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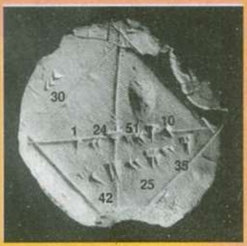
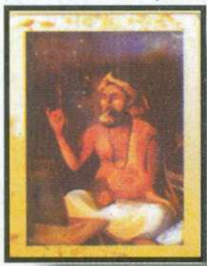
Indira Gandhi National Open University
School of Computer and Information Sciences

BCS-054
Computer Oriented Numerical Techniques



Babylonian clay tablet YBC 7289 for value of $\sqrt{2}$(c.1800-1600 BC)

Aryabhatta (c. 476-550 AD)



Varahmihira (c. 505-587 AD)

Babylonian Mathematical clay tablet Pimpton 322 (c.1800 BC)

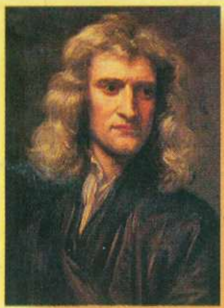
Vedic Period: Sulbha/ Shulbha sutras composed by **Baudhayana** (c. 800 BCE), **Manav** (c. 750 BCE), **Apastamba** (c. 600 BCE) and **Katyayana** (3rd Century, BCE) contain, among other mathematical contributions, examples of Pythagorean triples and formula for the value of $\sqrt{2}$ giving accuracy up to 5 places after the decimal.

Brahmagupta (c. 598-670 AD)

Muhammad ibn Musa al-Khwarizmi (c.780-c.850 AD)

A page from Hisab al-jabar w' al Muqabala by al-Khwarizmi (c. 820 AD)

Fragment from Euclid's Elements (c.100 AD)

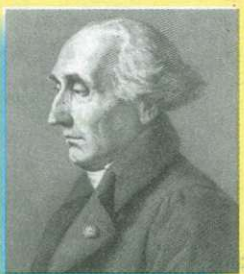


Isaac Newton (1642-1727 AD)

Leonhard Euler (1707-1783 AD)

A page from Bakhshali Manuscript (Between 2nd Cent. BC, 3rd Cent. AD)

A page from (Chinese): The Nine Chapters on the Mathematical Art (c. 2nd Cent. AD)



Joseph-Louis Lagrange (1736-1813 AD)

Johann Carl Friedrich Gauss (1777-1855 AD)

A page from Ganita-Kaumudi (1356 AD) on Magic Squares

Differentiation, Integration and Differential Equations

3

"शिक्षा मानव को बन्धनों से मुक्त करती है और आज के युग में तो यह लोकतंत्र की भावना का आधार भी है। जन्म तथा अन्य कारणों से उत्पन्न जाति एवं वर्गगत विषमताओं को दूर करते हुए मनुष्य को इन सबसे ऊपर उठाती है।"

– इन्दिरा गांधी

"Education is a liberating force, and in our age it is also a democratising force, cutting across the barriers of caste and class, smoothing out inequalities imposed by birth and other circumstances."

- Indira Gandhi

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Indira Gandhi National Open University
School of Computer and Information Sciences

BCS-054
COMPUTER ORIENTED NUMERICAL TECHNIQUES

Block

3

DIFFERENTIATION, INTEGRATION AND DIFFERENTIAL EQUATIONS

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BLOCK INTRODUCTION

This block also has three units, about respectively, Numerical Differentiation, Numerical Integration and numerical solutions of ordinary differential equations.

Though, there are a number of methods for differentiation, yet, this being an introductory course; **in Unit 1,** we discuss only methods of differentiation based on interpolation. We discuss methods of numerical differentiation, first, for uniformly spaced tabular points and then for non-uniformly spaced tabular points, all based on Interpolation. The methods are illustrated through a number of examples.

In Unit 2, we discuss the topic of numerical integration. First, we discuss the methodology used in the methods discussed in the unit. Then, based on this methodology, three methods for numerical integration are discussed, which respectively use: Rectangular Rule, Trapezoidal Rule and Simpson's (1/3) Rule. Finally, we discuss the issue of errors introduced by each of the methods.

In Unit 3, we discuss the topic of numerical solution of special type of ordinary differential equations. For this purpose, first, we define a number of required concepts, including that of Initial Value Problem (IVP/ ivp). Then, we describe (general) methodology; used by the methods, discussed in the unit, for numerical solutions of IVPs; in two parts. The following four methods, based on the methodology, are discussed: Euler's Method, Improved Euler's Method, Runge Kutta Method of Order 2 and Runge Kutta Method of Order 4.

UNIT 1 NUMERICAL DIFFERENTIATION

Structure

1.0 INTRODUCTION

Derivatives and higher order derivatives play very important role in scientific investigations, particularly, in respect of measuring/approximating rate of changes in some (dependent) variables with respect to some other (independent) variables, occurring in some scientific phenomena. There are a number of analytical methods in calculus for finding the derivatives of dependent variables/functions. However, these methods can be applied only when the (dependent) variables can be expressed in some closed form, like $y = x^2 + \log(x)$, etc. Most of the time, the available scientific data either does not explicitly exhibit such a closed form relationship, or, in many cases, even cannot be expressed in such a form at all. The data, generally, is available in tabular form. In such cases, and otherwise also, the numerical methods, to be discussed in this unit, many times prove to be quite useful.

Before discussing various methods for the purpose, let us look into some basic issues in respect of our subject matter. Let us recall a bit of our knowledge of co-ordinate geometry. The statement *'An arbitrary point P on a line has co-ordinate x'*, has the underlying assumptions : (i) there is a point of reference, or origin, say O, which has been assigned value 0 (ii) there is a unit of measurement, that is taken as 1. Then the statement *'An arbitrary point P on a line has co-ordinate x'* means P is x units away from the origin

O (0) (1) P (x)

In order to explain what we intend to do, let us assume a particular value of x is 22.

Instead of 0 and 1, we now take x_0, say 2, (x_0 times of the unit mentioned above) as new point of reference, and $h = x_1 - x_0$ (again in terms of the unit mentioned above) as new unit of measurement, where x_1 (say 6) is some point other than x_0. Then, for the assumed values, $h = x_1 - x_0 = 6 - 2 = 4$. Also, the same point P, in the new system, will have co-ordinate, some number, say, p *(under our assumptions, p=5)*. As the point considered is the same P, therefore, $x = x_0 + ph$.

O (0) x_0 h $x_1(1)$ P (p)

Thus, an arbitrary point P, earlier denoted by (the variable) x, is now denoted by (the variable) p (and, h being the new unit). Also, x and p are related by the equation

$$\mathbf{x = x_0 + ph} \qquad \ldots \text{(i)}$$

Further, if y is a dependent variable, which is a function of (independent variable) x, then, in view of equation (i), y can be thought as a function of variable p. Thus, we can talk of dy/dp, instead of dy/dx. Also, from (i), we get, **dp/dx = (1/h)**.

The relevance of the above discussion lies in the fact that the derivative of a function, the subject matter of this unit, will be discussed in terms of the derivative of some polynomial, obtained as an approximation of the given function, by any of the methods discussed under **Interpolation** in **Unit 2 of Block 2**, namely, Newton's FD Formula, Newton's BD Formula, Stirling's Formula, Bessel's Formula, in the case of equidistant tabular points and by any of the methods, namely, Lagrange's Method, Newton's Divided Difference Method, etc., in the general case.

Before coming to the discussion of finding the value of a derivative of a dependent variable; in the rest of the section, we recall the formulae from Unit 2, about approximating the value of a dependent variable y (not the derivative of y), which include **Newton's** Forward Difference **(FD)**, **Newton's** Backward Difference **(BD).**

For this purpose, let us suppose that **we have to compute the value of y corresponding to the independent variable x, or the new independent variable p, where,** from Eq. (i) we have $\mathbf{x = x_0 + ph}$ **or** $\mathbf{p = (x - x_0)/h}$. We denote the value of y at x_0 by y_0 i.e., $y_0 = y(x_0)$. We also recall Shift operator E, defined as $\mathbf{E\,y(x) = y(x + h)}$ **or** $\mathbf{E\,(p) = y(p+1)}$ **so that** $\mathbf{y = y\,(x) = y\,(x_0 + ph) = E^p\,y\,(x_0) = E^p\,y_0}$. In the rest of the section, the independent variable will be p only. Expressing E in terms of Δ or ∇ we get FD and BD interpolation formulas respectively.

(i) Newton's FD Formula

$$y_p = E^p y_0 = (1+\Delta)^p y_0 = \left(1 + p\Delta + \frac{p(p-1)}{2!}\Delta^2 + \frac{p(p-1)(p-2)}{3!}\Delta^3 + \ldots\right) y_0$$

$$= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots$$

This formula is used to compute the value of y near the upper end **(the beginning)** of the table such that $0 < p < 1$.

(ii) Newton's BD Formula

$$y_p = E^p y_0 = (1-\nabla)^{-p} y_0 = \left(1 + p\nabla + \frac{p(p+1)}{2!}\nabla^2 + \frac{p(p+1)(p+2)}{3!}\nabla^3 + \ldots\right) y_0$$

$$= y_0 + p\nabla y_0 + \frac{p(p+1)}{2!}\nabla^2 y_0 + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_0 + \ldots$$

This formula is used near the lower end **(the last part)** of the table choosing x_0 such that $-1 < p < 0$. The CD formulas are used to find the value in the middle of the table. Next, two popular CD formulas, i.e., in terms of δ, are given below :

(iii) Stirling's Formula

$$y_p = y_0 + p\mu\delta y_0 + \frac{p^2}{2!}\delta^2 y_0 + \binom{p+1}{3}\mu\delta^3 y_0 + \frac{p^2(p^2-1^2)}{4!}\delta^4 y_0 + \ldots$$

$$= y_0 + p\frac{1}{2}\left(\delta y_{\frac{1}{2}} + \delta y_{-\frac{1}{2}}\right) + \frac{p^2}{2!}\delta^2 y_0 + \frac{(p+1)p(p-1)}{3!}\frac{1}{2}\left(\delta^3 y_{\frac{1}{2}} + \delta^3 y_{-\frac{1}{2}}\right)$$

$$+ \frac{p^2(p^2-1^2)}{4!}\delta^4 y_0 + \ldots$$

This formula should be used for $-0.25 \le p \le 0.25$.

(iv) Bessel's Formula

$$y_p = \mu y_{\frac{1}{2}} + \left(p - \frac{1}{2}\right)\delta y_{\frac{1}{2}} + \binom{p}{2}\mu\delta^2 y_{\frac{1}{2}} + \binom{p}{2}\frac{1}{3}\left(p - \frac{1}{2}\right)\delta^3 y_{\frac{1}{2}} + \binom{p+1}{4}\mu\delta^4 y_{\frac{1}{2}} + \ldots$$

$$= \frac{1}{2}(y_0 + y_1) + \left(p - \frac{1}{2}\right)\delta y_{\frac{1}{2}} + \binom{p}{2}.\frac{1}{2}(\delta^2 y_1 + \delta^2 y_0) + \binom{p}{2}\frac{1}{3}\left(p - \frac{1}{2}\right)\delta^3 y_{\frac{1}{2}}$$

$$+ \binom{p+1}{4}\frac{1}{2}(\delta^4 y_1 + \delta^4 y_0) + \ldots$$

This formula is used for $-0.25 \le p \le 0.75$.

Next, we state formula of interpolation, when the tabular points x_i's need not be equidistant.

Lagrange's Method

In Lagrange's method a polynomial is fitted passing through all the given points. Obviously, if (n + 1) points (x_i, y_i), i = 0 (1) are given, a polynomial of highest degree n can be fitted which is represented as,

$$y(x) = L_0(x)y_0 + L_1(x)y_1 + \ldots + L_i(x)y_i + \ldots + L_n(x)y_n$$

where $L_i(x) = \dfrac{(x - x_0)(x - x_1)\ldots(x - x_{i-1})(x - x_{i+1})\ldots(x - x_n)}{(x_i - x_0)(x_i - x_1)\ldots(x_i - x_{i-1})(x_i - x_{i+1})\ldots(x_i - x_n)}$;

$L_i(x), i = 0(1)n$ are known as Lagrange's coefficients and each of them is a polynomial of degree n. It should be noted that they satisfy a property,

$$L_i(x_j) = 1,\ i = j$$
$$= 0,\ i \neq j$$

After having set the background for discussion, let us have a brief look on the objectives of the unit, before discussing the subject matter in details.

1.1 OBJECTIVES

After going through this unit, you should be able to

- state the formulae for finding derivatives and higher order derivatives based on FD, BD, Stirling's Difference and Bessel's Difference Formulae, when the tabular points are equi-distant;
- use these formulae in finding derivatives and higher order derivatives when the tabular points are equi-distant;
- state the formulae for finding derivatives and higher order derivatives based on Lagrange's Difference Method, when the tabular points are not necessarily equi-spaced; and
- use these formulae in finding derivatives and higher order derivatives when the tabular points are not necessarily equi-spaced.

1.2 METHODS BASED ON INTERPOLATION – UNIFORMLY SPACED TABULAR POINTS

Let us suppose that (n + 1) function values $y_i = f(x_i)$ are given for $x = x_i$, i = 0 (1) n and that $x_0 < x_1 < x_2 < \ldots < x_n$ and $x_i - x_{i-1} = h$, i = 1 (1) n.

Assuming that f(x) is differentiable in the interval $I = [x_0, x_n]$ we can compute the derivatives of f(x) at any point x in the interval I by differentiating some appropriate interpolation formula. From our discussion in **1.0 : Introduction**, we recall the following :

The symbol y denotes the dependent variable with x denoting independent variable, or p denoting the new independent variable. Also, from Eq. (i) we have $x = x_0 + ph$ or $p = (x - x_0)/h$. The value of y at x_0 is denoted by y_0 i.e., $y_0 = y(x_0)$. We also recall Shift operator E, defined as $E\ y(x) = y(x + h)$ or $E\ (p) = y\ (p + 1)$ so that $y = y(x) = y\ (x_0 + ph) = E^p\ y\ (x_0) = E^p\ y_0$. **In the rest of the section, the independent variable will be p only.**

From $p = (x - x_0)/h$, we get, $dp/dx = 1/h$

Hence, $dy/dx = (dy/dp).\ (dp/dx) = (1/h)\ dy/dp$.

and $$\frac{d^2y}{dx^2} = \frac{1}{h^2} \cdot \frac{d^2y}{dp^2}, \text{ etc.}$$

Newton's FD Formula, for y= y(p), y as a function of p, is

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots$$

This formula is used to compute the value of y near the upper end (the beginning) of the table such that $0 < p < 1$.

(i) Differentiating FD interpolation formula we get $(0 \le p < 1)$,

$$dy/dx = (dy/dp).\ (dp/dx) = (1/h)\ dy/dp$$
$$= (1/h)\ \{\Delta y_0 + ((2p - 1)/2)\ \Delta^2 y_0 + ((3p^2 - 6p + 2)/6)\ \Delta^3 y_0 + \ldots\} \quad \ldots (A)$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\{\Delta^2 y_0 + (p-1)\ \Delta^3 y_0 + \ldots\}$$

(ii) Differentiating BD formula we get $(-1 < p \le 0)$

$$\frac{dy}{dx} = \frac{1}{h}\left\{\nabla y_0 + \frac{2p+1}{1}\nabla^2 y_0 + \frac{3p^2+6p+2}{6}\nabla^3 y_0 + \ldots\right\} \quad \ldots (B)$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\{\nabla^2 y_0 + (p+1)\ \nabla^3 y_0 + \ldots\}$$

(iii) Differentiation of Stirling's formula gives $(-0.5 \le p \le 0.5)$,

$$\frac{dy}{dx} = \frac{1}{h}\left\{\mu\delta y_0 + p\delta^2 y_0 + \frac{3p^2-1}{6}\mu\delta^3 y_0 + \frac{2p^3-p}{12}\delta^4 y_0 + \ldots\right\} \quad \ldots (C)$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left\{\delta^2 y_0 + p\,\mu\delta^3 y_0 + \frac{6p^2-1}{12}\delta^4 y_0 + \ldots\right\}$$

(iv) By differentiating Bessel's formula we get $(0.25 \le p \le 0.75)$

$$\frac{dy}{dx} = \frac{1}{h}\left\{\delta y_{\frac{1}{2}} + \frac{2p-1}{2}\delta^2 y_{\frac{1}{2}} + \frac{6p^2-6p+1}{12}\delta^3 y_{\frac{1}{2}} + \frac{2p^3-3p^2-p+1}{12}\mu\delta^4 y_{\frac{1}{2}} + \ldots\right\} \quad \ldots (D)$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left\{\mu\delta^2 y_{\frac{1}{2}} + \frac{2p-1}{2}\delta^3 y_{\frac{1}{2}} + \frac{6p^2-6p-1}{12}\mu\delta^4 y_{\frac{1}{2}} + \ldots\right\}.$$

It must be remembered, as already indicated, that appropriate formula should be used i.e. FD formula at the upper end of the table, BD formula near the lower end of the table and CD formula in the middle of the table. Further, the point $x = x_0$ has to be chosen suitably according to the formula used. It may also be noted that to find a derivative at the tabular point $x = x_0$ the value of $p = 0$.

It may also be mentioned that in most of the cases we do not go beyond second or third differences in a formula for computing derivatives.

Example 1

The values of $y = \sqrt{x}$ are given below for x = 1.5 (0.5) 3.5.

X	1.5	2.0	2.5	3.0	3.5
$y = \sqrt{x}$	1.2247	1.4142	1.5811	1.7320	1.8708

Find y′ and y″ at x = 2.25 using FD formula.

Solution

The difference table is constructed as follows :

X	Y	1st diff.	2nd diff.	3rd diff.	4th diff.
1.5	1.2247				
		0.1895			
2.0	1.4142		– 0.0226		
		0.1669		0.0066	
2.5	1.5811		– 0.0160		– 0.0027
		0.1509		0.0039	
3.0	1.7320		– 0.0121		
		0.1388			
3.5	1.8708				

Choosing $x_0 = 2.0,\ p = \dfrac{2.25 - 2.0}{0.5} = 0.5$

$$y'(x) = \frac{1}{h}\left\{\Delta y_0 + \frac{2p-1}{2}\Delta^2 y_0 + \frac{3p^2 - 6p + 2}{6}\Delta^3 y_0\right\}$$

$$= \frac{1}{0.5}\left\{0.1669 + \frac{2\times 0.5 - 1}{2}\times(-0.0160) + \frac{3\times(0.5)^2 - 6\times(0.5) + 2}{6}\times(0.0039)\right\}$$

$$= 2\,\{0.1669 + 0 - 0.0001625\}$$

$$= 0.333475$$

$$y''(x) = \frac{1}{h^2}\{\Delta^2 y_0 + (p-1)\,\Delta^3 y_0\}$$

$$= \frac{1}{(0.5)^2}\{-0.0160 + (0.5-1)\times(0.0039)\}$$

$$= 4\,\{-0.0160 - 0.00195\}$$

$$= -0.0718$$

Example 2

From the data of the preceding example, compute y′(x) and y″(x) at the tabular point x = 2.0, using FD formula.

Solution

Choosing $x_0 = 2.0,\ p = 0$

$$y'(x) = \frac{1}{h}\left\{\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0\right\}$$

$$= \frac{1}{0.5}\left\{0.1669 - \frac{1}{2}\times(-0.0160) + \frac{1}{3}\times(0.0039)\right\}$$

$$= 2\,\{0.1669 + 0.0080 + 0.0013\}$$

$$= 0.3524$$

$$y''(x) = \frac{1}{h^2}\{\Delta^2 y_0 - \Delta^3 y_0\}$$

$$= 4\,\{-0.0160 - 0.0039\}$$

$$= -0.0796$$

In many practical problems, at the tabular points we approximate the first derivative by retaining the terms in the differentiation formula up to first difference only and to approximate second derivative, we retain terms in the differentiation formula up to second difference only. Thus, at the tabular point $x = x_i$ where $y = y_i$ the first and second derivatives can be computed by FD, BD and CD formulas as follows :

First derivative

FD : $$y_i' = \frac{1}{h}\Delta y_i = \frac{y_{i+1} - y_i}{h}$$

BD : $$y_i' = \frac{1}{h}\nabla y_i = \frac{y_i - y_{i-1}}{h}$$

CD : $$y' = \frac{1}{h}\delta y_i = \frac{y_{i+\frac{1}{2}} - y_{i-\frac{1}{2}}}{h}$$

Second Derivative

FD : $$y_i'' = \frac{1}{h^2}\Delta^2 y_i = \frac{y_{i+2} - y_{i+1} + y_i}{h^2}$$

BD : $$y_i'' = \frac{1}{h^2}\nabla^2 y_i = \frac{y_i - 2y_{i-1} + y_{i-2}}{h^2}$$

CD : $$y_i'' = \frac{1}{h^2}\delta^2 y_i = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$$

It should be noted that at the top-most point in the table, say $x = x_0$, only FD formula can be used and at the lower-most point, say $x = x_n$ only BD formula can be used while the data is provided for $x = x_i,\ i = 0\,(1)\,n$ and $x_i - x_{i+1} = h,\ i = 1\,(1)\,n$.

Example 3

The values of a function $y = f(x)$ are given below for $x = 1\,(1)\,5$.

x	1	2	3	4	5
y = f (x)	7	15	21	19	3

Find the value of x for which f (x) attains its maximum value. Use Stirling's formula for differentiation.

Solution

x	y	1st diff.	2nd diff.	3rd diff.
1	7			
		8		
2	15		− 2	
		6		− 6
3	21		− 8	
		− 2		− 6
4	19		− 14	
		− 16		
5	3			

We see from the table that maximum should occur in the neighbourhood of $x = 3$; hence we choose $x_0 = 3$, $h = 1$. Let us assume maximum occurs at some point $x = x_p$ which we have to compute. We find $\frac{dy}{dx}$ at $x = x_p$ in terms of p. Then we shall put $\frac{dy}{dx} = 0$ to get the value of p at which f(x) will have maximum value.

From Stirling's formula for first derivative,

$$y'(x) = \frac{1}{h}\left\{\mu\delta y_0 + p\delta^2 y_0 + \frac{3p^2 - 1}{6}\mu\delta^3 y_0\right\}$$

$$= \frac{1}{2}(6 - 2) + p(-8) + \frac{3p^2 - 1}{6}\left(\frac{-6-6}{2}\right)$$

$$= 3 - 8p - 3p^2$$

Putting $3p^2 + 8p - 3 = 0$ and solving for p gives $p = \frac{1}{3}$ or -3.

The point where f(x) attains its maximum is given by

$$x = x_p = x_0 + ph = 3 + \frac{1}{3} = \frac{10}{3} = 3.3333$$

We neglect $p = -3$ since it gives absurd value of x_p.

Further, we can also find the maximum value of f(x) at $x = \frac{10}{3}$ using Stirling's interpolation formula or Bessel's formula.

Example 4

Find the first and second derivatives of y= f(x) at x = 1.1 from the following tabulated values.

x	: 1.0	1.2	1.4	1.6	1.8	2.0
y = f(x)	: 0.0000	0.1280	0.5440	1.2960	2.4320	4.0000

Solution

We calculate the FDs as given in the table below :

X	y = f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
1.0	0.0					
		0.1280				
1.2	0.1280		0.2880			
		0.4160		0.0480		
1.4	0.5440		0.3360		0.0000	
		0.7520		0.0480		0.0000
1.6	1.2960		0.3840		0.0000	
		1.1360		0.0480		
1.8	2.4320		0.4320			
		1.5680				
2.0	4.0000					

Since, $x = x_0 + p\,h$, $x_0 = 1$, $h = 0.2$ and $x = 1.1$, we have $p = \dfrac{1.1-1}{0.2} = 0.5$

Substituting the value of p in formula (A), we get

$$dy/dx = (1/0.2)\ \{\Delta y_0 - (0.25/6)\ \Delta^3 y_0 + \ldots\}$$

Substituting the values of Δf_0 and $\Delta^3 f_0$, in the Eqn., from table given above, we get

$$f'(1.1) = (dy/dx)_{1.1} = 0.63$$

To obtain the second derivative, Differentiate formula (A) and obtain

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\{\Delta^2 y_0 + (p-1)\ \Delta^3 y_0 + \ldots\}$$

Thus $f''(1.1) = 6.6$

Note : If we construct a forward difference interpolating polynomial P(x), fitting the data given in Table above, we will find that $f(x) = P(x) = x^3 - 3x + 2$. Also, $f'(1.1) = 6.3$, $f''(1.1) = 6.6$. The values obtained from this equation or directly as done above have to be same as the interpolating polynomial is unique.

Example 5

Find $f'(x)$ at $x = 0.4$ from the following table of values.

x :	0.1	0.2	0.3	0.4	0.5
y = f(x) :	1. 10517	1.22140	1.34986	1.49182	256

Solution

Since we are required to find the derivative at the right-hand end point, we will use the backward difference formula. The backward difference table for the given data is given by

x	f(x)	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$
0.1	1.10517			
		0.11623		
0.2	1.22140		0.01223	
		0.12846		0.00127
0.3	1.34986		0.01350	
		0.14196		
0.4	1.49182			

Since $x_n = 0.4$, $h = 0.1$, $x = 0.4$, we get $p = 0$.

Substituting the value of p in formula (B), we get

$$f'(0.4) = \frac{1}{4}\left[\Delta f_3 + \frac{1}{2}\Delta^2 f_3 + \Delta^3 f_3\right] \quad (\text{ where } \nabla f_4 = \Delta f_3 \text{ etc.})$$

h

$$= \frac{1}{0.1}\left[0.14196 + \frac{0.0135}{2} + \frac{0.00127}{3}\right]$$

$$= 1.14913$$

Ex. 1 : The position f(x) of a particle moving in a line at different point of time x_k is given by the following table. Estimate the velocity and acceleration of the particle at points x = 1.5 and 3.5.

x	:	0	1	2	3	4
f(x)	:	− 25	− 9	0	7	15

Ex. 2 : Construct a difference table for the following data :

x	:	1.3	1.5	1.7	1.9	2.1	2.3	2.5
f(x)	:	3.669	4.482	5.474	6.686	8.166	9.974	12.182

Taking h = 0.2, compute $f'(1.5)$ and the error, if we are given $f(x) = e^x$.

1.3 METHODS BASED ON INTERPOLATION – NON-UNIFORMLY SPACED TABULAR POINTS

In this section, we derive a differentiation formula for the derivatives using non-uniformly spaced tabular points. That is, when the difference between any two consecutive tabular points is not uniform.

Lagrange's Method

From the Introduction in this unit and from Unit 2, we know that for a given function $y = f(x)$, for which $(n + 1)$ functional values $f(x_i)$ or y_i are given for $i = 0\ (1)\ n$, with x_i's not necessarily equally spaced; for approximation, by Lagrange's method, a polynomial can be fitted passing through all the given. Also, the fitted polynomial can be of highest degree n, and is represented as :

$$P_n(x) = L_0(x)y_0 + L_1(x)y_1 + \ldots + L_i(x)y_i + \ldots + L_n(x)y_n$$

where $L_i(x)=\dfrac{(x-x_0)(x-x_1)\ldots(x-x_{i-1})(x-x_{i+1})\ldots(x-x_n)}{(x_i-x_0)(x_i-x_1)\ldots(x_i-x_{i-1})(x_i-x_{i+1})\ldots(x_i-x_n)}$;

$L_i(x), i=0(1)n$ are known as Lagrange's coefficients and each of them is a polynomial of degree n. It should be noted that they satisfy a property,

$$L_i(x_j)=1,\ i=j$$
$$=0,\ i\neq j$$

The $L_i(x)$'s can also be written as

$$\mathbf{L_i(x) = \phi(x)/\{(x - x_i)\,\phi_1(x_i)\}}, \text{ where}$$

$$\phi(x) = (x - x_0)(x - x_1)\ldots(x - x_i)\ldots(x - x_n), \text{ and}$$

$$\phi_1(x_i) = (x_i - x_0)(x_i - x_1)\ldots\ldots(x_i - x_{i-1})(x_i - x_{I+1})\ldots(x_i - x_n)$$

Differentiating $P_n(x)$ w.r.t. x, we obtain

$$f'(x) \approx P'_n(x) = \sum^n_{i=0} L'_i(x_i)\, y_i$$

If we want to obtain the differentiation formulas for any higher order, say qth $(1 \le q \le n)$ order derivative, then we differentiate $P_n(x)$, q times and get

$$f^{(q)} \approx P^{(q)}_n(x) = \sum^n_{i=0} L^{(q)}_i(x_i)\, y_i$$

Let us consider the following examples.

Example 6

Find $y'(x) = f'(x)$ using Lagrange Interpolation for the data (x_k, y_k), $k = 0, 1$.

Solution

We know that $P_1(x)=L_0(x)f_0+L_1(x)f_1$

where $L_0(x)=\dfrac{x-x_1}{x_0-x_1}$ and $L_1(x)=\dfrac{x-x_0}{x_1-x_0}$

Now, $P'_1(x)=L'_0(x)f_0+L'_1(x)f_1$

and $L'_0(x)=\dfrac{1}{x_0-x_1}, L'_1(x)=\dfrac{1}{x_1-x_0}$

Hence, $f'(x)=P'_1(x)=\dfrac{f_0}{x_0-x_1}+\dfrac{f_1}{x_1-x_0}=\dfrac{(f_1-f_0)}{(x_1-x_0)}$

Example 7

Find $f'(x)$ and $f''(x)$ given f_0, f_1, f_2 at x_0, x_1, x_2 respectively, using Lagrange interpolation.

Solution

By Lagrange's interpolation formula

$$f(x)\approx P_2(x)=L_0(x)f_0+L_1(x)f_1+L_2(x)f_2$$

where, $L_0(x)=\dfrac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}$; $\quad L'_0(x)=\dfrac{2x-x_1-x_2}{(x_0-x_1)(x_0-x_2)}$

$L_1(x)=\dfrac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$; $\quad L'_1(x)=\dfrac{2x-x_0-x_2}{(x_1-x_0)(x_1-x_2)}$

$L_2(x)=\dfrac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$; $\quad L'_2(x)=\dfrac{2x-x_0-x_1}{(x_2-x_0)(x_2-x_1)}$

Hence, $f'(x)=P'_2(x)=L'_0(x)f_0+L'_1(x)f_1+L'_2(x)f_2$

And $P''_2(x)=L''_0(x)f_0+L''_1(x)f_1+L''_2(x)f_2$

$$=\frac{2f_0}{(x_0-x_1)(x_0-x_2)}+\frac{2f_1}{(x_1-x_0)(x_1-x_2)}+\frac{2f_2}{(x_2-x_1)}$$

Example 8

Given the following values of f(x) = In x, find the approximate value of

f ′ (2.0) and f ″(2.0).

x	:	2.0	2.2	2.6
f(x)	:	0.69315	0.78846	0.95551

Solution

Using the Lagrange's interpolation formula, we have

$$f'(x_0)=P'_2(x_0)=\frac{2x_0-x_1-x_2}{(x_0-x_1)(x_0-x_2)}f_0+\frac{x_0-x_2}{(x_1-x_0)(x_1-x_2)}f_1+\frac{x_0-x_1}{(x_2-x_0)(x_2-x_1)}f_2$$

(Some coefficients of f_1 and f_2 become zero at $x = x_0$.)

$\therefore$ we get

$$f'(2.0)=\frac{4-2.2-2.6}{(2-2.2)(2-2.6)}(0.69315)+\frac{2-2.6}{(2.2-2)(2.2-2.6)}(0.78846)$$

$$+\frac{2-2.2}{(2.6-2)(2.6-2.2)}(0.95551)=0.49619$$

The exact value of f ′ (2.0) = 0.5.

Similarly,

$$f''(x_0)=2\left[\frac{f_0}{(x_0-x_1)(x_0-x_2)}+\frac{f_1}{(x_1-x_0)(x_1-x_2)}+\frac{f_2}{(x_2-x_0)(x_2-x_1)}\right]$$

$$\therefore f''(2.0)=2\left[\frac{0.69315}{(2-2.2)(2-2.6)}+\frac{0.78846}{(2.2-2)(2.2-2.6)}+\frac{0.95551}{(2.6-2)(2.6-2.2)}\right]$$

$$=-0.19642$$

The exact value of f ″ (2.0) = − 0.25.

You may now try the following exercise.

Ex. 3 : Use Lagrange's interpolation to find f ′ (x) and f ″(x) at each of the values x = 2.5; 5.0 from the following table

x	:	1	2	3	4
f(x)	:	1	16	81	256

1.4 SUMMARY

In this unit, we defined a number formulae for approximating first order and higher order derivatives of a function $y = f(x)$, using various formulae like FD and BD. Lagrange's differences etc.

In the case when tabular points are equidistant, we derived the following four formulae :

(i) **Differentiating FD formula we get** (for $0 \le p < 1$)**, first two derivatives are approximated as**

$$dy/dx = (1/h)\{\Delta y_0 + ((2p-1)/2)\,\Delta^2 y_0 + ((3p^2-6p+2)/6)\,\Delta^3 y_0 + \ldots\} \quad \ldots (A)$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\{\Delta^2 y_0 + (p-1)\,\Delta^3 y_0 + \ldots\}$$

(ii) **Differentiating BD formula we get** (for $-1 < p \le 0$)**, first two derivatives are approximated as**

$$\frac{dy}{dx} = \frac{1}{h}\left\{\nabla y_0 + \frac{2p+1}{1}\nabla^2 y_0 + \frac{3p^2+6p+2}{6}\nabla^3 y_0 + \ldots\right\} \quad \ldots (B)$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\{\nabla^2 y_0 + (p+1)\,\nabla^3 y_0 + \ldots\}$$

(iii) **Differentiation of Stirling's formula** (for $-0.5 \le p \le 0.5$)**, first two derivatives are approximated as**

$$\frac{dy}{dx} = \frac{1}{h}\left\{\mu\delta y_0 + p\delta^2 y_0 + \frac{3p^2-1}{6}\mu\delta^3 y_0 + \frac{2p^3-p}{12}\delta^4 y_0 + \ldots\right\} \quad \ldots (C)$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left\{\delta^2 y_0 + p\,\mu\delta^3 y_0 + \frac{6p^2-1}{12}\delta^4 y_0 + \ldots\right\}$$

(iv) **By differentiating Bessel's formula** (for $0.25 \le p \le 0.75$)**, first two derivatives are approximated as**

$$\frac{dy}{dx} = \frac{1}{h}\left\{\delta y_{\frac{1}{2}} + \frac{2p-1}{2}\delta^2 y_{\frac{1}{2}} + \frac{6p^2-6p+1}{12}\delta^3 y_{\frac{1}{2}} + \frac{2p^3-3p^2-p+1}{12}\mu\delta^4 y_{\frac{1}{2}} + \ldots\right\} \ldots (D)$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left\{\mu\delta^2 y_{\frac{1}{2}} + \frac{2p-1}{2}\delta^3 y_{\frac{1}{2}} + \frac{6p^2-6p-1}{12}\mu\delta^4 y_{\frac{1}{2}} + \ldots\right\}$$

In the case when tabular points are not necessarily equidistant, we derived the following formula :

$f'(x) \approx P_n'(x) = \sum^n i = 0\; L_i'(x_i)\, y_i$, where

$\mathbf{L_i(x) = \phi(x)/\{(x - x_i)\,\phi_1(x_i)\}}$, **and**

$\phi(x) = (x - x_0)(x - x_1)\ldots(x - x_n)$, and

$\phi_1(x_i) = (x_i - x_0)(x_i - x_1)\ldots..(x_i - x_{i-1})(x_i - x_{I+1})\ldots(x_i - x_n)$.

After deriving each of the formulae, we illustrated through examples, how to use the formula in finding first and higher order derivatives for a function defined by a table.

1.5 SOLUTIONS/ANSWERS

Ex. 1 : We are required to find $f'(x)$ and $f''(x)$ at $x = 1.5$ and 3.5 which are off-step points. Using the Newton's forward difference formula with $x_0 = 0$, $x = 1.5$, $p = 1.5$, we get $f'(1.5) = 8.7915$ and $f''(1.5) = -4.0834$.

Using the backward difference formula with $x_n = 4$, $x = 3.5$, $s = -0.5$, we get $f'(3.5) = 7.393$ and $f''(3.5) = 1.917$.

Ex. 2 : The difference table for given problem is :

X	f(x)	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
1.3	3.669				
		0.813			
1.5	4.482		0.179		
		0.992		0.41	
1.7	5.474		0.220		0.007
		1.212		0.48	
1.9	6.686		0.268		0.012
		1.480		0.060	
2.1	8.166		0.328		0.012
		1.808		0.072	
2.3	9.974		0.400		
		2.208			
2.5	12.182				

Taking $x_0 = 1.5$ we see that $p = 0$ and we obtain from the interpolation formula

$$f'(1.5) = \frac{1}{h}\left[\Delta f_0 - \frac{\Delta^2 f_0}{2} + \frac{\Delta^3 f_0}{3} - \frac{\Delta^4 f_0}{4} + \ldots\right]$$

$$= \left[0.992 - \frac{0.220}{2} + \frac{0.048}{3} - \frac{0.012}{4}\right] = 4.475$$

Exact value is $e^{1.5} = 4.4817$ and error is $= 0.0067$.

Ex. 3 : In the given problem $x_0 = 1$, $x_1 = 2$, $x_2 = 3$, $x_3 = 4$ and $f_0 = 1$, $f_1 = 16$, $f_2 = 81$ and $f_3 = 256$.

Constructing the Lagrange fundamental polynomials, we get

$$L_0(x) = -\left(\frac{x^3 - 9x^2 + 26x - 24}{6}\right); L_1(x) = \left(\frac{x^3 - 8x^2 + 19x - 12}{2}\right)$$

$$L_2(x) = -\left(\frac{x^3 - 7x^2 + 14x - 8}{2}\right); L_3(x) = \left(\frac{x^3 - 6x^2 + 11x - 6}{6}\right)$$

$$P_3(x) = L'_0(x) f_0 + L'_1(x) f_1 + L'_2(x) f_2 + L'_3(x) f_3$$

$$P'_3(x) = L'_0(x) f_0 + L'_1(x) f_1 + L'_2(x) f_2 + L'_3(x) f_3$$

$$P''_3(x) = L''_0(x) f_0 + L''_1(x) f_1 + L''_2(x) f_2 + L''_3(x) f_3$$

We obtain after substitution,

$$P'_3(2.5)=62.4167; P''_3(2.5)=79; P'_3(5)=453.667; P''_3(5)=234.$$

The exact values of $f'(x)$ and $f''(x)$ are (from $f(x) = x^4$)

$$f'(2.5) = 62.5, f'(5) = 500; f''(2.5) = 75; f''(5) = 300.$$

UNIT 2 NUMERICAL INTEGRATION

Structure

2.0 INTRODUCTION

Definite integrals play an important role in scientific investigations and engineering applications. As an illustration in support of the statement, let us consider a simple problem of calculating life expectancy of an electronic equipment : The probability density function of the time to failure of an electronic component in a copier (in Hours) is

$$f(x) = (e^{-x/1000})/\,1000.$$

Determine the probability that a component lasts more than 3000 hours before failure.

Solution consists in finding the value of

$$1-\int_0^{3000}\frac{\left(e^{\frac{-x}{1000}}\right)}{1000} = 1-\left(-\,|\,e^{\frac{-x}{1000}}\,|_0^{3000}\right) = e^{-3} \approx 0.049787$$

However, in many applications, the function may not be expressible in a closed form as is in the example above, but may be available only in a tabular form. Or, even if, it can be expressed in closed form, it is in such a complicated form as $f(x) = (\sin x)/x$, which has no anti-derivative. In such situations, and otherwise also, numerical methods of integration may be quite useful. In this unit, we will study some of these methods.

As is true of any numerical methods, studied so far, for solving a mathematical problem, so is true also of methods of numerical integration : a numerical method, generally, does not provide exact solution, but only approximate solution. Hence, there is need for error analysis, which, however, is beyond the scope of the course.

You need to remember only the final formulae, and not the derivation of the formulae.

2.1 OBJECTIVES

After going through this unit, you should be able to

- explain the methodology of numerical integration;
- state, explain, derive and use rectangular rule of integration;
- state, explain, derive and use trapezoidal rule of integration; and
- state, explain, derive and use Simpson's rule of integration.

2.2 METHODOLOGY OF NUMERICAL INTEGRATION

We shall be interested in evaluating the definite integral $I = \int_a^b f(x)\,dx$, where the function f(x) is defined at each point in [a, b] and there is no discontinuity. We also assume that f(x) possesses same sign in [a, b]. It should be remembered that $\int_a^b f(x)\,dx$ represents the area enclosed by the curve y = f(x), the x-axis y = 0 and the vertical lines x = a and x = b. **To compute the area numerically is called 'quadrature'.**

The first step (of methodology) for evaluating the integral is to divide the interval [a, b] into suitable number of sub-intervals, say n, each of width h, such that $h = \frac{b-a}{n}$. Let us denote these points of division on x-axis as $a = x_0 < x_1 < x_2 < \ldots < x_n = b$ so that n sub-intervals may be defined as $[x_i, x_{i+1}]$, $i = 0(1)\,n-1$. The x_i's, the points on x-axis, are called **nodal or pivotal points**. The pivotal points are (or are chosen) such that the function values $f(x_i)$, $i = 0\,(1)\,n$, at these points, are known.

Various integration formulas are devised by approximating the integral over one sub-interval, over a (larger) sub-interval of 2 consecutive sub-intervals or, in general, over a (larger) sub-interval of k consecutive sub-intervals, at a time (say, $[x_i, x_{i+1}]$, $[x_{i+1}, x_{i+2}]$, …, $[x_{i+k-1}, x_{i+k}]$, for i =1, 1 + k, 1 + 2k, . . .). The required value of the definite integral over the entire interval [a, b] is sum of integrals over disjoint combination of these (larger, if k > 1) sub-intervals of k consecutive subintervals of the form $[x_i, x_{i+1}]$. The methods differ in the choice of k and in the choice of polynomial for representing the integral over the (larger, if $2 \le k$) sub-interval consisting of k consecutive sub-intervals $[x_i, x_{i+1}]$, $[x_{i+1}, x_{I+2}]$, . . . ,$[x_{i+k-1}, x_{i+k}]$, for i =1, 1 + k, 1 + 2k . . .

For example, in the case of trapezoidal rule discussed below, first, k is taken as 1. Then, on each interval $[x_i, x_{i+1}]$, for i = 0 (1)n, a polynomial representing a trapezoid, is fitted. In the case of Simpson's (1/3) rule, first, k is taken as 2. And, then, on each (double) interval $[x_i, x_{i+2}]$, for i = 0(2)n, a polynomial representing a parabola, is fitted. Obviously n (so that the number of nodal points is n + 1) and k have to be chosen so that n is an integral multiple of k, say mk = n, m an integer. That is, if a formula involving k consecutive intervals is used then it will be invoked m times to cover the interval [a, b], where m = (b – a)/kh. In this unit, we shall discuss formulas only when k = 1 and 2.

The advantage of fitting m times a polynomial of degree k over fitting a single polynomial of degree n = mk, is that computing m times a polynomial of degree k passing through (k + 1) points, is much simpler than computing a single polynomial of degree n = mk, passing through n + 1 = mk + 1 points.

2.3 RECTANGULAR RULE

We approximate the integral over an interval $[x_i, x_{i+1}] = h$ as,

$$\int_{x_i}^{x_{i+1}} f(x)\,dx = (x_{i+1} - x_i)\,f(x_i) = h\,f(x_i), \quad i = 0, 1, 2, \ldots, n-1$$

Adding over all the intervals and denoting $f(x_i) = f_i = y_i$ the formula may be written as,

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_n} y(x)\,dx = h\,\{y_0 + y_1 + y_2 + \ldots + y_{n-1}\}$$

Geometrically, the integral in each interval is represented by the area of a rectangle, formed by h, the length of the interval, and first value y_i of the two values y_i and y_{i+1}.

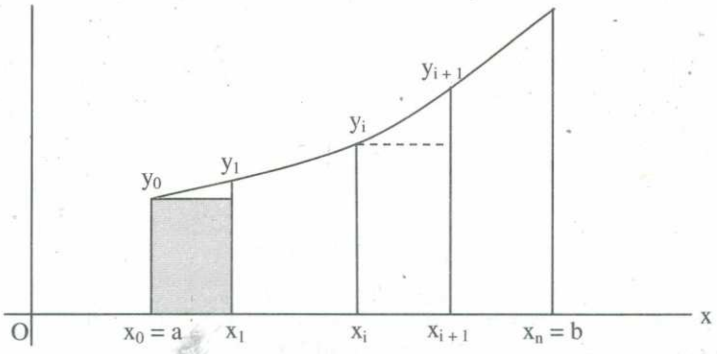

Figure 1 : Rectangular Rule

The rectangular rule is more of a theoretical curiosity, as it is hardly used in applications; specially, in view of the fact that, for k = 1, an almost equally simple, both theoretically and computationally, rule called trapezoidal rule is available. Next we discuss the trapezoidal rule.

Notation : We may denote by $I_R[f]$ the value of the integral of a function f, through rectangular rule. E_R [f] denotes error = Actual – I_R [f].

Remark : When more than two nodal points are involved, and at least area of two rectangles is involved in I_R, then the rule **may be called 'composite rectangular rule'. But in all cases we will just use the term 'rectangular rule'.**

Example 1

Evaluate $\int_0^1 e^{-x^2}\,dx$, using Rectangular rule.

Solution

The values of the function $f(x) = e^{-x^2}$ at x =0, 0.5 and 1 are

$f(0) = 1, f(0.5) = 0.7788, f(1) = 0.36788$

Taking h = 0.5 and using

(a) $I_R = h\,[f(0) + f(0.5)] = 0.5\,[1 + 0.7788] = 0.8894.$

2.4 TRAPEZOIDAL RULE

Trapezoidal Rule/Formula

We approximate the function f(x) in the interval $[x_i, x_{i+1}]$ by a straight line joining the points (x_i, y_i) and (x_{i+1}, y_{i+1}). (In case of rectangular rule, the rectangle is obtained by joining (x_i, y_i) and (x_{i+1}, y_i)). For convenience let us consider the integral in the first interval (x_0, x_1). The line joining (x_0, y_0) and (x_1, y_1) may be written from the Lagrange's formula as,

$$y(x) = \frac{x - x_1}{x_0 - x_1}\,y_0 + \frac{x - x_0}{x_1 - x_0}\,y_1. \text{ Also, } x_1 - x_0 = h.$$

Now,
$$\int_{x_0}^{x_1} f(x)\,dx = \int_{x_0}^{x_1}\left(\frac{x - x_1}{x_0 - x_1}\,y_0 + \frac{x - x_0}{x_1 - x_0}\,y_1\right)dx$$

$$= \frac{1}{2h}\left[-(x - x_1)^2\,y_0 + (x - x_0)^2\,y_1\right]_{x = x_0}^{x = x_1}$$

$$= \frac{h}{2}(y_0 + y_1)$$

Adding over all the intervals the Trapezoidal formula/rule may be written as,

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_n} y(x)\,dx \approx \frac{h}{2}\{(y_0 + y_1) + (y_1 + y_2) + \ldots + (y_{n-2} + y_{n-1}) + (y_{n-1} + y_n)\}$$

$$= \frac{h}{2}\{y_0 + 2(y_1 + y_2 + \ldots + y_{n-1}) + y_n\}$$

or, $$= h\left\{\frac{y_0 + y_n}{2} + (y_1 + y_2 + \ldots + y_{n-1})\right\} \qquad \ldots (A)$$

Remark : In case $2 \le n$, then number of nodal points is at least 3 and the number of intervals is at least 2. In this case, the trapezoidal rule is also referred to as **composite trapezoidal rule.**

Geometrically, the integral in an interval is approximated by the area of a trapezium (see Figure 2).

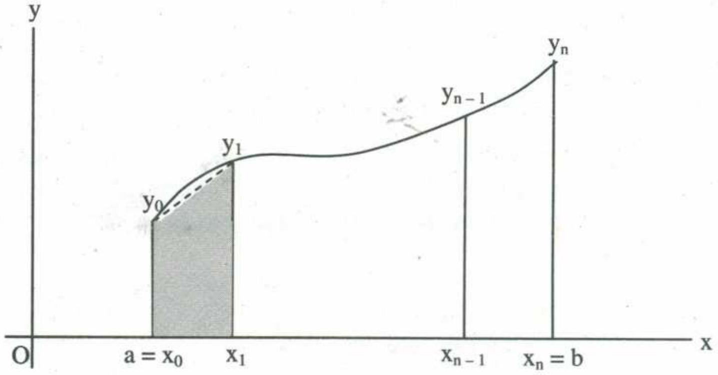

Figure 2 : Trapezoidal Rule

Notation : We may denote by $I_T[f]$, the quantity

$h\left\{\frac{y_0 + y_n}{2} + (y_1 + y_2 + \ldots + y_{n-1})\right\}$, **the value of the integral of a function f, through trapezoidal rule. $E_T[f]$ denotes error = Actual – $I_T[f]$.**

Example 2

Find the approximate value of

$$I = \int_0^1 \frac{dx}{1+x}$$

using trapezoidal rule with n = 1, the number of intervals.

Solution

$$y(x) = f(x) = \frac{1}{1+x}.$$

Here $x_0 = 0$, $x_1=1$, and $h = 1 - 0 = 1$, $y_0 = 1$, $y_1 = \frac{1}{2}$. From trapezoidal rule, we get

$$I \approx \frac{1}{2}\left(1 + \frac{1}{2}\right) = 0.75$$

Example 3

Evaluate the integral $I = \int_0^1 \frac{dx}{\sqrt{1+x^2}}$ by trapezoidal rule dividing the interval [0, 1] into five equal parts. Compute up to five decimals.

Solution

$n = 5;\ h = \frac{1-0}{5} = 0.2$

i	0	1	2	3	4	5
x	0	0.2	0.4	0.6	0.8	1.0
$y = \frac{1}{\sqrt{1+x^2}}$	1.0	0.98058	0.92848	0.85749	0.78087	0.70711

From Trapezoidal Rule;

$$I \approx \frac{h}{2}[y_0 + y_5 + 2(y_1 + y_2 + y_3 + y_4)]$$

$$= \frac{0.2}{2}[1.0 + 0.70711 + 2(0.98058 + 0.92848 + 0.85749 + 0.78087)]$$

$$= 0.1[1.70711 + 2 \times 3.54742]$$

$$= 0.88016$$

Example 4

Evaluate $\int_1^6 [2 + \sin(2\sqrt{x})]\,dx$ using Trapezoidal rule with 11 points.

Solution

$$y(x) = f(x) = (2 + \sin(2\sqrt{x})).$$

To generate 11 sample points, we use $h = \left[\frac{(6-1)}{10}\right] = 0.5$. The points are 1, 1.5, 2.0, 2.5, 3.0, 3.5, 4.0, 4.5, 5.0, 5.5 and 6.0. Using formula (A), we obtain

$$I = \frac{0.5}{2}\left[f(1) + f(6) + 2\{f(1.5) + f(2) + f(2.5) + f(3) + f(3.5) + f(4) + f(4.5) + f(5) + f(5.5)\}\right]$$

$$= 0.25[2.90929743 + 1.01735756]$$

$$+0.5[2.63815764 + 2.30807174 + 1.97931647 + 1.68305284 + 1.43530410 +$$

$$+1.24319750 + 1.10831775 + 1.02872220 + 1.00024140]$$

$$= 0.25[3.92665499] + 0.5[14.42438165]$$

$$= 0.98166375 + 7.21219083 = 8.19385457.$$

Ex. 1 : Evaluate the integral $\int_0^6 (x^2 + x + 2)dx$ using trapezoidal rule, with h = 1.0.

Ex. 2 : (i) Use the trapezoidal (using 2 nodal points for trapezoidal rule).

(a) $\int_1^2 \ln x\,dx$

(b) $\int_0^{0.1} x^{1/3}\, dx$

(c) $\int_0^{\pi/4} \tan x\, dx$

Note : Ex. 2 : (ii) is given at the end of Section 2.5.

2.5 SIMPSON'S 1/3 RULE

Simpson's 1/3rd Rule/Formula

In this case, the integral is evaluated over two intervals at a time, say $[x_0, x_1]$ and $[x_1, x_2]$. The function f(x) is approximated by a quadratic passing through the points (x_0, y_0) and (x_1, y_1) and (x_2, y_2). From Lagrange's formula we may write the quadratic as,

$$y(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} y_0 + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} y_1 + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} y_2$$

Integrating term by term, using $x_1 - x_0 = h = x_2 - x_1$ and $(x - x_1) = (x - x_2) + (x_2 - x_1) = (x - x_2) + h$; $(x - x_0) = (x - x_2) + (x_2 - x_0) = (x - x_2) + 2h$, we get

$$\int_{x_0}^{x_2} \frac{(x - x_1)(x - x_2)}{(-h)(-2h)} dx = \frac{h}{3}$$

$$\int_{x_0}^{x_2} \frac{(x - x_0)(x - x_2)}{h(-h)} dx = \frac{4}{3} h$$

$$\int_{x_0}^{x_2} \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} dx = \frac{h}{3}$$

Hence we get,

$$\int_{x_0}^{x_2} f(x)\, dx = \int_{x_0}^{x_2} y(x)\, dx = \frac{h}{3} y_0 + \frac{4h}{3} y_1 + \frac{h}{3} y_2$$

$$= \frac{h}{3}(y_0 + 4y_1 + y_2)$$

Applying this formula over next two intervals and then next two and so on for $\frac{n}{2}$ times and adding we get

$$\int_a^b f(x)\, dx = \int_{x_0}^{x_n} y(x)\, dx = \int_{x_0}^{x_2} y(x)\, dx = \int_{x_2}^{x_4} y(x)\, dx + \ldots + \int_{x_{n-2}}^{x_n} y(x)\, dx$$

$$= \frac{h}{3}[(y_0 + 4y_1 + y_2) + (y_2 + 4y_3 + y_4) + \ldots + (y_{n-2} + 4y_{n-1} + y_n)]$$

$$= \frac{h}{3}[y_0 + y_n + 4(y_1 + y_3 + \ldots + y_{n-1}) + 2(y_2 + y_4 + \ldots + y_{n-2})] \quad \ldots (B)$$

Obviously n should be chosen as a multiple of 2 i.e. an even number for applying this formula.

Remarks :

(1) In the Simpson's rule, number (n+1) of nodal points must be odd and at least three.

(2) **Composite Simpson's rule :** If number of parabolas/quadratic polynomials that can be fitted is at least 2(hence, the number of nodal points is at least 5 and number of sub-intervals is at least 4) then, the rule is also called composite Simposn's rule.

(3) The Simpson 1/3 rule is very popular. It integrates exactly a cubic function or, equivalently, gives the area under a segment of cubic curve. On the other hand, the trapezoidal rule integrates exactly a linear function only.

Notation : **We may denote by $I_S[f]$ the value of the integral of a function f, through Simpson's rule. $E_S[f]$ = error = Actual – $I_S[f]$.**

Example 5

Evaluate the integral $I = \int_0^{0.8} \frac{dx}{\sqrt{1+x}}$ by Simpson's 1/3rd rule dividing the interval [0, 0.8] to 4 equal sub-intervals. Compute up to five places of decimal only.

Solution

$$n = 4;\ h = \frac{0.8 - 0}{4} = 0.2$$

I	0	1	2	3	4
X	0	0.2	0.4	0.6	0.8
$y = \frac{1}{\sqrt{1+x}}$	1.0	0.91287	0.84515	0.79057	0.74536

From Simpson's 1/3rd Rule

$$I = \int_0^{0.8} ydx = \frac{h}{3}[(y_0 + 4y_1 + y_2) + (y_2 + 4y_3 + y_4)]$$

$$= \frac{h}{3}[y_0 + y_4 + 4(y_1 + y_3) + 2 \times y_2)]$$

$$= \frac{0.2}{3}[1.0 + 0.74536 + 4(0.91287 + 0.79051) + 2 \times 0.84515)]$$

$$= \frac{0.2}{3}[1.74536 + 4 \times 1.70344 + 1.69030]$$

$$= 0.68329$$

Example 6

Find the approximate value of $I = \int_0^1 \frac{dx}{1+x}$ using Simpson's rule (3 nodal points for Simpson's rule).

Solution

Here $x_0 = 0$, $x_1 = 0.5$, $x_2 = 1$ and $h = \frac{1}{2}$.

Using Simpson's rule we have

$$I_s[f] = \frac{h}{3}[f(0) + 4f(0.5) + f(1)] = \frac{1}{6}\left[1 + \frac{8}{3} + 0.5\right] = 0.694445$$

Example 7

Calculate the value of the integral

$$\int_4^{5.2} \log x \, dx \text{ by}$$

(a) Trapezoidal rule

(b) Simpson's $\frac{1}{3}$ rule

Assume h = 0.2. The values of log x for each point of sub-division are given below :

X	$f(x) = \log x$
$x_0 = 4$	$f(0) = 1.3862944$
$x_0 + h = 4.2$	$f(1) = 1.4350845$
$x_0 + 2h = 4.4$	$f(2) = 1.4816045$
$x_0 + 3h = 4.6$	$f(3) = 1.5260563$
$x_0 + 4h = 4.8$	$f(4) = 1.5686159$
$x_0 + 5h = 5.0$	$f(5) = 1.6094379$
$x_0 + 6h = 5.2$	$f(6) = 1.6486586$

Solution

Taking $h = 0.2$, divide the range of integration (4, 5.2) into six equal parts.

(a) By Trapezoidal rule, we have

$$\int_4^{5.2} \log x dx = \frac{h}{2}[f(0) + f(6) + 2\{f(1) + f(2) + f(3) + f(4) + f(5)\}]$$

$$= \frac{0.2}{2}[3.034953 + 2 \times 7.6207991]$$

$$= 0.1(18.276551) = 1.8276551$$

(b) By Simpson's 1/3 rule, we have

$$\int_4^{5.2} \log x \, dx = \frac{h}{3}[f(0) + f(6) + 4\{f(1) + f(3) + f(5)\} + 2\{f(2) + f(4)\}]$$

$$= \frac{0.2}{3}[3.034953 + 4(4.5705787) + 2(3.0502204)]$$

$$= \frac{0.2}{3}[3.034953 + 18.282315 + 6.1004408]$$

$$= \frac{0.2}{3} \times 27.417709 = 1.8278472.$$

Example 8

Evaluate $\int_0^1 \frac{dx}{1+x}$ using

(a) Composite trapezoidal rule and (b) composite Simpson's rule with 2, 4 and 8 subintervals.

Solution

We give in Table below the values of f (x) with $h = \frac{1}{8}$ from x = 0 to x = 1.

Table

x :	0	1/8	2/8	3/8	4/8	5/8	6/8	7/8	1
f(x) :	1 f_0	8/9 f_1	8/10 f_2	8/11 f_3	8/12 f_4	8/13 f_5	8/14 f_6	8/15 f_7	8/16 f_8

If n = 2 then h = 0.5 and the ordinates f_0, f_4 and f_8 are to be used

We get $I_T[f]=\frac{1}{4}[f_0+2f_4+f_8]=\frac{17}{24}=0.708333$

$$I_s[f]=\frac{1}{6}[f_0+4f_1+f_8]=\frac{25}{36}=0.694444$$

If N = 4 then h = 0.25 and the ordinates f_0, f_2, f_4, f_6, f_8 are to be used.

We have $I_T[f]=\frac{1}{8}[f_0+f_8+2(f_2+f_4+f_6)]=0.697024$

$$I_T[f]=\frac{1}{12}[f_0+f_8+4(f_2+f_6)+2\ f_4]=0.693254$$

If N = 8 then h = 1/8 and all the ordinates in Table 1 are to be used.

We obtain $I_T[f]=\frac{1}{16}[f_0+f_8+2(f_1+f_2+f_3+f_4+f_5+f_6+f_7)]=0.694122$

$$I_s[f]=\frac{1}{24}[f_0+f_8+4(f_1+f_3+f_5+f_7)+2(f_2+f_4+f_6)]=0.693147$$

Example 9

Evaluate $\int_1^6 [2+\sin(2\sqrt{x})]\,dx$ using Simpsons rule with 11 points.

Solution

To generate 11 sample points we use $h=(6-1)/10=0.5$, the points are same as in Example 4. Using the Simpson's formula, we get

$$I = \frac{1}{6}[f(1) + f(6) + 4\{f(1.5) + f(2.5) + f(3.5) + f(4.5) + f(5.5)\}$$

$$+ 2\{f(2) + f(3) + f(4) + f(5)\}]$$

$$I = \frac{1}{6}[2.90929743+1.01735756]$$

$$+ 2.0\{2.30807174+1.68305284+1.24319750+1.02872220\}$$

$$+4.0\{2.63815764+1.97931647+1.43530410+1.10831775+1.00024140\}$$

$$=\frac{1}{6}[(3.92665499)+2.0(6.26304429)+4.0(8.16133735)]=8.18301550.$$

Example 10

Evaluate $\int_0^1 e^{-x^2}\,dx$, using

(a) trapezoidal rule and b) Simpson's rule (each with minimum number of nodal points). If the exact value of the integral is 0.74682 correct to 5 decimal places, find the error in these rules.

Solution

The values of the function $f(x) = e^{-x^2}$ at $x = 0,\ 0.5$ and 1 are

$$f(0) = 1,\ f(0.5) = 0.7788,\ f(1) = 0.36788$$

(a) $I_T[f] = \frac{h}{2}[f_0 + f_1]$ we get $I_T[f] = \frac{1}{2}(1 + 0.36788) = 0.68394.$

(b) Taking h = 0.5 and using Simpson's rule, we get

$$I_S[f] = \frac{h}{3}[f_0 + 4f_1 + f_3] = \frac{h}{3}[f(0) + 4f(0.5) + f(1)] = 0.74718$$

Exact value of the integral is 0.74682.

The errors in these rules are given by $E_T[f] = 0.06288, E_s[t] = -0.00036.$

Ex. 2 : (ii) Use the Simpson's rule to approximate the following integrals (3 nodal points for Simpson's rule).

(a) $\int_1^2 \ln x \, dx$.

(b) $\int_0^{0.1} x^{1/3} \, dx$,

(c) $\int_0^{\pi/4} \tan x \, dx$

Ex. 3 : Find an approximation to $\int_{1.1}^{1.5} e^x \, dx$, using

(a) the trapezoidal rule with h = 0.4

(b) Simpson's rule with h = 0.2

Ex. 4 : Evaluate $\int_0^1 \frac{dx}{1+x^2}$ by subdividing the interval (0, 1) into 6 equal parts and using.

(a) Trapezoidal rule,

(b) Simpson's rule. Hence find the value of π and actual errors.

Ex. 5 : A function f(x) is given by the table

x	:	1.0	1.5	2.0	2.5	3.0
f(x)	:	1.000	2.875	7.000	14.125	25.000

Find the integral of f(x) using, (a) trapezoidal rule, (b) Simpson's rule, using h – 0.1.

Ex. 6 : The speedometer reading of a car moving on a straight road is given by the following table Estimate the distance traveled by the car in 12 minutes using (a) Trapezoidal rule (b) Simpson's rule.

Time (minutes)	:	0	2	4	6	8	10	12
Speedo meter Reading	:	0	15	25	40	45	20	0

Ex. 7 : Evaluate $\int_{0.2}^{0.4} (\sin x - \ln x + e^x)\, dx$

using (a) Trapezoidal rule and (b) Simpson's rule taking h = 0.1.

Ex. 8 : Find the approximate value of $I = \int_0^1 \frac{dx}{1+x}$ using (i) Trapezoidal rule (two points) and (ii) Simpson's 1/3 rule (three points). Obtain a bound for the error in the Trapezoidal Rule. The exact value of I = In 2 = 0.693147 correct to six decimal places.

Ex. 9 : Evaluate the integral $I = \int_0^1 \frac{dx}{1+x}$ using (i) composite trapezoidal rule, and (ii) composite Simpson's rule, with 2, 4 and 8 equal subintervals.

2.6 ERROR IN VARIOUS INTEGRATION FORMULAS

The maximum error in various integration formulas in the evaluation of the integral

$I = \int_a^b f(x)\, dx$ is

(i) E_R [f] in Rectangular Rule : $\frac{(b-a)}{2} f'(\xi)$

(ii) E_T [f] in Trapezoidal Rule : $-\frac{(b-a)\, h^2}{12} f''(\xi)$

(iii) E_S [f] in Simpson's 1/3rd Rule : $-\frac{(b-a)\, h^4}{180} f^{iv}(\xi)$

where $x = \xi$ is some point in [a, b] for which $f'(x)$ or $f''(x)$ or $f^{(iv)}(x)$ has maximum numerical value.

2.7 SUMMARY

In this unit, we discussed various methods of evaluating the definite integral

$1 = \int_a^b f(x)\, dx$ where the function f(x) is defined at each point in [a, b] and there is no discontinuity. For the methods discussed, we also assumed that f(x) possesses same sign in [a, b]. All the discussed methods follow the same methodology, or essentially the same sequence of steps, outlined below :

The first step for evaluating the integral is to divide the interval [a, b] into suitable number of sub-intervals, say n, each of width h, such that $h = \frac{b-a}{n}$. If the points of division on x-axis are denoted as: $a = x_0 < x_1 < x_2 < \ldots < x_n = b$, then the function values $f(x_i)$, $i = 0\,(1)\,n$, at these points, are assumed to be known. Points x_i are known as **nodal/pivotal points**. Each pair of successive points defines a sub-interval, which may be denoted as $[x_i, x_{i+1}]$, $i = 0(1)\,n-1$.

(Second step)Various integration formulas are devised by approximating the integral over one sub-interval, over a (larger) sub-interval of 2 consecutive sub-intervals or, in general, over a (larger) sub-interval of k consecutive sub-intervals, at a time (say, $[x_i, x_{i+1}]$, $[x_{i+1}, x_{i+2}]$, . . . ,$[x_{i+k-1}, x_{i+k}]$, for i =1, 1 + k, 1 + 2k, . . .). The required value of the definite integral over the entire interval [a, b] is sum of integrals over disjoint combination of these (larger, if k > 1) sub-intervals of k consecutive subintervals of the form $[x_i, x_{i+1}]$, covering whole of [a, b].

The methods differ in the choice of k and in the choice of polynomial for representing the integral over the (larger, if $2 \le k$) sub-interval consisting of k consecutive sub-intervals $[x_i, x_{i+1}]$, $[x_{i+1}, x_{i+2}]$, . . . ,$[x_{i+k-1}, x_{i+k}]$, for i = 1, 1 + k, 1 + 2k . . .

The methods discussed in the unit are :

(i) **Rectangular rule**, obtained by taking k = 1 and the polynomial, which represents a horizontal straight line joining (x_i, y_i) and (x_{i+1}, y_i) (i.e., forming a rectangle with other boundaries),

(ii) **Trapezoidal rule**, obtained by taking k = 1 and the polynomial, which represents a straight line joining (x_i, y_i) and (x_{i+1}, y_{i+1}) (i.e., forming a trapezoid with other boundaries),

(iii) **Simpson's 1/3 rule**, obtained by taking k = 2 and the polynomial, which represents a parabola passing through 3 points (x_i, y_i), (x_{i+1}, y_{i+1}) and (x_{i+2}, y_{i+2}).

2.6 SOLUTIONS/ANSWERS

Ex. 1 : Here $x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 4, x_5 = 5, x_6 = 6, h = 1$. Therefore,

$$I = \frac{h}{2}[f(0) + 2f(1) + 2f(2) + 2f(3) + 2f(4) + 2f(5) + f(6)]$$

$$= \frac{1}{2}[2+2(4)+2(8)+2(14)+2(22)+2(32)+44]=103.$$

Ex. 2 : **(i) and (ii) combined**

(a) $I_T[f]=\frac{h}{2}[f_0 + f_1]=0.346574$

$$I_s[f]=\frac{h}{3}[f_0 + 4f_1 + f_2]$$

$$=\frac{0.5}{3}[4\ln 1.5+\ln 2]=0.385835$$

(b) $I_T[f]=0.023208,\ |E_T[f]|=\text{none}.$

$I_S[f]=0.032296,\ |E_S[f]|=\text{none}.$

(c) $I_T[f]=0.39270,\ |E_T[f]|=0.161.$

$I_S[f]=0.34778,\ |E_S[f]|=0.00831.$

Ex. 3 : $I_T[f]=1.49718$

$I_S[f]=1.47754$

Ex. 4 : With h = 1/6, the values of $f(x) = \dfrac{1}{1+x^2}$

From x = 0 to 1 are given by table :

x	:	0	1/6	2/6	3/6	4/6	5/6	1
f(x)	:	1	0.972973	0.9	0.8	0.692308	0.590167	0.5

Now $I_T[f]=\frac{h}{2}\left[f_0+f_6+2(f_1+f_2+f_3+f_4+f_5)\right]$

$= 0.784241$

$I_S[f]=\frac{h}{3}\left[f_0+f_6+4\,(f_1+f_3+f_5)+2(f_2+f_4)\right]$

$= 0.785398$

$\int_0^1 \frac{dx}{1+x^2}=\left[\tan^{-1}x\right]_0^1=\frac{\pi}{4}.$ Exact $\pi = 3.141593$

Value of π from $I_T[f]=4\times 0.784241=3.136964$

Error in calculating π by $I_T[f]$ is $E_T[f]=0.004629$

Value of π from $I_s[f]=4\times 0.785398=3.141592$

Error in π by $I_s[f]$ is $E_s[f]=1.0\times10^{-6}$

Ex. 5 : $I_T[f]=\left(\frac{h}{2}\right)\left[f_0+f_4+2(f_1+f_2+f_3)\right]$

$=(1/4)[1+25+2(2.875+7+14.125)]=18.5$

$I_s[f]=\left(\frac{h}{3}\right)\left[f_0+f_4+2f_2+4(f_1+f_3)\right]$

$= (1/6)[1+25+2\times 7+(2.875+14.125)]=18$

Ex. 6 : Let $v_0=0,\ v_1=15, v_2=25, v_3=40, v_4=45, v_5=20, v_6=0,\ h = 2$. Then

$I=\int_0^{12} v\,dt,$

$I_T[v]=\left(\frac{h}{2}\right)[v_0+v_6+2(v_1+v_2+v_3+v_4+v_5)]=290$

$I_s[v]=\frac{880}{30}=293.33.$

Ex. 7 : The values of $f(x) = \sin x - \ln x + e^x$ are

$f(0.2)=3.02951, f(0.3)=2.849352, f(0.4)=2.797534$

$I_T[f]=\left(\frac{0.1}{2}\right)[f(0.2)+2f(0.3)+f(0.4)]=0.57629$

$$I_S[f]=\left(\frac{0.1}{3}\right)[f(0.2)+4f(0.3)+f(0.4)]=0.574148$$

Exact value = 0.574056

$E_T = 2.234 \times 10^{-3}$

$E_s = 9.2 \times 10^{-5}$

Ex. 8 : (1) Using the trapezoidal rule, we have

$$I=\frac{1}{2}\left(1+\frac{1}{2}\right)=0.75 \text{ Error}=0.75-0.693147=0.056853$$

Using the Simpson's rule, we have

$$I\simeq\frac{1}{6}\left(1+\frac{8}{3}+\frac{1}{2}\right)=\frac{25}{36}=0.001297.$$

Error = 0.75 − 0.693147 = 0.056853.

Ex. 9 : (1) When n = 2, we have h = 0.5, Nodes are 0,0.5 and 1

We obtain

$$I_T=\frac{1}{4}\left[f(0)+2f(0.5)+f(1)\right]=\frac{1}{4}\left(1+\frac{4}{3}+\frac{1}{2}\right)=0.708333$$

$$I_S=\frac{1}{6}\left[f(0)+4f(0.5)+f(1)\right]=\frac{1}{6}\left(1+\frac{8}{3}+\frac{1}{2}\right)=0.694444$$

n = 4: h = 0.25, Nodes are 0, 0.25, 0.5, 0.75 and 1. We obtain

$$I_T=\frac{1}{8}\left[f(0)+2\{f(0.25)+f(0.5)+f(0.75)\}+f(1)\right]=0.697024$$

$$I_S=\frac{1}{12}\left[f(0)+4f(0.25)+2f(0.5)+4f(0.75)+f(1)\right]=0.693254$$

n = 8: h = 0.125, nodes are 0, 0.125, 0.25, . . . , 1.0.

We have eight subintervals for trapezoidal rule and four subintervals for Simpson's rule. We get

$$I_T=\frac{1}{16}\left[f(0)+2\sum_{i=1}^{7}f\left(\frac{i}{8}\right)+f(1)\right]=0.694122$$

$$I_S=\frac{1}{24}\left[f(0)+4\sum_{i=1}^{4}f\left(\frac{2i-1}{8}\right)+2\sum_{i=1}^{3}f\left(\frac{2i}{8}\right)+f(1)\right]=0.693155.$$

The exact value of the integral is $I = 0.693147$.

UNIT 3 ORDINARY DIFFERENTIAL EQUATIONS (ODEs)

Structure

3.0 INTRODUCTION

The essential fact is simply that all the pictures which science now draws of nature, and which alone seem capable of according with observational fact, are mathematical pictures. . . .

Sir James Jeans

Differential equations occur in almost every scientific discipline and play a significant role in modeling problems from almost every discipline of human experience. Their significance can be visualized from their ability to mathematically describe, or model, real-life situations, concerning diverse disciplines of, viz., demography, ecology, chemical kinetics, architecture, physics, mechanical engineering, quantum mechanics, electrical engineering, civil engineering, meteorology, and a relatively new science called *chaos*.

In this context, it may be noticed that even the same differential equation may be important to several disciplines, although for different reasons. For example, demographers, ecologists, and mathematical biologists would immediately recognize the equation : **dp/dt = rp,** as the *Malthusian law of population growth*. It is used to predict populations of certain kinds of organisms reproducing under ideal conditions – whereas physicists, chemists, and nuclear engineers would be more inclined to regard the equation as a mathematical portrayal, or model, of radioactive decay. Even many economists and mathematically minded investors would recognize this differential equation, but in a totally different context: it also models future balances of investments earning interest at rates compounded continuously.

However, a large number of classes of differential equations are not solvable by conventional analytical methods giving closed form solutions. Numerical methods for solving differential equations can be useful in such situations and otherwise also.

3.1 OBJECTIVES

After studying this unit, you should be able to

- classify differential equations into various types : ordinary, partial, linear, non-linear, of a particular order, and of a particular degree;

- define and explain the concepts: canonical form of a differential equation, initial condition, initial condition problem (IVP), boundary condition, boundary condition problem (BCP), general solution and particular solution of a differential equation; and
- solve ordinary differential equation(ODE) using
  - o Euler's method
  - o Improved Euler's Method
  - o Runge Kutta Method of Order 2
  - o Runge Kutta Method of Order 4.

3.2 DEFINITIONS OF BASIC CONCEPTS

(A Portion of this Section has been adapted from Section 3.2 of CS-71.)

Before we start discussion of various methods of solving differential equations, let us consider definitions of some basic concepts from the discipline.

Definition : An equation involving one or more variables, to be called dependent variables and their derivatives (at least one) with respect to one or more variables, to be called independent variables, is called a **differential equation.** For example,

$$(4x + y)\frac{dy}{dx} = 2y + x \qquad \ldots (1)$$

$$3x\frac{\partial z}{\partial x} + 4y\frac{\partial z}{\partial y} - z = y + 2z \qquad \ldots (2)$$

are differential equations. In Eq. (1), y is the only dependent variable and x is the only independent variable, whereas in Eq. (2), z is the only dependent variable, and x and y are independent variables.

Differential equations of the form (1), involving derivatives w.r.t. a single independent variable, are called **ordinary differential equations (ODEs)** whereas, those of the form (2) involving derivatives w.r.t. two or more independent variables are called **partial differential equations (PDEs)**.

Definitions: The **order** of a differential equation is the order of the highest order derivative appearing in the equation and its **degree** is the highest exponent of the highest order derivative after the equation has been rationalized i.e., after it has been expressed in the form free from radicals and any fractional power of the derivatives or negative power. For example, the equation

$$7\left(\frac{d^3y}{dx^3}\right)^2 + 12\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 3x^2\left(\frac{dy}{dx}\right)^3 = 0 \qquad \ldots (3)$$

is of **third** order and **second** degree. Equation

$$y = 2x\frac{dy}{dx} + \frac{15}{dy/dx}$$

is of **first** order and **second** degree as it can be written in the form

$$y\frac{dy}{dx} = 2x\left(\frac{dy}{dx}\right)^2 + 15, \qquad \ldots (4)$$

after rationalization, free from fractional power derivatives, etc.

Definition : When the dependent variable and its derivatives occur in the first degree only and not as higher powers or as products with each other; the equation is said to be **linear,** otherwise it is said to be **nonlinear.**

The Equation $2\dfrac{d^2y}{dx^2}+4y=3x^2$ is a linear ODE. On the other hand, the differential equation $(x + y)^2 \dfrac{dy}{dx}=1$ is a nonlinear ODE, because, it involves products like $y\left(\dfrac{dy}{dx}\right)$ and $y^2\left(\dfrac{dy}{dx}\right)$.

Similarly, $\dfrac{\partial^2 z}{\partial x^2}+\dfrac{\partial^2 z}{\partial y^2}-\left(\dfrac{\partial^2 z}{\partial x\,\partial y}\right)^2=0$, is a nonlinear PDE, as it involves $\left(\dfrac{\partial^2 z}{\partial x\,\partial y}\right)^2$.

In this unit, only Ordinary Differential Equations (ODEs) will be discussed.

The general form of a linear ODE of order n can be expressed in the form

$$L[y] \equiv a_0(x)\, y^{(n)}(x) + a_1(x)\, y^{(n-1)}(x) + \ldots + a_{n-1}(x)\, y'(x) + a_n(x)\, y(x) = r(x) \qquad \ldots (5)$$

where $r(x)$, $a_i(x)$, $i = 0, 1, 2, \ldots, n$ are known functions of x only and

$$L = a_0(x)\frac{d^n}{dx^n} + a_1(x)\frac{d^{n-1}}{dx^{n-1}} + \ldots + a_{n-1}(x)\frac{d}{dx} + a_n(x),$$

is the linear differential operator. The general nonlinear ODE of order n can be written as

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0 \qquad \ldots (6)$$

or, $$y^{(n)} = f(x, y, y', y'', \ldots, y^{(n-1)}). \qquad \ldots (7)$$

Eq. (7) is called a **canonical representation** of Eq. (6). In such a form, the highest order derivative is expressed in terms of lower order derivatives and the independent variable.

By the **solution of an ODE** of the form $F(x, y, y', y'', \ldots, y^{(n)}) = 0$ or, of the form $y^{(n)} = f(x, y, y', y'', \ldots, y^{(n-1)})$, we mean, a relation, say, $y = g(x)$, free of derivatives; **such that if** throughout, y is replaced by g(x), in the differential equation, then the equation is satisfied.

The **general solution** of an nth order ODE contains n arbitrary constants. In order to determine these arbitrary constants, we require n conditions.

These constraints can be determined and a **particular solution** can be obtained if n suitable conditions on y and/or its derivatives are prescribed. In numerical analysis, we are concerned with finding only the particular solution.

Definitions : If all these conditions, to determine the values of arbitrary constants involved in the solution of a differential equation of order n, are given **at one point,** then these conditions are known as **initial conditions** and the differential equation together with the initial conditions is called an **initial value problem** (IVP)/(i.v.p.).

> The nth order IVP along with associated initial conditions can be written as
>
> $y^{(n)}(x) = f(x, y, y', y'', \ldots, y^{(n-1)})$ (differential equation)
>
> $y^{(p)}(x_0) = y_0^{(p)}$, $p = 0, 1, 2, \ldots, n - 1$. (n associated conditions) $\ldots$ (8)

Definitions : If the n conditions are prescribed **at more than one point** then these conditions are known as **boundary conditions.** These conditions are prescribed usually at two points, say x_0 and x_1 and we are required to find the solution y(x) between $x_0 < x < x_1$. The differential equation together with the boundary conditions is then known as a **boundary value problem** (BVP).

In this course, we study only IVPs.

It may be noted that the independent variable is generally denoted either by x or by t, however, there is no such restriction, i.e., any other symbol also may be used for the purpose.

3.3 METHODOLOGY FOR NUMERICAL SOLUTIONS OF IVPs – PART (i)

Earlier, in Section 2.2 of this block, we discussed: Methodology of Numerical Integration. A **methodology** is a sequence of general steps, which (i.e., the sequence) forms an outline of many of the methods used for solving a particular type of problems. **Here, we discuss first two steps of the methodology for solving IVPs. Rest of the steps will be discussed later.**

Step1 of the Methodology :

In order to solve an IVP involving n^{th} order differential equation, transform the IVP into n IVPs, each involving only first order derivatives, and attempt to solve only these IVPs involving only first order derivatives.

The justification for doing so is based on the following argument

Consider an i. v. p. of order n,

$$\frac{d^n y}{dx^n} = f(x, y, y', y'', \ldots, y^{n-1})$$

when the n initial conditions are prescribed at $x = x_0$ as

$$y(x_0) = y_0, y'(x_0) = y'_0, y''(x_0) = y''_0, \ldots, y^{n-1}(x_0) = y_0^{n-1}$$

Introducing new variable $z_1, z_2, \ldots, z_{n-1}$ such that

$$z_1 = \frac{dy}{dx}, z_2 = \frac{d^2y}{dx^2}, \ldots, z_{n-1} = \frac{dy^{n-1}}{dx^{n-1}}$$

The above o.d.e. of order n reduces to the following n o.d.e.'s each of order one,

$$\frac{dy}{dx} = z_1$$

$$\frac{dz_1}{dx} = z_2$$

$$\vdots$$

$$\frac{dz_{n-1}}{dx} = f(x, y, z_1, x_2, \ldots, x_{n-1})$$

The above n first order equations have to be solved simultaneously for $y, z_1, z_2, \ldots, z_{n-1}$, with prescribed n initial conditions, namely, $y(x_0), z_1(x_0), z_2(x_0), \ldots, z_{n-1}(x_0)$.

Summarizing, the first step of the methodology is to replace the given IVP of order n by equivalent set of n IVPs involving only first order differential equations, and then attempt solutions of these differential equations.

Hence, in order to solve an IVP of order n, basically we need to discuss the methods for solving an IVP involving only first order equation, say $\frac{dy}{dx} = f(x, y)$ with initial condition $y(x_0) = y_0$.

Step 2

Thus, from Step1, the problem now is to solve the IVP:

$$y' = f(x, y), \; y(x_0) = y_0 \qquad \ldots (10)$$

In order to explain Step 2 of the methodology, let us recall **what is meant by solution** of a first order differential equation, first, (i) in general, then (ii) **at a point and finally (iii) over an interval** *(the definitions can be easily generalized to an arbitrary ODE)*:

Let y be a dependent variable, which is a function of an independent variable x. This fact is denoted as $y = g(x)$, for some g, or may be denoted simply as y(x). The value of y for a given value of x, say x= 3, is denoted as y(3). In general, the value of y for x = a or $x = x_0$ is denoted respectively as y(a) and $y(x_0)$. Generally, in such cases, we call y(3), y(a) or $y(x_0)$, respectively, as value at point 3, at point a, or at point x_0.

For a given first order differential equation $dy/dx = f(x, y)$, we may note that in the initial stage of the problem, it is not known (explicitly/implicitly) what is the functional relation, between y and x, free of derivatives, like $y = 5x^3 + \log x + 9$, or, in general, like $y = g(x)$. In other words, g is **not** known. By the **solution of an ODE** $dy/dx = f(x, y)$**,** we mean, a relation, say, $y = g(x)$, free of derivatives**; such that if** throughout, y is replaced by g(x), in the differential equation $dy/dx = f(x, y)$, then the equation is satisfied. Finding such a general functional relationship, in the case of most of the differential equations, is generally, quite difficult or even impossible.

For a given IVP $dy/dx = f(x, y)$, $y(x_0) = y_0$, and for a given point $c > x_0$, we attempt to find y (c), such that $y'(c) = f(c, y(c))$. **Finding y(c), constitutes a solution of the given differential equation at x = c.** Finally, by solution of the differential equation $dy/dx = f(x, y)$ over an interval [a, b], we mean solution of the differential equation for every point of the interval.

In the IVPs, as a part of the definition of IVP, as is done in (10), it is assumed that solution of the differential equation, at some point, generally denoted by x_0 is known, i.e., $y_0 = y(x_0)$ is known so that $y'(x_0)$ is calculated as $f(x_0, y_0)$.

Main part of Step 2 : For IVP (10), solutions are attempted only for points $x > x_0$. It may be that we are given some $b > x_0$, and solutions are to be found over $[x_0, b]$, or it may be that no such b is given. In that case, we assume that solutions are to be found for all $x > x_0$. In either of these cases,

Step 2 of the methodology consists of

(i) determining a suitable step size h > 0 either through calculation or by assumption, so that solutions are attempted only at points : $x_0 + h$, $x_0 + 2h$, $x_0 + 3h, \ldots, (x_0 + nh) \ldots$ *(If, the solution is required at any other point, then, additionally, some other method including interpolation may be used.).* Let $x_n = (x_0 + nh)$, with n = 1, 2, 3, . . . **These x_n's are called grid points or mesh points.**

(ii) Solution of the IVP (10) is attempted in a step-by-step manner as follows : First, solve (10) at $x_0 + h$; use this solution in finding solution at $x_0 + 2h$, and so on, i.e. for attempting solution at $x_0 + (i + 1) h$, we must have found solutions at $x_0 + h$, $x_0 + 2h, \ldots, x_0 + i h$, and then we use these solutions in attempting solution at $x_0 + (i + 1) h$.

The rest of the steps of the methodology will be discussed after description of two methods, which follow the methodology. First, we discuss some methods based on these two steps.

3.4 EULER'S METHOD

Let **y (x_1) denote exact solution** and let **y_1 denote the computed solution** at $x_1 = x_0 + h$ of the i. v. p. of order 1 :

$$y'(x) = (dy/dx) = f(x, y)$$

with initial condition at

$$x = x_0 \text{ as } y(x_0) = y_0 \qquad \ldots (10)$$

Then from Taylor's series,

$$y(x_1) = y(x_0 + h) = y(x_0) + h\,y'(x_0) + \frac{h^2}{2!}\,y''(x_0) + \ldots$$

Truncating the Taylor's series after second term we approximate the value of y (x_1) as : $y_1 = y_0 + h\,y'(x_0) = y_0 + hf\,(x_0, y_0)$; using $y'(x_0) = f(x_0, y_0)$.

This is Euler's formula and error in the formula is given by $\frac{h^2}{2}\,y''(\xi)$ where $x_0 \le \xi \le x_1$.

This is why, Euler's method is called **first-order method (order of a method to be defined later)**

> Summarizing, for the i. v. p. of order 1 :
>
> $y'(x) = (dy/dx) = f(x, y)$ with initial condition at $x = x_0$ as $y(x_0) = y_0$,
>
> For any general point $x = x_{n+1}$, the **Euler's formula** may be written as,
>
> $\mathbf{y_{n+1} = y_n + h\,f\,(x_n, y_n)}$,
>
> while the error in the formula would be $(h^2/2)\,y''(\xi)$, $x_n \leqq \xi \leqq x_{n+1}$, in the computation of y_{n+1} from y_n.

Example 1

Use Euler method to find the solution of $y' = f(t, y) = t + y$, given $y(0) = 1$.

Find the solution on [0, 0.8] with h = 0.2.

(note again: t is taken as independent variable, instead of x. Given diff. eq. is $y' = t + y = f(t, y)$. Hence, $f(t, y) = t + y$ and f_0 = value of f at (t = 0)= (0 + 1) = 1).

Solution

We have $\mathbf{y_{n+1} = y_n + h\,f_n}$

$$y(0.2) \approx y_1 = y_0 + (0.2)\,f_0$$
$$= 1 + (0.2)\,[0 + 1] = 1.2$$

$$y(0.4) \approx y_2 = y_1 + (0.2)\,f_1$$
$$= 1.2 + (0.2)\,[0.2 + 1.2]$$
$$= 1.48$$

$$y(0.6) \approx y_3 = y_2 + (0.2)\,f_2$$
$$= 1.48 + (0.2)\,[0.4 + 1.48]$$
$$= 1.856$$

$$y(0.8) \approx y_4 = y_3 + (0.2)\,f_3$$
$$= 1.856 + (0.2)\,[0.6 + 1.856]$$
$$= 2.3472$$

Example 2

Solve the differential equation $y' = t + y$, $y(0) = 1$; $t \in [0, 1]$ by Euler's method using $h = 0.1$. If the exact value is $y(1) = 3.436564$, find the exact error.

*(**Note again** : t is taken as independent variable, instead of x. Given diff. Eq. is $y' = t + y = f(t, y)$. Hence, $f(t, y) = t + y$ and $y_0 = f_0 =$ value of f at $(t = 0) = (0 + 1) = 1$).*

Solution

Euler's method is

$$y_{n+1} = y_n + hy'_n = y_n + h\, f(t_n, y_n) = y_n + h\,(t_n + y_n)$$

For the given problem, we have

$$\mathbf{y_{n+1} = y_n + h\,[t_n + y_n]}$$
$$\mathbf{= (1 + h)\, y_n + ht_n}$$

$h = 0.1$, $y(0) = 1$,

$y_1 = (1 + h)\, y_0 + (0.1)\, 0 =$

$(1 + 0.1)(1) + (0.1)(0) = 1.1$

$y_2 = (1.1)(1.1) + (0.1)(0.1) = 1.22$,

$y_3 = 1.362$, $\quad y_4 = 1.5282$,

$y_5 = 1.72102$, $\quad y_6 = 1.943122$,

$y_7 = 2.197434$, $\quad y_8 = 2.487178$

$y_9 = 2.815895$, $\quad y_{10} = 3.187485 \approx y(1)$.

Exact error $= 3.436564 - 3.187485$

$= 0.249079$

Example 3

Using the Euler's method tabulate the solution of the IVP

$$y' = -2ty^2,\ y(0) = 1$$

in the interval [0,1] taking $h = 0.2, 0.1$.

Solution

We know, for i.v.p. of order 1 : $\mathbf{y'(x) = f(x, y)}$ with initial condition at $x = x_0$ as $y(x_0) = y_0$, for any general point $x = x_{i+1}$, **the Euler's formula** may be written as

$$\mathbf{y_{i+1} = y_i + h\, f(x_i, y_i)}$$

In this problem, $f(t, y) = -2ty^2$

At a general point $x = x_{i+1}$, **the Euler's formula, in this case,** may be written as,

$$\mathbf{y_{i+1} = y_i + h\, f(x_i, y_i)} = y_i - 2h\, t_i\, y_i^2, \text{ i.e.,}$$

$$\mathbf{y_{i+1}} = y_i - 2h\, t_i\, y_i^2$$

Starting with $t_0 = 0$, $y_0 = 1$, we obtain the following table of values for $h = 0.2$.

Table : h = 0.2

T	Y (t)
0.2	1
0.4	0.92
0.6	0.78458
0.8	0.63684
1.0	0.50706

Thus, y (1.0) = 0.50706 with h = 0.2

Similarly, starting with $t_0 = 0$, $y_0 = 1$, we obtain the following table of values for h = 0.1.

Table : h = 0.1

T	y(t)	T	y(t)
0.1	1.0	0.6	0.75715
0.2	0.98	0.7	0.68835
0.3	0.94158	0.8	0.62202
0.4	0.88839	0.9	0.56011
0.5	0.82525	1.0	0.50364

y (1.0) = 0.50364 with h = 0.1.

Remark : Since the Euler's method is of O(h), it requires h to be very small to obtain the desired accuracy. Hence, very often, the number of steps to be carried out becomes very large. In such cases, we need higher order methods to obtain the required accuracy in a limited number of steps.

Example 4

For the IVP : $y'+2y=2-e^{-4t}$, $y(0)=1$. Use the Euler's method with step size h = 0.1 to find approximate values of the solution at t = 0.1, 0.2, 0.3, 0.4, and the percentage errors in the solutions. The solution of the differential equation is

$$y(t)=1+\frac{1}{2}e^{-4t}-\frac{1}{2}e^{-2t}.$$

Solution

In order to use Euler's Method, we first need to rewrite the differential equation as $y'=2-e^{-4t}-2y$.

Hence, $f(t,y)=2-e^{-4t}-2y$. The initial values are $t_0 = 0$ and $y_0 = 1$ (given).

Euler's method gives $y_{j+1}=y_j+hf(t_j, y_j)$.

We have for j = 0, and h = 0.1.

$$f_0=f(t_0,y_0)=f(0,1)=2-e^{-4(0)}-2(1)=-1$$

$$y_1 = y_0 + h\, f_0 = 1 + (0.1)(-1) = 0.9$$

Therefore, the approximation to the solution at $t_1 = 0.1$

$$y(0.1)\approx y_1=0.9.$$

According to our convention, y (0.1), y (0.2), . . . etc. denote exact values and y_1, y_2, . . . etc. denote calculated values.

For *j* = 1, we get,

$$f_1=f(0.1,0.9)=2-e^{-4(0.1)}-2(0.9)=-0.470320046,$$

$$y_2=y_1+hf_1=0.9+(0.1)(-0.470320046)=0.852967995$$

Therefore, the approximation to the solution at $t_2 = 0.2$ is $y(0.2)\approx y_2=0.852967995$.

For j = 2, 3, 4 we obtain the following values.

$f_2=-0.155264954,$	$y_3=0.837441500$
$f_3=0.023922788,$	$y_3=0.839833779$
$f_4=0.1184359245,$	$y_5=0.851677371$

The percentage error in the numerical solutions is defined by

Percentage error $= \left|\dfrac{\text{exact value} - \text{approximate}}{\text{exact value}}\right| \times 100$. The results are presented in the following table :

t_n	Approximation	Exact	Error
$t_0 = 0$	$y_0 = 1$	$y(0) = 1$	0 %
$t_1 = 0.1$	$y_1 = 0.9$	$y(0.1) = 0.925794646$	2.79 %
$t_2 = 0.2$	$y_2 = 0.852967995$	$y(0.2) = 0.889504459$	4.11 %
$t_3 = 0.3$	$y_3 = 0.837441500$	$y(0.3) = 0.876191288$	4.42 %
$t_4 = 0.4$	$y_4 = 0.839833779$	$y(0.4) = 0.876283777$	4.16 %
$t_5 = 0.5$	$y_5 = 0.851677371$	$y(0.5) = 0.883727921$	3.63 %

Example 5

Repeat the previous example with the approximations at t = 1, t = 2, t = 3, t = 4 and t = 5. Use h = 0.1, h = 0.05, h = 0.01, h = 0.005, and h = 0.001 for the approximations.

Solution

Below are two tables, one gives approximations to the solution and the other gives the errors for each approximation. We'll leave the computational details to you to check.

Approximations						
Time	**Exact**	***h*** **= 0.1**	***h*** **= 0.05**	***h*** **= 0.01**	***h*** **= 0.005**	***h*** **= 0.001**
$T = 1$	0.9414902	0.9313244	0.9364698	0.9404994	0.9409957	0.9413914
$T = 2$	0.9910099	0.9913681	0.9911126	0.9910193	0.9910139	0.9910106
$T = 3$	0.9987637	0.9990501	0.9988982	0.9987890	0.9987763	0.9987662
$T = 4$	0.9998323	0.9998976	0.9998657	0.9998390	0.9998357	0.9998330
$T = 5$	0.9999773	0.9999890	0.9999837	0.9999786	0.9999780	0.9999774

Percentage Errors					
Time	***h*** **= 0.1**	***h*** **= 0.05**	***h*** **= 0.01**	***h*** **= 0.005**	***h*** **= 0.001**
$t = 1$	1.08 %	0.53 %	0.105 %	0.053 %	0.0105 %
$t = 2$	0.036 %	0.010 %	0.00094 %	0.00041 %	0.0000703 %
$t = 3$	0.029 %	0.013 %	0.0025 %	0.0013 %	0.00025 %
$t = 4$	0.0065 %	0.0033 %	0.00067 %	0.00034 %	0.000067 %
$t = 5$	0.0012 %	0.00064 %	0.00013 %	0.000068 %	0.000014 %

We can see from these tables that as h decreases, the accuracy of the approximations are improved.

Example 6

Solve the initial value problem to compute approximation for y (0.1), y (0.2) using Euler's material with h = 0.1.

$\dfrac{dy}{dt} + 2y = 3e^{-4t}, y(0) = 1$. Compare with the exact solution $y(t) = \dfrac{(5e^{-2t} - 3e^{-4t})}{2}$.

Solution

Euler's method gives $y_{j+1} = y_j + hf(t_j, y_j)$

where, $f(t, y) = 3e^{-4t} - 2y$.

For $j = 0$,

We get, $y_1 = y_0 + hf(t_0, y_0)$.

Now, $f(t_0, y_0) = f(0,1) = 3e^0 - 2 = 1$.

We obtain, $y(0.1) \approx y_1 = 1 + (0.1)(1) = 1.1$

The exact solution is $y(0.1) \approx 1.0413$ and the absolute error is 0.0587.

For the next time Step $j = 1$,

We get $y_2 = y_1 + hf(t_1, y_1)$.

Now, $f(t_1, y_1) = (0.1, 1.1) = 3e^{-0.4} - 2(1.1) = -0.189$

We obtain $y(0.2) \approx y_2 = 1.1 + 0.1(-0.189) = 1.0811$.

The exact solution is $y(0.2) \approx 1.0018$ and the absolute error is 0.0793.

Solve the following IVPs under Ex. 1 to Ex. 5 using Euler's method

Ex. 1 : $y' - y = -\frac{1}{2}e^{1/2}\sin(5t) + 5e^{1/2}\cos(5t), \qquad y(0) = 0$

Use Euler's Method to find the approximation to the solution at $t = 1$. Use $h = 0.1$, $h = 0.05$ for the approximations and find the percentage errors.

Ex. 2 : $y' = 1 - 2xy$, $y(0.2) = 0.1948$. Find $y(0.4)$ with $h = 0.2$.

Ex. 3 : $y' = \frac{1}{x^2 - 4y}$, $y(4) = 4$. Find $y(4.1)$ taking $h = 0.1$.

Ex. 4 : $y' = \frac{y - x}{y + x}$, $y(0) = 1$. Find $y(0.1)$ with $h = 0.1$.

Ex. 5 : $y' = 1 + y^2$, $y(0) = 1$. Find $y(0.6)$ taking $h = 0.2$ and $h = 0.1$.

3.5 IMPROVED EULER'S METHOD

Let us suppose that y_1 has been computed from the Euler's formula as above and let it be denoted as,

$$y_1^* = y_0 + hf(x_0, y_0)$$

We compute $y'(x) = \frac{dy}{dx} = f(x, y)$ at $x = x_1$ where $y(x_1) = y_1^*$ i.e. $\frac{dy}{dx} = f(x_1, y_1^*)$.

The value of y_1 is then improved in the Euler's formula by taking the average value of $\frac{dy}{dx}$ for (x_0, y_0) and (x_1, y_1^*). Then modified (improved) Euler's formula may be written as, $y_1 = y_0 + \frac{h}{2}\{f(x_0, y_0) + f(x_1, y_1^*)\}$

At a general point $x = x_{n+1}$, the Improved Euler formula/method becomes

$$\left.\begin{array}{l} y^*_{n+1} = y_0 + hf\,(x_n, y_n) \\ y_{n+1} = y_n + \dfrac{h}{2}\{f\,(x_n, y_n) + f\,(x_{n+1}, y^*_{n+1})\} \end{array}\right\}$$

Example 7

Using modified Euler's method, find the value of y for x = 0.1 and 0.2 from the differential equation $\dfrac{dy}{dx} = x^2 + y^2 - 2;\ y(0) = 1$. Compute upto 5 places of decimal only.

Solution

Euler's modified formula is,

$$\left.\begin{array}{l} y^*_{n+1} = y_n + hf\,(x_n, y_n) \\ y_{n+1} = y_n + \dfrac{h}{2}\{f\,(x_n, y_n) + f\,(x_{n+1}, y^*_{n+1})\} \end{array}\right\}$$

Here, $h = 0.1;\ f(x, y) = x^2 + y^2 - 2,\ x_0 = 0;\ y_0 = 1;\ x_1 = 0.1$

$f(x_0, y_0) = 0 + 1 - 2 = -1.0$

$y^*_1 = y_0 + hf\,(x_0, y_0) = 1 + 0.1 \times (-1.0) = 0.9$

$f(x_1, y^*_1) = 0.1^2 + 0.9^2 - 2 = -1.18$

$$y_1 = y_0 + \frac{h}{2}\{f\,(x_0, y_0) + f\,(x_1, y^*_1)\}$$

$$= 1.0 + \frac{0.1}{2}\{-1.0 - 1.18\} = 0.891$$

For $x = x_2 = x_1 + h = 0.2;\ x_1 = 0.1,\ y_1 = 0.891$

$f(x_1, y_1) = 0.1^2 + 0.891^2 - 2 = -1.19612$

$y^*_2 = y_1 + hf\,(x_1, y_1)$

$= 0.891 + 0.1 \times (-1.19612) = 0.77139$

$f(x_2, y^*_2) = 0.2^2 + 0.77139^2 - 2 = -1.36496$

$$y_2 = y_1 + \frac{h}{2}\{f\,(x_1, y_1) + f\,(x_2, y^*_2)\}$$

$$= 0.891 + \frac{0.1}{2}\{-1.19612 - 1.36496\} = 0.76295.$$

3.6 METHODOLOGY FOR NUMERICAL SOLUTIONS OF IVPs – PART (ii)

We recall that in order to solve the IVP :

$$\mathbf{y' = f(x, y)\,,\ y(x_0) = y_0} \qquad \ldots (10)$$

for any general point $x = x_{n+1}$,

the Euler's formula may be written as,

$$\mathbf{y_{n+1} = y_n + h\,f\,(x_n, y_n),} \qquad \ldots (11)$$

The Improved Euler formula/method becomes

$$\left.\begin{aligned} y^*_{n+1} &= y_0 + h f(x_n, y_n) \\ y_{n+1} &= y_n + \frac{h}{2}\{f(x_n, y_n) + f(x_{n+1}, y^*_{n+1})\} \end{aligned}\right\} \quad \ldots (12)$$

From the above, we find that in both of these methods, the solution at grid point x_{n+1} is obtained from the solution at grid point x_n from the following general formula :

$$\mathbf{y_{n+1} = y_n + h\,\phi\,(x_n, y_n, h)} \quad \ldots (13)$$

where, in Euler's method, $\phi\,(x_n, y_n, h) = f\,(x_n, y_n)$, whereas, in Improved Euler's method, $\phi\,(x_n, y_n, h)$ **is obtained through the two steps** (refer to Eq. (12) above) : first y^*_{n+1} is obtained and then its value is used in calculating $\phi\,(x_n, y_n, h)$. The advantage of Improved Euler's method is that if we expand y_{n+1} in terms of y_n through Taylor's series, then the terms up to coefficients of h^2 agree and the error is of $O(h^3)$.

It has been found that, with some restrictions on f(x, y) of Eq. (10), **by using p appropriate steps,** *(as two in the case of Improved Euler's method)* methods may be devised, so that terms up to the coefficient of h^p in the computation of y_{n+1} by the method, coincide with the corresponding terms in the expansion of y_{n+1} by Taylor's series. The error, due to the method, is of order $O(h^{p+1})$. **Such a method using p steps, with** error of order $O(h^{p+1})$, **is called a method of order p.**

Step 3 of the methodology consists of finding appropriate p-step methods of order p to calculate $\phi\,(x_n, y_n, h)$, **so that error in computation of y_{n+1} by the method is** $O(h^{p+1})$. Two German mathematicians Runge and Kutta have been pioneers in this field, and in discovering methods using Step 3 of the methodology. Hence, these methods are called **Runge-Kutta (R-K) methods**. It may be noted that even for a particular order, say 2, there is not necessarily a single method, but a group of methods, which is called R-K group of methods of the order.

Next, we study **some R-K methods of order 2 and of order 4 (only), suggested by them.**

3.7 RUNGE-KUTTA (R-K) METHOD OF ORDER 2

We have seen that **modified/improved Euler's method** gives value of y_{n+1} at a general point $x = x_{n+1}$, through the following steps

$$\boxed{\left.\begin{aligned} y^*_{n+1} &= y_0 + h f(x_n, y_n) \\ y_{n+1} &= y_n + \frac{h}{2}\{f(x_n, y_n) + f(x_{n+1}, y^*_{n+1})\} \end{aligned}\right\}}$$

The method involves two steps. Also, the error in the method is $O(h^3)$. Hence, it is second order method. And, it is a two-step method.

A general R-K Method is a two-step method of Order 2, developed as follows

Let the value of y, say y_o, be known at $x = x_o$.

The other values of y_{n+1} in the general R-K method of order 2, at $x = x_{n+1}$, are obtained as follows :

$$\mathbf{y_{n+1} = y_n + k, \quad for\ n = 0, 1, 2 \ldots}$$

where, k is obtained as follows :

$$\left.\begin{aligned} & k_1 = h\,f\,(x_n, y_n)\,; \\ & k_2 = h\,f\,(x_n + \alpha h,\, y_n + \beta\,k_1) \\ \text{and} \quad & k = w_1 k_1 + w_2 k_2 \end{aligned}\right\} \quad \ldots (A)$$

And the parameters α, β and weights w_1 and w_2 are determined so that y_{n+1} agrees with Taylor's expansion of $y(x_n + h)$ up to powers of h^2, and hence, it becomes a method of order 2, i.e., error in computation of y_{n+1} is of $O(h^3)$.

Improved Euler's method can be obtained as R-K method of order 2, by taking in Eq. (A), $\alpha = 1$, $\beta = 1$, $w_1 = (1/2) = w_2$ to arrive at the following formula :

$$\mathbf{y_{n+1} = y_n + k, \quad for\ n = 0, 1, 2 \ldots}$$

where, k is obtained as follows:

$$\left.\begin{aligned} & k_1 = h\,f(x_n, y_n)\,; \\ & k_2 = h\,f(x_n + h, y_n + k_1) \\ \text{and}\quad & k = (1/2)(k_1 + k_2) \end{aligned}\right\} \qquad \ldots (B)$$

Looking closely, it is same as Modified Euler's formula, though derived differently. **Improved Euler's Method is also called Modified Euler's Method and also called Improved Tangent Method.**

There are other R-K Methods of Order 2, including Heun's method and Optimal method. But, we will not discuss these methods.

Example 8

Solve the IVP $y' = -ty^2, y(2) = 1$ and find y(2.1) and y(2.2) with h = 0.1 using Improved tangent method/modified Euler method.

Compare the results with the exact solution $y(t) = \dfrac{2}{t^2 - 2}$.

Solution

We have the exact values

y(2.1) = 0.82988 and y(2.2) = 0.70422

Improved tangent method is

$$y_{n+1} = y_n + K_2$$

$$K_1 = hf(t_n, y_n)$$

$$K_2 = hf\left(t_n + \frac{h}{2} y_n + \frac{K_1}{2}\right).$$

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For this problem $f(t, y) = -t\,y^2$ and

$$K_1 = (0.1)[(-2)(1)] = -0.2$$

$$K_2 = (0.1)\left[(-2.05)\ (1 - 0.1)^2\right] = -0.16605$$

y(2.1) = 1 – 0.16605 = 0.83395

Taking $t_1 = 2.1$ and $y_1 = 0.83395$, we have

$$K_1 = hf(t_1, y_1) = (0.1)\left[(-2.1)(0.83395)^2\right] = -0.146049$$

$$K_2 = hf\left(t_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right).$$

$$= (0.1)\left[-(2.15)(0.83395 - 0.0730245)^2\right] = -0.124487$$

$y(2.2) = y_1 + k_2 = 0.83395 - 0.124487 = 0.70946.$

3.8 RUNGE-KUTTA (R-K) METHOD OF ORDER 4

Next, we discuss, the fourth order R-K method or R-K method of order 4, which has an accuracy of O (h^4). It is also called classical R-K method and is the most popular for solving first order o.d.e's. Let us suppose that value of y_n is already known for $x = x_n$. Then the value y_{n+1} at $x = x_{n+1}$ is computed in four stages as follows:

$$k_1 = hf\,(x_n, y_n)\ ;\ k_2 = hf\left(x_n + \frac{1}{2}h,\ y_n + \frac{1}{2}k_1\right)$$

$$k_3 = hf\left(x_n + \frac{1}{2}h,\ y_n + \frac{1}{2}k_2\right)\ ;\ k_4 = hf\,(x_n + h,\ y_n + k_3)$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$y_{n+1} = y_n + k$$

As mentioned above, this method is also called Classical R-K method of order 4.

Of course, there are other R-K methods of order 4, including R-K-Gill method. But, in this course, we will not consider any other R-K method of order 4.

Example 9

Using fourth order R-K method compute the values of y for x = 0.2 and x = 0.4 taking a step size of 0.2 from the differential equation, $\frac{dy}{dx} = x\,(1 + y),\ y\,(0) = 1$. Compute upto 4 decimals only.

Solution

$$h = 0.2\ ;\ f\,(x, y) = x\,(1 + y)\ ;\ x_0 = 0,\ y_0 = 1\ ;\ x_1 = x_0 + h = 0.2$$

$$k_1 = 0.2 \times 0 \times (1 + 1) = 0$$

$$k_2 = 0.2 \times f\,(0 + 0.1, 1 + 0) = 0.2 \times f\,(0.1, 1) = 0.2 \times \{0.1\,(1 + 1)\} = 0.04$$

$$k_3 = 0.2 \times f\,(0+0.1, 1+0.02) = 0.2\times f\,(0.1, 1.02) = 0.2\times\{0.1\,(1+1.02)\} = 0.0404$$

$$k_4 = 0.2\times f\,(0+0.2, 1+0.0404) = 0.2\times f\,(0.2, 1.0404) = 0.2\times\{0.2\,(1+1.0404)\} = 0.0816$$

$$k = \frac{1}{6}(0 + 2 \times 0.04 + 2 \times 0.0404 + 0.0816) = 0.0404$$

$$y_1 = y_0 + k = 1 + 0.0404 = 1.0404$$

$$x_1 = 0.2,\ y_1 = 1.0404\ ;\ x_2 = 0.4.\ y_2 = ?\ ;\ h = 0.2$$

$$k_1 = 0.2 \times f\,(0.2, 1.0404) = 0.2 \times 0.2\,(1 + 1.0404) = 0.0816$$

$$k_2 = 0.2 \times f\,(0.2 + 0.1, 1.0404 + 0.0408) = 0.2 \times 0.3\,(1 + 1.0812) = 0.1249$$

$$k_3 = 0.2 \times f\,(0.2 + 0.1, 1.0404 + 0.0624) = 0.2 \times 0.3\,(1 + 1.1028) = 0.1262$$

$$k_4 = 0.2 \times f\,(0.2 + 0.2, 1.0404 + 0.1262) = 0.2 \times 0.4\,(1 + 1.1666) = 0.1733$$

$$k = \frac{1}{6}(0.0816 + 2 \times 0.1249 + 2 \times 0.1262 + 0.1733) = \frac{1}{6}(0.7571) = 0.1262$$

$$y_2 = y_1 + k = 1.0404 + 0.1262 = 1.1666$$

Example 10

Solve the IVP $y' = t + y$, $y(0) = 1$ by Runge-Kutta method of order 4 for $t \in (0,0.5)$ with $h = 0.1$. Also find the error at $t = 0.5$, if the exact solution is $y(t) = 2e^t - t - 1$.

Solution

Initially, $y(t_0) = y(0) = 1$

We have

$$K_1 = hf(t_0, y_0) = (0.1)[0+1] = 0.1$$

$$K_2 = hf\left(t_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right) = (0.1)[0.05 + 1 + 0.05] = 0.11$$

$$K_3 = hf\left(t_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right) = (0.1)[0.05 + 1 + 0.055] = 0.1105$$

$$K_4 = hf(t_0 + h, y_0 + K_3) = (0.1)[0.1 + 1 + 0.1105] = 0.12105$$

$$y_1 = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$$= 1 + \frac{1}{6}[1 + 0.22 + 0.2210 + 0.12105] = 1.11034167$$

Taking $t_1 = 0.1$ and $y_1 = 1.11034167$, we repeat the process.

$$K_1 = hf(t_1, y_1) = (0.1)[0.1 + 1.11034167] = 0.121034167$$

$$K_2 = hf\left(t_1 + \frac{h}{2}, y_1 + \frac{K_1}{2}\right)$$

$$= (0.1)\left[0.1 + 0.05 + 1.1103416 + \frac{(0.121034167)}{2}\right] = 0.132085875$$

$$K_3 = hf\left(t_1 + \frac{h}{2}, y_1 + \frac{K_2}{2}\right)$$

$$= (0.1)\left[0.1 + 0.05 + 1.1103417 + \frac{(0.132085875)}{2}\right] = 0.132638461$$

$$K_4 = hf(t_1 + h, y_1 + K_3) = (0.1)[0.1 + 0.05 + 1.11034167 + \frac{(0.132085875)}{2}$$

$$= 0.144303013$$

$$y_2 = y_1 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$$= 1.11034167 + \frac{1}{6}[(0.121034167 + 2(0.132085875) + 2(0.132638461)$$

$$+ 0.1443030131 = 1.24280514$$

Rest of the values y_3, y_4 and y_5 are given in Table below.

Table

t_n	y_n
0.0	1
0.1	1.11034167
0.2	1.24280514
0.3	1.39971699
0.4	1.58364848
0.5	1.79744128

Now the exact solution is

$$y(t)=2e^{t}-t-1$$

Error at t = 0.5 is

$$y(0.5)-y_5=(2e^{0.5}-0.5-1)-\frac{1.79}{44128}$$
$$=1.79744254-1.79744128$$
$$=0.000001261$$
$$=0.13\times10^{-5}.$$

Example 11

Use Runge-Kutta method to solve the IVP $y'=\frac{(t-y)}{2}$ on [0, 0, 2] with y (0) = 1. Compare the solutions with h = 0.2 and 0.1.

Solution

$h = 0.2$, $k_1 = -0.1$, $k_2 = -0.085$, $k_3 = -0.08575$,

$k_4 = -0.07(425$, $u(0.2) \approx 0.9145125$; $h = 0.1$, $u(0.1) \approx 0.95035$,

$u(0.2) \approx 0.911726$.

Example 12

Solve the IVP $y' = 2y+3e^t, y(0)=0$ using classical R-K method of order 4.

Find y (0.1), y(0.2), y(0.3) taking h = 0.1. Also find the errors at t = 0.3, if the exact solution is $y(t) = 3(e^{2t}-e^t)$.

Solution

Classical R-K method is

$$y_{n+1} = y_n + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

Here $t_0 = 0$, $y_0 = 0$, $h = 0.1$

$$K_1 = h\,f(t_0, y_0) = 0.3$$

$$K_2 = h\,f\left(t_0+\frac{h}{2}, y_0+\frac{K_1}{2}\right) = 0.3453813289$$

$$K_3 = h\,f\left(t_0+\frac{h}{2}, y_0+\frac{K_2}{2}\right) = 0.3499194618$$

$$K_4 = h\,f(t_0+h, y_0+K_3) = 0.4015351678$$

$$y_1 = 0.3486894582$$

Taking $t_1 = 0.1, y_1 = 0.3486894582$, we repeat the process and obtain

$K_1 = 0.4012891671$	$K_2 = 0.4584170812$
$K_3 = 0.4641298726$	$K_4 = 0.6887058455$
$y(0.2) = 0.8112570941$	

Taking $t_2 = 0.2$, $y_2 = 0.837870944$ and repeating the process we get

$K_1 = 0.53399502$	$K_2 = 0.579481565$
$K_3 = 0.61072997$	$K_4 = 0.694677825$
$\therefore$ y (0.3) = 1.416807999	

From the exact solution we get

$$y(0.3) = 1.416779978$$

Error in classical R-K method (at t = 0.3) = 02802×10^{-04}

Example 13

Solve the initial value problem $u' = -2tu^2$ with $u(0) = 1$ and $h = 0.2$ on the interval [0, 1]. Use the fourth order classical Runge-Kutta method.

Solution

We have $X_0 = 1$, $u_0 = 1$, $h = 0.2$, and $f(t, u) = -2tu^2$

For n = 0

$$k_1 = hf(t_0, u_0) = 0$$

$$k_2 = hf\left(t_0 + \frac{h}{2}, u_0 + \frac{1}{2}k_1\right) = -0.04$$

$$k_3 = hf\left(t_0 + \frac{h}{2}, u_0 + \frac{1}{2}k_2\right) = -0.038416$$

$$k_4 = hf(t_0 + h, u_0 + k_3) = -0.0739715$$

$$u(0.2) \approx u_1 = 1 + \frac{1}{6}[0 - 0.08 - 0.076832 - 0.0739715] = 0.9615328$$

For $n = 1$

$$t_1 = 0.2, u_1 = 0.9615328$$

$$k_1 = hf(t_1, u_1) = -0.0739636$$

$$k_2 = hf\left(t_1 + \frac{h}{2}, u_1 + \frac{1}{2}k_1\right) = -0.1025754$$

$$k_3 = hf\left(t_1 + \frac{h}{2}, u_1 + \frac{1}{2}k_2\right) = -0.0994255$$

$$k_4 = hf(t_1 + h, u_1 + k_3) = -0.1189166$$

$$u(0.4) \approx u_2 = 0.9615328 + \frac{1}{6}[-0.0739636 - 0.2051508 - 0.1988510 - 0.1189166]$$

$$= 0.8620525$$

Similarly, we get, $u(0.6) \approx u_3 = 0.7352784$

$$u(0.8) \approx u_4 = 0.6097519,\ u(1.0) \approx u_5 = 0.5000073$$

Example 14

A ball which is at temperature 1200 K is allowed to cool down in air at an ambient temperature of 300 K. Assuming heat is lost only due to radiation, the differential equation for the temperature of the ball is given by

$$\frac{d\theta}{dt} = -2.2067 \times 10^{-12} (\theta^4 - 81 \times 10^8), \qquad \theta(0) = 1200\ K.$$

Find the temperature at t = 480 seconds using Runge-Kutta 4th order method. Assume a step size of h = 240 sec.

Solution

From the given differential equation, we have,

$$f\ (t, \theta) = -\ 2.2067 \times 10^{-12}\ (\theta^4\ -\ 81 \times 10^8)$$

Runge-Kutta method gives $\theta_{n+1} = \theta_n + \frac{h}{6}\ (k_1 + 2k_2 + 2k_3 + k_4)$

For n = 0, we have $t_0 = 0$, $\theta_0 = \theta\ (0) = 1200$

$$k_1 = f(t_0, \theta_0) = f(0,1200) = -2.2067 \times 10^{-12}(1200^4 - 81 \times 10^8) = -4.5579$$

$$k_2 = f\left(t_0 + \frac{1}{2}h, \theta_0 + \frac{1}{2}k_1 h\right) = f\left(0 + \frac{1}{2}(240), 1200 + \frac{1}{2}(-4.5579)240\right)$$

$$= f(120, 653.05)$$

$$= -2.2067 \times 10^{-12}(653.05^4 - 81 \times 10^8)$$

$$= -0.38347$$

$$k_3 = f\left(t_0 + \frac{1}{2}h\theta_0 + \frac{1}{2}k_2 h\right) = f\left(0 + \frac{1}{2}(240), 1200 + \frac{1}{2}(-038347)240\right)$$

$$= f(120, 1154.0)$$

$$= 2.2067 = 10^{-12}(1154.0^4 - 81 \times 10^8) = -3.8954$$

$$k_4 = f\left(t_0 + h, \theta_0 + hk_3\right) = f\left(0 + 240, 1200 + 240\right)(-3.8954) = f(120, 265.10)$$

$$= 2.2067 \times 10^{-12}(265.10^4 - 81 \times 10^8) = 0.0069750$$

We obtain,

$$\theta(240) \approx \boldsymbol{\theta_1} = \theta_0 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1200 + \frac{240}{6}(-4.5579 + 2(-0.38347) + 2(-3.8954) + (0.0069750))$$

$$\boldsymbol{\theta_1} = 1200 + 40(-13.1087) = 675.65$$

At the next step, For n = 1, we have, $t_1 = 240$, $y_1 = 675.65$

We obtain,

$$k_1 = f(t_1, \theta_1) = -0.44199$$

$$k_2 = f\left(t_1 + \frac{1}{2}h, \theta_1 + \frac{1}{2}k_1 h\right) = -0.31372$$

$$k_3 = f\left(t_1 + \frac{1}{2}h, \theta_1 + \frac{1}{2}k_2 h\right) = -0.34775$$

$$k_4 = f\left(t_1 + h, \theta_1 + k_3 h\right) = -0.25351$$

$$\theta(480) \approx \boldsymbol{\theta_2} = \theta_1 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 675.65 + \frac{240}{6}\ (-0.44199 + 2(-0.31372) + 2(-0.34775) + (-0.25351))$$

$$= 675.65 + 40(-2.0184) = 594.91$$

Solve the following IVPs using R-K method of order 4 :

Ex. 6 : $y' = \frac{y-t}{y+t}, y(0)=1.$ Find y(0.5) taking h = 0.5.

Ex. 7 : $y' = 1 - 2ty$, y(0.2) = 0.1948. Find y(0.4) taking h = 0.2.

Ex. 8 : $10ty' + y^2 = 0, y(4) = 1.$ Find y(4.2) taking h = 0.2. Find the error given the exact solution is $y(t) = \frac{1}{c + 0.1 \text{in } t'}$ where c = 0.86137

Ex. 9 : $y' = \frac{1}{t^2} - \frac{y}{t} - y^2, y(1) = -1.$ Find y (1.3) taking h = 0.1. Given the exact solution to be $y(t) = \frac{1}{t'}$ find the error at t = 1.3.

Ex. 10 : Using Runge-Kutta method of order 4, find y(0.2) given that $y' = 3x + y/2$, y(0)= 1 taking h = 0.1.

Ex. 11 : Using Runge-Kutta method of order 4, compute $y(0.2)$ and $y(0.4)$ for the IVP $10y' = x^2 + y^2, y(0) = 1,$ taking h = 0.1.

Ex. 12 : Apply Runge-Kutta fourth order method to find an approximate value of y when $x = 0.2$ given that $y' = x + y$ with y(0) = 1 and h = 0.2.

3.9 SUMMARY

In this unit we have covered the following points :

In this unit, we have (I) defined a number of concepts, (II) explained a methodology for solving IVPs and, finally, (III) discussed a number of methods for solving IVPs.

(I) **Definitions :** Differential equations of the form (1), involving derivatives w.r.t. a single independent variable are called **ordinary differential equations (ODEs)** whereas, those of the form (2) involving derivatives w.r.t. two or more independent variables are called **partial differential equations (PDEs)**.

The **order** of a differential equation is the order of the highest order derivative appearing in the equation and its **degree** is the highest exponent of the highest order derivative after the equation has been rationalized i.e., after it has been expressed in the form free from radicals and any fractional power of the derivatives or negative power. For example, the equation

$7\left(\frac{d^3y}{dx^3}\right)^2 + 12\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 3x^2\left(\frac{dy}{dx}\right)^3 = 0$ is of **third** order and **second** degree.

The general ODE of order n can be written as

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0 \qquad \ldots (6)$$

or, $$y^{(n)} = f(x, y, y', y'', \ldots, y^{(n-1)}). \qquad \ldots (7)$$

Eq. (7) is called a **canonical representation** of Eq. (6). In such a form, the highest order derivative is expressed in terms of lower order derivatives and the independent variable.

By the **solution of an ODE** of the form $F(x, y, y', y'', \ldots, y^{(n)}) = 0$ or, $y^{(n)} = f(x, y, y', y'', \ldots, y^{(n-1)})$, we mean, a relation, say, y = g(x), free of derivatives; **such that if** throughout, y is replaced by g(x), in the differential equation, then the equation is satisfied.

The **general solution** of an nth order ODE contains n arbitrary constants. In order to determine these arbitrary constants, we require n conditions.

These constraints can be determined and a **particular solution** can be obtained if n suitable conditions on y and/or its derivatives are prescribed. In numerical analysis, we are concerned with finding only the particular solution.

When the dependent variable and its derivatives occur in the first degree only and not as higher powers or as products with each other; the equation is said to be **linear,** otherwise it is said to be **nonlinear.** The Equation $2\frac{d^2y}{dx^2}+4y=3x^2$ is a linear ODE.

If all these conditions, to determine the values of arbitrary constants involved in the solution of a differential equation of order n, are given **at one point,** then these conditions are known as **initial conditions** and the differential equation together with the initial conditions is called an **initial value problem** (IVP).

> The nth order IVP along with associated initial conditions can be written as
>
> $y^{(n)}(x) = f(x, y, y', y'', \ldots, y^{(n-1)})$ (differential equation)
>
> $y^{(p)}(x_0) = y_0^{(p)}$, $p = 0, 1, 2, \ldots, n-1$. (n associated conditions)

If the n conditions are prescribed **at more than one point** then these conditions are known as **boundary conditions.** These conditions are prescribed usually at two points, say t_0 and t_1 and we are required to find the solution y(t) between $t_o < t < t_1$. The differential equation together with the boundary conditions is then known as a **boundary value problem** (BVP).

In this course, we discuss solutions of only IVP's, defined explicitly in the box above.

(II) Three-step methodology to solve an IVP :

Step 1 of the Methodology :

In order to solve an IVP involving n^{th} order differential equation, transform the IVP into n IVP's, each involving only first order derivatives, and attempt to solve only these IVP's involving only first order derivatives.

Step 2 of the methodology consists of

(i) a suitable step size $h > 0$ is either calculated or assumed, so that solutions are attempted only at points : $x_0 + h$, $x_0 + 2h$, $x_0 + 3h$, . . . , $(x_0 + nh)$ . . . *(If the solution is required at any other point, then some other method including interpolation may be used.)* Let $x_n = (x_0 + n\,h)$, with n = 1, 2, 3 .... **These x_n's are called grid points or mesh points.**

(ii) Solution of the IVP (10) is attempted in a step-by-step manner as follows : Solve (10) at $x_0 + h$, use the solution in finding solution at $x_0 + 2h$, and so on, i.e. while attempting solution at $x_0 + (i + 1)\,h$, we must have found solutions at $x_0 + h$, $x_0 + 2h$, . . . , $x_0 + ih$, and then use these solutions in attempting solution at $x_0 + (i + 1)\,h$.

Step 3 of the methodology consists of

(i) **the assumption that y_{n+1} can be obtained from y_n, using the general rule:**

$$y_{n+1} = y_n + h\,\phi\,(x_n, y_n, h)$$

(ii) **finding appropriate p-step methods of order p to calculate $\phi\,(x_n, y_n, h)$, so that error in computation of y_{n+1} by the method is $O(h^{p+1})$.**

Two German mathematicians Runge and Kutta, conceived and developed the ideas behind the methods, are called **Runge-Kutta (R-K) methods**.

Definition : A **method** for solving an i.v.p., **is said to be of order p,** if it yields a solution of the i.v.p, so that, for some positive integer p, the first (p + 1) terms in the calculation of y_{n+1} from y_n, coincide with the corresponding terms in the Taylor's expansion of y_{n+1} in terms of y_n and its derivatives. The error then is of $O(h^{p+1})$.

(II) Various methods for solving an i.v.p. of order 1 : $y'(x) = (dy/dx) = f(x, y)$ with initial **condition at $x = x_0$ as $y(x_0) = y_0$**; in which **y_{n+1} is calculated using**

$$y_{n+1} = y_n + h\,\phi\,(x_n, y_n, h).$$

The following methods based on the idea differ from each other in what value is determined and assigned to (x_n, y_n, h)

(i) Euler's method/formula may be written as :

$$y_{n+1} = y_n + h\,f\,(x_n, y_n),$$

so that $\phi(x_n, y_n, h)$ is taken as $f(x_n, y_n)$. In this method, the error in the formula would be $(h^2/2)\,y''(\xi)$, $x_i \leqq \xi \leqq x_{i+1}$, in the computation of y_{n+1} from y_n. This is why the method is first-order method.

(ii) Improved Euler formula/method is a two-step method. At a general point $x = x_{n+1}$, formula becomes

$$\boxed{\left.\begin{aligned} y^*_{n+1} &= y_0 + hf\,(x_n, y_n) \\ y_{n+1} &= y_0 + \frac{h}{2}\{f\,(x_n, y_n) + f\,(x_{n+1}, y^*_{n+1})\} \end{aligned}\right\}}$$

so that $\phi(x_n, y_n, h)$ is taken as $(1/2)\{f(x_n, y_n)$ $+ f(x_{n+1}, y^*_{n+1})\}$. In this method, the error in the formula would be $O(h^3)$ in the computation of y_{n+1} from y_n. This is why the method is fsecond-order method.

For the R-K methods, summarized below, it may be noted that even for a particular order, say 2, there is not necessarily a single method, but a group of methods, which is called R-K group of methods of the order.

(iii) A general R-K Method is a two-step method of Order 2, developed as follows :

Let the value of y, say y_o, be known at $x = x_o$.

The other values of y_{n+1} in the general R-K method of order 2, at $x = x_{n+1}$, are obtained as follows :

$$y_{n+1} = y_n + k, \quad \text{for } n = 0, 1, 2, \ldots$$

where, k is obtained as follows :

$$\left.\begin{aligned} k_1 &= h\,f\,(x_n, y_n); \\ k_2 &= h\,f\,(x_n + \alpha\,h, y_n + \beta\,k_1) \\ \text{and} \quad k &= w_1k_1 + w_2k_2 \end{aligned}\right\} \quad \ldots (A)$$

And the parameters α, β and weights w_1 and w_2 are determined so that y_{n+1} agrees with Taylor's expansion of $y(x_n + h)$ up to powers of h^2, and hence, it becomes a method of order 2, i.e., error in computation of y_{n+1} is of $O(h^3)$.

Improved Euler's method can be obtained as R-K method of order 2, by taking in (A), $\alpha = 1$, $\beta = 1$, $w_1 = (1/2) = w_2$.

(iv) Classical R-K method, a special case of general R-K method of order 4, and the most popular for solving first order o.d.e's. is defined as follows :

Let us suppose that value of y_n is already known for $x = x_n$. Then the value y_{n+1} at $x = x_{n+1}$ is computed in four stages as follows :

$$k_1 = hf\,(x_n, y_n)\ ;\ k_2 = hf\left(x_n + \frac{1}{2}h,\ y_n + \frac{1}{2}k_1\right)$$

$$k_3 = hf\left(x_n + \frac{1}{2}h,\ y_n + \frac{1}{2}k_2\right)\ ;\ k_4 = hf\,(x_n + h,\ y_n + k_3)$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$y_{n+1} = y_n + k.$$

3.10 SOLUTIONS/ANSWERS

Ex. 1 : The solution to the linear first order differential equation is $y(t) = e^{t/2}\sin(5t)$. Approximations at t = 1 are : with h = 0.1, y(1) = – 0.97167, and with h = 0.05, y(1) = – 1.26512. The percentage errors are 38.54 %, 19.98 % respectively.

Ex. 2 : Euler's method is

$$y_{k+1} = y_k + hf_k = y_k + h\,(1 - 2x_k\,y_k)$$

$$y(0.4) = 0.1948 + (0.2)\,(1 - 2 \times 0.2 \times 0.1948)$$

$$= 0.379216.$$

Ex. 3 : $y' = \dfrac{1}{x^2 + y}$, $y(4) = 4$, $y'(4) = 0.05$

$$y(4.1) = y(4) + hy'(4)$$

$$= 4 + (0.1)\,(0.05) = 4.005.$$

Ex. 4 : Euler's method $y' = (y - x)/(y + x)$, $y(0) = 1$, $y'(0) = 1$

$$y(0.1) = 1 + (0.1)\,(1) = 1.1$$

Ex. 5 : Euler's method is $y_{k+1} = h + y_k + hy_k^2$.

Starting with $t_0 = 0$ and $y_0 = 1$, we have the following tables of values :

Table : h = 0.2

T	Y(t)
0.2	1.4
0.4	1.992
0.6	2.986

∴ $y(0.6) = 2.9856.$

Table : h = 0.1

T	Y(t)
0.1	1.2
0.2	1.444
0.3	1.7525
0.4	2.1596
0.5	2.7260
0.6	3.5691

$\therefore$ $y(0.6) = 3.5691$

$\therefore$ $y(0.6) = 1.99171$ with $h = 0.05$.

Ex. 6 : $K_1 = 0.5$, $K_2 = 0.333333$

$K_3 = 0.3235294118$, $K_4 = 0.2258064516$

$y(0.5) = 1.33992199$.

Ex. 7 : $K_1 = 0.184416$, $K_2 = 0.16555904$

$K_3 = 0.1666904576$, $K_4 = 0.1421615268$

$y(0.4) = 0.3599794203$.

Ex. 8 : $K_1 = -0.005$, $K_2 = -0.004853689024$

$K_3 = -0.0048544$, $K_4 = -0.004715784587$

$y(4.2) = 0.9951446726$.

Exact $y(4.2) = 0.995145231$, Error $= 0.559 \times 10^{-6}$

Ex. 9 : $K_1 = 0.1$, $K_2 = 0.09092913832$

$K_3 = 0.9049729525$, $K_4 = 0.8260717517$

$y(1.1) = -0.909089993$

$K_1 = 0.08264471138$, $K_2 = 0.07577035491$

$K_3 = 0.07547152415$, $K_4 = 0.06942067502$

$y(1.2) = -0.8333318022$

$K_1 = 0.06944457204$, $K_2 = 0.06411104536$

$K_3 = 0.06389773475$, $K_4 = 0.0591559551$

$y(1.3) = -0.7692307692$

Exact $y(1.3) = -0.7692287876$

Error $= 0.19816 \times 10^{-5}$

Ex. 10 : 1.1749.

Ex. 11 : 1.0207, 1.0438.

Ex. 12 : $k_1 = 0.2$, $k_2 = 0.2400$, $k_3 = 0.2440$, $k_4 = 0.2888$, $y(0.2) \approx 0.2468$.

NOTES

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