_1 + 2u_2 = \frac{5}{192}\ ,\ u_1 - 4u_2 = -\frac{34}{384},$$

whose solution is $u_1 = 1/192,\ u_2 = u_3 = 9/384$.

***

You may now try the following exercises.

E2) Find the solution of $\nabla^2 u = 0$ in R subject to the given R and boundary conditions using the five point formulas.

a) R : square $0 \le x \le 1,\ 0 \le y \le 1.u(x, y) = x - y$ on Γ. Assume $h = 1/3$.

b) R : rectangle $0 \le x \le 3,\ 0 \le y \le 2.\ u(x, y) = x^2 - 3y^2$ on Γ. Assume $h = 1.0$.

c) R : triangle $0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le x + y \le 1.\ u(x, y) = x^2 - y^2$ on Γ. Assume $h = 1/3$.

d) R : rectangle $0 \le x \le 5/6,\ 0 \le y \le 1.u(x, y) = x - y$ on Γ. Assume $h = 1/3$.

E3) Find the solution of $\nabla^2 u = G(x, y)$ in R subject to the given R, G and boundary conditions, using the five point formula.

a) R : square $0 \le x \le 1, 0 \le y \le 1. G(x, y) = 2x + 3y. u(x, y) = x - y$ on Γ. Assume $h = 1/3$.

b) R : rectangle $0 \le x \le 3, 0 \le y \le 2$. $G(x, y) = x - y$. $u(x, y) = x^2 - 3y^2$ on Γ. Assume $h = 1.0$.

c) R : triangle $0 \le x \le 1, 0 \le y \le 1$, $0 \le x + y \le 1. G(x, y) = x^2 + y^2$. $u(x, y) = x^2 - y^2$ on Γ. Assume $h = 1/3$.

d) R : rectangle $0 \le x \le 5/6, 0 \le y \le 1. G(x, y) = 2x + 3y. u(x, y) = x - y$ on Γ. Assume $h = 1/3$,

E4) Using the Taylor series expansions, show that the five point formula is of order $O(h_1^2 + h_2^2)$ or $O(h^2)$.

Treatment of mixed boundary conditions

If the mixed boundary condition is prescribed at a nodal point on the boundary, then the solution at this point is also to be determined. To apply the mixed boundary condition at a nodal point on the boundary, we need an approximation to the normal derivative $\partial u / \partial n$. If the region R is a square or a rectangle, then the normal to the boundary is in the direction of x or y axis (see Fig.10). Since the order of the formula being used at the nodal points inside R is of order $O(h_1^2 + h_2^2)$ or $O(h^2)$, it is preferable to use same order approximations to $\partial u / \partial x$ or $\partial u / \partial y$, so that the totality of the difference equations are of order $O(h_1^2 + h_2^2)$ or $O(h^2)$. However, in the case of curved boundaries like a circle, we may use $O(h)$ approximation. $O(h^2)$ approximations to the first partial derivative are given as follows (see Eqns.(4), (6), (8)).

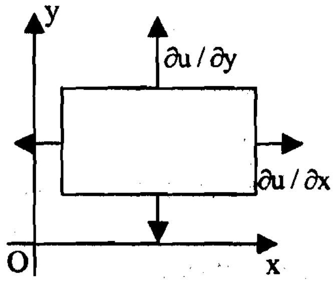


Fig.10: Normal to the boundary

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{1}{2h_1}[-3u(x_i, y_j) + 4u(x_{i+1}, y_j) - u(x_{i+2}, y_j)] \tag{31}$$

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{1}{2h_1}[3u(x_i, y_j) - 4u(x_{i-1}, y_j) + u(x_{i-2}, y_j)] \tag{32}$$

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{1}{2h_1}[u(x_{i+1}, y_j) - u(x_{i-1}, y_j)] \tag{33}$$

Similar expressions hold for $\partial u / \partial y$. Approximations (31) and (32) are one sided approximations (uses the nodal point on the boundary and nodal points inside R) and approximation (33) uses a nodal point outside the region. The nodal point which is outside R can be eliminated by using both the difference approximation (as if it is a point inside) and the boundary approximation (33) at that nodal point.

Let us illustrate this procedure through an example.

Example 4: Find the solution of $\nabla^2 u = 0$ in R, where R is the square $0 \le x \le 1$, $0 \le y \le 1$, subject to the boundary conditions

$$u(x, y) = x^2 - y^2 \text{ on } x = 0, y = 0, y = 1; u + \frac{\partial u}{\partial x} = x^2 + 2x - y^2 \text{ on } x = 1$$

using the five point formula. Use central difference approximation in the boundary condition. Assume uniform step length $h = 1/2$ along the axes.

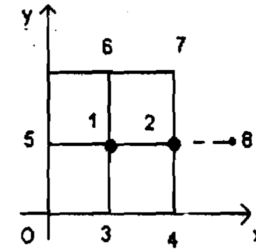


Fig.11: Mesh in Example 4

Solution: The mesh, with $h = 1/2$, is as given in Fig.11. There are two unknowns, at the nodal points 1 and 2 having coordinates $(1/2, 1/2), (1, 1/2)$. Denote the unknowns at these points as u_1, u_2. Note that u_1 is inside R and u_2 is on the boundary. If we apply the central difference approximation (33) at the point 2, then we require an external point $8(u_8)$, which is at a distance of h from 2 (along the line parallel to x-axis). Using the boundary conditions, we get

$$u_3(1/2,0)=1/4, u_4(1,0)=1, u_5(0,1/2)=-1/4,$$
$$u_6(1/2,1)=-3/4, u_7(1,1)=0.$$

We obtain the following difference equations.

At 1: The step lengths on all four sides are the same $(h=1/2)$.

$$u_2+u_6+u_5+u_3-4u_1=0,$$

or, $$u_2-\frac{3}{4}-\frac{1}{4}+\frac{1}{4}-4u_1=0, \text{ or } -4u_1+u_2=\frac{3}{4}.$$

At 2: Using the central difference approximation (33) to $\partial u/\partial x$, and the boundary condition at the point $(1,1/2)$, we get

$$u+\frac{\partial u}{\partial x}=u_2+\frac{1}{2h}(u_8-u_1)=u_2+u_8-u_1=1+2-\frac{1}{4}=\frac{11}{4}.$$

Applying the finite difference scheme at 2, we get

$$u_8+u_7+u_1+u_4-4u_2=0$$

Eliminating u_8, we get

$$u_1-u_2+\frac{11}{4}+0+u_1+1-4u_2=0, \text{ or } 2u_1-5u_2=-\frac{15}{4}.$$

The solution is $u_1=0$, $u_2=3/4$.

***

Example 5: Find the solution of $\nabla^2 u=12x$, $0\le x\le 5$, $0\le y\le 4$, subject to the boundary conditions $u(x,y)=x^3+3xy^2$ on $x=1$, $y=0$, $y=1$ and

$2u+\frac{\partial u}{\partial x}=3y^2$ on $x=0$, using the five point formula. Use central difference approximation in the boundary condition. Assume uniform step length $h=2.0$ along the axes.

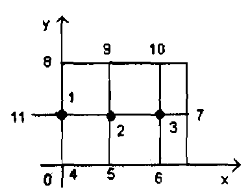


Fig.12: Mesh in Example 5

Solution: The mesh, with $h=2.0$, is as given in Fig. 12. There are three unknowns 1, 2, 3 at the nodal points $(0,2),(2,2),(4,2)$. Denote the unknowns at these points as u_1, u_2, u_3. Using the boundary conditions, we have

$$u_4(0,0)=0,\ u_5(2,0)=8,\ u_6(4,0)=64,$$
$$u_7(5,2)=185,\ u_8(0,4)=0,$$
$$u_9(2,4)=104,\ u_{10}(4,4)=256.$$

We obtain the following difference equations.

At 2: Step lengths on all four sides are the same $(h=2.0)$.

$$u_1+u_5+u_3+u_9-4u_2=24h^2=96,$$

or, $$u_1-4u_2+u_3=96-8-104=-16.$$

At 3: At the point $(4,2)$, we use the formula (25) since $h_1=1.0,\ h_2=h_3=h_4=2.0$. We have

$$a_0=-2\left[\frac{1}{h_1h_3}+\frac{1}{h_2h_4}\right]=-2\left[\frac{1}{2}+\frac{1}{4}\right]=-\frac{3}{2},\ a_1=\frac{2}{h_1(h_1+h_3)}=\frac{2}{3},$$

$$a_2=\frac{2}{h_2(h_2+h_4)}=\frac{2}{8}=\frac{1}{4},\ a_3=\frac{2}{h_3(h_1+h_3)}=\frac{2}{6}=\frac{1}{3},$$

$$a_4=\frac{2}{h_4(h_2+h_4)}=\frac{2}{8}=\frac{1}{4}.$$

Therefore, we have at the point $(4,2)$

$$a_0u_3+a_1u_7+a_2u_{10}+a_3u_2+a_4u_6=48,$$

or, $$-\frac{3}{2}u_3+\frac{2}{3}(185)+\frac{1}{4}(256)+\frac{1}{3}u_2+\frac{1}{4}(64)=48,$$

or, $$\frac{1}{3}u_2-\frac{3}{2}u_3=48-\frac{370}{3}-64-16=-\frac{466}{3},\ 2u_2-9u_3=-932.$$

At 1: Using the central difference approximation (33) for $\partial u/\partial x$, and boundary conditions at the point $(0, 2)$, we get

$$2u_1 + \frac{1}{4}(u_2 - u_{11}) = 12, \text{ or } u_{11} = 8u_1 + u_2 - 48.$$

Applying the finite difference scheme at 1 and eliminating u_{11}, we get

$$u_2 + u_8 + u_{11} + u_4 - 4u_1 = 4(0) = 0,$$

or, $\quad u_2 + (8u_1 + u_2 - 48) - 4u_1 = 0, \text{ or } 2u_1 + u_2 = 24.$

We have the system of equations

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 2 & -9 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 24 \\ -16 \\ -932 \end{bmatrix}$$

The solution of this system is $u_1 = -3.3766,\ u_2 = 30.7532,\ u_3 = 110.3896.$

You may now try the following exercises.

E5) Find the solution of $\nabla^2 u = 0$ in R, where R is the square $0 \le x \le 1,\ 0 \le y \le 1$, subject to the boundary conditions

$$u(x, y) = x^2 - y^2 \text{ on } x = 0,\ y = 0, y = 1; u + \frac{\partial u}{\partial x} = x^2 + 2x - y^2 \text{ on } x = 1$$

using the five point formula. Use central difference approximation in the boundary condition. Assume uniform step length $h = 1/3$ along the axes.

E6) Find the solution of $\nabla^2 u = 0$ in R, where R is the square $0 \le x \le 1,\ 0 \le y \le 1$, subject to the boundary conditions

$$u(x, y) = x + y \text{ on } x = 0, y = 0, y = 1; u + \frac{\partial u}{\partial x} = 1 + x + y \text{ on } x = 1$$

using the five point formula. Use central difference approximation in the boundary condition. Assume uniform step length $h = 1/2$ along the axes.

E7) Find the solution $\nabla^2 u = x^2 + y^2,\ 0 \le x \le 1, 0 \le y \le 1$, subject to the boundary conditions $u(x, y) = (x^4 + y^4)/12$ on $x = 1, y = 0, y = 1$ and

$$12u + \frac{\partial u}{\partial x} = x^4 + y^4 + \frac{1}{3}x^3, \text{ on } x = 0,$$ using the five point formula. Use central difference approximation in the boundary condition. Assume uniform step length $h = 1/2$ along the axes.

E8) The **nine-point formula**, with the same step length along x and y axis (see Fig.13), for the solution of

(a) the Laplace equation $\nabla^2 u = 0$ in R is given by

$$4S_1 + S_2 - 20u_{i,j} = 0.$$

(b) the Poisson equation $\nabla^2 u = G(x, y)$ in R is given by

$$4S_1 + S_2 - 20u_{i,j} = 6h^2\left[G_{i,j} + \frac{h^2}{12}\nabla^2 G_{i,j}\right]$$

where $S_1 = u_{i+1,j} + u_{i-1,j} + u_{i,j-1} + u_{i,j+1}$, and

$$S_2 = u_{i+1,j+1} + u_{i-1,j+1} + u_{i+1,j-1} + u_{i-1,j-1}.$$

Using Taylor series expansions, show that the above formulas are of order $O(h^4)$.

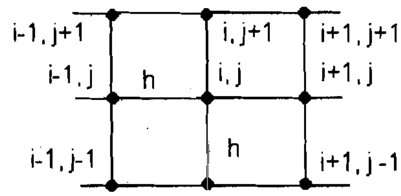


Fig.13: Mesh for Nine point formula

E9). Use the formulas in E8) to solve the problems in the sets E2)a and E3)a.

In the above section, we have derived difference methods for the solution of the elliptic boundary value problems. We shall derive in the next two sections methods for the solution of parabolic and hyperbolic equations.

Let us first consider methods for the solution of parabolic equation.

11.4 DIFFERENCE METHODS FOR PARABOLIC EQUATIONS

Here we shall be illustrating the methods considering the one dimensional heat conduction equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{34}$$

in a region $R(a \le x \le b)$ and $0 \le t \le T$, with suitable initial and boundary conditions. Superimpose on this region a rectangular network of mesh lines defined by $x_i = a + ih,\ i = 0,1,2,\ldots,M;\ x_0 = a, x_M = b,\ h = (b-a)/M;\ t_n = nk,\ n = 0,1,2,\ldots$ At any point (i, n), we denote the exact solution by $u(x_i, t_n)$ and the numerical solution by u_i^n.

Now, approximate

$$\left(\frac{\partial u}{\partial t}\right)_i^n \approx \frac{1}{k}\Delta_t u_i^n = \frac{1}{k}\left[u_i^{n+1} - u_i^n\right] \qquad \text{(see Eqn.(17))} \tag{35}$$

and $$\left(\frac{\partial^2 u}{\partial x^2}\right)_i^n \approx \frac{1}{h^2}\delta_x^2 u_i^n = \frac{1}{h^2}\left[u_{i+1}^n - 2u_i^n + u_{i-1}^n\right] \qquad \text{(see Eqn.(16))} \tag{36}$$

Then, applying Eqn.(34) at the point (i,n), we obtain the formula

$$\frac{1}{k}\left[u_i^{n+1} - u_i^n\right] = \frac{1}{h^2}\left[u_{i+1}^n - 2u_i^n + u_{i-1}^n\right]$$

or, $$u_i^{n+1} = u_i^n + \lambda\left[u_{i+1}^n - 2u_i^n + u_{i-1}^n\right] \tag{37}$$

where $\lambda = k/h^2$ is called the **mesh ratio parameter**. The computational molecule is given in Fig.14. The formula is called a **two level formula** as it involves values on two levels of time, n and n+1. The formula computes explicitly, the solution on the current level n+1, using the values on the previous level n. Hence, it is an **explicit method** called the **Schmidt method**.

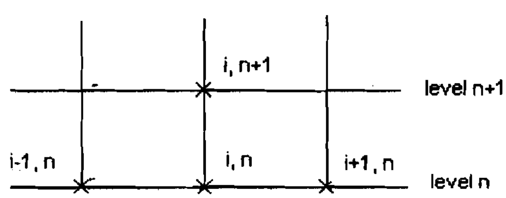


Fig.14: Schmidt method

Now, approximate

$$\left(\frac{\partial u}{\partial t}\right)_i^n \approx \frac{1}{k}\nabla_t u_i^n = \frac{1}{k}\left[u_i^n - u_i^{n-1}\right] \qquad \text{(see Eqn.(18))} \tag{38}$$

and use the same approximation for $\frac{\partial^2 u}{\partial x^2}$. Then applying Eqn.(34) at $(i, n+1)$, we obtain the formula

$$-\lambda u_{i-1}^{n+1} + (1+2\lambda)u_i^{n+1} - \lambda u_{i-1}^{n+1} = u_i^n \tag{39}$$

where $\lambda = k/h^2$ is the **mesh ratio parameter**. The computational molecule is given in Fig.15. The formula is a **two level formula**. The formula contains three unknowns on the current level $n+1$, which are u_{i+1}^{n+1}, u_i^{n+1}, u_{i-1}^{n+1}. Setting $i = 0,1,2,\ldots,M$, or $i = 1,2,\ldots,M-1$, depending on the given boundary conditions, we obtain a system of equations for the required unknowns. Therefore, the solution is being computed implicitly and the method is an **implicit method** called the **Laasonen method**. It is sometimes called a **purely implicit method.**

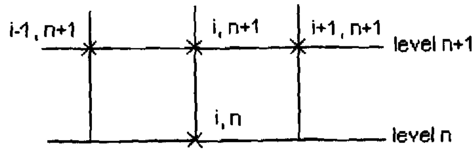


Fig.15: Laasonen method

Now, we write an approximation to Eqn.(34) at $\left(i,\ n+\frac{1}{2}\right)$ as

$$\frac{1}{k}\delta_t u_i^{n+\frac{1}{2}} = \frac{1}{h^2}\delta_x^2 u_i^{n+\frac{1}{2}} = \frac{1}{2h^2}\left[\delta_x^2 u_i^{n+1} + \delta_x^2 u_i^n\right]$$

(40)

The right hand side is the mean of δ_x^2 on the two levels $n, n+1$. Simplifying, we get

$$\left(1-\frac{\lambda}{2}\delta_x^2\right)u_i^{n+1} = \left(1+\frac{\lambda}{2}\delta_x^2\right)u_i^n$$

or,
$$-\frac{\lambda}{2}u_{i-1}^{n+1} + (1+\lambda)\,u_i^{n+1} - \frac{\lambda}{2}u_{i+1}^{n+1} = \frac{\lambda}{2}u_{i-1}^n + (1-\lambda)u_i^n + \frac{\lambda}{2}u_{i+1}^n \tag{41}$$

This formula is a two level implicit formula. It is called the **Crank-Nicolson method.** It is one of the most famous methods as the concept involved is used to derive difference formulas for many partial differential equations. The computational molecule is given in Fig.16.

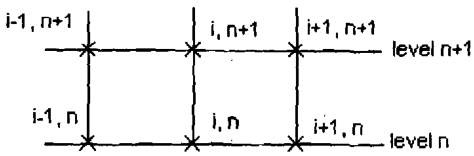


Fig.16: Crank-Nicolson method

It is interesting to note that the above three methods for the solution of the one-dimensional heat equation can be combined as

$$u_i^{n+1} = u_i^n + \lambda\left[\theta\ \delta_x^2\ u_i^{n+1} + (1-\theta)\ \delta_x^2\ u_i^n\right] \tag{42}$$

where $0 \le \theta \le 1$. When $\theta = 0, 1, 1/2$, we get Schmidt method, Laasonen method and Crank-Nicolson method respectively.

Truncation error and order of a method

Consider the formula (37). The truncation error T_i^n at the node $(i, n+1)$ is defined as

$$T_i^n = u(x_i, t_{n+1}) - u(x_i, t_n) - \lambda[u(x_{i-1}, t_n) - 2u(x_i, t_n) + u(x_{i+1}, t_n)] \tag{43}$$

The order of the method is defined by $k^{-1}T_i^n$. (Note that we have multiplied the equations by k to obtain the formulas).

Substituting the Taylor series expansions of each term about (x_i, t_n) on the right hand side of (43) and simplifying, we obtain the truncation error (T.E). However, it would be easier to obtain the T.E, if we find the contributions from the terms $\Delta_t u(x_i, t_n), \delta_x^2\, u(x_i, t_n)$. We have

$$\delta_x^2 u(x_i,t_n) = u(x_{i+1},t_n) - 2u(x_i,t_n) + u(x_{i-1},t_n)$$

$$= \left[\left\{u + h\frac{\partial u}{\partial x} + \frac{h^2}{2}\frac{\partial^2 u}{\partial x^2} + \cdots\right\} - 2u + \left\{u - h\frac{\partial u}{\partial x} + \frac{h^2}{2}\frac{\partial^2 u}{\partial x^2} - \cdots\right\}\right](x_i,t_n)$$

$$= \left[h^2\frac{\partial^2 u}{\partial x^2} + \frac{h^4}{12}\frac{\partial^4 u}{\partial x^4} + \cdots\right](x_i,t_n)$$

$$\Delta_t u(x_i,t_n) = u(x_i,t_{n+1}) - u(x_i,t_n)$$

$$= \left[k\frac{\partial u}{\partial t} + \frac{k^2}{2}\frac{\partial^2 u}{\partial t^2} + \cdots\right](x_i,t_n)$$

Hence,

$$T_i^n = \Delta_t u(x_i,t_n) - \lambda\delta_x^2 u(x_i,t_n)$$

$$= \left[\left\{k\frac{\partial u}{\partial t} + \frac{k^2}{2}\frac{\partial^2 u}{\partial t^2} + \cdots\right\} - \frac{k}{h^2}\left\{h^2\frac{\partial^2 u}{\partial x^2} + \frac{h^4}{12}\frac{\partial^4}{\partial x^4} + \cdots\right\}\right]_{(x_i,t_n)}$$

$$= \left[k\left(\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2}\right) + \frac{k^2}{2}\frac{\partial^2 u}{\partial t^2} + \cdots - \frac{kh^2}{12}\frac{\partial^4 u}{\partial x^4} + \cdots\right\}\Bigg]_{(x_i,\,t_n)}$$

Since from the differential equation

$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 u}{\partial x^2}\right) = \frac{\partial^4 u}{\partial x^4}$, we obtain the T.E as

$$T_i^n = \left[\frac{k}{12}(6k - h^2)\frac{\partial^4 u}{\partial x^4} + \cdots\right]_{(x_i,t_n)} = O(k^2 + kh^2) \qquad (44)$$

The T.E. is of order $O(k^2 + kh^2)$ and hence the order of the method is $k^{-1}T_i^n = O(k + h^2)$. For a fixed value of λ, that is $\lambda = k/h^2$, we have $k = \lambda h^2$ or $k = O(h^2)$. Hence, for a fixed λ, the method behaves like an $O(h^2)$ method. Now, for $\lambda = 1/6$, or $k = h^2/6$, the first term on the right hand side in (44) vanishes. Therefore, for this value of λ, the T.E is of order $O(k^3 + kh^4)$ or, the order of the method is $O(k^2 + h^4)$.

Example 6: Find the solution of the heat conduction equation subject to the given initial and boundary conditions

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},\ 0 \le x \le 1,\ \ u(x,0) = \sin(\pi x),\ 0 \le x \le 1,\ u(0,t) = 0 = u(1,t)$$

using (i) Schmidt method, (ii) Laasonen method and (iii) Crank-Nicolson method, with $h = 1/3$ and $\lambda = 1/6$. Integrate for two time levels. Find the maximum absolute errors on each time level, if the exact solution is $u(x,t) = e^{-\pi^2 t}\sin(\pi x)$.

Solution: The mesh is given in Fig.17.

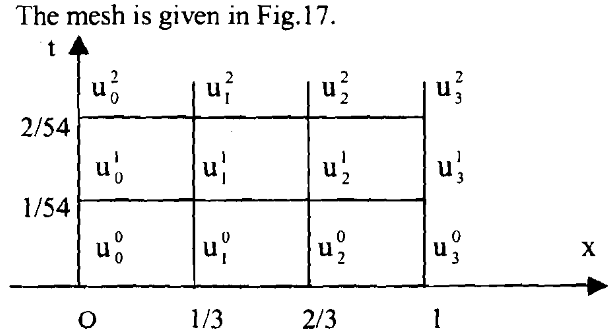


Fig.17: Mesh in Example 6

The nodal points are given by $x_i = ih = \frac{i}{3}, i = 0,1,2,3,$ and $t_n = nk = \frac{n}{54}, n = 0,1,2$.

There are two unknowns on each time level, on the lines $x = 1/3$ and $x = 2/3$.
Boundary conditions give $u_0^n = 0 = u_3^n$, $n = 0, 1, 2$.
Initial condition gives

$$u_0^0 = u(0,0) = 0,\ u_1^0 = u\left(\frac{1}{3}, 0\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2},$$

$$u_2^0 = u\left(\frac{2}{3}, 0\right) = \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2},\ u_3^0 = u(1,0) = \sin\pi = 0.$$

(i) Schmidt method For $\lambda = 1/6$, we have the method as

$$u_i^{n+1} = u_i^n + \frac{1}{6}(u_{i-1}^n - 2u_i^n + u_{i+1}^n)$$

$$= \frac{1}{6}(u_{i-1}^n + 4u_i^n + u_{i+1}^n).$$

On level 1, $n = 0$: $u_i^1 = \frac{1}{6}(u_{i-1}^0 + 4u_i^0 + u_{i+1}^0)$, $i = 1, 2$.

For $i = 1$: $u_1^1 = \frac{1}{6}[u_0^0 + 4u_1^0 + u_2^0]$

$$= \frac{1}{6}\left[0 + 4\left(\frac{\sqrt{3}}{2}\right) + \frac{\sqrt{3}}{2}\right] = \frac{5\sqrt{3}}{12} = 0.72169.$$

For $i = 2$: $u_2^1 = \frac{1}{6}(u_1^0 + 4u_2^0 + u_3^0)$

$$= \frac{1}{6}\left[\frac{\sqrt{3}}{2} + 4\left(\frac{\sqrt{3}}{2}\right) + 0\right] = \frac{5\sqrt{3}}{12} = 0.72169.$$

We have $t_1 = k = 1/54$.

Exact solution is $u\left(\frac{\pi}{3}, \frac{1}{54}\right) = u\left(\frac{2\pi}{3}, \frac{1}{54}\right) = e^{-\pi^2/54}\left(\frac{\sqrt{3}}{2}\right) = 0.72136.$

Maximum absolute error on level 1 $= |\ 0.72169 - 0.72136\ | = 0.00033.$

On level 2, $n = 1$: $u_i^2 = \frac{1}{6}(u_{i-1}^1 + 4u_i^1 + u_{i+1}^1)$, $i = 1, 2$.

For $i = 1$: $u_1^2 = \frac{1}{6}(u_0^1 + 4u_1^1 + u_2^1)$

$$= \frac{1}{6}[0 + 4(0.72169) + 0.72169] = 0.60141.$$

We find $u_2^2 = u_1^2 = 0.60141$.
We have $t_2 = 2k = 1/27$.

Exact solution is $u\left(\frac{\pi}{3}, \frac{2}{54}\right) = u\left(\frac{2\pi}{3}, \frac{2}{54}\right) = e^{-\pi^2/27}\left(\frac{\sqrt{3}}{2}\right) = 0.60087.$

Maximum absolute error on level $2 = |0.60141 - 0.60087| = 0.00054.$

Note that for $\lambda = 1/6$, Schmidt method is of order $O(k^2 + h^4)$.

(ii) Laasonen method For $\lambda = 1/6$, we have the method as

$$-u_{i-1}^{n+1} + 8u_i^{n+1} - u_{i+1}^{n+1} = 6u_i^n.$$

On level 1, $n = 0$: $-u_{i-1}^1 + 8u_i^1 - u_{i+1}^1 = 6u_i^0$, $i = 1, 2$

For $i = 1$: $-u_0^1 + 8u_1^1 - u_2^1 = 6u_1^0 = 6\left(\frac{\sqrt{3}}{2}\right) = 3\sqrt{3}$

or, $\quad 8u_1^1 - u_2^1 = 3\sqrt{3}$, since $u_0^1 = 0$.

For $i = 2$: $-u_1^1 + 8u_2^1 - u_3^1 = 6u_2^0 = 3\sqrt{3}$

or, $\quad -u_1^1 + 8u_2^1 = 3\sqrt{3}$, since $u_3^1 = 0$.

We have the system of equations

$$\begin{bmatrix} 8 & -1 \\ -1 & 8 \end{bmatrix}\begin{bmatrix} u_1^1 \\ u_2^1 \end{bmatrix} = \begin{bmatrix} 3\sqrt{3} \\ 3\sqrt{3} \end{bmatrix}$$

Subtracting the two equations, we get $u_1^1 = u_2^1$.

Hence, $u_1^1 = u_2^1 = \dfrac{3\sqrt{3}}{7} = 0.74231$.

Maximum absolute error on level $1 = |0.74231 - 0.72136| = 0.02095$.

On level 2, $n = 1$: $\quad -u_{i-1}^2 + 8u_i^2 - u_{i+1}^2 = 6u_i^1, i = 1, 2$.

For $i = 1$: $\quad -u_0^2 + 8u_1^2 - u_2^2 = 6u_1^1 = 4.45386$

or, $\quad 8u_1^2 - u_2^2 = 4.45386$, since $u_0^2 = 0$.

For $i = 2$: $\quad -u_1^2 + 8u_2^2 - u_3^2 = 6u_2^1 = 4.45386$

or, $\quad -u_1^2 + 8u_2^2 = 4.45386$, since $u_3^2 = 0$.

We obtain $u_1^2 = u_2^2 = 0.63627$.

Maximum absolute error on level $2 = |0.63627 - 0.60087| = 0.0354$.

(iii) Crank-Nicolson Method For $\lambda = 1/6$, we have the method as

$$-u_{i-1}^{n+1} + 14u_i^{n+1} - u_{i+1}^{n+1} = u_{i-1}^n + 10u_i^n + u_{i+1}^n.$$

On level 1, $n = 0: -u_{i-1}^1 + 14u_i^1 - u_i^1 = u_{i-1}^0 + 10u_i^0 + u_{i+1}^0, i = 1, 2$.

For $i = 1$: $-u_0^1 + 14u_1^1 - u_2^1 = u_0^0 + 10u_1^0 + u_2^0$

or, $\quad 14u_1^1 - u_2^1 = \dfrac{11\sqrt{3}}{2}$.

For $i = 2$: $-u_1^1 + 14u_2^1 - u_3^1 = u_1^0 + 10u_2^0 + u_3^0$

or, $\quad -u_1^1 + 14u_2^1 = \dfrac{11\sqrt{3}}{2}$.

We have the system of equations

$$\begin{bmatrix} 14 & -1 \\ -1 & 14 \end{bmatrix}\begin{bmatrix} u_1^1 \\ u_2^1 \end{bmatrix} = \begin{bmatrix} 11\sqrt{3}/2 \\ 11\sqrt{3}/2 \end{bmatrix}$$

The solution of the system is $u_1^1 = u_2^1 = \dfrac{11\sqrt{3}}{26} = 0.73279$.

Maximum absolute error on level $1 = |0.73279 - 0.72136| = 0.01143$.

On level 2, $n = 1: -u_{i-1}^2 + 14u_i^2 - u_{i+1}^2 = u_{i-1}^1 + 10u_i^1 + u_{i+1}^1, i = 1, 2$.

For $i = 1$: $-u_0^2 + 14u_1^2 - u_2^2 = u_0^1 + 10u_1^1 + u_2^1 = 11(0.73279) = 8.06069$,

or, $\quad 14u_1^2 - u_2^2 = 8.06069$.

For $i = 2$: $-u_1^2 + 14u_2^2 - u_3^2 = u_1^1 + 10u_2^1 + u_3^1 = 11(0.73279) = 8.06069$,

or, $\quad -u_1^2 + 14u_2^2 = 8.06069$.

We have the system of equations

$$\begin{bmatrix} 14 & -1 \\ -1 & 14 \end{bmatrix}\begin{bmatrix} u_1^2 \\ u_2^2 \end{bmatrix} = \begin{bmatrix} 8.06069 \\ 8.06069 \end{bmatrix}$$

The solution of this system is $u_1^2 = u_2^2 = \dfrac{8.06069}{13} = 0.62005$.

Maximum absolute error on level $2 = |0.62005 - 0.60087| = 0.0918$.

You may now try the following exercises.

E10) Show that the Laasonen method is of order $O(k + h^2)$.

E11) Show that the Crank-Nicolson method is of order $O(k^2 + h^2)$.

E12) Find the solution of the following initial boundary value problem, subject to the given initial and boundary conditions.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, u(x,0) = 2x \text{ for } x \in [0,1/2] \text{ and } 2(1-x) \text{ for } [1/2,1],$$
$$u(0,t) = 0 = u(1,t).$$

Use $h = 0.2$. (Note the symmetry in the problem and the solution about $x = 1/2$). Solve using (i) Schmidt method with $\lambda = 1/6$ and $1/4$. (ii) Laasonen method with $\lambda = 1/2$. (iii) Crank-Nicolson method with $\lambda = 1/2$. Integrate for two time levels.

E13) Find the solution of the following initial boundary value problem, subject to the given initial and boundary conditions.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \; 0 \le x \le 1, \; u(x,0) = \sin(2\pi x), \; 0 \le x \le 1, \; u(0,t) = 0 = u(1,t).$$

Assume $h = 0.25$. Solve using (i) Schmidt method with $\lambda = 1/6$ and $1/4$. (ii) Laasonen method with $\lambda = 0.6$. (iii) Crank-Nicloson method with $\lambda = 0.6$. Integrate for two time levels.

After having derived difference methods for elliptic and parabolic equations, we shall now derive methods for hyperbolic equations.

11.5 DIFFERENCE METHODS FOR HYPERBOLIC EQUATIONS

Let us consider the one dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \tag{45}$$

in a region $R(a \le x \le b)$ and $0 \le t \le T$, subject to the initial conditions $u(x,0) = f(x), (\partial u / \partial t)\,(x,0) = g(x)$ and suitable boundary conditions. Superimpose on this region a rectangular network of mesh lines defined by

$x_i = a + ih, i = 0,1,2,\ldots, M; x_0 = a, x_M = b, h = (b-a)/M;\; t_n = nk,\; n = 0,1,2,\ldots$

At any point (i,n), we denote the exact solution by $u(x_i, t_n)$ and the numerical solution by u_i^n.

Now, approximate

$$\left(\frac{\partial^2 u}{\partial t^2}\right)_i^n \approx \frac{1}{k^2}\delta_t^2 u_i^n = \frac{1}{k^2}[u_i^{n+1} - 2u_i^n + u_i^{n-1}] \tag{46}$$

and
$$\left(\frac{\partial^2 u}{\partial x^2}\right)_i^n \approx \frac{1}{h^2}\delta_x^2 u_i^n = \frac{1}{h^2}[u_{i+1}^n - 2u_i^n + u_{i-1}^n] \tag{47}$$

Then, we obtain the formula

$$\delta_t^2 u_i^n = r^2 \delta_x^2 u_i^n$$

where $r = k/h$ is called the **mesh ratio parameter**. Simplifying, we get

$$u_i^{n+1} - 2u_i^n + u_i^{n-1} = r^2[u_{i+1}^n - 2u_i^n + u_{i-1}^n]$$

or,
$$u_i^{n+1} = 2u_i^n - u_i^{n-1} + r^2[u_{i+1}^n - 2u_i^n + u_{i-1}^n] \tag{48}$$

The computational molecule is given in Fig.18. Note that a minimum of three levels is required. The formula computes explicitly, the solution on the current level $n+1$, using the values on the previous levels n and $n-1$.

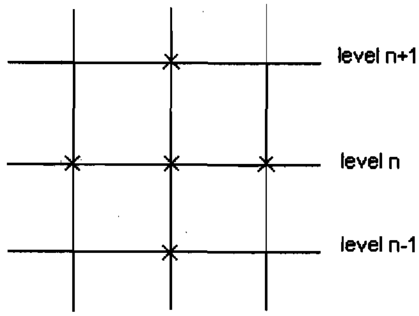


Fig.18: Computational molecule for explicit method (48)

Truncation error

We have

$$\delta_x^2 u(x_i, t_n) = u(x_{i+1}, t_n) - 2u(x_i, t_n) + u(x_{i-1}, t_n)$$

$$= \left[h^2 \frac{\partial^2 u}{\partial x^2} + \frac{h^4}{12} \frac{\partial^4 u}{\partial x^4} + \cdots \right]_{(x_i, t_n)}$$

$$\delta_t^2 u(x_i, t_n) = u(x_i, t_{n+1}) - 2u(x_i, t_n) + u(x_i, t_{n-1})$$

$$= \left[k^2 \frac{\partial^2 u}{\partial t^2} + \frac{k^4}{12} \frac{\partial^4 u}{\partial t^4} + \cdots \right]_{(x_i, t_n)}$$

Hence

$$T_i^n = \delta_t^2 u(x_i, t_n) - r^2 \delta_x^2 u(x_i, t_n)$$

$$= \left[\left\{ k^2 \frac{\partial^2 u}{\partial t^2} + \frac{k^4}{12} \frac{\partial^4 u}{\partial t^4} + \cdots \right\} - \frac{k^2}{h^2} \left\{ h^2 \frac{\partial^2 u}{\partial x^2} + \frac{h^4}{12} \frac{\partial^4 u}{\partial x^4} + \cdots \right\} \right]_{(x_i, t_n)}$$

$$= \left[k^2 \left(\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} \right) + \frac{k^2}{12} \left(k^2 \frac{\partial^4 u}{\partial x^4} - h^2 \frac{\partial^4 u}{\partial x^4} \right) + \cdots \right]_{(x_i, t_n)}$$

$$= \left[\frac{k^2}{12} \left(k^2 \frac{\partial^4 u}{\partial t^4} - h^2 \frac{\partial^4 u}{\partial x^4} \right) + \cdots \right]_{(x_i, t_n)} = O(k^4 + k^2 h^2) \qquad (49)$$

where we have used the differential Eqn.(45).

The T.E is of order $O(k^4 + k^2 h^2)$ and hence the order of the method is $k^{-2} T_i^n = O(k^2 + h^2)$. (Note that we have multiplied the equation by k^2 to obtain the method). For a fixed value of r, that is, $r = k/h$, we have $k = rh$ or $k = O(h)$. Hence, for a fixed value of r, the method behaves like an $O(h^2)$ method.

Implementation of the method

To start the computations, we need the data on two lines $n = 0$ and $n = -1$. At $n = 0$, initial condition is prescribed. The information required on the line $n = -1$ is obtained by using suitable approximation to $\frac{\partial u}{\partial t} = g(x)$, at $t = 0$. We may use the approximation

$$\frac{\partial u}{\partial t} = \frac{1}{2k} \delta_{2t} u + O(k^2), \text{ or } u_i^{-1} = u_i^1 - 2kg_i. \qquad (50)$$

The value of u_i^{-1} obtained from Eqn.(50) may be used in Eqn.(48) for $n = 0$.

Using the ideas introduced in the previous section (see Eqn.(42)), we write a general three level scheme as

$$\delta_t^2 u_i^n = r^2 \delta_x^2 [\theta u_i^{n+1} + (1 - 2\theta) u_i^n + \theta u_i^{n-1}] \qquad (51)$$

For $\theta = 0$, we get the explicit method (48). For $\theta \neq 0$, we get an implicit method. The method is of order $O(k^2 + h^2)$ for all values of θ. For $\theta = 1/2$, we get the method

$$\delta_t^2 u_i^n = \frac{r^2}{2}\delta_x^2[u_i^{n+1} + u_i^{n-1}] \tag{52}$$

and for $\theta = 1$, we get the method

$$\delta_t^2 u_i^n = r^2\delta_x^2[u_i^{n+1} - u_i^n + u_i^{n-1}] \tag{53}$$

Example 7: Find the solution of the initial-boundary value problem

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2},\ 0 \le x \le 1$$

$$u(x,0) = \sin(\pi x), 0 \le x \le 1;\ \frac{\partial u}{\partial t}(x,0) = 0, 0 \le x \le 1,$$

$$u(0,t) = 0,\ u(1,t) = 0, t > 0$$

by using the (i) explicit scheme (48) and (ii) the implicit scheme (52). Use the central difference approximation to the derivative to obtain initial condition. Assume $h = 1/4$, $r = 1/2$. Integrate for one time step. Compare with the exact solution $u(x,t) = \sin(\pi\ x)\cos(\pi\ t)$.

Solution: For $h = 1/4, r = 1/2$, we have $k = rh = 1/8$. The nodal points are given by $x_i = ih$, $i = 0,1,2,3,4,$ and $t_n = nk, n = 0,1,2,\ldots$.

The initial conditions give the values $u_i^0 = \sin(\pi\ i/4)$, $i = 0,1,2,3,4$. The boundary conditions give $u_i^n = 0$, $i = 4$. The mesh is given in Fig.19.

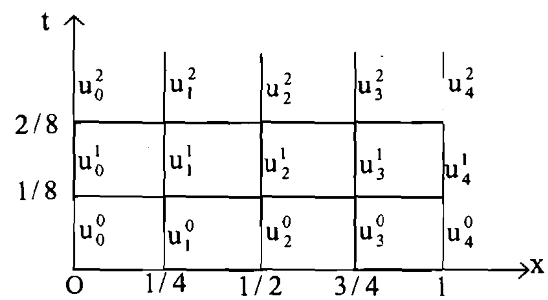


Fig.19: Mesh in Example 7

(i) The explicit method (48) gives

$$u_i^{n+1} = \frac{3}{2}u_i^n + \frac{1}{4}(u_{i-1}^n + u_{i+1}^n) - u_i^{n-1}, i = 1,2,3.$$

Since, $u_t(x,0) = 0$, we get $u_i^{-1} = u_i^1$, $i = 1,2,3$.

For $n = 0$, we get

$$u_i^1 = \frac{3}{2}u_i^0 + \frac{1}{4}(u_{i-1}^0 + u_{i+1}^0) - u_i^{-1}$$

or, $$u_i^1 = \frac{3}{4}u_i^0 + \frac{1}{8}(u_{i-1}^0 + u_{i+1}^0).$$

Substituting the initial values, we get

$$u_1^1 = \frac{3}{4}u_1^0 + \frac{1}{8}(u_0^0 + u_2^0) = \frac{3}{4}\sin\left(\frac{\pi}{4}\right) + \frac{1}{8}(0+1) = 0.65533,$$

$$u_2^1 = \frac{3}{4}u_2^0 + \frac{1}{8}(u_1^0 + u_3^0) = \frac{3}{4} + \frac{1}{8}\left[\sin\left(\frac{\pi}{4}\right) + \sin\left(\frac{3\pi}{4}\right)\right] = 0.92678,$$

$$u_3^1 = \frac{3}{4}u_3^0 + \frac{1}{8}(u_2^0 + u_4^0) = \frac{3}{4}\sin\left(\frac{3\pi}{4}\right) + \frac{1}{8}(1+0) = 0.65593.$$

we have $t_1 = k = 1/8$.

The exact solution is

$$u\left(\frac{1}{4},\frac{1}{8}\right)=u\left(\frac{3}{4},\frac{1}{8}\right)=0.65328,\ u\left(\frac{1}{2},\frac{1}{8}\right)=0.92388.$$

(ii) The implicit method is given by

$$u_i^{n+1}=2u_i^n-u_i^{n-1}+\frac{r^2}{2}\delta_x^2\,[u_i^{n+1}+u_i^{n-1}]$$

or, $$-\frac{1}{8}u_{i-1}^{n+1}+\frac{5}{4}u_i^{n+1}-\frac{1}{8}u_{i+1}^{n+1}=2u_i^n+\frac{1}{8}u_{i-1}^{n-1}-\frac{5}{4}u_i^{n-1}+\frac{1}{8}u_{i+1}^{n-1}.$$

From the initial condition $u_t(x,0)=0$, we get $u_i^{-1}=u_i^1$. Hence, we get

$$-\frac{1}{4}u_{i-1}^1+\frac{10}{4}u_i^1-\frac{1}{4}u_{i+1}^1=2u_i^0.$$

For $i=1,2,3,$ we get the system of equations

$$-\frac{1}{4}u_0^1+\frac{10}{4}u_1^1-\frac{1}{4}u_2^1=2u_1^0=2\sin(\pi/4)=\sqrt{2}$$

$$-\frac{1}{4}u_1^1+\frac{10}{4}u_2^1-\frac{1}{4}u_3^1=2u_2^0=2\sin(\pi/2)=2$$

$$-\frac{1}{4}u_2^1+\frac{10}{4}u_3^1-\frac{1}{4}u_4^1=2u_3^0=2\sin\left(\frac{3\pi}{4}\right)=\sqrt{2}.$$

Substituting $u_0^1=0, u_4^1=0,$ we get

$$\begin{bmatrix}10 & -1 & 0\\ -1 & 10 & -1\\ 0 & -1 & 10\end{bmatrix}\begin{bmatrix}u_1^1\\ u_2^1\\ u_3^1\end{bmatrix}=\begin{bmatrix}4\sqrt{2}\\ 8\\ 4\sqrt{2}\end{bmatrix}$$

The solution of this system is (solve by Gauss elimination)

$$u_1^1=u_3^1=0.65886,\ u_2^1=0.93177.$$

Try the following exercises now.

E14) Show that the method (51) is of order k^2+h^2.

E15) Solve Example 7, with $h=1/4$ and $r=1/3$. Integrate for one time step. Compare with the exact solution $u(x,t)=\sin(\pi x)\cos(\pi t)$.

In the previous two sections, we have derived difference methods for elliptic boundary value problems and initial value problems (IVP). It is important to know whether these methods always converge to the exact solution or not. If not, then what are the conditions under which the methods may converge? Moreover, in the case of IVPs, we perform computations for a very large number of time steps (theoretically infinite cycles of computations). Therefore, it is important to know whether all the errors (truncation error, round of errors, and other errors) are under control, that is, either they decay or at least bounded. To answer these question we shall now discuss consistency, convergence and stability for IVPs.

11.6 CONSISTENCY, CONVERGENCE AND STABILITY OF THE METHODS

A method for solving an IVP is said to be **consistent** if $T.E \to 0$ as $h \to 0$ and $k \to 0$ and the finite difference approximation converges to the corresponding differential equation. Consider the one-dimensional parabolic equation

$$\frac{\partial u}{\partial t}=Lu$$

where L is the differential operator in the space directions and let $L_h u$ be the difference approximation to Lu. Then, the consistency condition can be written as

$$\left\| \frac{u(x,t+k)-u(x,t)}{k} - L_h u \right\| \to 0 \text{ as } h \to 0, k \to 0.$$

Consistency does not guarantee that the solution of the difference equation approximates the solution of the differential equation. If the solution of the difference equation tends to the solution of the differential equation as $h \to 0$ and $k \to 0$, then the difference equation is said to be **convergent**.

Numerical solutions always contain round off and other errors like computational mistakes. The cumulative effect of all these errors should be controlled in order to have meaningful results. If these errors do not increase exponentially with increasing number of time steps, then the method is said to be **stable**. This means that none of the components of the initial solution, say, $u(x,0) = f(x)$ should be amplified infinitely during the infinite cycles of computation. We often test a method only for consistency and stability due to the following theorem of Lax:

For a properly posed IVP and a finite difference equation to it that satisfies the consistency condition, stability is the ***necessary*** *and* ***sufficient*** *condition for convergence of the solution.*

We now discuss, very briefly, the analysis of stability.

Von Neumann (Fourier series) stability analysis

We are considering a constant coefficient partial differential equation in one space dimension and a consistent difference approximation to it. Let $\overline{u}_i^n$ be the numerical solution of this difference equation. Then, we can write

$$\overline{u}_i^n = u_i^n + \varepsilon_i^n \tag{54}$$

where ε_i^n is the error due to round off and/or a computational mistake. We can show that the difference equation itself governs the propagation of errors, that is, the difference equation can be considered as the error equation. Further, for difference equations with constant coefficients, the error can be expanded in a finite Fourier series. The error in the initial condition can be expanded as

$$\varepsilon_m^0 = \sum_j A_{jo} e^{i\beta_j mh} \tag{55}$$

where $\beta_j = j\pi/(b-a)$. We now seek a solution of the error equation such that it reduces to Eqn.(55) at time $t = 0$ (that is for $n = 0$). We write

$$\varepsilon_m^n = \sum_j A_{jn} \xi^n(j) e^{i\beta_j mh} \tag{56}$$

where ξ is an arbitrary real or complex number. Since, for linear homogeneous equations, the sum of linearly independent solutions is also a solution, it is sufficient to consider a single term as

$$\varepsilon_m^n = A\xi^n(j) e^{i\beta mh} \tag{57}$$

where β is a real number and A is an arbitrary constant. In order that the initial error ε_m^0 does not grow as n increases, it is necessary and sufficient that

$$|\xi| \le 1 \tag{58}$$

If this condition is satisfied then the difference scheme is said to be **stable**. The number ξ is called the **amplification factor** of the scheme.

Let us discuss the stability of the methods that we have derived earlier.

Consider the explicit scheme (37) for the solution of the one dimensional heat conduction equation. The error satisfies the equation (since the error equation is the same as the difference scheme)

$$\varepsilon_m^{n+1} = \varepsilon_m^n + \lambda[\varepsilon_{m+1}^n - 2\varepsilon_m^n + \varepsilon_{m-1}^n] \tag{59}$$

(We are using m in place of i to avoid confusion with the imaginary number i).

Substituting ε_m^n given by Eqn.(57) in Eqn.(59), we get

$$A\xi^{n+1}e^{i\beta mh} = A\xi^{n}e^{i\beta mh} + \lambda[A\xi^{n}e^{i\beta(m+1)h} - 2A\xi^{n}e^{i\beta mh} + A\xi^{n}e^{i\beta(m-1)h}]$$

Cancelling $A\xi^{n}e^{i\beta mh}$ on both sides, we obtain the amplification factor as

$$\xi = 1 + \lambda[e^{i\beta h} - 2 + e^{-i\beta h}] = 1 + \lambda[2\cos(\beta h) - 2]$$
$$= 1 + 2\lambda[\cos(\beta h) - 1] = 1 - 4\lambda\sin^2(\beta h/2)$$

The method is stable if

$$|\xi| \le 1, \text{ or, } |1 - 4\lambda\sin^2(\beta h/2)| \le 1$$

or, $\quad -1 \le 1 - 4\lambda\sin^2(\beta h/2) \le 1.$

The right inequality is satisfied automatically. The left inequality gives

$$-1 \le 1 - 4\lambda\sin^2(\beta h/2), \text{ or } \lambda \le \frac{1}{2\sin^2(\beta h/2)}.$$

Since $0 \le \sin^2(\beta h/2) \le 1$, we get the stability condition as $0 < \lambda \le 1/2$, that is, the mesh ratio satisfies the condition $(k/h^2) \le (1/2)$. For any value of λ greater that $1/2$, the computations would blow up.

Consider, now, the Laasonen method (39). Substituting ε_m^n given by Eqn.(57) in Eqn.(39), we get

$$-\lambda A\xi^{n+1}e^{i\beta(m-1)h} + (1+2\lambda)A\xi^{n+1}e^{i\beta mh} - \lambda A\xi^{n+1}e^{i\beta(m+1)h} = A\xi^{n}e^{i\beta mh}.$$

Canceling $A\xi^{n}e^{i\beta mh}$ on both sides, we obtain

$$[1 - \lambda(e^{-i\beta h} - 2 + e^{i\beta h})]\xi = 1, \text{ or } [1 - \lambda\{2\cos(\beta h) - 2\}]\xi = 1$$

or, $\quad [1 + 4\lambda\sin^2(\beta h/2)]\xi = 1, \text{ or } \xi = \dfrac{1}{1 + 4\lambda\sin^2(\beta h/2)}.$

Since $\lambda > 0$, we find that $|\xi| \le 1$ always. Hence, the Laasonen scheme is stable for all values of the mesh ratio parameter λ. Such a scheme is called an **unconditionally stable** scheme.

You may now try the following exercises.

E16) Show that the Crank-Nicolson scheme (41) for the solution of the one dimensional heat equation is unconditionally stable.

E17) The heat conduction equation $u_t = u_{xx}$ is approximated by

$$\frac{1}{2k}(u_m^{n+1} - u_m^{n-1}) = \frac{1}{h^2}(u_{m-1}^n - 2u_m^n + u_{m+1}^n)$$

(i) Determine the truncation error of the method.
(ii) Investigate the stability using the Von Neumann method.

E18) The heat conduction equation $u_t = u_{xx}$ is approximated by

$$u_i^{n+1} = u_i^n + \lambda[\theta\delta_x^2 u_i^{n+1} + (1-\theta)\delta_x^2 u_i^n], \quad 0 \le \theta \le 1.$$

Investigate the stability using the Von Neumann method.

E19) The wave equation $u_{tt} = u_{xx}$ is approximated by

$$\delta_t^2 u_i^n = \frac{r^2}{2}\delta_x^2[u_i^{n+1} + u_i^{n-1}].$$

where $r = k/h$. Investigate the stability using the Von Neumann method.

E20) The wave equation $u_{tt} = u_{xx}$ is approximated by

$$\delta_t^2 u_i^n = r^2\delta_x^2[\theta u_i^{n+1} + (1-2\theta)u_i^n + \theta u_i^{n-1}]$$

where $r = k/h$. Investigate the stability using the Von Neumann method.

We now end this unit by giving a summary of what we have covered in it.

11.7 SUMMARY

We have covered the following points in this unit.

1. Finite difference method for solving

   a) Laplace equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

   b) Poisson equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = G(x, y)$

   c) General elliptic equation

   $$A\frac{\partial^2 u}{\partial x^2} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G$$

   are derived.

2. Solved the above equations under Dirichlet or mixed boundary conditions using the five point formula in a given region R.

3. Derived finite difference methods for solving parabolic and hyperbolic equations.

4. Solved the one-dimensional heat equation and wave equation under different types of initial and boundary conditions using explicit or implicit finite difference methods.

5. Discussed the Von Neumann stability analysis to test the stability of finite difference methods.

11.8 SOLUTIONS/ANSWERS

E1) We use the following Taylor series expansions

$$u(x+h, y, t) = u(x, y, t) + h\frac{\partial u}{\partial x} + \frac{h^2}{2}\frac{\partial^2 u}{\partial x^2} + \frac{h^3}{6}\frac{\partial^3 u}{\partial x^3} + \frac{h^4}{24}\frac{\partial^4 u}{\partial x^4} + \cdots$$

$$u(x+2h, y, t) = u(x, y, t) + 2h\frac{\partial u}{\partial x} + \frac{(2h)^2}{2}\frac{\partial^2 u}{\partial x^2} + \frac{(2h)^3}{6}\frac{\partial^3 u}{\partial x^3} + \frac{(2h)^4}{24}\frac{\partial^4 u}{\partial x^4} + \cdots$$

$$u(x-h, y, t) = u(x, y, t) - h\frac{\partial u}{\partial x} + \frac{h^2}{2}\frac{\partial^2 u}{\partial x^2} - \frac{h^3}{6}\frac{\partial^3 u}{\partial x^3} + \frac{h^4}{24}\frac{\partial^4 u}{\partial x^4} - \cdots$$

$$u(x-2h, y, t) = u(x, y, t) - 2h\frac{\partial u}{\partial x} + \frac{(2h)^2}{2}\frac{\partial^2 u}{\partial x^2} - \frac{(2h)^3}{6}\frac{\partial^3 u}{\partial x^3} + \frac{(2h)^4}{24}\frac{\partial^4 u}{\partial x^4} - \cdots$$

$$u(x, y, t+k) = u(x, y, t) + k\frac{\partial u}{\partial t} + \frac{k^2}{2}\frac{\partial^2 u}{\partial t^2} + \cdots$$

$$u(x, y, t-k) = u(x, y, t) - k\frac{\partial u}{\partial t} + \frac{k^2}{2}\frac{\partial^2 u}{\partial t^2} + \cdots$$

substituting the above relations in the formulas (12) and (18), we obtain the orders as

formula (12) : $\dfrac{h}{2}\dfrac{\partial^2 u}{\partial x^2} + \cdots = O(h)$, formula (13) : $-\dfrac{h}{2}\dfrac{\partial^2 u}{\partial x^2} + \cdots = O(h)$,

formula (14) : $h\dfrac{\partial^3 u}{\partial x^3} + \cdots = O(h)$, formula (15) : $-h\dfrac{\partial^3 u}{\partial x^3} + \cdots = O(h)$,

formula (16) : $\dfrac{h^2}{12}\dfrac{\partial^4 u}{\partial x^4} + \cdots = O(h^2)$, formula (17) : $\dfrac{k}{2}\dfrac{\partial^2 u}{\partial t^2} + \cdots = O(k)$,

formula(18) : $-\dfrac{k}{2}\dfrac{\partial^2 u}{\partial t^2} + \cdots = O(k)$.

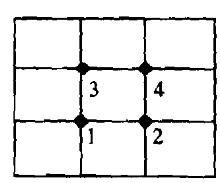


Fig.20

E2) a) Equispaced on all four sides for the points 1, 2, 3 and 4. (see Fig.20)

At 1: $-4u_1 + u_2 + u_3 = 0$

At 2: $u_1 - 4u_2 + u_4 = -4/3$

At 3: $u_1 - 4u_3 + u_4 = 4/3$

At 4: $u_2 + u_3 - 4u_4 = 0$

The solution is $u_1 = 0, u_2 = 1/3,\ u_3 = -1/3,\ u_4 = 0$.

b) Equispaced on all sides for both the points 1 and 2. (see Fig.21)

At 1: $-4u_1 + u_2 = -(1-11-3) = 13$.

At 2: $u_1 - 4u_2 = -(4+6-8) = -2$.

The solution is $u_1 = -10/3,\ u_2 = -1/3$.

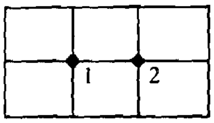


Fig.21

c) Equispaced on all sides for the point 1. (see Fig.22)

$$-4u_1 + \frac{1}{3} - \frac{1}{3} - \frac{1}{9} + \frac{1}{9} = 0 \text{ gives } u_1 = 0$$

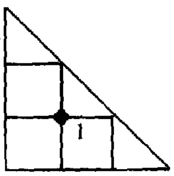


Fig.22

d) Equispaced on all sides for the points 1 and 3. (See Fig.23) At the points 2 and 4, the mesh spacing on the right hand side is $1/6$, while the other mesh spacing are $1/3$. We use formula (25) at the points 2 and 4.

At 1: $-4u_1 + u_2 + u_3 = 0$

At 2: $12u_1 - 54u_2 + 9u_4 = -18$

At 3: $u_1 - 4u_3 + u_4 = 4/3$

At 4: $9u_2 + 12u_3 - 54u_4 = -1$.

The solution is $u_1 = 0,\ u_2 = 1/3,\ u_3 = -1/3,\ u_4 = 0$.

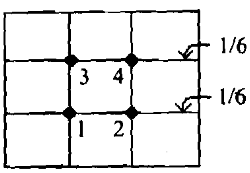


Fig.23

E3) a) See Fig.20. We use the formula (24) with $h = 1/3$ and $G(x, y) = 2x + 3y$.

At 1: $-4u_1 + u_2 + u_3 = 5/27$

At 2: $u_1 - 4u_2 + u_4 = -29/27$

At 3: $u_1 - 4u_3 + u_4 = 44/27$

At 4: $u_2 + u_3 - 4u_4 = 10/27$

The solution is $u_1 = -\dfrac{25}{216},\ u_2 = \dfrac{43}{216},\ u_3 = -\dfrac{103}{216}, u_4 = -\dfrac{35}{216}$

b) See Fig.21, with $h = 1.0$ and $G(x, y) = x - y$

At 1: $-4u_1 + u_2 = 13$

At 2: $u_1 - 4u_2 = -1$

The solution is $u_1 = -17/5,\ u_2 = -3/5$.

c) See Fig.22, with $h = 1/3$ and $G(x, y) = x^2 + y^2$

At 1: $-4u_1 + \dfrac{1}{3} - \dfrac{1}{3} - \dfrac{1}{9} + \dfrac{1}{9} = \dfrac{1}{9}\left(\dfrac{1}{9} + \dfrac{1}{9}\right)$, or $-4u_1 = \dfrac{2}{81}$

We get, $u_1 = -1/162$.

d) See Fig.23, with $h = 1/3$ and $G(x, y) = 2x + 3y$.

At 1: $-4u_1 + u_2 + u_3 = 5/27$

At 2: $12u_1 - 54u2 + 9u_4 = -479/27$

At 3: $u_1 - 4u_3 + u_4 = 44/27$

At 4: $9u_2 + 12u_3 - 54u_4 = -17/27$.

The solution is $u_1 = -0.078902,\ u_2 = 0.305289,\ u_3 = -0.435703,$ $u_4 = -0.034281$ (use Gauss elimination).

E4) For the formula (24), we have

$$\begin{aligned} T.E &= u(x_{i+1}, y_j) + u(x_{i-1}, y_j) + u(x_i, y_{j+1}) + u(x_i, y_{j-1}) \\ &\quad - 4u(x_i, y_j) - h^2 G(x_i, y_j) \end{aligned}$$

$$=[e^{hD_x}+e^{-hD_x}+e^{hD_y}+e^{-hD_y}-4]u(x_i,y_j)-h^2G(x_i,y_j)$$

$$=\left[h^2(D_{xx}+D_{yy})+\frac{h^4}{12}(D_{xxxx}+D_{yyyy})+\cdots\right]u(x_i,y_j)-h^2G(x_i,y_j)$$

where, $D_x=\frac{\partial}{\partial x}$, $D_{xx}=\frac{\partial^2}{\partial x^2}$, etc. Using the differential equation $u_{xx}+u_{yy}=G(x,y)$, we get

$$T.E=\left[\frac{h^4}{12}(D_{xxxx}+D_{yyyy})+\cdots\right]u(x_i,y_j)$$

Therefore, $T.E=O(h^4)$ and order of the method $=h^{-2}(T.E)=O(h^2)$. Similarly for the formula (3), we obtain

$T.E=O(h_1^4+h_1^2h_2^2)$ and order of the method $=h_1^{-2}(T.E)=O(h_1^2+h_2^2)$.

E5) We have the following equations (see Fig.24)

At 1: $-4u_1+u_2+u_4=0$

At 2: $u_1-4u_2+u_3+u_5=-4/9$

At 3: Using central difference approximation, we obtain from the boundary condition

$$u_{11}=\frac{52}{27}-\frac{2}{3}u_3+u_2$$

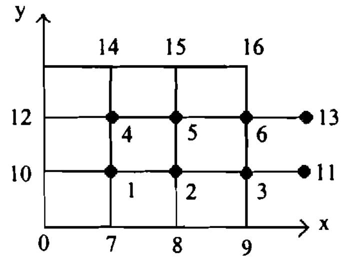


Fig.24

Using the five point formula at 3 and eliminating u_{11}, we get

$$2u_2-\frac{14}{3}u_3+u_6=-79/27$$

At 4: $u_1-4u_4+u_5=12/9$

At 5: $u_2+u_4-4u_5+u_6=5/9$

At 6: Using the central difference approximation, we obtain from the boundary condition

$$u_{13}=\frac{46}{27}+u_5-\frac{2}{3}u_6.$$

Using the five point formula at 6 and eliminating u_{13}, we get

$$u_3+2u_5-\frac{14}{3}u_6=-\frac{46}{27}$$

Solving the above system by Gauss elimination, we obtain the solution as

$u_1=0.000087,\ u_2=0.333555,\ u_3=0.889267,$

$u_4=-0.333207,\ u_5=0.0004172,\ u_6=0.556880.$

E6) We have the following equations (see Fig.25)

At 1: $-4u_1+u_2=-5/2$

At 2: Using the central difference approximation, we obtain from the boundary condition

$$u_3=u_1-u_2+\frac{5}{2}$$

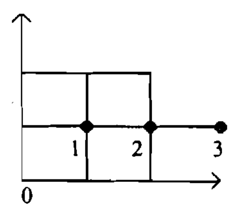


Fig.25

Using the five points formula at 2 and eliminating u_3, we get

$$2u_1-5u_2=-\frac{11}{2}$$

The solution is $u_1=1,\ u_2=3/2$

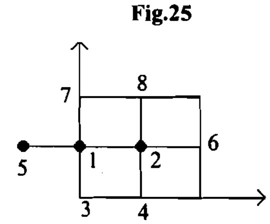


Fig.26

E7) We have the following equations (see Fig.26)

At 1: From boundary condition:

$$12u_1 + (u_2 - u_5) = \frac{1}{16},$$

or, $\quad u_5 = 12u_1 + u_2 - \frac{1}{16}$

Using the five point formula at 1

$$u_2 + \frac{1}{12} + \left(12u_1 + u_2 - \frac{1}{16}\right) - 4u_1 = \frac{1}{4}\left(\frac{1}{4}\right), \text{ or } 4u_1 + u_2 = \frac{1}{48}$$

At 2: $\frac{1}{192} + \frac{17}{192} + u_1 + \frac{17}{192} - 4u_2 = \frac{1}{2}\left(\frac{1}{4}\right)$, or $u_1 - 4u_2 = -\frac{11}{192}$.

The solution is $u_1 = \frac{5}{3264}, u_2 = \frac{1}{68}$.

E8) $S_1 = u(x_i + h, y_j) + u(x_i - h, y_j) + u(x_i, y_j + h) + u(x_i, y_j - h)$

$$= \left[4 + h^2(D_x^2 + D_y^2) + \frac{h^4}{12}(D_x^4 + D_y^4) + \cdots\right] u(x_i, y_j)$$

where $D_x^2 = \frac{\partial^2}{\partial x^2}$, $D_x^4 = \frac{\partial^4}{\partial x^4}$ etc.

$S_2 = u(x_i + h, y_j + h) + u(x_i - h, yi + h) + u(x_i + h, y_j - h) + u(x_i - h, y_j - h)$

$$= \left[4 + 2h^2(D_x^2 + D_y^2) + \frac{h^4}{6}(D_x^4 + 6D_x^2 D_y^2 + D_y^4) + \cdots\right] u(x_i, y_j)$$

$$4S_1 + S_2 - 20u_{i,j} = \left[6h^2\nabla^2 u + \frac{h^4}{2}\nabla^4 u + O(h^6)\right]_{i,j}$$

For $\nabla^2 u = 0$: $4S_1 + S_2 - 20u_{i,j} = O(h^6)$.

Hence, T.E $= O(h^6)$ and the order of the method is 4 or $O(h^4)$.

For $\nabla^2 u = G$: $4S_1 + S_2 - 20u_{i,j} = 6h^2 G_{i,j} + \frac{h^4}{2}\nabla^2 G_{i,j} + O(h^6)$.

Hence, T.E $= O(h^6)$ and the order of the method is 4 or $O(h^4)$.

E9) a) Refer Fig.20. We have $u = x - y$ on Γ and $h = 1/3$. We have the following equations

At 1: $\quad -20u_1 + 4u_2 + 4u_3 + u_4 = 0$

At 2: $\quad 4u_1 - 20u_2 + u_3 + 4u_4 = -7$

At 3: $\quad 4u_1 + u_2 - 20u_3 + 4u_4 = 7$

At 4: $\quad u_1 + 4u_2 + 4u_3 - 20u_4 = 0$

The solution is $u_1 = 0,\ u_2 = 1/3,\ u_3 = -1/3,\ u_4 = 0$.

b) Refer again to Fig.20. We have $G = 2x + 3y$, $u = x - y$ on Γ and $h = 1/3$. Also, $\nabla^2 G = 0$. We have the following equations.

$$4S_1 + S_2 - 20u_{i,j} = \frac{2}{3}G_{i,j}.$$

At 1: $\quad -20u_1 + 4u_2 + 4u_3 + u_4 = 10/9$

At 2: $\quad 4u_1 - 20u_2 + u_3 + 4u_4 = 77/9$

At 3: $\quad 4u_1 + u_2 - 20u_3 + 4u_4 = 79/9$

At 4: $\quad u_1 + 4u_2 + 4u_3 - 20u_4 = 20/9$

The solution is

$u_1 = -0.31361,\ u_2 = -0.59404,\ u_3 = -0.60462, u_4 = -0.36652$.

E10) Using the Taylor series expansions, we get

$$T.E = u(x_i, t_{n+1}) - \lambda[u(x_{i+1}, t_{n+1}) - 2(x_i, t_{n+1}) + u(x_{i-1}, t_{n+1})] - u(x_i, t_n)$$

$$= \left[k\left(\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2}\right) + \frac{k^2}{2}\left(\frac{\partial^2 u}{\partial t^2} - 2\frac{\partial^3 u}{\partial x^2 \partial t}\right) - \frac{kh^2}{12}\frac{\partial^4 u}{\partial x^4} + \cdots \right]_i^n$$

Using the differential equation, we get the order of the method as

$\frac{1}{k}(T.E) = O(k + h^2)$.

E11) Using Taylor series expansions, we get

$$T.E = \left[u(x_i, t_{n+1}) - \frac{k}{2h^2}\{u(x_{i+1}, t_{n+1}) - 2(x_i, t_{n+1}) + u(x_{i-1}, t_{n+1})\} \right]$$

$$- \left[u(x_i, t_n) + \frac{k}{2h^2}\{u(x_{i+1}, t_n) - 2u(x_i, t_n) + u(x_{i-1}, t_n)\} \right]$$

$$= \left[k\left(\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2}\right) + \frac{k^2}{2}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2}\right) - \frac{kh^2}{12}\frac{\partial^4 u}{\partial x^4} - \frac{k^3}{4}\frac{\partial^4 u}{\partial x^2 \partial t^2} + \cdots \right]_i^n$$

Using the differential equation, we get

$T.E = O(k^3 + kh^2)$, and the order of the method is $\frac{1}{k}(T.E) = O(k^2 + h^2)$.

E12) Because of symmetry of the differential equation and initial conditions about $x = \frac{1}{2}$, we have $u(0.2) = u(0.8)$ and $u(0.4) = u(0.6)$ (see Fig.27). It is sufficient to solve for $u(0.2)$ and $u(0.4)$. The conditions give

$u_0^0 = 0,\ u_1^0 = 0.4,\ u_2^0 = 0.8 = u_3^0,\ u_0^n = 0 = u_5^n,\ n \geq 0$.

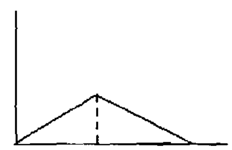


Fig.27: Initial conditions

Schmidt method $\lambda = \frac{1}{6}$ $u_i^{n+1} = u_i^n + \frac{1}{6}(u_{i+1}^n - 2u_i^n + u_{i-1}^n)$.

We get $u_1^1 = 0.4,\ u_2^1 = 0.73333,\ u_1^2 = 0.38889,\ u_2^2 = 0.677775$.

Similarly, solve for $\lambda = 1/4$.

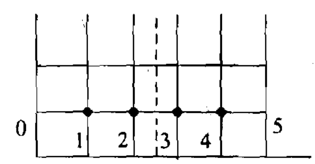


Fig.27: Mesh

Laasonen method $\lambda = \frac{1}{2}$: $-\frac{1}{2}u_{i-1}^{n+1} + 2u_i^{n+1} - \frac{1}{2}u_{i+1}^{n+1} = u_i^n$.

$n = 0$: For $i = 1$ and $i = 2$, we get

$$2u_1^1 - \frac{1}{2}u_2^1 = 0.4,\ -\frac{1}{2}u_1^1 + \frac{3}{2}u_2^1 = 0.8$$

whose solution is $u_1^1 = 0.363638,\ u_2^1 = 0.65455$.

$n = 1$: For $i = 1$ and $i = 2$, we get

$$2u_1^1 - \frac{1}{2}u_2^1 = 0.363638,\ -\frac{1}{2}u_1^2 + \frac{3}{2}u_2^2 = 0.65455$$

whose solution is $u_1^2 = 0.31736,\ u_2^2 = 0.54215$.

Crank-Nicolson method $\lambda = \frac{1}{2}$.

$$-\frac{1}{4}u_{i-1}^{n+1} + \frac{3}{2}u_i^{n+1} - \frac{1}{4}u_{i+1}^{n+1} = \frac{1}{4}u_{i-1}^n + \frac{1}{2}u_i^n + \frac{1}{4}u_{i+1}^n$$

$n = 0$: For $i = 1$ and $i = 2$, we get

$$\frac{3}{2}u_1^1 - \frac{1}{4}u_2^1 = 0.4,\ -\frac{1}{4}u_1^1 + \frac{5}{4}u_2^1 = 0.7$$

whose solution is $u_1^1 = 0.37241,\ u_2^1 = 0.63448$.

$n = 1$: For $i = 1$ and $i = 2$, we get

$$\frac{3}{2}u_1^2 - \frac{1}{4}u_2^2 = 0.344825,\ -\frac{1}{4}u_1^2 + \frac{5}{4}u_2^2 = 0.56896$$

whose solution is $u_1^2 = 0.31629,\ u_2^2 = 0.51843$.

E13) See Fig.28. The conditions give

$$u_0^0 = 0,\ u_1^0 = 1,\ u_2^0 = 0,\ u_3^0 = -1,\ u_4^0 = 0,\ u_0^n = 0 = u_4^n.$$

Schmidt method $\lambda = \frac{1}{6}$, $u_i^{n+1} = u_i^n + \frac{1}{6}(u_{i-1}^n - 2u_i^n + u_{i+1}^n)$.

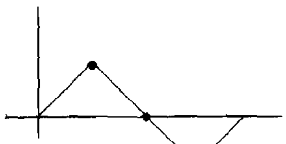


Fig.28. Initial condition

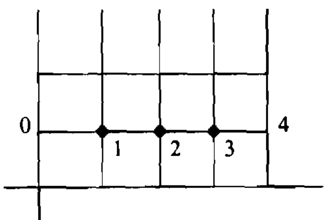


Fig.28. Mesh

We obtain the following results: $u_1^1 = \frac{2}{3},\ u_2^1 = 0,\ u_3^1 = -\frac{2}{3}$.

$u_1^2 = \frac{4}{9},\ u_2^2 = 0,\ u_3^2 = -\frac{4}{9}$.

Laasonen method $\lambda = 0.6$: $-0.6u_{i-1}^{n+1} + 2.2u_i^n - 0.6u_{i+1}^n = u_i^n$.

$n = 0$: $2.2u_1^1 - 0.6u_2^1 = 1,\ -0.6u_1^1 + 2.2u_2^1 - 0.6u_3^1 = 0,$

$-0.6u_2^1 + 2.2u_3^1 = -1.$

Solution is $u_1^1 = 0.454545,\ u_2^1 = 0,\ u_3^1 = -u_1^1 = 0.454545$.

Similarly, obtain for level 2.

Crank-Nicolson method $\lambda = 0.6$.

$$-0.3u_{i-1}^{n+1} + 1.6u_i^{n+1} - 0.3u_{i+1}^{n+1} = 0.3u_{i-1}^n + 0.4u_i^n + 0.3u_{i+1}^n$$

$n = 0: 1.6u_1^1 - 0.3u_2^1 = 0.4,\ -0.3u_1^1 + 1.6u_2^1 - 0.3u_3^1 = 0.$

$-0.3u_2^1 + 1.6u_3^1 = -0.4$.

Solution is $u_1^1 = 0.25,\ u_2^1 = 0,\ u_3^1 = -u_1^1 = 0.25$.

Similarly, obtain for level 2.

E14) $u_{tt} = u_{xx}$. The method is

$$\delta_t^2 u_i^n = r^2\delta_x^2\left[\theta u_i^{n+1} + (1-2\theta)u_i^n + \theta u_i^{n-1}\right]$$

$$= r^2\delta_x^2\left[u_i^n + \theta\delta_t^2 u_i^n\right]$$

Using Taylor series expansions, we get

$$\delta_t^2 u_i^n = \left[k^2 D_t^2 + \frac{k^4}{12}D_t^4 + \cdots\right]u_i^n$$

$$\delta_x^2 u_i^n = \left[h^2 D_x^2 + \frac{h^4}{12}D_x^4 + \cdots\right]u_i^n, \text{ where } D_t^2 = \frac{\partial^2}{\partial t^2}, D_x^2 = \frac{\partial^2}{\partial x^2}, \text{ etc.}$$

$$T.E = \left[k^2(D_t^2 - D_x^2) + k^2\left(\frac{1}{12} - \theta\right)D_x^4 - \frac{h^2}{12}D_x^4 + \cdots\right]u_i^n$$

Using the differential equation $u_{tt} = u_{xx}$, that is $\frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial x^2}$, we get

$$\text{order} = \frac{1}{k^2}(T.E)$$

$$= \left[k^2\left(\frac{1}{2} - \theta\right)D_x^4 - \frac{h^2}{12}D_x^4 + \cdots\right]u_i^n = O(k^2 + h^2).$$

E15) $k = \frac{1}{12}$. Initial and boundary conditions give

$$u_i^0 = \sin\left(\frac{\pi i}{4}\right), i = 0,1,2,3,4;\ u_0^n = 0 = u_4^n,\ n \geq 1.$$

(see Fig.19). Also, $u_i^{-1} = u_i^1$.

Explicit method $u_i^{n+1} = 2u_i^n - u_i^{n-1} + \frac{1}{9}(u_{i-1}^n + u_{i+1}^n)$.

$n = 0 : u_i^1 = 2u_i^0 - u_i^{-1} + \frac{1}{9}(u_{i-1}^0 + u_{i+1}^n)$

or $\quad u_i^1 = u_i^0 + \frac{1}{18}(u_{i-1}^0 + u_{i+1}^0)$.

We get $u_1^1 \doteq 0.76266,\ u_2^1 = 1.07857,\ u_3^1 = 0.76266$.

Exact solution : $u\left(\frac{1}{4}, \frac{1}{12}\right) = \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{12}\right) = 0.68301$

$u\left(\frac{3}{4}, \frac{1}{12}\right) = 0.68301,\ u\left(\frac{1}{2}, \frac{1}{12}\right) = \cos\left(\frac{\pi}{12}\right) = 0.96593$.

Implicit method

$$u_i^{n+1} = 2u_i^n - u_i^{n-1} + \frac{1}{18}\left[\delta_x^2 u_i^{n+1} + \delta_x^2 u_i^{n-1}\right] - \frac{1}{18}u_{i-1}^{n+1} + \frac{10}{9}u_i^{n+1} - \frac{1}{18}u_{i+1}^{n+1}$$

$$= 2u_i^n - u_i^{n-1} + \frac{1}{18}(u_{i-1}^{n-1} - 2u_i^{n-1} + u_{i+1}^{n-1})$$

using the initial condition for $n = 0,\ u_i^{-1} = u_i^1$, we get the system

$$20u_1^1 - u_2^1 = \frac{18}{\sqrt{2}},\ -u_1^1 + 20u_2^1 - u_3^1 = 18,\ -u_2^1 + 20u_3^1 = \frac{18}{\sqrt{2}}$$

The solution is $u_1^1 = u_3^1 = 0.68482,\ u_2^1 = 0.96848$.

E16) Substitute $\epsilon_m^n = A\xi^n e^{i\beta mh}$. (Note that the given difference scheme is the same as the error equation). The method is

$$\left(1 - \frac{\lambda}{2}\delta_x^2\right)u_m^{n+1} = \left(1 + \frac{\lambda}{2}\delta_x^2\right)u_m^n$$

substituting for ϵ_m^n, we get

$$\delta_x^2 \epsilon_m^{n+1} = A\xi^{n+1}e^{i\beta mh}\left[e^{\beta h} - 2 + e^{-\beta h}\right]$$

$$= A\xi^{n+1}e^{i\beta mh}\left[2\cos\beta - 2\right] = -4A\xi^{n+1}e^{i\beta mh}\sin^2\left(\frac{\beta h}{2}\right)$$

Similarly, $\delta_x^2 \epsilon_m^n = -4A\xi^n e^{i\beta mh}\sin^2\left(\frac{\beta h}{2}\right)$.

Substituting in the method and simplifying, we get

$$\left[1 + 2\lambda\sin^2\left(\frac{\beta h}{2}\right)\right]A\xi^n e^{i\beta mh} = \left[1 - 2\lambda\sin^2\left(\frac{\beta h}{2}\right)\right]A\xi^n e^{i\beta mh}$$

or, $\quad \xi = \dfrac{1 - 2\lambda\sin^2(\beta h/2)}{1 + 2\lambda\sin^2(\beta h/2)} \le 1$, always

since $\sin^2\left(\frac{\beta h}{2}\right) \ge 0$ and $\lambda > 0$. Therefore, Crank-Nicolson scheme is unconditionally stable.

E17) The given method is $u_m^{n+1} - u_m^{n-1} = 2\frac{k}{h^2}\delta_x^2 u_m^n$

Expanding in Taylor series, we get

$$\text{T.E} = \left[2k\frac{\partial u}{\partial t} + \frac{k^3}{3}D_t^3 + \cdots\right]u_m^n - 2\frac{k}{h^2}\left[h^2 D_x^2 + \frac{h^4}{12}D_x^4 + \cdots\right]u_m^n$$

$$= \left[2k\left(\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2}\right) + \frac{k^3}{3}\frac{\partial^3 u}{\partial t^3} - \frac{kh^2}{6}\frac{\partial^4 u}{\partial x^4} + \cdots\right]_m^n$$

$$= O(k^3 + kh^2)$$

Substituting $\epsilon_m^n = A\xi^n e^{i\beta mh}$, we get the characteristic equation as

$$\left[\xi - \xi^{-1}\right]A\xi^n e^{i\beta mh} = 2\lambda\left[-4\sin^2\phi\right]A\xi^n e^{i\beta mh}$$

or, $\xi^2 + 8\lambda \sin^2 \theta\xi - 1 = 0$, where $\phi = \dfrac{\beta h}{2}$.

We get, $\xi = \dfrac{1}{2}\left[-8\lambda \sin^2 \phi \pm \sqrt{64\lambda^2 \sin^4 \phi + 4}\right]$

Now, $64\lambda^2 \sin^4 \phi + 4 \geq 4$. Therefore, there is at least one root (with signs of both the terms in the numerator taken as negative) for which $|\xi| > 1$. Therefore, the method is unstable.

E18) Replace suffix i by m. Substituting $\epsilon_m^n = A\xi^n e^{i\beta mh}$ and canceling $A\xi^n e^{i\beta mh}$, we get

$$[1 + 4\lambda\theta \sin^2 \phi]\xi = 1 - 4\lambda(1-\theta)\sin^2 \phi, \text{ where } \phi = \frac{\beta h}{2}.$$

or, $\xi = \dfrac{1 - 4\lambda(1-\theta)s^2}{1 + 4\lambda\theta s}$, where $s = \sin\phi$.

For stability, we require $|\xi| \leq 1$, or

$$-1 \leq \frac{1 - 4\lambda(1-\theta)s^2}{1 + 4\lambda\theta s} \leq 1$$

or, $-1 - 4\lambda\theta s \leq 1 - 4\lambda(1-\theta)s^2 \leq 1 + 4\lambda\theta s^2$

The right inequality gives $\lambda > 0$, which is true. The left inequality gives

$$-2 - 4\lambda(2\theta - 1)s^2 \leq 0.$$

For $\theta \geq \dfrac{1}{2}$, this result is always true. Hence, the method is unconditionally stable for $\theta \geq \dfrac{1}{2}$. For $\theta < \dfrac{1}{2}$, we write $-2 + 4\lambda(1-2\theta)s^2 \leq 0$.

Using the maximum value of $\sin\phi$, we get $-2 + 4\lambda(1-2\theta) \leq 0$, or

$\lambda \leq \dfrac{1}{2(1-2\theta)}$, which is the stability condition. For $\theta = 0$ (explicit scheme), we get $\lambda \leq \dfrac{1}{2}$.

E19) Substituting $\epsilon_m^n = A\xi^n e^{i\beta mh}$, we get the characteristic equation as

$$(\xi - 2 + \xi^{-1}) = r^2(-2s^2)(\xi + \xi^{-1}), \text{ where } s = \sin\left(\frac{\beta h}{2}\right)$$

or, $\xi^2(1 + 2r^2s^2) - 2\xi + (1 + 2r^2\xi^2) = 0$

or, $\xi^2 - 2B\xi + 1 = 0$, where $B = \dfrac{1}{1 + 2r^2s^2}$

The roots are $\xi = \left[B \pm \sqrt{B^2 - 1}\right]$.

If $|B| > 1$, one of the roots is always greater than 1.

If $|B| = 1$, $|\xi| = 1$.

If $|B| < 1$, then ξ is a complex pair $\xi = B \pm (\sqrt{B^2 - 1})\, i$, whose magnitude is $|\xi| \leq 1$. Hence, for stability, we require $|B| \leq 1$. Therefore, we require

$$-1 \leq \frac{1}{1 + 2r^2s^2} \leq 1, \text{ or } -1 - 2r^2s^2 \leq 1 \leq 1 + 2r^2s^2$$

The right inequality is always true. The left inequality gives $-2 - 2r^2s^2 \leq 0$ which is also true. Hence, the method is unconditionally stable.

E20) Following problem E19), the characteristic equation is obtained as

$$\xi^2 - 2B\xi + 1 = 0, \text{ where } B = 1 - \frac{2r^2s^2}{1+4\theta r^2s^2}, \quad s = \sin\left(\frac{\beta h}{2}\right).$$

For stability, we require $|B| \leq 1$ or,

$$-1 \leq 1 - \frac{2r^2s^2}{1+4\theta r^2s^2} \leq 1$$

The right inequality is always true. The left inequality gives

$$-2 \leq -\frac{2r^2s^2}{1+4\theta\, r^2s^2} \quad \text{or} \quad 1 + r^2s^2(4\theta - 1) \geq 0.$$

This is always satisfied for $\theta \geq \frac{1}{4}$. Therefore, the method is unconditionally stable for $\theta \geq \frac{1}{4}$. For $0 < \theta < \frac{1}{4}$, we get the stability condition as $r^2 \leq \frac{1}{1-4\theta}$.

—x—

PRACTICAL ASSIGNEMENTS

Sessions 4 & 5:

1. Write a program to solve
   (a) Laplace equation $\nabla^2 u = 0$ on (i) a square of side a (ii) a rectangle of side a, b.
   (b) Poisson equation $\nabla^2 u = G(x, y)$ on (i) a square of side a (ii) a rectangle of side a, b.

   Use Dirichlet boundary conditions. Use a method of second order and uniform step length h along x and y directions. Solve the resulting equations by Gauss elimination method. Test your program on exercises E2) and E3).

2. Write a program to solve

   $$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 \le x \le a, \quad t \ge 0$$

   $$u(x, 0) = f(x), \quad u(0, t) = A, \quad u(a, t) = B.$$

   using

   (i) Schmidt method
   (ii) Laasonen method
   (iii) Crank-Nicolson method.

   Input h, k and number of steps n. Solve the tridiagonal systems in (ii) and (iii) above by using Gauss elimination method. Test your program on exercises E12) and E13).

3. Write a program to solve

   $$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad 0 \le x \le 1, \quad t \ge 0.$$

   $$u(x, 0) = f(x), \quad \frac{\partial u}{\partial t}(x, 0) = g(x)$$

   $$u(0, t) = A, \quad u(1, t) = B.$$

   using

   (i) explicit scheme
   (ii) implicit scheme.

   Input h, k and number of steps. Solve the tridiagonal system in (ii) using Gauss elimination method. Use central difference approximation to $\left(\frac{\partial u}{\partial t}\right)(x, 0)$. Test your program on Example 7 and E15).

UNIT 12 FINITE ELEMENT METHODS

Structure **Page No.**

12.1 INTRODUCTION

In Unit 11, we have discussed the finite difference method for finding solutions of partial differential equations. In particular, we have solved the Laplace equation, Poisson equation and one-dimensional heat and wave equations. In this unit, we shall introduce another popular numerical method called finite element method for solving partial differential equations. Finite element methods are applicable to both initial and boundary value problems of ordinary and partial differential equations. However, we shall confine our discussion to the boundary value problems and in particular, discuss finite element methods for solving Laplace and Poisson equations in two dimensions. Finite element methods have some advantages over finite difference methods. One advantage of the finite element method over finite difference method is the relative ease with which the boundary conditions of the problem are handled. Irregularly shaped boundaries are handled in a natural manner by the finite element method in comparison to the finite difference method, where special formulas have to be developed for the boundaries.

In Sec.12.2, we have started the discussion by defining triangular and rectangular finite elements usually used in the case of two-dimensional problems. In Sec.12.3 Galerkin's finite element method which is a weighted residual method is discussed and its application to the Dirichlet boundary value problem for Poisson equation and Laplace equation is illustrated. Variational formulation of the Laplace and Poisson equations are also given in this section.

Objectives

After studying this unit you should be able to

- discretize a given two dimensional domain using triangular and rectangular finite elements;
- derive finite element Galerkin method for the solution of Laplace and Poisson equations with Dirichlet boundary conditions.

12.2 FINITE ELEMENTS

In finite element methods, we generate difference equations by using the variational principle or weighted residual methods. The closed domain R, where the given partial differential equation holds, is divided into a finite number of non-overlapping subdomains $R_1, R_2, \ldots, R_M$. These subdomains are called the finite elements. We use the straight line elements in the one dimensional case (see Fig.1), that is in solving ordinary differential equations. In Fig.1, the interval $[a, b]$ is subdivided into three straight line elements e_1, e_2, e_3. Generally, in two dimensions, we use the triangular or rectangular elements (see Figs.2, 3). In Fig.2, we have eight triangular elements numbered $e_1, e_2, \ldots, e_8$. In Fig.3, we have four rectangular elements numbered e_1, e_2, e_3, e_4. The **curved boundaries** are handled in a natural manner.

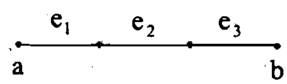


Fig.1: Straight line element

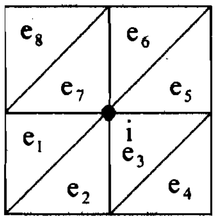


Fig.2: Triangular element

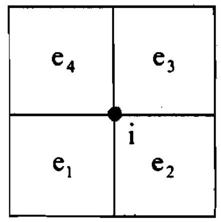


Fig.3: Rectangular element

In each of the elements e_i, we approximate the solution by a function say w, which is continuous and defined in terms of the nodal values belonging to that element. At the boundaries of the elements called the **interfaces**, the inter element conditions are to be satisfied. The general requirement at the interface is that the approximating function and its partial derivatives upto order one less than the highest order derivative occurring in the partial differential equation or its variational form must be continuous. The approximate solution w is then substituted in the differential equation and the weighted residual method is applied. Alternatively, solution w is substituted in the variational form of the partial differential equation. This gives rise to a system of linear or non-linear difference equations. The solution of this system gives the approximate solution of the partial differential equation at the nodal points in R. For simple networks, the difference equations derived by the finite difference and finite element methods are identical. As we have mentioned earlier, in this unit, we shall consider only the solution of Laplace and Poisson equations in two dimensions. The simple finite elements that can be used in this case are triangular and rectangular elements.

Triangular and Rectangular Elements

As you know, line segment elements (see Fig.1) are used to solve ordinary differential equations. For solving partial differential equations, the two basic elements that are used are triangular or rectangular elements. That is, the two dimensional domain R is subdivided into triangular or rectangular elements.

Assemblage of triangles can always represent a two dimensional domain of any shape. Normally, we use equilateral triangles.

Triangular element

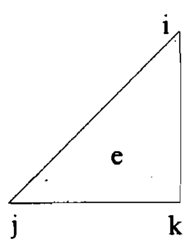


Fig.4: Triangular element

Consider the triangular element with corners at $(x_i, y_i), (x_j, y_j), (x_k, y_k)$, taken in the anti-clockwise direction. This type of element is called a three node triangle. We have three degrees of freedom (three nodal values at the corners), (see Fig.4). We write a linear approximation inside each element of the form

$$u^{(e)}(x, y) = a_1 + a_2 x + a_3 y \tag{1}$$

At the nodes, we get

$$u_i = a_1 + a_2 x_i + a_3 y_i$$

$$u_j = a_1 + a_2 x_j + a_3 y_j$$

$$u_k = a_1 + a_2 x_k + a_3 y_k$$

For the solution of a_1, a_2, a_3, we have the system

$$\begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} u_i \\ u_j \\ u_k \end{bmatrix}$$

Using Cramer's rule, the solution of this system is obtained as

$$a_1 = \frac{\Delta_1}{2\Delta},\ a_2 = \frac{\Delta_2}{2\Delta},\ a_3 = \frac{\Delta_3}{2\Delta}$$

where Δ is the area of the triangle and

$$2\Delta = 2(\text{area of triangle}) = \begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{vmatrix}$$

$$= (x_i - x_k)(y_j - y_k) - (x_j - x_k)(y_i - y_k)$$

$$\Delta_1 = \begin{vmatrix} u_i & x_i & y_i \\ u_j & x_j & y_j \\ u_k & x_k & y_k \end{vmatrix} = u_i(x_j y_k - x_k y_j) - u_j(x_i y_k - y_i x_k) + u_k(x_i y_j - x_j y_i)$$

$$\Delta_2 = \begin{vmatrix} 1 & u_i & y_i \\ 1 & u_j & y_j \\ 1 & u_k & y_k \end{vmatrix} = u_i(y_j - y_k) + u_j(y_k - y_i) + u_k(y_i - y_j)$$

$$\Delta_3 = \begin{vmatrix} 1 & x_i & u_i \\ 1 & x_j & u_j \\ 1 & x_k & u_k \end{vmatrix} = u_i(x_k - x_j) + u_j(x_i - x_k) + u_k(x_j - x_i)$$

Substituting the values of a_1, a_2, a_3 in Eqn.(1) and simplifying we obtain the approximation as

$$u^{(e)}(x,y) = \frac{1}{2\Delta}(\Delta_1 + \Delta_2 x + \Delta_3 y) \qquad (2)$$
$$= N_i^{(e)}(x,y)u_i + N_j^{(e)}(x,y)u_j + N_k^{(e)}(x,y)u_k$$

where,

$$N_i^{(e)}(x,y) = \frac{1}{2\Delta}\left[(x_j y_k - x_k y_j) + (y_j - y_k)x + (x_k - x_j)y\right],$$
$$N_j^{(e)}(x,y) = \frac{1}{2\Delta}\left[(x_k y_i - x_i y_k) + (y_k - y_i)x + (x_i - x_k)y\right], \qquad (3)$$
$$N_k^{(e)}(x,y) = \frac{1}{2\Delta}\left[(x_i y_j - x_j y_i) + (y_i - y_j)x + (x_j - x_i)y\right].$$

$N_i^{(e)}, N_j^{(e)}, N_k^{(e)}$ are called the **shape functions** of the approximation. Setting $x = x_i$, $y = y_i$ in Eqn.(2), we get

$$u^{(e)}(x_i, y_i) \equiv u_i$$
$$= N_i^{(e)}(x_i, y_i)u_i + N_j^{(e)}(x_i, y_i)u_j + N_k^{(e)}(x_i, y_i)u_k$$

Hence, $N_i^{(e)}(x_i, y_i) = 1$,

$N_j^{(e)}(x_i, y_i) = 0$,

$N_k^{(e)}(x_i, y_i) = 0$.

Similarly, substituting $x = x_j, y = y_j$ and $x = x_k, y = y_k$ in Eqn(2), we find that the shape functions satisfy

$$N_i^{(e)}(x_i, y_i) = 1 = N_j^{(e)}(x_j, y_j) = N_k^{(e)}(x_k, y_k)$$

and $N_m^{(e)}(x_r, y_r) = 0$, for $m \neq r$, $r = i, j, k$.

We can verify these results directly from Eqn(3) also.
In vector notation, we can write Eqn.(2) as

$$u^{(e)}(x,y) = [N_i^{(e)} \;\; N_j^{(e)} \;\; N_k^{(e)}] \begin{bmatrix} u_i \\ u_j \\ u_k \end{bmatrix} = [N^{(e)}]\{u^{(e)}\} \qquad (4)$$

where $[N^{(e)}] = [N_i^{(e)} \;\; N_j^{(e)} \;\; N_k^{(e)}]$ and $\{u^{(e)}\} = [u_i \;\; u_j \;\; u_k]^T$.

If the domain contains K elements, the representation of the solution variable over the whole domain is given by

$$u(x,y)=\sum_{e=1}^{K} u^{(e)}(x,y)=\sum_{e=1}^{K}[N^{(e)}]\,\{u^{(e)}\} \tag{5}$$

On the interfaces, that is, the sides of the triangles, $u(x,y)$ has no discontinuities. The elements that we are considering here are called **conforming elements.** Two adjoining elements are conforming if the value of u is determined uniquely on their common edge.

Rectangular element

Let the domain R be divided into rectangular elements. Assume that the sides of the element are parallel to x and y axes respectively. Since there are four vertices for a rectangle a piecewise linear approximation of the type (1), (which has only three parameters) is not possible. Hence, we choose the piecewise polynomial in the form

$$u(x,y)=a_1+a_2x+a_3y+a_4xy \tag{6}$$

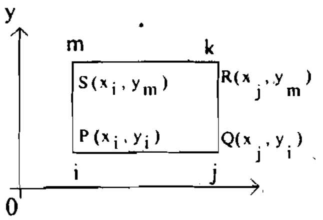


Fig.5: Rectangular element

Let i, j, k and m denote the corners $P(x_i,y_i)$, $Q\,(x_j,y_i)$, $R(x_j,y_m)$ and $S(x_i,y_m)$, (see Fig. 5). We call these the nodes of the rectangle. At the nodes, (setting $(x,y)=(x_i,y_i)$, $(x,y)=(x_j,y_i)$, $(x,y)=(x_j,y_m)$, $(x,y)=(x_i,y_m)$, in Eqn.(6)), we get

$$u_i=a_1+a_2x_i+a_3y_i+a_4x_iy_i \tag{7}$$

$$u_j=a_1+a_2x_j+a_3y_i+a_4x_jy_i \tag{8}$$

$$u_k=a_1+a_2x_j+a_3y_m+a_4x_jy_m \tag{9}$$

$$u_m=a_1+a_2x_i+a_3y_m+a_4x_iy_m \tag{10}$$

Subtracting Eqn.(10) from Eqn.(7), we get

$$(y_i-y_m)\,a_3+x_i(y_i-y_m)\,a_4=u_i-u_m$$

or,

$$a_3+a_4x_i=\frac{u_i-u_m}{y_i-y_m} \tag{11}$$

Subtracting Eqn.(9) from Eqn.(8), we get

$$a_3(y_i-y_m)+x_j(y_i-y_m)\,a_4=u_j-u_k$$

or,

$$a_3+a_4x_j=\frac{u_j-u_k}{y_i-y_m} \tag{12}$$

Subtracting Eqn.(12) from Eqn.(11), we obtain

$$a_4(x_i-x_j)=\frac{u_i-u_m-u_j+u_x}{y_i-y_m}$$

or,

$$a_4=\frac{u_i-u_m-u_j+u_k}{(x_i-x_j)\,(y_i-y_m)}$$

Hence, $a_3=\dfrac{u_i-u_m}{y_i-y_m}-a_4x_i$.

Subtracting Eqn.(8) from Eqn.(7), we obtain

$$a_2(x_i-x_j)+a_4y_i(x_i-x_j)=u_i-u_j.$$

Hence, $a_2=\dfrac{u_i-u_j}{x_i-x_j}-a_4y_i$.

From Eqn.(7), we get

$$a_1=u_i-a_2x_i-a_3y_i-a_4x_iy_i$$

Substituting these values of a_1, a_2, a_3 and a_4 in Eqn.(6) and simplifying, we get

$$\begin{aligned}u(x,y)&=a_1+a_2x+a_3y+a_4xy\\&=N_i^{(e)}u_i+N_j^{(e)}u_j+N_k^{(e)}u_k+N_m^{(e)}u_m\end{aligned} \tag{13}$$

where,

$$N_i^{(e)}=\frac{(x-x_j)\,(y-y_m)}{(x_i-x_j)\,(y_i-y_m)},\quad N_j^{(e)}=\frac{(x-x_i)\,(y-y_m)}{(x_j-x_i)\,(y_i-y_m)},$$

$$N_k^{(e)} = \frac{(x - x_i)(y - y_i)}{(x_j - x_i)(y_m - y_i)}, \quad N_m^{(e)} = \frac{(x - x_j)(y - y_i)}{(x_i - x_j)(y_m - y_i)}. \tag{14}$$

Note that the nodes are $P(x_i, y_i), Q(x_j, y_i), R(x_j, y_m)$ and $S(x_i, y_m)$.

At the node $P(x_i, y_i)$, we have

$$N_i^{(e)}(x_i, y_i) = 1,\ N_j^{(e)}(x_i, y_i) = 0,\ N_k^{(e)}(x_i, y_i) = 0,\ N_m^{(e)}(x_i, y_i) = 0.$$

At $Q(x_j, y_i)$, we have

$$N_i^{(e)}(x_j, y_i) = 0,\ N_j^{(e)}(x_j, y_i) = 1,\ N_k^{(e)}(x_j, y_i) = 0,\ N_m^{(e)}(x_j, y_i) = 0.$$

At $R(x_j, y_m)$, we get

$$N_i^{(e)}(x_j, y_m) = 0,\ N_j^{(e)}(x_j, y_m) = 0,\ N_k^{(e)}(x_j, y_m) = 1,\ N_m^{(e)}(x_j, y_m) = 0.$$

At $S(x_i, y_m)$, we get

$$N_i^{(e)}(x_i, y_m) = 0,\ N_j^{(e)}(x_i, y_m) = 0,\ N_k^{(e)}(x_i, y_m) = 0,\ N_m^{(e)}(x_i, y_m) = 1.$$

Therefore, the shape functions have the value 1 at the node where it is defined and have the value 0 at all the other nodes.

In general, when the sides of the rectangle are not parallel to the axes, the shape functions satisfy the condition

$$\begin{aligned} N_r^{(e)}(x_s, y_s) &= 1, \text{ for } r = s \\ &= 0, \text{ for } r \neq s. \end{aligned}$$

Note that the shape functions derived in Eqn.(14) are the Lagrange bivariate interpolations. It is important that the interpolation polynomial $u(x, y)$ is not confused with the shape or interpolation functions $N^{(e)}$. The interpolation polynomial $u(x, y)$ applies to an element, whereas $N_r^{(e)}(x, y)$ is the shape functions for u_r in an element.

Let R be a two-dimensional domain. Consider a typical node i, called an **apex node**. It is the node which is common to all the neighbouring elements. The nodes marked • in Fig.2 and 3 are apex nodes. In Fig.2, the apex i is common to six triangular elements. In Fig.3, the apex i is common to four rectangular elements. The piecewise approximating function $u(x, y)$ over the whole domain R can be written as

$$u(x, y) = \sum_{i=1}^{M} N_i(x, y) u_i \tag{15}$$

where M is the number of nodes contained in R, N_i are the interpolating functions and u_i are the values at the nodes. The shape functions $N_i(x, y)$ are defined by

$$\begin{aligned} N_i(x, y) &= N_i^{(e)}(x, y), \text{ given by Eqns.(3) or (14) if } (x, y) \text{ is in } e \text{ and has } i \text{ as an apex} \\ &= 0, \text{ otherwise.} \end{aligned}$$

Since we have defined some elements that can be used in the two dimensional domain, we are ready to introduce finite element methods for the solution of Laplace and Poisson equations. Finite element method can be applied either to extremize the variational form of the partial differential equation or directly through the use of a weighted residual method. Galerkin's method uses the weighted residual approach. In the next section, we shall discuss the Galerkin method and derive the finite element Galerkin method. Here we shall be giving the method for two-dimensional boundary value problem although the method can be generalized to any dimension.

12.3 GALERKIN METHOD

Consider the boundary value problem

$$L(u) = f(x, y);\ x, y \in R \tag{16}$$

$$B_i[u] = g_i(x, y);\ x, y \in \Gamma \tag{17}$$

where L is the second order linear differential operator, Γ is the boundary of R and B_i are the boundary conditions. We assume the solution of the partial differential equation in the form

$$u(x,y) \approx w(x,y,a_1,a_2,\ldots,a_m) = \phi_0(x,y) + \sum_{i=1}^{m} a_i \phi_i(x,y) \tag{18}$$

which depends on the parameters $a_1, a_2, \ldots, a_m$ and $\phi_i(x,y)$ are basis functions satisfying the boundary conditions and taken from a complete set of functions (can be taken as polynomials, orthogonal functions etc).

Normally, $\phi_0(x,y)$ satisfies the inhomogeneous boundary conditions and $\phi_i(x,y), i = 1,2,\ldots,m$ satisfy the homogeneous boundary conditions

$$B_i[\phi_0] = g_i \ , \ B_i[\phi_j] = 0 \ , j = 1,2,\ldots,m.$$

The error in satisfying the differential equation is given by

$$E(x,y,\mathbf{a}) = L[w(x,y,\mathbf{a})] - f(x,y) \tag{19}$$

where $\mathbf{a} = [a_1, a_2, \ldots, a_m]^T$.

In the Galerkin method, the error is made orthogonal over the whole domain, to the functions $\phi_i(x,y),\ i = 1,2,\ldots,m$. In other words,

$$\iint_R E(x,y,\mathbf{a})\phi_j(x,y)\,dx\,dy = 0,\ j = 1,2,\ldots,m \tag{20}$$

This gives a $m \times m$ system of equations for the solution of $a_1, a_2, \ldots, a_m$.

Let us now consider the application of the method to the solution of the boundary value problem defined by Eqns.(16), (17). Let k be the number of nodes in an element. We write the approximation in an element e as

$$u(x,y) = \sum_{i=1}^{k} N_i u_i = N^{(e)} u^{(e)} \tag{21}$$

where $N^{(e)} = [N_1 \ N_2 \ldots N_k]$ and $u^{(e)} = [u_1 \ u_2 \ldots u_k]^T$. The residual is

$$E = L(u) - f$$

Here, we choose the weight function as N_i. Therefore, we require

$$\iint_R [L(u) - f]\, N_i\, dx\, dy = 0,\ i = 1,2,\ldots,k \tag{22}$$

Since the region is divided into M elements, similar equation holds for each element

$$\iint_{e_j} [L(u) - f] N_i\, dx\ dy = 0,\ i = 1,2,\ldots,k,\ \ j = 1,2,\ldots,M \tag{23}$$

where N_i are now the shape functions in the element and u_i are the values of the variable at the nodes in the element. The elemental equations are assembled (the contribution of all the common elements at an apex i) and the fixed boundary conditions are applied.

Let us consider the application of this method to the Dirichlet boundary value problem for the Poisson equation defined by

$$Lu + G(x,y) = u_{xx} + u_{yy} + G(x,y) = 0 \ \text{ in } \ R \tag{24}$$

$$u(x,y) = 0 \ \text{ on } \Gamma \tag{25}$$

Let R contain M elements, each element with k nodes. We write the approximation in the whole domain as (see Eqn.(15))

$$u(x,y) = \sum_{i=1}^{M} N_i(x,y)\, u_i \tag{26}$$

Substituting $u(x,y)$ in the Eqn.(24) and using Eqn.(22), we obtain

$$\sum_{i=1}^{M}\left[\iint_R \{(N_i)_{xx}+(N_i)_{yy}\}N_j dx\,dy\right]u_i+\iint_R G(x,y)N_j dx\,dy=0, j=1,2,\ldots,M \qquad (27)$$

Let R be the domain as given in Fig.6.

R is bounded by the curves CDA and ABC.

Equation of curve CDA : $x=h_1(y)$.

Equation of curve ABC: $x=h_2(y)$.

Therefore, R is defined as

$$R:\quad p_1\le y\le p_2\ ,\ h_1(y)\le x\le h_2(y) \qquad (28)$$

We can also say that R is bounded by the curves DAB and BCD.

Equation of curve DAB : $y=g_1(x)$.

Equation of curve BCD : $y=g_2(x)$.

Therefore, R is also defined as

$$R:\quad q_1\le x\le q_2\ ,\ g_1(x)\le y\le g_2(x) \qquad (29)$$

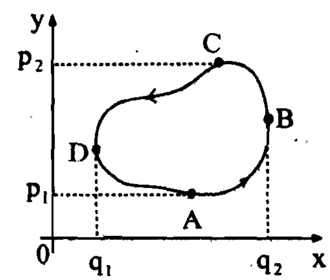


Fig.6

Integrating the first term of Eqn.(27) and using the definition of R as given by Eqn.(28) ,we get (see. Fig.6)

$$\iint_R (N_i)_{xx}N_j dx\,dy=\int_{y=p_1}^{p_2}\int_{x=h_1(y)}^{h_2(y)}(N_i)_{xx}N_j dx\,dy$$

$$=\int_{p_1}^{p_2}[(N_i)_x N_j]_{x=h_1(y)}^{h_2(y)}dy-\iint_R (N_i)_x(N_j)_x dx\,dy$$

$$=\oint_\Gamma (N_i)_x N_j d\Gamma-\iint_R (N_i)_x(N_j)_x\,dx\,dy \qquad (30)$$

The first integral is integration along the boundary Γ (line integral).

Similarly, using the definition of R as given by Eqn.(29), the second term of Eqn.(27) gives

$$\iint_R (N_i)_{yy}N_j dx\,dy=\oint_\Gamma (N_i)_y N_j\,d\Gamma-\iint_R (N_i)_y(N_j)_y dx\,dy \qquad (31)$$

The contributions of the first term on the right hand side of Eqns(30) and (31) are zero for all the elements inside. Only the elements which have a part of the boundary as its sides, have contributions from the natural boundary conditions. When no natural boundary conditions are given, the contributions of the first terms are taken as zero. For the boundary value problems that we are studying, we take the contribution of these terms as zero. Thus, from Eqns.(27), (30) and (31), we obtain

$$\sum_{i=1}^{M}\left[\iint_R \{(N_i)_x(N_j)_x+(N_i)_y(N_j)_y\}dx\,dy\right]u_i=\iint_R G\,N_j dx\,dy,\ j=1,2,\ldots,M. \qquad (32)$$

For the Laplace equation $\nabla^2 u=0$, we have $G(x,y)=0$ on the right hand side of Eqn.(32). The integrals are evaluated for each element. The element equations are then assembled. The equation at each node i (apex i) is written from the contribution of all the elements common to it. The coefficient matrix of the final system is a band matrix.

We shall now illustrate the method through examples.

Example 1: Find the solution of the boundary value problem

$$\nabla^2 u=x^2+y^2,\ 0\le x\le 1,\quad 0\le y\le 1$$

$$u=\frac{1}{12}(x^4+y^4)\ \text{on the boundary}$$

using the Galerkin method with (i) triangular elements, (ii) rectangular elements and one internal node $(h=1/2)$.

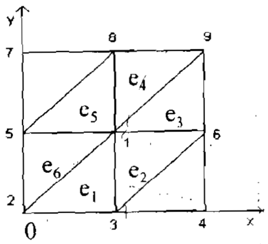


Fig.7. Triangular elements

Solution: (i) **Triangular elements:** The mesh is given in Fig.7. At the apex 1, six triangular elements e_1, e_2, e_3, e_4, e_5 and e_6 contribute.

The boundary values are

$$u_2 = u(0,0) = 0,\ u_3 = u\left(\frac{1}{2}, 0\right) = \frac{1}{192} = u_5,$$

$$u_6 = u\left(1, \frac{1}{2}\right) = \frac{17}{192} = u_8,\ u_9 = u(1,1) = \frac{1}{6}.$$

Comparing the given equation with Eqn.(24) we have in this case $G(x,y) = -(x^2+y^2)$. We use Eqn.(32), for obtaining the difference equation at the node 1. We have

$$\sum_{m=1}^{M}\left[\iint_R \{(N_m)_x (N_i)_x + (N_m)_y (N_i)_y\}\, dx\, dy\right] u_m + \iint_R (x^2+y^2) N_i\, dx\, dy \qquad (33)$$

Let us now obtain the contribution of each element. The apex node is always denoted by i and the remaining nodes are numbered as j, k in the anti-clockwise direction.

The shape functions N_i, N_j and N_k are given as follows (see Eqn.(3))

$$N_i = \frac{1}{2\Delta}[(x_j y_k - x_k y_j) + (y_j - y_k)x + (x_k - x_j)y]$$

$$N_j = \frac{1}{2\Delta}[(x_k y_i - x_i y_k) + (y_k - y_i)x + (x_i - x_k)y]$$

$$N_k = \frac{1}{2\Delta}[(x_i y_j - x_j y_i) + (y_i - y_j)x + (x_j - x_i)y]$$

where Δ = area of the triangle.

Now, we evaluate the integrals in Eqn.(33) for each element.

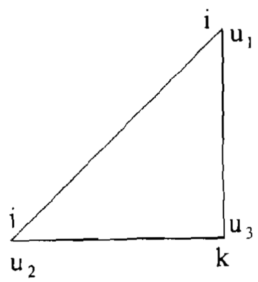


Fig.8

Element e_1: $i = \left(\frac{1}{2}, \frac{1}{2}\right)$, $j = (0,0)$, $k = \left(\frac{1}{2}, 0\right)$, $\Delta = \frac{1}{8}$. (see Fig.8)

$$N_i = 4\left[\frac{1}{2}y\right] = 2y,\ N_j = 4\left[\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{-1}{2}\right)x\right] = 1 - 2x$$

$$N_k = 4\left[\frac{1}{2}x - \frac{1}{2}y\right] = 2(x-y).$$

We have $(N_i)_x = 0$, $(N_i)_y = 2$, $(N_j)_x = -2$, $(N_j)_y = 0$,

$(N_k)_x = 2$, $(N_k)_y = -2$.

Substituting the above values in Eqn(33), we obtain

$$\left[\iint_{e_1} 4\, dx\, dy\right] u_1 + \left[\iint_{e_1} (0)\, dx\, dy\right] u_2 + \left[\iint_{e_1} -4\, dx\, dy\right] u_3 + \iint_{e_1} (x^2+y^2)(2y)\, dx\, dy$$

Now, $\displaystyle\iint_{e_1} f(x,y)\, dx\, dy = \int_{x=0}^{1/2}\left(\int_{y=0}^{x} f(x,y)\, dy\right) dx$

Carrying out the integrations, we get

$$4\left(\frac{1}{8}\right)u_1 - 4\left(\frac{1}{8}\right)u_3 + 2\left(\frac{3}{4}\right)\left(\frac{1}{160}\right) = \frac{1}{2}u_1 - \frac{1}{2}\left(\frac{1}{192}\right) + \frac{3}{320} = \frac{1}{2}u_1 + \frac{13}{1920}$$

Element e_2: $i=\left(\frac{1}{2},\frac{1}{2}\right)$, $j=\left(\frac{1}{2},0\right)$, $k=\left(1,\frac{1}{2}\right)$, $\Delta=\frac{1}{8}$. (see Fig.9)

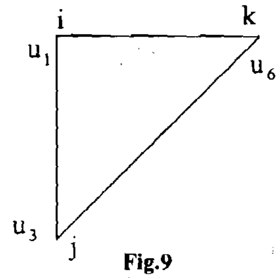


Fig.9

$$N_i=4\left[\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)+\left(\frac{-1}{2}\right)x+\left(1-\frac{1}{2}\right)y\right]=1-2x+2y,$$

$$N_j=4\left[\frac{1}{2}-\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)+\left(\frac{1}{2}-1\right)y\right]=1-2y,$$

$$N_k=4\left[-\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)x\right]=-1+2x.$$

We have, $(N_i)_x=-2,\ (N_i)_y=2,\ (N_j)_x=0,(N_j)_y=-2,(N_k)_x=2,(N_k)_y=0$.

Substituting these values in Eqn.(33), we obtain

$$\left[\iint_{e_2}\{-2)\ (-2)+(2)\ (2)\}dx\,dy\right]u_1+\left[\iint_{e_2}(-2)\ (2)dx\,dy\right]u_3$$

$$+\left[\iint_{e_2}(2)\ (-2)dx\,dy\right]u_6+\iint_{e_2}(x^2+y^2)\ (1-2x+2y)dx\,dy$$

Now, $\iint_{e_2}f(x,y)dx\,dy=\int_{x=1/2}^{1}\left[\int_{y=x-1/2}^{1/2}f(x,y)dy\right]dx$

Carrying out the integrations, we get

$$(8)\left(\frac{1}{8}\right)u_1-4\left(\frac{1}{8}\right)\left(\frac{1}{192}\right)-4\left(\frac{1}{8}\right)\left(\frac{17}{192}\right)+\frac{11}{480}=u_1-\frac{23}{960}.$$

Element e_3: $i=\left(\frac{1}{2},\frac{1}{2}\right)$, $j=\left(1,\frac{1}{2}\right)$, $k=(1,1)$, $\Delta=\frac{1}{8}$. (see Fig.10)

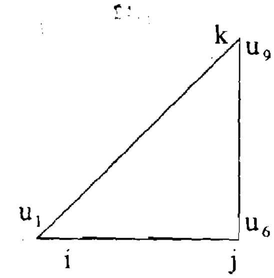


Fig.10

$$N_i=4\left[1-\frac{1}{2}+\left(\frac{1}{2}-1\right)x\right]=2(1-x),$$

$$N_j=4\left[\left(1-\frac{1}{2}\right)x+\left(\frac{1}{2}-1\right)y\right]=2(x-y),$$

$$N_k=4\left[\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)-\frac{1}{2}+\left(1-\frac{1}{2}\right)y\right]=-1+2y.$$

We have $(N_i)_x=-2,\ (N_i)_y=0,\ (N_j)_x=2,(N_j)_y=-2,(N_k)_x=0,(N_k)_y=2$.

Substituting these values in Eqn.(33), we obtain

$$\left[\iint_{e_3}4dx\,dy\right]u_1+\left[\iint_{e_3}-4dx\,dy\right]u_6+(0)u_9+2\iint_{e_3}(x^2+y^2)(1-x)\,dx\,dy$$

Now, $\iint_{e_3}f(x,y)dx\,dy=\int_{x=1/2}^{1}\left[\int_{y=1/2}^{x}f(x,y)dy\right]dx$

Carrying out the integrations, we get

$$(4)\left(\frac{1}{8}\right)u_1-4\left(\frac{1}{8}\right)u_6+\left(\frac{39}{960}\right)=\frac{1}{2}u_1-\frac{1}{2}\left(\frac{17}{192}\right)+\frac{39}{960}=\frac{1}{2}u_1-\frac{7}{1920}.$$

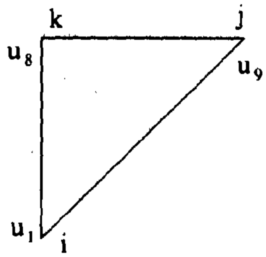


Fig.11

Element e_4: $i=\left(\frac{1}{2},\frac{1}{2}\right)$, $j=(1,1)$, $k=\left(\frac{1}{2},1\right)$, $\Delta=\frac{1}{8}$. (see Fig.11)

We obtain

$$N_i = 2(1-y),\ N_j = -1+2x,\ N_k = 2(-x+y)$$

and $\quad (N_i)_x = 0,\ (N_i)_y = -2,\ (N_j)_x = 2, (N_j)_y = 0, (N_k)_x = -2, (N_k)_y = 2.$

Substituting the above values in Eqn.(33), we obtain

$$\left[\iint_{e_4} 4dx\,dy\right]u_1 + (0)u_9\left[\iint_{e_4} -4dx\,dy\right]u_8 + \iint_{e_4}(x^2+y^2)2\,(1-y)\,dx\,dy$$

Now, $\displaystyle\iint_{e_4} f(x,y)dx\,dy = \int_{x=1/2}^{1}\left[\int_{y=x}^{1} f(x,y)dy\right]dx.$

Carrying out the integrations, we get

$$(4)\left(\frac{1}{8}\right)u_1 - 4\left(\frac{1}{8}\right)\left(\frac{17}{192}\right)+\frac{39}{960}=\frac{1}{2}u_1-\frac{7}{1920}.$$

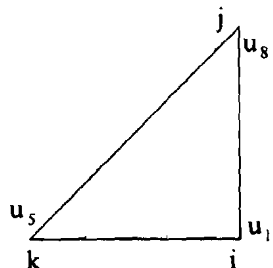


Fig.12

Element e_5: $i=\left(\frac{1}{2},\frac{1}{2}\right)$, $j=\left(\frac{1}{2},1\right)$, $k=\left(0,\frac{1}{2}\right)$, $\Delta=\frac{1}{8}$. (see Fig.12)

We obtain

$$N_i = 1+2x-2y,\ N_j = -1+2y,\ N_k = 1-2x.$$

and $\quad (N_i)_x = 2,\ (N_i)_y = -2,\ (N_j)_x = 0, (N_j)_y = 2, (N_k)_x = -2, (N_k)_y = 0.$

Substituting these values in Eqn.(33), we obtain

$$\left[\iint_{e_5} 8dx\,dy\right]u_1 + \left[\iint_{e_5} -4dx\,dy\right]u_8 + \left[\iint_{e_5} -4dx\,dy\right]u_5$$
$$+\iint_{e_5}(x^2+y^2)(1+2x-2y)\,dx\,dy$$

Now, $\displaystyle\iint_{e_5} f(x,y)dx\,dy = \int_{x=0}^{1/2}\left[\int_{y=1/2}^{(1+2x)/2} f(x,y)dy\right]dx$

Carrying out the integrations, we get

$$(8)\left(\frac{1}{8}\right)u_1 - 4\left(\frac{1}{8}\right)\left(\frac{17}{192}\right)-4\left(\frac{1}{8}\right)\left(\frac{1}{192}\right)+\frac{11}{480}=u_1-\frac{23}{960}.$$

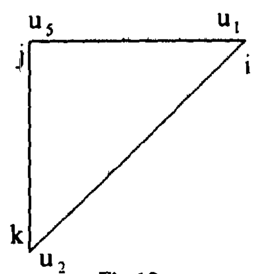


Fig.13

Element e_6: $i=\left(\frac{1}{2},\frac{1}{2}\right)$, $j=\left(0,\frac{1}{2}\right)$, $k=(0,0)$, $\Delta=\frac{1}{8}$. (see Fig.13)

We obtain

$$N_i = 2x,\ N_j = -2x+2y,\ N_k = 1-2y.$$

And $\quad (N_i)_x = 2,\ (N_i)_y = 0,\ (N_j)_x = -2, (N_j)_y = 2, (N_k)_x = 0, (N_k)_y = -2.$

Substituting these values in Eqn.(33), we obtain

$$\left[\iint_{e_6} 4dx\,dy\right]u_1 + \left[\iint_{e_6} -4dx\,dy\right]u_5 + (0)u_2\iint_{e_6}(x^2+y^2)2x\,dx\,dy$$

Now, $\iint\limits_{e_6} f(x,y)\,dx\,dy = \int\limits_{x=0}^{1/2}\left[\int\limits_{y=x}^{1/2} f(x,y)\,dy\right]dx$

Carrying out the integrations, we get

$$(4)\left(\frac{1}{8}\right)u_1 - 4\left(\frac{1}{8}\right)\left(\frac{1}{192}\right) + \frac{9}{960} = \frac{1}{2}u_1 + \frac{13}{1920}.$$

Adding all the contributions, we obtain the difference equation at the node 1 as

$$\left(\frac{1}{2}+1+\frac{1}{2}+\frac{1}{2}+1+\frac{1}{2}\right)u_1 + \left(\frac{13}{1920}-\frac{23}{960}-\frac{7}{1920}-\frac{7}{1920}-\frac{23}{960}+\frac{13}{1920}\right)=0$$

or, $\quad 4u_1 - \frac{1}{24} = 0.$

Hence, $u_1 = 1/96$.

It may be noted that for the given problem, $u(x,y) = (x^4 + y^4)/12$ satisfies both the differential equation and the boundary condition. Hence, $u = (x^4+y^4)/12$ is the exact solution. Note that we have obtained (coincidentally) the finite element solution which is the same as the exact solution $u\left(\frac{1}{2},\frac{1}{2}\right) = \frac{1}{96}$.

Now we solve Example 1 using rectangular elements.

(ii) **Rectangular elements:** The mesh is given in Fig.14. At the apex 1, four rectangular elements contribute. The boundary values are same as earlier, and in addition we have $u_4 = u(1,0) = \frac{1}{12} = u_7$. Let us now obtain the contribution of each element. The apex node is always denoted by i and the remaining nodes j, k, m are numbered in the anti-clockwise direction. The shape functions N_i, N_j, N_k and N_m are given as follows (see Eqn.(14)).

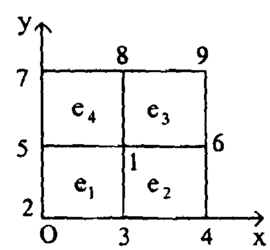


Fig.14: Rectangular elements

$$N_i = \frac{(x-x_j)(y-y_m)}{(x_i-x_j)(y_i-y_m)},\quad N_j = \frac{(x-x_i)(y-y_m)}{(x_j-x_i)(y_j-y_m)},$$

$$N_k = \frac{(x-x_i)(y-y_i)}{(x_j-x_i)(y_m-y_i)},\quad N_m = \frac{(x-x_j)(y-y_i)}{(x_i-x_j)(y_m-y_i)}$$

Element e_1: $i = \left(\frac{1}{2},\frac{1}{2}\right)$, $j = \left(0,\frac{1}{2}\right)$, $k = (0,0)$, $m = \left(\frac{1}{2}, 0\right)$ (see Fig.15)

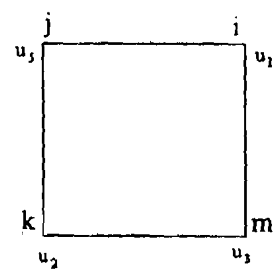


Fig.15

$$N_i = \frac{(x-0)(y-0)}{(1/2)(1/2)} = 4xy,\; N_j = \frac{\left(x, \frac{1}{2}\right)(y-0)}{(-1/2)(1/2)} = -2(2x-1)y$$

$$N_k = \frac{\left(x-\frac{1}{2}\right)\left(y-\frac{1}{2}\right)}{(-1/2)(-1/2)} = (2x-1)(2y-1)$$

$$N_m = \frac{(x-0)\left(y-\frac{1}{2}\right)}{(1/2)(-1/2)} = -2x(2y-1)$$

substituting these values in Eqn.(33), we obtain

$$\left[\iint_{e_1} 16(x^2+y^2)\,dx\,dy\right]u_1+\left[\iint_{e_1}\{(-4y)(4y)-2(2x-1)(4x)\}dx\,dy\right]u_5$$

$$+\left[\iint_{e_1}\{2(2y-1)4y+2(2x-1)2x\}\right]u_2+\left[\iint_{e_1}\{-2(2y-1)\,(4y)-4x(4x)\}dx\,dy\right]u_3$$

$$+\iint_{e_1}(x^2+y^2)\,(4xy)\,dx\,dy$$

Now, $\displaystyle\iint_{e_1} f(x,y)\,dx\,dy=\int_{x=0}^{1/2}\int_{y=0}^{1/2} f(x,y)\,dx\,dy$

Carrying out the integrations, we get

$$16\left(\frac{1}{24}\right)u_1-\frac{1}{6}\left(\frac{1}{192}\right)-0-\frac{1}{6}\left(\frac{1}{192}\right)+\frac{1}{64}=\frac{2}{3}u_1+\frac{1}{72}.$$

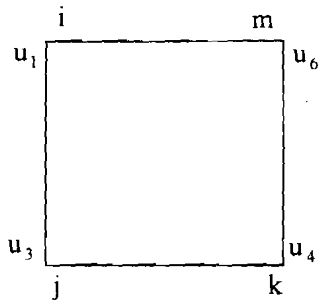


Fig.16

Element e_2 **:** $i=\left(\frac{1}{2},\frac{1}{2}\right)$, $j=\left(\frac{1}{2},0\right)$, $k=(1,\,0)$, $m=\left(1,\,\frac{1}{2}\right)$ (see Fig.16).

$$N_i=-4(x-1)y\,,\,N_j=2(x-1)\,(2y-1)$$

$$N_k=-(2x-1)\,(2y-1),\ N_m=2(2x-1)y.$$

Eqn.(33), yields

$$\left[\iint_{e_2}\{(-4y)^2+16(x-1)^2\}dxdy\right]u_1+\left[\iint_{e_2}\{-2(2y-1)\,(4y)+4(x-1)(-4)\,(x-1)\}dx\,dy\right]u_3$$

$$+\left[\iint_{e_2}\{-2(2y-1)(-4y)-2(2x-1)(-4)(x-1)\}dx\,dy\right]u_4$$

$$+\left[\iint_{e_2}\{4y(-4y)+2(2x-1)\,(-4)\,(x-1)\}dx\,dy\right]u_6+\iint_{e_2}(x^2+y^2)\,(-4)\,(x-1)\,y\,dxdy$$

Now, $\displaystyle\iint_{e_2} f(x,y)\,dx\,dy=\int_{x=1/2}^{1}\left[\int_{y=0}^{1/2}(x,y)\,dy\right]dx$

Carrying out the integrations, we get

$$\frac{2}{3}u_1-\frac{1}{6}\left(\frac{1}{192}\right)-\left(\frac{1}{3}\right)\left(\frac{1}{12}\right)-\frac{1}{6}\left(\frac{17}{192}\right)+\frac{7}{192}=\frac{2}{3}u_1-\frac{1}{144}.$$

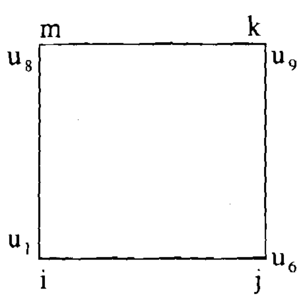


Fig.17

Element e_3**:** $i=\left(\frac{1}{2},\frac{1}{2}\right)$, $j=\left(1,\,\frac{1}{2}\right)$, $k=(1,\,1)$, $m=\left(\frac{1}{2},1\right)$ (see Fig.17)

$$N_i=-4(x-1)\,(y-1)\,,\,N_j=-2(x-1)\,(y-1)$$

$$N_k=(2x-1)\,(2y-1),\ N_m=-2(x-1)\,(2y-1).$$

Eqn.(33), yields

$$\left[\iint_{e_3}\{16(y-1)^2+16(x-1)^2\}dxdy\right]u_1+\left[\iint_{e_3}\{-16(y-1)^2-2\,(2x-1)(4)\,(x-1)\}dx\,dy\right]u_6$$

$$+\left[\iint_{e_3}\{2(2y-1)(4)\,(y-1)+2(2x-1)(4)(x-1)\}dx\,dy\right]u_9$$

$$+\left[\iint_{e_3}\{-2(2y-1)\,(4)(y-1)-4\,(x-1)\,(4)\,(x-1)\}dx\,dy\right]u_8$$

$$+\iint_{e_3}(x^2+y^2)\,(4)\,(x-1)\,(y-1)\,dxdy$$

Carrying out the integrations, we get

$$\frac{2}{3}u_1 - \frac{1}{6}\left(\frac{17}{192}\right) - \left(\frac{1}{3}\right)\left(\frac{1}{6}\right) - \frac{1}{6}\left(\frac{17}{192}\right) + \frac{11}{192} = \frac{2}{3}u_1 - \frac{1}{36}.$$

Element e_4 **:** $i = \left(\frac{1}{2}, \frac{1}{2}\right),\ j = \left(\frac{1}{2}, 1\right), k = (0,1),\ m = \left(0, \frac{1}{2}\right)$ (see Fig.18).

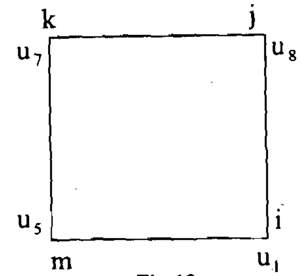


Fig.18

We obtain

$$N_i = -4(y-1)\ (y-1)\ , N_j = 2(2y-1)$$
$$N_k = -(2x-1)\ (2y-1),\ N_m = 2(2x-1)\ (y-1).$$

Eqn.(33) gives

$$\left[\iint_{e_4}\{16(y-1)^2 + 16x^2\}dxdy\right]u_1 + \left[\iint_{e_4}\{2(2y-1)(-4)\ (y-1) + 4x(-4x)\}dx\,dy\right]u_8$$
$$+\left[\iint_{e_4}\{-2(2y-1)(-4)\ (y-1) - 2(2x-1)(-4x)\}dx\,dy\right]u_7$$
$$+\left[\iint_{e_4}\{4(y-1)\ (-4)(y-1) + 2\ (2x-1)\ (-4x)\}dx\,dy\right]u_5$$
$$+\iint_{e_4}(x^2 + y^2)\ (-4)x\ \ (y-1)\ dxdy$$

Carrying out the integrations, we get

$$\frac{2}{3}u_1 - \frac{1}{6}\left(\frac{17}{192}\right) - \frac{1}{3}\left(\frac{1}{12}\right) - \frac{1}{6}\left(\frac{1}{192}\right) + \frac{7}{192} = \frac{2}{3}u_1 - \frac{1}{144}.$$

Adding all the contributions, we obtain the difference equation at the node 1 as

$$\left(\frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \frac{2}{3}\right)u_1 + \frac{1}{72} - \frac{1}{144} - \frac{1}{36} - \frac{1}{144} = 0$$

or, $\frac{8}{3}u_1 - \frac{1}{36} = 0$

We obtain $u_1 = \frac{1}{96}$ which is the same as the exact solution.

Remark: In Example 1, coincidentally, the finite element solution obtained in both the cases are the same as the exact solution. However, in practice, large number of elements (triangular or rectangular) are required in order to obtain accurate solutions which in turn, increase the number of computations and level of difficulty for obtaining the elemental equations manually. Normally, we solve the problem with say, M elements. We compute the problem again by increasing the number of elements. Convergence of the solution values at the nodes can be studied to stop the computations.

As we have mentioned earlier finite element methods can be applied to extremise the variational form of partial differential equations. We now give here the variational form of Laplace and Poisson equation.

Variational Forms

First, consider a functional in one independent variable x in the form

$$I(u) = \int_{x_1}^{x_2} F(x, u, u_x, u_{xx})\,dx \qquad (34)$$

The problem is to find $u(x)$, called an **extremal**, so as to extremise the functional in Eqn.(34). From the theory of variations, we can show that $u(x)$ that extremises (34) is also the solution of the **Euler-Lagrange** equation

$$\frac{\partial F}{\partial u}-\frac{d}{dx}\left(\frac{\partial F}{\partial u_x}\right)+\frac{d^2}{dx^2}\left(\frac{\partial F}{\partial u_{xx}}\right)=0 \qquad (35)$$

Therefore, the solution of the partial differential Eqn.(35) extremises the functional (34) and the function extremising (34) is the solution of the partial differential Eqn.(35). Geometric considerations can be used to find whether it gives a minimum of Eqn.(34) or not. Now, consider a functional in two independent variable x, y in the form

$$I(u)=\iint_R F(x,y,u,u_x,u_y,u_{xx},u_{xy},u_{yy})\,dA \qquad (36)$$

The corresponding **Euler-Lagrange** equation can be written as

$$\frac{\partial F}{\partial u}-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)+\frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial u_{xx}}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{\partial F}{\partial u_{yy}}\right)+\frac{\partial^2}{\partial x\partial y}\left(\frac{\partial F}{\partial u_{xy}}\right)=0 \qquad (37)$$

In deriving the variational form and hence the Euler-Lagrange form, we require that the following conditions are satisfied.

$$\left[\frac{\partial F}{\partial u_x}-\frac{d}{dx}\left(\frac{\partial F}{\partial u_{xx}}\right)\right]_{x_1}^{x_2}=0 \text{ and } \left[\frac{\partial F}{\partial u_{xx}}\right]_{x_1}^{x_2}=0.$$

These conditions are called the natural boundary conditions of the problem.

Example 2: Consider a functional as

$$I(u)=\iint_R\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2-2u\,f(x,y)\right]dA \qquad (38)$$

Then, using Eqn.(37), the **Euler-Lagrange** equation is given by

$$-2f-\frac{\partial}{\partial x}\left[2\frac{\partial u}{\partial x}\right]-\frac{\partial}{\partial y}\left[2\frac{\partial u}{\partial y}\right]=0 \text{ or } u_{xx}+u_{yy}+f(x,y)=0 \qquad (39)$$

which is the Poisson equation.
If we set $f=0$, then we get

$$I(u)=\iint_R\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]dA \qquad (40)$$

and its **Euler-Lagrange equation** is the Laplace equation $u_{xx}+u_{yy}=0$.

Eqns.(40) and (38) are respectively called the variational formulations of the Laplace equation $u_{xx}+u_{yy}=0$ and the Poisson equation $u_{xx}+u_{yy}+f(x,y)=0$.

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Note that the variational form contains partial derivatives that are one order lower than the order of the partial differential equation. This is an advantage of the variational formulation. However, in practice, it is not easy to write the variational form of a given partial differential equation. In the case of a boundary value problem as defined by Eqn.(16) **if the operator L is self adjoint, then application of the classical variational principle and the Galerkin's method lead to the same matrix system.** Thus, for the Laplace and Poisson equations, both these methods yield the same result. However, if the differential equation considered is not self adjoint, then the difference equations obtained by the two methods are different. In this unit, since we are discussing the solution of only the Laplace and Poisson equations, application of Galerkin's method is sufficient for our purpose.

You may now try the following exercises.

E1) Find the solution of the boundary value problem

$$\nabla^2 u=4,\ 0\le x\le 1,\ 0\le y\le 1$$

$$u=x^2+y^2 \text{, on the boundary}$$

using the Galerkin method with (i) triangular elements, (ii) rectangular elements and one internal node $(h = 1/2)$.

E2) Find the solution of the boundary value problem

$$\nabla^2 u = x^2 + 2y^2, 0 \le x \le 1,\ 0 \le y \le 1$$

$$u = \frac{1}{12}(x^4 + 2y^4), \text{ on the boundary}$$

using the Galerkin method with (i) triangular elements (ii) rectangular elements and one internal node $\left(h = \frac{1}{2}\right)$..

E3) Find the solution of the boundary value problem

$$\nabla^2 u = x + y,\ 0 \le x \le 1,\ 0 \le y \le 1$$

$$u = \frac{1}{6}(x^3 + y^3), \text{ on the boundary}$$

using the Galerkin's method with rectangular elements and one internal node $(h = 1/2)$.

E4) Find the solution of the boundary value problem

$$\nabla^2 u = 0,\ 0 \le x \le 1,\ 0 \le y \le 1$$

$$u = x + y, \text{on the boundary}$$

using the Galerkin method with triangular elements and one internal node $(h = 1/2)$.

We now end this unit by giving a summary of what we have covered in it.

12.4 SUMMARY

In this Unit, we have learnt the following points.

1. In finite element methods of solving a partial differential equation in a domain R, we generate difference equations by dividing R into a finite number of non overlapping sub-domains called finite elements.
2. Straight line elements are used in the case of one-dimensional problems whereas, triangular or rectangular elements are used in two dimensions.
3. The required solution is approximated by piecewise continuous polynomials defined in terms of nodal values at the vertices. At each nodal point, difference equation is obtained. The solution of these equations gives the numerical solution of the differential equation.
4. In finite element methods, difference equations can be generated by using the variational principle or weighted residual methods.
5. Finite element Galerkin method is a weighted residual method and does not require the variational form of the given differential equation.
6. If the given pde is self-adjoint then application of the variational principle and the Galerkin's method lead to the same matrix system.

12.5 SOLUTIONS/ANSWERS

E1) $u_1 = u\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}$.

E2) $u_1 = u\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{64}$.

E3) $u_1 = u\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{24}$.

E4) $u_1 = u\left(\frac{1}{2}, \frac{1}{2}\right) = 1$.

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