

**M. SC. (MATHEMATICS WITH
APPLICATIONS IN COMPUTER
SCIENCE) [M. SC. (MACS)]
Term-End Examination
December, 2025**

MMT-006 : FUNCTIONAL ANALYSIS

Time : 2 Hours

Maximum Marks : 50

Weightage : 70%

Note : (i) *Question No. 1 is compulsory.*

(ii) *Answer any **four** questions from the remaining.*

1. State True or False, with justification, for each statement : 5×2=10

(a) $\mathbf{R}^2$ with any norm is complete.

- (b) 0 is not an eigen value of any non-zero compact operator on an infinite-dimensional Banach space.
 - (c) If M is a closed subspace of a Hilbert space, there is a subspace N such that $M = N^\perp$.
 - (d) Every linear subspace of $X \times X$ is the graph of a linear map.
 - (e) X' separates points of X for any normed space X .
2. (a) Let M be a closed subspace of a Banach space X . Prove that there is a closed subspace N such that $X = M \oplus N$ if and only if there is a bounded projection P with $R(P) = M$. 4
- (b) Show that a compact self-adjoint operator has an eigen value. 3
- (c) Prove that the dual space of C_0 is l^1 . 3

3. (a) If $\{e_i\}$ is an orthonormal set in a Hilbert space H such that

$$\|x\|^2 = \sum | \langle x, e_i \rangle |^2, \forall x \in H,$$

then prove that $\{e_i\}$ is an orthonormal basis. 3

- (b) Show that for any bounded linear operator A on a Banach space X , the

series $\sum_0^{\infty} \frac{A^n}{n!}$ converges to a bounded

invertible operator. 2+2

- (c) Prove that a normed space in which every absolutely convergent series is convergent is complete. 3

4. (a) If M is a subspace of a normed space X , show that M' is linearly isometric to $X'/M^{\perp}$. 4

- (b) Define a positive operator. Let A be bounded self-adjoint operators on a Hilbert space H . Show that : 2+4

- (i) A^2 is a positive operator.
- (ii) If A is a positive operator, then $I + A$ is invertible.

5. (a) Let X be a normed space. Let : 3

$$B(x, s) = \{y \in X : \|x - y\| > s\}$$

Show that :

$$\overline{B(x, s)} = \{y \in X : \|x - y\| \leq s\}$$

- (b) State open mapping theorem. Give an example to show that the open mapping theorem may not hold if the normed spaces are not Banach spaces. 4
- (c) Calculate the adjoint of the operator S defined on l^2 by $S(x) = (x(2), x(3), \dots)$.

3

6. (a) If A is a bounded operator on an infinite dimensional Banach space X such that $\|Ax\| \geq C\|x\|$ for all $x \in X$ and some $C > 0$, prove that A cannot be compact. 4
- (b) Define an approximate eigen value and give an example. 1+2
- (c) Prove that the space $C^1[0, 1]$ is complete with the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$. 3

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