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M. SC. (MATHEMATICS WITH APPLICATIONS IN COMPUTER SCIENCE) [M. SC. (MACS)]

Term-End Examination

December, 2024

MMT-008 : PROBABILITY AND STATISTICS

Time : 3 Hours *Maximum Marks : 100*

Weightage : 50%

Note : *(i) Question No. 8 is compulsory. Attempt any* ***six*** *questions from question nos.* ***1*** *to* ***7****.*

(ii) Use of scientific and non-programmable calculator is allowed.

(iii) Symbols have their usual meanings.

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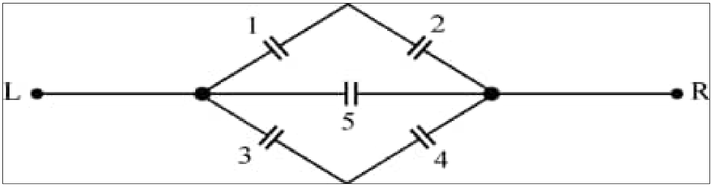
1. (a) Consider the following system consisting of five relays : 1, 2, 3, 4 and 5 between the

terminals L and R. Let the probability of the closing of each relay of the circuit be given by *p*. If all the relays function independently, what is the probability that current exists between the terminals L and R ? 6



(b) Let the random variables X and Y have the joint probability density function (p.d.f.) :

$$f_{X,Y}(x, y) = \frac{2}{n(n+1)}, \quad y = 1, 2, \ldots, x;$$
$$x = 1, 2, \ldots, n$$

Then, compute :

(i) The marginal P.d.f.'s f_X and f_Y.

(ii) The conditional pdf's $f_{X|Y}(\cdot \mid y)$ and $f_{Y|X}(\cdot \mid x)$.

(iii) The conditional expectation $E(X \mid Y = y)$. 9

2. (a) Let the random vector X follows $N_4(\mu, \Sigma)$, where $\mu = (2, 1 - 1 - 2)'$ and

$$\Sigma = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -2 & -1 \\ 1 & -2 & 9 & -1 \\ 1 & -1 & -1 & 16 \end{bmatrix}$$

Find the mean and variance of $P'X$ and $\text{Cov}(P'X, Q'X)$ if $P' = (1\ 1\ 2\ 1)$ and $Q' = (1 - 2\ 2\ 1)$. 8

(b) Define the higher order transition probabilities in a Markov chain. In this context, state and prove the Chapman-Kolmogorov equation which is satisfied by the transition probabilities of a Markov chain. 7

3. (a) In a X-ray clinic, patients arrive at a mean rate of 6 per hour and the technician takes their X-ray at a mean rate of 8 per hour. On the basis of past records, it is assumed

that the service time follows some unspecified positive skewed unimodal distribution with standard deviation of 0.083 hour (approximately 5 minutes). If there is only one technician to serve the patients and the queue capacity is infinite, find the values of the following queuing characteristics : 6

(i) Average queue length

(ii) Average waiting time in the queue

(iii) Average waiting time in the system

(b) In a clinical study of concentration change in fatty acid (FA) and blood glucose (G) in 12 schizophrenic patients and 13 normal persons, the following records were obtained :

	Mean Change	
	Schizophrenics	Normals
G (in mg %)	– 25.6	– 31.1
FA (in m-equiv/litre)	– 0.06	– 0.15

Pooled Covariance matrix

$$S = \begin{bmatrix} 302 & 0.292 \\ 0.292 & 0.0085 \end{bmatrix}$$

Assuming that both FA and G follow a bivariate normal distribution, test the hypothesis that the normal and schizophrenic mean vectors are equal. [You many like to use the following values :

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$F_{0.05,\,2,\,22} = 3.44$, $F_{0.05,\,2,\,12} = 3.88$]

4\. (a) Let the random variables X and Y represent the temperature of a certain object in degrees Celsius and Fahrenheit respectively. Then it is known that $Y = \frac{9}{5}X + 32$ and $X = \frac{5}{9}Y - \frac{160}{9}$.

(i) If $Y \sim N(\mu, \sigma^2)$, determine the distribution of X.

(ii) If $P[90 \le Y \le 95] = 0.95$, then also $P[a \le X \le b] = 0.95$ for some $a < b$.

Determine the numbers a and b. 8

(b) Let $X \sim N_3(\mu, \Sigma)$ with $\mu = \begin{bmatrix} -3 \\ 1 \\ 4 \end{bmatrix}$ and

$\Sigma = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. Then determine which of the following random variables are independent : 7

(i) (X_1, X_2) and X_3

(ii) $(X_1 + X_2)/3$ and X_3

5. (a) An insurance company has three claims adjusters in its branch office. People with claims against the company are found to arrive in a Poisson fashion at an average rate of 20 per 8-hour day. The amount of time an adjuster spends with a claimant is found to have a negative exponential distribution, with mean service time 40 minutes. Claimants are processed in the order of their appearance : 7

(i) How many hours in a week can an adjuster expect to spend with claimants ?

(ii) How much time, on an average, does a claimant spend in the branch office ?

(b) What are the Renewal Processes ? Give some examples of renewal processes. Mentioning the defining properties of the Poisson process, obtain the probability of n events in a time interval of length t. Also show that inter-arrival times between two successive arrivals in Poisson process follows an exponential distribution. 8

6. (a) Let us consider the vector $Y = (y_1, y_2, y_3, y_4)$ following $N_4(\mu, \Sigma)$ with $\mu' = (2\ 6\ 3 - 3)$ and $\Sigma = \begin{bmatrix} 16 & 0 & 3 & 0 \\ 0 & 9 & 0 & 2 \\ 3 & 0 & 4 & 1 \\ 0 & 2 & 1 & 9 \end{bmatrix}$. 7

(i) Obtain the marginal distribution of $\begin{pmatrix} y_1 \\ y_3 \end{pmatrix}$.

(ii) Obtain the conditional distribution of $\begin{pmatrix} y_3 \\ y_4 \end{pmatrix}$ given $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1.5 \\ -2.6 \end{pmatrix}$.

(iii) Find the correlation coefficient between y_1 and y_3.

(b) The transition probability matrix of a two-state Markov chain is given by :

$$P = \begin{bmatrix} (1-\theta) & \theta \\ \beta & (1-\beta) \end{bmatrix}, \quad 0 \le \theta, \beta \le 1$$

$$|1-\theta-\beta| < 1$$

Show that the n-step transition probabilities are given by : 8

$$P^{(n)} = A + (1-\theta-\beta)^n B,$$

where :

$$A = \begin{bmatrix} \beta/(\theta+\beta) & \theta/(\theta+\beta) \\ \beta/(\theta+\beta) & \theta/(\theta+\beta) \end{bmatrix}$$

and $B = \begin{bmatrix} \theta/(\theta+\beta) & -\theta/(\theta+\beta) \\ \beta/(\theta+\beta) & -\beta/(\theta+\beta) \end{bmatrix}$.

7. (a) The life of a certain part in a new automobile is a random variable X whose p.d.f. is negative exponential with parameter $\lambda = 0.005$ days. Then : 8

(i) Find the expected life of the part.

(ii) If the automobile comes with a spare part whose life is a random variable Y, distributed as X and independent of it, find the p.d.f. of the combined life of the part and its spare.

(iii) Find the probability that $X + Y \geq 500$ days.

(b) Consider the function *f* defined by :

$$f(x, y) = \begin{cases} \dfrac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}, & \text{for } (x, y) \text{ outside the square } [-1,1] \times [-1,1] \\ \dfrac{1}{2\pi} e^{-\frac{x^2+y^2}{2}} + \dfrac{1}{2\pi e} x^3 y^3, & \text{for } (x, y) \text{ in the square } [-1,1] \times [-1,1] \end{cases}$$

(i) Show that *f* is a non-bivariate normal p.d.f.

(ii) Show that both marginals are N (0, 1) p.d.f.'s. 7

8. State whether the following statements are True *or* False. Justify your answer with a short proof or a counter example : 10

(i) Consider an experiment of tossing of a coin of 10 rupees and 5 rupees. Let us define the events :

E_1 : Head on 10 rupees coin

E_2 : Head on 5 rupees coin

E_3 : Coins match

The events are mutually independent.

(ii) Let random variable $x \sim$ negative exponential distribution with parameter λ, that is :

$$f(x) = \lambda e^{-\lambda x}, \quad x > 0, \quad \lambda > 0$$

then $P(X > s + t \mid X > s) = P(X > t)$ for some $s, \; t > 0$.

(iii) For the three states 0, 1, 2 the Markov chain has the t. p. m. :

$$\begin{bmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{bmatrix}$$

with initial distribution $P[X_0 = i] = \frac{1}{3}$, $i = 0, 1, 2$. Then $P[X_2 = 1, X_0 = 0] = \frac{5}{38}$.

(iv) In the Poisson renewal process, N_t, the probability of number of events in $(0, t]$ is given by :

$$P[N_t = r] = F_r(t) - F_{r+1}(t);$$

$F_r(t) = 1 - P[S_r > t]$, S_r being the time of rth event = $X_1 + X_2 + \ldots\ldots + X_r$.

(v) In $(M/M/1) : (\infty / FIFO)$ queueing model, the average queue length is $\frac{\lambda^2}{\mu^2}(\mu - \lambda)^2$.

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