Indira Gandhi National Open University
School of Sciences

MMTE-001
GRAPH THEORY

Block

2

GRAPH TRACING AND COLOURING

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BLOCK INTRODUCTION

In the previous block, we familiarised you with some fundamental concepts in graph theory, which you will need throughout this block and the next block.

In **Unit 5** of this block, we will focus our attention on Eulerian and Hamiltonian graphs. We will see some sufficient and some necessary conditions for a graph to be Hamiltonian. We shall also see the Chinese Postman Problem and the Travelling Salesman Problem, as two applications.

Unit 6 deals with colouring of graphs, which involves colouring of vertices and colouring of edges. We shall discuss how vertex-coloring leads to the concept of vertex-chromatic number, and of k-chromatic graphs. Similarly, edge-coloring leads to the concepts of k-edge-chromatic graphs. We shall discuss some properties of these graphs. We shall also introduce the concept of line graph, and shall discuss the role of line graphs in finding the edge-chromatic number of a graph.

In **Unit 7,** we discuss planar graphs. The study of the concept of planarity was motivated by the famous Four Colour problem. A detailed account of the problem can be found on Wikipedia. Here we will investigate the characteristics of planar graphs. We shall also discuss the related concepts such as dual graph and colouring of planar graphs and give a proof of the Five Colour Theorem, and state the Four-Colour Theorem.

NOTATIONS AND SYMBOLS (Used in Block 2)

$\chi(G)$	the chromatic number of G
$\chi'(G)$	the edge-chromatic number of G
$\alpha(G)$	the independence number of G
$L(G)$	the line graph of G
$G \cdot e$	the contraction of G by e
$\theta(G)$	the thickness of G
$\nu(G)$	the crossing number of G
G^*	the dual of a plane graph G

UNIT 5

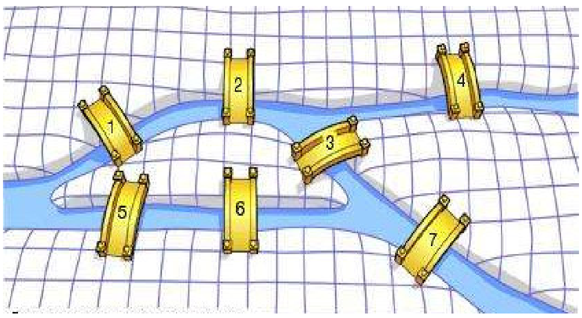
EULERIAN AND HAMILTONIAN GRAPHS

Structure **Page Nos.**

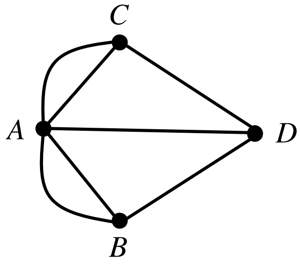
5.1 INTRODUCTION

Suppose you go to a new city as a salesperson. You should naturally like to familiarise yourself with all the important routes. One way to do this is to buy a map of the city and go around the city. If you do this without proper planning you may pass through some of the routes more than once. To avoid this, you would need to sit down and plan your route. The most efficient way would use every route only once. But is it possible to find such a route?

This question is so natural that you may not be surprised to know that a similar question was raised more than 250 years ago. Königsberg was a city that was known as Prussia those days. The Pregal river flowed through this city forming two islands, namely, A and B as shown in Fig. 1(a).



(a)



(b)

Fig. 1: A schematic diagram of Könisberg

The two islands and the rest of the city, denoted by the land masses C and D were connected to each other by seven bridges. Some of the citizens used to amuse themselves with the following question: Is it possible to go around the city using each bridge exactly once?

In 1736, the great Swiss Mathematician Leonard Euler, which we have already met in Unit 1, answered this question in negation by converting this into a problem of graph theory. The multigraph modelling the problem is shown in Fig. 1.(b). One can see by manually traversing the multigraph that there is no way to complete the traversal going through all the vertices and all the edges without repeating any edge. A multigraph that admits such a traversal is called an Eulerian multigraph as you will see in Section 5.2. You will also see a criterion for Eulerian multigraphs, and Fluery's Algorithm in the same section.

In Section 5.3, we shall discuss the edge-traceable graphs and Chinese Postman Problem.

A mathematical puzzle invented by Sir William Hamilton in 1859 involves finding a cycle containing all the vertices of a graph. A graph that contains such a cycle is called a Hamiltonian graph in the honour of Hamilton. In Section 5.4 we will give some necessary and sufficient conditions for a graph to be Hamiltonian. Finally, in Section 5.5 we will discuss the Travelling Salesman Problems.

Objectives

After reading this unit, you should be able to:

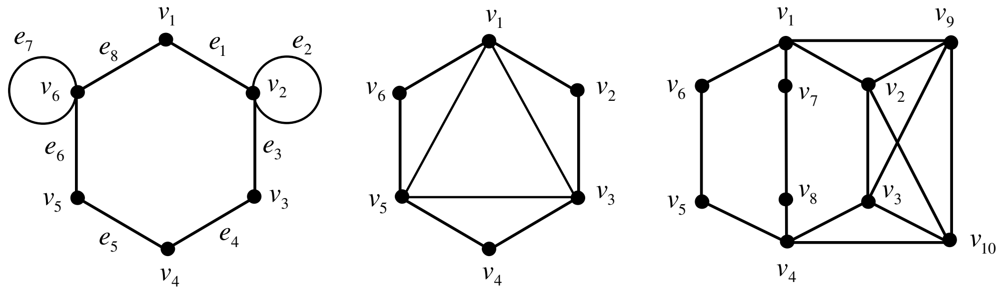
- check whether a given graph or multigraph is Eulerian or not;
- apply Fleury's Algorithm to find an Eulerian circuit in an Eulerian graph;
- check whether a given graph is edge-traceable or not;
- check whether a given graph satisfies certain necessary conditions for being Hamiltonian;
- check whether a given graph satisfies certain sufficient conditions for being Hamiltonian;
- find a minimum-weight Hamiltonian cycle in a weighted complete graph.

5.2 EULERIAN GRAPHS

Recall from Unit 1 that a trail is a walk in which no edge is repeated. A closed trail, also called a circuit, is a trail whose starting vertex and end vertex are the same. Now consider the following definition.

Definition: Let G be a multigraph. A circuit containing all the edges and all the vertices of G is called an **Eulerian circuit** of G, and G is called **Eulerian** if it contains an Eulerian circuit.

It is evident that an Eulerian multigraph is connected because an Eulerian circuit contains all the vertices and edges. The simplest class of Eulerian graphs is cycles, for example C_6. Now consider the following multigraphs G_1, G_2 and G_3.



G_1 G_2 G_3

We can see that $C=(v_1,e_1,v_2,e_2,v_2,e_3,v_3,e_4,v_4,e_5,v_5,e_6,v_6,e_7,v_6,e_8,v_1)$ is an Eulerian circuit in G_1. Hence G_1 is Eulerian.

Likewise G_2 is also Eulerian because we can start at the vertex v_1, traverse the inner triangle, come back to v_1 and traverse the outer cycle to reach back to v_1. In the same way show that G_3 is Eulerian.

You must have realised that producing an Eulerian circuit or showing its non-existence could be a difficult task, especially when the multigraph at hand has large size. We may find some clever way of tracing an Eulerian circuit. So, we need a necessary and sufficient condition for a multigraph to be Eulerian. The next theorem gives such a condition.

Theorem 1: A connected multigraph G is Eulerian if and only if the degree of each of its vertices is even.

Proof: Consider an Eulerian multigraph G and, an Eulerian circuit C in G. An Eulerian circuit C is a traversal of all the edges of a graph once and only once, starting and ending at the same vertex. We can repeat vertices as many times as we want, but we can never repeat an edge once it is traversed. Every time we arrive at a vertex during our traversal of G, we enter via one edge and exit via another. Thus there must be an even number of edges at every vertex. Therefore, every vertex of G has even degree.

To prove the converse, consider a connected multigraph G in which each vertex has even degree. We will now prove that G contains an Eulerian circuit by the Principle of Mathematical Induction on the number of edges in G. Suppose that the number of edges in G is 0. Since the edge set is empty the statement that there is an Eulerian circuit containing all the edges is vacuously true. Assume that all the multigraphs with fewer edges than G that have all vertices of even degree are Eulerian. Note that G has no vertex of degree 0 since it is connected. So, all the vertices have degree at least 2. We can start from an arbitrary point $u=u_1$ and trace a Eulerian circuit C as follows:

We choose any edge u_0u_1 incident at u_0. Since u_1 has degree at least 2, there is another edge incident at u_1, say u_1u_2. We go on tracing a circuit like this, always making sure that we enter and leave any vertex by different edges. During the course of tracing C, we may pass through u_0 several times. The process ends when we reach u_0 and find that there is no unused edge to leave u_0. If the circuit we have obtained contains all the edges, we are done. Otherwise, we remove this circuit from G and call the resulting (possibly disconnected) graph H. All the vertices in each of the components of H have even degree and all the components have fewer edges that G. So all the components are Eulerian.

We now get an Eulerian circuit in G as follows: We start from any vertex u on the circuit C and traverse all the edges of C, with the condition that as soon as we reach a vertex v of a component of H we traverse the Eulerian circuit in that component, eventually returning to u. We would have used each of the edges only once, that is, we have obtained an Eulerian circuit. ■

Thus to find whether a given multigraph is Eulerian or not we only need to know the vertex degrees. For instance, the Petersen graph is not Eulerian because it has vertices of odd degree. Likewise n-wheel is also not Eulerian as it has vertices of degree 3. We shall now discuss an algorithm that constructs an Eulerian circuit in a given Eulerian graph.

Recall that an edge e of a connected multigraph G is a bridge of G if $G-e$ is disconnected.

Fluery's Algorithm:

The algorithm constructs an Eulerian circuit in an Eulerion multigraph by extending a trail starting from any vertex. The procedure of Fluery's Algorithm is given below.

Input: An Eulerian multigraph G

Output: The Eulerian circuit of G.

i) Select a vertex u of G randomly.

ii) At each stage, traverse any available edge, choosing a bridge only if there is no alternative.

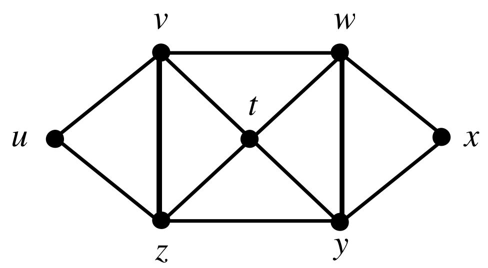
iii) After traversing each edge, erase it (also erase resulted isolated vertex), and then choose another available edge.

iv) Stop when there are no more edges.

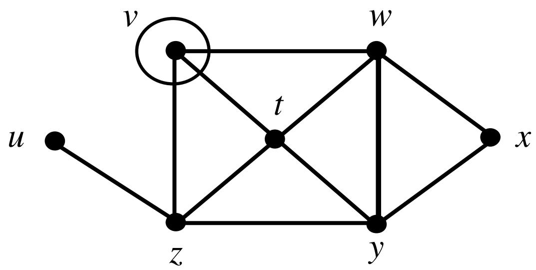
This algorithm is very easy to apply. At each stage, we choose a bridge only as a last resort – this qualification is clearly essential, since once we have traversed a bridge we cannot return to the part of the graph we have just left.

Let us consider an example.

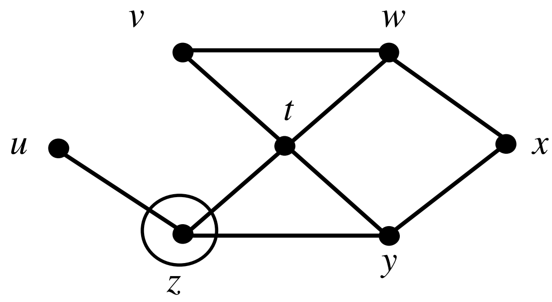
Example 1: Use Fluery's Algorithm to find an Eulerian circuit starting with u in the following graph.



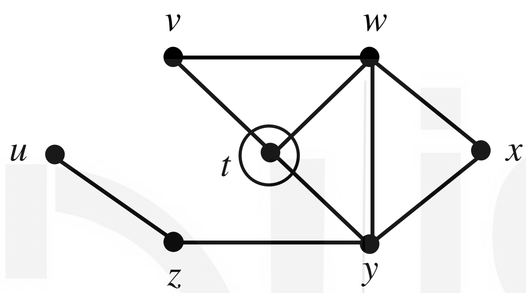
Solution: Beginning with the vertex u, we traverse the edge uv as uv is not a bridge. So, we get $C=(u,uv,v)$. Then uv is deleted, and we obtain the following graph.



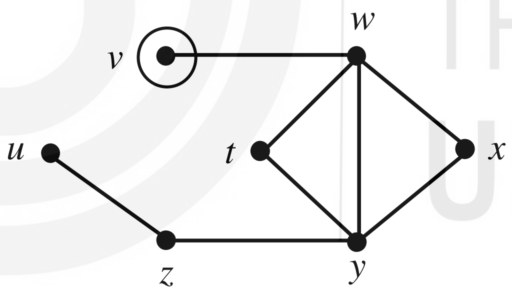
Now, let us traverse the edge $vz,$ to get $C=(u,uv,v,vz,z).$ Deleting vz we get the following graph.



We cannot traverse edge uz (which is a bridge), so we must use either zt or zy. Traversing zt we get $C=(u,uv,v,vz,z,zt,t).$ After deleting zt the graph obtained is as follows:

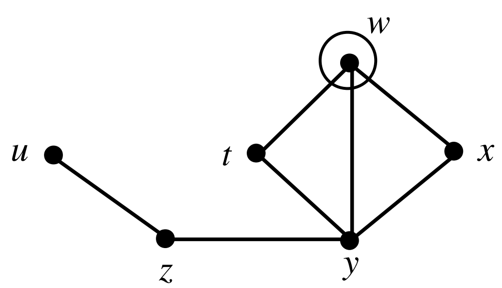


Now we traverse the edge tv to get $C=(u,uv,v,vz,z,zt,t,tv,v).$ The graph resulting from the removal of tv is as follows:

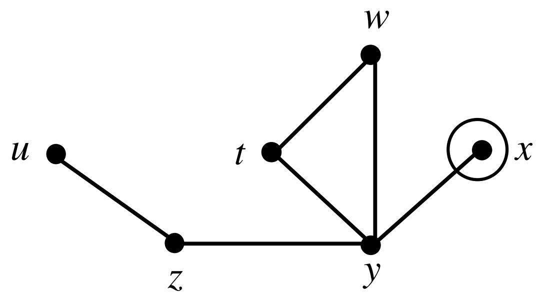


Now from v we have to traverse the bridge vw as there is no alternative. Hence $C=(u,uv,v,vz,z,zt,t,tv,v,vw,w).$

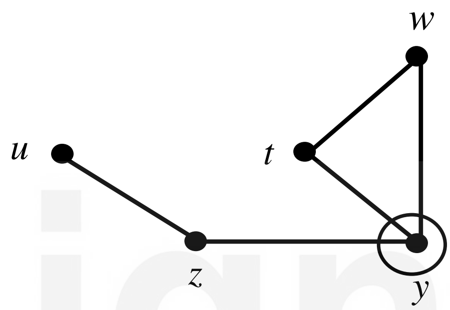
Deleting the edge $vw,$ and also the resulting isolated vertex $v,$ we get the graph as follows:



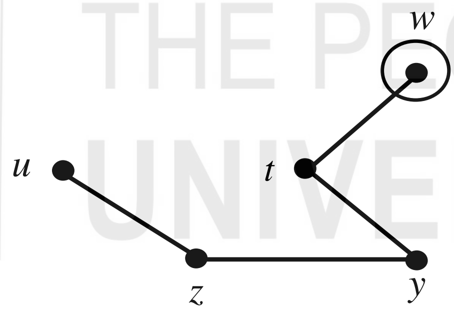
Now we traverse the edge wx to get $C=(u,uv,v,vz,z,zt,t,tv,v,vw,w,wx,x).$ After deleting wx the graph obtained is as follows:



Now we have no alternative but to traverse the edge xy, and hence $C = (u, uv, v, vz, z, zt, t, tv, v, vw, w, wx, x, xy, y)$. The graph after deleting xy, and the resulting isolated vertex x, is given below.



From y we cannot traverse the bridge yz as there are other edges available. We traverse yw to get $C = (u, uv, v, vz, z, zt, t, tv, v, vw, w, wx, x, xy, y, yw, w)$. The graph after deleting yw is given below.



Now all the edges in the graph above are bridges, and so traversing them in order we get

$$C = (u, uv, v, vz, z, zt, t, tv, v, vw, w, wx, x, xy, y, yw, w, wt, t, ty, y, yz, z, zu, u)$$

which is the desired Eulerian circuit.

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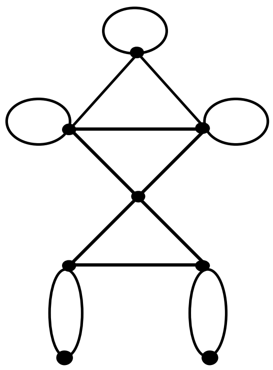
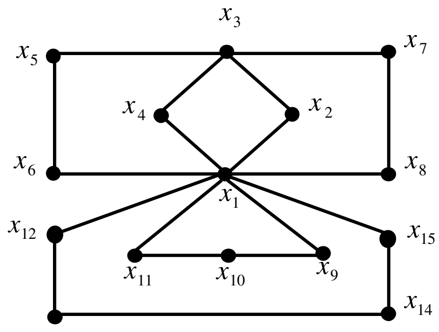
Remark 1: If G is an Eulerian graph with m edges, then Fleury's Algorithm stops after exactly m steps. When it stops, we are back at the starting vertex. So, we get an Eulerian circuit of the graph G. It can be proved that Fleury's Algorithm always yields an Eulerian circuit. Due to the complexity of the proof, we omit it.

Now, here are some exercises for you to try.

E1) Check whether the following multigraphs are Eulerian or not.

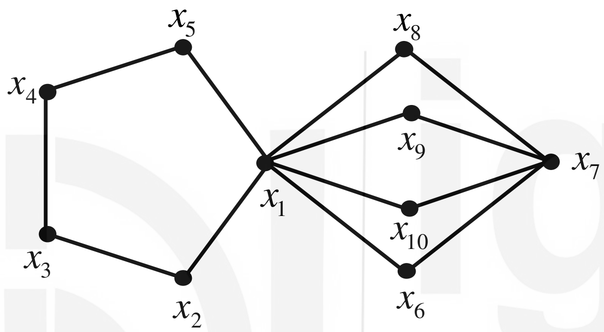


G_1



G_2

E2) Show that the following graph is Eulerian. Also, find an Eulerian circuit in it by Fluery's Algorithm.



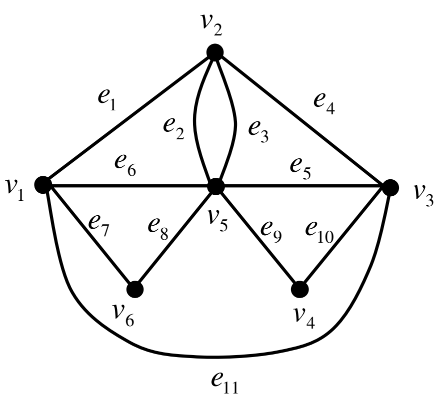
E3) i) For what values of n, is K_n Eulerian?

ii) For what values of m and n, is $K_{m,n}$ Eulerian?

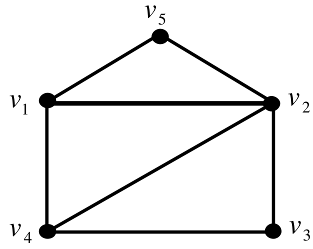
iii) For what value of n is Q_n Eulerian?

E4) Show that, in an Eulerian graph, an Eulerian circuit can be traced starting from any vertex.

E5) Use Fluery's Algorithm to find an Eulerian circuit starting at v_1 in the following multigraph.



Recall that an **open trail** is a trail whose endpoints are distinct. For example, $(v_1, v_4, v_2, v_1, v_5, v_2, v_3, v_4)$ is an open trail in the graph given below.

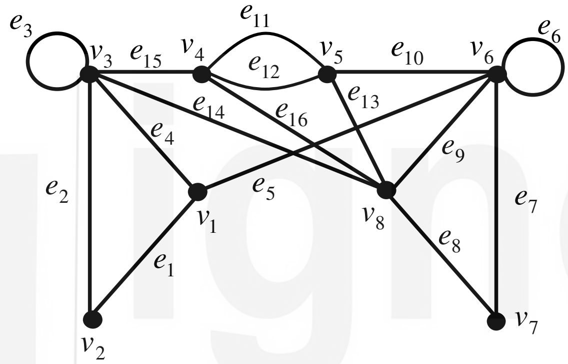


Definition: A connected multigraph G is said to be **semi-Eulerian** if G contains an open trail that includes every edge of G.

For instance, the graph given above is semi-Eulerian.

Let us now consider another example.

Example 2: Show that the multigraph given below is semi-Eulerian.



Solution: Consider the walk

$(v_1, e_1, v_2, e_2, v_3, e_3, v_3, e_4, v_1, e_5, v_6, e_6, v_6, e_7, v_7, e_8, v_8, e_9, v_6, e_{10}, v_5, e_{11}, v_4, e_{12},$

$v_5, e_{13}, v_8, e_{14}, v_3, e_{15}, v_4, e_{16}, v_8).$

This contains all the sixteen edges of the multigraph, and the end vertices are distinct. Since no edge is repeated, this walk is an open trail containing all the edges of the multigraph. So, the multigraph is **semi-Eulerian.**

***

Just like Eulerian multigraphs, there is a necessary and sufficient condition for a semi-Eulerian multigraph. It is given below.

Theorem 2: A connected multigraph G is semi-Eulerian if and only if it has exactly two vertices of odd degree.

Proof: Let G be a semi-Eulerian multigraph. Let u and v be the starting and ending vertices of the open trail that contains every edge of $G.$ If we add an edge e joining the vertices u and $v,$ we get an Eulerian graph in which, by Theorem 1, every vertex must have even degree. If we now recover G by removing the edge $e,$ we see that u and v are the only vertices of odd degree.

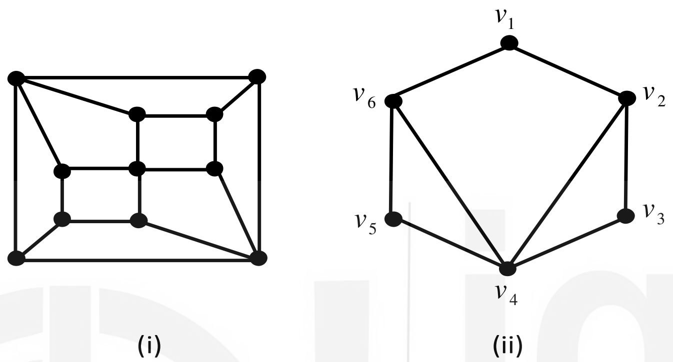
Conversely, suppose that G has exactly two vertices u and v of odd degree. If we add an edge e joining the vertices u and $v,$ we get a connected multigraph in which every vertex has even degree. By Theorem 1, this multigraph must be Eulerian, and so must posses an Eulerian circuit. Removal of e from this, produces an open trail which includes every edge of G. So G is semi-Eulerian. ■

It follows from the above proof that two vertices of odd degree must be the starting and ending vertices of any open trail which includes every edge of G.

Now we can easily see why the multigraph in Example 2 is semi-Eulerian. It has exactly two odd vertices, namely v_1 and v_8.

Now try the following exercise.

E6) Are the following graphs Eulerian? Are they semi-Eulerian? Justify our answers.



Next, we shall discuss an application of Eulerian multigraphs. It is known as the Chinese Postman Problem.

5.3 CHINESE POSTMAN PROBLEM

An important problem which has appeared in various guises is the so-called Chinese Postman Problem. The word ‘Chinese’ refers to the problem, not the postman. The problem was formulated in 1962 by Meigu Guan a Chinese mathematician. It can be stated as follows:

Chinese Postman Problem (CPP): A postman wishes to deliver mail along all the streets in his area, and then return to the post office. How can the route be planned so as to cover the smallest total distance?

First, let us model the problem as a weighted graph (G,W). Each house in the area assigned to the postman represents a vertex, and each street between two houses represents an edge. Each edge e is then assigned a nonnegative weight $W(e)$. Now, CPP can be formulated as follows:

“Given a connected weighted graph (G,W), find a closed walk C in G such that C contains all the edge of G, and $W(C)$ is minimum.”

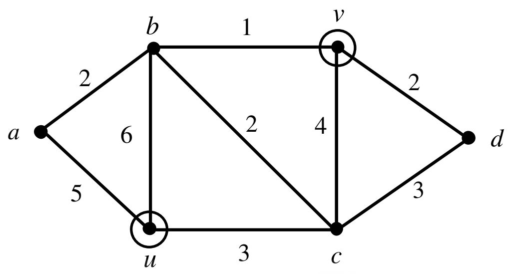
If the map of the postman’s area happens to correspond to an Eulerian graph, then there is no difficulty with this problem – the postman can simply choose an Eulerian circuit (using Fleury’s Algorithm, if necessary), and such a circuit will involve the smallest total distance. However the problem arises when the graph is not Eulerian. In that case, the postman needs to visit some parts of the route more than once, and would like to minimise the amount of retraversal. That is, the closed walk C, we are interested in, may have some edges repeated. But which edge need to be repeated so as $W(C)$ is minimum?

You have seen that a non-Eulerian graph has at least two odd vertices. We shall describe a solution to CPP only for those non-Eulerian graphs that contain exactly two odd vertices. The solution is given below.

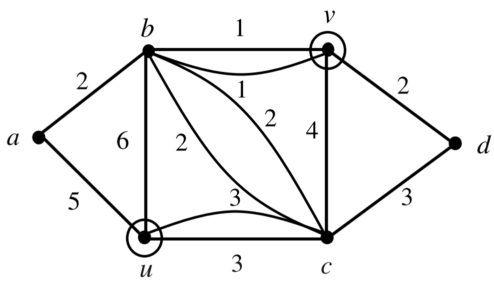
"Given a weighted graph (G,W), let (u,v) be the unique pair of odd vertices in G. Find a shortest (u,v)-path P in (G,W). Construct a weighted multigraph G^* by duplicating each edge e of P. Then G^* is Eulerian, and an Eulerian circuit in G^* can be found using Fluery's Algorithm."

To illustrate the method, let us consider the following weighted graph G with just two vertices u and v of odd degree.

Note that **duplicating** an edge e of a weighted multigraph means adding an edge parallel to e with the same weight as $W(e)$.



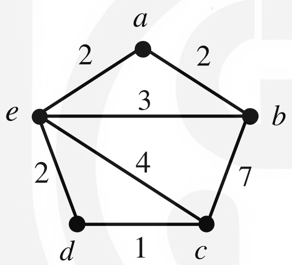
G



G^*

The shortest (u,v)-path in G is $P=(u,c,b,v)$ with $W(P)=6.$ Duplicating each edge of $P,$ we get G^* shown above right.

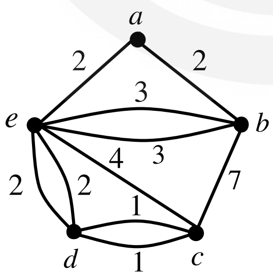
For graphs with more than two vertices of odd degree, we can adopt this method, linking such vertices with shortest paths. Unfortunately, the details of this procedure are too complicated to be included here.



G

Example 3: Solve the Chinese Postman Problem for the graph G given in left margin.

Solution: Note that G has exactly two vertices of odd degree namely, b and $c.$ The shortest (b,c)-path is (b,e,d,c) which has length 6. Now duplicating the edges in this path, we obtain the Eulerian multigraph, G^* shown in the left margin.



G^*

Applying Fuery's Algorithm on G^* we get the Eulerian circuit as (a,b,c,d,e,d,c,e,b,e,a) of length 27.

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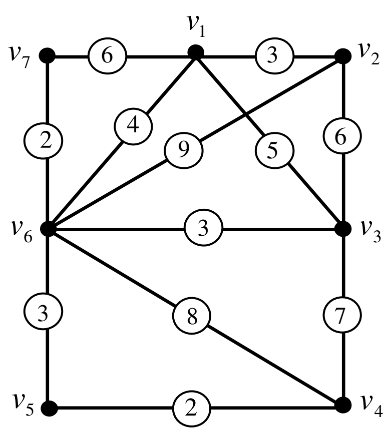
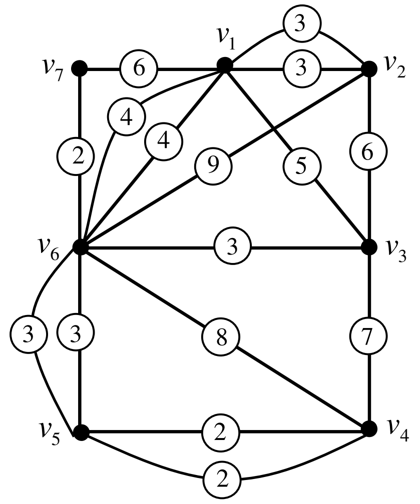
Example 4: Solve the Chinese Postman Problem for the following graph.



G^*

Solution: The two vertices of odd degree in G are v_2 and v_4. The shortest (v_2, v_4)-path is $(v_2, v_1, v_6, v_5, v_4)$ which has length 12. Duplicating each edge of this path we get the multigraph G^* as shown below.



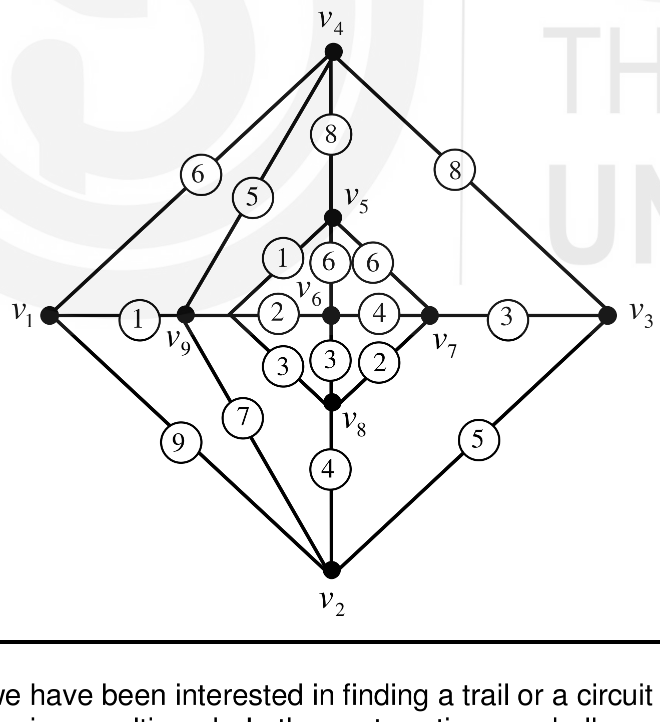
G^*

Using Fluery's Algorithm the Eulerian circuit in G^* is
$(v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_1, v_3, v_6, v_4, v_5, v_6, v_1, v_2, v_6, v_1)$.

***

Now try the following exercise.

E7) Solve the Chinese Postman Problem for the following graph.



Thus far we have been interested in finding a trail or a circuit that cover all the edges of a given multigraph. In the next section we shall concentrate on the graphs that contain a cycle passing through all the vertices.

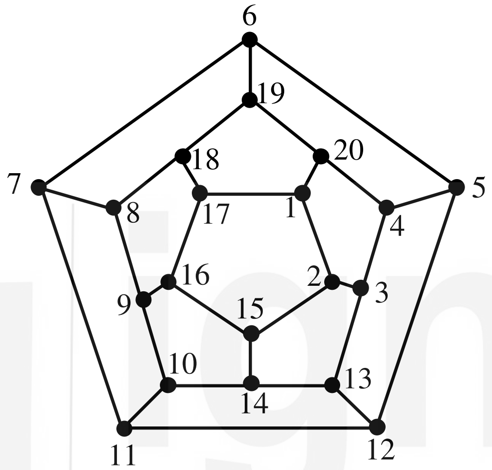
5.4 HAMILTONIAN GRAPHS

Suppose a transport company operates bus services between 10 different places. There are places with no direct bus service between them, but there is always a route between any two places that go through the other places. In this situation, the company wants to offer a round trip that passes through each of the cities **exactly once**. Is it possible?

Let us formulate this question as a problem in graph theory. Let us represent the places by vertices. Two vertices are adjacent if a bus route directly connects the corresponding places. Since it is possible to go from one place to another, the graph we get is a connected graph. Now, the question is:

Is there a cycle in the graph that contains every vertex?

A similar question was the basis of the mathematical game described by Sir Willian Hamilton (1805-1865) in 1857. In this game, he took a regular dodecahedron a graph of which is drawn below.



Each of its 20 vertices is supposed to represent a city of the world. A player is supposed to find a 'world tour' starting with any vertex, visiting every vertex exactly once before reaching back to the starting vertex. This amounted to finding a cycle covering all the vertices of the regular dodecahedron. The cycle $(1,2,3,\ldots,20,1)$ is such a cycle.

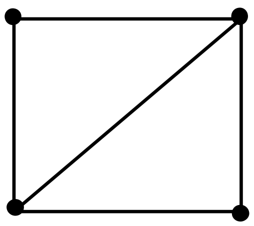
In other words, a Hamiltonian cycle is just a spanning cycle, i.e., a spanning subgraph that is a cycle.

Let us now look at the formal definition.

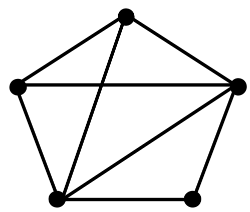
Definition: A cycle C in a graph G is called a **Hamiltonian cycle** if C contains all the vertices of G. A graph is called **Hamiltonian** if it contains a Hamiltonian cycle. A graph is called **non-Hamiltonian** if it does not contain any Hamiltonian cycle.

Thus the graph given above is Hamiltonian.

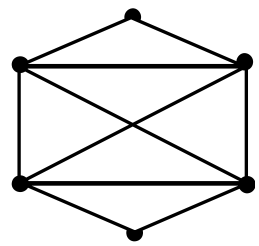
Can you think of some more examples of Hamiltonian graphs? Of course, you can easily see that C_n and K_n are Hamiltonian. The following graphs are also Hamiltonian.



G_1



G_2



G_3

Note that G_1 is obtained by adding an edge to C_4. Likewise, G_2 and G_3 are obtained by adding some edges to C_5 and C_6, respectively. In fact, if you add any number of edges to a given Hamiltonian graph, the resulting graph is Hamiltonian.

Are there any non-Hamiltonian graphs? Trees are obvious examples. Since they don't have any cycles, they cannot have a spanning cycle.

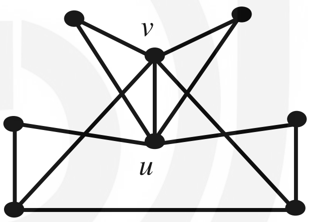
Note that, by definition, a Hamiltonian graph contains a cycle containing all the vertices. So, a Hamiltonian graph cannot have leaves. Below we give another property of Hamiltonian graphs.

Theorem 3: If G is a Hamiltonian graph, then $c(G-S) \leq |S|$ for every $\emptyset \neq S \subset V(G)$.

Proof: Consider a Hamiltonian graph G, with a Hamiltonian cycle C. When we traverse C, at each pass through a component of $G-S$ we enter in S at an unused vertex. Thus S must have at least as many vertices as the number of components of $G-S$. ■

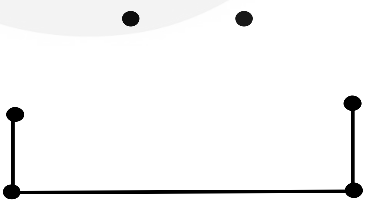
Theorem 3 essentially gives us a criterion of non-Hamiltonicity. Let us now discuss some examples.

Example 5: Is the following graph G Hamiltonian?



G

Solution: Let $S = \{u, v\}$. Then $G-S$ is as drawn below.



$G - S$

Since $c(G-S) = 3 > |S|$, it follows that G is not Hamiltonian.

***

Example 6: Show that a Hamiltonian graph has no cut-vertices..

Solution: We prove the contrapositive of the statement. Let G be a graph with a cut-vertex v. Let $S = \{v\}$. Then $G-S$ has more components than G. If G is disconnected, G is non-Hamiltonian. Otherwise, $c(G) = 1$. Then $c(G-S) > 1 = |S|$. Hence by Theorem 3, G is non-Hamiltonian.

***

Example 7: Show that $K_{m,n}$ is Hamiltonian iff $m = n$.

Solution: The statement says that $m = n$ is necessary as well as sufficient for $K_{m,n}$ to be Hamiltonian. We shall first prove the necessary condition. So, let $m \neq n$. Without loss of generality assume that $m < n$. Let (X, Y) be a bipartion of $K_{m,n}$ with $|X| = m$, and $|Y| = n$. Now we can see that $K_{m,n} - X$ has n components. So, $c(K_{m,n} - X) > |X|$. Hence by Theorem 3, $K_{m,n}$ is not Hamiltonian.

Now let us assume that $m = n$. Let $\{u_1, u_2, \ldots, u_n\}$ and $\{v_1, v_2, \ldots, v_n\}$ be the partite sets of $K_{m,n}$, i.e., of $K_{n,n}$. Then $(u_1, v_1, u_2, v_2, \ldots, u_n, v_n, u_1)$ is a Hamiltonian cycle of $K_{m,n}$. Consequently, $K_{m,n}$ is Hamiltonian.

***

Now try the following exercises.

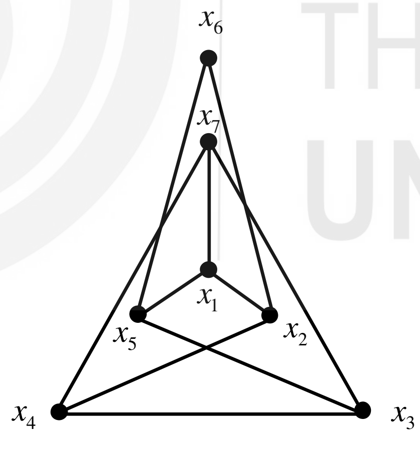
E8) Construct a non-Hamiltonian graph on 5 vertices, with maximum size.

E9) Find a graph which is Hamiltonian but not Eulerian.

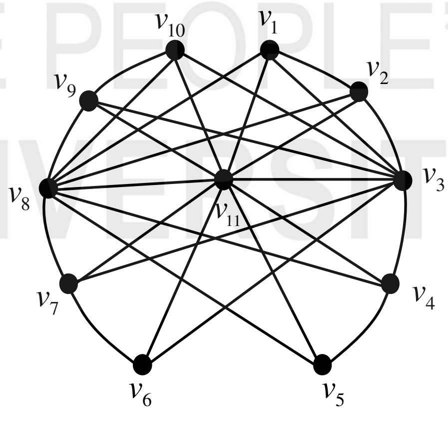
E10) Find a graph which is Eulerian but not Hamiltonian.

E11) Find a Hamiltonian cycle in the hypercube Q_3.

E12) Check whether the following graphs are Hamiltonian or not.



G_1



G_2

So far, we have seen some necessary conditions for a graph to be Hamiltonian. They are helpful if we want to show that a given graph is non-Hamiltonian. To show that a given graph is Hamiltonian, we need some sufficient conditions.

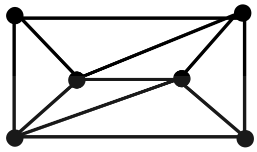
At first sight, the problem of deciding whether or not a given graph is Hamiltonian seems very similar to the problem of deciding whether or not it is Eulerian and we might expect there to be a simple necessary and sufficient condition for a graph to be Hamiltonian, analogous to that of Theorem 1. However, no such condition is known and the search for necessary or sufficient conditions for a graph to be Hamiltonian is a major area of study in graph theory today.

You know that adding edges to a Hamiltonian graph results in a Hamiltonian graph. So, graphs with many edges are more likely to be Hamiltonian than graphs with fewer edges. We can make this idea precise in various ways. Two of the most important of these are the following sufficient conditions of G.A. Dirac and O. Ore, published in 1952 and 1960, respectively.

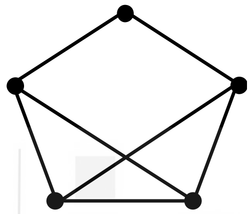
Theorem 4 (Dirac's Theorem): Let G be a graph with $n(G) \geq 3$. If $\delta(G) \geq \frac{n}{2}$, then G is Hamiltonian.

Theorem 5 (Ore's Theorem): Let G be a graph with $n(G) \geq 3$. If for every pair (u, v) of nonadjacent vertices $d(u) + d(v) \geq n(G)$, then G is Hamiltonian.

We illustrate the use of these theorems by considering the following graphs.



G_1



G_2

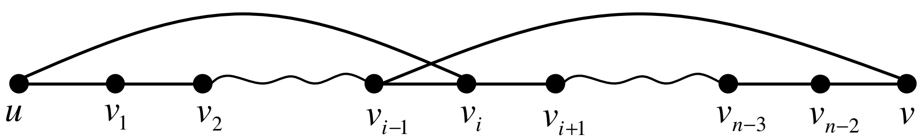
We have $\delta(G_1) = 3$ and $n(G_1) = 6$, which imply $\delta(G_1) \geq \frac{n(G_1)}{2}$. So, G_1 is Hamiltonian by Dirac's Theorem. On the other hand, G_2 does not satisfy the premise of Dirac's Theorem, so Dirac's Theorem does not apply. However, $d(u) + d(v) \geq n(G_2)$ for all pairs of non-adjacent vertices u and v, so G_2 is Hamiltonian by Ore's Theorem.

Note that $\delta(G) \geq \frac{n(G)}{2}$ implies $d(u) + d(v) \geq \frac{n(G)}{2} + \frac{n(G)}{2} = n(G)$ for each pair of vertices u and v. It follows that Dirac's Theorem can be deduced from Ore's Theorem. So we prove only Ore's Theorem.

Proof of Ore's Theorem: We shall prove this result by contradiction. Suppose that there exists an n-vertex non-Hamiltonian graph G in which $d(u) + d(v) \geq n$ for each pair of non-adjacent vertices u and v. Without loss of generality, let G be maximal non-Hamiltonian. This means G contains a path passing through each vertex (such a path is called a **Hamiltonian path**), say $(u, v_1, v_2, v_3, ..., v_{n-3}, v_{n-2}, v)$. By adding only one edge uv, we can make a Hamiltonian cycle.

A maximal non-Hamiltonian graph is a graph that is non-Hamiltonian but it becomes Hamiltonian by just adding an edge between a pair of nonadjacent vertices of it.

Now, we consider an observation that 'if v_i is adjacent to u then v_{i-1} cannot be adjacent to v'. This statement must be true, otherwise we get a Hamiltonian cycle $(u, v_1, v_2, ..., v_{i-1}, v, v_{n-2}, v_{n-3}, ..., v_{i+1}, v_i, u)$ as shown below.



This implies that v which has degree obviously less than or equal to $(n-1)$ as G is a graph, has one more constraint that it cannot be adjacent to those vertices of G which are adjacent to u.

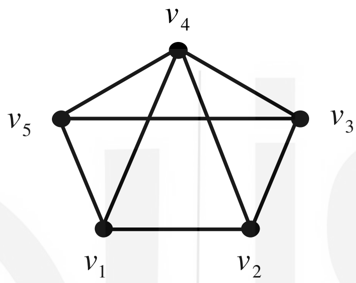
Hence, $d(v) \le (n-1) - d(u) \Rightarrow d(u) + d(v) \le n-1.$

This contradicts the hypothesis. Hence Ore's Theorem is proved. ■

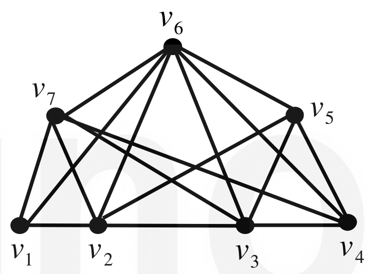
Remark 2: Note that Theorem 4 and Theorem 5 are just sufficient conditions. They are not at all necessary. For example, $C_n, n > 4$, is always Hamiltonian but C_n is a 2-regular graph, and therefore, $d(u) + d(v) = 4 < n$ always.

Let us look at some examples to see the use of Dirac's and Ore's Theorems.

Example 8: Check whether the conditions of Dirac's Theorem and Ore's Theorem hold for following graphs.



G_1



G_2

Solution: Note that $n(G_1) = 5,$ and $\delta(G_1) = 3.$ So $\delta(G_1) \ge \frac{n(G_1)}{2}.$ Hence G_1 satisfies Dirac's condition, and hence Ore's condition.
For G_2, we have $n(G_2) = 7,$ and $\delta(G_2) = 3.$ Hence G_2 does not satisfy Dirac's condition. Let us now check whether G_2 satisfies Ore's condition or not. We have

$$v_1 \not\leftrightarrow v_3 \Rightarrow d(v_1) + d(v_3) = 8 \ge 7$$

$$v_1 \not\leftrightarrow v_4 \Rightarrow d(v_1) + d(v_4) = 7 \ge 7$$

$$v_1 \not\leftrightarrow v_5 \Rightarrow d(v_1) + d(v_5) = 7 \ge 7$$

$$v_2 \not\leftrightarrow v_4 \Rightarrow d(v_2) + d(v_4) = 9 \ge 7$$

$$v_5 \not\leftrightarrow v_7 \Rightarrow d(v_5) + d(v_7) = 9 \ge 7$$

Thus, G_2 satisfies Ore's condition.

***

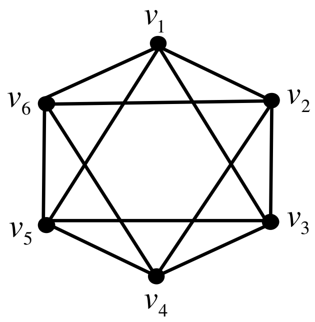
Now try the following exercises to test your understanding.

E13) Give an example of each of the following:

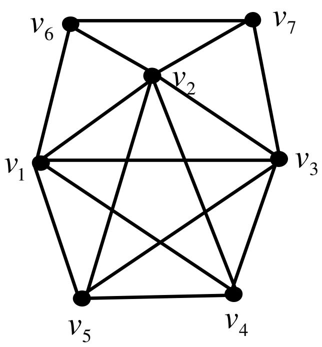
i) A Hamiltonian graph which does not satisfy the conditions of Ore's theorem;

ii) A non-Hamiltonian graph G with minimum degree $\frac{(n(G)-1)}{2}$.

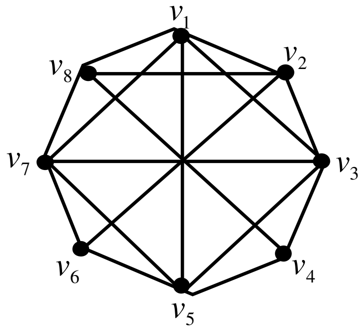
E14) Which of the following graphs satisfy Dirac's or Ore's conditions? Which of them are Hamiltonian?



i)



ii)



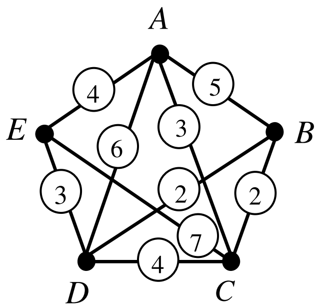
iii)

As we have said before there are no conditions that are necessary as well as sufficient for a graph to be Hamiltonian. Thus in general, it is difficult to prove that a given graph is Hamiltonian. For example, the **Petersen graph is not Hamiltonian**, but it is not easy to show this. In next section, we shall discuss Travelling salesman problem.

5.5 TRAVELLING SALESMAN PROBLEM (TSP)

A travelling salesman wants to visit a number of cities and return to the starting point, in such a way that each city is visited exactly once and the total distance covered is as small as possible. Given the various distances between the cities, what route should be chosen?

In principle, we can solve this problem by looking at all possible routes and choosing the one which involves the least total distance. For example, if there are five cities A, B, C, D and E, and if the connections between them are as shown in the graph given below then a traveling salesman at A should visit the cities in the order $ACBDEA$ (or in reverse order $AEDBCA$), covering a total distance of 14.



Unfortunately, as soon as we increase the number of cities, we run into difficulties, since there is no known algorithm which provides a simple and efficient solution for the problem. Although there are several ad-hoc procedures which can be used to give approximate solutions, a full solution effectively involves looking at all possible routes and choosing the shortest, which is a computationally difficult task.

Let us now formally state the Travelling Salesman Problem.

Travelling Salesperson Problem (TSP): For given a weighted complete graph, find a minimum-weight Hamiltonian cycle in it.

Although till date there is no algorithm which can solve TSP, there are many algorithms that give approximate solutions. That is, these algorithms can produce Hamiltonian cycles with quite small weights.

Here we shall discuss one such algorithm.

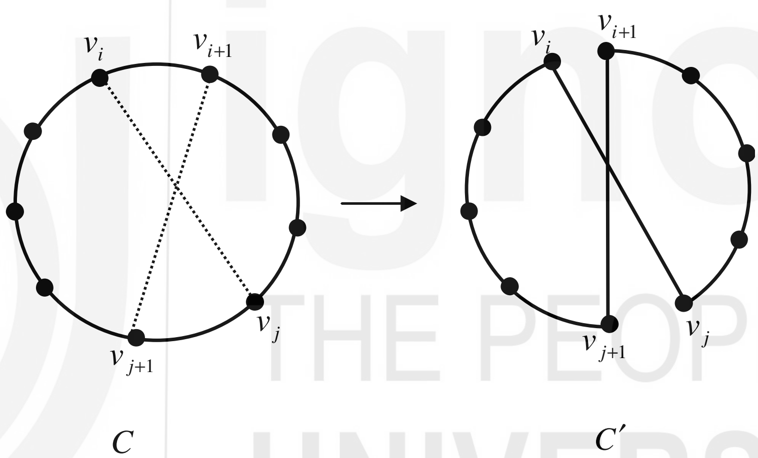
Consider a weighted complete graph (G,W) on n vertices. Let $C=(v_1,...,v_n,v_1)$ be a Hamiltonian cycle in $G.$ For a fixed $i,$ first check whether there is a j such that

$$W(v_iv_{i+1})+W(v_jv_{j+1})>W(v_i\,v_j)+W(v_{i+1}\,v_{j+1})$$

If this inequality holds, then replace the cycle C by

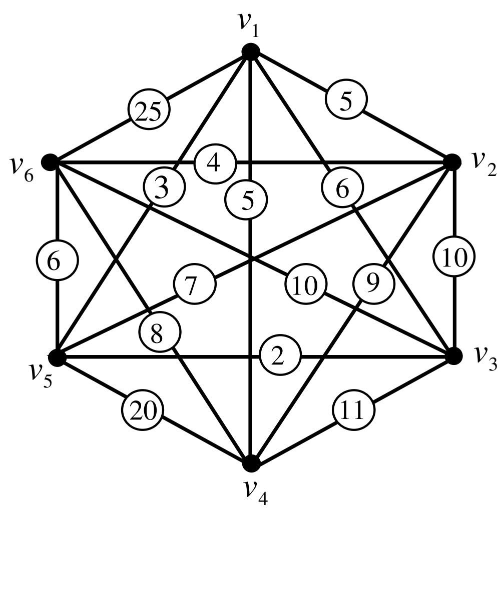
$$C'=(v_1,...,v_i,v_j,v_{j-1},...,v_{i+1},v_{j+1},v_{j+2},..,v_nv_1).$$

See the following illustration.



Clearly, the weight of the cycle C' is strictly less than that of cycle C. After performing a sequence of such modifications, one is left with a cycle whose weight cannot be reduced further by this process. Of course, there is no guarantee that the resulting cycle will have the least possible weight. There may be other cycles with lower weight. But it will often be fairly good. Let us consider an example of this process.

Example 9: Consider the following weighted copy of K_6.

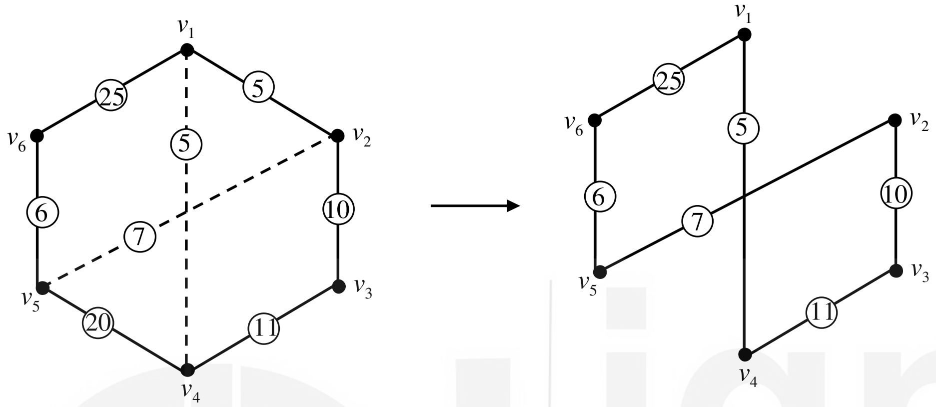


Starting with the cycle $C = (v_1, v_2, v_3, v_4, v_5, v_6, v_1)$, find a Hamiltonian cycle having weight smaller than 35.

Solution: Note that $W(C) = 77.$ Look at the edges $v_1 v_2$ and $v_4 v_5$ of C. We have $W(v_1 v_2) = 5, W(v_4 v_5) = 20$ and $W(v_1 v_4) = 5, W(v_2 v_5) = 7.$ So

$$W(v_1 v_2) + W(v_4 v_5) > W(v_1 v_4) + W(v_2 v_5).$$

So, we modify the cycle C to $C' = (v_1, v_4, v_3, v_2, v_5, v_6, v_1)$. Here, $W(C') = 64.$ (See the picture below.)

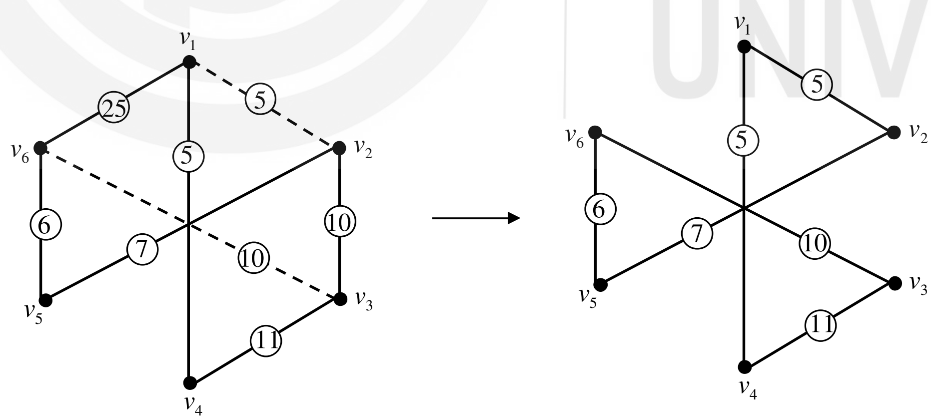


C C'

Now look at the edges $v_3 v_2$ and $v_6 v_1$ in C'. We have $W(v_3 v_2) = 10, W(v_6 v_1) = 25$ and $W(v_3 v_6) = 10, W(v_2 v_1) = 5.$ So

$$W(v_3 v_2) + W(v_6 v_1) > W(v_3 v_6) + W(v_2 v_1).$$

See the picture below.



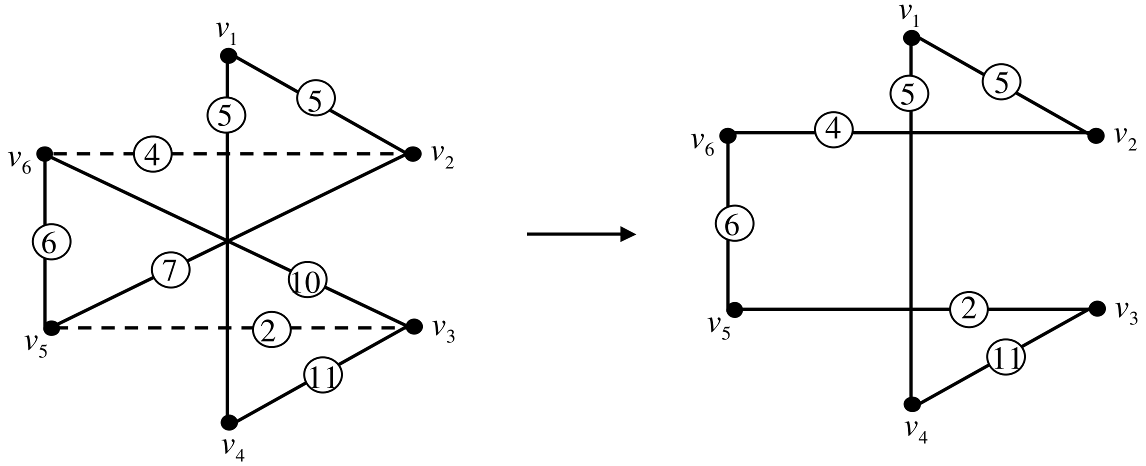
C' C''

Thus C' transforms into the cycle $C'' = (v_1, v_4, v_3, v_6, v_5, v_2, v_1)$, and $W(C'') = 44.$

Now look at the edges $v_3 v_6$ and $v_5 v_2$ in C'' which have weights 10 and 7, respectively. We have

$$W(v_3 v_6) + W(v_5 v_2) > W(v_3 v_5) + W(v_6 v_2).$$

See the picture below.



C'' C'''

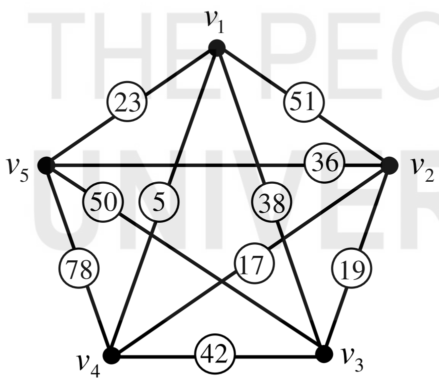
Thus C'' transforms into $C''' = (v_1, v_4, v_3, v_5, v_6, v_2, v_1)$. We have $W(C''') = 33$ which is smaller than 35. So we stop here.

***

Here are some related exercises for you to try.

E15) Carry out the reduction step once more on the cycle obtained in Example 9 to get a Hamiltonian cycle with lesser weight.

E16) Start with the cycle $(v_1, v_2, v_3, v_4, v_5, v_1)$ in the following weighted complete graph K_5, and perform the reduction step twice to get a Hamiltonian cycle with smaller weight.



We have now reached the end of this unit. Let us briefly summarise what we have studied in this unit.

5.6 SUMMARY

In this unit we have covered the following points.

1. Discussed Eulerian graphs, and their characterisation.

2. Discussed semi-Eulerian graphs and their characterisation.

3. Discussed the Chinese Postman Problem, and gave a solution of it for a special case.

4. Discussed what are Hamiltonian graphs and the properties possessed by them.

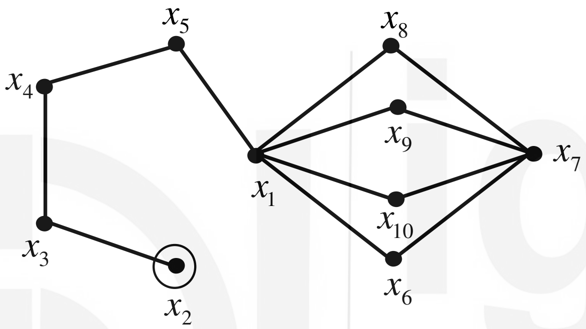
5. Discussed some sufficient and some necessary conditions for Hamiltonian graphs.

6. Discussed the Travelling Salesman Problem, and offered an algorithm for obtaining Hamiltonian cycles with smaller weights.

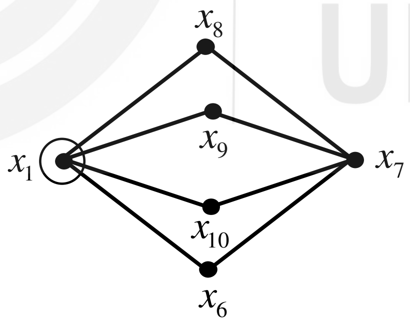
5.7 SOLUTIONS/ANSWERS

E1) We can see that the multigraph G_1 has two vertices of degree 5, which is odd. Therefore, G_1 is not Eulerian. The vertices in the graph G_2 have all even degree. Therefore, G_2 is Eulerian.

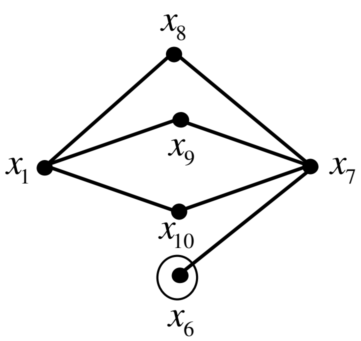
E2) Since all the vertices in the graph are even, the graph is Eulerian. Let us now apply Fluery's Algorithm, beginning at the vertex x_1. Since x_2 is not a bridge, we traverse the edge $x_1 x_2$. Thus we have $C = (x_1, x_1 x_2, x_2)$. Deleting the edge $x_1 x_2$ we get the following graph.



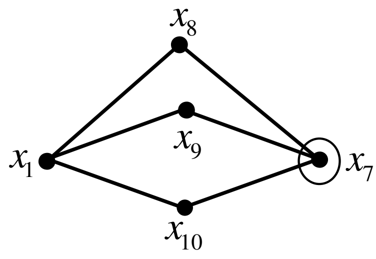
Now we can see that the edges $x_2 x_3, x_3 x_4, x_4 x_5$ and $x_5 x_1$ are bridges, and so traversing and deleting them in order we get $C = (x_1, x_1 x_2, x_2, x_2 x_3, x_3, x_3 x_4, x_4, x_4 x_5, x_5, x_5 x_1, x_1)$, and the graph as follows:



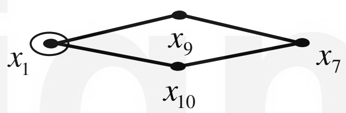
Now let us traverse the edge $x_1 x_6$ to get $C = (x_1, x_1 x_2, x_2, x_2 x_3, x_3, x_3 x_4, x_4, x_4 x_5, x_5, x_5 x_1, x_1, x_1 x_6, x_6)$. Deleting $x_1 x_6$ we get the following graph.



Now we traverse the edge $x_6 x_7$ to get
$C = (x_1, x_1 x_2, x_2, x_2 x_3, x_3, x_3 x_4, x_4, x_4 x_5, x_5, x_5 x_1, x_1, x_1 x_6, x_6, x_6 x_7, x_7)$.
The graph after deleting the edge $x_6 x_7$, and the resulting isolated vertex x_6, is given below.



Now we traverse the edge $x_7 x_8$, and then the bridge $x_8 x_1$ so that
$C = (x_1, x_1 x_2, x_2, x_2 x_3, x_3, x_3 x_4, x_4, x_4 x_5, x_5, x_5 x_1, x_1, x_1 x_6, x_6, x_6 x_7,$
$x_7, x_7 x_8, x_8, x_8 x_1, x_1)$. The graph after deleting $x_7 x_8, x_8$ and $x_8 x_1$ is given below.



Now we traverse the edge $x_1 x_9$, and the then the subsequent bridges $x_9 x_7, x_7 x_{10}$ and $x_{10} x_1$ to get

$C = (x_1, x_1 x_2, x_2, x_2 x_3, x_3, x_3 x_4, x_4, x_4 x_5, x_5, x_5 x_1, x_1, x_1 x_6, x_6, x_6 x_7,$

$x_7, x_7 x_8, x_8, x_8 x_1, x_1, x_1 x_9, x_9, x_9 x_7, x_7, x_7 x_{10}, x_{10}, x_{10} x_1, x_1)$ which is an Eulerian circuit.

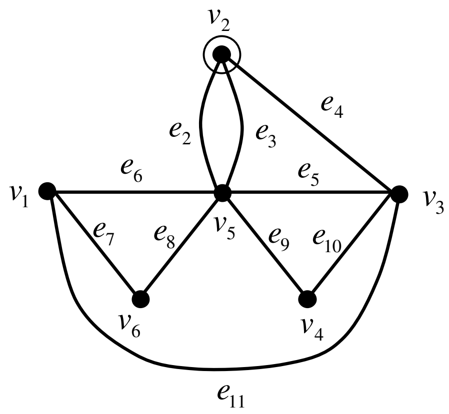
E3) i) K_n is an $(n-1)$-regular graph. So, it is Eulerian when $n-1$ is even, i.e., n is odd.

ii) $K_{n,m}$ has all vertices of degree m or n. So, it is Eulerian when m and n are both even.

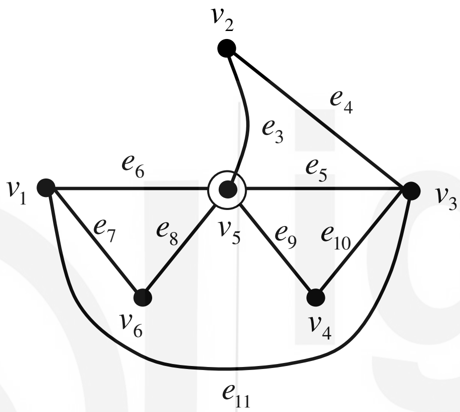
iii) We know that Q_n is n-regular. Therefore, Q_n is Eulerian if n even.

E4) Suppose G is an Eulerian graph and $(v_0, v_1, ..., v_n, v_0)$ is an Eulerian circuit in it. Let $x = v_i$ be any vertex in G. Then, the circuit $(v_i, v_{i+1}, ..., v_n, v_0, v_1, ..., v_{i-1}, v_i)$ is also Eulerian, which starts and ends at v_i.

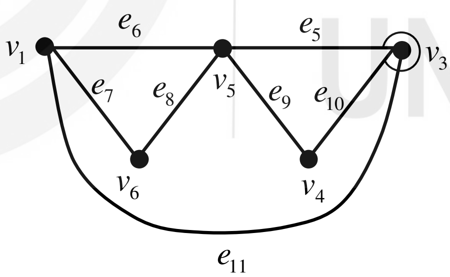
E5) Beginning at v_1 we traverse e_1. So we have $C = (v_1, e_1, v_2)$. After removing e_1 the resulting multigraph is



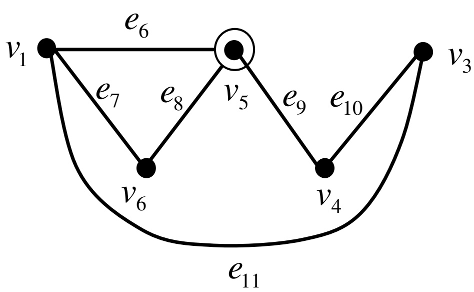
From v_2 we traverse e_2, so $C = (v_1, e_1, v_2, e_2, v_5)$. The graph obtained by removing e_2 is given below.



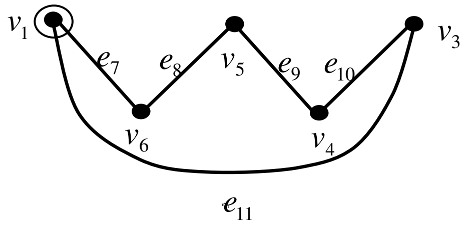
Now we traverse e_3 and then v_2 and e_4 to get $C = (v_1, e_1, v_2, e_2, v_5, e_3, v_2, e_4, v_3)$. The graph obtained after deleting e_3, v_2 and e_4 is:



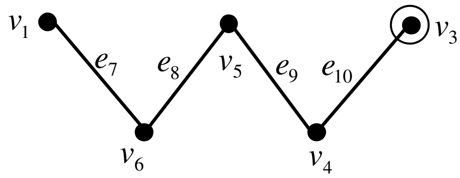
Now, we traverse e_5 to get $C = (v_1, e_1, v_2, e_2, v_5, e_3, v_2, e_4, v_3, e_5, v_5)$. The graph after removing e_5 is:



Next, we traverse e_6 to get $C = (v_1, e_1, v_2, e_2, v_5, e_3, v_2, e_4, v_3, e_5, v_5, e_6, v_1)$. The graph after removing e_6 is:



From v_1 we traverse e_{11} to get $C = (v_1, e_1, v_2, e_2, v_5, e_3, v_2, e_4, v_3, e_5, v_5, v_6, v_1, e_{11}, v_3)$. After removing e_{11} the resulting graph is:

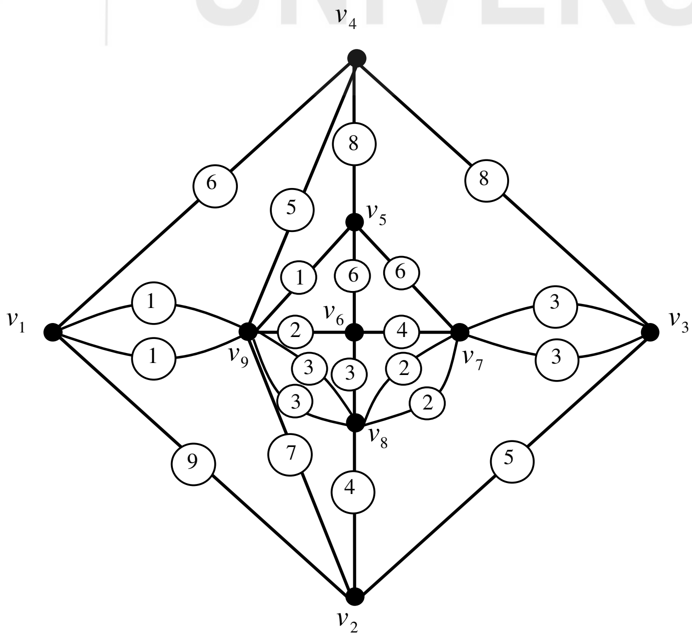


Now we can see that all the edges e_{10}, e_9, e_8 and e_7 are bridges in the graph above. Traversing them in order we get the Eulerian circuit $C = (v_1, e_1, v_2, e_2, v_5, e_3, v_2, e_4, v_3, e_5, v_5, e_6, v_1, e_{11}, v_3, e_{10}, v_4, e_9, v_5, e_8, v_6, e_7, v_1)$.

E6 i) The graph has some odd vertices. So it is not Eulerian. Since the number of odd vertices is more than 2, it is neither semi-Eulerian.

ii) This graph is also not Eulerian as it has the odd vertices v_2 and v_6. Since v_2 and v_6 are the only odd vertices, it is semi-Eulerian.

E7) The graph has exactly two odd vertices, namely, v_1 and v_3. The shortest (v_1, v_3)-path is $(v_1, v_9, v_8, v_7, v_3)$ which has length 9. Duplicating the edges of this path, we get the multigraph G^* as shown below.

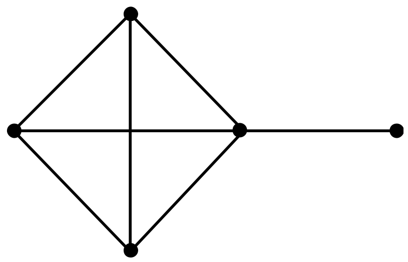


G^*

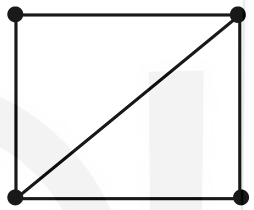
Apply Fleury's Algorithm on G^* we get the following Eulerian circuit
$(v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_3, v_7, v_8, v_9, v_5, v_7, v_8, v_9, v_2, v_8, v_6, v_9, v_4, v_1, v_9, v_1)$

It has length
$9+5+8+8+6+4+3+3+2+3+1+6+2+3+7+4+3+2+9+5+6+1+1=92.$

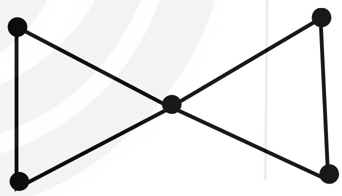
E8) The required graph is given below. Observe that adding any edge to it results in a Hamiltonian graph.



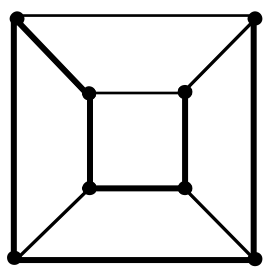
E9) There are plenty of such graphs. One such is given below.



E10) There are also many such graphs. For instance, look at the one given below.

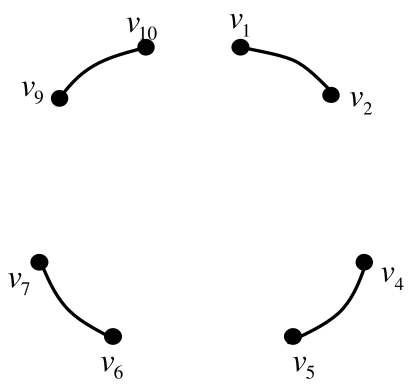


E11) The bold edges in the following copy of Q_3 form a Hamiltonian cycle.



E12) G_1 is Hamiltonian because it has the Hamiltonian cycle $(x_1, x_7, x_4, x_3, x_5, x_6, x_2, x_1)$.

If we remove the set of vertices $S = \{v_3, v_8, v_{11}\}$ from G_2, the resulting graph $G_2 - S$ is:



$G_2 - S$

Clearly $c(G_2 - S) = 4 \not\leq |S|$. Therefore, G_2 is not Hamiltonian.

E13) i) For example, consider the graph Q_3 which is Hamiltonian. But nonadjacent vertex pairs in Q_3 do not have degree at least 8.

ii) One such graph is given in the solution of E10.

E14) i) This is a 4-regular graph and $n(G) = 6$. Hence $\delta(G) \geq \frac{n(G)}{2}$.
Thus by Dirac's Theorem, G is Hamiltonian. Also, note that G satisfies the Ore's condition.

ii) Here $n(G) = 7$. The vertices v_6 and v_7 have degree $3 < \frac{7}{2}$.
Therefore, Dirac's criterion does not apply. However, the only pair of non-adjacent vertices in this graph are

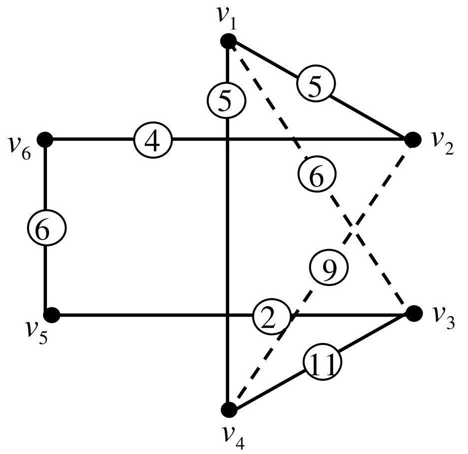
$$(v_1, v_7), (v_3, v_6), (v_4, v_6), (v_4, v_7), (v_5, v_6), (v_5, v_7).$$

All these pairs have degree sum at least 7. Thus, Ore's condition is satisfied. So, this graph is Hamiltonian.

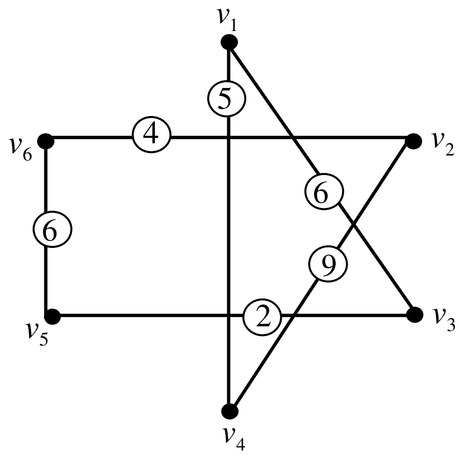
iii) Here $n(G) = 8$, and the graph is 5-regular. So, Dirac's criterion is satisfied. Hence, by Dirac's Theorem, this graph is also Hamiltonian. Clearly, G also satisfied Ore's condition.

E15) Look at the edge $v_4 v_3$ and $v_2 v_1$ in the cycle $C''' = (v_1, v_4, v_3, v_5, v_6, v_2, v_1)$.
We have $W(v_4 v_3) = 11, W(v_2 v_1) = 5$ and $W(v_4 v_2) = 9, W(v_3 v_1) = 6$. Thus $W(v_4 v_3) + W(v_2 v_1) > W(v_4 v_2) + W(v_3 v_1)$.

See the picture below.



C'''

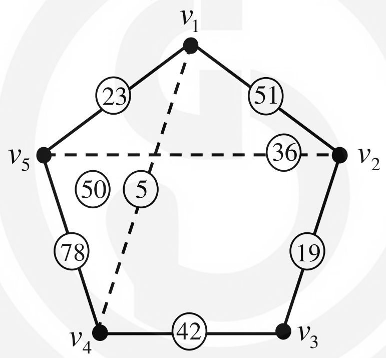


$C^{(iv)}$

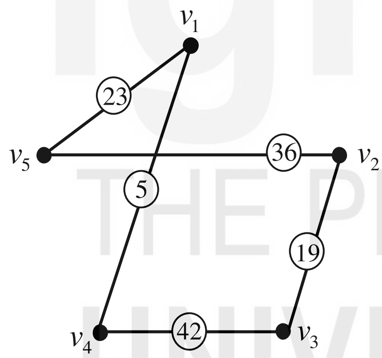
So, we transform the cycle C''' into $C^{(iv)} = (v_1, v_4, v_2, v_6, v_5, v_3, v_1)$. Here $W(C^{(iv)}) = 32$.

E16) We have the cycle $C = (v_1, v_2, v_3, v_4, v_5, v_1)$, with $W(C) = 213$. Look at the edge $v_1 v_2$ and $v_4 v_5$. We $W(v_1 v_4) = 5, W(v_2 v_5) = 36$. So, $W(v_1 v_2) + W(v_4 v_5) > W(v_1 v_4) + W(v_2 v_5)$.

Look at the picture below.



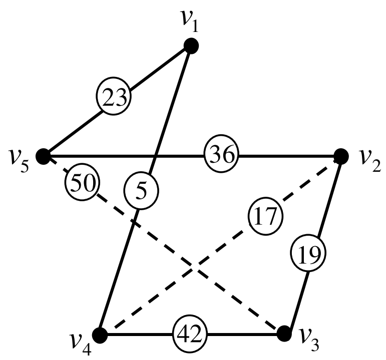
C



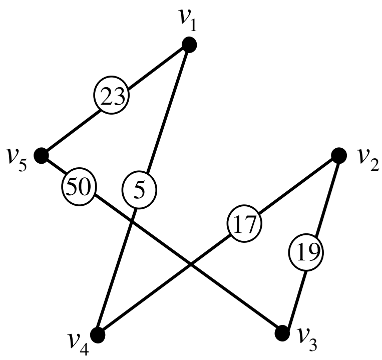
C'

So, we modify C to $C' = (v_1, v_4, v_3, v_2, v_5, v_1)$, and $W(C') = 125$. Now in C' look at the edges $v_4 v_3$ and $v_2 v_5$. Here $W(v_4 v_3) = 42, W(v_2 v_5) = 36$ and $W(v_4 v_2) = 17, W(v_3 v_5) = 50$. Thus $W(v_4 v_3) + W(v_2 v_5) > W(v_4 v_2) + W(v_3 v_5)$.

Look at the picture below.



C'



C''

Therefore, we transform C' into $C'' = (v_1, v_4, v_2, v_3, v_5, v_1)$. Here $W(C'') = 114$ which is smaller than $W(C')$.

UNIT 6

COLOURING OF GRAPHS

Structure **Page No**

6.1 INTRODUCTION

You must have seen political maps of India with different states coloured differently to distinguish between them. Have you ever wondered what is the minimum number of colours required to colour the map so that any two states with a common boundary are given two different colours? This problem of finding the minimum number of colours needed to colour a given map is called the map colouring problem.

We can formulate the problem in terms of graph theory. We can construct a graph in such a way that each state of India corresponds to a vertex of a graph and if two states are adjacent, the corresponding vertices are also adjacent. So, we have to colour the vertices of the graph in such a way that any pair of adjacent vertices have different colours. In the map colouring problem, we ask for the minimum number of colours needed to carry out such a colouring.

A proper vertex colouring is an assignment of colours to the vertices of graph such that adjacent vertices are assigned different colours. In this unit, we shall consider a number of problems involving the colouring of vertices or edges of a graph.

Vertex colouring is discussed in Section 6.2. This section also deals with the notion of chromatic number of a graph, and some bounds on chromatic number. Loops and multiple edges make no impact on vertex colouring. So in this unit only simple graphs are considered. In Sec. 6.3, we study line graphs and edge coloring. In Sec. 6.5, we end our unit with a brief discussion of applications of colouring.

Objectives

After studying this unit, you should be able to:

- compute the chromatic number of small graphs;
- compute some upper and lower bounds for the chromatic number of a graph;
- describe edge-colouring of a graph, and compute is the edge-chromatic numbers of shall graphs;
- state and use Books' theorem;
- apply a greedy coloring algorithm;
- explain and apply Mycielski's construction;
- state 5-colours theorem;
- apply the concept graph coloring to solve many problems like time tabling, channel assignment, routing and scheduling.

6.2 VERTEX- COLOURING

In this section, we start our discussion with vertex colouring. In the subsection 6.2.1, we define vertex colouring and give some examples. In subsection 6.2.2, we will prove some simple results on the bounds on the minimum number of colours needed to colour the vertices of a given graph.

6.2.1 Definition and Examples

We shall begin with an example.

Example 1: (Storing Chemicals): A chemical manufacturer wishes to store chemicals in a warehouse. Some chemicals react violently when in contact with each other, and the manufacturer decides to divide the warehouse into a number of areas so as to separate dangerous pairs of chemicals. In the following table, an asterisk indicates those pairs of chemicals that must be kept apart. How to find the smallest number of areas needed to store these chemicals safely?

Table 1

	a	b	c	d	e	f	g
a	_	*	*	*	_	_	*
b	*	_	*	*	*	_	*
c	*	*	_	*	_	*	_
d	*	*	*	_	_	*	_
e	_	*	_	_	_	_	_
f	_	_	*	*	_	_	*
g	*	*	_	_	_	*	_

Solution: Let us represent the situation by a graph (see Fig. 1(a)). Here, vertices are chemicals, and the edges between two chemicals indicate pairs of chemicals that must be kept apart.

Here the problem is to determine the smallest number of areas needed to store the chemicals safely. We note that the chemicals a, b, c and d must be in separate areas, so at least four areas are necessary. To indicate this we assign numbers $\boxed{1},\boxed{2},\boxed{3},$ and $\boxed{4}$ to a,b,c and d, respectively. We call these numbers "colours" shown next to the vertex label in the graph in Fig. 1(b). Now we can assign colour 1 to e as a and e are nonadjacent. Similarly, f gets colour 2, and g gets colour 3.

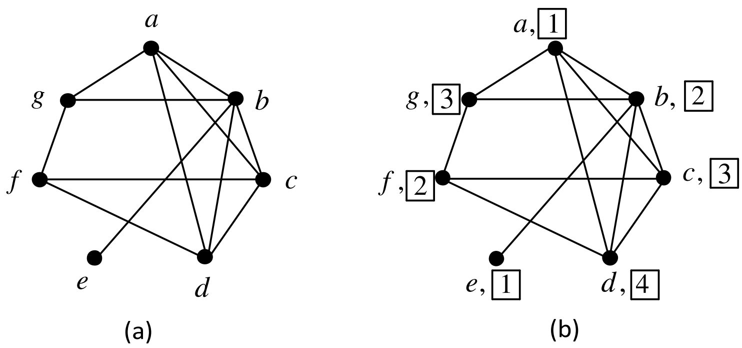


Fig. 1

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Of Course, other colour assignments are also possible. What we have shown is just a way of classifying vertices of the graph. Here we get four classes namely $\{a,e\},\{b,f\},\{c,g\},\{d\}$.

Let us now pause, and have a formal definition before going further.

Definition: A ***k* - vertex-colouring** of a graph G is an assignment of k colours to each of the vertices of G in such a way that no two adjacent vertices have the same colour. A graph G is called ***k* – vertex-colourable** if there exists a $k-$vertex colouring of G. The minimum k such that G is k-vertex colourable is called the **vertex-chromatic number of *G* ,** usually denoted as $\chi(G)$.

In this section, we are discussing only vertex-colouring. So, we will use the terms '$k-$colouring', and '$k-$colourable' respectively. We will say that a graph is $k-$chromatic if it has chromatic number k.

Note that in Fig. 1, we required only four colours, and so we could have used actual names of colours such as red, green, blue and orange. But this practice is not feasible for large graphs. Also the numbers $1,2,3,$ etc are used often for vertex labelling. Therefore we shall use $\boxed{1},\boxed{2},...,\boxed{3},$ etc for colours.

A colouring that uses the minimum possible number of colours is called an **optimal colouring**.

Let us now look at some examples.

Example 2: Colour the graphs in Fig. 2 with the minimum possible number of colours, and find the chromatic numbers of these graphs.

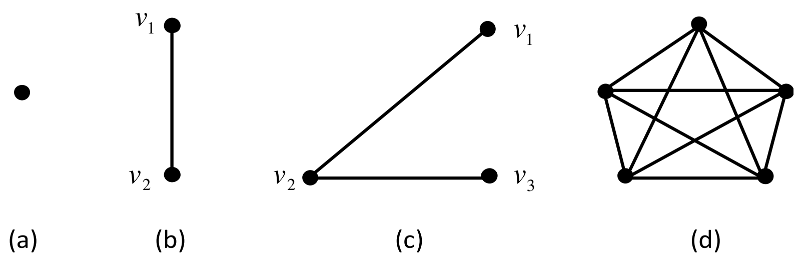


Fig. 2

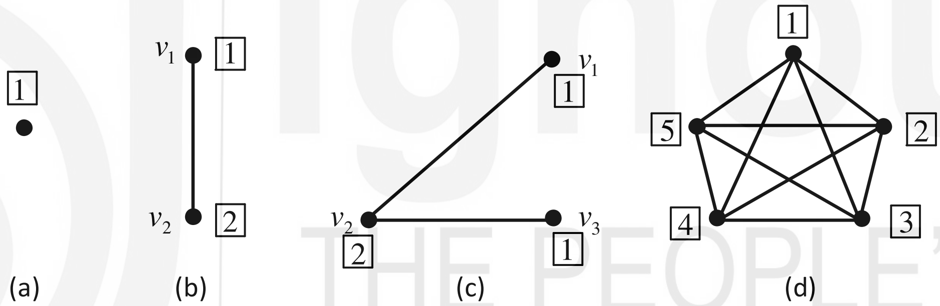
Solution: In Fig. 3(a), K_1, has just one vertex. Let us colour this with $\boxed{1}$. Thus, this graph is 1-colourable and its chromatic number is 1.

In Fig. 2(b), the graph is K_2 which has two adjacent vertices. We assign $\boxed{1}$ to the vertex v_1 and $\boxed{2}$ to the vertex v_2. Thus, we have a 2-colouring. Is there a 1-colouring? No? Since two vertices are adjacent, we need two colours. Hence $\chi(K_2) = 2$.

In Fig. 2(c), the graph has three vertices, and we can colour them with three different colours. Can we also have a 2-colouring? Notice that, v_1 and v_3 are not adjacent, we can colour them with the same colour, say $\boxed{1}$. The vertex v_2 is adjacent to both v_1 and v_3. So, we cannot assign $\boxed{1}$ to v_2. Let us assign $\boxed{2}$ to v_2. So, we have a 2-colouring. Since we cannot have a 1-colouring,this graph has chromatic number 2.

The graph in Fig. 2 (d), is K_5. Here, every two vertices are adjacent, so we need as many colours as there are vertices, that is, we need five colours. So, K_5 has chromatic number 5, i.e., $\chi(K_5) = 5$.

Thus, the vertex-colourings of the graphs of Fig. 2 are shown below.



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Remark 1: In the above example, we have seen that the chromatic number of K_1 is 1. More generally, if a graph consists of isolated vertices, its chromatic number is 1. Conversely, if the chromatic number of a graph is 1, it consists of isolated vertices.

Also, we saw that the chromatic number of K_5 is 5. More generally, the chromatic number of K_n is n, i.e. , $\chi(K_n) = n$, because every two vertices are adjacent in K_n.

Let us consider another example.

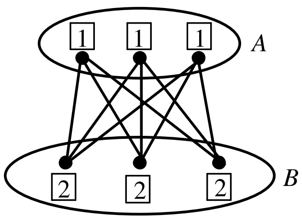

Fig. 3

Example 3: Find the chromatic number of a non-null bipartite graph.

Solution: Let G be a non-null bipartite graph. Then the vertex set of G can be partitioned into two non-empty disjoint subsets A and B such that any two vertices in a given set are non-adjacent. We get a 2-colouring of G by assigning $\boxed{1}$ to all the vertices in A and $\boxed{2}$ to all the vertices in B. (This is illustrated in a particular case in Fig. 3). Further, note that, since A and B are non empty and since the edge set of G is non empty, at least one vertex in A is adjacent to a vertex in B and these two vertices must have different colours. So, we cannot manage with less than two colours. So, $\chi(G) = 2$.

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Remark 2: We saw in Example 3 that the chromatic number of a non-null bipartite graph is 2. The converse is also true. In a 2- colourable graph G , we can partition the edge set of G into two non empty sets A and B as follows:

$A = \{v \in V(G) | v$ is assigned the colour $\boxed{1}\}$

$B = \{v \in V(G) | v$ is assigned the colour $\boxed{2}\}$

By the definition of colouring no two vertices in A are adjacent, and similarly for B . Since A and B are disjoint, G is bipartite by definition.

Here are some exercises to test your understanding.

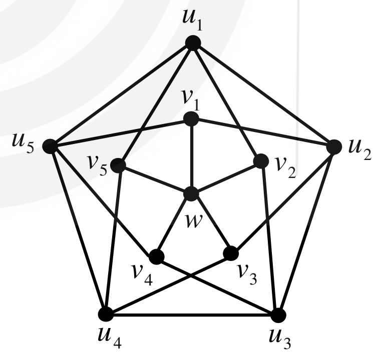
E1) What is the chromatic number of a tree with at least two vertices?

E2) What is the chromatic number of an even cycle $C_{2n}, n \geq 2$?

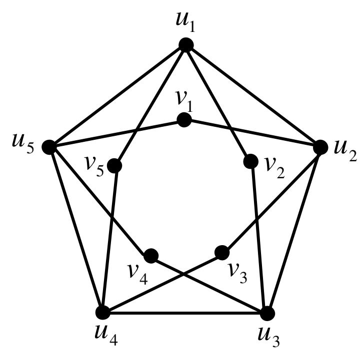
E3) Is an odd cycle $2-$colourable? What is $\chi(C_{2n+1}),$ for $n \geq 1$?

If a graph is k -colourable, are all its subgraphs k -colourable? Let us see. Let G be a k -colourable graph and H be any subgraph of G. We assign to each vertex of H the same colour that we assigned to it, considered as a vertex of G . If two vertices are non-adjacent in G , they are non-adjacent in H and therefore this gives a colouring of H . In other words, $\chi(H) \leq k = \chi(G)$ for every subgraph H of G . We can also restate this statement in the following form. If a graph G has a subgraph H with chromatic number k , the chromatic number of G must be at least k . This fact helps us in finding the chromatic number of a graph sometimes. We illustrate this in the next example.

Example 4: Find the chromatic number of the Grötzsch graph drawn below.



Solution: If we remove the vertex w from the graph given above, we get the following subgraph H.



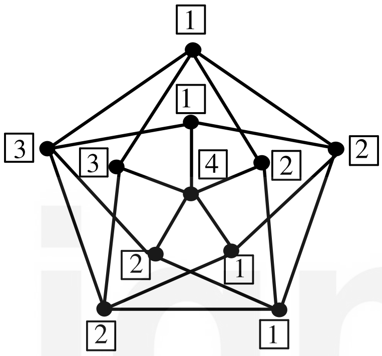
H

Since $\{u_1, u_2, u_3, u_4, u_5\}$ induces an odd cycle, it requires three colours, say $\boxed{1}$, $\boxed{2}$ and $\boxed{3}$. We assign these colours as follows:

$$u_1 \to \boxed{1}, u_2 \to \boxed{2}, u_3 \to \boxed{1}, u_4 \to \boxed{2}, u_5 \to \boxed{3}$$

Note that for each i, $v_i \not\leftrightarrow u_i$. Therefore, for each i, v_i can get the same colour as u_i. Thus all the three colours are used on $\{v_1, v_2, v_3, v_4, v_5\}$. Now w is adjacent to each of $\{v_1, v_2, v_3, v_4, v_5\}$. So w cannot be coloured with any of the colours, $\boxed{1}$, $\boxed{2}$ and $\boxed{3}$. Therefore, w needs another colour, say $\boxed{4}$. Consequently, the chromatic number of the Grötzsch graph is at least 4.

Now a 4-colouring is shown below.



Thus, the chromatic number of the Grötzsch graph is 4.

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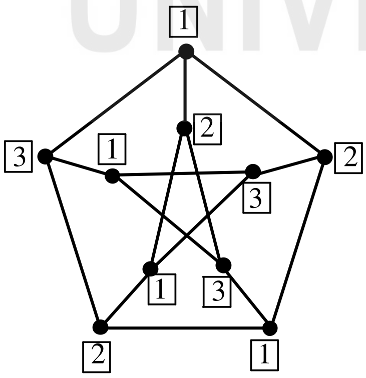
Example 5: Find the chromatic number of the Petersen graph.

Solution: Note that the Petersen graph contains a 5-cycle which is an odd cycle. So, the chromatic number of the Petersen graph is at least 3. A 3-colouring of the Petersen graph is shown below.



This proves that the chromatic number of the Petersen graph is 3.

***

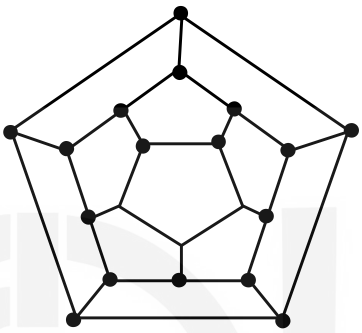
In the above examples and problems, we have used the fact that if H is a subgraph of a graph G then $\chi(H) \le \chi(G)$. This leads to the following result.

Proposition 1: For every graph G, $\chi(G) \ge \omega(G)$.

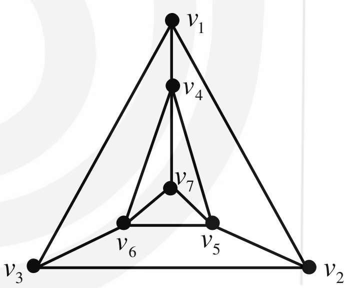
Proof: Try yourself.

From Proposition 1 it follows that if G contains a copy of K_n, then $\chi(G) \geq n$. However, the converse is not true, i.e. if a graph has chromatic number at least n, it need not have a subgraph K_n. Peterson graph provides a counter-example for this. As we have seen, the chromatic number of Peterson graph is 3 and it does not contain a subgraph K_3. In 1955, Mycielski proved that, for any interger k, there exists a k-chromatic graph without triangles. The proof of this statement is beyond the scope of this course. However, it is not difficult to prove the much weaker result that if the chromatic number of a connected graph is greater than 2, it contains an odd cycle. We leave this as an exercise for you, along with some more exercises, to test your understanding of the material we have covered so far.

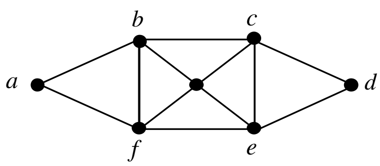
E4) Find a 3-colouring of the graph given below. Also find its chromatic number.



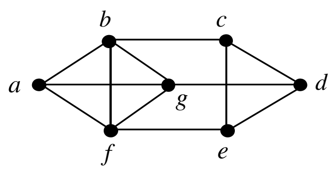
E5) Find the chromatic number of the following graph.



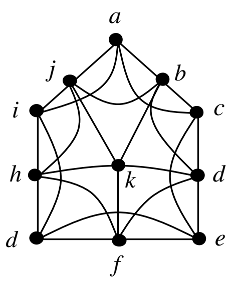
E6) Determine the chromatic number of each of the following graphs:



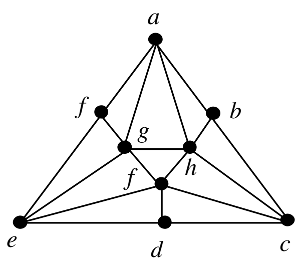
(a)



(b)



(c)



(d)

E7) Show that if $\chi(G) \geq 3$ for a graph $G,$ then G contains an odd cycle. Does the converse hold?

E8) Construct a graph, different from a complete graph, which has chromatic number 5.

Recall that a 2-colourable graph is bipartite. To prove this, we had put all the vertices having the same colour in a single set. There are two colours and so we got two disjoint subsets of the vertex set.

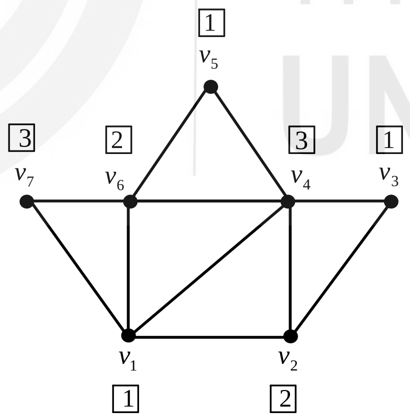
What does happen when a graph G is k-colourable? Do we get k mutually disjoint subsets of $V(G)$? Let us see the following definition.

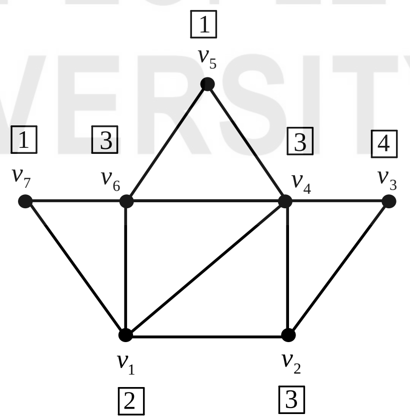
Definition: For a k – colouring of a graph $G,$ consider the set $S_i = \{x \in V(G) | x$ is assigned the colour $\boxed{i}\}$, for $1 \leq i \leq k$. Clearly, $S_i \cap S_j = \emptyset$ for every $i \neq j$ and $V(G) = S_1 \cup ... \cup S_k$. If $\chi(G) = k,$ each of the k colours is assigned to some vertex. So none of these subsets is empty. Therefore, we get a partition of the vertex set $V(G)$ into k mutually disjoint non-empty subsets. The subsets $S_1,..., S_k$ are called **the colour classes** of G given by the colouring.

So, the colour classes of a 2-colourable graph give a bipartition of the vertex set of the graph, making it bipartite.

Let us now look at some examples of colour classes.

Example 6: Below given are two different colourings of the same graph. Write the colour classes in for each colouring. Also tell which colouring is optimal.





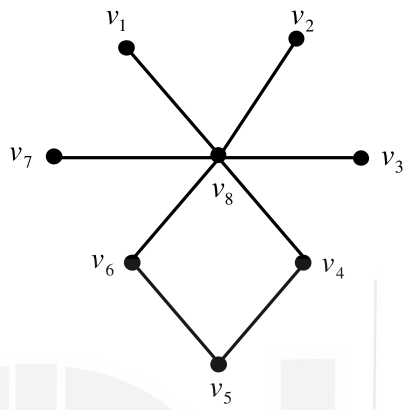
Solution: The colouring on the left assigns colour $\boxed{1}$ to $v_1, v_5.$ So, let $S_1 = \{v_1, v_5\}.$ The colour $\boxed{2}$ is assigned to v_2 and $v_6.$ So, let $S_2 = \{v_2, v_6\}.$ The colour $\boxed{3}$ is assigned to v_4 and $v_7,$ and so let $S_3 = \{v_4, v_7\}.$ Thus this colouring has 3 colour classes namely S_1, S_2 and $S_3.$ Similarly, the colour classes of the colouring on the right are $\{v_5, v_7\}, \{v_1\}, \{v_2, v_4, v_6\}$ and $\{v_3\}.$ Note that the graph has a triangle, and hence no colouring uses fewer than 3 colours. Since the colouring on the left uses 3 colours, it is an optimal colouring.

***

We have seen that any coloring of a graph generates a set of colour classes of the graph. Since all the vertices in a colour class get the same colour, it follows that **each colour class** is an independent set.

An independent set is called **maximal** if it is not contained in any other independent set. The number of vertices in a largest independent set of a graph G, is called the **independence number** of G, and it is denoted by $\alpha(G)$.

Example 7: Find all the maximal independent sets in the following graph G. Also, find $\alpha(G)$.



G

Solution: We can see that G has the following maximal independence sets:

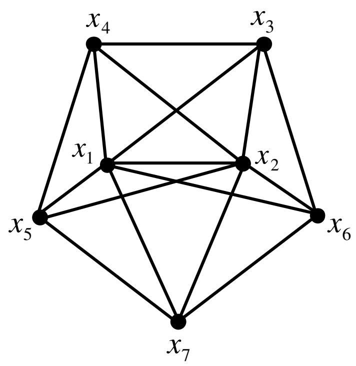
$\{v_8, v_5\}, \{v_5, v_1, v_2, v_3, v_7\}, \{v_1, v_2, v_3, v_4, v_6, v_7\}$

We check that $\{v_8, v_5\}$ is a maximal independent set. This is easy to see because all the other remaining vertices of G are adjacent to one of these two vertices. So, if any more vertices are added, the resulting set will no longer be an independent set. You can check that the other two sets are also maximal independent sets in the same way. Since the largest independent set has 6 vertices, $\alpha(G) = 6$.

***

Now try the following exercises:

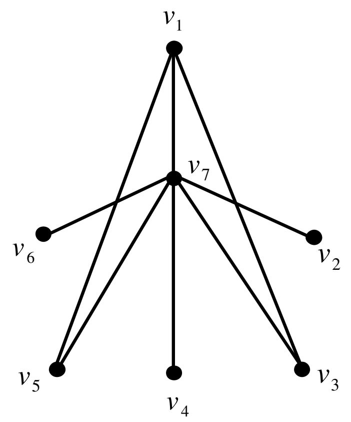
E9) Colour the following graph in two different ways and give the colour classes in each of the cases.



E10) Any two maximal independent sets are disjoint. True or false?

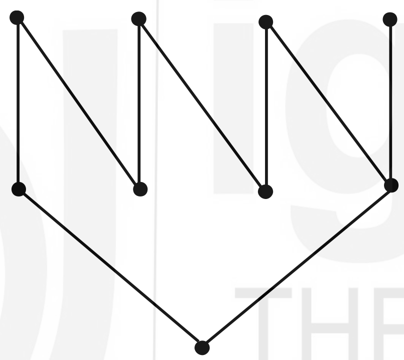
E11) What are the values of $\alpha(K_n)$ and $\alpha(C_n)$ for all n?

E12) Find an independent set of cardinality 4 in the graph given below:

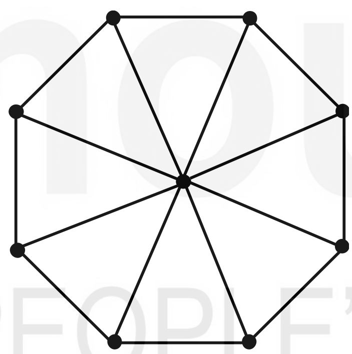


E13) Let $S_1, S_2, \ldots S_k$ be the colour classes of a graph G with respect to an optimal colouring. Show that $\alpha(G) \geq \max_{1 \leq i \leq k} |S_i|$.

E14) Find the independence number of the graphs given below.



i)

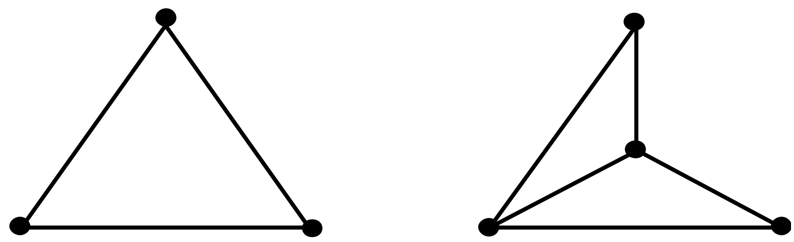


ii)

6.2.2 Bounds for Chromatic Numbers

In this sub-section we will prove some bounds for the chromatic number of a graph in terms of maximum degree. For this, we need the concept of k-critical graphs.

We will introduce you to this concept through an example. Consider the graph K_4. If we remove a vertex or an edge, we get a graph isomorphic to one of the graphs given below.



Both the graphs above have chromatic number 3, as you can easily verify.

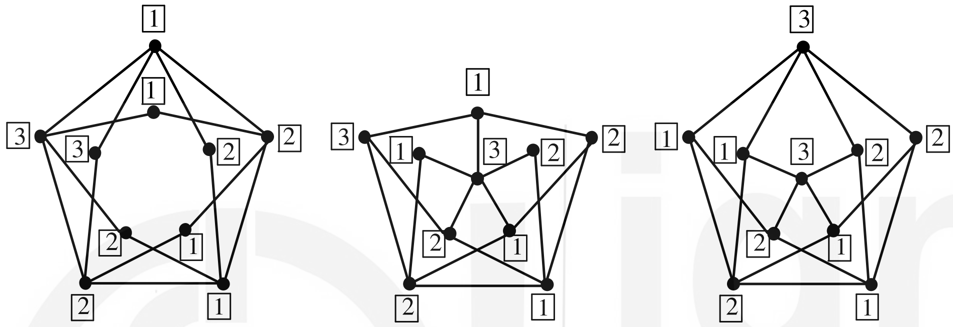
Also, any other proper subgraph of K_4 is contained in one of these graphs. So, the chromatic number of any subgraph of K_4 is strictly less than that of K_4 which is 4. This leads to the following definition.

Definition: A graph G is said to be k**-critical** or **critically** k**-chromatic** if $\chi(G) = k$ and $\chi(H) < k$ for every proper subgraph H of G.

Thus, in the discussion above we have shown that K_4 is 4-critical. Let us look at one more example.

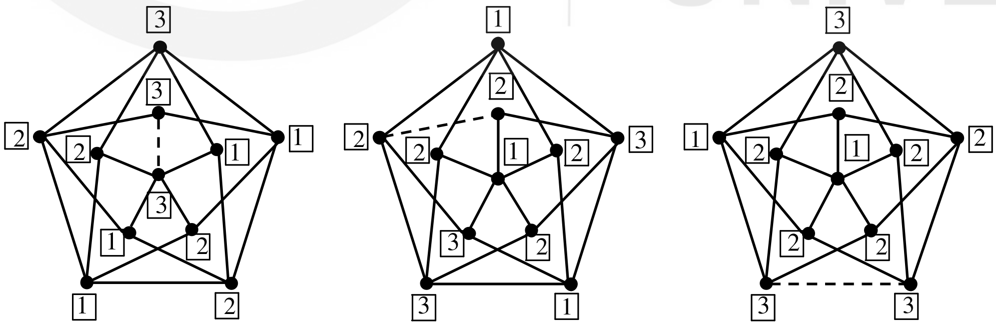
Example 8: Show that the Grötzsch graph is 4-critical.

Solution: Let G be the Grötzsch graph. From Example 4, $\chi(G) = 4$. Let us remove a vertex of this graph. Depending on the vertex, we get a subgraph isomorphic to one of the following three graphs:



From the colouring given in these graphs, it is clear that each of these graphs is 3-colourable. Moreover, all of them contain 5-cycles, that is, they are not 2-colourable. Thus, their chromatic number is 3, which is smaller than $\chi(G)$. This means that $\chi(G-v) < \chi(G)$, for every vertex v of the graph G.

Now, if we remove just one edge of G, without removing any vertex, we get a subgraph isomorphic to one of the graphs given below. The dotted lines indicate the edges we have deleted.

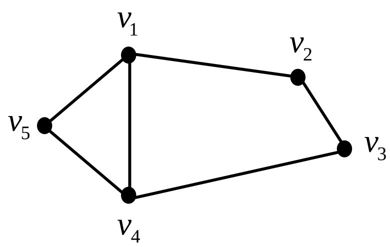


From the colouring of above graphs, it is clear that each of them is 3-colourable. Moreover, these graphs contain 5-cycles and hence have chromatic number 3. Thus, $\chi(G-e) < \chi(G)$, for every edge e of G. But then, every proper subgraph of G is, in fact, a subgraph of one of the above six graphs. Thus, $\chi(H) < \chi(G)$, for every proper subgraph H of G. So, the Grötzsch graph is 4-critical.

***

Now, consider a graph with chromatic number k. Need it be k-critical? The following example will help you answer this question.

Example 9: Show that the graph given below is 3-chromatic, but it is not 3-critical.



Solution: We can assign $\boxed{1}$ to v_1 and v_3, $\boxed{2}$ to v_2 and v_4, $\boxed{3}$ to v_5. This gives us a 3-colouring of the graph. This graph is 3-chromatic because it contains a 3-cycle (v_1, v_4, v_5, v_1). If we remove the vertex v_2, the resulting graph still has chromatic number 3 because it still contains the 3-cycle (v_1, v_4, v_5, v_1). So, G has a subgraph which has the same chromatic number as G. So, it cannot be 3-critical.

***

Now we know that a graph with chromatic number k needn't be k-critical, you may wonder if it will contain a k-critical subgraph. The following result tells us that this is true.

Theorem 1: Let G be a graph with chromatic number k. Then, it has a subgraph which is k-critical.

Proof: Consider a graph G with $\chi(G) = k$. If G is k-critical, we are done. Otherwise, G has a proper subgraph H_1 with $\chi(H_1) = k$. If H_1 is k-critical, we are done. Otherwise, H_1 has a proper subgraph H_2 with $\chi(H_2) = k$. We repeat the process. In the worst case, we will be left with a k-chromatic subgraph of G on k vertices. If we remove any vertex from this graph, we will get a graph on $k-1$ vertices which is $(k-1)$-colorable. So, the k-chromatic subgraph on k vertices that we have obtained is k-critical. ■

We now discuss an example that illustrates Theorem 1.

Example 10: Find a 3-critical subgraph of the graph given in Example 9.

Solution: On removing the vertices v_2 and v_3, we get a graph isomorphic to K_3. This is 3-critical.

***

Here are some exercises for you to try!

E15) Every acyclic graph is 2-critical. True or false? Justify.

E16) Show that K_n is an n-critical graph.

E17) Show that the Petersen graph is not 3-critical.

E 18) Find a 3-critical subgraph of the Petersen graph.

Let us now make a table of the values of $\chi(G)$ and $\Delta(G)$ for some of the examples we have discussed so far to see if we can find a relationship between these two quantities.

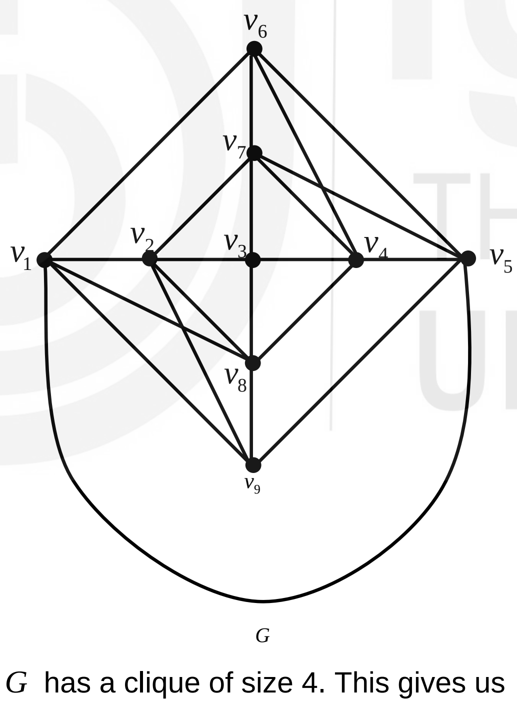
G	$\chi(G)$	$\Delta(G)$
Grötzsch graph	4	5
Petersen graph	3	3
Dodecahedron	3	3
C_5	3	2
K_4	4	3
K_5	5	4

Observe that, except for C_5, K_4 and K_5, all other graphs satisfy the relation $\chi(G) \leq \Delta(G)$. In 1941, R.I. Brooks proved the following result.

Theorem 2 (Brooks' Theorem): Let G be a connected graph which is neither an odd cycle nor a complete graph. Then, $\chi(G) \leq \Delta(G)$.

We will not prove Theorem 2 in this course. Let us now illustrate the application of Brook's Theorem through an example.

Example 11: Find the chromatic number of the following graph G.



G

Solution: Note that G has a clique of size 4. This gives us $\chi(G) \geq 4.$ We have $\Delta(G) = 5.$ Therefore, by Brooks' Theorem, $\chi(G) \leq 5.$ Thus we have either $\chi(G) = 4$ or $\chi(G) = 5.$ Now we shall show that $\chi(G) \neq 4.$ Consider the cycle $C = (v_1, v_2, v_3, v_4, v_5, v_1).$ Since C is an odd cycle, C requires 3 colours. Let $S = \{v_6, v_7, v_8, v_9\}.$ Note that each of the vertices v_7 and v_8 has 4 neighbours in $C.$ In fact,

$$V(C) \subseteq N_G(v_7) \cup N_G(v_8).$$

This means v_7 and v_8 must get a colour different from the colours of the vertices of $C.$ But $v_6 \leftrightarrow v_7$ and $v_8 \leftrightarrow v_9.$ So the vertices in S must get at least two colours, different from those of the vertices in $C.$ Consequently,

$\chi(G) \neq 4.$ Therefore, $\chi(G) = 5.$

***

Remark 3: The bound given by Brooks' Theorem may not be as good as it was in Example 11. For example, in the case of $K_{1,n}$, the maximum degree is n, but the chromatic number is just 2. So the difference $\chi(K_{1,n}) - \Delta(K_{1,n}) = n-2$, is large when n is large.

Now consider the following lemma, which gives us an important property of k-critical graphs.

Lemma 1: If G is a k-critical graph then $\delta(G) \geq k-1.$

Proof: If possible, let G be a k-critical graph with $\delta(G) < k-1.$ Let $v \in V(G)$ be such that $\delta(G) = d_G(v).$ Since G is k-critical, $\chi(G-v) \leq (k-1).$ That is, $G-v$ has a $(k-1)$-colouring. Since $d_G(v) < (k-1),$ the vertex v is adjacent to fewer than $k-1$ vertices. So, there is at least one colour, among the $k-1$ colours, that is not assigned to any of the vertices adjacent to $v.$ We can assign this colour to v to get a $k-1$ colouring of $G.$ This contradicts the fact that $\chi(G) = k.$ Thus our assumption is wrong, and therefore, $\delta(G) \geq (k-1).$ ■

Corollary 1: Every k-chromatic graph G has at least k vertices of degree $\geq (k-1)$.

Proof: Let G be a k-chromatic graph, that is, $\chi(G) = k$. Let H be a k-critical subgraph of G. Thus, $|V(H)| \geq k$ and $\delta(H) \geq (k-1)$. This means that every vertex v of H satisfy the property that $d_G(v) \geq d_H(v) \geq \delta(H) \geq (k-1)$. There are at least k such vertices. This proves the result. ■

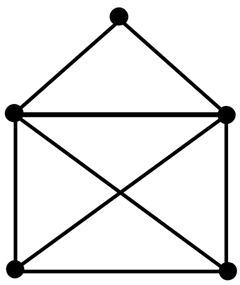
We shall now prove a weaker bound on the chromatic number. It is stated below.

Theorem 3: For every graph $G,$ $\chi(G) \leq \Delta(G)+1.$

Proof: Let G be a k-chromatic graph. Using Corollary 1 to Lemma 1, choose a vertex $v \in V(G),$ such that $d_G(v) \geq (k-1)$. But then, $\Delta(G) \geq d_G(v) \geq (k-1),$ that is $\chi(G) = k \leq \Delta(G)+1.$ ■

Now try the following exercise.

E19) Is the following graph 4-critical? Justify.

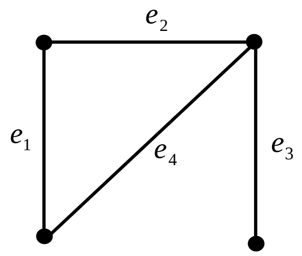


6.3 LINE GRAPHS AND EDGE-COLOURING

We shall now discuss the concept of edge-colouring, which is analogous to vertex-colouring. But, first, let us introduce what we mean by "line graph".

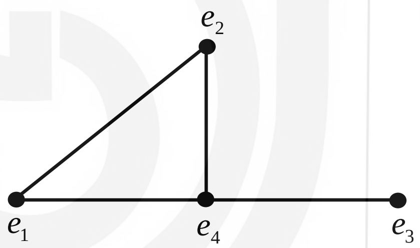
Definition: The **line graph** of a graph G is the graph $L(G)$ with $V(L(G)) = E(G)$, and any two vertices in $L(G)$ are adjacent if they are incident on the same vertex of G.

For example, consider the graph G given below.



G

Here $E(G) = \{e_1, e_2, e_3, e_4\}$. So, $V(L(G)) = \{e_1, e_2, e_3, e_4\}$. Since e_1 and e_2 are incident on the same vertex of G, $e_1 e_2 \in E(L(G))$. Similarly, the other edges of $L(G)$ are $e_1 e_4, e_2 e_3$, and $e_2 e_4$. Thus, we can draw $L(G)$ as below.



$L(G)$

Incidentally, in this case we have $L(G) \cong G$. However, this does not happen in general.

Let us now see how the sizes of a graph and its line graph are related.

Example 12: Let G be a graph with size m. If m' is the size of $L(G)$, then show that $m + m' = \frac{1}{2} \sum_{v \in V(G)} d_G(v)^2$.

Solution: Note that every vertex v of G contributes $\binom{d_G(v)}{2}$ edges of $L(G)$.

Therefore, $L(G)$ has $\sum_{v \in V(G)} \binom{d_G(v)}{2}$ edges. That is,

$$m' = \sum_{v \in V(G)} \binom{d_G(v)}{2}$$

$$=\frac{1}{2}\left[\sum_{v\in V(G)} d_G(v)^2 - \sum_{v\in V(G)} d_G(v)\right]$$

$$=\frac{1}{2}\sum_{v\in V(G)} d_G(v)^2 - m.$$

Thus, $m+m'=\frac{1}{2}\sum_{v\in V(G)} d_G(v)^2.$

***

Consider an edge $e=uv$ of a graph G. Then e is a vertex of $L(G)$. We can determine the degree of e in $L(G)$ in terms of the degrees of u and v in G. Each edge of G incident on u, apart from e, represents a vertex in $L(G)$ adjacent to e. Thus u contributes $d_G(u)-1$ towards the degree of e. Similarly, v contributes $d_G(v)-1$ towards the degree of e. Therefore, $d_{L(G)}(e)=d_G(u)+d_G(v)-2.$ This is an important relation. Let us now consider the following example.

Example 13: Show that if G is a regular graph, then so is $L(G)$. What about the converse?

Solution: Let G be a k-regular graph. Pick any vertex e in $L(G)$. Then $e=uv$ for some $u,v\in V(G)$. Now from the relation given above, we have

$$d_{L(G)}(e)=d_G(u)+d_G(v)-2=k+k-2=2k-2.$$

This proves that $L(G)$ is $(2k-2)$-regular.

The converse statement is : "If $L(G)$ is regular, then so is G." However, the converse is not true. A counter-example is $K_{1,3}$. You can see that $L(K_{1,3})=K_3$, which is regular, but $K_{1,3}$ is not.

***

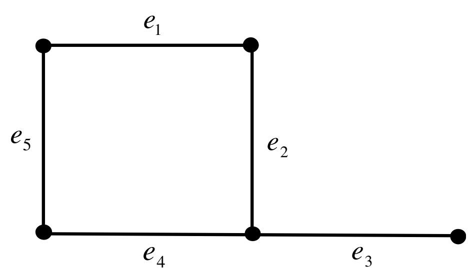
Now let us define edge-colouring.

Definition: A k **-edge-colouring** of a graph G is an assignment of k colours to all the edges of G in such a way that no two edges incident on the same vertex have the same colour. A graph G is called k **-edge-colourable** if there exists a k-edge-colouring of G. The minimum k such that G is k-edge-colourable is called the **edge-chromatic number** of G, and is usually denoted by $\chi'(G)$.

Symbolically,

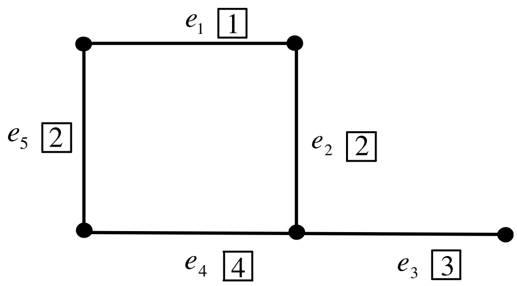
$$\chi'(G)=\min\,\{k \mid G \text{ is } k\text{-edge-colourable.}\}$$

Consider the following graph, for instance.



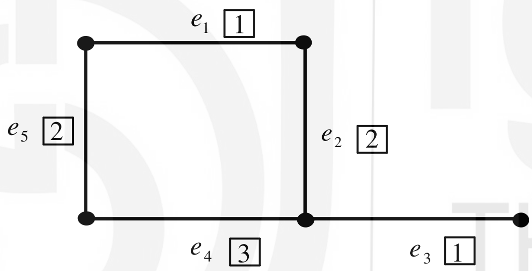
G

Let us begin by assigning e_1 the colour $\boxed{1}$. Note that the edges e_2, e_3 and e_4 are incident on the same vertex, and hence each of them must get a different colour. Let $\boxed{2}$, $\boxed{3}$ and $\boxed{4}$ be the colours assigned to them in order. The remaining edge e_5 shares endpoints with the edges e_1 and e_4. Therefore, $\boxed{e_5}$ must get a colour distinct from $\boxed{1}$ and $\boxed{4}$. We can assign $\boxed{2}$ to e_5. Thus we get the following 4-edge-colouring of G.



Is the 4-edge-colouring of G shown above minimal? i.e., does it use the minimum number of colours?

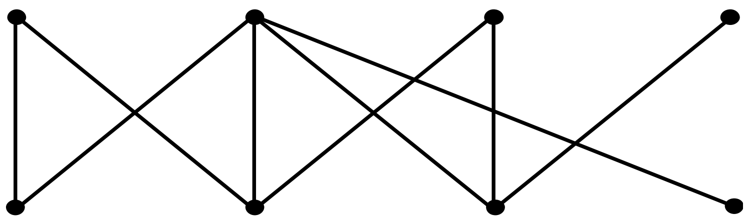
The answer is no. A 3-edge-colouring is shown below.



However, it is evident that there is no 2-edge-colouring of G, because in any edge-colouring the edges e_2, e_3 and e_4 must get three different colours. Thus, we have $\chi'(G) = 3$.

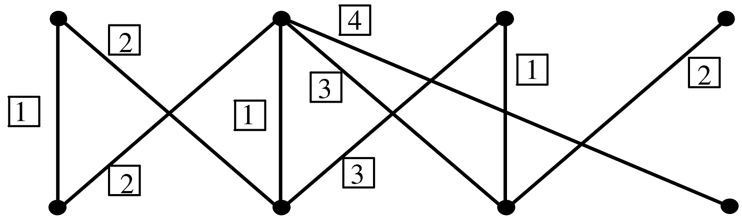
Remark 4: In fact, for any graph G, $\chi'(G) \geq \Delta(G)$. What is more interesting is the class of graphs for which $\chi'(G) = \Delta(G)$. Let us consider the following examples.

Example 14: Find $\chi'(G)$, for the graph G given below.



G

Solution: Since $\Delta(G) = 4$, from Remark 4, it follows that $\chi'(G) \geq 4$. A 4-edge-colouring of G is given below.



Therefore, $\chi'(G) = 4.$

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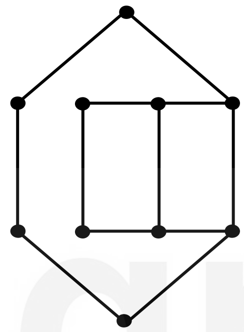
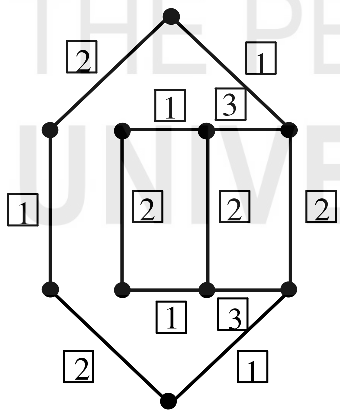
Example 15: What is the edge-chromatic number of the following graph G?



G

Solution: We have $\Delta(G) = 3.$ So, $\chi'(G) \geq 3.$ Now look at the following 3-edge-colouring of $G.$



Thus, $\chi'(G) = 3.$

***

You might have observed that the graphs given in Examples 14 and 15 are bipartite. The following theorem tells us that if a graph is bipartite, then its edge-chromatic number is equal to its maximum degree.

Theorem 4: If G is a bipartite graph, then $\chi'(G) = \Delta(G).$

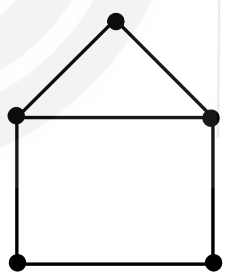
Proof: We shall apply the Principle of Mathematical Induction, on the size of a bipartite graph. When G is a bipartite graph with just one edge, the result holds. So, let us assume that the result holds for all bipartite graphs with size smaller than m for some $m > 1.$ Now consider a bipartite graph G with size $m.$ Let $\Delta(G) = k.$ Pick an edge $e = uv$ of $G.$ Let $H = G - e.$

The induction hypothesis, then, implies that $\chi'(H) = \Delta(H)$. But $\Delta(H) \le \Delta(G) = k$. So, $\chi'(H) \le k$. Consider any k-edge-colouring of H. Note that both u and v have degree at most $k-1$ in H, and there are k colours on the edges of H. So there is some colour i missing at u, meaning that no edge incident on u has colour i. But i cannot be missing at v, otherwise we could assign i to the edge e to get a k-edge-colouring of G. Similarly, there is some edge j missing at v, but present at u. Clearly, $i \ne j$. Now define H_{ij} to be the subgraph of H that is **induced by the edges** coloured i or j. Then there are two cases—either both u and v lie in the same component of $H_{i,j}$ or do not.

When both u and v are in the same component of $H_{i,j}$, let P be a (u,v)-path in that component. Then P cannot have even length, because otherwise $P+e$ would become an odd cycle which would contradict the fact that G is bipartite. Therefore, P has odd length. This means, the first and the last edges of P have the same colour. But this is also a contradiction because the only possible colour of the last edge of P is i, and $i \ne j$. Therefore, u and v lie in distinct components of $H_{i,j}$.

Now we can interchange the colours i and j on the component of $H_{i,j}$ that contains u. This makes the colour j missing on u. Since j is already missing on v, we can assign j to e to get a k-edge-colouring of G. This gives $\chi'(G) \le k$. Since $\chi'(G) \ge k$ is always true, we have $\chi'(G) = k$. Thus the result holds for all the bipartite graphs. ■

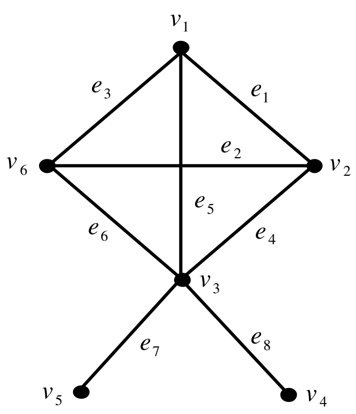
It is important to note that the condition of Theorem 4 that G is bipartite is sufficient only. There are plenty of graphs which are not bipartite, yet satisfy the conclusion of Theorem 4. One such graph is given below.



Just verify that for this graph, $\chi'(G) = \Delta(G)$ holds. Now let us see how large can $\chi'(G)$ be from $\Delta(G)$, for any graph G. The following theorem due to Vising and Gupta settles this.

Theorem 5 (Vising-Gupta Theorem): For any graph G, $\chi'(G) \le \Delta(G) + 1$.

We shall not prove this theorem here. Let us consider a graph where $\chi'(G)$ achieves the upper bound $\Delta(G) + 1$.



G

Example 16: Find the edge-chromatic number of the graph G given in the right margin.

Solution: Note that $\Delta(G) = 5$. So, $5 \le \chi'(G) \le 6$. Now we shall show that $\chi'(G) \ne 5$. The edges incident on v_3 require 5 different colours. The edges

e_1, e_2, e_3 form a triangle, and hence they require 3 different colours. But every edge in $\{e_1, e_2, e_3\}$ is adjacent to some edge in $\{e_4, e_5, e_6\}$. Therefore, any edge-colouring of G uses at least 6 colours. Consequently, $\chi'(G) = 6.$

***

Now you should try the following exercises.

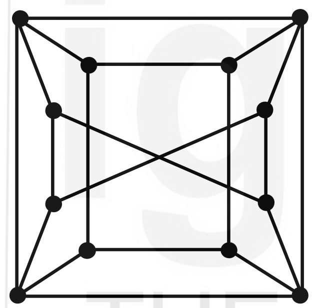
E20) If e is a cut-edge of a graph G, then e is a cut-vertex of $L(G)$. Prove or disprove.

E21) Find $L(P_n)$, $L(C_n)$ and $L(S_n)$ for all $n \geq 3.$

E22) Give an example of a graph G such that $m(L(G)) < m(G).$

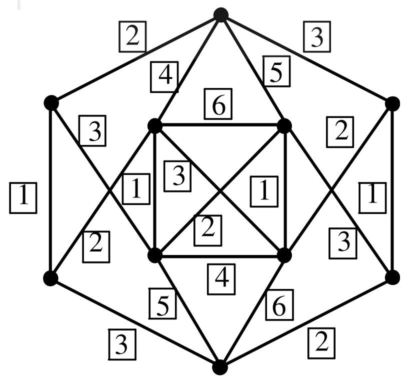
E23) Show that for any graph G, $\chi'(G) = \chi(L(G)).$

E24) Find the edge-chromatic number of the following graph.



E25) Show that for any (n,m)-graph G, $\chi'(G) \geq \dfrac{2m}{n}.$

E26) Does the edge-colouring shown in the following graph use the minimum number of colours? Justify your answer.



E27) For all $n \geq 1$, $\chi'(K_n) = n - 1.$ True or false? Justify.

E28) Show that for all $m, n \geq 1$, $\chi'(K_{m,n}) = \max\{m, n\}.$

Thus far we have discussed the concepts of vertex-colouring and edge-colouring, and certain results. The next we shall discus some applications of the concepts.

6.4 APPLICATIONS

In this section we shall discuss about some applications of graph colouring.

6.4.1 Timetabling Problem

Making of timetabling in schools, colleges or universities is, often, a difficult task as it involves a complicated process. Usually, a timetable is made for a week consisting of five or six working days. A teacher is supposed to teach for a fixed number of periods per week. Likewise, each class is supposed to be taught for a fixed number of periods per week. Assume that a school has m teachers say $t_1, t_2, \ldots t_m$, and n classes say, $c_1, c_2 \ldots, c_n$. Further, for each i, j, let p_{ij} be the number of periods the teacher t_i has to take for the class c_j. Thus a general timetable of the school can be represented by the following table.

	c_1	c_2	$\cdots$	c_n
t_1	p_{11}	p_{12}	$\cdots$	p_{1n}
t_2	p_{21}	p_{22}	$\cdots$	p_{2n}
$\vdots$	$\vdots$			
t_m	p_{m1}	p_{m2}	$\cdots$	p_{mn}

Equivalently, this information can also be represented by a bipartite multigraph with bipartition (X, Y), where $X = \{t_1, t_2, \ldots, t_m\}$, $Y = \{c_1, c_2, \ldots, c_n\}$ and there are p_{ij} edges with endpoints t_i and c_j. Note that in a period a teacher can teach at most one class. Similarly, in a period a class can be taught by at most one teacher. Thus, the timetable for one period seeks to find a largest set of pairwise nonadjacent edges. All such edges can be assigned the same colour, say 1. Then these edges can be removed, and in the remaining multigraph we can again find a largest set of nonadjacent edges. This new set can be assigned colour 2. We can continue this process as long as the remaining multigraph is non-null.

6.4.2 Channel Assignment

A cellular network (or mobile network) is a network distributed over land areas called “cells”, each served by at least one fixed-location transceiver known as a “cell site” or “base station”. A cellular network can be represented by a graph, where the vertices are the cells, and two vertices are adjacent if the corresponding cells are linked by radio signals. Normally, each cell uses a fixed set of radio frequencies, called channels which is different from the set of radio frequencies of the adjacent cells. Since the channels are limited, cells reuse the same channels. When two adjacent cells receive the same channel, network interference occurs. Channel assignment is the process that allocates cells to the channels of a cellular network. The main focus of the channel assignment problem is to find strategies that give maximum channel reuse without violating the interference constraints so that the blocking is minimal. There are different methods to solve this problem. One such method is vertex-colouring in which channels are treated as colours to be assigned to the cells.

6.5 SUMMARY

In this we have covered the following points.

1. Definition and examples of vertex-colouring

2. We have defined what is the chromatic number of a graph, and discusses some bounds on chromatic number.

3. We have defined the independence number of a graph.

4. We have discussed the concept of k -critical graphs and some of their properties.

5. The concept of line graphs is introduced along with the concept of edge-colouring

6. The statements of Brooks' Theorem, and Vising-Gupta Theorem are given.

7. We have finally discussed some applications of graph colouring such as time tabling and channel assignment problems.

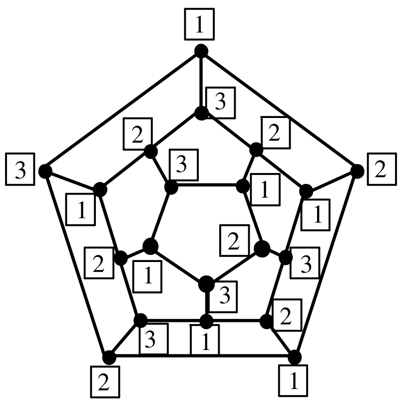
6.6 SOLUTIONS/ANSWERS

E1) A tree is a bipartite graph. Therefore, its chromatic number is at most 2. A tree with at least 2 leaves is connected, and hence requires at least 2 colours. Thus, a tree with at least two leaves is 2-chramatic.

E2) Even cycles are bipartite graphs. Therefore, they have chromatic number 2, i.e., $\chi(C_{2n}) = 2$ for all n.

E3) We get a 3-colouring of C_{2n+1} as follows: Let $\{v_1, v_2, ..., v_{2n+1}\}$ be the vertex set of C_{2n+1}. We assign $\boxed{1}$ to all the vertices in the set $\{v_i \in V(C_{2n+1}) | i$ is odd, $1 \le i \le 2n\}$ and $\boxed{2}$ to all the vertices in the set $\{v_i | i$ is even, $2 \le i \le 2n\}$. Now, v_{2n+1} is adjacent to both v_1 and v_{2n}. So, we cannot assign $\boxed{1}$ or $\boxed{2}$ to this vertex. Therefore, we assign the third colour $\boxed{3}$ to v_{2n+1}. Since C_{2n+1} is non-null, and non-bipartite, $\chi(C_{2n+1}) \ge 3$. Thus, $\chi(C_{2n+1}) = 3$, for all n.

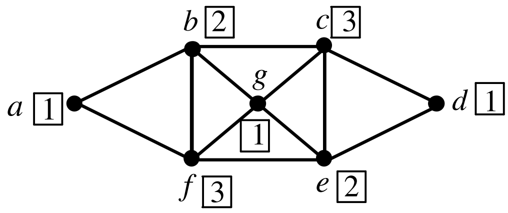
E4) A 3-colouring of the graph is given below:



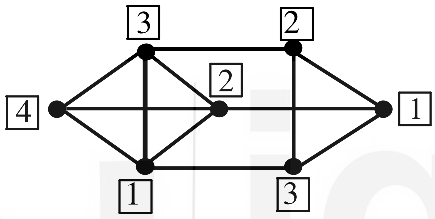
It has a cycle of length 5 as a subgraph, which is an odd cycle. This implies the chromatic number of this graph is 3.

E5) Note that the graph induced by $\{v_4, v_5, v_6, v_7\}$ is K_4. So, it has a clique of size 4 and therefore we need at least 4 colours. We get a 4-colouring by assigning $\boxed{1}$ to v_1, $\boxed{2}$ to v_2, $\boxed{3}$ to v_3, $\boxed{2}$ to v_4, $\boxed{3}$ to v_5, $\boxed{1}$ to v_6 and $\boxed{4}$ to v_7. Therefore, the chromatic number of the graph is 4.

E8) a) The graph contains a triangle, hence its chromatic number is at least 3. Below given is a 3-colouring of this graph.

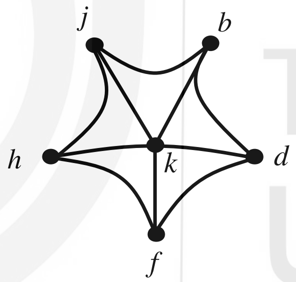


b) The graph contains a clique of size 4, namely the set $\{a, b, g, f\}$. Therefore, its chromatic number is at least 4. A 4-colouring is given below.



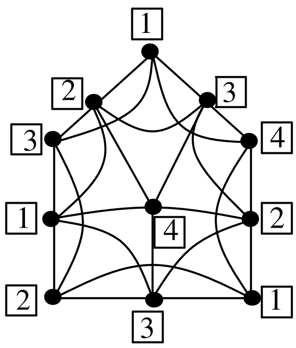
Therefore, the graph has chromatic number 4.

c) Consider the following subgraph H of the given graph.



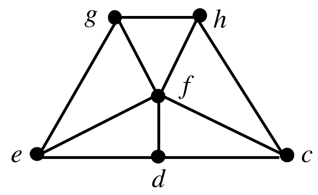
H

The cycle (b, d, f, h, j, b) of H is an odd cycle, and hence requires 3 colours. The vertex k is adjacent to all the vertices on this cycle. So, k must get a different colour. This means $\chi(H) \geq 4$. Consequently, the given graph has chromatic number at least 4. A 4-colouring of the graph is given below.



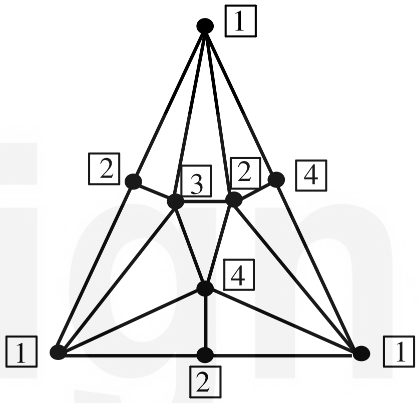
Therefore, the chromatic number of the graph is 4.

d) Consider the following subgraph H of the given graph.



H

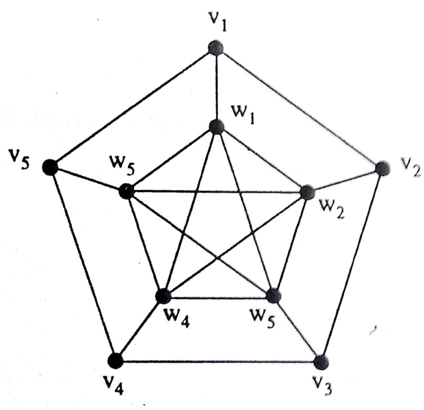
H has an odd cycle, namely $C = (c, d, e, g, h, c)$. Since C requires 3 colours, and f is adjacent to each vertex of C, it follows that $\chi(H) \geq 4$. Consequently, the given graph also has chromatic number at least 4. Now, a 4-colouring is shown below.



Therefore, the given graph has the chromatic number 4.

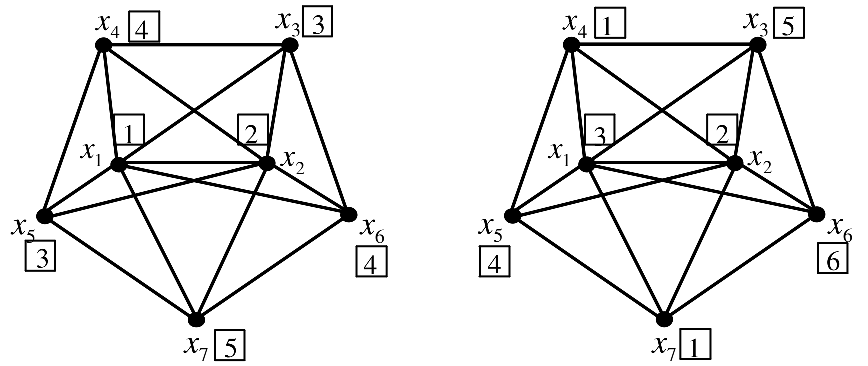
E7) Since it has chromatic number greater than 2, it cannot be bipartite. So, it must contain an odd cycle. The converse statement is–If G contains an odd cycle, then $\chi(G) \geq 3$. The converse is true because an odd cycle requires at least 3 colours, and hence $\chi(G) \geq 3$.

E8) Consider the following graph.



It contains a clique of size 5, namely $\{w_1, w_2, w_3, w_4, w_5\}$. So, its chromatic number is at least 5. We first give a 5-colouring to the subgraph isomorphic to K_5 by assigning $\boxed{i}$ to w_i, $1 \leq i \leq 5$. Next, we assign $\boxed{2}$ to v_1, $\boxed{3}$ to v_2, $\boxed{1}$ to v_3, $\boxed{2}$ to v_4 and $\boxed{1}$ to v_5. Consequently, the graph is 5-chromatic.

E9) Two different colourings of the graph are given below.



The colour classes for the colouring on the left are $\{x_1\},\{x_2\},\{x_7\}$, $\{x_3,x_5\},\{x_4,x_6\}$. The colour classes for the coloring on the right are $\{x_1\},\{x_2\},x_3,\{x_4,x_7\}$ and $\{x_5,x_6\}$.

E10) False. For a counterexample, consider the disjoint union of K_1 and K_2.

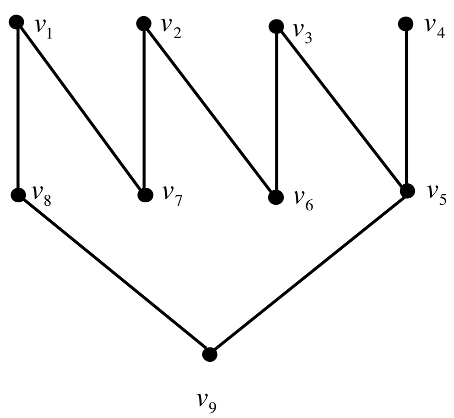
E11) Each maximal independent set of K_n contains just one vertex. Therefore, $\alpha(K_n)=1.$

Now let us compute $\alpha(C_n)$. Let $C_n=(v_1,v_2,v_3,\ldots,v_n,v_1)$. If n is even, then the maximal independent sets of C_n are $\{v_1,v_3,v_5,\ldots,v_{n-1}\}$ and $\{v_2,v_4,v_6,\ldots,v_n\}$. Each of them has size $\frac{n}{2}$. So $\alpha(C_n)=\frac{n}{2}$, when n is even. Now assume that n is odd. Then $\{v_1,v_3,v_5,\ldots,v_{n-2}\}$ and $\{v_2,v_4,v_6,\ldots,v_{n-1}\}$ are the only maximal independent sets, and each of the consists of $\frac{n-1}{2}$ vertices. So, in this case $\alpha(C_n)=\frac{n-1}{2}$.

E12) $\{v_1,v_2,v_4,v_6\}$ is one such set.

E13) Let S be a largest independent set of G. Since S_i is an independent set $|S_i|\le|S|$ for each i. Therefore, $\max\limits_{1\le i\le k}|S_i|\le|S|$. This implies $\alpha(G)\ge\max\limits_{1\le i\le k}|S_i|$.

E14) i) Let us label the vertices of the given graph as follows.



Then all the maximal independent sets of this graph are $\{v_1, v_2, v_3, v_4, v_9\}$ and $\{v_5, v_6, v_7, v_8\}$. Consequently, the independence number of the graph is 5.

ii) The given graph has a cycle C_8. We know that $\alpha(C_8) = 4$. The remaining vertex is adjacent to all the vertices of this cycle. Therefore, the independence number of the graph is 4.

E15) False. A counter-example is the graph $K_{1,2}$ which is acyclic but not 2-critical.

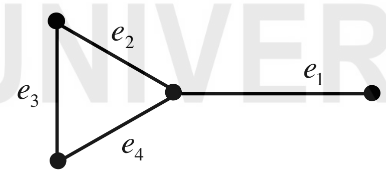
E16) If we remove any vertex of K_n, we get K_{n-1} which has chromatic number $n-1$. Let $\{v_1, v_2, v_3, \ldots, v_n\}$ denote the vertex set of K_n. Let us remove an edge from K_n. Renumbering the edges if necessary, we can assume that the edge we have removed is v_1v_n. Then, we get an $n-1$ coloring as follows: Assign $\boxed{1}$ to both v_1 and v_n. For each $v_i, 2 \le i \le n-1$, assign $\boxed{i}$.

E17) In Example 5, we have seen that the Petersen graph is 3-chromatic. It has a 5-cycle as a proper subgraph which is also 3-chromatic. Therefore, the Petersen graph is not 3-critical.

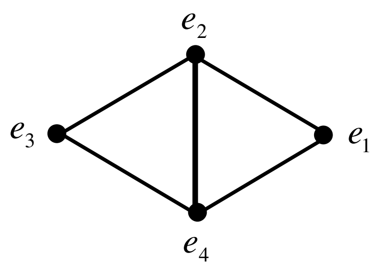
E18) The Petersen graph contains a 5-cycle which is 3-critical.

E19) The given graph has K_4 as a proper subgraph. Since $\chi(K_4) = 4$, the given graph is not 4-critical.

E20) The statement is false. For a counterexample consider the following graph G.



The edge e_1 is a cut-edge of G. However, e_1 is not a cut-vertex of $L(G)$ as you can see from the graph given below.



$L(G)$

E21) Note that P_n has $n-1$ edges. Hence $L(P_n)$ has $n-1$ vertices. Further $L(P_n) = P_{n-1}$.

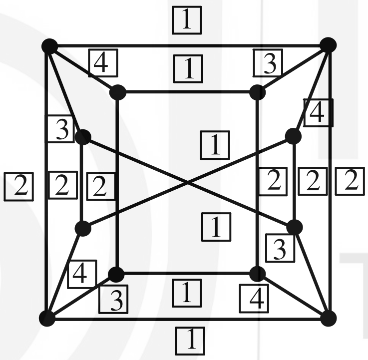
We can write $C_n = P_n + e$, where the endpoints of the edge e are precisely the endpoints of P_n. Since $L(P_n) = P_{n-1}$ and e becomes a vertex in $L(C_n)$ adjacent to the endpoints of P_{n-1}, we get $L(C_n) = C_n$.

Note that S_n has n vertices, and $n-1$ edges. All these $n-1$ edges share one endpoint. Thus these $n-1$ edges form a clique in $L(S_n)$, that is, $L(S_n) = K_{n-1}$.

E22) There are several such graphs, for instance P_n for $n \geq 2$.

E23) Note that in any edge-colouring of G, the adjacent edges get distinct colours. We know that two edges in G are adjacent if and only if they are adjacent in $L(G)$ as vertices. Thus, any k-edge-colouring of G corresponds to a k-vertex-colouring of $L(G)$. Therefore, $\chi'(G) = \chi(L(G))$.

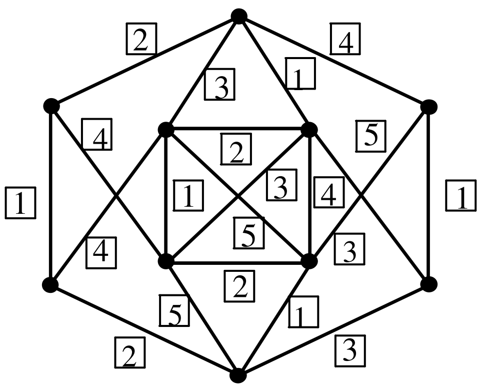
E24) The given graph has the maximum degree 4, and hence its edge-chromatic number is at least 4. A 4-edge-colouring of the graph is shown below.



Therefore, the edge-chromatic number of the graph is 4.

E25) Consider any (n,m)-graph G. Note that $\frac{2m}{n}$ is the average degree of a vertex in G, whereas $\Delta(G)$ is the maximum degree of G. Therefore, $\frac{2m}{n} \leq \Delta(G)$. Since we know that $\Delta(G) \leq \chi'(G)$, the inequality $\frac{2m}{n} \leq \chi'(G)$ follows.

E26) No. A 5-edge-colouring of the graph is given below.



E27) False. A counter-example is the graph K_3. We have $\chi'(K_3) = 3 \neq 2$.

E28) Since $K_{m,n}$ is a bipartite graph,

$$\chi'(K_{m,n}) = \Delta(K_{m,n}) = \max\{m, n\}.$$

UNIT 7

PLANAR GRAPHS

Structure — Page No

7.1 INTRODUCTION

In this unit we introduce you to another type of graphs called planar graphs. In earlier days the study of the concept of planarity was motivated by famous problems such as the Four–Colour Problem and the Utility Problem. Now a days this concept is also applied in the design of circuit layouts on silicon chips! The concept of planarity is also very useful in many practical situations such as electric connection without an extra layer of insulation in the design of printed circuit board. Therefore, in this unit we shall study plane graphs which help us investigate the characteristics of planar graphs.

In Section 7.2, we study plane, planar and nonplanar graphs. In section 7.3, we study the Euler's Formula, which relates the numbers of vertices, edges and faces of a plane drawing of a planar graph and study some restrictions on the number of edges and degrees of vertices on a planar graph.

Section 7.4 is divoted to some characterisations of planar graphs. In this section, we introduce the notions of subdivision and contraction of a graph, and study results due to Kuratowski and Wagner, to check planarity. In Section 7.5, we study about Printed Circuits Board (PCB) layouts Problem, and introduce the concepts of thickness and crossing. Section 7.6 is divoted to dual graphs. We end this unit by giving an overview of the Map Colouring Problem and the Four-Colour Theorem.

Objectives

After studying this unit, you should be able to

- understand the difference between plane and planar graphs;
- check whether a given graph is planar or not;
- understand why K_5 and $K_{3,3}$ are not planar;
- identify the faces of a plane graph, and compute the face degrees;
- state, prove and apply Euler's Formula;
- use Kuratowski's and Wagner's Theorems to check planarity;
- understand the PCB Problem;
- find the thickness and crossing number of some small graphs;
- draw the dual graph of a plane graph;
- define the face-colouring of a plane graph, and check whether a given graph is k-face-colourable or not.

7.2 PLANAR GRAPHS

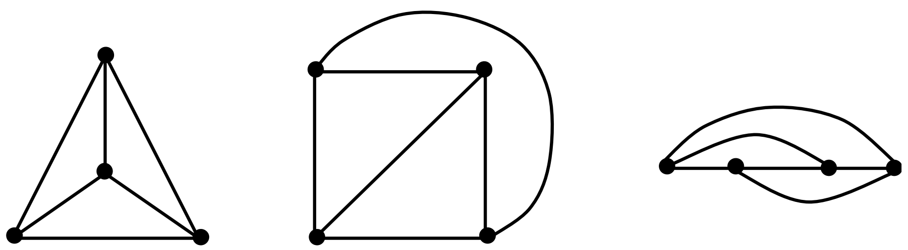
In this section, we shall study plane and planar graphs. In transistor radios and television sets, you must have seen printed circuit boards. These boards have slots for various components and these slots are connected to each other. The connections between these slots must be made in such a way that no two connections cross each other. Given an electronic circuit, is it always possible to design a printed circuit board corresponding to it?

This can be formulated as a problem in graph theory. We replace the electronic components by vertices and the connections between them by edges. If the resulting graph can be drawn in such a way that no two of the edges cross each other except at the vertices, then we can design a printed circuit board for the given circuit. Graphs that can be drawn in this way are called planar graphs. A formal definition is here.

An **edge-crossing** is a point of intersection of two edges, different from their endpoints.

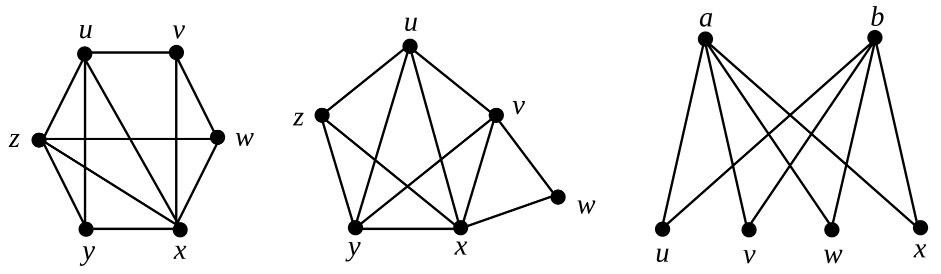
Definition: A graph G is called a **planar graph** if it can be drawn in the plane in such a way that every two edges either do not intersect or their points of intersection are their endpoints. Any such drawing is called a **plane drawing** (or **plane embedding**) of G. A graph G is called **nonplanar** if no plane drawing of G exists. A plane drawing of a graph is also known as a **plane graph.**

For example, the graph K_4 is planar, since it can be drawn in the plane without edge-crossings. The following are three plane drawings of K_4.



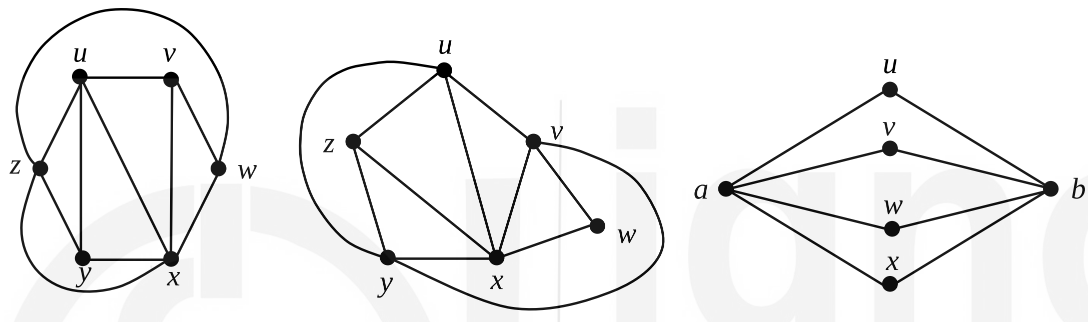
Let us look at some more examples.

Example 1: Show that the following graphs are planar, by finding a plane drawing of each.



(a) (b) (c)

Solution: Note that the above graphs can be redrawn, in order, as follows.



(a) (b) (c)

Each of these new graphs is a plane drawing of the corresponding graph above. Therefore, the given graphs are planar.

***

Example 2: Show that the star graph S_n is planar.

Solution: The graph $S_n, n \geq 2,$ has n vertices. One of the vertices, say $v,$ is adjacent to all the remaining $n-1$ vertices. Thus a plane drawing of S_n can be obtained by placing v at the centre of a circle, and the remaining vertices on the circumference of the circle. Therefore, S_n is planar.

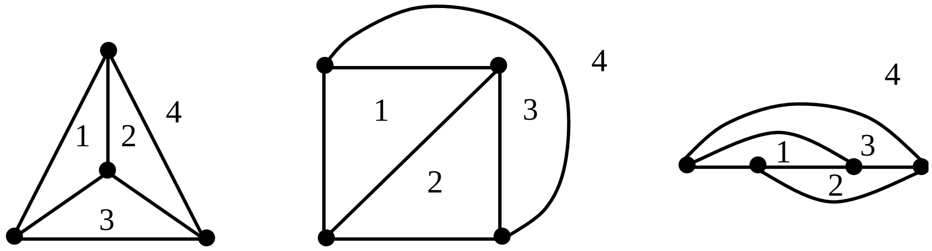
***

Example 3: Show that every subgraph of a planar graph is planar.

Solution: If G is a planar graph, then we can draw G in the plane without crossings. If we now remove the vertices and edges not included in the subgraph, then we obtain a plane drawing of the subgraph. Therefore, the result holds.

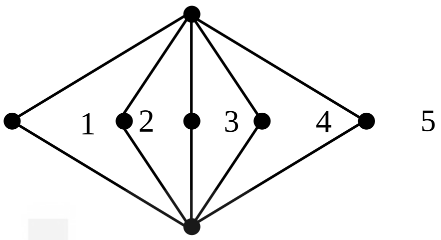
***

Every plane drawing of a planar graph divides the plane into a number of regions. For example, any plane drawing of K_4 divides the plane into four regions of which exactly one is unbounded. Three plane drawings of K_4 are drawn below. In each of these drawings the region marked “4” is unbounded.



Plane drawings of K_4

Similarly, any plane drawing of $K_{2,5}$ divides the plane into five regions. In the drawing below the region marked "5" is unbounded.

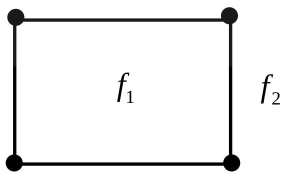


A Plane drawing of $K_{2,5}$

We make these ideas precise as follows.

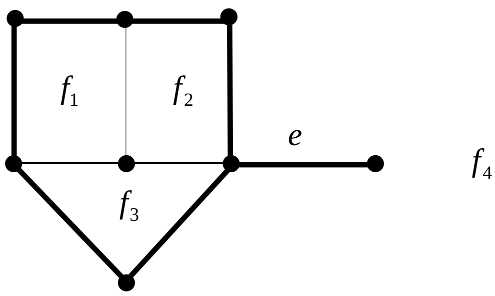
Definition: Let G be a plane graph. Then G divides the set of points of the plane not lying on G into maximal regions, called the **faces** of G. Exactly one of these faces is unbounded. The **boundary** of a face f of G is the subgraph of G that bounds f.

It is clear that two distinct faces of a plane graph do not have any point in common, however they may share the same boundary. For instance, the plane graph G given below has two faces, namely f_1 and f_2. Both these faces have the same boundary G.



G

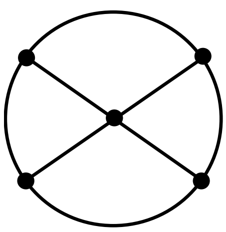
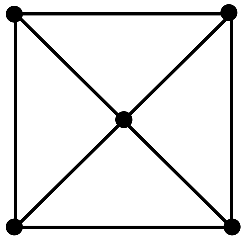
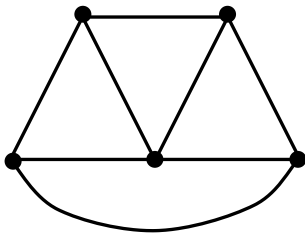
Also, note that every edge of a plane graph need not lie on the boundaries of two faces. For instance, consider the following plane graph G.



G

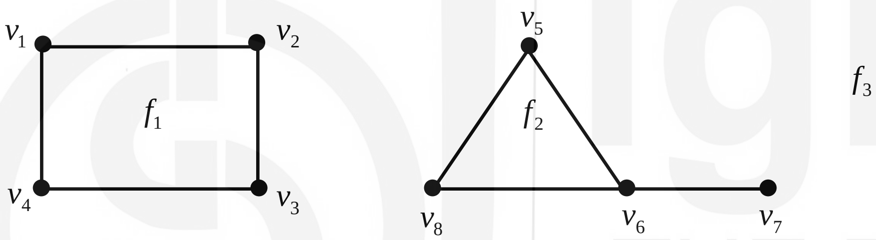
The edge e of this graph lies on the boundary of the face f_4, which is shown is bold edges. The boundaries of f_1, f_2 and f_3 do not contain e.

Note that if G_1 and G_2 are two isomorphic plane graphs, then G_1 and G_2 have the same number of faces. In other words, the number of faces of a plane drawing of a planar graph is uniquely determined. For instance, look at any of the following plane drawing of the planar graph W_5.

They are pairwise isomorphic, and each has exactly five faces.

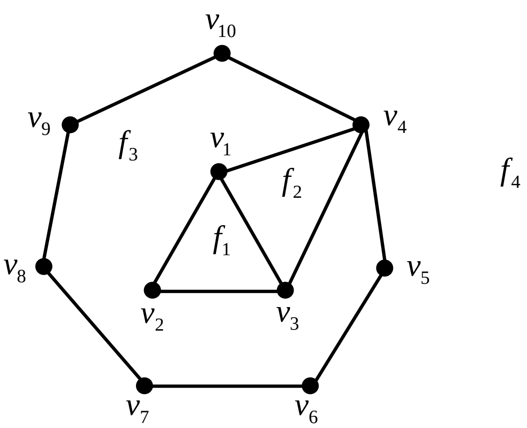
By now you must have observed that to determine the face of a plane graph, one essentially needs to take a closed walk on its boundary. In case the graph is disconnected, one closed walk may not be enough for every face. For example, consider the graph given below.



This graph has three faces f_1, f_2 and f_3. The face f_3 is bounded by two closed walks, namely $(v_1, v_2, v_3, v_4, v_1)$ and $(v_5, v_6, v_7, v_6, v_8, v_5)$. Thus three are a total of 9 edges that enclose the face f_3, where the edge $v_6 v_7$ is counted twice. This leads to the following definition.

Definition: Let G be a plane graph, and let f be a face of G. Then the **degree** of f, denoted by $d(f)$, is the number of edges encountered in a closed walk (or walks) around the boundary of f.

For example, consider the plane graph G given below.



G

There are four faces, namely f_1, f_2, f_3 and f_4 in G. The face f_1 is bounded by the closed walk (v_1, v_2, v_3, v_1), which has 3 edges. Therefore, $d(f_1) = 3$. Similarly, $d(f_2) = 3$. The face f_3 is bounded by the closed walk $(v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_4, v_1)$, which contains 11 edges. Therefore, $d(f_3) = 11$. The closed walk $(v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_4)$ bounds the face f_4, and so $d(f_4) = 7$.

If we now sum all the face degrees, we get $\sum_{i=1}^{4} d(f_i) = 3 + 3 + 11 + 7 = 24$, which is exactly twice the number of edges in G.

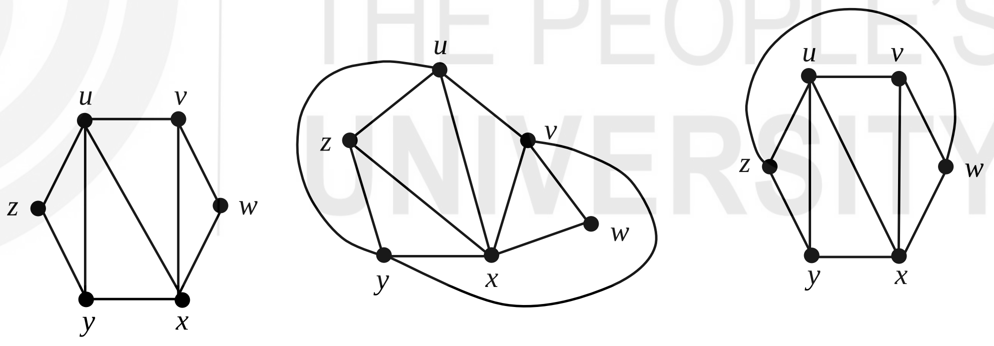
This makes us suspect that the Handshaking Lemma for graphs has "face analogue" for a plane graph. This is indeed the case, as stated below.

Theorem 1 (Handshaking Lemma for Planar Graphs): In any plane drawing of a planar graph, the sum of all the face degrees is equal to twice the number of edges. That is, if $f_1, f_2, \ldots, f_r$ are the faces of a plane drawing of a planar graph G, then

$$\sum_{i=1}^{r} d(f_i) = 2m(G).$$

Proof: In any plane drawing of a planar graph, each edge has two sides (which may lie on the boundary of a single face or on the boundaries of two different faces), so it contributes exactly 2 to the sum of the face degrees. Hence, the result follows. ■

Example 4: Verify Theorem 1 for each of the following plane graphs.



(a) (b) (a)

Solution: (a) There are 8 edges and 4 faces with face degrees 3, 3, 4, 6. Thus we have $3 + 3 + 4 + 6 = 16 = 2 \times 8$.

(b) There are 11 edges and 7 faces in this graph, with face degrees 3, 3, 3, 3, 3, 3, 4. Therefore, $3 + 3 + 3 + 3 + 3 + 3 + 4 = 22 = 2 \times 11$.

(c) There are 10 edges and 6 faces in this graph, with face degrees 3, 3, 3, 3, 4, 4. Therefore, $3 + 3 + 3 + 3 + 4 + 4 = 20 = 2 \times 10$.

***

There are some exercises for you to try.

E1) Consider any face f of a plane graph G. Let S be a set of vertices lying on the boundary of f. Then the subgraph induced by S is a subgraph of the boundary of f. True or false? Justify.

E2) Show that a graph is planar iff each of its components is planar.

E3) For which values of n is the cycle graph C_n planar?

E4) Show that the wheel graph W_n is planar for all $n \geq 4$.

E5) Show that every proper subgraph of K_5 is planar.

E6) Every graph that contains a nonplanar subgraph is nonplanar. True or False?

E7) If G and H are two planar graphs sharing exactly one vertex, then show that $G \cup H$ is also planar.

E8) Show that every tree is a planar graph.

Thus far we have seen the definition and examples of plane and planar graphs. In the next section we shall discuss an important formula for such graphs.

7.3 EULER'S FORMULA

In this section we introduce the Euler's Formula, which relates the number of vertices, edges and faces of a plane drawing of a planar graph. We begin with an example.

Example 5: For each of the plane graphs in Example 4, count the numbers of vertices (n), edges (m) and faces (r), and find the value of $n - m + r$.

Solution: (a) There are 6 vertices, 8 edges and 4 faces, so the required value is $6 - 8 + 4 = 2$.

(b) There are 6 vertices, 11 edges and 7 faces, so the required value is $6 - 11 + 7 = 2$.

(c) There are 6 vertices, 10 edges and 6 faces, so the required value is $6 - 10 + 6 = 2$.

***

In Example 5, we have seen that for each of the plane drawings under consideration, $n - m + r = 2$.

This equation holds for any plane drawing of a connected planar graph, and is known as Euler's Formula. It tells us that each plane drawing of a given connected planar graph with n vertices and m edges must have the same number of faces, $2 - n + m$.

Theorem 2 (Euler's Formula): Let G be a connected planar graph, and let n, m and r denote, respectively, the numbers of vertices, edges and faces in a plane drawing of G. Then $n - m + r = 2$.

Proof: A plane drawing of any connected planar graph G can be constructed by taking a spanning tree of G and adding edges to it, one at a time, until a plane drawing of G is obtained.

We prove Euler's Formula by showing that:

i) for any spanning tree, $n - m + r = 2$;

ii) adding an edge does not change the value of $n - m + r$.

First, we prove statement (i). Let T be any spanning tree of G; then we can draw T in the plane without crossings. Since T has n vertices and $n-1$ edges, and there is only 1 face the (unbounded one), we have

$$n - m + r = n - (n-1) + 1 = 2,$$

as required.

Now we prove statement (ii) by successively adding to T the remaining edges of G until a plane drawing of G is obtained. At each stage the added edge joins two nonadjacent vertices.

Since we have a plane drawing of a subgraph of G, the added edge cuts an existing face in two. This leaves n unchanged, increases m by 1, and increases r by 1, and so leaves $n - m + r$ unchanged. Since $n - m + r = 2$ throughout the process, the result follows. ■

Example 6: Verify the Euler's Formula for each of the following planar graphs:

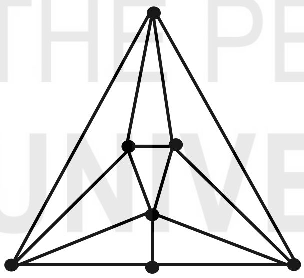
a) the octahedron graph

b) $W_n, n \geq 4$

c) $K_{2,6}$

d) the graph formed from the vertices and edges of a $k \times k$ square lattice.

Solution: (a) A plane drawing of the octahedron graph is given below.



There are 6 vertices, 12 edges and 8 faces, and

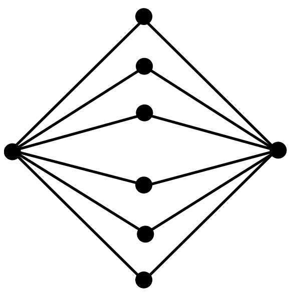
$$n - m + r = 6 - 12 + 8 = 2.$$

(b) Note that W_n has n vertices, and $2n-2$ edges. Any plane drawing of W_n has n faces. Therefore,

$$n - (2n-2) + n = 2,$$

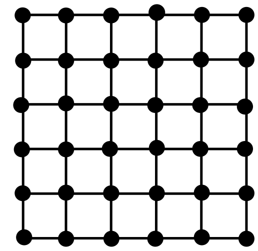
which verifies the Euler's Formula.

(c) A plane drawing of $K_{2,6}$ is given below.



$K_{2,6}$

Here $n=8, m=12$ and $r=6$. Therefore, $n-m+r=2$.



A 5×5 square lattice

(d) A $k\times k$ square lattice has $(k+1)^2$ corners which correspond to the vertices of the graph. There are $2k(k+1)$ edges in it. It has k^2 squares which correspond to the bounded faces of a plane drawing of it. Thus its plane drawing has k^2+1 faces in all. Now, we have

$$n-m+r=(k+1)^2-2k(k+1)+(k^2+1)=2.$$

***

Next, we show how Euler's Formula can be used to prove that certain graphs are nonplanar. We first derive two corollaries of Theorem 2 that give upper bounds for the number of edges of a planar graph.

Corollary 1: Let G be a planar graph with $n(\geq 3)$ vertices and m edges. Then $m\leq 3n-6$.

Proof: Assume, without loss of generality, that G is connected. (Why?) Consider a plane drawing of G with r faces. Since a graph has no loops or multiple edges, the degree of each face is at least 3. It follows from the Handshaking Lemma for Planar Graphs that $3r\leq 2m$. Substituting r from Euler's Formula $r=m-n+2$, we obtain $3m-3n+6\leq 2m$, and hence $m\leq 3n-6$, as required. ■

Using Corollary 1, we can prove that the complete graph K_5 is nonplanar.

Example 7: Show that K_5 is nonplanar.

Solution: We prove this by contradiction. Suppose that K_5 is planar. Since K_5 is a connected graph with 5 vertices and 10 edges, it follows from Corollary 1 that $10\leq(3\times 5)-6=9$, which is false. This shows that K_5 is nonplanar.

***

However, note that we cannot use Corollary 1 to prove that $K_{3,3}$ is nonplanar, as $K_{3,3}$ has 6 vertices and 9 edges, and the inequality $9\leq(3\times 6)-6$ is true. Instead, we shall use the following corollary to prove that $K_{3,3}$ is nonplanar.

Corollary 2: Let G be a planar graph with $n(\geq 3)$ vertices, m edges and no triangles. Then $m\leq 2n-4$.

Proof: Consider a plane drawing of G with r faces. The degree of each face of such a drawing is at least 4 as G has no triangles. It follows from the Handshaking Lemma for Planar Graphs that $4r\leq 2m$, so $2r\leq m$. Substituting r from the Euler's Formula $r=m-n+2$, we obtain $2m-2n+4\leq m$, and hence $m\leq 2n-4$, as required. ■

Example 8: Show that $K_{3,3}$ is nonplanar.

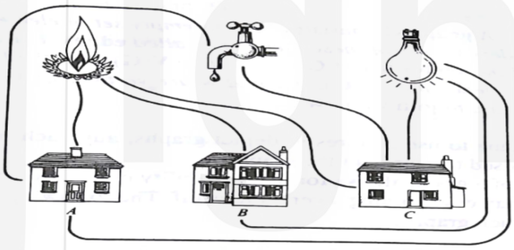
Solution: We prove this also by contradiction. Suppose that $K_{3,3}$ is planar. Since $K_{3,3}$ is a connected graph with 6 vertices, 9 edges and no triangles, it follows from Corollary 2 that $9 \le (2\times 6)-4=8,$ which is false. Therefore, $K_{3,3}$ is nonplanar.

***

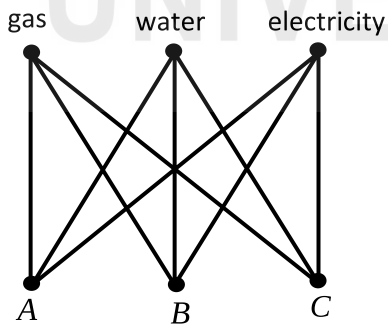
Corollary 3: If G is a planar graph, then G contains at least one vertex of degree 5 or less.

Proof: Let G be a planar (n,m)- graph. If $n \le 2,$ there is nothing to prove. So, let $n \ge 3.$ Let us assume that all vertices of G have degree greater than or equal to 6. That is, $d(v) \ge 6$ for all $v \in V(G).$ Therefore, $\sum_{v\in V(G)} d(v) \ge 6n,$ which implies $2m \ge 6n.$ That is, $m \ge 3n.$ But by Corollary 1, we know that $m \le 3n-6,$ which is a contradiction. Hence our assumption is wrong. So there exists at least one vertex of degree 5 or less in $G.$ ■

Example 9 (Utilities Problem): We wish to connect three houses A, B and $C,$ to three utilities, gas, water, and electricity. For safety reasons it is necessary that the various connections should not cross each other. Can the connections be made? The following picture shows how eight of the nine connections can be drawn, but how about the ninth?



Solution: We can represent the connections by means of the following graph, where the vertices correspond to the three houses and the three utilities. This is the complete bipartite graph $K_{3,3}$ with the sets $\{A, B, C\}$ and {gas, water, electricity} as its partite sets.



Finding a solution of this problem is equivalent to finding a plane drawing of $K_{3,3}$. But $K_{3,3}$ is nonplanar, so it has no plane drawing. Hence, the utilities problem has no solution.

***

Now, try the following exercises.

E9) a) Let G be a connected planar graph with $n(\ge 5)$ vertices, m edges and girth 5. Then prove that $m \le \frac{5}{3}(n-2).$

(Hint: Use the method of proof of Corollaries 1 and 2.)

Recall that the girth of a graph is the length of a shortest cycle in it.

b) Hence show that the Petersen graph is nonplanar.

E10) By giving a plane drawing, show that Q_3 is planar. Is Q_4 planar? Justify.

E11) Every nonplanar graph contains at least one nonplanar proper subgraph. True or false? Justify.

E12) Give an example of each of the following or explain why no such graph exists.

a) a planar graph in which each vertex has degree 5;

b) a planar graph in which each vertex has degree 6 or more.

E13) Does there exists a planar $(6,12)$-graph? If yes, draw such a graph.

E14) Let G be a planar graph with n vertices, m edges, r faces and k components. Show that $n-m+r=k+1$.

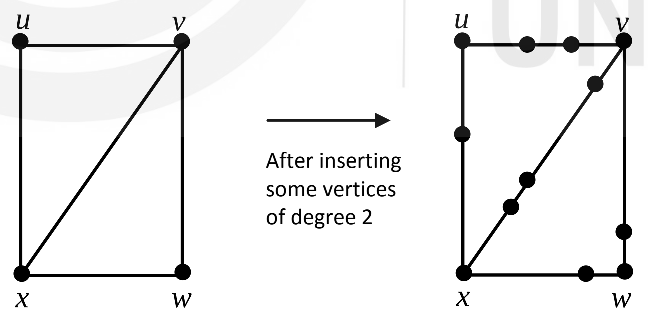
E15) Give an example of a 4-regular planar graph on 6 vertices. Is this graph maximal planar? Why?

A graph is said to be **maximal planar** if it is planar and has the maximum number of edges on a fixed vertex set.

The restrictions on the number of edges and degrees of vertices of a planar graph given in Corollaries 1, 2 and 3 are useful for showing that certain graphs are nonplanar. For example, we used them to show that K_5 and $K_{3,3}$ are nonplanar. Unfortunately, the method does not work for all graphs. For example, the Petersen graph satisfy these inequalities but is nonplanar. Hence, we turn our attention to other ways of determining whether a given graph is planar.

7.4 CHARACTERISATION OF PLANAR GRAPHS

In this section, we describe two methods for determining planarity. The first method involves the "insertion" of vertices of degree 2 into the edges of a graph G, as shown below.



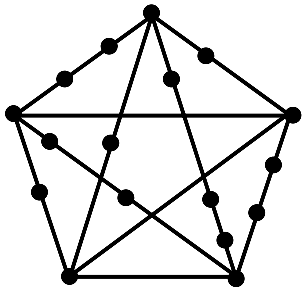
G A subdivision of G

Definition: Let G be a graph. The **subdivision** of an edge uv of G is the operation of replacing uv with a (u,v)-path P such that $V(G)\cap V(P)=\{u,v\}$. A graph H is called a **subdivision** of G if H is obtained by subdividing some edges of G.

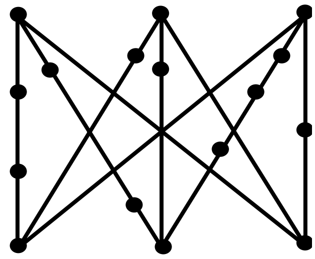
Consider a graph G and a subdivision H of G. If G is planar, then one can obtain a plane drawing of H by just adding new vertices on the existing edges of G, as this operation does not create any edge crossing. On the other hand, if G is nonplanar, then a new vertex cannot be added at an existing edge - crossing. This means an edge-crossing in G remains an edge-crossing in H as well. And hence, H is nonplanar. This leads to the following lemma.

Lemma 1: Every subdivision of a planar graph is planar, and every subdivision of a nonplanar graph in nonplanar.

For example, the following graphs are nonplanar, since the first graph is a subdivision of K_5, and the second one is a subdivision of $K_{3,3}$.



A subdivision of K_5



A subdivision of $K_{3,3}$

The next corollary follows immediately from Lemma 1.

Corollary 1: If G is a planar graph, then G contains no subdivision of K_5 or $K_{3,3}$.

You may be wondering why we are so concerned with K_5 and $K_{3,3}$ and their subdivisions. The reason is that the converse of Corollary 1 is also true. That is, if a graph G contains no subdivision of K_5 or $K_{3,3}$, then G is planar.

This result, proved in 1930, is due to Polish mathematician Kazimierz Kuratowski. We state it formally but omit the proof, which is rather long and complicated.

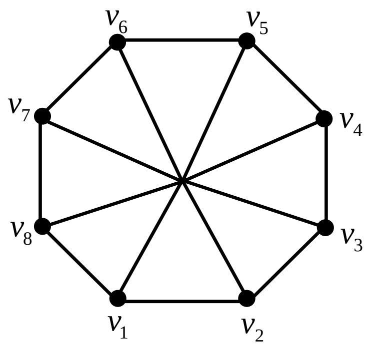
Theorem 3 (Kuratowski's Theorem): A graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$.

Remark 2: The graphs K_5 and $K_{3,3}$ are the two fundamental nonplanar graphs, sometimes also called **Kuratowski's graphs.** The reason is that every nonplanar graph can be obtained by a sequence of edge subdivisions or vertex additions from a Kuratowski's graph. In fact, Kuratowski's Theorem can be reworded as follows.

"A graph is nonplanar iff it contains a subdivision of a Kuratowski's graph."

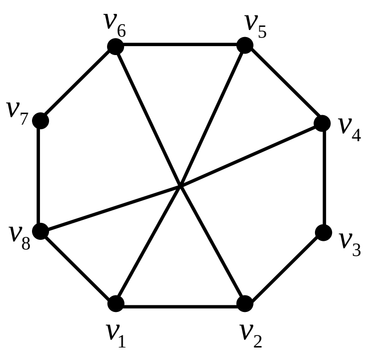
Let us look at an example.

Example 10: Use Kuratowski's Theorem to show that the following graph is nonplanar.



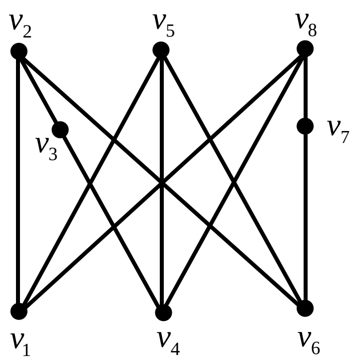
G

Solution: Deleting $v_3 v_7$ from G we get a subgraph H which is a subdivision of $K_{3,3}$ as shown below.



H

$\cong$



A subdivision of $K_{3,3}$

Thus G contains a subdivision of $K_{3,3}$. Therefore, by the Kuratowski's Theorem, G is nonplanar.

***

Another characterisation of planar graphs involves the notion of "contracting" an edge. This is done by bringing one endpoint of the edge closer and closer to the other endpoint until they coincide, and then replacing any resulting parallel edges by a single edge. Below we present this notion formally.

Definition: Let G be a graph and $e = uv \in E(G)$. The **contraction** of e is the operation on G to obtain a new graph $G \cdot e$ by replacing the vertices u and v by a new vertex p_{uv} such that all the neighbours of u and v in G are the neighbours of p_{uv} in $G \cdot e$.

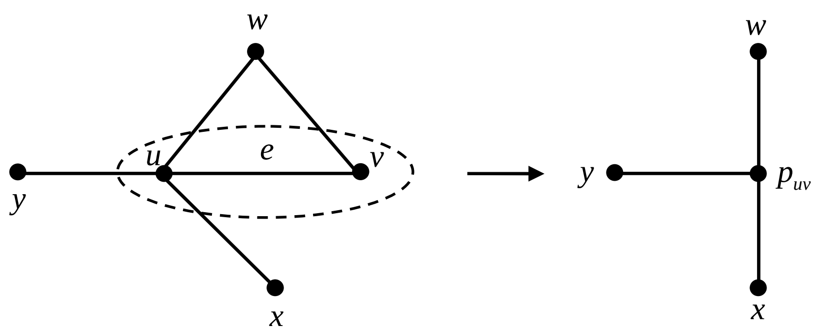
Definition: A graph H is said to be a **contraction** of a graph G if there is a sequence of graphs $(G_0, G_1, G_2, \ldots, G_k)$ such that

i) $G_0 = G$,

ii) $G_{i+1} = G_i \cdot e$ for all $0 \le i \le k-1$,

iii) $G_k = H$.

For instance look at the graphs G and $G \cdot e$ given below.



G $\qquad$ $G \cdot e$

We now state the following analogue of Kuratowski's Theorem due to Wagner (1937). We omit the proof.

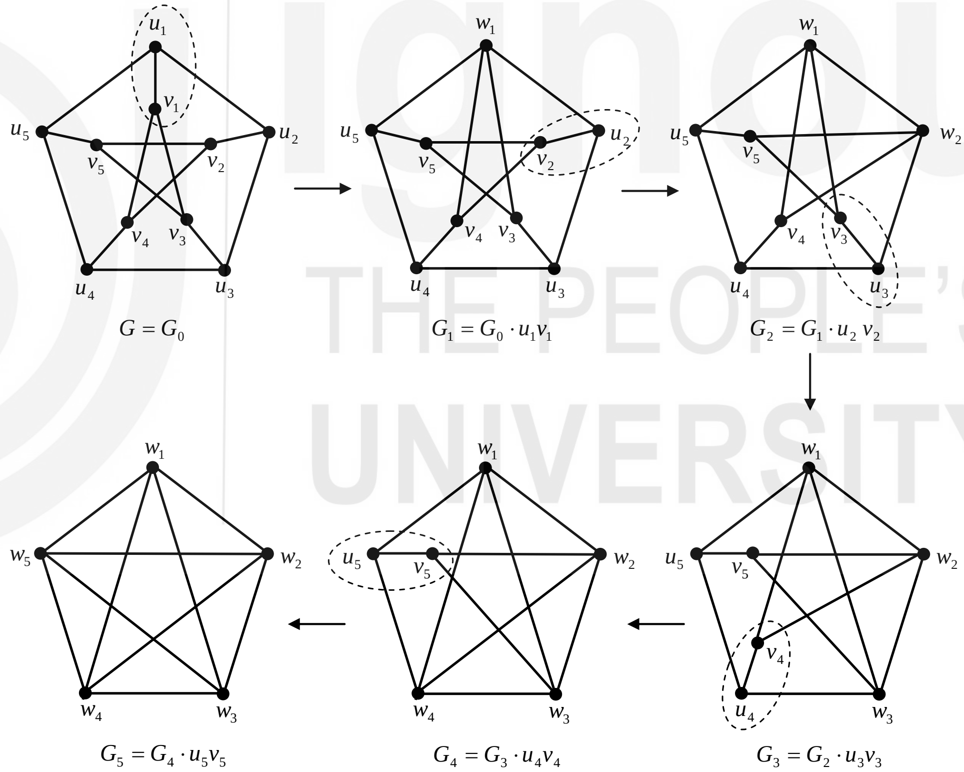
Theorem 4 (Wagner's Theorem): A graph is planar if and only if it contains no subgraph that has K_5 or $K_{3,3}$ as a contraction.

The importance of Theorems 3 and 4 is that they give necessary and sufficient conditions for a graph to be planar in graph-theoretic terms (subgraphs, subdivision, contraction of a graph), rather than in geometric terms (crossing, drawing in the plane). However, Theorems 3 and 4 do not provide an easy way of showing that a given graph is planar, since this would involve looking at a large number of subgraphs and verifying that none of them is a subdivision of K_5 or $K_{3,3}$, or contains K_5 or $K_{3,3}$ as a contraction. For this reason, no currently used algorithm for testing the planarity of a graph is based on either of these theorems.

Let us consider an example.

Example 11: Show that K_5 is a contraction of the Petersen graph. Hence, deduce that the Petersen graph is nonplanar.

Solution: Starting from the Petersen graph $G,$ we construct a sequence $(G_0 = G, G_1, G_2, G_3, G_4, G_5 = K_5)$ as shown below.

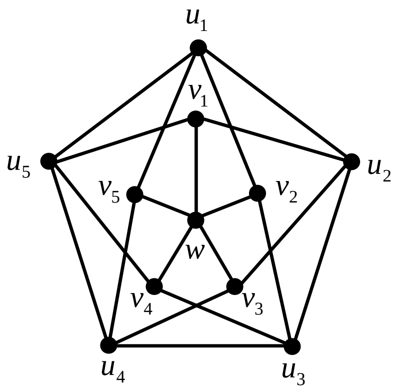


We have obtained $G_5 = K_5.$ Therefore, K_5 is a contraction of the Petersen graph. Consequently, by the Wagner's Theorem, the Petersen graph is nonplanar.

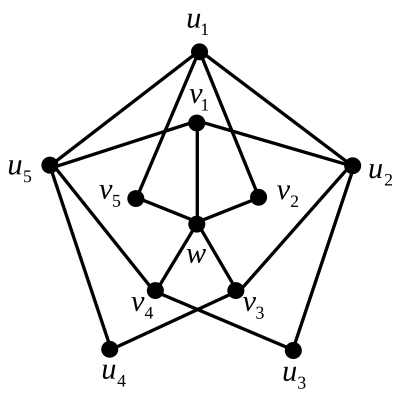
***

Example 12: Show that $K_{3,3}$ is a contraction of the Grötzsch graph. Hence conclude that the Grötzsch graph is nonplanar.

Solution: Given on the left below is the Grötzsch graph. If we remove the edges $v_2 u_3, u_3 v_4$ and $u_4 v_5$ we get the subgraph H as shown on the right below.

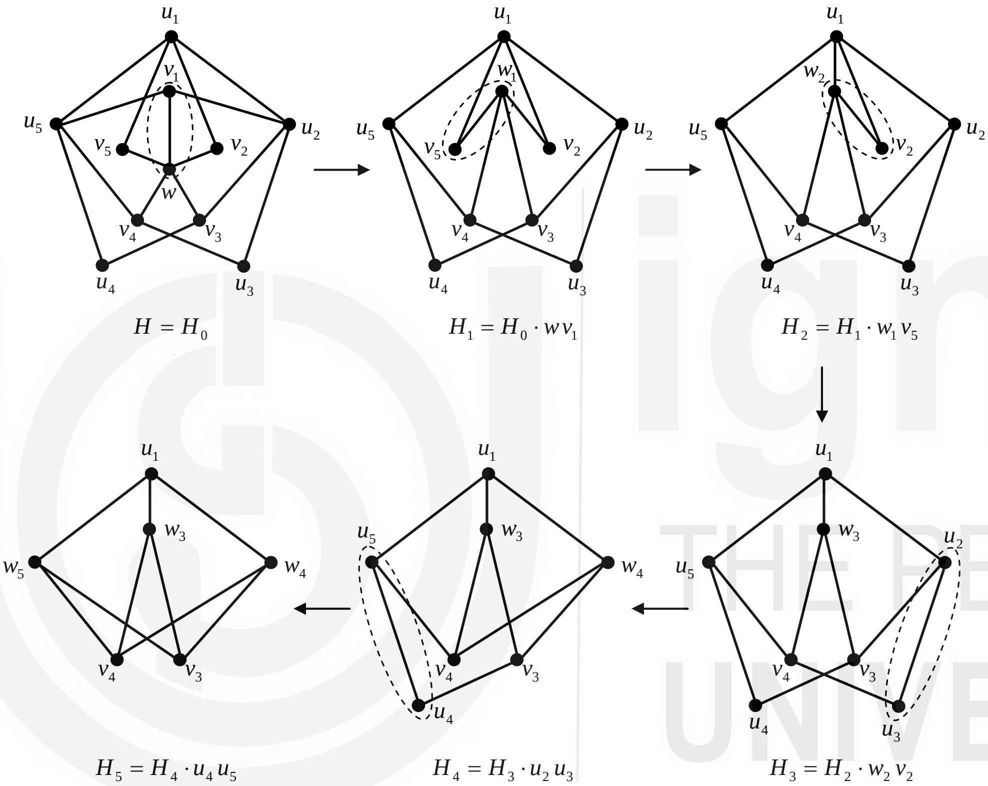


G



H

Now we get $K_{3,3}$ as a contraction of H as follows.



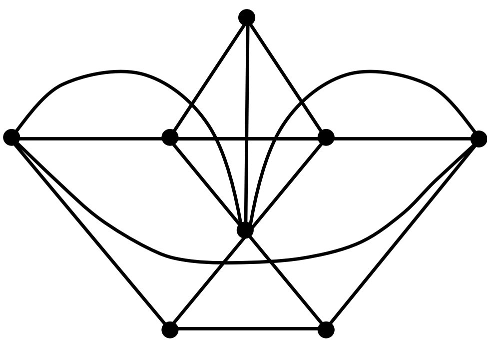
Now it is clear that $H_5 = K_{3,3}$. Thus H is a subgraph of $G,$ which has $K_{3,3}$ as a contraction. Therefore, by Theorem 4, G is nonplanar.

***

Now, try following exercises.

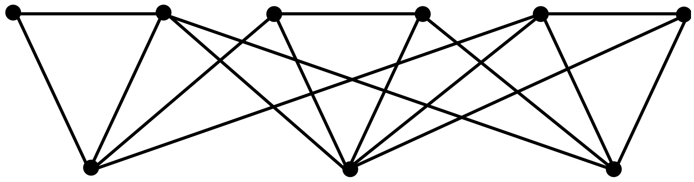
E16) Let H be a subdivision of a graph $G,$ and K be a subdivision of H. Then K is a subdivision of G. True or false? Justify.

E17) Give a plane drawing of the graph given below, if it exists. Otherwise, explain why no plane drawing exists.



E18) If H is a contraction of a graph G, then $\Delta(H) \leq \Delta(G)$. True or False? Justify.

E19) Is the following graph planar? Justify your answer.



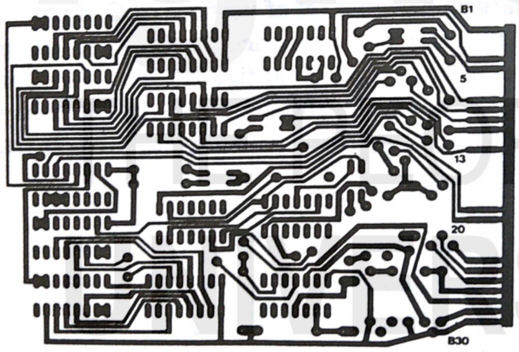
E20) Does there exist planar graph with 7 vertices and 12 edges such that each of its faces is bounded by exactly 3 edges? Why?

There are many real life applications of planarity. One of the most important applications is in Printed Circuit Board (PCB) layouts that we are going to discuss in the next section.

7.5 PCB PROBLEM

In this section, we shall discuss about the Printed Circuit Board Layout Problem, often shortened as PCB Problem. We shall also see how this problem gives rise to the concepts of thickness and crossing.

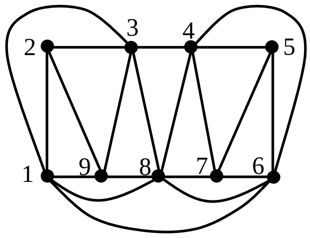
In printed circuits, electronic components are connected by conducting strips printed directly onto a flat board of insulating material. A sample PCB is shown below.



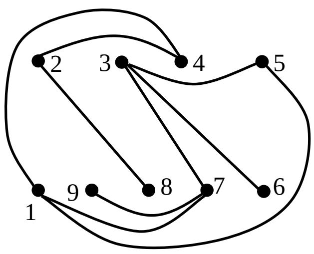
Such printed connectors are not allowed to cross, since this would lead to undesirable electrical contact at crossing points. Circuits in which many crossings are unavoidable may be printed on several boards (layers) that are then sandwiched together. Each board consists of a printed circuit without crossings. Now, the question arises is – what is the smallest number of such layers need for a given circuit?

We illustrate this problem with a particular example.

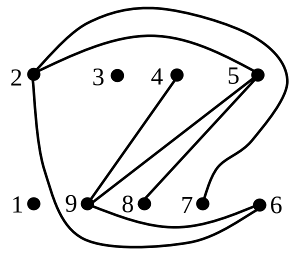
Consider a printed circuit that has 36 interconnections and is represented by the complete graph K_9. It is impossible to arrange all these interconnections in one layer, or even two; three layers are needed, and a solution is given below.



layer 1



layer 2



layer 3

Note that each edge of K_9 is included on just one of the layers – for example, the edge 28 appears on layer 2, and the edge 69 appears on layer 3.

Each of these three graphs is a planar graph. So the PCB problem, indeed, seeks a decomposition of a graph into a minimum number of planar subgraphs. In the case of K_9, the three planar subgraphs corresponding to the three layers shown above are induced by the following sets, respectively.

$\{12, 13, 16, 18, 19, 23, 29, 34, 38, 39, 45, 46, 47, 48, 56, 57, 67, 68, 78, 89\}$,

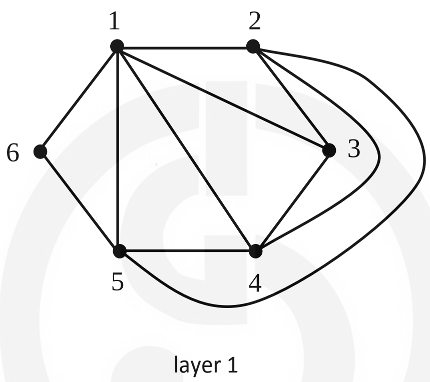
$\{14, 15, 17, 24, 28, 35, 36, 37, 79\}$,

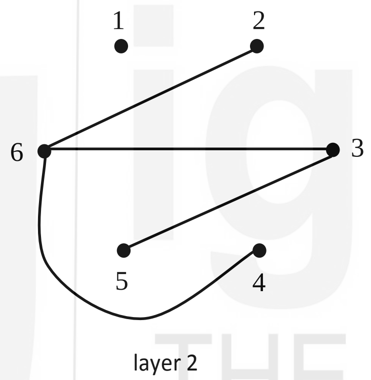
$\{25, 26, 27, 49, 58, 59, 69\}$.

Let us consider one more example.

Example 13: Show that K_6 can be 'printed' in two layers and write down a corresponding edge decomposition.

Solution: The graph K_6 can be printed in two layers, as follows.





The corresponding sets of edges are

$\{12, 13, 14, 15, 16, 23, 24, 25, 34, 45, 56\}$, $\{26, 35, 36, 46\}$.

***

Note that a graph may have many decompositions into planar subgraphs. However here we are concerned with the minimum number of such subgraphs in a decomposition. This leads us to define the thickness of a graph.

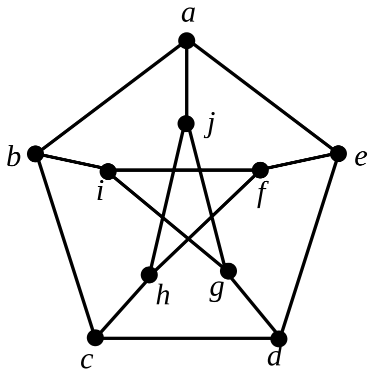
Definition: The **thickness** of a graph G is the minimum number of planar subgraphs whose union is G and, the **crossing number** of graph G is the **minimum number** of crossings in drawing of G in the plane.

We shall write $\theta(G)$ for the thickness of G, and $\nu(G)$ for the crossing number of G. Thickness and crossing number tell us how far a graph deviates from planarity.

By definition, the thickness of a planar graph is 1 and crossing number of a planar graph is zero. The thickness of each of the Kuratowski's graphs K_5 and $K_{3,3}$ is, clearly, 2 and their crossing number is 1.

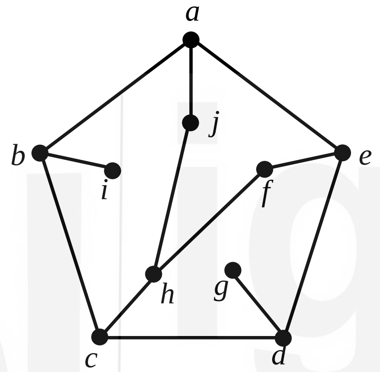
Example 14: Find the thickness of the Petersen graph.

Solution: Consider a drawing of the Petersen graph as given below.

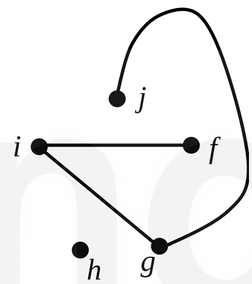


G

We know that G is nonplanar. Therefore, $\theta(G) \geq 2.$ Now look at the following planar subgraphs G' and G'' whose union is $G.$ That is, $\{G', G''\}$ is a decomposition of G into planar subgraphs. Thus $\theta(G) = 2.$



G'



G''

***

In general, there is no known formula that gives the thickness of a graph. However, we can easily obtain a lower bound for thickness that often coincides with the correct value. This is given below.

Theorem 5: Let G be an (n, m)-graph, with $n \geq 3.$ Then

a) $\theta(G) \geq \left\lceil \dfrac{m}{3n-6} \right\rceil,$

b) $\theta(G) \geq \left\lceil \dfrac{m}{2n-4} \right\rceil,$ if G has no triangles.

Proof: a) If $\theta(G) = k,$ then G has a minimum of k planar subgraphs whose union is $G.$ Each of these subgraphs can have at most $3n-6$ edges. (See Corollary 1.) Thus k subgraphs can have at most $k.\ (3n-6)$ edges altogether. That is, $m \leq k(3n-6),$ which gives $k \geq \dfrac{m}{3n-6}.$

However, $\theta(G)$ is an integer, so $\theta(G) \geq \left\lceil \dfrac{m}{3n-6} \right\rceil.$

b) The proof is similar to part (a). Use Corollary 2. ■

We can now deduce lower bounds for the thickness of K_n and that of $K_{r,s}.$

Theorem 6: a) $\theta(K_n) \geq \left\lfloor \frac{n+7}{6} \right\rfloor$ for all $n \geq 1$.

b) $\theta(K_{m,n}) \geq \left\lceil \frac{mn}{2m+2n-4} \right\rceil$, for all $m, n \geq 1$.

Proof: (a) The result holds if $n=1$ or $n=2$. So assume $n \geq 3$. Then $m = \frac{1}{2}n(n-1)$. It follows from part (a) of Theorem 5 that

$$\theta(K_n) \geq \left\lceil \frac{\frac{1}{2}n(n-1)}{3n-6} \right\rceil$$

$$= \left\lfloor \frac{\frac{1}{2}n(n-1)+(3n-6)-1}{3n-6} \right\rfloor \quad \left(\because \left\lceil \frac{p}{q} \right\rceil = \left\lfloor \frac{p+q-1}{q} \right\rfloor, \forall\ p, q \in \mathbb{N}.\right)$$

$$= \left\lfloor \frac{n^2+5n-14}{6(n-2)} \right\rfloor$$

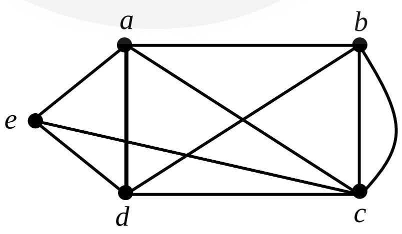
$$= \left\lfloor \frac{n+7}{6} \right\rfloor$$

b) Try yourself. ■

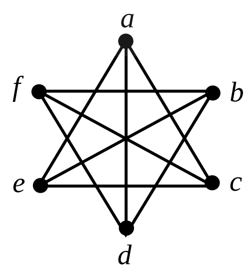
So, to sum up, although we cannot solve the PCB problem in general, we have obtained a lower bound for the solution, and this bound coincides with the correct value surprisingly often.

Now try the following exercises.

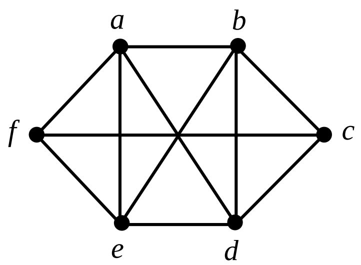
E21) Find the thickness and crossing number of each of the following graphs:



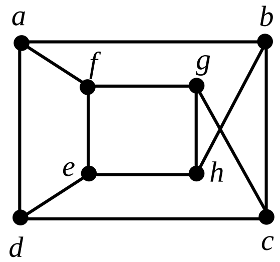
(i)



(ii)



(iii)



(iv)

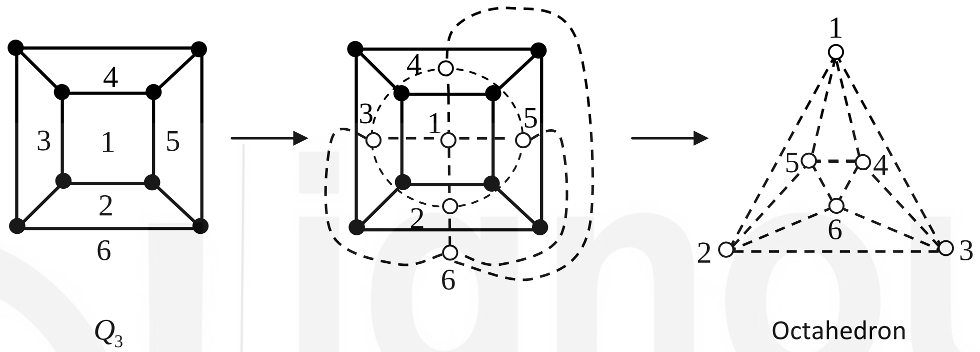
[Hint: Check for planarity first by redrawing the graphs].

E22) Show that $\theta(K_{n,n}) = \left\lfloor \frac{n+5}{4} \right\rfloor$ for all $n \geq 1$.

By now you must have gained some understanding of plane and planar graphs. The next concept we are going to introduce is about the dual of a plane graph.

7.6 DUALITY

We now introduce the idea of duality for plane graphs. To illustrate this idea, we consider the graph of the cube Q_3. If we place a new vertex within each face (including the unbounded one) and join the pairs of new vertices that lie in adjacent faces, we obtain the graph of the octahedron as follows.

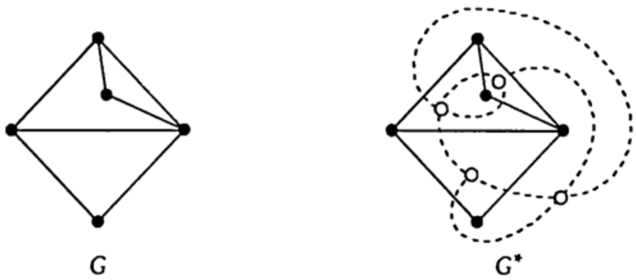


(The new vertices are represented by small circles, and the edges joining them are indicated by dashed lines.) The graph we have obtained is the "dual" of Q_3. A formal definition is here.

Definition: Let G be a plane graph. Then the **dual** of G, denoted by G^* is the graph constructed as follows.

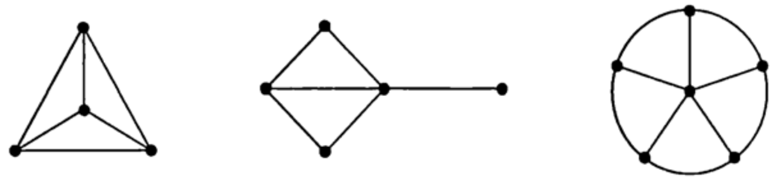
Draw a point in each face of G, and call then the vertices of G^*. For each edge e of G draw an edge e^* joining the vertices of G^* in the faces on either side of e: these are the edges of G^*.(Note that if any edge e of G lies on the boundary of exactly on face of G, then e^* is a loop of G^*. Thus G^* may be a multigraph.)

For instance, look at the graph G and its dual G^* drawn below.



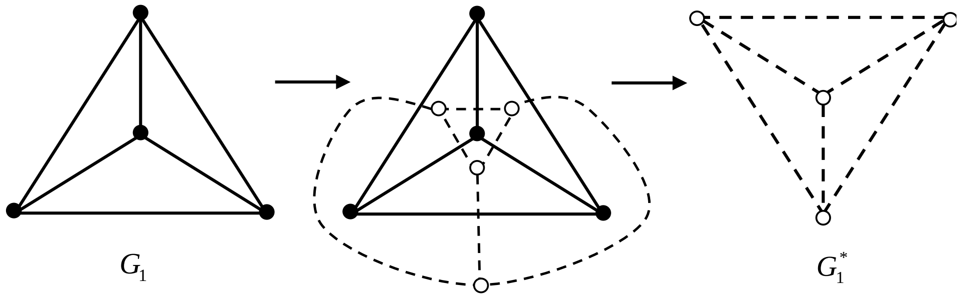
It turns out that the dual of a plane graph is a planar graph. Let us look at one more example.

Example 15: Draw the dual of each of the following plane graphs.

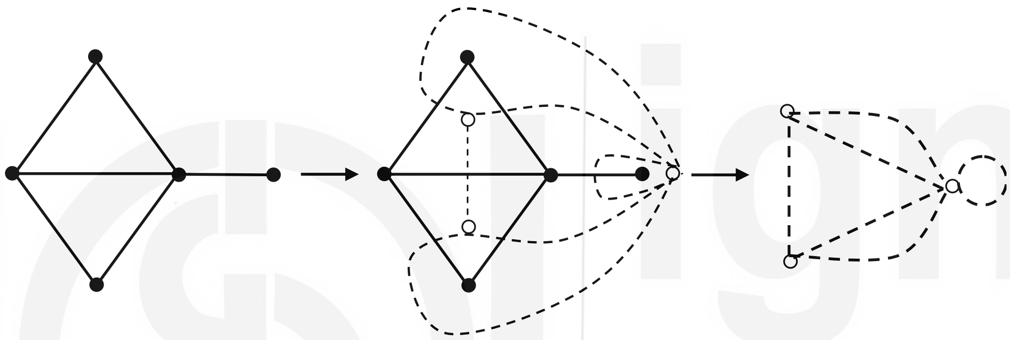


G_1 G_2 G_3

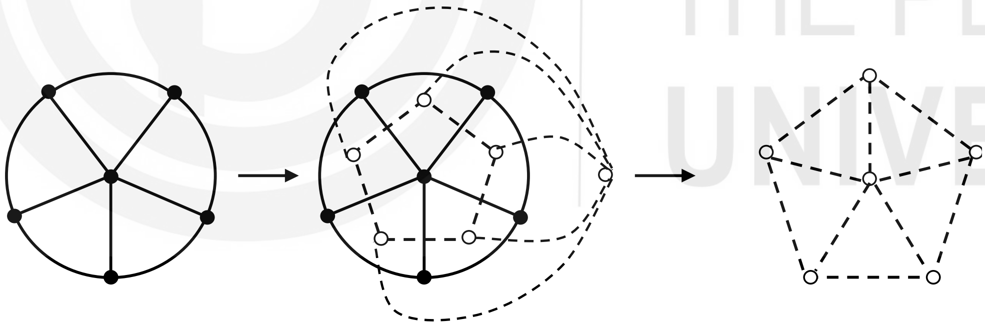
Solution: The duals of G_1, G_2 and G_3 are constructed as follows.



G_1 G_1^*



G_2 G_2^*



G_3 G_3^*

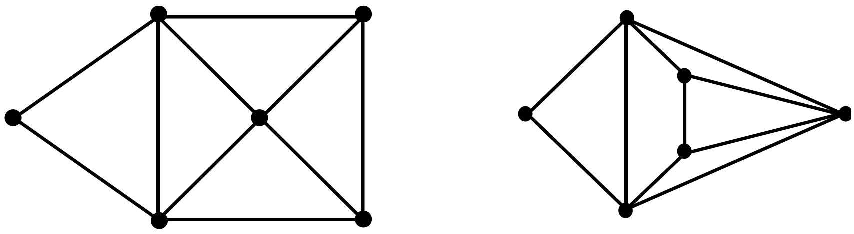
***

Observe that in the example above we have $G_1 \cong G_1^*$ and $G_3 \cong G_3^*$. Such graphs are "self-dual" as defined below.

Definition: A plane graph G is called **self-dual** if $G \cong G^*$.

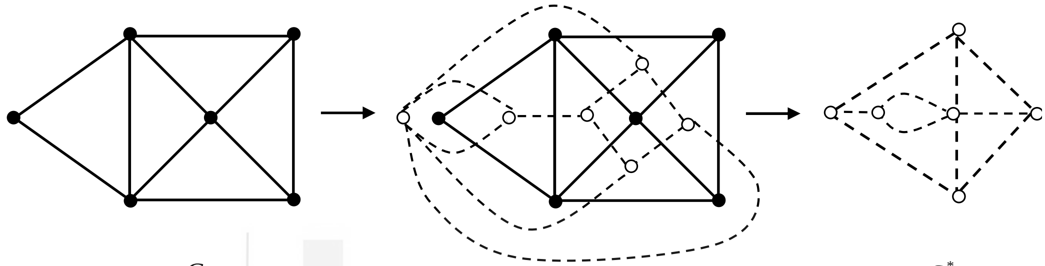
You can clearly verify that the wheel graph W_n is self-dual for all $n \geq 4$. The next example shows that the duals of two isomorphic plane graphs need not be isomorphic.

Example 16: Show that the following plane graphs are isomorphic, but their duals are not.

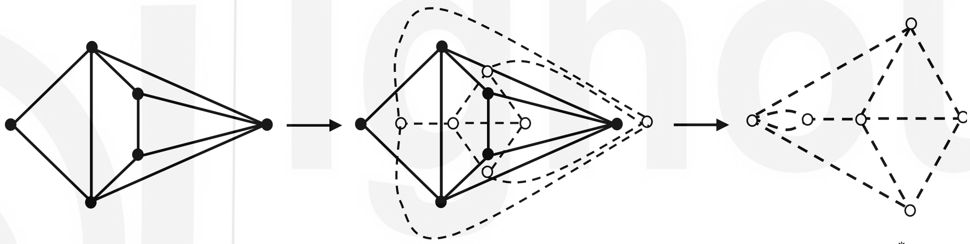


G_1 G_2

Solution: Establishing an isomorphism between G_1 and G_2 is easier. (Just find.) Let us draw the duals of G_1 and G_2.



G_1 G_1^*



G_2 G_2^*

Now observe that $G_1^* \ncong G_2^*$.

***

Now we shall show that the dual of a plane graph is always connected, no matter whether the original graph is connected or not.

Theorem 7: For every plane graph $G,$ its dual G^* is connected.

Proof: Let G be any plane graph. Pick any two vertices u and v in G^*. Then there is a curve in the plane that joins u and $v,$ which does not pass through any of the vertices of $G.$ This curve, of course, passes through the faces and edges of $G,$ alternately. But the faces of G correspond to the vertices of G^*, and the edges of G correspond to the edges of G^*. Thus, we get an alternating sequence of vertices and edges of G^* with endpoints u and $v,$ which is, indeed, a (u,v)-walk in G^*. Therefore, G^* is connected. ■

Now consider the following results.

Proposition: Let G be a connected plane graph. Then the number of faces of G^* is equal to the number of vertices of $G.$

Proof: Assume that G has n vertices, m edges and r faces. Also let G^* have n^* vertices, m^* edges and r^* faces. Since both G and G^* are connected, by the Euler's Formula, $n - m + r = 2$ and $n^* - m^* + r^* = 2.$

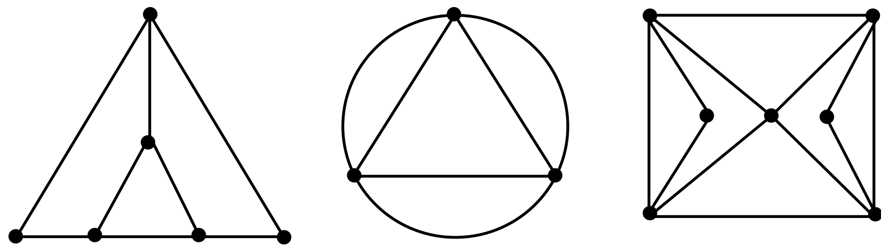
By definition, we have $n^* = r$ and $m^* = m.$ Substituting these values in the above equations, we get $r^* = n.$ ■

Finally we state the following result, without proof.

Theorem 8: If G is a connected plane graph, then $(G^*)^* \cong G.$

Now try the following exercises.

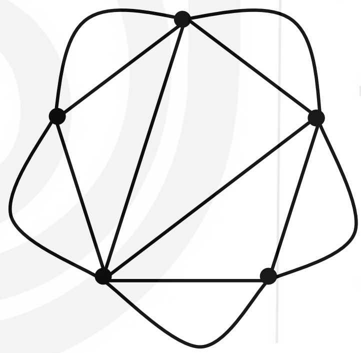
E23) Find dual of the following graphs:



(i) (ii) (iii)

E24) If a plane graph G has a cut-vertex, then G^* also has a cut-vertex. True or false? Justify.

E25) Without actually drawing $G^*,$ find the number of vertices, edges and faces of $G^*,$ where G is the multigraph given below:



E26) Under what conditions on a plane graph, is its dual a graph?

Next, we shall discuss the colouring of plane graphs.

7.7 MAP COLOURING PROBLEM

Recall that in Unit 6 we have discussed the concept of graph colouring, namely the vertex-colouring and edge-colouring. In case of plane graph, there is yet another way to colour a graph! It is the face-colouring. We shall define what face-colouring of a plane graph means, and how it is related to the map colouring problem.

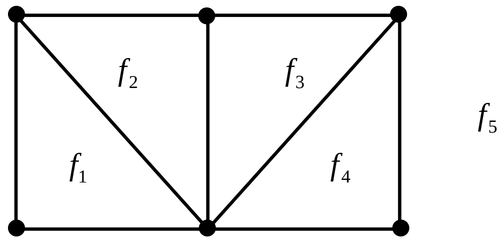
Next, we shall discuss the Map Coloring Problem. We will show that this can be reduced to coloring of planar graphs. We shall also show that any planar graph can be vertex coloured with at must five colours.

Indeed, any planar graph can be vertex-coloured with just 4 colours. However, this remained a problem (known as four Colour Problem) for a long time, until

1979 when Appel and Haken gave an algorithmic proof. They took nearly 1200 hours of computer time on some fastest available computers of that time to eventually conclude that no more than 4 colours were needed.

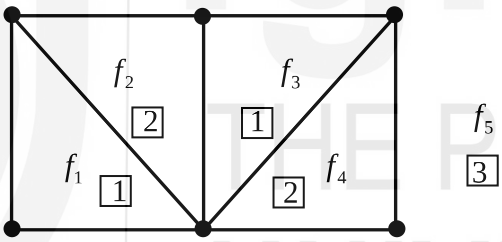
Let us begin with the following definition.

Definition: Let G be a plane graph. A k**-face-colouring** of G is an assignment of k colours to the faces of G in such a way that no two faces with a common boundary get the same colour. If there exists a k-face colouring for G, then G is said to be k**-face-colourable**.
For instance look at the following graph G.



G

There are five faces in G, namely f_1, f_2, f_3, f_4 and f_5, where f_5 is unbounded. The face f_1 shares boundary with f_2 as well as f_5. Thus f_1 must get a colour different from those of f_2 and f_5. A 3-face-colouring of G is given below.



A 3-face-colouring of G.

Now let us look an important relationship between the face-colouring and vertex-colouring.

Theorem 9: Let G be a connected plane graph with no cut-edges. Then G is k-face-colourable iff G^* is k-vertex-colourable.

Proof: Assume that G is k-face-colourable. This means, there is a face-colouring of G that assigns k colours to the faces of G. Recall that the faces of G corresponds to the vertices of G^*. Moreover, two faces of G share a boundary edge iff the corresponding vertices of G^* are adjacent. Since any two faces with a common boundary edge get distinct colours, any two adjacent vertices in G^* get two distinct colours. Thus G^* has a k-vertex-colouring.
Now assume that G^* has a k-vertex-colouring, and reverse the arguments used above to conclude that G has a k-face-colouring.

Now we define what we mean by a map.

Definition: A map is a connected plane graph with no cut-edges.

Thus in the language of maps, the statements of Theorem 10 is as follows:

"A map G is k-face-colourable iff G^* is k-vertex-colourable."

This means the face-colouring of a map is equivalent to the vertex colouring of its dual.

Now, we present the statement of Four-Colour Theorem.

Theorem 10 (Four-Colour Theorem): Every planar graph can be coloured with 4 colours. In other words, if G is a planar graph, then $\chi(G) \leq 4$.

7.8 SUMMARY

In this unit, we have covered the following points:

1. We have described what are plane graph, and planar graph.

2. Any plane drawing of a planar graph G divides the set of points of the plane not lying on G into **regions**, called **faces**.

3. Graphs $\boldsymbol{K_5}$ and $K_{3,3}$ are two special non-planar graphs, called Kuratowski's graphs.

4. We have seen the Euler's Formula that relates the number of vertices, edges and faces of any plane drawing of a connected planar graph.

5. **Kuratowski's Theorem:** A graph is planar if and only if it contains no subdivision of K_5 and $K_{3,3}$.

6. **Wanger's Theorem:** A graph is a planar graph if and only if it contains no subgraph that has $\boldsymbol{K_5}$ or $K_{3,3}$ as a contraction.

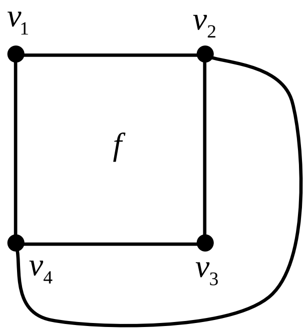
7. We have seen the definitions of thickness and crossing number of a graph

8. We have discussed the PCB layout problem.

9. We have defined the concept of the dual of a plane graph, and finally gave an overview of the Map Colouring Problem.

7.9 SOLUTIONS/ANSWERS

E1) The statement is false. For a counter-example consider the plane graph G and its face f given below.



G

Let $S = \{v_1, v_2, v_3, v_4\}$. Then S induces G, which is not a subgraph of the boundary of f.

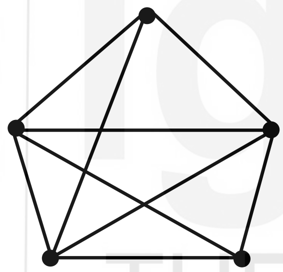
E2) Let G be a planar graph, and H be a component of G. Then H is a subgraph of G. Now by Example 3, H is planar. Therefore, every component of G is planar.

Conversely, assume that a graph G has k components each of them being planar. Then we can draw k mutually disjoint disks in the plane. Now draw, one by one, plane embeddings of these components inside these disks one embedding per disk. This way we get a plane embedding of G. Hence, G is planar.

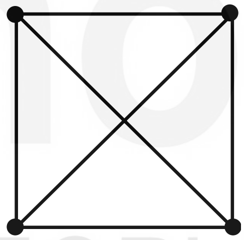
E3) For $n \geq 3$, we can draw the vertices of C_n on a circle, and add edges joining the consecutive vertices. This gives us a plane embedding of C_n.

E4) Try yourself. (See Example 2.)

E5) We know that every proper subgraph of K_5 is a subgraph of one of the following graphs.



$K_5 - e$



K_4

Since both $K_5 - e$ and K_4 are planar, every proper subgraph of K_5 is planar.

E6) True. This is the contrapositive of the statement given in Example 3.

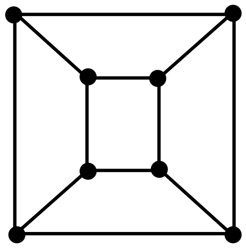
E7) Consider two planar graphs G and H sharing exactly one vertex, say v. Draw two disks meeting exactly at one point in the plane. Then place the vertex v at the point of contact of the disks. Now draw the remaining vertices of a plane drawing of G inside one disk, and of H inside another disk. This gives us a plane drawing of $G \cup H$, and hence $G \cup H$ is planar.

E8) We shall prove it by using the Principle of Mathematical induction on the order of a tree. It is clear that a tree of order 1 is planar. A tree of order 2 is also planar. Now assume that all trees of order n are planar for some $n \geq 2$. Consider a tree T with order $n+1$. Let v be a leaf of T, and u be the neighbour of v in T. Then $T - v$ is a tree with order n. Therefore, by the induction hypothesis, $T - v$ is planar. Now T can be obtained by taking the union of $T - v$ with the edge uv. So, by E7, T is planar. Therefore, by the Principle of Mathematical induction, all trees are planar.

E9) a) Try yourself.

b) The Petersen graph has 10 vertices and 15 edges. If it is planar, then by part a) we must have $15 \le \frac{5}{3}(10-2)$. But this inequality is false. Therefore, the Petersen graph is nonplanar.

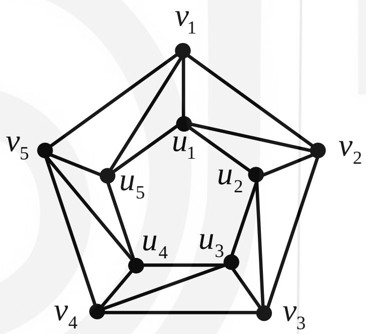
E10) Q_3 is planar because the following is a plane drawing of Q_3.



Recall that Q_4 has $2^4 = 16$ vertices, and $4 \times 2^3 = 32$ edges. We also know that Q_4 is bipartite, and hence has no triangles. Now if Q_4 is planar, then by Corollary 2, $32 \le 2 \times 16 - 4$. but this inequality does not hold. Hnece Q_4 is nonplanar.

E11) False. See E5.

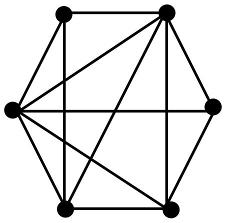
E12) a) Look at the following graph.



Each vertex in this graph has degree 4. Now add two vertices x and y to this graph in such a way that x is the neighbour of u_1, u_2, u_3, u_4 and u_5; and y is a neighbour of v_1, v_2, v_3, v_4 and v_5. This gives us a desired planar graph.

b) No such graph exists due to Corollary 3.

E13) One such graph is given below.



E14) Let n_i, m_i, r_i denote the number of vertices, number of edges, and the number of faces, respectively in the i[th] component of G. Note that one face is common in all the components. Therefore,

$$r = \sum_{i=1}^{k} r_i - (k-1).$$

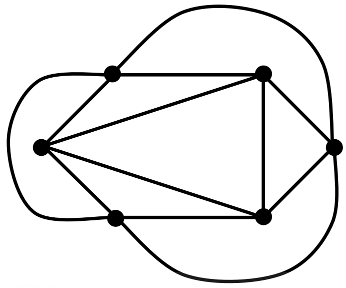
Now applying the Euler's Formula for each component we get $n_i - m_i + r_i = 2,$ i.e., $r_i = m_i - n_i + 2$ for all $i = 1, 2, \ldots, k.$ Therefore,

$$r = \sum_{i=1}^{k} (m_i - n_i + 2) - (k - 1)$$

$$= m - n + 2k - k + 1 = m - n + k + 1.$$

This gives us $n - m + r = k + 1.$

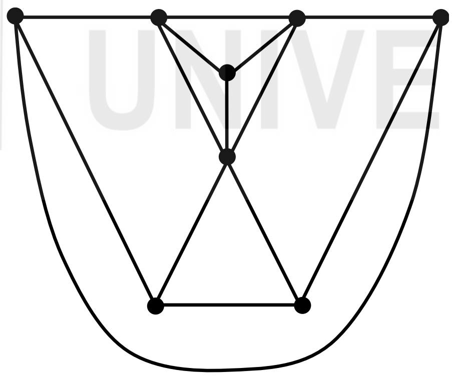
E15) A 4-regular planar graph on 6 vertices is drawn below.



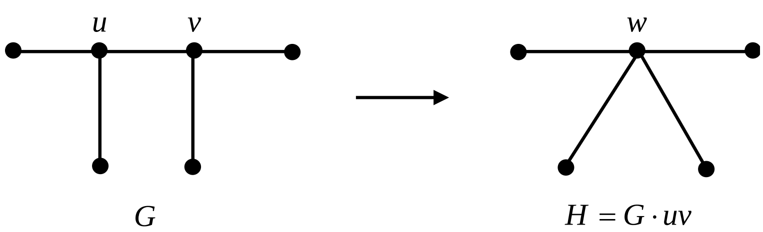
We know that $3n - 6$ is the maximum number of edges in a planar graph on n vertices. Here $n = 6$ and $m = 12.$ So $m = 3n - 6.$ Therefore, this graph is maximal planar.

E16) Note that every edge uv of G either belongs to H or transforms into a (u, v)-path in $H.$ Similarly, every edge xy of H either belongs to $K,$ or transforms into an (x, y)-path in $K.$ Thus every edge uv of G either belongs to $K,$ or transforms into a (u, v)-path in $K.$ Therefore, K is a subdivision of $G.$

E17) A plane drawing is given below.

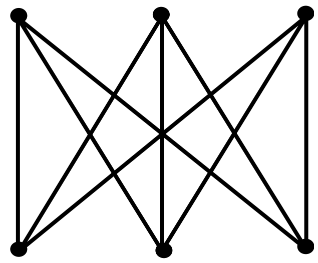


E18) False. A counter-example is given below.



Here H is a contraction of $G.$ But $\Delta(H) \not\leq \Delta(G).$

E19) Observe that the given graph can be contracted into the follow graph.

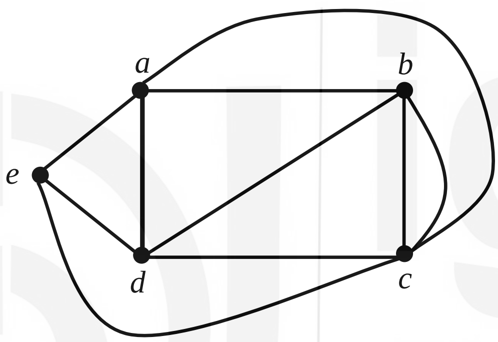


This is $K_{3,3}$. Hence the given graph is not planar.

E20) Note that if G is a planar graph with $n = 7$ vertices, and $m = 12$ edges, then any plane drawing of G has $r = m - n + 2 = 12 - 7 + 2 = 7$ faces. Now each face is bounded by exactly 3 edges, which means that each face has degree 3. Then by the Handshaking Lemma for Planar Graphs, we have $3 \times 7 = 2 \times 12$, which is not true.

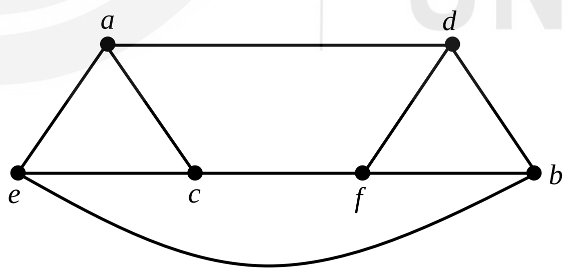
Therefore, no such graph exists.

E21) i) The given graph can be redrawn as follows.

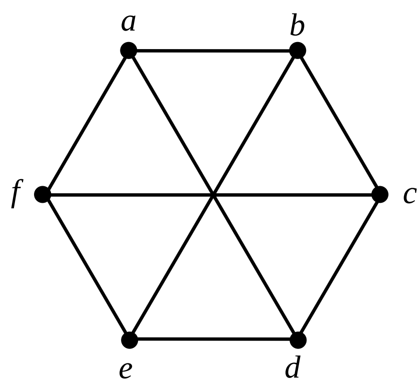


So, the graph is planar. Hence its thickness is 1, and the crossing number is 0.

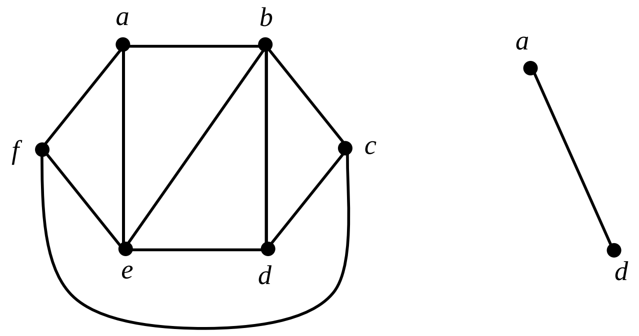
ii) The thickness and the crossing numbers of this graph are also 1 and 0, respectively, as it has a following plane drawing.



iii) Note that the given graph G (say) contains the following subgraph.



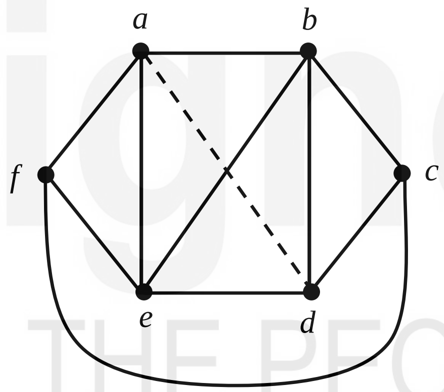
This is isomorphic to $K_{3,3}$. Thus G is nonplanar, and hence has thickness at least 2. Two planar subgraphs of G are drawn below.



G_1 G_2

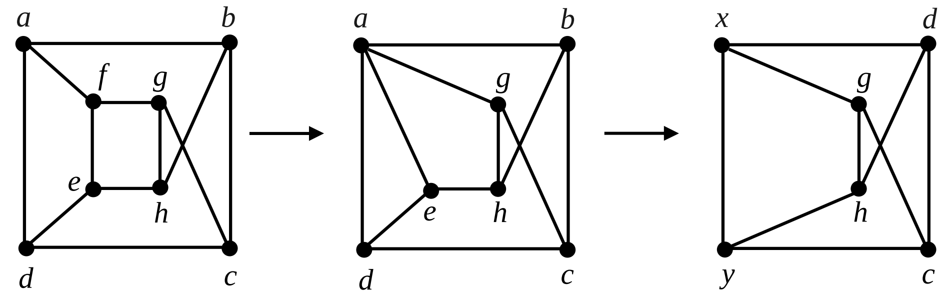
Therefore, $\theta(G) = 2$.

In order to find $\nu(G)$, note that $\nu(G) \geq 1$. A drawing of G with one edge-crossing is given below.



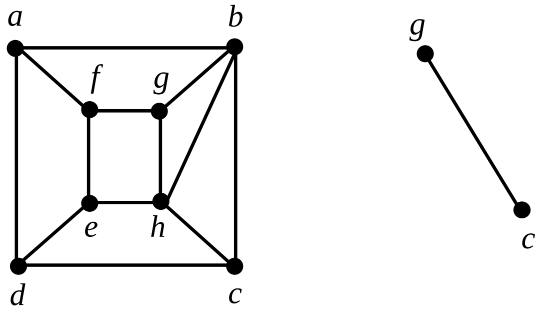
Consequently, $\nu(G) = 1$.

iv) We can contract the given graph into $K_{3,3}$ as follows.



$G_0 = G$ $G_1 = G_0 \cdot af$ $G_2 = G_1 \cdot de$

Observe the $G_2 \cong K_{3,3}$. Thus G contains $K_{3,3}$ as a contraction. Therefore, by the Wagner's Theorem G is nonplanar. This implies $\theta(G) \geq 2$. Two planar subgraphs forming a decomposition of G are given below.



H_1 H_2

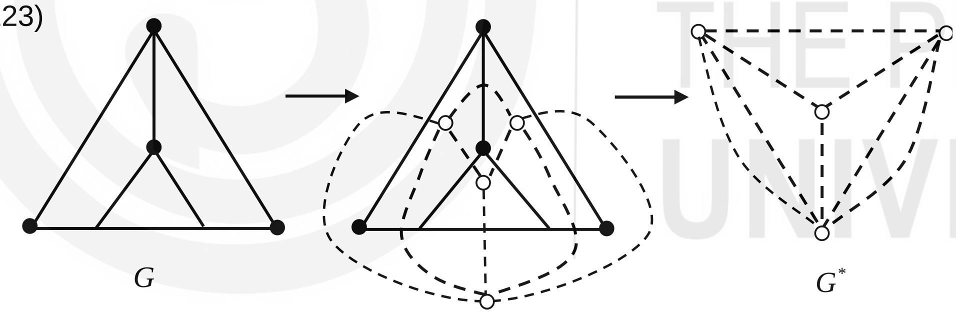
Consequently, $\theta(G) = 2.$

Since G is nonplanar, $\nu(G) \geq 1.$ The given drawing of G has one edge-crossing. Therefore, $\nu(G) = 1.$

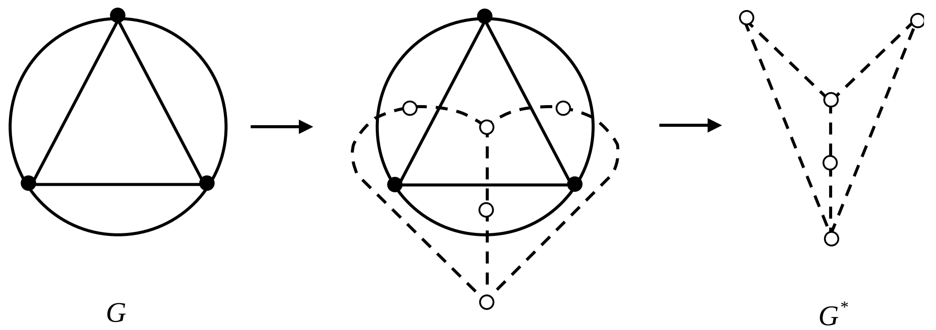
E22) From Theorem 6(b), we get

$$\begin{aligned}\theta(K_{n,n}) &= \left\lceil \frac{n^2}{4n-4} \right\rceil \\ &= \left\lfloor \frac{n^2+4n-4-1}{4n-4} \right\rfloor \\ &= \left\lfloor \frac{(n-1)(n+5)}{4(n-1)} \right\rfloor \\ &= \left\lfloor \frac{n+5}{4} \right\rfloor.\end{aligned}$$

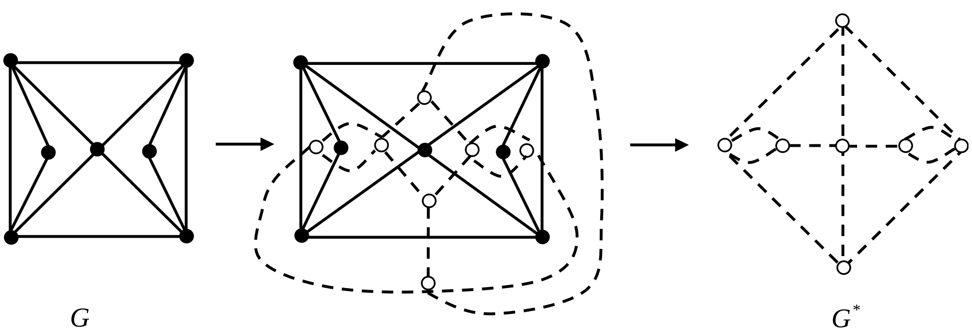
E23)



G G^*



G G^*



G G^*

E24) False. For instance, look at the graph G and its dual G^* drawn below.

G G^*

Here G has a cut-vertex, but G^* does not have any cut-vertex.

E25) The multigraph G has $n=5$ vertices, $m=12$ edges and $r=9$ faces. Therefore, G^* has $n^*=r=9$ vertices, $m^*=m=12$ edges and $r^*=n=5$ faces.

E26) Observe that a cut-edge in G corresponds to a loop in G^*. Also, if any two faces of G share more than one boundary edges, then the corresponding vertices in G^* will be the endpoints of parallel edges. Thus for G^* to be a graph, G must have not share more than one boundary edge.