UNIT 7 RENEWAL PROCESSES-I

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7.1 INTRODUCTION

Many situations arise in day-to-day life, whether at home or in the office, where a certain event recurs again and again. Further, the time interval between two consecutive occurrences, though not fixed, follows the same probabilistic pattern. Study of such processes helps in a better understanding of these events. Such processes are called renewal processes. We will start the study of these processes in Sec. 7.2 of this unit. In Sec. 7.3 and 7.4, we shall cite few examples to understand the renewal processes. In Sec. 7.5, we shall discuss the distribution of S_n, the time of the n-th occurrence of the event, and the distribution of renewal process N_t, the number of occurrences till time t. In the last section of this unit, we shall discuss the renewal equation. By the end of the unit this is what you should be able to do.

Objectives

After studying this unit, you should be able to:

- identify a renewal process;
- find the law of the process (S_n or N_t) in a few simple cases;
- use the renewal argument to find certain quantities of interest;
- find the renewal function;
- derive the renewal equation.

7.2 DEFINITION

Let us start with a question. If a bulb lasts for 12 hours, then how many bulbs are required for one week continuous lighting? The answer of this seems very simple with various assumptions like all the bulbs are alike, all the bulbs are having same life time, also one bulb is used at anytime, i.e., no simultaneous use of bulbs, etc. In this situation, the life times of bulbs are random variables with common cumulative distribution function (cdf) $F(x)$. This can now be considered as a stochastic model, in which all the random variables are mutually independent and non-negative. In other words, this is the study of properties of a sequence $\{x_n\}$ of iid non-negative random variables, which is a representation of the life times of the bulbs that are being renewed (or replaced). The life time distribution of the renewed bulb is same as that of the previous bulb. The sequence $\{x_n\}$ is said to constitute a renewal sequence which the assumption that the failure of the bulbs is complete and sudden and are replaced instantaneously.

Consider an event whose time of occurrence is a random variable X with cumulative distribution function (cdf) F. i.e. $F(x) = P(X \le x)$. Since $X \ge 0$, F will satisfy

$F(0) = 0$.

Suppose this event occurs repeatedly and such that the occurrences are independent of one another. Such a process can be modelled via an i.i.d. sequence of random variables $X_1, X_2, \ldots$ with common cdf F, where X_n can be thought of as the time between the $(n-1)$-th and the n-th of the event.

Let the partial sums of the sequence be S_n, which is given by

$$S_0 = 0,\ S_n = \sum_{i=1}^{n} X_i,\ n = 1, 2, 3, \ldots \qquad (1)$$

Here S_n denotes the total time elapsed before the n-th event occurs, this is, time instant of n-th occurrence of the event and is known as renewal epochs.

X_i is the interval between occurrences of two successive renewals, which is called the renewal period of the process.

The number of occurrences of the event in any given time interval can be counted. Let N_t be the number of renewals upto time t and $\{N_t : t \ge 0\}$ denote this counting process, which can be defined as follows.

$$N_0 = 0,\ N_t = \inf\{k : S_{k+1} > t\} \qquad (2)$$

This process $\{N_t : t \ge 0\}$ is called a renewal process. Note that for any $t > 0$, exactly k occurrences take place upto time t if and only if $S_k \le t < S_{k+1}$, i.e.

$$\{N_t = k\} = \{S_k \le t < S_{k+1}\} \qquad (3)$$

$\{N_t : t \ge 0\}$, $\{X_n\}$ and $\{S_n\}$ can be used to define a renewal process and also can be determined from Eqn. (1) and Eqn. (2).

Note that there is an equivalent way of defining N_t which is given in the following exercise.

E1) Let $\{\hat{N}_t : t \ge 0\}$ be defined by

$$\hat{N}_0 = 0,\ \hat{N}_t = \sup\{k : S_k \le t\}.$$

Show that $\hat{N}$ coincides with the renewal process N.

Let us now define renewal process in formal definition.

Definition 1 (Renewal Process): The renewal process $\{N_t : t \ge 0\}$ is a non-negative integer valued counting process that counts the number of occurrences of an event during the time interval (0,t], where the times between occurrences of two successive events are non-negative, independent, identically distributed random variables.

Definition 2 (Renewal Sequence): Let $\{N_t : t \ge 0\}$ be a renewal process and let S_n, $n = 1, 2, \ldots$ be the time of occurrence of the n-th event so that Eqn.(2) holds and let $S_0 \equiv 0$. Then the sequence $\{S_n\}$ is called a renewal sequence.

There are lots of processes in real life which can be modelled as renewal processes. The variety of situations are replacement, queueing, maintenance, simulation, manpower studies, demography, reliability, inventory control, etc. In the next section, we start off with the most basic example, viz., that of a Poisson process.

7.3 AN EXAMPLE: POISSON PROCESS

The Poisson process has already been introduced in Unit 5 as an example of a pure birth process. It is also a renewal process as we will see below. First we will give an alternative definition of the Poisson process.

Definition 3: A process N_t is a Poisson process with parameter λ if

1. $N_0 \equiv 0$.
2. For $t > 0$, N_t has a Poisson distribution with parameter λt.
3. For $s, t > 0$,

$$N_{t+s} - N_s \overset{d}{=} N_t$$

and is independent of $\{N_u : 0 \le u \le s\}$.

Here $V \overset{d}{=} W$ means random variables V and W have the same probability distribution.

The last property is called the **stationary independent increments** property of the Poisson Process.

Example 1 (Poisson Process): Consider the counting process when the interoccurrence times are i.i.d. random variables which are exponentially distributed with parameter $\lambda > 0$. Let $X_1, X_2, \ldots$ denote this sequence. Then the probability density function (pdf) of X_1 (and hence of all X_i) is

$$f(x) = \begin{cases} \lambda e^{-\lambda x} ; & x > 0 \\ 0 ; & \text{otherwise} \end{cases}$$

and its cdf is given by

$$F(x) = \begin{cases} 1 - e^{-\lambda x} ; & x \ge 0 \\ 0 ; & \text{otherwise.} \end{cases}$$

Further, it is well known that for every $n \ge 1$, S_n (defined by (Eqn.(1)) is a gamma random variable with parameters n and λ. Thus the cdf of S_n is given by

$$F_n(x) = \begin{cases} 1 - \sum_{j=0}^{n-1} \dfrac{e^{-\lambda x}(\lambda x)^j}{j!}, & x \ge 0 \\ 0, & \text{otherwise.} \end{cases} \qquad (4)$$

In this case we can find the distribution of the renewal process $\{N_t : t \ge 0\}$ as follows. Fix $t > 0$, then

$$P(N_t = 0) = P(S_1 > t) = P(X_1 > t) = e^{-\lambda t} \qquad (5)$$

Further for any positive integer k, using Eqn.(3) and Eqn.(4), we get

$$\begin{aligned} P(N_t = k) &= P(S_k \le t < S_{k+1}) \\ &= P(S_{k+1} > t) - P(S_k > t) \\ &= \sum_{j=0}^{k} \frac{e^{-\lambda t}(\lambda t)^j}{j!} - \sum_{j=0}^{k-1} \frac{e^{-\lambda t}(\lambda t)^j}{j!} \\ &= \frac{e^{-\lambda t}(\lambda t)^k}{k!}. \end{aligned} \qquad (6)$$

Eqns (5) and (6) now imply that for every $t > 0$, N_t is a Poisson random variable with parameter (λt). To show that N_t is a Poison process with parameter λ we will show that N has stationary independent increments. See Definition 3. For this, we need the following memory less property of the exponential distribution: If $Y \sim \exp(\alpha)$, then

$$P(Y > s+t \mid Y > s) = P(Y > t) = e^{-\alpha t} \tag{7}$$

Now fix $s, t > 0$ and consider $N_{s+t} - N_s$ which counts the number of occurrences of the event in the time interval $(s, s,+t]$. By definition of S and N (see Definitions 1 and 2), the time of occurrence of the first event after s is S_{N_s+1} and S_{N_s} is the time of occurrence of the most recent or last event before time s. Then,

$$\begin{aligned}
&P(N_{s+t} - N_s = 0 \mid N_s = k, S_{N_s} = x, N_u : 0 \le u \le s) \\
&\qquad = P(S_{N_s+1} - s > t \mid N_s = k,\, S_{N_s} = x, N_u : 0 \le u \le s) \\
&\qquad = P(S_k + X_{k+1} - s > t \mid N_s = k,\, S_{N_s} = x, N_u : 0 \le u \le s) \\
&\qquad = P(X_{k+1} > s + t - x \mid X_{k+1} > s - x) \\
&\qquad = e^{-\lambda t}.
\end{aligned} \tag{8}$$

The last equality above follows from the memoryless property of X_{k+1}. Similarly, for $i \ge 1$, we can write

$$\begin{aligned}
&P(N_{s+t} - N_s = i \mid N_s = k, S_{N_s} = x, N_u : 0 \le u \le s) \\
&= P(S_k + X_{k+1} + \cdots + X_{k+i+1} - s > t, \\
&\quad S_k + X_{k+1} + \cdots + X_{k+i} - s \le t \mid N_s = k, S_{N_s} = x, N_u : 0 \le u \le s) \\
&= P(X_{k+1} + \cdots + X_{k+i+1} > s + t - x \\
&\quad X_{k+1} + \cdots + X_{k+i} \le s + t - x \mid X_{k+1} > s - x) \\
&= P(X_{k+1} + \cdots + X_{k+i+1} > s + t - x \mid X_{k+1} > s - x) \\
&\quad - P(X_{k+1} + \cdots + X_{k+i} \le s + t - x \mid X_{k+1} > s - x) \\
&= \sum_{j=0}^{i} \frac{e^{-\lambda t}(\lambda t)^j}{j!} - \sum_{j=0}^{i-1} \frac{e^{-\lambda t}(\lambda t)^j}{j!} \\
&= \frac{e^{-\lambda t}(\lambda t)^i}{i!},
\end{aligned} \tag{9}$$

where once again the last equality follows from the memoryless property of X_{k+1} and the fact that the X_j's are i.i.d. exponential (λ). Thus, the increments $N_{s+t} - N_s$ are independent of $\{N_u : 0 \le u \le s\}$ and have the same distribution as N_t for all $s, t > 0$. Thus, the renewal process N_t satisfies all the properties of definition 3, and hence is a Poisson Process with parameter λ.

As mentioned earlier, Poisson process is the most basic example of a renewal process. Many of the results that we will study in later units in this block can be first viewed in terms of the Poisson process. In the next section, we will look at more examples of renewal processes.

7.4 SOME MORE EXAMPLES

In the last section, we focused on a specific example of a renewal process when the time between successive events are i.i.d. exponential random variables. The Poisson process is a very important example and we studied it in some detail. In this section, we will discuss some more examples. It is left to the reader to check that the processes mentioned in these examples are indeed renewal processes.

Clearly if the interoccurrence time distribution, i.e. the distribution of X_i, changes we will get a different renewal process. Note that it is possible that X_i takes value 0. In fact, $P(X_i = 0) = F(0)$ can be strictly greater than 0. Note, further, that $X_n = 0$ means that the $S_n = S_{n-1}$ so that the $(n-1)$th and n-th events occur simultaneously. We will give a couple of examples where X_i is a discrete random variable.

Example 2 (Negative Binomial Process): Let X_n be i.i.d. Bernoulli $(p), 0 < p < 1$, i.e.

$$P(X_n = 0) = 1 - p, P(x_n = 1) = p.$$

then the corresponding renewal process $\{N_t, t = 0, 1, 2, \ldots\}$ is called a negative binomial process.

Example 3 (Binomial Process): Let X_n be i.i.d. each having geometric distribution

$$P(X_n = i) = (1-p)p^{i-1}, \qquad i = 1, 2, 3, \ldots$$

where $0 < p < 1$. Then the corresponding renewal process $\{N_t, t = 0, 1, 2, \ldots\}$ is called the binomial process.

Now we will see a few real life situations where such processes arise. We will study some of these processes in detail later.

Example 4 (Recurrent Events of Markov Chains): Let $\{Y_n : n = 0, 1, 2, \ldots\}$ be a Markov chain with state space $\Gamma = \{0, 1, 2, \ldots\}$ and such that $Y_0 = 0$. Let S_n be the number of transitions or the time for the n-th visit of the chain Y_n to 0. Here S_0 can be thought of as the time of the zero-th visit to zero which is 0. Thus

$$S_n = \min\{k > S_{n-1} : Y_k = 0\}.$$

Then it follows from the Markov property that the sequence $\{X_n\}$ defined by

$$X_n = S_n - S_{n-1},\ n = 1, 2, \ldots$$

is an i.i.d. sequence. Hence S_n is a renewal sequence and the associated counting process $\{N_t, t = 0, 1, 2, \ldots\}$ is a renewal process.

Example 5 (A Queueing Example): Consider a bank counter where a clerk attends to the customers who are standing in a queue one after another. Assume that customers arrive at the counter at random times in an independent fashion so that the inter-arrival times are i.i.d. exponential. Also assume that the successive service times for the customers also an i.i.d sequence of random variables. Then there are several different renewal processes associated with this system. We give a few examples below.

1. The process N_t^1 which denotes the total number of arrivals upto time t is a Poisson renewal process.

2. The process N_t^2 which counts the number of busy periods completed upto (and including) time t is also a renewal process. More precisely, let $\{Q_t : t \geq 0\}$ be the queue-length process which just denotes the number of persons in the queue at time t. This obviously equals the number of arrivals minus the number of services completed by the bank clerk upto time t. Assume $Q_0 = 0$ and let $S_0 = 0$. Define

$$S_s = \inf\{t > S_{n-1} : Q_t = 0\},\ X_n = S_n - S_{n-1}.$$

Here S_n denotes the time when the n-th busy period gets over, X_n is the time duration of the n-th busy period. S_n forms a renewal sequence and the associated counting process N_t^2 is a renewal process.

3. The process N_t^3 which counts the start of successive busy periods is also a renewal process, if $Q_0 = 1$ and the service has just started.

Example 6 (Counter Process): Consider a system where electrical impulses or signals are arriving and are being recorded on a recording device (counter). If the successive electrical impulses are assumed to be arriving in an i.i.d. fashion, then the process N_t^1 which counts the number of signals arriving in the time interval $(0, t]$ is clearly a renewal process.

Further, due to mechanical restrictions, suppose that as soon as a signal is recorded the counter gets locked for a small but fixed amount of time and any signal arriving in that time period will not be recorded. Then the process N_t^2 which counts the number of recorded signals in the time interval $(0, t]$ is also a renewal process.

Example 7 (Replacement Models): Let $X_1, X_2, \ldots$ denote the lifetimes of a particular type of components used in a machine. We will assume that the lifetimes are i.i.d. Suppose that whenever a component fails, it is replaced immediately by the next one. Then clearly the process which counts the number of components that fail upto (and including) time t is a renewal process.

Now, many times for preventive maintenance it is economical to replace the component at some fixed age, say T, even if it has not failed till then. Of course, if the component fails earlier, it is replaced on failure. Then each of the following processes is a renewal process.

1. number of failures in the interval $(0, t]$.
2. number of planned preventive replacements in the interval $(0, t]$.
3. number of replacements in the interval $(0, t]$.

Example 8 (Upper Records of a Renewal Process): Let $\xi_1, \xi_2, \ldots$ be i.i.d. random variables with $E(\xi_1) \geq 0$, so that the random variables take non-negative values with positive probability. Let

$$U_0 = 0,\ U_n = \sum_{i=1}^{n} \xi_i, \qquad n \geq 1,$$

and let S_n be the upper record times for the sequence U_n defined by

$$S_0 = 0, S_n = \inf\{k : U_k > U_{S_{k-1}}, k = 1, 2, \ldots\}$$

Then S_n is called the n-th upper record time of the sequence $\{U_n\}$. This new sequence is also a renewal sequence and the associated counting process is a renewal process.

Now, try the following exercises.

E2) Let N_t be the renewal process in Example 2. Then show that N_n has a negative binomial distribution for all non-negative integers n.

E3) For the renewal process N_t of Example 3, show that N_n has a binomial distribution for all non-negative integers n.

So far, we have discussed the renewal process with examples. Now in this section, we shall focus on the distribution of the renewal times and the renewal process.

7.5 DISTRIBUTION OF THE RENEWAL TIMES AND THE RENEWAL PROCESS

By now you have seen that renewal processes arise in a variety of situations. We will now study some of its properties. We will need the following definition.

Definition 4 (Convolution of Functions): Let g_1 and g_2 be any two non-negative functions. The convolution of g_1 and g_2 is a function denoted by $g_1 * g_2$ and defined by

$$g_1 * g_2(x) = \int_0^x g_1(x-y)g_2(y)dy.$$

Note that $g_1 * g_2 = g_2 * g_1$.

Now consider the renewal sequence S_n defined by Definition (1). Assume that the times duration of occurrence between two successive events, X_i's, are i.i.d. with common cumulative distribution function F. Assume, $F(0)=0$.

Independence of X_i's and Eqn.(1) imply that the joint distribution of $S_1, S_2, \dots, S_n$, for any n is completely determined by the cdf F.

Now let F_n denote the cdf of S_n. Note that $F_1 \equiv F$, since $S_1 = X_1$. Now using the fact that S_n is a sum of two independent random variables X_n and S_{n-1} we can get the cdf F_n by convoluting F_{n-1} and F as follows.

$$\begin{aligned} F_n(x) &= P(S_n \le x) \\ &= \int_0^x P(S_{n-1} \le x-y)\ dF_1(y) \\ &= \int_0^x F_{n-1}(x-y)\ dF(y) \\ &= F_{n-1} * F(x). \end{aligned} \tag{10}$$

Iterating the above we get that F_n is a n-fold convolution of F with itself. We will denote this by

$$F_n(x) = F^{*n}(x) \tag{11}$$

Let N_t be the renewal process associated with the sequence S_n. Then the probability law of the renewal process is also completely characterized by the cdf F. In fact, for

$$0 < t_1 < t_2 < \cdots < t_n \text{ and } 0 \le k_1 \le k_2 \le \cdots \le k_n$$

$$\begin{aligned} &P(N_{t_1} = k_1, N_{t_2} = k_2, \dots, N_{t_n} = k_n) \\ &\quad = P(S_{k_1} \le t_1 < S_{k_1+1}, S_{k_2} \le t_2 < S_{k_2+1}, \dots, S_{k_n} \le t_n < S_{k_n+1}) \end{aligned} \tag{12}$$

The assertion now follows since the RHS is completely determined by F. We will restate it in the form of a theorem.

Theorem 1: The distribution of the processes $\{S_n : n = 1,2,\dots\}$ and $\{N_t, t \ge 0\}$ is completely determined by the F, where F is the cdf of the interoccurrence times. Further, Eqn.(3) implies

$$P(N_t \ge k) = P(S_k \le t) = F_k(t), \qquad t \ge 0,\ k = 0, 1, 2, \dots,\ F_0(t) = 1 \tag{13}$$

Thus,

$$P(N_t = k) = P(N_t \ge k) - P(N_t \ge k+1) = F_k(t) - F_{k+1}(t) \tag{14}$$

We have already used this last assertion in Eqn.(6) while studying the Poisson renewal process.

We have already commented that as soon as an event occurs the process starts afresh or is renewed. This notion can be made more precise as follows. Note that the random variable N_t denotes the number of occurrences of the event in the time interval $(0,t]$. Similarly, N_{t+X_1} will denote the number of occurrences of the event in the time interval $(0,t+X_1]$. This last number is at least 1 since the first occurrence has taken place at time X_1. If we discount for that occurrence or in other words look at $N_{t+X_1}-1$, then we will get the number of occurrences of the event in the time interval $(X_1,\ t+X_1]$. As the process is a renewal process, therefore the following theorem looks logical.

Theorem 2: The processes $\{N_t,\ t\geq 0\}$ and $\{N_{t+X_1}-1, t\geq 0\}$ have the same probability law.

Proof: We have to show that the finite dimensional distributions of the two processes are the same. First note that since X_i's are i.i.d., $S_n - X_1 = \sum_{i=2}^{n} X_i$ has the same law as that of $S_{n-1} = \sum_{i=1}^{n-1} X_i$. Similarly

$$(S_1,S_2,\ldots,S_n) \overset{d}{=} (S_2-X_1,S_3-X_1,\ldots,S_{n+1}-X_1). \tag{15}$$

Now fix $0<t_1<t_2<\ldots<t_n$ and integers $0\leq k_1\leq k_2\leq\ldots\leq k_n$

$$\begin{aligned}
&P(N_{t_1+X_1}-1=k_1, N_{t_2+X_1}-1=k_2,\ldots,N_{t_n+X_1}-1=k_n)\\
&=P(N_{t_1+X_1}=k_1+1, N_{t_2+X_1}=k_2+1,\ldots,N_{t_n+X_1}=k_n+1)\\
&=P(S_{k_1+1}\leq t_1+X_1<S_{k_1+2},\ldots,S_{k_n+1}\leq t_n+X_1<S_{k_n+2})\\
&=P(S_{k_1+1}-X_1\leq t_1<S_{k_1+2}-X_1,\ldots,S_{k_n+1}-X_1\leq t_n<S_{k_n+2}-X_1)\\
&=P(S_{k_1}\leq t_1<S_{k_1+1}, S_{k_2}\leq t_2<S_{k_2+1},\ldots,S_{k_n}\leq t_n<S_{k_n+1})\\
&=P(N_{t_1}=k_1, N_{t_2}=k_2,\ldots,N_{t_n}=k_n).
\end{aligned} \tag{16}$$

where the second last equality follows from Eqn. (15) above. This completes the proof of the theorem.

Then above theorem is also helpful in deriving integral equalities concerning some quantities which are naturally associated with a renewal process. We illustrate the point with an example below.

Example 9: Let $p_n(t)=P(N_t=n)$. We will use the above theorem to show that

$$p_n(t)=\int_0^t p_{n-1}(t-s)dF(s) \text{ for all } n\geq 1. \tag{17}$$

Fix $n>0$. We will condition on the time of first renewal which is same as the time of the first occurrence of the concerned event. Note that if $S_1>t$ iff $N_t=0$. If, however, $S_1<t$ then $\{N_t=n\}$ if and only if there are exactly $n-1$ occurrences of the event in the time interval $(S_1,t]$. But, Theorem 2 implies that the conditional probability of the above event given $S_1<t$ is exactly same as that of the event where exactly $n-1$ events occur in the time interval $(0,t-S_1]$. Summarising we can write

$$P(N_t=n\mid S_1=s)=\begin{cases}0 & \text{if } s>t\\ p_{n-1}(t-s) & \text{if } s\leq t.\end{cases}$$

Thus, unconditioning on S_1 we get

$$p_n(t) = \int_0^{\infty} P(N_t = n \mid S_1 = s)dF(s)$$
$$= \int_0^{t} p_{n-1}(t-s)dF(s).$$

The exercise given below is the discrete time version of the example above and can be solved similarly. The trick is to condition on the time of first occurrence and use Theorem 2.

Note that in discrete time version $F(0)$ may be positive.

Try an exercise.

E4) In Examples 9 above also assume that X_i's are discrete random variables with

$$P(X_1 = i) = q_i$$

Then show that

$$p_n(j) = \sum_{l=0}^{j} p_{n-1}(j-l)q_l. \quad (18)$$

So far we have discussed renewal process with various examples and distributions S_n and N_t. In this section, we shall learn about the renewal equation.

7.6 RENEWAL EQUATION

We will continue the study of renewal processes. An important ingredient is the renewal equation which is an equation satisfied by the renewal function . Before we study renewal equation, let us first define the renewal function formally in the definition given below.

Definition 5 (Renewal Function): The renewal function M_t is defined as the expected number of renewals upto time t, i.e.

$$M_t = E(N_t) \quad (19)$$

The renewal function plays a very important role in renewal theory. We will see that it completely characterizes the renewal process. Note that the Eqn.(19) along with Eqn.(14) gives

$$M_t = \sum_{k=0}^{\infty} kP(N_t = k)$$
$$= \sum_{k=0}^{\infty} k(F_k(t) - F_{k+1}(t))$$
$$= \sum_{k=0}^{\infty} F_k(t). \quad (20)$$

Eqn.(20) can be used to compute the renewal function M_t. We can also use Theorem 2 for this purpose as follows.

First, let us look at an example.

Example 10: For the Poisson process with rate λ, the renewal function is

$$M_t = E(N_t) = \lambda t.$$

Now, try to find the solution of the exercises given below.

E5) Find the renewal function for the Negative Binomial process of Example 2.

E6) Find the renewal function for the Binomial Process of Example 3.

In continuation to renewal function, let us discuss now renewal argument.

As we did for solving Example 9 and (E4) we will condition on the first renewal time S_1. If $s > t$ then $N_t = 0$ with probability one and hence $M_t = 0$. However, if $S_1 \le t$, then by Theorem 2 the expected number of renewals in the time interval $(S_1, t]$ given $S_1 = s$ coincides with M_{t-s}, the expected number of renewals in the time interval $(0, t-s]$. Since one renewal has already occurred at time s, we get that, conditioned on $S_1 \le t$ the expectation of N_t is summarising, we get

$$E(N_t \mid S_1 = s) = \begin{cases} 0 & \text{if } s > t \\ 1 + M_{t-s} & \text{if } s \le t. \end{cases} \tag{21}$$

Unconditioning on S_1 or in other words taking expectation with respect to S_1 we get

$$\begin{aligned} M_t &= \int_0^\infty E(N_t \mid S_1 = s)dF(s) \\ &= \int_0^t [1 + M_{t-s}]dF(s) \\ &= F(t) + \int_0^t M_{t-s}dF(s). \end{aligned} \tag{22}$$

The above method where we conditioned on the first renewal time S_1 and then integrated with respect to S_1 is called the **renewal argument**.

Definition 6 (Renewal Equation): The above Eqn. (22) is called the renewal equation.

The renewal equation plays a very important role in the study of renewal processes.

Now try the following exercises.

E7) Consider Example 6, the counter process. Suppose that the electrical impulses are arriving at time which are i.i.d. exponential (λ) random variables. Assume that the counter gets locked for a fixed time $\varepsilon > 0$ on the arrival of every signal (whether it is recorded or not). Also assume that to begin with (at time 0) the counter is similarly locked. Let N_t^1 and N_t^2 be as defined in the example and let $Y_1^i, Y_2^i, \ldots, i = 1, 2$ denote the corresponding inter-arrival times. Find the distribution of Y_j^2. Find the corresponding renewal functions M_t^1 and M_t^2.

E8) Consider a system with two components which are arranged in series. The system fails if either of the two components fail. On failure, a component is replaced instantaneously. Suppose each component works independently and has life and exponential life time distribution with parameter λ. Let N_t denotes the number of failures for the system in the time interval $[0, t)$. Find the distribution of N_t and also the renewal function.

E9) A particular component in a system is very critical. Hence there is a built-in redundancy – there are two identical components in the system. However, only one of them is used at a time. Whenever the first one fails the system is automatically switched on to the second (redundant) component. The system fails whenever the second component also fails. At that time both the components are replaced instantaneously.

Assume that the life time distribution of the component is exponential with parameter λ. Express this system in a "renewal process" framework. Find the inter-occurrence time distribution.

Now before ending this unit let us go over its main points.

7.7 SUMMARY

In this unit, you studied the following:

1. There are many situations that we come across where an event recurs again and again and such that the time between two consecutive occurrences are i.i.d. The number of occurrences of the event upto time t can be modeled as a process N_t, $t \geq 0$ which is called the **renewal process.** The sequence of the time of occurrences of the event S_n is called the **renewal sequence.**

2. **Poisson process** is the basic example of a renewal process. We have seen several other examples of renewal process.

3. We have studied the properties of the distribution of S_n and N_t. We now know that the distributions are completely determined by the distribution function F of the interoccurrence time. We have also learnt how to condition on the first occurrence time to deduce properties of the renewal process. Such a method is called the **renewal argument.**

4. The expected value of numbers of renewals, M_t is called the **renewal function.** We have derived the **renewal equation** satisfied by M_t.

7.8 SOLUTIONS/ANSWERS

E1) Note that $\{\hat{N}_t = k\} = \{S_k \leq t < S_{k+1}\} = \{N_t = k\}$.

E2) By definition

$$\{K : S_k \leq t < S_{k+1}).$$

Since each X_i takes only two possible values 0 or 1, and since S_k is the time of occurrence of the k-th event, we get $\{S_k \leq n < S_{k+1}\}$ occurs if and only if **(i)** $X_k = 1$ and **(ii)** out of the first $(k-1)X_i$'s exactly $n-1$ are 1's and the rest 0. Hence

$$P(N_n = k) = \binom{k-1}{n-1} p^n (1-p)^{k-n}, k = n, n+1, n+2, \ldots$$

Thus, N_n has a negative binomial distribution with parameters n and p.

E3) Note that X_i has a geometric distribution with parameter $(1-p)$. Thus, it follows that S_k has a negative binomial distribution with parameters k and $(1-p)$. Once again we need to look at the event $\{S_k \leq n < S_{k+1}\}$. This is same as the event in which the first n Bernoulli trials result in exactly k successes and $n-k$ failures while the $(k+1)$ st success occurs at the j-th trail for some $j > n$, the intermediate $j-n-1$ trials having resulted in failures. Thus,

$$P(N_n = k) = P(S_k \le n < S_{k+1})$$
$$= \binom{n}{k}(1-p)^k p^{n-k} \sum_{j=n+1}^{\infty} (1-p)p^{j-n-1}$$
$$= \binom{n}{k}(1-p)^k p^{n-k}.$$

Thus, N_n has a binomial distribution with parameters n and $(1-p)$.

E4) Note that when $n = o$

$$p_0(j) = P(N_j = 0) = P(X_1 > j) = 1 - \sum_{l=0}^{j} q_l, \quad j = 0,1,2,\ldots$$

Using Theorem 2 we get for $n \ge 1$ and $j \ge 0$

$$P\left(N_j = n \mid s_1 = l\right) = \begin{cases} 0 & \text{if } \ l = j+1, j+2, \ldots \\ p_{n-1}(j-l) & \text{if } \ l = 0,1,2,\ldots,j. \end{cases}$$

Unconditioning we get

$$p_n(j) = \sum_{l=0}^{j} p_{n-1}(j-l) q_l \,.$$

E5) Since the distribution of X_i is discrete, the events occur only at integral times. Hence $N_t = N_{[t]}$ where $[t]$ denotes the largest integer smaller than or equal to t. Since $N_{[t]}$ has a negative binomial distribution with parameters $[t]$ and p we get

$$M_t = \frac{[t]}{p}.$$

E6) Once again $N_t = N_{[t]}$ and further the distribution is binomial with parameters $[t]$ and $1-p$. Thus,

$$M_t = (1-p)[t], t \ge 0.$$

E7) Note Y_j^2 denotes the time between the $(j-1)$-th and j-th recorded signals. The random variables Y_j^2 and i.i.d. Further, and arriving signal gets recorded if and only if it has arrived after a gap of time more than ε, i.e., if and only if $Y_j^2 > \varepsilon$. Thus

$$P(Y_1^2 \le x) = P(X \le x \mid X > \varepsilon) = 1 - e^{-\lambda(x-\varepsilon)}, \ x \ge \varepsilon$$

where X is a exponential random variable with parameter λ and denotes the time of arrival of the impulse.

E8) The failure time distribution is the distribution of the minimum of two i.i.d. exponential (λ) random variables. We know that this is also exponential (2λ). Thus, the renewal process N_t is a Poisson process with parameter 2λ and hence the renewal function is given by $M_t = E(N_t) = 2\lambda t$.

E9) Let X^1 denotes the failure time of the first component and X^2 that of the second component. The system fails at time $X^1 + X^2$. Let $\left(X_j^1, X_j^2\right): j = 1, 2, \ldots$ denote the successive failure times of the two components. The renewal event is the failure of the system. Then the associated renewal sequence is given by

$$S_n = \sum_{j=1}^{n} \left(X_j^1 + X_j^2 \right).$$

Let N_t denotes the associated renewal process.

Now, we want to find the distribution of $X_1^1 + X_1^2$. Since X_1^1, X_1^2 are i.i.d. exponential (λ) the inter-occurrence time distribution is gamma with parameters 2 and λ and its cdf is given by Eqn. (4) for $n = 2$.

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UNIT 8 RENEWAL PROCESSES-II

Structure **Page No**

8.1 INTRODUCTION

In this unit, we will study the renewal function M_t in more detail. We will begin the unit by finding the Laplace transform of the function in Sec. 8.2. This will be used to show that the renewal function completely characterises the renewal process. We will then study the elementary and the basic renewal theorems and see some of their applications. We will also define the residual time γ_t and study its asymptotic distribution. By the end of the unit, this is what you should be able to do.

Objectives

After studying this unit, you should be able to:

- define the Laplace transform of the renewal function and compute it in some special cases;
- use the renewal argument to evaluate $E[S_{N_t+1}]$ in terms of the renewal function M_t and the mean interoccurrence time μ;
- identify renewal type equations and find the solutions to these equations;
- use the Basic renewal theorem in applications;
- find the asymptotic distributions of the residual time γ_t, the current age δ_t and the total age β_t.

8.2 LAPLACE TRANSFORM OF THE RENEWAL FUNCTION

In Section 7.6, we defined the renewal function and showed that it satisfies the renewal Eqn. (22). It is clear from Theorem 1 in Sec. 7.5 (see also Eqn. (20) in Sec. 7.6) that M_t is completely determined by F, the cumulative distribution function of the interoccurrence times.

In this section, we will study some more properties of the renewal function M_t. We will start by deriving the Laplace transform of M_t. For this purpose we define the Laplace transform of a function in the form needed for our discussion.

Definition 1 (Laplace Transform): Let $h:[0, \infty) \rightarrow [0, \infty)$ be a non-decreasing function. The Laplace transform (or Laplace Stiltjes transform) of h, denoted by $\tilde{h}$, is a function of t defined by

$$\tilde{h}(t) = \int_0^\infty e^{-ts} dh(s),$$

for values of t for which the improper integral on the right hand side converges. It is known that the Laplace transform of a function completely characterises the function. In other words the following is true: Let $\tilde{h}_1$ and $\tilde{h}_2$ be the Laplace transforms of h_1 and h_2 respectively. If $\tilde{h}_1 = \tilde{h}_2$ then $h_1 = h_2$.

Example 1: Let X be a non-negative random variable with distribution function H. Then H is clearly a non-negative and non-decreasing function. Thus,

$$\tilde{H}(t) = \int_0^\infty e^{-ts} dH(s) = E[e^{-tX}].$$

Thus, the Laplace transform of the cdf H evaluated at $t > 0$ is just the moment generating function of the random variable X evaluated at $(-t)$.

Thus, $\tilde{H}(t)$ exists if moment generating function $H(s)$ exists.

Example 2: Let Z_1 and Z_2 be two independent random variables with cdf's H_1 and H_2, respectively. Let $\tilde{H}_1$ and $\tilde{H}_2$ be the corresponding Laplace transforms. We have seen that if H is the cdf of $Z_1 + Z_2$ then $H = H_1 * H_2$. Then using Example 1 above the properties of moment generating functions we get

$$\tilde{H}(t) = E\left[e^{-t(Z_1+Z_2)}\right] = E\left[e^{-tZ_1}\right]E\left[e^{-tZ_2}\right] = (\tilde{H}_1)(t)\,(\tilde{H}_2)(t).$$

Iterating the above we can write for $H = H_1 * H_2 * \ldots * H_n$,

$$\tilde{H}(t) = \prod_{j=1}^{n} (\tilde{H}_j)(t)$$

as the Laplace transform of sum of n independent random variables.

In particular if the H_i's are all the same or in other words if the underlying random variables are identically distributed with distribution function H, then $H = [H_1]^{*n}$ and hence

$$\tilde{H}(t) = [(\tilde{H}_1)(t)]^n.$$

We now want to calculate the Laplace transform of the renewal function M_t. Note that since $M_t = E(N_t)$ and N_t counts the number of occurrences of the recurring event, $\tilde{M}_t$ is non-negative non decreasing function. Thus $\tilde{M}_t$, is defined and Definition 1 can be used to calculate it. Recall the definition of F_n, the n-fold convolution of F. Then by using Eqn. (20) of Sec.7.6 and Example 2 above, we get

$$\begin{aligned}
\tilde{M}_t &= \int_0^\infty e^{-ts} dM_s \\
&= \int_0^\infty e^{-ts} d\left(\sum_{k=1}^{\infty} F_k(s)\right) \\
&= \sum_{k=1}^{\infty} \int_0^\infty e^{-ts} dF_k(s) \\
&= \sum_{k=1}^{\infty} (\tilde{F}_k)t \\
&= \sum_{k=1}^{\infty} [\tilde{F}(t)]^k
\end{aligned}$$

$$= \frac{\tilde{F}(t)}{1-\tilde{F}(t)}. \tag{1}$$

Thus, if we know $\tilde{F}(t)$, the transform of F, then Eqn. (1) can be used to calculate $\tilde{M}_t$. The converse is also true which leads us to the following theorem.

Theorem 1: The renewal function $\{M_t : t \geq 0\}$ completely characterizes the renewal process $\{N_t : t \geq 0\}$.

Proof: Clearly by Eqn. (1) if the renewal function is completely known then so is $\{\tilde{M}_t : t \geq 0\}$. Further it follows from Eqn. (1) that

$$\tilde{F}(t) = \frac{\tilde{M}_t}{1+\tilde{M}_t}. \tag{2}$$

Thus, $\{\tilde{F}(t) : t \geq 0\}$ is completely known. Since Laplace transform of a function completely determines the function, we get that $\{F(t) : t \geq 0\}$ is completely determine. Theorem of Sec.7.5 now implies that the distribution of $\{N_t : t \geq 0\}$ is completely determined. This completes the proof.

We recall the renewal Eqn. (22) of Sec. 7.6, which is

$$M_t = F(t) + \int_0^t M_{t-s} dF(s).$$

We can rewrite this using the convolution notation as

$$M_t = F(t) + (M * F)_t. \tag{3}$$

Remark: Note that if the distribution function F has a density f then

$$\int_0^t M_{t-s}\, dF(s) = \int_0^t M_{t-s}\, f(s)\, ds = (M * f)_t$$

where the convolution appearing in the last term above is as in Definition 4 of Sec. 7.5. However we will also continue to use the convolution notation for distribution functions as in Eqn. (3).

If we take the Laplace transform on both sides of Eqn. (3), we get

$$\tilde{M}_t = \tilde{F}(t) + \tilde{M}_t \tilde{F}(t). \tag{4}$$

Note that this is same as Eqn. (2).

We will now calculate the Laplace transform of M_t in some special cases.

Example 3: Let the interoccurrence times be exponentially distributed with rate $\lambda > 0$. Then we know that $\{N_t : t \geq 0\}$ is a Poisson process with rate λ and hence $M_t = \lambda t$. Thus,

$$\tilde{M}_t = \int_0^\infty e^{-ts} d(\lambda s) = \int_0^\infty e^{-ts} \lambda d(s) = \frac{\lambda}{t}.$$

Note that one can also use Eqn. (1) to compute the above. We leave this as an exercise.

Example 4: Let N_t be the binomial process introduced in Example 3 and E6) of Unit 7. Then $N_t \equiv N_{[t]}$ is a Binomial random variable with parameters $[t]$ and $(1-p)$. Here $[t]$ denotes the largest integer smaller than or equal to t. Thus, the

Laplace transform of M_t is the moment generating function (m.g.f.) (evaluated at $(-t)$) of a binomial random variable and, thus, we get

$$\tilde{M}_t = [1+(1-p)\,(e^{-t}-1)]^{[t]} = [p+(1-p)\,e^{-t}]^{[t]}$$

Example 5: Let N_t be the negative binomial process introduced in Example 2 and E5 of Unit 7. Then N_t is negative binomial random variable with parameters $[t]$ and p. As in Example 4 we can find the m.g.f. of the random variable in order to find the Laplace transform of M_t. However, in this case the interoccurrence distribution is very simple with the corresponding random variable taking only two values 0 and 1 with probabilities $1-p$ and p respectively. Thus, $\tilde{F}(t)$ is just the m.g.f. of this simple random variable and equals $(1-p)+pe^{-t}$. Thus, using Eqn. (1) we get

$$\tilde{M}_t = \frac{\tilde{F}(t)}{1-\tilde{F}(t)} = \begin{cases} \dfrac{(1-p)+pe^{-t}}{p-pe^{-t}} & \text{if } t>0 \\ 1 & \text{if } t=0 \end{cases}.$$

Let us now try to solve the following exercises.

E1) For Example 3 compute $\tilde{F}(t)$ and use this to calculate $\tilde{M}_t$.

E2) Let $\{X_n : n \geq 1\}$ be an i.i.d. sequence of interoccurrence times with common probability mass function given by

$$P(X_n = 0) = 0.2,\ P(X_n = 1) = 0.3,\ P(X_n = 2) = 0.5.$$

Let N_t, $t \geq 0$ be the corresponding renewal process. Find the Laplace transform $\tilde{M}_t$ of the renewal function M_t.

E3) Let $\{X_n : n \geq 1\}$ be an i.i.d. sequence of interoccurrence times with common probability density function given by

$$f(x) = \begin{cases} e^{-(x-1)} & \text{if } x>1 \\ 0 & \text{otherwise} \end{cases}.$$

Let N_t, $t \geq 0$ be the corresponding renewal process. Find the Laplace transform $\tilde{M}_t$ of the renewal function M_t.

Though we have calculated the renewal function and its Laplace transform in a few cases, it is not always easy to compute these for all the renewal processes. Thus, we study the asymptotic behaviour of M_t as $t \to \infty$ which we will do in the next section.

8.3 ELEMENTARY RENEWAL THEOREM

At the end of the last subsection we remarked that it is not always easy to compute the renewal function M_t for all finite t. However, in this subsection, we will see that by studying the asymptotic properties of the function we can approximately calculate the function for large values of t. We begin with a couple of exercises which will be used in the proof of Theorem 2 below.

E4) Show that for every fixed $t>0$, $F_n(t)$ is monotonically decreasing in n. Use this to show that for all nonnegative integers n, r, k with $0 \leq k \leq r-1$

$$F_{nr+k}(t) \leq [F_r(t)]^n F_k(t),$$

where we define $F_0(t)=1$.

E5) Let $F(0)<1$. Use E4) to show that for every $t>0$ there exists an r such that $F_r(t)<1$.

Now we can prove the following result. We assume that $F(0)<1$.

Theorem 2: $M_t<\infty$ for all $t\geq 0$.

Proof: Fix $t>0$. By E5) there exists a r such that $F_r(t)<1$. Then for this r, using Eqn. (20), we get

$$\begin{aligned}
M_t &= \sum_{j=1}^{\infty} F_j(t) \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{r-1} F_{nr+k}(t)-F_0(t) \\
&\leq \sum_{n=0}^{\infty}\sum_{k=0}^{r-1} [F_r(t)]^n F_k(t)-F_0(t) \\
&= \frac{1}{1-F_r(t)}\sum_{k=0}^{r-1} F_k(t)-F_0(t) \\
&< \infty, \text{ where we define } F_0(t)=1
\end{aligned}$$

This proves the theorem.

We will now prove an auxiliary lemma.

Lemma 1: Assume that $E(X_1)<\infty$. Then

$$E(S_{N_t+1}]=E(X_1)\,(M_t+1).$$

Proof: Note that $S_{N_t+1}>t$ denotes the time of first occurrence of the event after time t. We will prove the lemma using the renewal argument. Let $A_t=E[S_{N_t+1}]$. The conditioning on X_1, the time of occurrence of the first event, we get

$$E[S_{N_t+1}\mid X_1=x]=\begin{cases} x & \text{if } x>t \\ x+A_{t-x} & \text{if } x\leq t \end{cases}.$$

Then

$$\begin{aligned}
A_t &= E[S_{N_t+1}] \\
&= \int_0^{\infty} E[S_{N_t+1}\mid X_1=x]\,dF(x) \\
&= \int_0^{t} [x+A_{t-x}]\,dF(x)+\int_t^{\infty} x\,dF(x) \\
&= \int_0^{\infty} x\,dF(x)+\int_0^{t} A_{t-x}\,dF(x) \\
&= E(X_1)+\int_0^{t} A_{t-x}\,dF(x) \\
&= E(X_1)+(A*F)_t.
\end{aligned}$$

Now taking Laplace transforms on both sides and noting that the Laplace transform of a constant function is itself we get

$$\tilde{A}_t = E(X_1) + \tilde{A}_t \tilde{F}(t).$$

Thus,

$$\tilde{A}_t = \frac{E(X_1)}{1-\tilde{F}(t)}$$
$$= E(X_1)\left[1+\frac{\tilde{F}(t)}{1-\tilde{F}(t)}\right]$$
$$= E(X_1)\,[1+\tilde{M}_t].$$

Since Laplace transforms of non-negative functions characterise the functions and since the right hand side in the above equation is the Laplace transform of the function $g(t) = E(X_1)\,[1+M_t]$, we get

$$A_t = E(X_1)[1+M_t], \quad t \ge 0$$

and this proves the result.

Alternatively, this lemma can be proved by the following argument $S_{N_t+1} = \sum_{i=1}^{N_t+1} X_i$,

where X_i being i.i.d.

$$E(S_{N_t+1}) = E\,E(S_{N_t+1}/N_t)$$
$$= E(N_t+1)\,E(X_i)$$
$$= E(X_i)\,(M_t+1).$$

Now we will state and prove our first asymptotic result, the Elementary Renewal Theorem. We restate all the assumptions in the statement of the theorem.

Theorem 3: Let $X_1, X_2, \ldots$ be i.i.d. non-negative random variables with common distribution function F. Let $E(X_1) = \mu < \infty$. Let S_n be the corresponding renewal sequence and let N_t be the associated renewal process. Then for the renewal function $M_t = E(N_t)$ the following holds.

$$\lim_{t\to\infty} \frac{M_t}{t} = \frac{1}{\mu}.$$

Proof: Since by definition, S_{N_t+1} is the time instant of the first occurrence of the event after time t we have $S_{N_t+1} > t$. Now using Lemma 1, we get

$$t < E[S_{N_t+1}] = \mu[1+M_t].$$

Dividing both side by μt and rearranging terms, we get

$$\frac{M_t}{t} > \frac{1}{\mu} - \frac{1}{t}.$$

Taking lim inf as $t \to \infty$ on both sides, we get

$$\liminf_{t\to\infty} \frac{M_t}{t} \ge \frac{1}{\mu}. \tag{5}$$

Now fix $c > 0$. Let X_i^c be the truncated random variables defined by

$$X_i^c = \begin{cases} X_i & \text{if } X_i \le c \\ c & \text{if } X_i > c \end{cases}.$$

Then note that the random variables $X_1^c, X_2^c, \ldots$ form another i.i.d. sequence with $E(X_1^c) = \mu^c$ (say). Let the associated renewal sequence, renewal process and renewal function be S_n^c, N_t^c and M_t^c respectively. Since $S_{N_t^c}^c \le t$ and $X_i^c \le c$ for all i, we get

$$S^c_{N^c_t+1} = S^c_{N^c_t} + X^c_{N^c_t+1} \le t + c.$$

Note that since $X^c_i \le X_i$ we get that $N^c_t \ge N_t$ which implies $M^c_t \ge M_t$. Then once again using Lemma 1 we have

$$t + c \ge E[S^c_{N^c_t+1}] = \mu^c[1 + M^c_t] \ge \mu^c[1 + M_t].$$

Thus, dividing by $t\mu^c$

$$\frac{M_t}{t} \le \frac{1}{\mu^c} + \frac{c}{t\mu^c} - \frac{1}{t}.$$

Taking limsup as $t \to \infty$ on both sides, we get for all $c > 0$

$$\limsup_{t\to\infty} \frac{M_t}{t} \le \frac{1}{\mu^c}. \qquad (6)$$

Finally since μ^c is the mean of the non-negative random variable X^c_1 and hence can be got by

$$\mu^c = \int_0^c P(X^c_1 > x)\,dx = \int_0^c (1 - F(x))\,dx,$$

Therefore

$$\lim_{c\to\infty} \mu^c = \int_0^\infty (1 - F(x))\,dx = \mu.$$

Thus, taking limit as $c \to \infty$ in Eqn. (6) we get

$$\limsup_{t\to\infty} \frac{M_t}{t} \le \frac{1}{\mu}. \qquad (7)$$

Now Eqn. (5) and Eqn. (7) together imply

$$\lim_{t\to\infty} \frac{M_t}{t} = \frac{1}{\mu}$$

and this completes the proof.

If follows directly from the elementary renewal theorem that the average number of renewals upto time t, namely M_t increases to ∞ as $t \to \infty$. It also gives the rate at which M_t increases.

In the next section, we will state a more general version of the above theorem called the Basic Renewal Theorem.

8.4 BASIC RENEWAL THEOREM

In the previous section, we studied the Elementary renewal theorem. The same type of arguments give a more general theorem which we will study in this section. We recall the definition of the convolution $g_1 * g_2$ of two functions g_1 and g_2 given in Unit 7.

Let a_t be a bounded function. As usual $F(t)$ will denote the cdf of the interoccurrence time and M_t the renewal function. We look at the following **renewal-type equation** (compare with the renewal equation, Eqn. (3)).

$$A_t = a_t + (A * F)_t. \qquad (8)$$

We are interested in finding solution A_t of Eqn. (8). The following theorem identifies such solution.

Theorem 4: Suppose a_t is a bounded function. There exists one and only one function A_t which is bounded on all finite intervals and which satisfies Eqn. (8). The unique solution is given by

$$A_t = a_t + \int_0^t a_{t-r} dM_r = a_t + (a * M)_t \tag{9}$$

where M_t is the renewal function.

Proof: Let A_t be defined by Eqn. (9). We will show that this is a solution of Eqn. (8). Using Eqn. (20) of Sec. 7.6 and Eqn. (9), we get

$$\begin{aligned}
A_t &= a_t + (a * M)_t = a_t + \left[a * \left(\sum_{k=1}^{\infty} F_k\right)\right]_t \\
&= a_t + \left[a * \left(F + \sum_{k=2}^{\infty} F_k\right)\right]_t \\
&= a_t + (a * F)_t + \left[a * \left(\sum_{k=2}^{\infty} (F_k)\right)\right]_t \\
&= a_t + (a * F)_t + \left[a * \left(\sum_{k=2}^{\infty} (F_{k-1} * F)\right)\right]_t \\
&= a_t + \left(\left(a + \left[a * \sum_{k=1}^{\infty} F_k\right]\right) * F\right)_t \\
&= a_t + ((a + a * M) * F)_t \\
&= a_t + (A * F)_t
\end{aligned}$$

which is Eqn. (8). Moreover, since a_t is bounded and M_t is non-decreasing we have for any $0 < T < \infty$

$$\sup_{0 \le t \le T} A_t \le \sup_{0 \le t \le T} |a_t| \left[1 + \int_0^T dM_r\right] \le \sup_{0 \le t \le T} |a_t| [1 + M_T] < \infty .$$

Thus, A_t is bounded on bounded intervals. Thus, Eqn. (9) gives one solution in the requisite class.

Now let B be any solution of Eqn. (8) which is bounded on bounded intervals. We have

$$\begin{aligned}
B_t &= a_t + (B * F)_t \\
&= a_t + ((a + B * F) * F)_t \\
&= a_t + (a * F)_t + (B * F_2)_t \\
&= \dots \\
&= a_t + \left(a * \left(\sum_{k=1}^{n-1} F_k\right)\right)_t + (B * F_n)_t
\end{aligned} \tag{10}$$

for every $n \ge 1$. Further

$$|(B * F_n)_t| = \left|\int_0^t B_{t-r} dF_n(r)\right| \le \left(\sup_{0 \le r \le t} |B_r|\right) F_n(t) .$$

Using Theorem 2 and Eqn. (20) of Sec. 7.6, we get that $F_n(t) \to 0$ as $n \to \infty$ for every $t > 0$. Hence

$$\lim_{n \to \infty} |(B * F_n)_t| = 0 \text{ for all } t > 0 . \tag{11}$$

Also

$$\lim_{n\to\infty}\left(a*\left(\sum_{k=1}^{n-1}F_k\right)\right)_t=\left(a*\left(\sum_{k=1}^{\infty}F_k\right)\right)_t=(a*M)_t. \qquad (12)$$

Thus, combining Eqns. (10) – (12), we get

$$B_t=a_t+\lim_{n\to\infty}\left(a*\left(\sum_{k=1}^{n-1}F_k\right)\right)_t+\lim_{n\to\infty}(B*F_n)_t=a_t+(a*M)_t.$$

This completes the proof.

As an immediate application we have the following exercise.

E6) Prove Lemma 1 using the above theorem.

To state the Basic renewal theorem we need to introduce one definition.

Definition 2 (Arithmetic Distribution Function): A distribution function G is said to be arithmetic if there exists a $\lambda > 0$ such that the set of all discontinuity points of G is a subset of $\{0, \pm\lambda, \pm 2\lambda, \ldots\}$.

The largest such λ is called the **span** of G.

Now we will state the Basic Renewal theorem without proof. This generalizes Theorem 4 to the solutions of the more general renewal type equations.

Theorem 5 (Basic Renewal Theorem): Let F be the distribution function of a positive random variable with mean $\mu < \infty$. Suppose that a_t is Riemann integrable and A_t is the solution of the renewal type equation

$$A_t = a_t + (A*F)_t$$

1. If F is not arithmetic, then

$$\lim_{t-\infty} A_t = \frac{1}{\mu}\int_0^\infty a_x dx.$$

2. If F is arithmetic with span λ, then for all $c > 0$,

$$\lim_{n-\infty} A_{c+n\lambda} = \frac{\lambda}{\mu}\sum_{n=0}^{\infty} a_{c+n\lambda}$$

We note the following Corollary which is also called the **Blackwell's renewal theorem.**

Corollary 1: Let $h > 0$ be fixed. If F is not arithmetic then

$$\lim_{t\to\infty}[M_{t+h} - M_t] = \frac{h}{\mu}.$$

Proof: Define

$$a_t = \begin{cases} 1 & \text{if } 0 \le t < h \\ 0 & \text{if } h \le t. \end{cases}$$

Substituting in Eqn. (8), we get for $t > h$

$$A_t = 0 + \int_{t-h}^{t} dM_r = M_t - M_{t-h}$$

Further $\int_0^\infty a_r dr = h$. Thus, the corollary follows from Theorem 5.

The Basic renewal theorem is useful in many applications. We will see one such application in the next section.

8.5 ASYMPTOTIC DISTRIBUTION OF THE RESIDUAL TIME

In this section, we will study the asymptotics of various quantities associated with a renewal process. We will use the renewal theorems that we have studied in the last two sections for this purpose.

Let $X_1, X_2, \ldots$ be the sequence of inter-arrival times which are assumed to be i.i.d. with common cdf F. Let S_n be the associated renewal sequence and let N_t be the corresponding renewal process.

Thus, S_{N_t} denotes the time of the last occurrence of the event before time t and $S_{N_{t+1}}$ denotes the time of the 'next' occurrence. We define a few more related quantities.

Definition 3: The residual time at time t (or excess life) is defined by

$$\gamma_t = S_{N_{t+1}} - t \tag{13}$$

and is the time left for the next occurrence as seen at time t.

The current life (or age) at time t is the random variable defined by

$$\delta_t = t - S_{N_t} \tag{14}$$

The total life β_t is defined as

$$\beta_t = \gamma_t + \beta_t = S_{N_{t+1}} - S_{N_t} \tag{15}$$

We will see these quantities in the Poisson process example.

Example 6: Suppose that the renewal process N_t is a Poisson process with parameter λ. Then the residual time γ_t is an exponential random variable with parameter λ as can be seen using the stationary increments property of N_t as follows.

$$P(\gamma_t > x) = P(N_{t+x} - N_t = 0) = P(N_x = 0) = e^{-\lambda x}$$

A similar argument can be used to find the distribution of δ_t

Now try the following exercise.

E7) Find the cdf of δ_t.

Moreover, using the independent increment property of N_t we also get that δ_t and γ_t are independent random variables.

While finding the distributions (or even the joint distribution) of δ_t and γ_t was quite easy in the case of the Poisson process, this is certainly not the case in general. As in the case of the renewal function, we look at the asymptotic properties of these random variables. We have the following lemma.

Recall that μ denotes the mean interoccurrence time, i.e. $\mu = E(X_1)$. We assume $\mu < \infty$. We also assume for the rest of the subsection that F is continuous. Then, clearly F in non-arithmetic.

Lemma 2: For every $z > 0$

$$\lim_{t\to\infty} P(\gamma_t > z) = \frac{1}{\mu}\int_z^{\infty}(1-F(s))ds.$$

Proof: For every $z > 0$ define the function A_t^z by

$$A_t^z = P(\gamma_t > z).$$

Then as usual conditioning on the first renewal occurring at $s > 0$, we get

$$P(\gamma_t > z \mid X_1 = s) = \begin{cases} 1 & \text{if} \quad s > t+z \\ 0 & \text{if} \quad t+z \ge s > t \\ A_{t-s}^z & \text{if} \quad t \ge s. \end{cases}$$

Unconditioning the above, we get

$$\begin{aligned} A_t^z &= \int_0^{\infty} P(\gamma_t > z \mid X_1 = s)dF(s) \\ &= \int_0^t A_{t-s}^z dF(s) + \int_t^{t+z} 0dF(s) + \int_{t+z}^{\infty} 1dF(s) \\ &= 1 - F(t+z) + \int_0^t A_{t-s}^z dF(s). \end{aligned}$$

This is a renewal type equation as in Eqn. (8). Further the function $(1-F(t+z))$ as a function of t is clearly bounded. Hence by Theorem 4 its unique solution is given by

$$A_t^z = 1 - F(t+z) + \int_0^t (1 - F(t+z-s))dM_s \tag{16}$$

But, Eqn. (16) may not be easy to compute. But note that since F is the cdf of a nonnegative random variable X_1, we have

$$\int_0^{\infty}(1-F(s))ds = \int_0^{\infty} P(X_1 > s)ds = E(X_1) = \mu < \infty.$$

Then

$$\int_0^{\infty}(1-F(t+z))dt = \int_z^{\infty}(1-F(s))ds < \infty.$$

Thus, the function $(1-F(t+z))$ is Riemann integrable and the Basic renewal theorem can be applied. Theorem 5 now implies

$$\lim_{t\to\infty} A_t^z = \frac{1}{\mu}\int_0^{\infty}(1-F(t+z))dt = \frac{1}{\mu}\int_z^{\infty}(1-F(s))ds.$$

This completes the proof.

An immediate consequence of the above Lemma is the following Corollary which gives the asymptotic joint distribution of γ_t and δ_t.

Corollary 2: For every $y > 0, z > 0$

$$\lim_{t\to\infty} P(\delta_t > y, \gamma_t > z) = \frac{1}{\mu}\int_{y+z}^{\infty}(1-F(s))ds.$$

Proof: Note that for any $y > 0, z > 0$ and any $t > 0$

$$\{\gamma_t > z, \delta_t > y\} = \{\gamma_{t-y} > z + y\}.$$

Thus, using Lemma 2, we get

$$\lim_{t\to\infty} P(\gamma_t > z, \delta_t > y) = \lim_{t\to\infty} P(\gamma_{t-y} > z + y) = \frac{1}{\mu}\int_{y+z}^{\infty}(1-F(s))ds.$$

We further get that asymptotically γ_t and δ_t behave similarly.

Corollary 3: For every $y > 0$,

$$\lim_{t\to\infty} P(\delta_t > y) = \frac{1}{\mu}\int_y^\infty (1-F(s))ds.$$

Proof: The proof follows immediately from Corollary 2 since

$$\begin{aligned}\lim_{t\to\infty} P(\delta_t > y) &= \lim_{t\to\infty}\lim_{z\downarrow 0} P(\gamma_t > z, \delta_t > y)\\ &= \lim_{z\downarrow 0}\lim_{t\to\infty} P(\gamma_t > z, \delta_t > y)\\ &= \lim_{z\downarrow 0}\frac{1}{\mu}\int_{z+y}^\infty (1-F(s))ds\\ &= \frac{1}{\mu}\int_y^\infty (1-F(s))ds.\end{aligned}$$

Finally, we find the asymptotic distribution of the total lifetime β_t as a consequence of the Basic renewal theorem.

Lemma 3: For every $x > 0$

$$\lim_{t\to\infty} P(\beta_t > x) = \frac{1}{\mu}\int_x^\infty sdF(s).$$

Proof: Fix $x > 0$. Define $B_t^x = P(\beta_t > x)$. Once again employing the familiar renewal argument, we get

$$P(\beta_t > x \mid X_1 = s) = \begin{cases} 1 & \text{if} \quad s > \max(t,x) \\ B_{t-s}^x & \text{if} \quad t \geq s \\ 0 & \text{otherwise.}\end{cases}$$

Unconditioning the above, we get

$$\begin{aligned}B_t^x &= \int_0^\infty P(\beta_t > x \mid X_1 = s)dF(s)\\ &= \int_0^t B_{t-s}^x dF(s) + \int_{\max(t,x)}^\infty 1\, dF(s)\\ &= 1 - F(\max(t,x)) + \int_0^t B_{t-s}^x dF(s).\end{aligned}$$

The basic renewal theorem now gives

$$\begin{aligned}\lim_{t\to\infty} B_t^x &= \lim_{t\to\infty} P(\beta_t > x)\\ &= \frac{1}{\mu}\int_0^\infty [1 - F(\max(t,x))]dt\\ &= \frac{1}{\mu}\left[\int_0^x [1-F(x)]dt + \int_x^\infty [1-F(t)]dt\right]\\ &= \frac{1}{\mu}\int_x^\infty sdF(s).\end{aligned}$$

The last equality above follows using integration by parts.

Now try the following exercises.

E8) Refer to E9) of Unit 7, where a system is fitted with two identical components – only one of which is used at a time. The system fails when both the components fail. Assuming the life time distribution of the component is exponential with parameter λ, find the Laplace transform of the interoccurrence time distribution.

E9) Refer to E8). Find the renewal function.

E10) In the above exercise, find the long term average number of failures.

Now before ending this unit, let us go over its main points.

8.6 SUMMARY

In this unit, we studied the following:

1. We defined the Laplace transform of a non-negative function and computed the Laplace transform of the renewal function M_t, which is denoted by $\tilde{M}_t$.

2. We saw the relationship between $\tilde{M}_t$ and $\tilde{F}_t$. With this we were able to conclude that the renewal function uniquely characterizes the distribution of a renewal process.

3. We computed $\tilde{M}_t$ in some special cases.

4. We proved the relation $E\left[S_{N_{t+1}}\right] = E(X_1)(M_t + 1)$.

5. We proved the elementary renewal theorem.

6. We looked at renewal type equations and found the unique solutions to such equations. We also stated the Basic renewal theorem.

7. We applied the basis renewal theorem to find asymptotic distributions of the residual time γ_t, the current age δ_t and the total age β_t.

8.7 SOLUTIONS/ANSWERS

E1) Here $F(t) = 1 - e^{-\lambda t}, t \geq 0$. Hence

$$\tilde{F} = \int_0^\infty e^{-ts} dF(s) = \int_0^\infty e^{-ts}(\lambda e^{\lambda s})ds = \frac{\lambda}{\lambda + t}$$

Thus,

$$\tilde{M}_t = \frac{\tilde{F}_t}{1 - \tilde{F}_t} = \frac{\lambda}{t}.$$

E2) As in Example 5, we will first find $\tilde{F}_t$. Note

$$\tilde{F}_t = E\left[e^{-tX_1}\right] = 0.2 + 0.3e^{-t} + 0.5e^{-2t}$$

Thus,

$$\tilde{M}_t = \frac{\tilde{F}_t}{1 - \tilde{F}_t} = \frac{0.2 + 0.3e^{-t} + 0.5e^{-2t}}{0.8 - 0.3e^{-t} - 0.5e^{-2t}}.$$

E3) Here

$$\tilde{F}_t = E\left[e^{-tX_1}\right] = \int_1^\infty e^{-tx} e^{-(x-1)} dx$$

$$= \frac{e^{-x(t+1)+1}}{t+1}\Bigg|_{x=1}^{x=\infty}$$

$$= \frac{e^{-t}}{t+1}.$$

Thus, $\tilde{M}_t = \frac{\tilde{F}_{t'}}{1-\tilde{F}_t} = \frac{e^{-t}}{t+1-e^{-t}}.$

E4) By definition for $1 \le m \le n$

$$F_n(t) = [F(t)]^{*n} = \left[F(t)^{*(n-m)} * [F(t)]^{*m}\right]$$

$$= F_{n-m}(t) * F_m(t) = \int_0^t F_{n-m}(t-s)dF_m(s)$$

$$\le F_{n-m}(t)F_m(t) \le F_m(t)$$

The last inequalities follows since F_{n-m} is a cumulative distribution function and hence non-decreasing. This proves the first part.

Using the second last inequality above, we get

$$F_{nr+k}(t) \le F_{(n-1)r+k}(t)F_r(t).$$

Now iterating this inequality n times, we get

$$F_{nr+k}(t) \le F_k(t)[F_r(t)]^n.$$

where we define $F_0(t) = 1$.

E5) Since $F(0) < 1$, we get using right continuity of F that there exists a $t_0 > 0$ such that $F(t_0) < 1$. Hence for all $0 \le t \le t_0, F(t) < 1$.

Now recall that F_2 is the cdf of $X_1 + X_2$ where X_1, X_2 are i.i.d. with common distribution function F. Thus, we have

$$P(X_1 + X_2 > 2t_0) \ge P(X_1 > t_0, X_2 > t_0) = P(X_1 > t_0)P(X_2 > t_0).$$

This is same as $1 - F_2(2t_0) \ge [1 - F(t_0)]^2 > 0$.

Hence for $0 \le t \le 2t_0$

$$F_2(t) \le F_2(2t_0) < 1.$$

By induction, we get

$$F_{2^j}(t) \le F_{2^j}(2^j t_0) < 1 \text{ for all } 0 \le t \le 2^j t_0.$$

Now using the monotone property of the sequence F_n proved in E4), we get the result.

E6) In the proof of the Lemma 1 we have shown that $A_t = E\left[S_{N_{t+1}}\right]$ satisfies the renewal type equation

$$A_t = E(X_1) + (A * F)_t.$$

This is same as Eqn.(8) with $a_t \equiv E(X_1)$. By Theorem 4 the unique solution is given by Eqn.(9) which in this case becomes

$$A_t = E(X_1) + \int_0^t E(X_1)dM_\tau = E(X_1)[1 + M_t].$$

E7) By Eqn. (14) it is evident that $\delta_t \le t$. Thus, $P(\delta_t \le x) = 1$ for all $x \ge t$. Now let, $x < t$. Then

$$P(\delta_t > x) = P(N_t - N_{t-x} = 0) = P(N_x = 0) = e^{-\lambda x}$$

E8) Let F denotes the cdf of an exponential random variable with parameter λ and let us denote the distribution of $X^1 + X^2$ by G, i.e., G is the cdf of a gamma random variable with parameters 2 and λ. Then $G = F * F$. Let the corresponding renewal functions be denoted by M_t^F and M_t^G. We know that $M_t^F = \lambda t$. (See Example 10 of Unit 7).

Further, by Example 2 (of Unit 8) the corresponding Laplace transforms are given by $\tilde{G} = \tilde{F}^2$. Now

$$\tilde{F}(t) = \int_0^\infty e^{-ts} dF(s) = \int_0^\infty \lambda e^{-ts} e - \lambda s ds = \frac{\lambda}{t+\lambda}.$$

Thus,

$$\tilde{G}(t) = \left(\frac{\lambda}{t+\lambda}\right)^2.$$

E9) As we have seen earlier

$$\tilde{G}(t) = \left(\frac{\lambda}{t+\lambda}\right)^2.$$

Now Eqn. (1) (of this unit) implies

$$\tilde{M}^G(t) = \frac{\tilde{G}(t)}{1-\tilde{G}(t)} = \frac{\lambda}{t} \cdot \frac{\lambda}{t+2\lambda}.$$

We have already noted (see Example 3) that $\tilde{M}^F(t) = \lambda / t$. Thus, we get that $\tilde{M}^G(t)$ is a product of two Laplace transform $\tilde{M}^F(t)$ and $\tilde{H}(t)$ where

$$\tilde{H}(y) = \frac{2\lambda}{t+2\lambda}.$$

Note $\tilde{M}^G(t) = \dfrac{\tilde{M}^F(t)\tilde{H}(t)}{2}$. Thus, we conclude using Example 2 that

$M_t^G = H_t * M_t^F$ where $H_t = 1 - e^{-2\lambda t}$. Hence

$$M_t^G = \frac{1}{2}\int_0^\infty (1 - e^{-2\lambda(t-s)})\lambda ds = \frac{1}{2}\left[\lambda t = \frac{1}{2}(1 - e^{-2\lambda t})\right].$$

E10) The required quantity is given by the elementary renewal theorem to be

$$\lim_{t\to\infty} \frac{M_t^G}{t} = \frac{\lambda}{2}.$$

This is expected since the long term average number of failures will be half the number when there is no redundant component built into the system.

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UNIT 9 RENEWAL PROCESSES-III

Structure **Page No**

9.1 INTRODUCTION

In this unit, we shall study a variation of the renewal process that we have been looking at till now. In particular we shall look at the delayed renewal process in Sec. 9.2. This can be thought of as the renewal process which one starts observing not at a renewal time but sometime between two renewals. In Sec. 9.3, we shall derive the renewal function of the delayed renewal process. As a special case we shall also study the equilibrium renewal process in Sec. 9.4. By the end of the unit this is what you should be able to do.

Objectives

After studying this unit, you should be able to:

- identify a delayed renewal process and its distributions and to know some other properties;
- find the renewal function M_t^D of the delayed renewal process in terms of the interoccurrence distribution F and G;
- find the relationship between M_t^D and the renewal function of the associated standard renewal process;
- find asymptotic behaviour of the renewal function M_t^D;
- define the equilibrium renewal process and understand its exact renewal function;
- show that the equilibrium renewal process has stationary increments.

9.2 DELAYED RENEWAL PROCESS

In this section, we shall study the delayed renewal process. To understand the difference with what we have studied in the last two units it will be helpful to think of a delayed renewal process as a usual renewal process N_t which is started at an intermediate time or at a time between two renewals. In such a case, the interoccurrence times for the new renewal process will not be i.i.d. In particular, the time upto the first occurrence of the event will not have the same distribution as the other interoccurrence times. We will make this notion more precise in the Example 1 given after formal definitions of delayed renewal sequence and delayed renewal process. First we start by defining the delayed renewal sequence.

Definition 1 (Delayed Renewal Sequence): Suppose $X_1, X_2, \cdots$ is a sequence of nonnegative random variables such that $X_2, X_3, \cdots$ are i.i.d. with common cdf F and where X_1 is a non-negative random variable with cdf G. X_i denotes the time between the $(i-1)$-th and the i-th occurrence of the renewal event. We assume that $F(0)<1$ and $G(0)<1$. Define

$$S_0^D = 0, \; S_n^D = \sum_{i=1}^{n} X_i \qquad (1)$$

The sequence $\{S_n^D\}$ is called the **delayed renewal sequence.** As before S_n^D denotes the total time elapsed before the n-th occurrence of the event.

Definition 2 (Delayed Renewal Process): Let $\{N_t^D : t \geq 0\}$ denotes the counting process for the delayed renewal sequence S_n^D, i.e.

$$N_0^D = 0, \; N_t^D = \inf\{k : S_{k+1}^D > t\} \qquad (2)$$

This process $\{N_t^D : t \geq 0\}$ is called the delayed renewal process.

Example 1: Let $\{N_t : t \geq 0\}$ be a standard or usual renewal process with interoccurrence time distribution given by the cdf F. Suppose we start observing the process only from time $s > 0$. Then the, new observed process can be written as $N_t^s = N_{t+s} - N_s$ for all $t > 0$. In other words N_t^s is the number of renewals of the original process in the time interval $[s,t]$. Since whatever has happened before time s remains unobserved, we think as if the time s is the starting time point 0 for the new process $\{N_t^s : t \geq 0\}$. Note that N_t^s also denotes the number of renewals of the new process in the time interval $[0,t]$.

Then N_t^s is a delayed renewal process. To see this we argue as follows. Clearly if $X_1, X_2, X_3, \ldots$ denote the interoccurrence times for N_t^s, then $X_2, X_3, \ldots$ are i.i.d. with common cdf F. However, X_1 is exactly the random variable γ_s defined in Eqn.(13) of Sec.8.5 and which denotes the excess life at time s of the original process $\{N_t : t \geq 0\}$. Hence X_1 has the same distribution as that of γ_s which in general will be different from F.

Example 2: Let $\{N_t^D : t \geq 0\}$ be a delayed renewal process as defined in definitions 1 and 2. Let $X_1, X_2, \ldots$ be the corresponding interoccurrence times. Let $N_t = N_{t+X_1}^D - 1$ for all $t \geq 0$. Then N_t is just the number of renewals in the time interval $[X_1, t + X_1]$. Clearly the corresponding interoccurrence times are $X_2, X_3, \cdots$ respectively. But now that sequence of interoccurrence times is i.i.d by definition. Thus, the process N_t is a (standard) renewal process. Thus, the delayed renewal process behaves like a standard renewal process after a delay of time X_1. This is also the reason for the name **delayed renewal process**.

Let us try the following exercise.

E1) Let $\{N_t : t \geq 0\}$ be a Poisson process. Then show that for every $s > 0$ the delayed renewal process $\{N_t^s : t \geq 0\}$ defined in Example 1 above is also a Poisson process and hence is a (standard) renewal process.

Example 2 suggests that the delayed renewal process will inherit a lot of the properties of the standard renewal process. We have the first theorem. We will once again use the convolution notation introduced earlier. We also recall that for every $n, F_n = F^{*n}$ (see Eqn. (11) of Unit 7).

Theorem 1: Let $\{N_t^D : t \geq 0\}$ be a delayed renewal process with the distribution of the first occurrence time given by cdf G and the remaining interoccurrence times being i.i.d. with cdf F. Then

$$P(N_t^D = 0) = 1 - G(t) \tag{3}$$

$$P(N_t^D = k) = (G * F_{k-1})_t - (G * F_k)_t \qquad k \geq 1. \tag{4}$$

Proof: Note that

$$P(N_t^D = 0) = P(X_1 > t) = 1 - G(t)$$

and the first assertion is proved. Now

$$P(N_t^D = k) = P(N_t^D \geq k) - P(N_t^D \geq k+1) = P(S_k^D \leq t) - P(S_{k+1}^D \leq t) \tag{5}$$

Further

$$\begin{aligned} P(S_2^D \leq t) &= P(X_1 + X_2 \leq t) \\ &= \int_0^\infty P(X_1 \leq t - s) dF(s) \\ &= \int_0^t G(t-s) dF(s) \\ &= (G * F)_t . \end{aligned}$$

Using an induction argument, we get that

$$P(S_k^D \leq t) = (G * F_{k-1})_t . \tag{6}$$

Assertion (4) now follows from (5).

In the next section, we shall study more properties of the delayed renewal process.

9.3 RENEWAL FUNCTION OF THE DELAYED RENEWAL PROCESS

As in the previous section, we will denote a delayed renewal process by $\{N_t^D : t \geq 0\}$. We want to study more properties of this process. Let $X_1, X_2, \ldots$ be the interoccurrence times of this process. By definition the X_i's form an independent sequence. Throughout the remainder of the unit G will denote the cdf of X_1 while the common cdf of $X_2, X_3, \ldots$ will be denoted by F. Further we will denote by $\{N_t : t \geq 0\}$ the (standard) renewal process with common interoccurrence distribution F.

Let

$$M_t^D = E(N_t^D) \tag{7}$$

be the renewal function of the delayed renewal process. As before M_t will denote the renewal function of the renewal process N_t.

Theorem 2: The renewal functions M_t^D and M_t are related by

$$M_t^D = G(t) + (G * M)_t . \tag{8}$$

Proof: Using the usual renewal argument, we get

$$E(N_t^D / X_1 = s) = \begin{cases} 0 & \text{if } s > t \\ 1 + M_{t-s} & \text{if } s \le t \end{cases}.$$

Integrating with respect to the distribution of X_1, we get

$$\begin{aligned} M_t^D &= \int_0^\infty E(N_t^D | X_1 = s) dG(s) \\ &= \int_0^t [1 + M_{t-s}]\, dG(s) \\ &= G(t) + \int_0^t M_{t-s} dG(s) \\ &= G(t) + (G * M)_t. \end{aligned}$$

This proves the result.

As an immediate consequence we have the result as in the next exercise. You can try to prove it yourself.

E2) Show that $M_t^D < \infty$ for all $t \ge 0$.

Note that Eqn. (8) of the present unit is different from the renewal type Eqn. (8) of Unit 8, since now the function M_t^D appears on the left hand side while M_t appears on the right hand side. We nevertheless take Laplace transform (as defined in Unit 8) on both sides to get

$$\tilde{M}_t^D = \tilde{G}(t) + \tilde{G}(t)\tilde{M}_t \tag{9}$$

Substituting for $\tilde{M}_t$ using Eqn. (1) of Unit 8, we get

$$\begin{aligned} \tilde{M}_t^D &= \tilde{G}(t) + \tilde{G}(t)\frac{\tilde{F}(t)}{1 - \tilde{F}(t)} \\ &= \tilde{G}(t)\left[1 + \frac{\tilde{F}(t)}{1 - \tilde{F}(t)}\right] \\ &= \frac{\tilde{G}(t)}{1 - \tilde{F}(t)}. \end{aligned} \tag{10}$$

As was the case with M_t, exact evaluation of M_t^D is not always possible. We have the following theorem which gives the asymptotic behaviour of M_t^D. We will state the theorem without proof. You can try the proof which is exactly along the lines of the proof of Theorem 3 of Unit 8. We will, however, prove the theorem in a special case a little later.

Theorem 3: Let $\{N_t^D : t \ge 0\}$ be a delayed renewal process such that the mean $\mu = E(X_2) < \infty$. Then

$$\lim_{t \to \infty} \frac{M_t^D}{t} = \frac{1}{\mu}. \tag{11}$$

We will prove the following limit theorem as an application of Theorem 2 above. Recall the Definition 2 of Unit 8 of an arithmetic (and non-arithmetic) function.

Theorem 4: Let $h > 0$ be fixed. If F is not an arithmetic function then

$$\lim_{t \to \infty} [M_t^D - M_{t-h}^D] = \frac{h}{\mu}.$$

Proof: Since Eqn. (8) holds for all t, and since $M * G = G * M$, we get

$$M_t^D = G(t) + \int_0^t M_{t-s} dG(s)$$

and for $t > h$

$$M_{t-h}^D = G(t-h) + \int_0^{t-h} M_{t-h-s} dG(s).$$

We can put $M_s \equiv 0$ for $s < 0$, then we get

$$\begin{aligned} M_t^D - M_{t-h}^D &= G(t) - G(t-h) + \int_0^t (M_{t-s} - M_{t-h-s}) dG(s) \\ &= G(t) - G(t-h) + \int_0^{t/2} (M_{t-s} - M_{t-h-s}) dG(s) \\ &\quad + \int_{t/2}^t (M_{t-s} - M_{t-h-s}) dG(s). \end{aligned}$$

Note that the integrand $(M_{t-s} - M_{t-h-s})$ is bounded for all $s \in [0,t]$ by

$$M_{t-s} - M_{t-h-s} \le M_h - 0 = M_h.$$

Further, Blackwell's renewal theorem (Corollary 1 of Sec. 8.4) implies that

$$\lim_{t \to \infty} [M_t - M_{t-h}] = \frac{h}{\mu}.$$

Since $t-h-s \ge 0$ for all $s \in [0, t-h]$ we can use this in the first integral above to get that

$$\begin{aligned} \lim_{t\to\infty} \int_0^{t/2} (M_{t-s} - M_{t-h-s})\, dG(s) &= \lim_{t\to\infty} \int_0^{\infty} I_{\{s \le t/2\}} (M_{t-s} - M_{t-h-s})\, dG(s) \\ &= \frac{h}{\mu} \int_0^{\infty} dG(s) = \frac{h}{\mu}. \end{aligned}$$

The interchange of limit and integral in the expression above is justified since the integrand remains bounded independent of t. Same is the case for the second integral and we get

$$\lim_{t\to\infty} \int_{t/2}^t (M_{t-s} - M_{t-h-s}) dG(s) = \frac{h}{\mu} \lim_{t\to\infty} (G(t) - G(t/2)) = 0.$$

Similarly $\lim_{t\to\infty}(G(t) - G(t-h)) = 0$. Combining all these we get the result.

In the case when F in not arithmetic, Theorem 4 can be used to prove Eqn. (11) given in Theorem 3 as follows. We need to use the following real analytic lemma which we state without proof.

Lemma 1: Let a_n be a sequence of real numbers such that $a_n \to a$ as $n \to \infty$. Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} a_k = \lim_{n\to\infty} a_n = a.$$

Now assume that F is non-arithmetic. For $n \ge 0$, setting $a_n = (M_{n+1}^D - M_n^D)$ and using Lemma 1, we get

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \left[M_{k+1}^D - M_k^D \right] = \lim_{n\to\infty} (M_{n+1}^D - M_n^D) = \frac{1}{\mu}.$$

The last equality follows from Theorem 4 above with $h = 1$. Moreover, the sum appearing on the left hand side above is a telescopic sum and equates to $M_n^D - M_0^D$. Thus, we get

$$\lim_{n\to\infty} \frac{1}{n} M_n^D = \lim_{n\to\infty} \frac{1}{n} M_0^D + \frac{1}{\mu} = \frac{1}{\mu} \tag{12}$$

Thus, Eqn. (11) is proved for all integer time points. Now let $t > 0$ be arbitrary. Let $[t]$ denotes the integer part of t. Since the numbers renewals are increasing in t, we get

$$\frac{M^D_{[t]}}{t} \le \frac{M^D_t}{t} \le \frac{M^D_{[t]+1}}{t}$$

Therefore

$$\frac{[t]}{t}\frac{M^D_{[t]}}{[t]} \le \frac{M^D_t}{t} \le \frac{M^D_{[t]+1}}{[t]+1}\frac{[t]+1}{t}.$$

Taking limit at $t \to \infty$ and using Eqn.(12), we get

$$\frac{1}{\mu} \le \lim_{t \to \infty} \frac{M^D_t}{t} \le \frac{1}{\mu}.$$

This proves Eqn. (11).

In the next section, we will study a special case of the delayed renewal process called the *equilibrium renewal process.*

9.4 EQUILIBRIUM RENEWAL PROCESS

In this section we will discuss a special case of the delayed renewal process namely the *equilibrium renewal process.* Recall that a delayed renewal process is one in which the interoccurrence times $X_2, X_3, \ldots$ are i.i.d. with cdf F, while time until the first renewal X_1, independent of the other X_i's possibly has a different cdf, say G. We also recall that μ denotes the mean of $X_j, j \ge 2$. Thus,

$$\mu = E(X_2) = \int_0^\infty (1 - F(s)), ds \tag{13}$$

We assume that $\mu < \infty$.

Suppose that while observing a renewal process, the last renewal happened so far back in the past that nobody remembers it. Then one can think of the renewal process as a delayed renewal process where the distribution function G of X_1 is same as the asymptotic distribution function of the excess life γ_t of the original renewal process. Recall that we had defined γ_t in Eqn. (13) of Unit 8 and its asymptotic distribution was given in Lemma 2 of that unit. In other words, we shall take

$$G(x) = \lim_{t \to \infty} P(\gamma_t \le x) = \frac{1}{\mu}\int_0^x (1 - F(s)), ds \tag{14}$$

Note that Eqn. (13) implies that G defined in Eqn. (14) is actually a valid cumulative distribution function.

We can now give the following definition.

Definition 3 (Equilibrium Renewal Process): Let $\{N^e_t : t \ge 0\}$ be a delayed renewal process with cdf G given by Eqn. (14). Then $\{N^e_t : t \ge 0\}$ is called an equilibrium renewal process.

As before we define the renewal function as the mean number of renewals, i.e.

$$M^e_t = E(N^e_t) \quad t \ge 0 \tag{15}$$

Then, we have the following theorem.

Theorem 5: The renewal function M_t^e Satisfies

$$M_t^e = \frac{t}{\mu} \text{for all } t \geq 0. \tag{16}$$

Proof: Taking Laplace transform on both sides of Eqn.(14), we get

$$\begin{aligned}
\tilde{G}(t) &= \int_0^\infty e^{-ts} dG(s) \\
&= \frac{1}{\mu}\int_0^\infty e^{-ts}(1 - F(s), ds \\
&= \frac{1}{\mu}\int_0^\infty e^{-ts}\int_s^\infty dF(r) ds \\
&= \frac{1}{\mu}\int_0^\infty \int_0^r e^{-ts} ds dF(r) \\
&= \frac{1}{\mu}\int_0^\infty \frac{1 - e^{-tr}}{t} dF(r) \\
&= \left(\frac{1}{\mu}\right)\left(\frac{1 - \tilde{F}(t)}{t}\right).
\end{aligned}$$

Since an equilibrium renewal process is also a delayed renewal process Eqn. (10) implies that

$$\tilde{M}_t^e = \frac{\tilde{G}(t)}{1 - \tilde{F}(t)} = \frac{1}{\mu t}.$$

Note that this coincides with the Laplace transform of the function $g(t) = t/\mu$. Since Laplace transforms characterise non-negative functions Eqn. (16) holds and the proof is complete.

Note that for the equilibrium renewal process Eqn. (16) holds for every time point t which is not the case in general for the standard or delayed renewal process for which the equality holds only asymptotically. Now we will study the distribution of the excess lifetime random variable γ_t^e for the equilibrium process $\{N_t^e : t \geq 0\}$. (Recall Eqn. (13) of Unit 8).

Theorem 6: For the equilibrium renewal process, the residual lifetime random variable γ_t^e satisfies for all $t, x \geq 0$

$$P(\gamma_t^e \leq x) = G(x) \tag{17}$$

where G is given by Eqn. (14).

Proof: Let $\{N_t : t \geq 0\}$ be the standard renewal process. Let γ_t be the residual lifetime random variable for this process as defined in Eqn. (13) of Unit 8. For every $x \geq 0$ define functions $A_t^{e,x}$ and A_t^x by

$$A_t^{e,x} = P(\gamma_t^e > x), \; A_t^x = P(\gamma_t > x).$$

Because of Eqn. (16) of Unit 8, we know that

$$A_t^x = 1 - F(t + x) + \int_0^t (1 - F(t + x - s)\, dM_s.$$

Setting $a_t^x = (1 - F(t + x))$ the above renewal type equation can be rewritten in convolution notation as

$$A_t^x = a_t^x + (M * a^x)_t \tag{18}$$

We will once again use the familiar conditioning argument. Note that

$$P(\gamma_t^e > x | X_1 = s) = \begin{cases} 1 & \text{if} \quad s > t + x \\ 0 & \text{if} \quad t + x \geq s > t \\ A_{t-s}^x & \text{if} \quad t \geq s. \end{cases}$$

Unconditioning with respect to X_1 (whose cdf is G), using Eqn. (18) and Eqn. (8), we get

$$\begin{aligned} A_t^{e,x} &= 1 - G(t+x) + \int_0^t A_{t-s}^x dG(s) \\ &= 1 - G(t+x) + (G * A^x)_t \\ &= 1 - G(t+x) + (G * a^x)_t + (G * M * a^x)_t \\ &= 1 - G(t+x) + (M^e * a^x)_t \\ &= 1 - G(t+x) + \int_0^t (1 - F(t+x-s))dM_s^e. \end{aligned}$$

But by Theorem 5 $M_t^e = t/\mu$ and hence

$$\begin{aligned} A_t^{e,x} &= 1 - G(t+x) + \frac{1}{\mu}\int_0^t (1 - F(t+x-s))ds \\ &= 1 - G(t+x) + \frac{1}{\mu}\int_x^{t+x} (1 - F(u))du \\ &= 1 - G(t+x) + G(t+x) - G(x) \\ &= 1 - G(x). \end{aligned}$$

This completes the proof.

Thus, at any time point t, the distribution of the excess time random variable in an equilibrium renewal process coincides with the *limiting distribution* of the excess time random variable in standard renewal process. The last two theorems also explain why this particular delayed renewal process (with G as in Eqn. (14)) is called an equilibrium renewal process. We end this section with an important result whose proof is left as an exercise.

E3) Show that the equilibrium renewal process $\{N_t^e : t \geq 0\}$ has stationary increments.

Let us now summarize this unit.

9.5 SUMMARY

In this unit, we studied the following:

1. We defined a delayed renewal process and also saw its connections with the standard renewal process.

2. We identified the distribution of the delayed renewal process in terms of the cdf's F and G.

3. We defined the renewal function M_t^D for the delayed renewal process. We also derived an equation satisfied by M_t^D and M_t, the renewal function for the associated standard renewal process.

4. We found the Laplace transform of M_t^D. We also studied some asymptotic properties of M_t^D.

5. We defined the equilibrium renewal process and found the exact expression for its renewal function. We also found the exact distribution of the excess lifetime random variable for the equilibrium renewal process. Finally, we saw that the equilibrium renewal process has stationary increments.

9.6 SOLUTIONS/ANSWERS

E1) We have seen in Example 1 that the distribution of X_1 is the same as the distribution of the excess time γ_s. Moreover, we saw in Example 6 that when $\{N_t : t \geq 0\}$ is a Poisson process with parameter λ, then γ_s is an exponential random variable with parameter λ. But in this case, this is exactly same as the common distribution of the interoccurrence times. Thus, in this case the interoccurrence times of the delayed renewal process are i.i.d exponential with parameter λ which means that the delayed renewal process is actually a standard renewal process. Moreover, we have already identified this process in Example 1 of Unit 7 to be a Poisson process with parameter λ.

E2) We already know that $M_t < \infty$ for all $t \geq 0$. Then using Eqn. (8) and the fact that $G(t) \leq 1$ for all t, we get

$$M_t^D = G(t) + (G * M)_t \leq 1 + M_t < \infty.$$

E3) Since N_t^e is also a delayed renewal process, Theorem 1 implies that the distribution is given by Eqn. (3) and Eqn. (4). In particular,

$$P(N_t^e = 0) = 1 - G(t), \quad P(N_t^e = k) = G * F_{k-1}(t) - G * F_k(t) \qquad k \geq 1$$

where G is now given by Eqn. (14). Now fix an $s > 0$ and consider the process

$$N_t^{e,s} = N_{s+t}^e - N_s^e$$

which counts the number of renewals after time s. Then as seen earlier in Example 1, $N_t^{e,s}$ is a delayed renewal process. Let Y_1 be the time of first occurrence of the event for this new process. Then Y_1 is exactly γ_s^e, the excess time random variable at time s for the equilibrium renewal process.
Now Theorem 6 implies that the cdf of γ_s is G given by Eqn. (14). Hence the distribution of Y_1 is G. This implies (by definition) that the delayed renewal process $N_t^{e,s}$ is also an equilibrium renewal process. In particular Eqn. (3) and Eqn. (4) hold for this process as well. Elaborating on this point, we get

$$P(N_{s+t}^e - N_s^e = 0) = 1 - G(t) = P(N_t^e = 0),$$

and

$$P(N_{s+t}^e - N_s^e = k) = G * F_{k-1}(t) - G * F_k(t) = P(N_t^e = 0) \qquad k \geq 1.$$

This shows that the equilibrium process has stationary increments.

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UNIT 10 RENEWAL PROCESSES-IV

Structure

10.1 INTRODUCTION

In this unit, we will apply the theory of renewal processes that we have studied so far to the replacement problem which can be described as follows. A component in a system, e.g. a bulb, a transistor, a machine, etc. fails at some random time. The component is replaced by another of the same type. We will assume that the replacement is done immediately or at the same instant that the component fails. This process is repeated as long as the system is in operation.

The lifetimes of the components can be considered to be an i.i.d. sequence of nonnegative random variables $X_1, X_2, \ldots$ with cdf

$$F(x) = P(X \leq x).$$

We will assume throughout this unit that the mean lifetime of a component, μ, $\mu = E(X_1)$, is finite.

We will look at the following three replacement policies and will use renewal theory to find the average number of renewals per unit time in the long run. There is a cost involved with each of these policies. We can use the results obtained to compare the policies and find an optimal one.

1. **Individual replacement policy:** The component is replaced immediately on failure.
2. **Age replacement policy:** The component is replaced immediately on failure or when it attains a fixed age $T > 0$, whichever occurs earlier.
3. **Block replacement policy:** The component is replaced immediately on failure and also at fixed time points $T, 2T, 3T, \ldots$.

We shall discuss the three replacement policies given above in Sec. 10.2, 10.3 and 10.4, respectively.

By the end of this unit, this is what you should be able to do.

Objectives

After studying this unit, you should be able to:

- understand individual replacement policy, age replacement policy and block replacement policy and identify the renewal processes associated with these policies;
- find the mean number of replacements per unit time in the long run for each of the policies;
- find the average long run cost per unit time for each of these policies;
- compare the policies.

10.2 INDIVIDUAL REPLACEMENT POLICY

Consider a component which is a part of some system. The component may fails after some random time and is immediately replaced by another identical component. Such a replacement policy is one which is usually employed and is called the individual replacement policy.

Let $X_1, X_2, \ldots$ denote the lifetimes of successive components which are put in service. By this we mean that the i^{th} component fails after being in service for time X_i. We assume that the lifetimes are independent and identically distributed nonnegative random variables with the common distribution function F. We will denote the common distribution function by F. To avoid trivialities we assume that $F(0) < 1$. Further the mean lifetime μ is assumed to be finite, i.e.

$$\mu = \int_0^\infty s dF(s) = E(X_k) < \infty .$$

We are interested in the number of components which fail during a time interval $[0, t]$. For the individual replacement policy this also coincides with the number of components replaced in the time interval $[0, t]$. Let N_t^I denotes this process. In other words

$$\begin{aligned} N_t^I &= \text{number of failures in the time interval } [0, t] \\ &= \text{number of replacements in the time interval } [0, t] \text{ under the individual} \\ &\quad \text{replacement policy} \qquad (1) \end{aligned}$$

Then clearly $\{N_t^I : t \geq 0\}$ is a renewal process with $X_1, X_2, \ldots$ being the corresponding interoccurrence times. In this case the renewal event is the event that a component fails and is replaced immediately. Let M_t^I be the renewal function. Then

$$M_t^I = E(N_t^I). \qquad (2)$$

As seen in Unit 7, finding the explicit value of this function at any time t is not always possible. In any case it will depend on the cdf F. However, the elementary renewal theorem (Theorem 3 of Unit 8) implies that

$$\lim_{t \to \infty} \frac{M_t^I}{t} = \frac{1}{\mu}. \qquad (3)$$

Thus, one can expect to replace the components at a mean rate of $1/\mu$ components per unit time in the long run. Typically, there is cost involved with every replacement. Let us assume that in the individual replacement policy this cost is c units per replacement. Thus, the expected cost of replacement in the individual replacement policy in the time interval $[0, t]$ will be cM_t^I. Then Eqn.(2) implies that the average long run cost will be c/μ per unit time.

In the long run, any replacement policy which replaces components before they actually fail will end up using more than $1/\mu$ components per unit time. However, sometimes such a policy has its advantages. This is particularly so, if the failure of a component not only stops the system but also adversely affects other components of the system which in turn leads to added costs. In the next section, we will look at one such policy, the age replacement policy.

10.3 AGE REPLACEMENT POLICY

Consider a situation in which failure of a component leads to some unwanted side effects whereby the other components of the systems also get affected. It is natural to

assume that the chance of failure of a component increases with its age. In such a case it might be advantageous to replace the component after a certain time or age.
Fix a $T>0$. Then in an **age replacement policy** a component is replaced upon its failure or upon its reaching age T, whichever happens earlier. We once again use the same set up as before. In other words the lifetimes of the components, denoted by $X_1, X_2, \ldots$ are assumed to be i.i.d. each with distribution function F. Let $\{N_t^{A,r} : t \geq 0\}$ be the counting process defined by

$$N_t^{A,r} = \text{number of replacements in the time interval } [0, t] \text{ under the age replacement policy.} \tag{4}$$

We reemphasize that replacement of a component does not mean the failure of the component in service. Thus, the above process is different from $\{N_t^{A,f} : t \geq 0\}$ defined below by

$$N_t^{A,f} = \text{number of failures in the time interval } [0, t] \text{ under the age replacement policy.} \tag{5}$$

Note that whatever happens between the $(i-1)^{th}$ and the i^{th} replacement depends only on the life length of the i^{th} component and hence only on X_i. Let Y_i denotes the time for which the i^{th} component is in service. Then the sequence $Y_1, Y_2, \ldots$ is an i.i.d. sequence. These are exactly the interoccurrence times for the process $\{N_t^{A,r} : t \geq 0\}$. Thus, we get that $\{N_t^{A,r} : t \geq 0\}$ is a renewal process.

Let us denote the common distribution function of Y_t's by $F^T(x)$. Note that the component will certainly be replaced at age T if it has not failed till that age. However, it will be replaced on failure before age T. Thus,

$$F^T(x) = P(Y_i \leq x) = \begin{cases} F(x) & \text{for } x < T \\ 1 & \text{for } x \geq T \end{cases}. \tag{6}$$

Further the mean renewal time, denoted by μ^T is given by

$$\mu^T = \int_0^\infty P(Y_i > x)\, dx = \int_0^\infty (1 - F^T(x))\, dx = \int_0^T (1 - F(x))\, dx. \tag{7}$$

This clearly implies that $\mu^T \leq \mu$. Let the renewal function be denoted by

$$M_t^{A,r} = E(N_t^{A,r}). \tag{8}$$

Then the elementary renewal theorem (See Theorem 3 of Unit 8) implies that

$$\lim_{t \to \infty} \frac{M_t^{A,r}}{t} = \frac{1}{\mu^T}. \tag{9}$$

Thus, one can expect to replace the components at a mean rate of $1/\mu^T$ components per unit time in the long run. In general, this will be greater than $\frac{1}{\mu}$, the long run mean rate of replacements in the individual replacement policy.

We can now do the same analysis for the process $\{N_t^{A,f} : t \geq 0\}$ defined in Eqn.(5). Let $Z_1, Z_2, \ldots$ denote the sequence of random variables which denotes the time between two actual successive failures. Clearly this sequence will also be an i.i.d. sequence which implies that $\{N_t^{A,f} : t \geq 0\}$ is a renewal process. Note that the random time Z_i consists of a sum of random number of time periods, say R_i, each of length

T, corresponding to the components which do not fail by the age T plus the last time period of length W_i corresponding to the age of the component that fails before time T. We can write

$$Z_i = R_i T + W_i. \tag{10}$$

The distribution of W_i is that of time to failure of a component conditioned on the event that its failure occurs before age T. Thus, for any i

$$P(W_i \le w) = P(X_i \le w \mid X_i \le T) = \begin{cases} \dfrac{F(w)}{F(T)} & 0 \le w \le T \\ 1 & w > T \end{cases}. \tag{11}$$

Also R_i takes the value k if and only if, in the i^{th} renewal interval, the first k components do not fail till age T and the $(K+1)^{th}$ component is the first component to fails before time T. For each component, the probability of survival till time T is $(1-F(T))$. Thus,

$$P(R_i = k) = (1-F(t))^k F(T),\ k = 0, 1, 2, \ldots \tag{12}$$

We want to apply the elementary renewal theorem to find the renewal function

$$M_t^{A,f} = E(N_t^{A,f}). \tag{13}$$

For this we need to find $E(Z_1)$. Using Eqn.(10)-(12) and the definition of μ^T in Eqn.(7), we get

$$\begin{aligned} E(Z_1) &= TE(R_1) + E(W_1) \\ &= T\sum_{k=0}^{\infty} k P(R_1 = k) + \int_0^{\infty} P(W_1 > w) \\ &= T\sum_{k=0}^{\infty} k F(T)(1-F(T))^k + \int_0^T \left(1 - \frac{F(w)}{F(T)}\right) dw \\ &= \frac{T(1-F(T))}{F(T)} + \frac{1}{F(T)}\int_0^T (F(T) - F(w))\,dw \\ &= \frac{1}{F(T)}\int_0^T (1-F(w))\,dw \\ &= \frac{\mu^T}{F(T)}. \end{aligned} \tag{14}$$

The elementary renewal theorem now says that

$$\lim_{t\to\infty} \frac{M_t^{A,f}}{t} = \frac{1}{E(Z_1)} = \frac{F(T)}{\mu^T}. \tag{15}$$

Thus, depending of the cdf F, the modified failure rate in the age replacement policy might give a possibly lower failure rate then $1/\mu$.

Now suppose that each replacement costs c_1 units of money. This is so whether or not the replacement is planned or due to failure of the component. However, whenever a failure occurs, then some extra cost c_2 is incurred. Define $C(T)$ to be the long run average cost per unit time for an age replacement policy as a function of T. Since, by Eqn.(9) and Eqn.(15) there are $1/\mu^T$ replacements on an average per unit time and $F(T)/\mu^T$ failures on an average per unit time in the long run, we get

$$C(t) = c_1 \frac{1}{\mu^T} + c_2 \frac{F(T)}{\mu^T} = \frac{c_1 + c_2 F(T)}{\int_0^T (1 - F(w))dw}. \quad (16)$$

For a given F and known values c_1 and c_2, one can find the optimal value of T which will minimize Eqn.(16).

Further the long range average costs per unit time for the individual and age replacement policies can be compared to choose one which is optimal.

Example 1: Suppose the lifetimes $X_1, X_2, \ldots$ are i.i.d. exponential random variables with parameter $\lambda > 0$. Let $T > 0$ and age replacement policy is to be employed.

a) Find μ^T.

b) Let $c_1 = 2$ and $c_2 = 3$. Find the long run average cost per unit time.

Solution: a) We have $F(x) = 1 - e^{-\lambda x}$ for $x \geq 0$. Using Eqn.(7), we get

$$\mu^T = \int_0^T (1 - F(x))dx = \int_0^T e^{-\lambda x} dx = \frac{1 - e^{-\lambda T}}{\lambda}.$$

b) Using Eqn.(16), we get

$$C(T) = \frac{c_1 + c_2 F(T)}{\mu^T} = \frac{\lambda(2 + 3(1 - e^{-\lambda T}))}{1 - e^{-\lambda T}} = 3\lambda + \frac{2\lambda}{1 - e^{-\lambda T}}.$$

Clearly the function $C(T)$ is minimized for $T = \infty$ when the second term above becomes zero. This suggests that when the lifetime distributions is exponential, the optimal policy is to replace only upon failure of a component.

You can now do the following exercise.

E1) Repeat Example 1 when the lifetimes are uniformly distributed on [0, 1] and $c_1 = 2$, $c_2 = 8$. Further, find a T that minimizes $C(T)$. For this case, which is a better policy in the long run in terms of cost – the individual replacement policy or the age replacement policy?

So far, we discussed the individual replacement policy and the age replacement policy. In this section, we shall discuss the block replacement policy.

10.4 BLOCK REPLACEMENT POLICY

Another replacement policy which implements planned replacement is the block replacement policy. In this for a fixed length of time $T > 0$ planned replacements take place at each of the times T, 2T, 3T, ... In addition, of course a component gets replaced upon failure.

Once again, let $X_1, X_2, \ldots$ denote the lifetimes of the successive components with common distribution function F and with finite mean μ. As in the previous section we consider two counting processes which are defined as follows.

$N_t^{B,r}$ = number of replacements in the time interval [0, t] under the block replacement policy. (17)

$$N_t^{B,f} = \text{number of failures in the time interval } [0, t] \text{ under the block replacement policy.} \tag{18}$$

As in the case of age replacement policy the two processes $\{N_t^{B,r} : t \geq 0\}$ and $\{N_t^{B,f} : t \geq 0\}$ are different. Let the corresponding renewal functions be denoted by

$$M_t^{B,r} = E(N_t^{B,r}),\ M_t^{B,f} = E(N_t^{B,f}) \tag{19}$$

respectively. We will first analyze the failure renewal process. Note that immediately after every planned replacement (at time $T, 2T, \ldots$), the process starts anew. Moreover, in every interval of the type $((j-1)T, jT],\ j = 1, 2, \ldots$ replacements will take place if and only if there is a failure. But this is exactly like the renewal process in the individual replacement policy.

Let $\{N_t^j : t \geq 0\},\ j = 1, 2, \ldots$ denote this process which counts the number of failures in the interval $((j-1)T, jT]$. Since the process starts afresh at time point jT due to planned replacements taking place at that time, the processes $\{N_t^j : t \geq 0\},\ j = 1, 2, \ldots$ are mutually independent. Note that the interoccurrence times for these processes are i.i.d. with cdf F, the lifetime distribution function. This also implies that

$$E(N_t^j) = M_t,\ \ j = 1, 2, \ldots \tag{20}$$

where M_t is exactly the same renewal function as defined in Eqn.(2).

We now want to find an expression for $N_t^{B,f}$ for $t > 0$. Note that t can be written as

$$t = nT + s \text{ for some } n = 0, 1, 2, \ldots,\ 0 \leq s < T.$$

Then $N_t^{B,f}$, the number of failures in the time interval $[0, t]$, coincides with the sum of the number of failures in each of the intervals $((j-1)T, jT],\ j = 1, 2, \ldots, n$ and the number of failures in the interval $(nT, t] = (nT, nT+s]$. i.e.

$$N_t^{B,f} = \sum_{j=1}^{n} N_T^j + N_s^{n+1}. \tag{21}$$

Hence

$$\begin{aligned} M_t^{B,f} = E(N_t^{B,f}) &= \sum_{j=1}^{n} E(N_T^j) + E(N_s^{n+1}) \\ &= \sum_{j=1}^{n} M_T + M_s \\ &= nM_T + M_s. \end{aligned} \tag{22}$$

Further, this implies

$$\begin{aligned} \lim_{t\to\infty} \frac{M_t^{B,f}}{t} &= \lim_{t\to\infty} \frac{nM_T}{t} + \lim_{t\to\infty} \frac{M_s}{t} \\ &= \lim_{n\to\infty} \frac{nM_T}{nT+s} + \lim_{n\to\infty} \frac{M_s}{nT+s} \\ &= \frac{M_T}{T}. \end{aligned} \tag{23}$$

Thus, the long run mean number of failures per unit time under the block replacement policy equals M_T / T, when planned replacements take place after every T units of time. The total number of replacements under this policy for $t = nT + s$ clearly equals the number of failures upto time t plus the number of planned replacements upto time t. Thus, we have the relation

$$N_t^{B,r} = N_t^{B,f} + [t/T]. \tag{24}$$

It now follows directly from Eqn.(23) and Eqn.(24) that

$$\lim_{t\to\infty} \frac{M_t^{B,r}}{t} = \lim_{t\to\infty} \frac{M_t^{B,f}}{t} + \lim_{t\to\infty} \frac{[t/T]}{t}$$
$$= \frac{M_T + 1}{T}. \tag{25}$$

As we had done earlier the long run average cost per unit time can now be computed. Once again, suppose that the cost of any replacement is c_1 and there is an additional cost c_2 whenever there is a failure. Then the long run average cost per unit time in the block replacement policy is

$$C(T) = (c_1 + c_2)\frac{M_T}{T} + c_1. \tag{26}$$

For a fixed c_1, c_2 one can find a T which will minimize $C(T)$ given in Eqn.(26). Further the optimal cost in the age replacement and block replacement policies can be compared with the cost of individual replacement policy to find a minimal cost policy.

Example 2: Let the lifetimes $X_1, X_2, \ldots$ are i.i.d. exponential random variables with parameters $\lambda > 0$. Find the long run average rate of replacements per unit time under the block replacement policy for a given $T > 0$.

Solution: The required rate is given by Eqn.(25) to be $(M_T + 1)/T$. For the given lifetime distribution, we had identified in Unit 7 the renewal process to be a Poisson process with parameter λ in Example 1. We had then derived the renewal function M_t in Example 10. We know that $M_t = \lambda t$. Thus, the require rate is $\lambda + (1/T)$.

We end the unit with the following exercises.

E2) A machine shop needs a certain kind of machine continuously. Whenever the machine fails it is replaced instantaneously. Assume that successive machine lifetimes are uniformly distributed over the interval [2, 5] years. Find the long-term rate of replacements.

E3) In the above exercise, assume that planned replacements take place every 3 years. So that a machine is replaced on failure or at the end of 3 years. Compute

a) long-term rate of replacements
b) long-term rate of failures
c) long-term rate of planned replacements.

Let us now summarize this unit.

10.5 SUMMARY

In this unit, we studied the following.

1. We studied the individual replacement policy and used the elementary renewal theorem to find the long run mean number of renewals per unit time. We also looked at the long run mean average cost per unit time under this policy.

2. We studied the replacement renewal process and the failure renewal process under the age replacement policy and the corresponding renewal functions in terms of the lifetime distribution F. We also looked at the long run average cost per unit time under this policy.

3. We saw how to use these results to compare the two policies of individual replacement and age replacement.

4. We studied the block replacement policy and found the long run mean number of replacements and failures respectively per unit time under this scheme.

10.6 SOLUTIONS/ANSWERS

E1) a) We have $F(x) = x$ for $0 \le x \le 1$. Then, for $T \in [0, 1]$, Eqn.(7) implies

$$\mu^T = \int_0^T (1 - F(x))dx = \int_0^T (1 - x)dx = T - \frac{T^2}{2}.$$

b) Using Eqn.(16), we get

$$C(T) = \frac{c_1 + c_2 F(T)}{\mu^T} = \frac{2 + 8T}{T - T^2/2}.$$

To find T which minimizes the above we find the derivative of $C(T)$ and equate it to zero to get

$$C'(T) = \frac{8(T - T^2/2) - (2 + 8T)(1 - T)}{(T - T^2/2)^2} = 0.$$

This leads to the quadratic equation

$$4T^2 + 2T - 2 = 0$$

whose two roots are given $T = 1/2$ and $T = -1$. Further one can check that $C''(1/2) > 0$ and hence we get that $T = 1/2$ minimizes $C(T)$. Thus, the minimum long range average cost per unit time in the age replacement policy is

$$C(1/2) = 16.$$

In the individual replacement policy, since a replacement takes place only upon failure, the cost of each replacement is $c_1 + c_2 = 2 + 8 = 10$. The long range mean number of replacements for this policy is given by Eqn.(3) and equals $1/\mu$ where μ is the mean for the distribution function F. In this example this equals $1/2$. Thus, the long range average cost per unit time is $10/(1/2) = 20$. Clearly the age replacement policy with $T = 1/2$ will be preferable.

E2) Since the mean interoccurrence time = mean of uniform $([2, 5]) = 7/2$, by elementary renewal theorem the long-term rate of replacements is $2/7$.

E3) (a) We are considering the age-replacement policy with $T = 3$. The life-time distribution is uniform, its cdf is given by

$$F(x) = \begin{cases} \dfrac{x-2}{3} & \text{for } 2 \le x \le 3 \\ 1 & \text{for } 3 \le x \end{cases}.$$

We use Eqn. (7) of this unit to calculate its mean. Thus,

$$\mu^T = \int_0^3 x dF^T(x) + 3P(Y_i = 3) = \int_0^3 \frac{5-x}{3} dx + 3\frac{2}{3} = \frac{=17}{6}.$$

Now the long-term rate of replacements is given by the elementary renewal theorem to be $6/17$.

b) The long-term rate of failures is given by Eqn. (15) and equals

$$F(3)/\mu^T = (1/3)(6/17) = 2/17.$$

c) The long-term rate of planned replacements can be calculated from the above and equals $(6-2)/17 = 4/17$

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