

**M. SC. (MATHEMATICS WITH
APPLICATIONS IN COMPUTER
SCIENCE) [M. SC. (MACS)]**

Term-End Examination

June, 2025

MMT-006 : FUNCTIONAL ANALYSIS

Time : 2 Hours

Maximum Marks : 50

Weightage : 70%

Note : (i) *Question No. 1 is compulsory.*

(ii) *Answer any **four** questions from the
remaining questions.*

1. State, with justification, whether each of the following statements is true or false : $5 \times 2 = 10$

(a) $[-2, 2] \times [-2, 2]$ is the unit ball for some norm on $\mathbf{R}^2$.

- (b) If M is a proper subspace of a Hilbert space H , then $M^\perp \neq (0)$.
- (c) Every non-zero linear functional on a normed space is an open map.
- (d) The dual of a separable normed space is separable.
- (e) For a non-zero compact operator, 0 is never an eigen value.
2. (a) State the Schwarz inequality and find conditions for equality to hold. $1+3=4$
- (b) Prove the completeness of the space $(l^1, \|\cdot\|_1)$. 3
- (c) If $\{A_n\}$ is a sequence of bounded linear operators on a Banach space X and if $A_n x \rightarrow Ax \forall x \in X$, show that A is linear and continuous. 3
3. (a) Show that the canonical map $J: X \rightarrow X''$ for a normed space X is a

linear isometry. Give an example,
where it is not onto. 2+2=4

- (b) A is a bounded linear operator on a Hilbert space H . If A is unitary, then for every orthonormal basis $\{u_i\}$ of H , $\{Au_i\}$ and $\{A^*u_i\}$ are both orthonormal bases for H . 3

- (c) For $f \in C[0,1]$ define $\phi(f) = \int_0^1 x f(x) dx$,
find $\|\phi\|$ for the norm $\|\cdot\|_\infty$ on $C[0, 1]$. 3

4. (a) If a Banach space is reflexive, then prove that its dual is reflexive. 3

- (b) Let A be a bounded self-adjoint operator on a Hilbert space H . Prove that : 3

$$\|A\| = \sup \{ | \langle Ax, x \rangle | : \|x\| = 1 \}$$

- (c) Show that a linear map T on a Banach space X is closed if and only if the graph $G(T)$ is complete in $X \times X$. 2+2=4

5. (a) State the Riesz representation theorem for Hilbert spaces. If f is a bounded linear functional on a Hilbert space H , then show that the representing vector of f is given by $\sum \overline{f(e_i)} e_i$, where $\{e_i\}$ is an orthonormal basis. 1+3=4
- (b) If $\{x_n\}$ is a sequence in a normed space X such that $\{f(x_n)\}$ is convergent for each $f \in X'$, prove that $\{x_n\}$ is bounded. 3
- (c) Let H be a Hilbert space and let $u, v \in H$. Define $Ax = \langle x, u \rangle v$. Calculate $\|A\|$. 3
6. (a) For $A, B \in M(n, \mathbf{R})$, the space of $n \times n$ real matrices, define $\langle A, B \rangle = \text{trace}(AB^t)$. Show that this gives an inner product and find an orthonormal basis. (B^t = transpose of B). 2+2=4

- (b) Let $\{x_1, \dots, x_n\}$ be a linearly independent set in a normed space X . Prove that there are $f_i \in X'$ such that $f_i(x_j) = \delta_{ij}$. 3

- (c) Prove that $l^1 \subset l^2$ and find $(l^1)^\perp$ in l^2 .

$$2+1=3$$

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