Indira Gandhi National
Open University
School of Sciences

BMTC-132
DIFFERENTIAL EQUATIONS

Block

1

FUNCTIONS OF TWO OR THREE VARIABLES

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DIFFERENTIAL EQUATIONS

Since the time of Isaac Newton (1642-1727) differential equations have been of fundamental importance in the application of mathematics to the physical sciences. Differential equations serve as mathematical models for many exciting "real-world" problems, not only in science and technology, but also in such diverse fields as economics, social sciences, psychology, demography etc. Rapid growth in the theory of differential equations and its applications to almost every branch of knowledge appears at an increasing rate, leading to a continued interest in the study by students in many disciplines. We are offering this course of differential equations to the students, entering the Bachelor's Degree Programme of UGC-CBCS, as a core subject.

It was Isaac Newton, the English mathematician and Gottfried Wilhelm Leibniz (1646-1716), the German mathematician who gave birth to the subject of differential equations in 1675. Leibniz forged a powerful tool of the integral sign and Newton classified differential equations of the first order. A large number of great mathematicians of the past three centuries such as Fermat, Newton, the Bernoullis, Euler, Lagrange, Charpit, Laplace, Gauss, Abel, Hamilton, Liouville, Riemaan, Poincare and others developed this subject and brought it to the present status.

In this course on differential equations, we shall be dealing with both ordinary as well as partial differential equations. For a better understanding of the concepts in the course a good knowledge of calculus is essential. The knowledge of our calculus course BMTC-131 will more or less fulfill your this requirement. Since BMTC-131only deals with a function of one variable, to fill up the gaps, we have added contents on functions of two and more variables in this course. Accordingly, we have divided the material in this course into four blocks.

Block 1 deals with functions of two or three real variables. In this block we are going to introduce you to the concepts of limit, continuity and differentiability of functions of more than one variable. You may recall, that when you studied these same concepts for functions of one variable, you had made use of the knowledge of the algebraic and topological structure of $\boldsymbol{R}$, the set of real numbers. You had used various properties of real numbers, the distance between two real numbers, and so on. So, to study these concepts for functions of two variables, we need to know the structure of $\boldsymbol{R}^2$, the set of points in a plane. If we want to study functions of three variables, we need to know the structure of $\boldsymbol{R}^3$, the set of points in space. We shall start with a brief discussion about these structures and discuss some algebraic and topological properties of $\boldsymbol{R}^2$ and $\boldsymbol{R}^3$. We shall then acquaint you with functions of more than one variable. The second unit introduces the concepts of limit and continuity for these functions. The remaining three units in this block are devoted to the discussion of different types of derivatives for functions of several variables. Though in this block we shall be dealing with real valued functions of three variables, the discussion can be easily extended to functions of any number of variables.

Blocks 2 and 3 are devoted to the ordinary differential equations (ODEs) and Block 4 deals with the total and partial differential equations (PDEs). In Block 2 we have started with the essentials and the basic definitions related to the study of differential equations. We have stated the conditions on the nature of solutions and conditions under which a unique solution of first order differential equations exists. There we have discussed various methods of solving first order first degree equations. Methods of finding solutions of equations with degree greater than one are also discussed for certain particular forms of the first order ODE. We have also formulated in this block some of the problems of physical and engineering interest in terms of first order linear differential equations.

Block 3 deals with ordinary differential equations of second and higher order. We have classified general linear differential equations into those with constant coefficients and those with variable coefficients and further classified these equations into homogeneous and non-homogeneous equations. The methods of solving homogeneous linear differential equations with constant coefficients have been discussed in this block. In the case of non-homogeneous equations with constant coefficients, the method of undetermined coefficients and the method of differential operators are given for finding particular integrals of the equations. The method of variation of parameters and the method of reduction of order, which can be applied for solving linear differential equations with constant as well as variable coefficients are also given here. We have laid specific stress on the applications of second order ordinary differential equations in this block.

In Block 4 which is the last block of this course, we move our focus to simultaneous, total and partial differential equations of the first order. The methods of solving simultaneous differential equations and some of their applications such as orthogonal trajectories of the system of curves on a given surface, particle motion in phase-space and electric circuits are discussed. Integrability condition for the total differential equation in three variables and the methods of solving it have been given. We have classified the first order PDEs into linear, semi-linear, quasi-linear and non-linear PDEs and discussed the various types of solutions/integrals of these PDEs giving the relation between these different integrals. The Lagrange's method of solving linear PDEs of the first order and the Charpit's method of solving non-linear PDE's of the first order have also been discussed in this block.

Each block consists of 4 or 5 units. Every unit has plenty of examples and exercises interspersed with the text. These examples will aid you to understand the theory and exercises are meant to help you check your progress. To master the various techniques presented in this course we advise you to put in a lot of practice and attempt all the exercises. The solutions/answers to the exercises in a unit are given at the end of the unit. We suggest that you look at them only after attempting the exercises. Miscellaneous exercises are given at the end of each block. These exercises will help you check your overall understanding of the various concepts you have learnt in that block. Each problem would require you to think and decide the method/technique to be used for solving it.

We are giving a list of various symbols and notations used in the text for your reference. As a part of continuous assessment, there will be one assignment based on the entire course, carrying a weightage of thirty per cent. The assignment will be available for download from the IGNOU website.

You may like to look up some more books on the subject and try to solve some exercises given in them. This will help you get a better grasp of the techniques discussed in this course. We are giving you a list of titles which will be **available in your Study Centre for reference purposes**.

Some Useful Books

1. *Calculus by GB Thomas or R.L Finney, Pearson Eduation.*

2. *A Text Book of Differential Equations by N. M. Kapur, Pitambar Book Depot.*

3. *Differential Equations with Applications and Historical Notes by George F. Simmons, Tata McGraw Hill.*

4. *Elements of Partial Differential Equations by Ian N. Sneddon, McGraw-Hill Book Company.*

BLOCK 1 FUNCTIONS OF TWO OR THREE VARIABLES

This is the first block of this course on Differential Equation. The purpose of this block is give you a foundation for studying Differential Equation course by introducing you to functions of more than one variables and extending the concepts studied for one variable function in Calculus to two or three variables. You are already familiar with some aspects of the calculus of functions of one variable from the calculus course BMTC-131. Newton and Leibniz are considered to be the founding fathers of calculus. The major developments in calculus took place in the seventeenth century. Later, in the eighteenth century, the basic concepts of calculus, for example, limit, continuity and differentiability were extended to functions of more than one variable. The need for studying functions of several variables arose when some mathematicians like Euler, Daniel Bernoulli, Fourier, Augustin-Louis Cauchy and d' Alembert were investigating some physical problems. Augustin-Louis Cauchy (1789-1857) was a dominant mathematical figure in Paris, the centre of the mathematical world in those days. Cauchy too has contributed to the development of several variables to a great extent.

This block is divided into five units. In **Unit 1** we begin with describing the structure of $\boldsymbol{R}^2$ and $\boldsymbol{R}^3$. We introduce the notion of a distance between two points for $\boldsymbol{R}^2$ and $\boldsymbol{R}^3$ and deduce its elementary properties. We define functions of 2 or 3 variables defined from $\boldsymbol{R}^2 \to \boldsymbol{R}$ and $\boldsymbol{R}^3 \to \boldsymbol{R}$ and give various examples of such functions. These functions are commonly termed as real-valued functions of several variables or simply functions of several variables. The concepts of level curves and level surfaces are also introduced which are used to describe respectively the graphs of 2 and 3 variables functions.

We shall define the concepts of limits and continuity for functions of several variables in **Unit 2**. We discuss the term simultaneous limits and repeated limits and give the connection between them.

In **Unit 3**, we extend the notion of differentiability from one variable to 2 or 3 variables. We first define the partial derivative of a function of several variables. The notion of partial derivatives does not fully generalise the concept of derivative of one variable function. Later the concept of differentiability for a function of several variables is defined. We discuss the connection between the differentiability, continuity and existence of partial derivatives.

The concepts of first order partial derivatives gives rise to the notion higher order partial derivative which is covered in **Unit 4**. We discuss some important theorems known as Young's, Schwarz and Euler's theorem which gives some sets of conditions under which the mixed partial derivatives become equal.

Unit 5, the last unit of this block, covers some applications of partial derivatives. We discuss chain rule to evaluate the partial derivatives of functions of 2 or 3 variables where each variable itself is a function of 2 or 3 variables. Here we also define homogeneous functions and explain Euler's theorem for homogenous functions.

The concepts which you are going to study in this course are bound to be a little more complex than the corresponding concepts for functions of one variable. But you will see that one variable case shows us the way in which these concepts could be generalised. So, each time we introduce a new concept we'll recall its parallel in the one variable case, and then see how it is extended to the several variables' case.

We have interspersed the text with a lot of solved examples. These examples will help you understand the theory better. We have given the answers of all the exercises in each unit at the end of the unit. As you will see, we often have to refer back to results or definitions from earlier units. For this we'll refer to sub-section y.z of Unit x as Sec. x.y.z, or to section y of Unit x as Sec. x.y. We'll also recall some results from our earlier course

BMTC-131 on calculus. We shall refer to a unit in this course as “Unit x of Calculus”. What we had said in our Calculus course remains true for this course too – to master the various techniques presented here, you will need to put in a lot of practice.

In case you want to seek some additional information about the concepts discussed here, or to solve some exercises, you can consult the following book:

***Calculus III* by Jerrold Marsden and Alan Weinstein, Springer**

This will be available in your study centre library.

We hope you find the techniques developed in this course useful in your further studies.

Notations and Symbols

$\boldsymbol{R}$	:	the set of real numbers
$\boldsymbol{R}^2\ (\boldsymbol{R}\times\boldsymbol{R})$	:	the Cartesian product of two copies of $\boldsymbol{R}$
$\boldsymbol{R}^3\ (\boldsymbol{R}\times\boldsymbol{R}\times\boldsymbol{R})$	:	the Cartesian product of three copies of $\boldsymbol{R}$
$\Rightarrow$	:	implies
$\subseteq$	:	is contained in
$\in$	:	belongs to
$<(\le)$	:	is less than (is less than or equal to)
$>(\ge)$	:	is greater than (is greater than or equal to)
$\therefore$	:	therefore
$\because$	:	because
i.e.	:	that is
$\forall$	:	for all
w.r.t.	:	with respect to
$\frac{d^n y}{dx^n},\ y^{(n)}$	:	nth order derivative of y w.r.t. x
$\frac{\partial^n f}{\partial x^n}, n=1,2,3$	:	nth order partial derivative of f w.r.t. x
$\propto$	:	is proportional to
$]\ ,\ [$	:	open interval
$[\ ,\ ]$	:	closed interval
$\lvert x \rvert$	:	absolute value of a number x
2D-System	:	2 dimensional coordinate system
3D-System	:	3 dimensional coordinate system
(r,θ,z)	:	a point in cylindrical coordinate system
(ρ,θ,ϕ)	:	a point in spherical coordinate system.
(x,y)	:	a point in 2D-system
(x,y,z)	:	a point in 3D-system
$\lim\limits_{\substack{x\to a\\ y\to b}} f(x,y)$	:	simultaneous limit of $f(x,y)$ as $(x,y)\to(a,b)$
$\lim\limits_{x\to a}\left(\lim\limits_{y\to b}\right)$	:	repeated limits

Greek Alphabets

α	:	Alpha
β	:	Beta
γ	:	Gamma
δ	:	Delta
ε	:	Epsilon
ζ	:	Zeta
η	:	Eta
θ	:	Theta
ι	:	Iota
λ	:	Lambda
μ	:	Mu
ν	:	Nu
ξ	:	exi
π	:	Pi
Π	:	(capital Pi)

ρ	:	Rho
$\sigma(\Sigma)$	:	Sigma (capital Sigma)
τ	:	Tou
χ	:	Chi
ψ	:	Psi
ϕ	:	Phi
ω	:	Omega

UNIT 1

R^2 AND R^3

Structure

Page No.

1.1 INTRODUCTION

This is the first unit of this block and the course. With this unit we start our discussion on three-dimensional space or 3D-space. We begin with a short introduction to the three-dimensional Cartesian coordinate system in Sec. 1.2 and acquaint you with some basic facts about points, lines and planes in this system. We discuss various ways of representing a line and a plane algebraically and geometrically. In Sec. 1.3 we discuss two other coordinate systems called cylindrical and spherical coordinate system for representing points in 3D space.

In Sec. 1.4 we establish a one-one relationship between the points in 2D space and in 3D-space and elements of the set $R^2 = R \times R$ and $R^3 = R \times R \times R$, called Cartesian products of R and discuss some algebraic and geometric properties of the sets. Lastly in Sec. 1.5 we discuss functions defined from R^3 or R^2 to R and the idea of the level curves and level surfaces.

The content covered in this unit will be frequently used in the rest of the course. Therefore we suggest that you do all the exercises in the unit as you come across them. Further please do not go to the next unit till you are sure that you have achieved the following objectives.

Objectives

After studying this unit you should be able to:

- represent a point in the 3D space;

- obtain the direction cosines and direction ratios of a line in 3D-space;
- obtain the equations of a line in canonical form, parametric form and or in two-point form;
- obtain the equations of a plane in three-point form, in intercept form or in normal form;
- find the distance between a point and a plane;
- find the angle between two lines, or between two planes, or between a line and a plane;
- find the point (or points) of intersection of two lines or of a line and a plane.

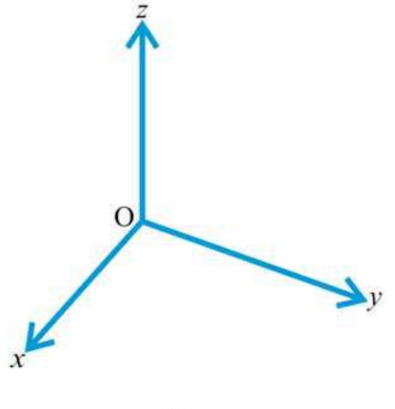
Now let us start our discussion of 3D-space.

1.2 THE CARTESIAN COORDINATE SYSTEM

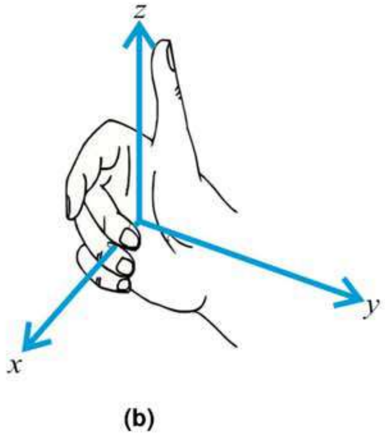
In this section we will discuss the three-dimensional Cartesian coordinate system which you might have studied in your school mathematics. We will explain how lines and planes are represented in this system by introducing the notions of direction cosines and direction ratios.

Recall that you are already familiar with the two-dimensional coordinate system from Unit 3 of the 1st semester Calculus course. Here we generalize the basic notions of two-dimensional system to three-dimensional system. You know that the two-dimensional coordinate system (2D-coordinate system, in brief) is represented by two mutually perpendicular lines. Analogous to this, we consider three mutually perpendicular lines in space for the three-dimensional coordinate system (3D- coordinate system or 3D-systems, in brief).

Let us take three mutually perpendicular lines in space which intersect in a point O (see Fig. 1 (a)). The point O is called the **origin**. This is physically possible because we are in a 3 dimensional space. The three mutually perpendicular lines are called coordinate axes and labeled the x-axis, y-axis and z-axis as shown in Fig. 1. Usually x-and y-axis are taken in a plane and z-axis orthogonal to the plane. The positive directions of the lines OX, OY and OZ are so chosen that if a right-handed screw (Fig. 1(b)) placed at O is rotated from OX to OY, it moves in the direction of OZ. The negative direction are shown as dotted lines in Fig. 1.



(a)



(b)

Fig. 1: The Cartesian coordinate axes in three dimensions. OX is the x-axis, OY-is the y-axis and OZ is the z-axis.

The three coordinate axes determine the three coordinate planes as shown in Fig. 2(a).

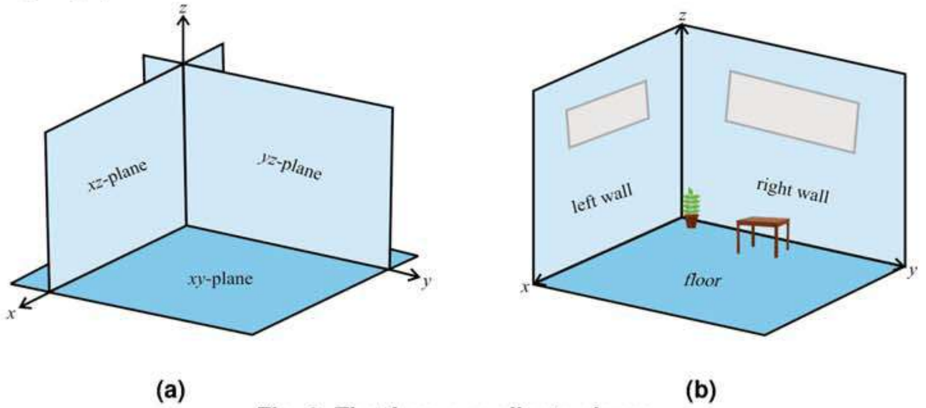


Fig. 2: The three coordinate planes.

The xy-plane is the plane that contains x-and y-axes; the yz-plane contains the y-and z-axes; the xz-plane contains the x-and z-axes. These three coordinate planes divide the 3D-space into eight parts, called **octants**.

To get a better visualization of the three planes, you may look at the bottom corner of a room and call the corner origin O. You choose the floor as the xy-plane, then the wall on your left is the xz-plane and on your right is the yz-plane. (See Fig. 2(b))

Next, we shall discuss how to locate a point in 3D-system.

Having fixed O as the point O on each of the axes, choose a point 1 on each of the axes (as shown in the Fig. 3). This amounts to choosing a scale on each axis endowing each point on the axes a number designation.

Given a point P in the space draw a line (please see the pink line in Fig. 3) perpendicular to the z-axis (in the plane determined by the z-axis and the point P). Mark this point as c where it meets the z-axis (see Fig. 3). Let c be the distance of C from the origin O on the z-axis. The distance c may be positive, zero, or negative depending on its location on z-axis. (C may be P itself if the point lies on the z-axis). Now draw a dotted line (line coloured in blue) through P perpendicular to the xy-plane). Let M denote the point of intersection (see Fig.3). (M is P itself if P lies on the xy-plane). Also draw lines (coloured blue and red respectively) from M perpendicular to x-axis and y-axis with meeting points as A and B, respectively. Let a and b be the distance of A and B from O on the x-axis and y-axis respectively. Then the coordinates of P is (a,b,c).

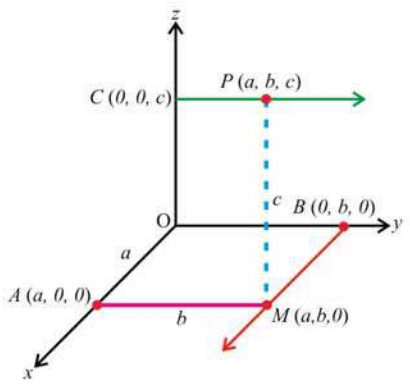
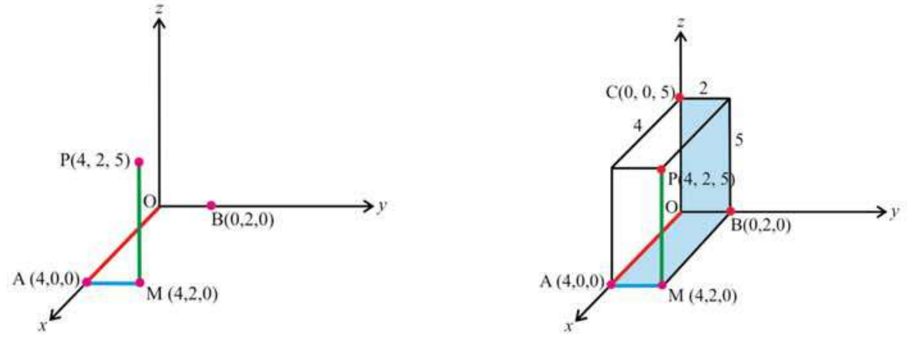


Fig. 3: Representation of a point P as (a,b,c).

The first letter in the triple which is ‘a’ is taken as the x-coordinate and second letter in the triple which is ‘b’ is taken as the y-coordinate and the third letter ‘c’ is the z-coordinate.

Alternatively, suppose we are given a triple (x, y, z). How do we find its location in 3D-system? For instance let us see how we locate the point $P(4,2,5)$. We follow the steps given below as shown in Fig. 4.



(a) **(b)**

Fig. 4

i) We first note that the x-coordinate is 4. Then we measure a distance 4 on the x-axis (the red coloured portion of the x-axis) and mark the point as $A(4,0,0)$.

ii) From A_1 draw a line parallel to the y-axis and measure the distance 2 units on it (the blue coloured line segment). Mark the point as $M(4,2,0)$.

iii) From A_2 draw a line parallel to z-axis and measure the distance 5 units on it (the green coloured line segment) and mark the point as $P(3,4,2)$.

Then P in Fig. 4 shows the required location of the point.

Infact corresponding to each point P we get a rectangular box with vertices as shown in Fig. 4 (b) and length of the sides as 4 units, 3 units and 2 units respectively. Then the point P is one of the vertices of the rectangle.
How do we locate points with coordinates as negative numbers? The figure Fig. 5 shows the location of the point $P(-4, 2, -5)$.

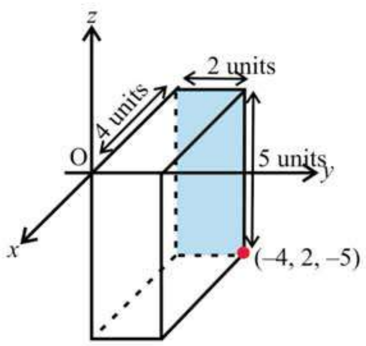


Fig. 5

So, for each point in 3D-space, there is an ordered triple (x, y, z) of real numbers. Conversely, given an ordered triple of real numbers, we can easily find a point P in space whose coordinates are the given triple. So there is a one-one correspondence between the points 3D-space and the set of ordered triple.

Here are some exercises.

E1) Locate the points $(1,1,0)$ and $(-4,-2,1)$ on a single set of 3D-coordinate axes.

E2) Suppose you start at the origin and move along the x-axis at a distance of 5 units in the negative direction reaching the point A and move horizontally along a line parallel to y-axis at a distance of 3 units in the positive direction reaching the point B and then move upward along a line parallel to z-axis at a distance of 6 units to reach a point C. What are the coordinates of A, B and C?

E3) Which one of the points $P(-5,-1,4)$, $Q(0,3,8)$ and $R(6,2,3)$ is closest to the xz-plane? Which point lies in the yz-plane. Give reasons for your answers.

These exercises would have helped you to get an idea about the location of the points in 3D-space.

Next, we will discuss position of lines in three-dimensional space and their representation.

You recall from the Calculus course, unit 3, that we can determine a line in two-dimensional space if we know a point lying on the line, and also the angle of inclination (or direction) of the line with x-axis (its slope).

In a similar way, we shall see how to represent a line in 3D-space we proceed as follows:

Let us consider the Cartesian coordinate system with O as the origin and OX, OY, OZ as the axes, and consider a line L which passes through O as given in Fig. 6 (a). Let the line L makes angles α, β and γ with the positive directions of the x, y and z-axes, respectively. The numbers $\cos\alpha, \cos\beta$ and $\cos\gamma$ are called the **direction cosines** of L.

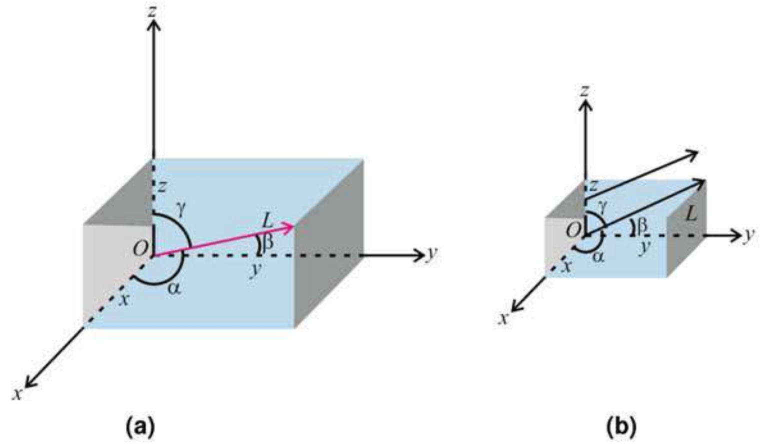


Fig. 6: $\cos\alpha, \cos\beta$ and $\cos\gamma$ are the direction cosines of the line L.

For example, the direction cosines of the x-axis are $\cos 0, \cos \pi/2, \cos \pi/2$, that is 1, 0, 0. Similarly that of the y-axis are 0, 1, 0 and that of the z-axis is 0, 0, 1.

Let us now consider a line L in space that need not pass through the origin O. Let L denote an arbitrary line in space. Given the coordinate system with origin at O, draw L', the unique line parallel to L and passing through O (see Fig. 6 (b)). Let $\cos\alpha, \cos\beta$ and $\cos\gamma$ be the direction cosines of L'. Then the direction cosines of L are the direction cosines of L'. Then L is uniquely determined by **any** point on L along with the direction cosines of L.

For example, the direction cosines of the line through $(1,1,1)$ and parallel to the $x-$ axis are $1,0,0$. Why? Note that here $\alpha = 0, \beta = \dfrac{\pi}{2}$ and $\gamma = \dfrac{\pi}{2}$.

We have the following definition:

Definition 1: Direction cosines of a line are the cosines of the angles between the line and the positive directions of the coordinate axes.

You might have noticed that the direction cosines depend on the coordinate system that we choose.

We shall now observe a simple property satisfied by the direction cosines of a line.

Let us take a line L passing through the origin O and let $P(x, y, z)$ be a point on L. Let direction cosines of the line L be $\cos\alpha, \cos\beta$ and $\cos\gamma$. Then you can see from Fig. 6 that

$$x = OP\ \cos\alpha,\ y = OP\ \cos\beta \text{ and } z = OP \cos\gamma \qquad \ldots (1)$$

Since $OP^2 = x^2 + y^2 + z^2 = OP^2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma)$, we find that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \qquad \ldots (2)$$

This simple property of the direction cosines of a line is useful in several ways, as you will see in the later units.

Note: The direction cosines are also denoted by the letters l, m and n where $l = \cos\alpha, m = \cos\beta$ and $n = \cos\gamma$. Correspondingly we get

$$l^2 + m^2 + n^2 = 1 \qquad \ldots (3)$$

Let us see some examples

Example 1: If a line makes angles $\dfrac{\pi}{4}$ and $\dfrac{\pi}{3}$ with x and y-axes, respectively, then what is the angle that it makes with z-axis?

Solution: We put $\alpha = \dfrac{\pi}{4}$ and $\beta = \dfrac{\pi}{3}$ in Eqn. (2), above. Then if γ is the angle that the line makes with z-axis, we get

$$\frac{1}{2}+\frac{1}{4}+\cos^2\gamma=1 \Rightarrow \cos\gamma=\pm\frac{1}{2} \Rightarrow \gamma=\frac{\pi}{3} \text{ or } \frac{2\pi}{3}$$

In this case you might have noticed that there are two lines that satisfy our hypothesis.

Don't be surprised! The following figure, Fig. 7, will help you to realize this. (Please note the changed in the orientation of x, y and z -axis).

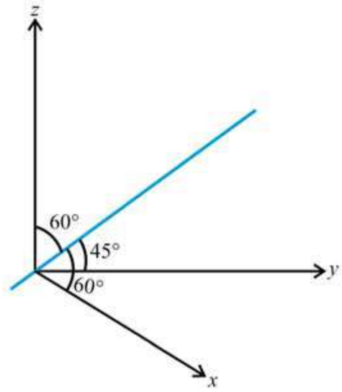


Fig. 7

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Here is another example:

Example 2: Find the direction cosines of the line which passes through the origin and the point $P(1,2,3)$.

Solution: From Eqn. (1), we know that $\cos\alpha=\frac{x}{OP}, \cos\beta=\frac{y}{OP}$ and $\cos\gamma=\frac{z}{OP}$. Now we apply a formula called "distance formula" for calculating the distance of OP. By the formula if the distance of any point $P(x,y,z)$ from the origin in given by $\sqrt{x^2+y^2+z^2}$. (We will derive the distance formula in Sec. 1.4).

> The distance formula for calculating the distance between two points (x_1,y_1,z_1) and (x_2,y_2,z_2) is $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

Here $(x,y,z)=(1,2,3)$. Therefore, $OP=\sqrt{1^2+2^2+3^2}=\sqrt{14}$

$\therefore \cos\alpha=\frac{1}{\sqrt{14}}, \cos\beta=\frac{2}{\sqrt{14}}$ and

Therefore $\alpha=\cos^{-1}\left(\frac{1}{\sqrt{14}}\right)$, $\beta=\cos^{-1}\left(\frac{2}{\sqrt{14}}\right)$ and $\gamma=\cos^{-1}\left(\frac{3}{\sqrt{14}}\right)$

***

From the example above, you might have realized that the computation of direction cosines can be tedius. Therefore for convenience we take numbers which are proportional to direction cosines. For example, the numbers 1,2,3 are proportional to the numbers $\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}$ and $\frac{3}{\sqrt{14}}$. Such numbers which are proportional to direction cosines are called direction ratios. Thus, if a,b,c are numbers proportional to the direction cosines

l, m and n respectively, then a, b, c are called **direction ratios of the line**.

Then we have $\frac{a}{l}=\frac{b}{m}=\frac{c}{n}=k$, say i.e $l=\frac{a}{k}, m=\frac{b}{k}, n=\frac{c}{k}$. If we substitute these values for l, m and n in Eqn. (3), we get

$$\frac{a^2}{k^2}+\frac{b^2}{k^2}+\frac{c^2}{k^2}=1$$

i.e. $\frac{a^2+b^2+c^2}{k^2}=1$

$\therefore k=\sqrt{a^2+b^2+c^2}$

This implies that $l=\frac{a}{\sqrt{a^2+b^2+c^2}}, m=\frac{b}{\sqrt{a^2+b^2+c^2}}, n=\frac{c}{\sqrt{a^2+b^2+c^2}}$

We formally define it now.

Definition 2: Three numbers a, b and c are called **direction ratios** of a line with direction cosines l, m and n, if $a=kl, b=km, c=kn,$ for some $k \in \boldsymbol{R}$.

Thus, any triple that is proportional to the direction cosines of a line are its direction ratios.

For example, if the direction cosines of a line are $\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2}$, then $\sqrt{2}, 1, 1$ are its direction ratios.

In the following example you will see an interesting fact.

Example 3: If L is a line passing through the origin, and let $P(a, b, c)$ be a point on it, then show that a, b, c are the direction ratios of L.

Solution: Let us look at Fig. 8 which shows a line passing through O (0, 0, 0) and $P(a, b, c)$. (Please note the change in the orientation of x, y and z-axis).

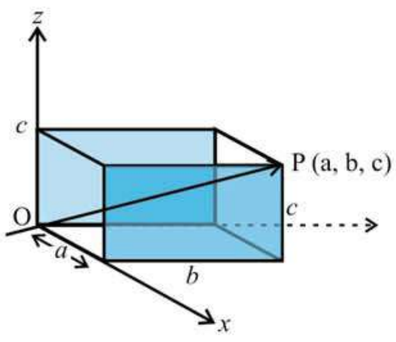


Fig. 8

Let $OP=r$. Then the direction cosines of L with respect to x, y and z-axes are $\cos\alpha=\frac{a}{r}, \cos\beta=\frac{b}{r}, \cos\gamma=\frac{c}{r}$ [recall from school trigonometry]. Therefore the direction ratios are given by a, b and c.

***

You can try these exercises now.

E4) If $\cos\alpha, \cos\beta$ and $\cos\gamma$ are the direction cosines of a line, then show that $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$.

E5) Find the direction cosines of the line OP where P is the point $(2, 2, -1)$.

We shall now discuss how the direction cosines or ratios of a line can be used to find the equation of the line in 3D-system.

Let us assume that the direction cosines of a line L are l, m and n, and also that a point $P(a, b, c)$ lies on it. To find the equation of L, we proceed as follows:

Let $Q(x, y, z)$ be any other point on L. Let us complete the cuboid with PQ as one of its diagonals (see Fig. 9).

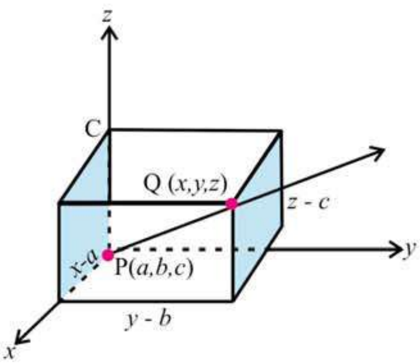


Fig. 9

Then $PA = x - a$, $PB = y - b$ and $PC = z - c$. Now, if $PQ = r$, you can see that

$\cos\alpha = \dfrac{x-a}{r}$, that is,

$l = \dfrac{x-a}{r}$. Similarly, $m = \dfrac{y-b}{r}$, $n = \dfrac{z-c}{r}$.

Thus, any point on the line satisfies the equations.

$$\frac{\boldsymbol{x-a}}{\boldsymbol{l}} = \frac{\boldsymbol{y-b}}{\boldsymbol{m}} = \frac{\boldsymbol{z-c}}{\boldsymbol{n}}. \qquad \ldots (4)$$

Since l, m and n may take zero value, the equation Eqn. (4) is a statement of proportions.

Note that Eqn. (4) consists of pairs of equations, given by

$$\frac{x-a}{l} = \frac{y-b}{m} \text{ and } \frac{y-b}{m} = \frac{z-c}{n},$$

or

$$\frac{x-a}{l} = \frac{z-c}{n} \text{ and } \frac{y-b}{m} = \frac{z-c}{n}$$

or

$$\frac{x-a}{l}=\frac{y-b}{m} \text{ and } \frac{x-a}{l}=\frac{z-c}{n}.$$

Note that we have three pairs of equations, each representing the line. In fact each pair of equations represent two planes whose intersection is the given line.

Conversely, any pair of equations of the form in Eqn. (4), represent a straight line passing through (a,b,c) and having direction ratios l, m and n.

This means that "A pair of linear equations represents a line". Eqn. (4) is called the **canonical form of the equations of a straight line.**
For example, the equations of the straight line passing through (1, 1, 1) and having direction cosines $\left(\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$ are

$$\frac{x-1}{1/\sqrt{3}}=\frac{y-1}{-1/\sqrt{3}}=\frac{z-1}{1/\sqrt{3}}, \text{ that is,}$$

$$\frac{x-1}{1}=\frac{y-1}{(-1)}=\frac{z-1}{1}.$$

Note that this is in the form of Eqn. (6), but 1, –1, 1 are **direction ratios** of the line, and not its direction cosines.

Remark 1: In Eqn. (4) you might have notice that the equations of the line passing through (a, b, c) and having direction cosines l, m, n can also be written as

$$\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n}=t \text{, say, where } t\in \boldsymbol{R}.$$

This means that $\frac{x-a}{l}=t,\ \frac{y-b}{m}=t,\ \frac{z-c}{n}=t$.

Thus we get three equations of the single variable t, given by

$$\boldsymbol{x=a+lt},\ \boldsymbol{y=b+mt},\ \boldsymbol{z=c+nt} \qquad \ldots(5)$$

The Equation in Eqn. (5) is another form of representation of the line passing through (a,b,c) and having direction cosines l, m and n.

The form in Eqn. (5) is called a **one-parametric form** of the equations of a line, in terms of the parameter t.

Please note that each value of the parameter t gives a point (x, y, z) on the line i.e. for a given value of t, Eqn (5) represents a point on the line having direction cosines proportional to l, m, n and passing through (a,b,c). That is why these equations are called parametric equations of the line L through the point $P(a,b,c)$.

In fact Eqn. (5) represents a family of lines parallel to each other.

Let us stop here and reflect on what we have discussed!!

There are two forms of equations of a line
i) **Canonical form** – This represents a pair of equations in two variables.

ii) **Parametric form** – This represents a system of three linear equations in one independent variable.

Let us now use Eqn. (5) to find another form of the equation of a line. Let two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ lie on a line L. Then, if l, m and n are its directions cosines of the line, then from Eqn. (5), any point (x, y, z) on the line satisfies the following equation.

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}. \qquad \ldots (6)$$

We also know that Q lies on L. Therefore we put $x = x_2, y = y_2$ and $z = z_2$ in the Eqn. (6). Hence we have

$$\frac{x_2 - x_1}{l} = \frac{y_2 - y_1}{m} = \frac{z_2 - z_1}{n} = k, \text{ say where } k \in \boldsymbol{R}.\mathrm{I} \qquad \ldots (7)$$

Thus we have $l = \dfrac{x_2 - x_1}{k}$, $m = \dfrac{y_2 - y_1}{k}$, $n = \dfrac{z_2 - z_1}{k}$.

Substituting for l, m and n in Eqn. (6), we get that

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}. \qquad \ldots (8)$$

This is another form of the equation of the line L and is called the **two-point form** of the equation of a line.

The third form of equation of a line

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}.$$

Can you recall the two point form of the equation of a line in 2D-space? You might have noticed that Eqn. (8) is a generalization of the two point form of the equation of a line in 2D-space to 3D-space.

For example, the equations of the line passing through (1, 2, 3) and (0, 1, 4) are $\dfrac{x-1}{-1} = \dfrac{y-2}{-1} = \dfrac{(z-3)}{1}$ *i.e.* $x - 1 = y - 2 = -(z-3)$.

Remark 2: Note that l, m, n can take zero values. In that case the expression $\dfrac{'a'}{0}$ is to be understood as 'a proportional to 0 (denoted by $a \alpha 0$) not as 'division of a by 0' (which is not defined). The Eqn. (6) and Eqn. (8) are only statements of proportion i.e. $\dfrac{x-a}{l} = \dfrac{y-b}{m} = \dfrac{z-c}{n}$ means $(x-a, y-b, z-c) \alpha (l, m, n)$. When $l = 0, m = 0$ and $n = 0$, then the line degenerates to a point.

Note that, while obtaining Eqn. (8) we have also shown the following:

If $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are two points lying on a line L, then $x_2 - x_1, y_2 - y_1$ and $z_2 - z_1$ are the direction ratios of L.

Let us see some examples.

Example 4: Show that the equation of a line passing through $(-3,5,3)$ and $(2,4,3)$ is $x+5y=22$, $z=3$.

Solution: i) Since the line passes through $(-3,5,3)$ and $(2,4,3)$, the equations are given by

$$\frac{x+3}{2-(-3)}=\frac{y-5}{4-5}=\frac{z-3}{z-3}=r \text{ , say, i.e.,}$$

This implies that $-(x+3)=5(y-5)$ and $z=3,$
Hence, the equations are $x+5y=22,\ z=3.$

***

Now you can try some exercises.

E6) Find the following:

i) The equation of the line joining the points $(-1,0,1)$ and $(1,2,3).$

ii) The equation of line with direction cosines $\frac{\sqrt{3}}{5},\frac{-\sqrt{6}}{5},\frac{4}{5}$ and passing through $(1,-1,2)$.

So far you have seen that a line in 3D-space can be represented by three forms namely

i) Canonical form.

ii) One parametric form.

iii) Two-point form.

Next we shall discuss how to represent a plane in 3D-space.

Let us first look at the coordinate planes.

Consider the xy -plane in Fig. 10 (a). The z-coordinate of every point in this space is 0. Conversely, any point whose z-coordinate is zero will be in the xy-plane. Thus, the equation $z=0$ describes the xy-plane.

Similarly, $z=3$ describes the plane which is parallel to the xy-plane and which is placed 3 units above it (See Fig. 10).

What is the equation of the yz-plane and xz -plane? Do you agree that they are given by $x=0$ and $y=0,$ respectively?

Another point we want you to notice about the planes is that their equations are linear in x, y and z. This suggests that a plane can be represented by

$$Ax+By+Cz+D=0 \qquad \ldots (9)$$

Infact Eqn.(9) represents the general linear equation of a plane where at least one of A, B, C is non-zero.

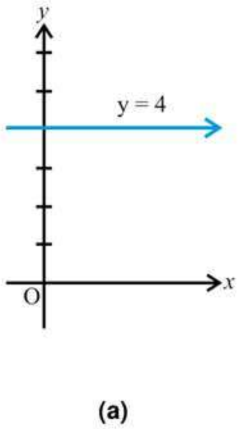
We are not giving more details of the fact given above, but will always use the fact that a plane is synonymous with a linear equation in 3 variables.

Thus, for example, we know that $2x + 5z = y$ represents a plane.

At this point we would like to make an important Note.

Note 1: In 2D-system a linear equation represents a line, while in 3D-system a linear equation represents a plane.

For example, $y = 4$ is a line in $\boldsymbol{R}^2$ (See Fig. 10 (a)), whereas it is a plane in $\boldsymbol{R}^3$ (See Fig. 10 (b)).



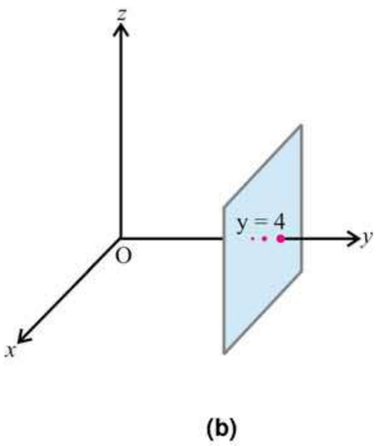


Fig. 10: The same equation represents a line in 2D-space and a plane in 3D-space.

We shall now state a result which gives the equation of a plane passing through three non-collinear points: The Proof of the theorem is omitted.

Theorem 1: The unique plane passing through three non-collinear points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is given by the determinant equation.

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0. \qquad \ldots (10)$$

Non-collinear points are points such that all of which do not lie on a line.

– ■ –

We will not prove this theorem here, but we shall use it in the later units. Hope that you are familiar with the method finding the determinant from +2 level Mathematics. Here is an example which helps you to test your memory!!!
[Those who are not familiar with the calculation of determinants can refer into any of the reference books mentioned in the course introduction].

Example 5: Find the equation of the plane which passes through the points (1, 1, 0), $(-2, 2, -1)$ and $(1, 2, 1)$.

Solution: To obtain the equation we find the determinant

$$\begin{vmatrix} x & y & z & 1 \\ 1 & 1 & 0 & 1 \\ -2 & 2 & -1 & 1 \\ 1 & 2 & 1 & 1 \end{vmatrix}$$

$$\Rightarrow x\begin{vmatrix}1 & 0 & 1\\ 2 & -1 & 1\\ 2 & 1 & 1\end{vmatrix} - y\begin{vmatrix}1 & 0 & 1\\ -2 & -1 & 1\\ 1 & 1 & 1\end{vmatrix} + z\begin{vmatrix}1 & 1 & 1\\ -2 & 2 & 1\\ 1 & 2 & 1\end{vmatrix} - \begin{vmatrix}1 & 1 & 0\\ -2 & 2 & -1\\ 1 & 2 & 1\end{vmatrix} = 0.$$

$$\Rightarrow 2x+3y-3z=5.$$

***

Here is another interesting example.

Example 6: Show that the equation of the plane which makes intercepts $2,-1,5$ on the x, y, z axes respectively is $\dfrac{x}{2}+\dfrac{y}{(-1)}+\dfrac{z}{5}=1.$

Solution: We first note that the plane makes intercept 2 on x-axis means that it passes through the point (2, 0, 0). In a similar way, the plane passes through $(0,-1,0)$ and $(0,0,5)$ on y-and z-axes respectively.

Substituting for these three points in the determinant equation, we get

$$\begin{vmatrix}x & y & z & 1\\ 2 & 0 & 0 & 1\\ 0 & -1 & 0 & 1\\ 0 & 0 & 5 & 1\end{vmatrix} = 0 \Rightarrow 5x-10y+2z=10$$

$$\Rightarrow \frac{x}{2}+\frac{y}{(-1)}+\frac{z}{5}=1.$$

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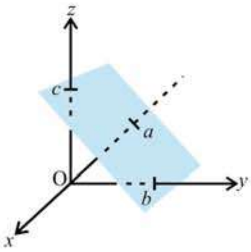


Fig. 11: The plane $\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}=1$.

Did you notice the relationship between the intercepts and the coefficients of the equation, while solving Example 6 above?

In general, you can see that the equation of the plane making intercepts a, b and c on the coordinate axes (see Fig. 11) is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1. \qquad \text{... (11)}$$

This is because (a, 0, 0), (0, b, 0) and (0, 0, c) lie on it.

Eqn. (11) is called the **intercept form** of the equation of a plane. You recall here the intercept form of an equation of a line in 2D-system.Then you see that Eqn. (11) is a generalization from 2D-system to 3D-system.

Here is an example

Example 7: Find the intercepts on the coordinate axes by the plane $2x-3y+5z=4.$

Solution: We rewrite the given equation of the plane as

$$\frac{x}{2}+\frac{y}{\left(-\frac{4}{3}\right)}+\frac{z}{\frac{4}{5}}=1.$$

Thus, the intercepts on the axes are $2,-\dfrac{4}{3}$ and $\dfrac{4}{5}$.

***

Why don't you try some exercises?

E7) Show that the four points $(0,-1,-1),(4,5,1),(3,9,4)$ and $(-4,4,4)$ are coplanar, that is, they lie on the same plane. (**Hint:** Obtain the equation of the plane passing through any three of the points, and check whether the fourth point lies on it.)

E8) Find the equation of the plane that passes through the points $R(1,3,2)$, $Q(3,-1,6)$ and $R(5,2,0)$.

In the next section we shall discuss two other coordinate systems.

1.3 THE CYLINDRICAL AND THE SPHERICAL COORDINATE SYSTEMS

In the last section we discussed about the Cartesian coordinate system. In this section we shall discuss two other coordinate systems-the cylindrical coordinate system and the spherical coordinate system. We shall first consider the cylindrical coordinate system.

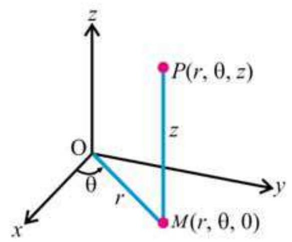


Fig. 12: Representation of a point (x, y, z) in terms of its cylindrical coordinates r, θ and z.

Recall that in unit 3 of the Calculus course, we introduced the polar coordinate system in order to give a convenient description of certain curves and regions in a plane. The cylindrical coordinate system is an extension of polar coordinates system which gives convenient descriptions of some commonly occurring surfaces and solids.

To start with, let us consider a point $P(x, y, z)$ in three-dimensional space. Let M be the foot of the perpendicular from P on the xy-plane. Then M is called the **projection** of P on the xy-plane (See Fig. 12). Then the distance $PM = |z|$. Note that M is a point in the xy-plane. Let the polar coordinates of M be (r, θ) (See Fig. 12). Then the cylindrical coordinates of P are (r, θ, z) where z is the length of the line PM which is the z-coordinate of P.

Remark: The equation $r = c$, a constant represents a cylinder in $\boldsymbol{R}^3$ (right circular cylinder of radius r and axis of symmetry as the z-axis.)
The equation $\theta = a$, a constant, represents a half plane along z-axis.
The equation $z = h$, a constant, represents a plane parallel to the xy-plane and passing through (o, o, h).

Thus a point in $\boldsymbol{R}^3$ in cylindrical coordinate system is conceived as the point of intersection (unique) of a cylinder, half plane and a plane, three mutually orthogonal surfaces.

Thus we have another system for representing points in 3-D space.

The conversion of points from one system to other is given by the following:

Conversion from Cartesian to Cylindrical and Vice-versa
If $P(x, y, z)$ is a point prepresented by the Cartesian coordinate system, then the representation of P in cylindrical coordinate system is given by the relation:

$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}, \quad z = z \qquad \text{... (12)}$$

Whereas if (r, θ, z) is the representation of a point P in the cylindrical system, then the representation of P in Cartesian system is given by the relation

$$x = r\cos\theta \quad y = r\sin\theta \quad z = z \qquad \text{... (13)}$$

Let us see some examples:

Example 8: Find the following.

a) Suppose that P is point with cylindrical coordinates $\left(2, \frac{2\pi}{3}, 1\right)$ then, find its Cartesian coordinates.

b) Find cylindrical coordinates of the point with Cartesian coordinates $(3, -3, -7)$.

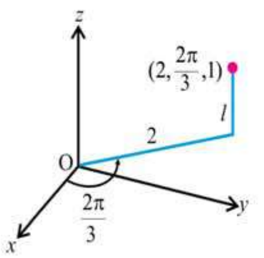


Fig. 13

Solution: a) The point P with cylindrical coordinates $\left(2, \frac{2\pi}{3}, 1\right)$ is plotted in Fig. 13. From Eqn. (13), its Cartesian coordinates are given by

$$x = 2\cos\frac{2\pi}{3} = 2\left(-\frac{1}{2}\right) = -1$$

$$y = 2\sin\frac{2\pi}{3} = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$$

$$z = 1$$

Thus, the point is $(-1, \sqrt{3}, 1)$ in Cartesian coordinate system.

b) From Eqn. (14) we have

$$r = \sqrt{3^2 + (-3)^2} = 3\sqrt{2}$$

$$\tan\theta = \frac{-3}{3} = -1 \quad \text{so} \quad \theta = \frac{7\pi}{4} + 2n\pi$$

$$z = -7$$

Therefore, one set of cylindrical coordinates is $(3\sqrt{2}, 7\pi/4, -7)$ and another is $(3\sqrt{2}, -\pi/4, -7)$. There are infinitely many choices with different values of n for polar coordinates.

***

Let us now look at the shape of some surfaces formed by the equations in cylindrical coordinate system in Cartesian system.

Example 9: Identify the surface given by the following:

i) $r = 7$

ii) $r = z$

iii) $r^2 + z^2 = 64$

Solution: Let us find one by one.

i) We know that the polar equation $r = 7$ represents a circle of radius 7 in 2D-system. Therefore for any z, z varying, we get circles of radius 7, piled up. What is the surface so formed by this? Can you guess? It is nothing but a cylinder of radius 7 centred on the z-axis. [See Fig. 14]

ii) The given equation is $r = z$. On squaring we get $r^2 = z^2$. From Eqn. (12) we have $r^2 = x^2 + y^2$. Therefore we get that $x^2 + y^2 = z^2$ i.e. $x^2 + y^2 - z^2 = 0$. This surface is called the hyperboloid of one sheet and is a three dimensional extension of a hyperbola.

iii) The given equation is $r^2 + z^2 = 64$. Putting $x^2 + y^2$ for r^2 in the given equation we get, $r^2 + z^2 = x^2 + y^2 + z^2 = 64$. It is now easy to identify the surface as sphere of radius 8.

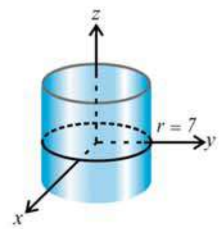


Fig. 14: The points whose cylindrical coordinate satisfy $r = 7$ forming a cylinder.

***

Remark: In the examples above, you might have observed that in cylindrical coordinates, certain cylinders have very simple equation $r = c$, a constant, compared to the equation $x^2 + y^2 = a^2$ in the Cartesian coordinate system. That is why the name "cylindrical coordinates".

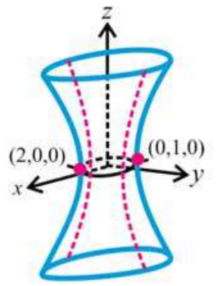


Fig. 15: Hyperboloid of one sheet.

Remark: Another interesting fact is that the cylindrical coordinates are useful in problems that involve symmetry about an axis, and the z-axis is chosen to coincide with the axis of symmetry.

You will become more familiar with the conversion between Cartesian and cylindrical coordinates, when you try the following exercises.

E9) The Cartesian coordinates of a point is $(-1, -\sqrt{3}, 2)$. What are its cylindrical coordinates?

E10) Find the Cartesian coordinates of the following points whose cylindrical coordinates are given by

i) $(3, \frac{\pi}{2}, 1)$

ii) $(4, -\frac{\pi}{3}, 5)$

Next we shall discuss another coordinate system known as spherical coordinate system. This is based on the polar coordinate system for 2D-system. It is easy to understand this with a diagram (See Fig. 16).

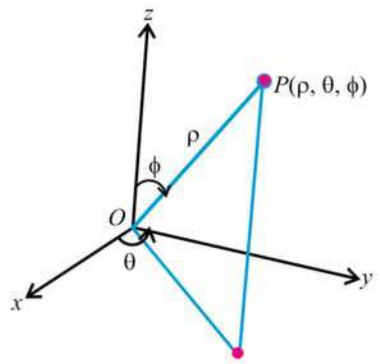


Fig. 16: The spherical coordinates of a point.

In Fig. 16, $P(x, y, z)$ is a point in the Cartesian coordinate system. Let us see what are its coordinates in spherical coordinate system. There are three parameters involved in the spherical coordinates of a point which are given by:

1. **Parameter** ρ : This is the distance $|OP|$ i.e. the distance from the origin to the point P. We take $\rho \geq 0$. (See Fig. 16)

2. **Parameter** θ **:** θ is as in the polar coordinate of the projection of the point P to the xy-plane (same as in the case of cylindrical coordinates). (See Fig. 16).

3. **Parameter** ϕ **:** This is the angle between the positive z-axis and the line OP. (See Fig. 16).

Then (ρ, θ, ϕ) is the spherical coordinates of P.

The **spherical coordinates** of a point P is denoted by (ρ, θ, ϕ) where $\rho = |OP|$ is the distance from the origin to P, θ is as in polar coordinates of the projection of the point to the xy-plane and ϕ is the angle between the positive z-axis and the line segment OP. Note that $\rho \geq 0, 0 \leq \theta \leq 2\pi,\ 0 \leq \phi \leq \pi$.

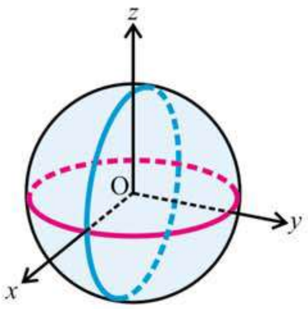


Fig. 17: $\rho = c, a$ **sphere**

Remark: Note that here any point say $(\rho_0, \theta_0, \phi_0)$, is the unique point of intersection of three mutually orthogonal surfaces, viz

1) The sphere with the centre of the origin and ρ_0 as its radius. This may be denoted by $\rho = \rho_0$.

2) The half plane $\theta = \theta_0$ along the $z-$axis.

3) The right circular cone $\phi = \phi_0$ symmetric about the $z-$axis.

The spherical coordinate system is especially useful in problems where there is symmetry about a point, and the origin is placed at this point. For example, the sphere with centre the origin and radius c has the simple equation $\rho = c$ (see Fig. 17); this is the reason for the name "spherical" coordinates.

Now we find the relationship between Cartesian, spherical and cylindrical coordinates. Let $OP = \rho$. We look at Fig. 18. Let $P(x, y, z)$ be a point in 3D-system as shown in Fig. 18. Let us draw a perpendicular from P to the z-axis meeting it at θ and let P' be the projection of the point on the xy-plane.

Let P' be the foot of the perpendicular from P on the xy-plane i.e. the projection of P on the xy-plane. Then from triangle OPQ and OPP', we have

$z = \rho \cos \phi$ and $r = \rho \sin \phi$

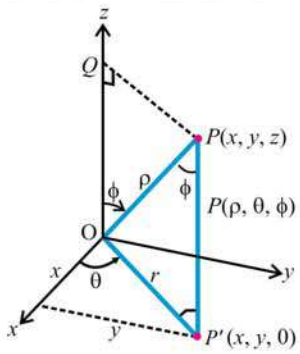


Fig. 18

We also have $x = r\cos\theta$ and $y = r\sin\theta$. Substituting $r = \rho\sin\phi$ for r in these equations, we get

$$x = \rho\sin\varphi\cos\theta,\ y = \rho\sin\varphi\sin\theta \text{ and } z = \rho\cos\phi \quad \text{... (14)}$$

Also, we have

$$\begin{aligned} x^2 + y^2 + z^2 &= r^2\cos^2\theta + r^2\sin^2\theta + \rho^2\cos^2\theta \quad \text{... (15)} \\ &= r^2 + \rho^2\cos^2\theta \\ &= \rho^2\sin^2\theta + \rho^2(\cos^2\theta) \\ &= \rho^2 \end{aligned}$$

We now state the conversion equations one by one.

1) To convert from **spherical coordinates to cylindrical coordinates** we use the relationship $r = \rho\sin\theta$, $\theta = \theta$, $z = \rho\cos\theta$.

2) To convert from **Cartesian coordinates to spherical coordinates**, we use the equations:

$$\rho = \sqrt{x^2 + y^2 + z^2}$$

$$\phi = \cos^{-1}\left(\frac{z}{\rho}\right)$$

$$\theta = \cos^{-1}\left(\frac{x}{\rho\sin\phi}\right)$$

3) To convert from **Spherical coordinates to Cartesian coordinate**, we use the equations

$$x = \rho\sin\phi\cos\theta$$

$$y = \rho\sin\phi\sin\theta$$

$$z = \rho\cos\phi.$$

Let us see some examples.

Example 10: The point $(2, \pi/4, \pi/3)$ is given in spherical coordinates. Find its Cartesian coordinates.

Solution: To find the Cartesian coordinates we put $\rho = 2, \theta = \dfrac{\pi}{4}$ and $\theta = \dfrac{\pi}{3}$ in the equation given in Eqn. (15). Then we have

$$x = \rho \sin\phi \cos\theta = 2\sin\frac{\pi}{3}\cos\frac{\pi}{4} = 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = \sqrt{\frac{3}{2}}$$

$$y = \rho \sin\phi \cos\theta = 2\sin\frac{\pi}{3}\cos\frac{\pi}{4} = 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = \sqrt{\frac{3}{2}}$$

$$z = \rho\cos\phi = 2\cos\frac{\pi}{2} = 2\left(\frac{1}{2}\right) = 1$$

Thus, the point $(2, \pi/4, \pi/3)$ is represented by $(\sqrt{3/2}, \sqrt{3/2}, 1)$ in Cartesian coordinates.

***

Example 11: The Cartesian coordinates of a point are given by $(0, 2\sqrt{3}, -2)$. Find its Spherical coordinates.

Solution: We use Eqn. 15 and get

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{0 + 12 + 4} = 4$$

To obtain the spherical coordinates we use the following equations and get

$$\cos\phi = \frac{z}{\rho} = \frac{-2}{4} = -\frac{1}{2} \quad \phi = \frac{2\pi}{3}$$

$$\cos\theta = \frac{x}{\rho\sin\phi} = 0. \qquad \therefore \theta = \frac{\pi}{2}$$

(Note that $\theta \neq 3\pi/2$ because $y = 2\sqrt{3} > 0$). Therefore, the spherical coordinates of the given point are $(4, \pi/2, 2\pi/3)$.

***

Why don't you try some exercises now.

E11) Convert the following points from spherical coordinates to Cartesian coordinates

i) $(1, 0, 0)$

ii) $\left(2, \dfrac{\pi}{3}, \dfrac{\pi}{4}\right)$

E12) Find the equation in Cartesian coordinates of a surface whose equation in spherical coordinates is given by $\rho = \cos\theta \sin\phi$

So far we have seen, that a point in space can be represented by three forms of coordinates – Cartesian, cylindrical and spherical. It is important to notice that in all the three representations, a point is associated with an ordered triple (a,b,c) of real numbers. Note that the order is important in the sense that we write the x-coordinate first, then the y-coordinate and the z-coordinate. Similarly for spherical and cylindrical also. That is, it is an ordered triple with specific meaning to the number appearing in a position.

Cartersian - $\boldsymbol{R}\times\boldsymbol{R}\times\boldsymbol{R}$
Cylindrical - $[0,\infty)\times[0,2\pi]\times\boldsymbol{R}$
Spherical - $[0,\infty)\times[0,2\pi]\times[-\pi,\pi]$

Recall that in the Calculus course you have learnt that such ordered pair of elements form a set called Cartesian product of real numbers which we denote by $\boldsymbol{R}^3=\{(x,y,z): x,y,z\in\boldsymbol{R}\}$ or $\boldsymbol{R}\times\boldsymbol{R}\times\boldsymbol{R}$ in Cartesian coordinates, $[0,\infty)\times[0,2\pi]\times\boldsymbol{R}$ in cylindrical coordinates and $[0,\infty)\times[0,2\pi]\times[-\pi,\pi]$ in spherical coordinates. In the next section we discuss Cartesian product, in detail. Thus, we learnt that there is a one – to – one correspondence between the element in $\boldsymbol{R}^3$ and points in 3D-space. We have noted that this association helps to study some geometrical properties of the set $\boldsymbol{R}^3$. Next we shall discuss some algebraic properties of $\boldsymbol{R}^3$.

1.4 CARTESIAN PRODUCTS R^n OF R AND PROPERTIES $(n=2,3)$

In this section we shall introduce you to sets which are called Cartesian products of Real numbers. We discuss some basic operations on these sets and also discuss some algebraic properties.

While studying Block 1 of the calculus course you have learnt that the sets of the type $X_1\times X_2\times...\times X_n$ are known as Cartesian products of the sets X_i, $i=1$ to n. Here we shall study the Cartesian products of the set $\boldsymbol{R}$. We consider only $\boldsymbol{R}\times\boldsymbol{R}$ and $\boldsymbol{R}\times\boldsymbol{R}\times\boldsymbol{R}$.

Let us begin with some definitions.

Definition 3: The Cartesian product $\boldsymbol{R}\times\boldsymbol{R}$ of the set of real numbers is set of all ordered pairs (x,y) where $x,y,\in\boldsymbol{R}$
We denote this by $\boldsymbol{R}\times\boldsymbol{R}$ or $\boldsymbol{R}^2$. Thus $\boldsymbol{R}^2=\{(x,y): x,y\in\boldsymbol{R}^2\}$.
For example, $(0,1.5),(\pi,-1),\left(\frac{1}{2},\sqrt{2},\right),\left(\sqrt{2},\frac{1}{2}\right)$ are all elements of $\boldsymbol{R}^2$.
Note that the ordered pair (x,y) is different from the ordered pair (y,x) if $x\neq y$ while the sets $\{x,y\}$ and $\{y,x\}$ are equal.

We now consider the products of three sets of $\boldsymbol{R}$.

Definition 4: The Cartesian products of 3 sets of $\boldsymbol{R}$ denoted by $\boldsymbol{R}\times\boldsymbol{R}\times\boldsymbol{R}$ is the set of all ordered tuples (x,y,z) where $x,y,z\in\boldsymbol{R}$. We denote this set by $\boldsymbol{R}^3$ also.

Thus $\boldsymbol{R}^3=\{(x,y,z): x,y,z\in\boldsymbol{R}\}$

Examples of elements of $\boldsymbol{R}^3$ are $\left(0,\frac{1}{\sqrt{3}},1\right)$, and $(-1,-2,\pi)$.

From the previous section you know that there is a one-one correspondence between the points in 3D-space and the elements in $\boldsymbol{R}^3$. For this reason, 3D-space is often denoted by the symbol $\boldsymbol{R}^3$. For a similar reason a 2D-plane is dented by $\boldsymbol{R}^2$ and a line by $\boldsymbol{R}$.

Note: Though we know that the Cartesian products can be defined for n copies of $\boldsymbol{R}$, **in this course we shall consider only $\boldsymbol{R}^2$ and $\boldsymbol{R}^3$.**

Next we define certain basic operations on $\boldsymbol{R}^3$ and their properties. You should note that all these definitions and results hold for $\boldsymbol{R}^2$ also by simply dropping the third coordinate.

We shall first consider "addition".

Definition (Addition) 5: If $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are two elements in $\boldsymbol{R}^3$, then the **sum $P+Q$** of P and Q is the element given by $P+Q=(x_1, y_1, z_1)+(x_2, y_2, z_2)=(x_1+x_2, y_1+y_2, z_1+z_2)$.

Thus to find the sum we add each coordinate in one element with the corresponding coordinates in the other element. This means that we can add any finite number of elements, through repeated addition.

Let us see some examples

Example 12: Add the following:

i) $(1,0,1)+(2,-3,4)$

ii) $(a,b,c)+(0,0,0)$

iii) $(x,y,z)+(-x,-y,-z)$

Solution: i) $(1,0,1)+(2,-3,4)=(1+2,0+(-3),1+4)$
$=(3,-3,5)$

ii) $(a,b,c)+(0,0,0)=(a+0,b+0,c+0)$
$=(a,b,c)$

Iii) $(x,y,z)+(-x,-y,-z)=(x-x,y-y,z-z)$
$=(0,0,0)$

***

While doing ii) in Example 12, you must have observed that, if the element (or point) $(0,0,0)$ is added to any other element of $\boldsymbol{R}$, then the sum is the element P only. Because of this $(0,0,0)$ is called the zero-element.

In (iii) of Example (8) above, you might have noticed that if P is the point (x,y,z) and if we denote $-P$, the point $(-x,-y,-z)$, then $P+(-P)=0$. This means that if we add P and $(-P)$, then we get the zero-element. We call $(-P)$ the **additive inverse** of P.

Try these exercises now.

E13) i) Add $(0,0,1)$ to the sum of $(1,0,0)$ and $(0,1,0)$.

ii) Find the sum of $(0.5,1),(-3,0)$ and $(1,0.5)$ and $(0,-3)$. Are they equal? Justify?

E14) Find the additive inverse of the following:

i) $\left(\frac{1}{2},\frac{-1}{3}\right)$

ii) $(-4.5,-8)$

E15) Find the sum of the additive inverse of $(-5x,4y,-z)$ and the zero-element.

In a similar way we define subtraction operation also.

Definition (Subtraction) 6: If $P(x_1,y_1,z_1)$ and $Q(x_2,y_2,z_2)$ are two elements of $\boldsymbol{R}^3$, then the difference of P and Q, denoted by $P-Q$ is the element given by

$$
\begin{aligned}
P-Q=(x_1,y_1,z_1)-(x_2,y_2,z_2)&=P+(-Q)\\
&=(x_1,y_1,z_1)+(-x_2-y_2,-z_2)\\
&=(x_1-x_2,y_1-y_2,z_1-z_2).
\end{aligned}
$$

For example, if $P=(2,2,-1)$ and $Q=(5,-3,2)$, then

$$
\begin{aligned}
P-Q&=P+(-Q)\\
&=P+(-Q)\\
&=(2,2,-1)+(-5,3,-2)\\
&=(-3,5,-3)
\end{aligned}
$$

Thus we saw that the operations of addition and subtraction can be easily extended from $\boldsymbol{R}$ to $\boldsymbol{R}^3$ (and $\boldsymbol{R}^2$). Note that in both these cases we added or subtracted the respective coordinate in the two elements. This method is called coordinate wise addition or coordinate wise subtraction.

Can we extend this method for multiplication also? What will happen when we multiply coordinate wise, let us see.

There are different ways that the multiplication is defined for $\boldsymbol{R}^3$ (or $\boldsymbol{R}^2$). Here we shall discuss an operation called "Scalar Multiplication" (The word "scalar" is used as a synonym for 'real numbers'). This product combines a real number and an element in $\boldsymbol{R}^3$. More precisely we have the following definition.

Definition(Scalar Multiplication) 7: If $P(x,y,z)$ is an element of $\boldsymbol{R}^3$, and let α, any element of $\boldsymbol{R}$, then the **scalar multiple of** P **by** α **is the element** $\alpha\mathrm{P}$ **given by**

$$\alpha P=\alpha(x,y,z)=(\alpha x,\alpha y,\alpha z)$$

Let us see some examples

Example 13: Find the following

i) $0.1(1,-1,4)+3(0,2,3)-\frac{1}{2}(2,4,6)$

ii) $3(x,-y,z)+(-2,4,1)$

Solution: Let us try one by one:

i) $0.1(1,-1,4)=(0.1,-0.1,0.4)$

$3(0,2,3)=(0,6,9)$

$\frac{1}{2}(2,4,6)=(1,2,3)$

$\therefore 0.1(1,-1,4)+3(0,2,3)-\frac{1}{2}(2,4,6)$

$=(0.1,-0.1,0.4)+(0,6,9)-(1,2,3)$

$=(-0.9,3.9,6.4)$

Next we find ii)

ii) $3(x,-y,z)=(3x,-3y,3z)$

Then we have

$$\begin{aligned} 3(x,-y,z)+(-2,4,1) &= (3x,-3y,3z)+(-2,4,1) \\ &= (3x-2,-3y+4,3z+1) \end{aligned}$$

***

The subtraction and scalar multiplication are defined for $\boldsymbol{R}^2$ just as those in $\boldsymbol{R}^3$, with third coordinate dropped off (!).

The following properties are easy to verify from the definition of the addition and scalar multiplication. We leave it as an exercise for you to check.

Properties: i) $\alpha\{(x_1,y_1,z_1)+(x_2,y_2,z_2)\}$
$=\alpha(x_1,y_1,z_1)+\alpha(x_2,y_2,z_2)$

ii) $(\alpha+\beta)(x_1,y_1,z_1)=\alpha(x_1,y_1,z_1)+\beta(x_1,y_1,z_1)$

iii) $(\alpha\beta)(x_1,y_1,z_1)=\alpha\{\beta(x_1,y_1,z_1)\}$

iv) $\alpha(x,y,z)=0$ for every $(x,y,z)\in\boldsymbol{R}^3$ if and only if $\alpha=0$.

Note: All the properties given above holds for $\boldsymbol{R}^2$ also.

Here is an exercise for you.

E16) Find $P-Q$ for the elements $\mathrm{P}=(2,2,-1)$ and $\mathrm{Q}=(5,-3,2)$ and check whether it is the same as $Q-P$.

E17) Verify the properties i) to iv) stated above.

Thus we have defined some basic arithmetic operations on $\boldsymbol{R}^3$ (and $\boldsymbol{R}^2$) and discussed some of their properties which are called algebraic properties of $\boldsymbol{R}^3$ (and $\boldsymbol{R}^2$).

Note: The concepts we have discussed in this section can easily be extended from three to any number of dimensions. We define $\boldsymbol{R}^n$, where n is a positive integer (possibly greater than three), as the set of all n-triples $(x_1, x_2, ..., x_n)$ where $x_i, i = 1...n$, are real numbers. For instance $(1, \sqrt{7}, \pi, -4) \in \boldsymbol{R}^4$.

We introduce some notations here.

Notation: Here onwards the elements in $\boldsymbol{R}^n$, $n > 1$ will be denoted as 'bold small letters $\boldsymbol{x, y, z,...}$. Whenever we need to distinguish elements from, $\boldsymbol{R}^3$ and $\boldsymbol{R}^2$, say, we denote any element of $\boldsymbol{R}^3$ as $\boldsymbol{x} = (x_1, x_2, x_3)$ and that of $\boldsymbol{R}^2$ as $\boldsymbol{x} = (x_1, x_2)$.

The algebraic properties discussed for $\boldsymbol{R}^3$ (and $\boldsymbol{R}^2$) help us to perform some basic arithmetic operations on $\boldsymbol{R}^3$ (and $\boldsymbol{R}^2$). Do these operations help us to find whether a point in 3D-space (or 2D-plane) is close to the origin or far from the origin. For instance, how do we calculate the distance between two points in space.

You know that for any two real numbers x, y the absolute value $|x - y| = \sqrt{(x-y)^2}$ gives the distance between the points represented by x and y on the real line. We extend this idea to define the distance between two points in $\boldsymbol{R}^2$ and $\boldsymbol{R}^3$. We use the notation $\| \, . \, \|$ instead of the absolute value $| \, . \, |$ and call it norm. Thus $\|x - y\|$ is called as 'the norm $x - y$'.

We formally define it now.

We know that, the length between two points in $\boldsymbol{R}$, gives the distance. Do you remember the notation that we used to denote the length of two points x and y on the real line. It is denoted by $|x - y|$ (we call it the absolute value of the difference between x and y. Note that we can also calculate the length by the formula

$$|x - y| = +\sqrt{(x-y)^2} \qquad \text{... (16)}$$

We now extend this formula given in Eqn. (17) to $\boldsymbol{R}^3$ (and $\boldsymbol{R}^2$).

If $\boldsymbol{x} = (x_1, x_2, x_3)$ and $\boldsymbol{y} = (y_1, y_2, y_3)$ are two elements in $\boldsymbol{R}^3$, then we denoted the distance between these two elements by

$$\|\boldsymbol{x} - \boldsymbol{y}\| = \|(x_1, x_2, x_3) - (y_1, y_2, y_3)\| = \|(x_1 - y_1, x_2 - y_2, x_3 - y_3)\|$$

$$= +\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}$$ (By taking clue from Eqn. (16)).

The expression on the R.H.S above is called the distance formula. We formally define it now.

Definition 8: If $\boldsymbol{x} = (x_1, x_2, x_3)$ and $\boldsymbol{y} = (y_1, y_2, y_3)$ are two elements in $\boldsymbol{R}^3$, then we define the distance $\|x - y\|$ between x and y as

$$\|\boldsymbol{x}-\boldsymbol{y}\| = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2} \qquad \text{... (17)}$$

Using the notation $\sum$ (- called sigma), we rewrite Eqn. (17) as

$$\|\boldsymbol{x}-\boldsymbol{y}\| = \sqrt{\sum_{i=1}^{3}(x_i-y_i)^2} \qquad \text{... (18)}$$

In $\boldsymbol{R}^2$, the distance between x and y is given by

$$\|\boldsymbol{x}-\boldsymbol{y}\| = \sqrt{\sum_{i=1}^{2}(x_i-y_i)^2} \qquad \text{... (19)}$$

Geometrically the distance between two points is shown in Fig. 19 (the red line).

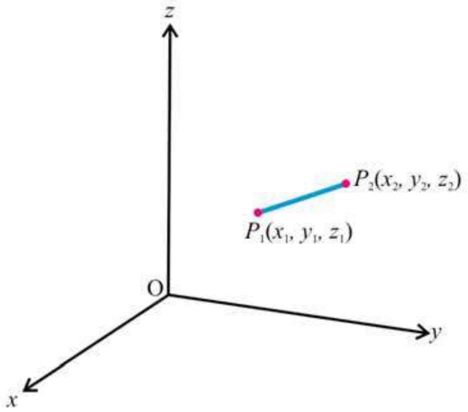


Fig. 19

To verify the distance formula we construct rectangular boxes as shown in Fig. 20 and apply Pythagoras Theorem. The details are omitted here. The following figure may help you in this regard.

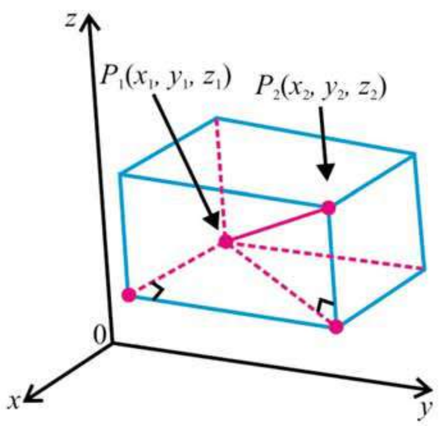


Fig. 20

Here are some examples to illustrate the formula.

Example 14: Find the distance between the points $\boldsymbol{a} = (1, -3, 7)$ and $\boldsymbol{b} = (2, -1, 5)$.

Solution: By the distance formula, the distance between $\boldsymbol{a}$ and $\boldsymbol{b}$ is given by

$$\|a-b\| = \sqrt{(1-2)^2 + (-3+1)^2 + (7-5)^2}$$
$$= \sqrt{1+4+4}$$
$$= 3.$$

[Be careful while dealing with negative numbers].

***

Example 15: Which one of the points a and b are closest to the point c where $a = (6,2,3)$, $b = (-5,-1,4)$ and $c = (0,3,8)$. Which of the points a, b and c lie in the yz-plane?

Solution: Let us calculate the distance of the points a and b from c.

$$\|a-c\| = \|(6,-1,-5)\| = \sqrt{6^2 + (-1)^2 + (-5)^2}$$
$$= \sqrt{36+1+25} = \sqrt{62}$$
$$\|b-c\| = \|(-5,-4,-4)\| = \sqrt{(-5)^2 + (-4)^2 + (-4)^2}$$
$$= \sqrt{25+16+16} = \sqrt{57}$$

Since the distance from c is less for b than a, the point b is close to c than a.

Also the x-coordinate of any point on the yz-plane is 0, therefore only the point **c** lies on the yz-plane.

***

To get more practice for applying the distance formula, why don't you try some exercises.

E18) Show that the triangle with vertices $P(-2,4,0)$, $Q(1,2,-1)$ and $R(-1,1,2)$ is an equilateral triangle.

Thus we have seen that the distance function $\|.\|$ helps us to calculate the distance between two points. You may note here that geometrically if the distance between two points is zero, then both the points coincide. We also know geometrically that the distance between **a** and **b** is the same as the distance between **b** and **a.** Both these properties can be stated in the following way.

Property 2: Let x and y be two points in R^3 (or R^2). Then the following holds:

i) $\|x-y\| = 0$ if and only if $x = y$

ii) $\|x - y\| = \|y - x\|$

The properties, above, can be verified directly from the definition.

The next property we are going to state is known as "triangle inequality". You know that, a triangle in a plane has the property that the sum of the lengths of any two sides of a triangle is always greater than or equal to the length of the remaining side. This means that if x, y, z are three points in R^3, then

$$\|x-y\| \le \|x-z\| + \|z-y\| \qquad ...(20)$$

The same is true for R^2 also. To prove this inequality, we have to take help of an important inequality known as **Cauchy-Schwarz (Cauchy is pronounced as koshi) inequality**. You will see in the later units that this inequality is very useful in the verification of many other important results.

We shall state it now.

Cauchy-Schwarz Inequality

If $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ are two elements in R^3, then the following holds:

$$\left|\sum_{i=1}^{3} a_i b_i\right| \le \sqrt{\sum_{i=1}^{3} a_i^2}\ \sqrt{\sum_{i=1}^{3} b_i^2} \qquad ...(21)$$

Similarly for R^2, we have the following:

$$\left|\sum_{i=1}^{2} a_i b_i\right| \le \sqrt{\sum_{i=1}^{2} a_i^2}\ \sqrt{\sum_{i=1}^{2} b_i^2} \qquad ...(22)$$

To help you to understanding the summations in (22) and (23), we do some examples.

Let us take two points in R^3 and verify the inequality.

Example 16: Verify Cauchy-Schwarz inequality for $a = (0, -1, \sqrt{3})$ and $b = (-3, 0, 4)$.

Solution: Here $n = 3$. Therefore we consider Eqn. (22). We are given that $a_1 = 0, a_2 = -1, a_3 = \sqrt{3}, b_1 = -3, b_2 = 0, b_3 = 4.$

We substitute these values in Eqn. (22) above and get

$$L.H.S = \sum_{i=1}^{3} a_i b_i$$

$$= a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$= 0 + 0 + 4\sqrt{3}$$

$$R.H.S = \sqrt{\sum_{i=1}^{3} a_i^2}\ \sqrt{\sum_{i=1}^{3} b_i^2}$$

$$= \sqrt{a_1^2 + a_2^2 + a_3^2}\ \sqrt{b_1^2 + b_2^2 + b_3^2}$$

$$= \sqrt{0+1+3}\ \sqrt{9+0+16}$$

$$= \sqrt{4}\ \sqrt{25}$$

$$= 2 \times 5 = 10.$$

Since $4\sqrt{3} \simeq 6.928 < 10$, L.H.S<R.H.S in (21) and hence the inequality is verified.

***

You can try with any other element in $\boldsymbol{R}^3$. Similarly for $\boldsymbol{R}^2$ also.

Let us now try to obtain the triangle inequality using Cauchy-Schwarz inequality.

From the Cauchy-Schwarz inequality we first deduce that if $\boldsymbol{x}, \boldsymbol{y}$ are any two elements in $\boldsymbol{R}^3$, then

$$\|\boldsymbol{x}+\boldsymbol{y}\| \le \|\boldsymbol{x}\| + \|\boldsymbol{y}\| \qquad ...(23)$$

The verification of the inequality in (23) is left as an exercise for you. We now deduce the triangle inequality given in (21) from the inequality in (24). Let us take any three points $\boldsymbol{x}, \boldsymbol{y}$ *and* z in $\boldsymbol{R}^3$. We write

$$\|\boldsymbol{x}-\boldsymbol{y}\| = \|\boldsymbol{x}-z+z-\boldsymbol{y}\| \qquad ...(24)$$

Apply the inequality (24) above to the Right Hand Side of (24) which gives that

$$\|\boldsymbol{x}-\boldsymbol{y}\| = \|\boldsymbol{x}-z+z-\boldsymbol{y}\| \le \|\boldsymbol{x}-z\| + \|z-\boldsymbol{y}\| \qquad ...(25)$$

Hence the triangle inequality given in (21) is verified.

It is now time for a break. Here are some exercises for you to check that how much you are able to grasp.

E19) Verify the inequality in (23) i.e. $\|\boldsymbol{x}+\boldsymbol{y}\| \le \|\boldsymbol{x}\| + \|\boldsymbol{y}\|,\ \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{R}^3$.

E20) Verify the properties (i) and (ii) of the norm function $\|.\|$.

Note: The two-dimensional and three dimensional sets $\boldsymbol{R}^2$ and $\boldsymbol{R}^3$ along with the $\|.\|$ distance function defined by the coordinate systems, are called Euclidean spaces of dimensions two and three respectively. It is named after the ancient Greek mathematician Euclid of Alexandria. The term "Euclidean" distinguishes these spaces from other spaces considered in Modern Geometry.

Next we shall consider some special subsets of $\boldsymbol{R}^3$ (and $\boldsymbol{R}^2$).

You know that the sets of the type $]a,b[=\{x \in \boldsymbol{R} : a < x < b\}$ where a and b are real numbers are called open intervals in $\boldsymbol{R}$. We now introduce sets in $\boldsymbol{R}^3$ which are analogues to open intervals in $\boldsymbol{R}$. The same holds for $\boldsymbol{R}^2$ also.

Definition 9: Let $\boldsymbol{a} \in \boldsymbol{R}^3$ and $r > 0$ be any real number. Then the set

$$S(\boldsymbol{a}, r) = \{\boldsymbol{x} \in \boldsymbol{R}^3 : \|\boldsymbol{x} - \boldsymbol{a}\| < r\} \qquad ...(26)$$

is called **open sphere** or **open ball** or **open disc** with centre $\mathbf{a}$ and radius r. This can also be written as

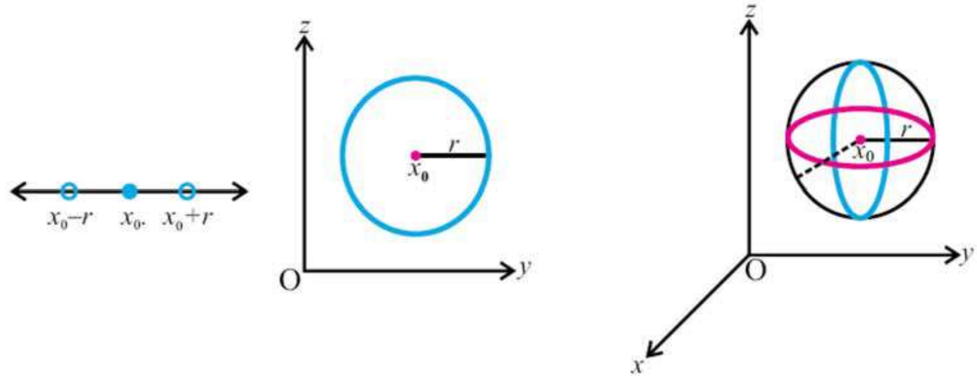
$$S(\boldsymbol{a},r)=\{x=(x_1,x_2,x_3)\in \boldsymbol{R}^3 : \sqrt{\sum_{i=1}^{3}(x_i-\boldsymbol{a}_i)^2}<r\} \text{ where } \boldsymbol{a}=(a_1,a_2,a_3) \quad \text{... (27)}$$

Similarly we can define an open ball or open disc with centre $\mathbf{a}$ and radius r in $\boldsymbol{R}^2$ given by

$$S(\boldsymbol{a},r)=\{x\in \boldsymbol{R}^2 : \|\boldsymbol{x}-\boldsymbol{a}\|<r\}$$

$$=\left\{(x_1,x_2)\in \boldsymbol{R}^2 : \sqrt{\sum_{i=1}^{2}(x_i-a_i)}<r\right\} \quad \text{where} \quad \begin{matrix} x=(x_1,x_2) \\ a=(a_1,a_2) \end{matrix}.$$

You may note that in $\boldsymbol{R}^2$, $S(a,r)$ is the interior of the disc in the plane with centre $\mathbf{a}$ and radius r. (Please see Fig. 21 (b)) and in $\boldsymbol{R}^3$, the set $S(\boldsymbol{a},r)$ is the interior of the ball with centre at $\boldsymbol{a}$ and radius r. (Please see Fig. 21 (c)). Note that here open means only the interior points are included, and not the boundary points. [In the figure the boundaries are shown by dotted lines.



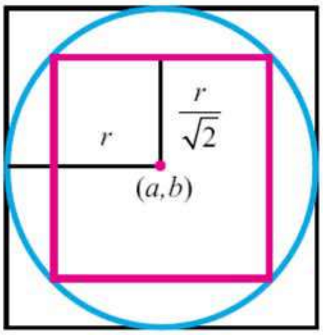
(a) $S(a,r)$ in $\boldsymbol{R}$ (b) $S(a,r)$ in $\boldsymbol{R}^2$ (c) $S(a,r)$ in $\boldsymbol{R}^3$

Fig. 21

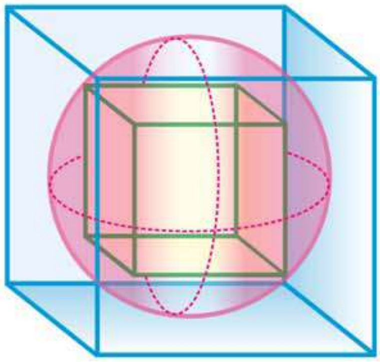
Next we shall define another class of sets in $\boldsymbol{R}^3$ (or $\boldsymbol{R}^2$) given by the following definition.

Definition 10: Let $\mathrm{a}=(a_1,a_2,a_3)\in \boldsymbol{R}^3$ and $r>0$. Define the set S given by
$S=\{x=(x_1,x_2,x_3)\in \boldsymbol{R}^3 : |x_1-a_1|<r, |x_2-a_2|<r, |x_3-a_3|<r\}$.
S is called an open cube in $\boldsymbol{R}^3$.

Similarly we can define open square in $\boldsymbol{R}^2$ analogous to open cubes in $\boldsymbol{R}^3$. Here is an interesting geometric relationship between open disc and open square (see Fig. 22 (a) and (b)). In the Fig. 22(a) you might have noticed that there is an open disc which contains an open square A_1 and contained in an open square A_2. Similarly in Fig. 22(b) there is an open sphere which contains an open cube and contained in an open cube. You can verify this with the definitions of open disc and open cube given by Fig. 21 (a) and Fig. 21 (b). We leave it as an exercise for you to verify.



(a) An open disc D containing an open square A_1 and contained in an open square A_2.



(b) Draw a cube in an open sphere.

Fig.22

Try these exercise now.

E21) Show that the open disc D with centre $\boldsymbol{a} = (a_1, a_2)$ and radius r in $\boldsymbol{R}^2$ contained in the open square A_1

$$A_1 = \{(x_1, x_2) : |x_1 - a_1| < r, |x_2 - a_2| < r\}$$

and contains the open square A_2

$$A_2 = \left\{(x_1, x_2) : |x_1 - a_1| < \frac{r}{\sqrt{2}}, |x_2 - a_2| < \frac{r}{\sqrt{2}}\right\}$$

E22) Show that the open sphere S with centre $\boldsymbol{a} = (a_1, a_2, a_3)$ and radius r in $\boldsymbol{R}^3$ is contained in the open cube C_1

$$C_1 = \{\boldsymbol{x} = (x_1, x_2, x_3) : |x_1 - a_1| < r, |x_2 - a_2| < r, |x_3 - a_3| < r\}$$

and contains the open cube C_2

$$C_2 = \left\{\boldsymbol{x} = (x_1, x_2, x_3) : |x_1 - a_1| < \frac{r}{\sqrt{3}}, |x_3 - a_3| < \frac{r}{\sqrt{3}}, |x_3 - a_3| < \frac{r}{\sqrt{3}}\right\}.$$

You are now familiar with certain subsets of Euclidean spaces $\boldsymbol{R}^3$ and $\boldsymbol{R}^2$ and their geometrical shapes. Now we shall turn our attention to functions defined on subsets of $\boldsymbol{R}^3$ and $\boldsymbol{R}^2$.

1.5 FUNCTIONS FROM R^n TO R $(n=2,3)$

In this subsection we shall introduce you to functions defined on the subsets of $\boldsymbol{R}^3$ (or $\boldsymbol{R}^2$). We shall familiarize you with the notions of graph, level curve and level surface for such functions.

You have already come across the definition of a function from one set to other set. Thus if X and Y are two non-empty sets, then a function from X to Y is a rule or correspondence which associates to each member for X, a unique member of Y. Here we shall consider functions for which X is a subset of $\boldsymbol{R}^3$ (or $\boldsymbol{R}^2$) and Y is a subset of $\boldsymbol{R}$. Such function are called **real-valued function of 3 variables (or 2 variables)**. In this unit we denote an element in $\boldsymbol{R}^3$ as (x, y, z) instead of (x_1, x_2, x_3). Similarly an element in $\boldsymbol{R}^2$ will be denoted (x, y) instead of (x_1, x_2). Both the notations are commonly used. Now we formally define it.

Definition 11: A real-value function f of three variables is a rule that assigns to each triple (x, y, z) in a set S of $\boldsymbol{R}^3$, a unique real number in $\boldsymbol{R}$ denoted by $f(x, y, z)$. The set S is the domain of f and its range is the set $\{f(x, y, z) : (x, y, z) \in S\}$. In a similar way, a function of two variables from $\boldsymbol{R}^2$ to $\boldsymbol{R}$ is defined. Sometimes we denoted a function f by $f(\boldsymbol{x}, \boldsymbol{y})$ to indicate that it is a two-variable function. Similarly for $\boldsymbol{R}^3$ also.

Note: (1) A function of **3 variables** means that the domain of the function is a subset of $\boldsymbol{R}^3$. Similarly for 2 variables it is a subset of $\boldsymbol{R}^2$.

(2) A real-valued function means that the range of the function is a subset of $\boldsymbol{R}$.

Let us see some examples

i) For $(x, y) \in \boldsymbol{R}^2$, define $f(x, y) = \sin x + \sin y$. Then f is a real-valued function of 2 variables defined on the whole of $\boldsymbol{R}^2$,

ii) Let S be the open sphere with centre at $(0,0,0)$ and radius 1 in $\boldsymbol{R}^3$. Then the function f defined by
$f(x, y, z) = \sqrt{1 - x^2 - y^2 - z^2}$
is a real-valued function of 3 variables with domain S.

iii) Let p_1, p_2 *and* p_3 denote the function from $\boldsymbol{R}^3$ to $\boldsymbol{R}$ defined by
$p_1(x, y, z) = x$
$p_2(x, y, z) = y$
$p_3(x, y, z) = z$
The functions p_1, p_2 *and* p_3 are real-valued functions defined on the whole of $\boldsymbol{R}^3$. They are called projection maps.

iv) Let $\mathrm{D} = [-1,1] \times [-1,1[$. For $(x, y) \in D$ define $f(x, y) = \sin^{-1} x \cos^{-1} y$. Then f is a real-valued function of two variables defined on D.

Apart from these, you will come across many examples of functions of 2 or 3 variables. Infact, such functions arise naturally in the study of real-life problems in all the areas. Some examples are given below.

1) The temperature T at a point on the surface of the earth at any given time on a particular place depend mainly on the longitude x and latitude y. In some study it has been represented by

   $T(x, y) = a x^2 y + b y$

   where a and b are constants.

   Here T is a real-valued function of two variables.

2) In a study on the growth of economy in a country, it was observed that the production output P is determined by the amount of labour L involved and the amount of capital C invested. The formula that was used for study was $P(L, K) = b L^{\alpha} K^{1-\alpha}$ where b and α are constants. Then P is a real-valued function of 2 variables L and K. The function P is called **Cobb-Douglas Production function**. This is an important function widely used in many situations, ranging from individual firms to global economic questions.

The examples mentioned above suggest that in some cases we can represent functions of two or three variables explicitly by algebraic expressions (or formulae). In some other cases one variable z will be given implicitly as a function of $z = f(x, y)$ by an equation of the form $F(x, y, z) = 0$. This means that $F(x, y, f(x, y) = 0$ for all (x, y) in the domain of f. Then F is called an **implicit function.**

Note: We often write $\boldsymbol{w = f(x, y, z)}$ to make explicit the value taken by a three variable function f at the generic point (x, y, z) and we write $z = f(x, y)$ for the value taken by a two variable function f at the generic point (x, y).

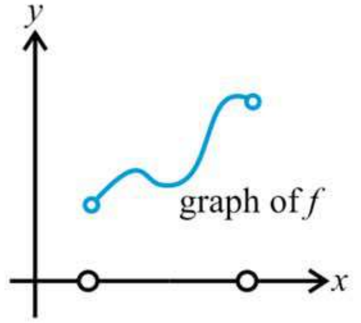
You know from the case of a one variable function that another way of visualizing the behaviour of a function f is to consider its graph. What is You know that for a one-variables function there is another way for visualizing the behaviour of a function by considering its graph. What is meant by graph of a two or three variable function? We shall now introduce you to the notions of graph and two other related notions- the level curves and level surfaces.

1.5.1 Level Sets and Graph of a Function

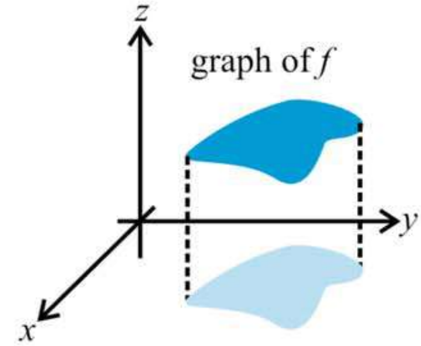
In this subsection we discuss the graph of a two variable function and introduce you to a related notion of level sets.

In the case of one variable, you know that, the graph of a function $f : \boldsymbol{R} \to \boldsymbol{R}$ is the subset of $\boldsymbol{R}^2$ consisting of all the points $(x, f(x))$ in the plane, $x \in \boldsymbol{R}$. The subset is usually considered as a curve in $\boldsymbol{R}^2$ as shown in Fig.23(a) in the margin. We represent it by $y = f(x)$.

Can you guess what will the graph of a two variable function be? You might have guessed that it is a surface in $\boldsymbol{R}^3$ as shown in Fig. 23 (b).



(a) Graph of a function of one variable



(b) Graph of a function of 2 variables

Fig. 23

Now we formally define the graph of a two-variable function.

Definition 12: If f is a function of two variables with domain D, then the graph of f is denoted by $G(f)$ is the set of all points (x, y, z) in $\boldsymbol{R}^3$ such that $z = f(x, y)$ i.e. $G(f) = \{(x, y, z) | z = f(x, y), (x, y) \in D\}$.

Let us now try to sketch the graphs of some simple functions.

Example 17: Sketch the graph of the function $f(x, y) = x - y + 2$.

Solution: By the definition, the graph of f is given by the equation

$$z = x - y + 2$$

You know that the equation represents a plane in $\boldsymbol{R}^3$. The graph is shown in Fig. 24 below

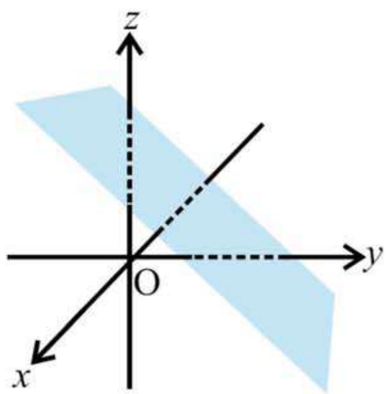


Fig.24: Graph of $f(x, y) = x - y + 2$

Example 18: Sketch the graph of the function $z = f(x, y) = x^2 + y^2$.

Solution: Suppose we set $z = c$, a constant. Then we get the circle $x^2 + y^2 = c$. Taking different values for c, say $1, 4, 9...$we get circles of radii $1, 2, 3$ respectively. The circles are actually sections of the surface $z = f(x, y)$ with the planes $z = 1^2, 2^2, 3^2.....$(see Fig. 25).

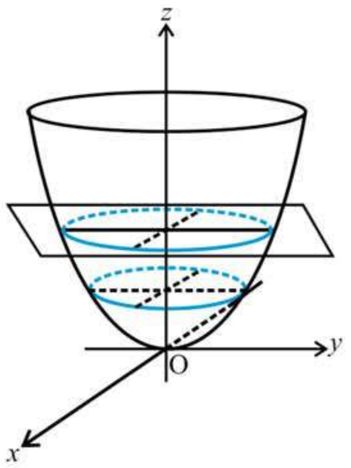


Fig. 25: Sections of the surface $z = x^2 + y^2$ by the plane $z = 1, 4$.

***

The example above suggests that in some cases if we consider the planar intersection of the surface with planes $z = k$, then we can visualize the shape of the surface. Are you convinced about this?

Try the following exercise

E23) Sketch the graph of the following functions:

i) $f(x, y) = 3x$

ii) $f(x, y) = x^2 - y^2$.

While attempting (i) and (ii) above, you might have observed that in the case of (i), the plane intersection with $z = k$ are lines and for (ii) the planar intersections are hyperbolas. You recall that you are already familiar with the three conics viz; Parabolas, ellipse and hyperbola from the Calculus course or even earlier at $10 + 2$ level.

The planar intersections which we discussed above are important and are given a special name. They are called level curves.

Definition (Level curve) 13: Let f be a function of two variables and c be a constant. The set of all (x, y) in the plane such that $f(x, y) = c$ is called a level curve of f.

This is essentially the orthogonal projection of the intersection of the surface $z = f(x, y)$ with the plane $z = c$ onto the $xy-$plane.

In the discussion above you have seen that the level curves of the $f(x, y) = x^2 + y^2$ are circles and that of $f(x, y) = x^2 - y^2$ are hyperbolas. Thus a graph of a two variable function can be obtained by piling up the level curves $f(x, y) = c$.

One common example of level curves occurs in topographic maps of mountain regions. The figure on next page (see Fig. 26) shows a weather map of the world indicating the average January temperature. The level curves are the curves that separate coloured bands.

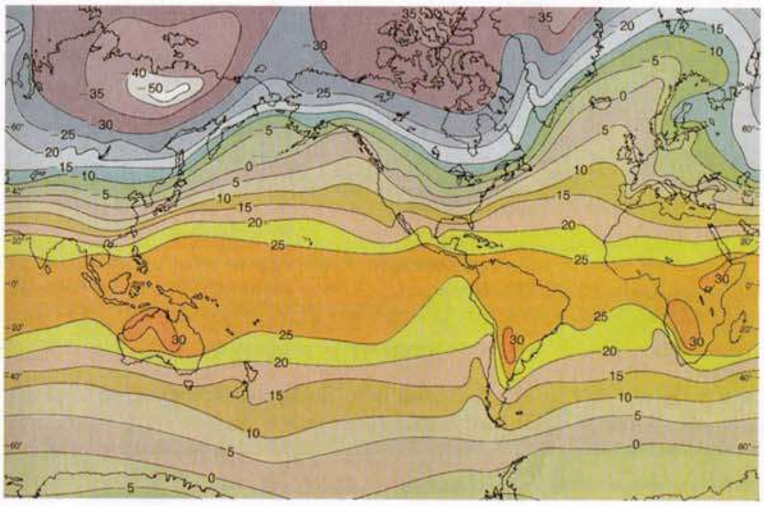


Fig. 26

So far we observed that if we draw the level curves, then we can visualize the graphs of a two-variable function.

We shall now consider graphs of three variable functions. According to the definition of a graph, it will be surface in the space $\boldsymbol{R}^4$ which we cannot visualize. But, if we consider the analogue of level curve and name it a level surface in $\mathbb{R}^3$ that might help us to visualize the graph. Let us see.
Here is a three variable function $f(x, y, z) = x - y + z + 2$. Now we have to consider $f(x, y, z) = c$ i.e.

$$x - y + z + 2 = c$$

i.e. $$x - y + z = c - 2$$

If we put different values for c, say $c = 1, c = 2$ and $c = 3$ etc, we get planes as shown in Fig. 27

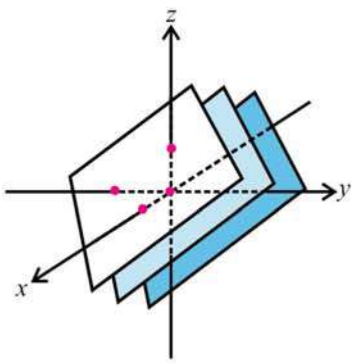


Fig. 27

Therefore we visualize the graph of this function as a surface obtained by piling up these planes in the four dimensional space.

These planes are known as the **level surfaces** of the function $f(x, y, z) = x - y + z + 2$.

We formally define it.

Definition 14: Let f be a function of three variables. The set of all points (x, y, z) in $\boldsymbol{R}^3$ such that $f(x, y, z) = c$ is called the level surface of f (with value c).

Let us see some examples

Example 19: Find the level surface of $f(x, y, z) = x^2 + y^2 + z^2 - 8$ for the values $c = 1, 8, 17$ etc. and sketch any of them.

Solution: The level surface for $c = 1$ is given by the equation

$$x^2 + y^2 + z^2 - 8 = 1$$

i.e. $\quad x^2 + y^2 + z^2 = 9.$

This is a sphere with radius 3.

Similarly for $c = 8$, we get the surface as

$$x^2 + y^2 + z^2 - 8 = 8$$

i.e. $\quad x^2 + y^2 + z^2 = 16$

This is also a sphere of radius 4.

For $c = 17$ we get the sphere $x^2 + y^2 + z^2 = 25.$

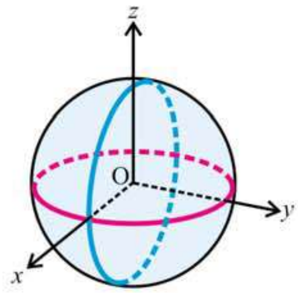


Fig. 28: Sphere of radius 3.

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Here are some exercises:

E24) Give a rough sketch of the level curves for the following functions for the values of c given alongside

i) $f(x, y) = x^2 + 4y^2, c = 0, 1, 4, 9$

ii) $f(x, y) = x / y, c = -3, -2, -1, 0, 1, 2, 3$

E25) Find the level surfaces of the given functions.

i) $f(x, y, z) = -x^2 - y^2 - z^2$

ii) $f(x, y, z) = x^2 + y^2$

With this we come to an end of this section and the unit.

1.6 SUMMARY

In this unit we have

1. explained the one-one relationship between the points in 2D-plane and 3D-plane respectively with the elements of the set $\boldsymbol{R}^2$ and $\boldsymbol{R}^3$.

2. defined the direction cosines and direction ratio of a line.

3. Introduction the canonical form, one parametric form and two point from of equation of a line.

4. discussed the cylindrical and the spherical coordinate systems and explained the conversion between the cylindrical, spherical and Cartesian coordinate systems.

5. explained the term Cartesian products of sets of real numbers and discussed the algebraic structure of $\boldsymbol{R}^2$ and $\boldsymbol{R}^3$.

6. defined real-valued functions of 2 or 3 variables i.e. functions defined from $\boldsymbol{R}^2 \to \boldsymbol{R}$ and $\boldsymbol{R}^3 \to \boldsymbol{R}$.

7. defined the sum, product, quotient and composite of functions from $\boldsymbol{R}^2 \to \boldsymbol{R}$ and $\boldsymbol{R}^3 \to \boldsymbol{R}$.

8. introduced level curves, level surfaces, respectively, for functions of two and three variables.

1.7 SOLUTIONS/ANSWERS

E1) We first consider the point $(1,1,0)$. For that we first locate the point $A_1(1,0,0)$ and then $A_2(1,1,0)$. Since the z-coordinate is 0, the point lies on the x, y-plane. Therefore A_2 is the required location of the point.

Next we consider the point $(-4,-2,1)$. For that we locate the points $A_1(-4,0,0)$ then $A_2(-4,-2,0)$ and then $A_3(-4,-2,1)$ by measuring the corresponding distance on the x, y and z-axis respectively. Then A_3 is the location of the required point.

E2) The coordinates of A is $(-5,0,0)$, B is $(-5,3,0)$ and C is $(-5,3,6)$.

E3) P is the point closest to the xz-plane. Q is the point lying on the yz-plane.

E4) We use the formula $\sin^2\alpha = 1-\cos^2\alpha$. Then we have

$$\begin{aligned}\sin^2\alpha+\sin^2\beta+\sin^2\gamma &= (1-\cos^2\alpha)+(1-\cos^2\beta)+(1-\cos^2\gamma)\\ &= 3-(\cos^2\alpha+\cos^2\beta+\cos^2\gamma)\\ &= 3-1=2.\end{aligned}$$

E5) The direction cosines are given by $\cos\alpha = \dfrac{x}{OP}, \cos\beta = \dfrac{y}{OP}$ and

$$\cos\gamma = \frac{z}{OP}.$$

Here $P(x,y,z) = (2,2,-1)$. By the distance formula

$$OP = \sqrt{2^2+2^2+1} = \sqrt{9} = 3.$$

$$\therefore \cos\alpha = \frac{2}{3}, \cos\beta = \frac{2}{3}, \cos\gamma = \frac{-1}{3}.$$

E6) i) $$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

Here $(x_1, y_1, z_1) = (-1, 0, 1)$

$(x_2, y_2, z_2) = (1, 2, 3)$

$$\frac{x+1}{2} = \frac{y}{2} = \frac{z-1}{2}$$

ii) The equation of the line is

$$\frac{x-1}{\sqrt{3}/5} = \frac{y+1}{-\sqrt{6}/5} = \frac{z-4}{4/5}$$

$$\frac{x-1}{\sqrt{3}} = \frac{y+1}{-\sqrt{6}} = \frac{z-4}{4}.$$

E7) The equation of the plane passing through the first three points is

$$\begin{vmatrix} x & y & z & 1 \\ 0 & -1 & -1 & 1 \\ 4 & 5 & 1 & 1 \\ 3 & 9 & 4 & 1 \end{vmatrix} = 0$$

$$\Rightarrow 5x - 7y + 11z + 4 = 0$$

If we put $x = -4$, $y = 4$ and $z = 4$ in the expression on the L.H.S of the above equation we get 0. That means $(-4, 4, 4)$ satisfies the equation.

This shows that the four points are coplanar.

E8) The equation of the plane that passes through the points $P(1,3,2), Q(3,-1,6)$ and $R(5,2,0)$ is given by

$$\begin{vmatrix} x & y & z & 1 \\ 1 & 3 & 2 & 1 \\ 3 & -1 & 6 & 1 \\ 5 & 2 & 0 & 1 \end{vmatrix} = 0$$

$$\Rightarrow x\begin{vmatrix} 3 & 2 & 1 \\ -1 & 6 & 1 \\ 2 & 0 & 1 \end{vmatrix} - y\begin{vmatrix} 1 & 2 & 1 \\ 3 & 6 & 1 \\ 5 & 2 & 0 \end{vmatrix} + z\begin{vmatrix} 1 & 3 & 1 \\ 3 & 6 & 1 \\ 5 & 2 & 1 \end{vmatrix}$$

$$\Rightarrow x(3\times 6 - 2\times -3 + 1\times -12) - y(1\times -2 - 2\times -5 + 1\times -24)$$
$$+ z(1\times 4 - 3\times -2\times -2 + 1\times -24) = 0$$

$$\Rightarrow x(18 + 6 - 12) - y(-2 + 10 - 24) + z(4 + 6 - 24) = 0$$

$$\Rightarrow 12x + 16y - 14z = 0$$

$\Rightarrow 6x + 8y - 7z = 0$. This gives the equation of the plane.

E9) The cylindrical coordinates are given by the equations

$$r = \sqrt{x^2 + y^2} = \sqrt{3+1} = 2$$

$$\tan\theta = \frac{y}{x} = \sqrt{3} \Rightarrow \theta = \frac{4\pi}{3} + 2n\pi$$

$$z = 2$$

Therefore one set of cylindrical coordinates are given by $\left(2, \frac{4\pi}{3}, 2\right)$.

E10) i) The Cartesian coordinates are given by

$$x = r\cos\theta, y = r\sin\theta, z$$

Here $r = 3, \theta = \frac{\pi}{2}$ and $z = 1$

Therefore the required coordinates are $(0,3,1)$.

ii) **Hint:** It can be shown similarly that $x = z, y = \frac{-4\sqrt{3}}{2}$ and $z = 5$.

The required coordinates in Cartesian coordinates are $(2, 2\sqrt{3}, 5)$.

E11) i) The Cartesian coordinates are given by the equations $x = \rho\sin\phi\cos\theta, y = \rho\sin\phi\sin\theta, z = \rho\cos\phi$. Here $\rho = 1, \phi = 0, \theta = 0$. Substituting these values in the equations above, we get $x = 0, y = 0, z = 1$.

Hence the Cartesian coordinates are $(0,0,1)$.

ii) The required coordinates are given by

$$x = 2\times\frac{1}{\sqrt{2}}\times\frac{1}{2},\ y = 2\times\frac{1}{\sqrt{2}}\times\frac{1}{2}, z = 2\times\frac{1}{\sqrt{2}}$$

The coordinates are $\left(\frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{\sqrt{2}}, \sqrt{2}\right)$.

E12) The Cartesian coordinates are given by the equation $x = \rho\sin\phi\cos\theta, y = \rho\sin\phi\sin\theta, z = \rho\cos\phi$. The equation of the surface in spherical coordinate is $\rho = \sin\theta\sin\phi$. Then we have $y = \rho\sin\phi\sin\theta = \rho^2 = x^2 + y^2 + z^2$.

i.e. $x^2 + y^2 + z^2 - y = 0$

i.e. $x^2 + \left(y - \frac{1}{2}\right)^2 - \frac{1}{4} + z^2 = 0.$

This equation represents a sphere whose centre is $\left(0, \frac{1}{2}, 0\right)$ and radius is $\frac{1}{2}$.

E13) i) The sum $(1,0,0) + (0,1,0) = (1,1,0)$ and therefore $(1,1,0) + (0,0,1) = (1,1,1)$.

ii) $(0.5,1) + (-3,0) = (-2.5,1)$
$(1,0.5) + (0,-3) = (1,-2.5)$.

The coordinates are not equal.

E14) The additive inverse of $\frac{1}{2}$ is $\left(-\frac{1}{2}\right)$ and the additive inverse of $\left(-\frac{1}{3}\right)$ is $\frac{1}{3}$. Therefore the additive inverse of $\left(\frac{1}{2},-\frac{1}{3},0\right)$ is $\left(-\frac{1}{2},\frac{1}{3},0\right)$ since $\left(\frac{1}{2},-\frac{1}{3},0\right)+\left(-\frac{1}{2},\frac{1}{3},0\right)=(0,0,0)$.

E15) The additive inverse of $(-5x,4y,-z)$ is $(5x,-4y,z)$.
Therefore $(5x,-4y,z)+(0,0,0)=(5x,-4y,z)$.

E16) i)
$$\begin{aligned} P-Q &= (2,2,-1)-(5,-3,2)\\ &= (-3,5,-3)\\ Q-P &= (5,-3,2)-(2,2,-1)\\ &= (3,-5,3)\end{aligned}$$
The coordinates are not equal.

E17) i)
$$\begin{aligned}\alpha\{(x_1,y_1,z_1)+(x_2,y_2,z_3)\} &= \alpha\{x_1+x_2,y_1+y_2,y_3+z_3\}\\ &= (\alpha(x_1+x_2),\alpha(y_1+y_2),\alpha(y_3+z_3))\\ &= (\alpha x_1+\alpha x_2,\alpha y_1+\alpha y_2,\alpha y_3+\alpha z_3)\\ &= (\alpha x_1,\alpha y_1,\alpha z_1)+(\alpha x_2,\alpha y_2,\alpha z_3)\\ &= \alpha(x_1,y_1,z_1)+\alpha(x_2,y_2,z_2)\end{aligned}$$

ii) and (iii) can be similarly tried

iv) Suppose that $(x,y,z)\,\forall\, x\in \boldsymbol{R}^3$. Choose $x=(1,0,0)$.
Then $ax=0\Rightarrow a(1,0,0)=0=(a,0,0)\Rightarrow a=0$
Consider $(x,y,z)\in \boldsymbol{R}^3$. Then $a(x,y,z)=(ax,ay,az)=0$, since $a=0$.

E18) To show that the triangle with vertices P, Q and R is an equilateral triangle, we have to show

$$\|P-Q\|=\|Q-R\|=\|R-P\|.$$

By the distance formula

$$\begin{aligned}\|P-Q\|^2 &= 3^2+2^2+1^2=14\\ \|Q-R\|^2 &= 2^2+1^2+3^2=14\\ \|R-P\|^2 &= 1^2+3^2+2^2=14\end{aligned}$$

Hence the triangle is an equilateral triangle.

E19) Let $\boldsymbol{x}=(x_1,x_2,x_3)$ and $\boldsymbol{y}=(y_1,y_2,y_3)$
$$\begin{aligned}\|x+y\|^2 &= \sum_{i=1}^{3}(x_i+y_i)^2\\ &= \sum_{i=1}^{3}x_i^2+\sum_{i=1}^{3}y_i^2+2\sum_{i=1}^{n}x_iy_i\end{aligned}$$

$$\leq \sum_{i=1}^{3} x_i^2 + \sum_{i=1}^{3} y_i^2 + 2\sqrt{\sum_{i=1}^{3} x_i^2}\sqrt{\sum_{i=1}^{3} y_i^2}$$ (in view of Cauchy's inequality)

Consequently

$$\|x+y\|^2 \leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\|$$

i.e. $\|x+y\|^2 \leq (\|x\|+\|y\|)^2$

or $\|x+y\| \leq \|x\|+\|y\|$

"$\Leftrightarrow$" stands for imply

"$\Leftarrow$" stands for implied by

E20) i) Let $x=(x_1, x_2, x_3)$ and $y=(y_1, y_2, y_3)$

$$\|x-y\|^2 = 0 \Leftrightarrow \sum_{i=1}^{3}(x_i - y_i)^2 = 0$$
$$\Leftrightarrow (x_i - y_i) = 0 \ \ \forall\, i$$
$$\Leftrightarrow x_i = y_i \ \ \forall\, i$$
$$\Leftrightarrow \boldsymbol{x} = \boldsymbol{y}$$

ii) $\|x-y\|^2 = \sum_{i=1}^{3}(x_i - y_i)^2$

$$= \sum_{i=1}^{3}(y_i - x_i)^2$$
$$= \|y-x\|^2$$

or $\|x-y\| = \|y-x\|$.

E21) If $(x_1, x_2) \in S$, then we know that $\sqrt{(x_1-a_1)^2 + (x_2-a_2)^2 < r}$ and therefore,

$$|x_1 - a_1| = \sqrt{(x_1-a_1)^2} < r$$
$$|x_2 - a_2| = \sqrt{(x_2-a_2)^2} < r$$

That is, $(x_1, x_2) \in A_1$. This means $S \subset A_1$

Now, if $(x_1, x_2) \in A_2$, then $|x_1 - a_1| < \frac{r}{\sqrt{2}}, |x_2 - a_2| < \frac{r}{\sqrt{2}}$, and

therefore, $(x_1-a_1)^2 + (x_2-a_2)^2 < \frac{r^2}{2} + \frac{r^2}{2} = r^2$

That is, $(x, y) \in S$. Thus $A_2 \subset S$.

See if you can do these exercises now.

E22) $S = \{x_1, x_2, x_3) \in R^3 \| (x-a_1, y-a_2, z-a_3)| < r\}$

Now, $(x_1, x_2, x_3) \in S \Rightarrow \sqrt{(x_1-a_1)^2 + (x_2-a_2)^2 + (x_3-a_3)} < r$

$$\Rightarrow (x_1-a_1)^2 + (x_2-a_2)^2 + (x_3-a_3)^2 < r^2$$
$$\Rightarrow (x_1-a_1)^2 < r^2, (x_2-y_2)^2 < r^2 \text{ and } (x_3-a_3)^2 < r^2$$
$$\Rightarrow \|x_1-a_1\| < r, \|x_2-y_2\| < r, \|x_3-a_3\| < r$$

$$\Rightarrow (x_1, x_2, x_3) \in C_1$$
$$\Rightarrow S \subset C_1$$

Now, $(x_1, x_2, x_3) \in C_2 \Rightarrow \|x_1 - a_1\| < \frac{r}{\sqrt{3}}, \|x_2 - a_2\| < \frac{r}{\sqrt{3}}, \|x_3 - a_3\| < \frac{r}{\sqrt{3}}$

$$\Rightarrow \sqrt{(x_1 - a_1)^2 + (x_2 - a_2)^2 + (x_3 - a_3)^2} < \sqrt{\frac{r^2}{3} + \frac{r^2}{3} + \frac{r^2}{3}} < r$$
$$\Rightarrow |(x_1 - a_1, x_2 - a_2, x_3 - a_3| < r$$
$$\Rightarrow (x_1, x_2, x_3) \in S$$
$$\Rightarrow C_2 \subset S.$$

E23) i) The equation represents the plane given in Fig. 29

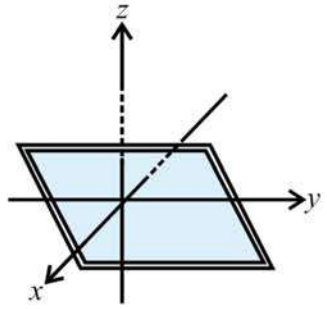
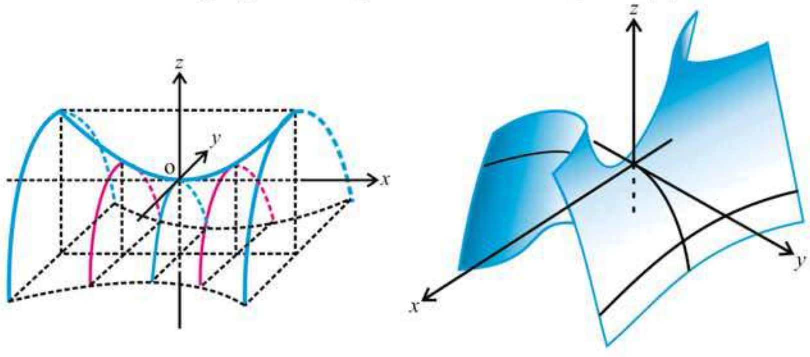


Fig. 29

ii) The equation is $z = x^2 - y^2$. By putting $x = k$, we get parabolas as shown in Fig. 30 (a). This figure is known as hyperbolic paraboloid. The graph is as given below in Fig. 30 (b)



(a) **(b)**

Fig. 30

E24) i) The level curves are shown in Fig. 31

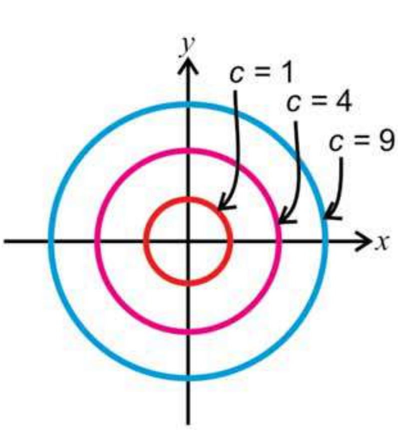


Fig. 31

ii) The level curves are shown in Fig. 32

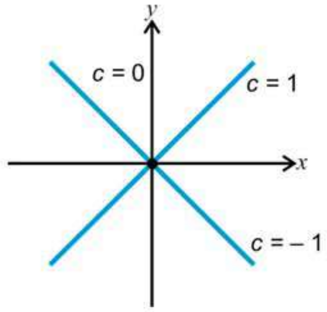


Fig. 32

E25) i) Level surfaces are spheres of radius $\sqrt{-c}$ if $c<0$; origin if $c=0$; and does not exist if $c>0$.

ii) Level surfaces are cylinders of radius $\sqrt{c}$ if $c>0$; z -axis if $c=0$; no level curve if $c<0$.

UNIT 2

LIMIT AND CONTINUITY

Structure — Page No.

2.1 INTRODUCTION

In the previous unit you have seen some examples of real-valued functions of several variables (i.e. 2 or 3 variables). In this unit we extend the notions of limit and continuity from one variable case to n variable case where $n = 2, 3$. We shall define these concepts for real-valued functions of 2 and 3 variables. You will see that the definitions of limit and continuity for functions of 2 and 3 variables are similar to those for functions of a single variable. In this unit, we shall also consider another type of limit called "repeated limit" for functions of 2 variables.

Wherever we are using the term "several variables" in this block, it means that we are considering only "2 or 3 variables".

Objectives

After reading this unit, you should be able to:

- define and evaluate limit of a real-valued function of 2 or 3 variables;
- state and use the rules of algebra of limits;
- check whether a function of several variables is continuous or not at a given point or at a set of points;
- evaluate the repeated limits for functions of two variables.

2.2 LIMITS OF REAL-VALUED FUNCTIONS

You are already familiar with the concept of limit for real-valued functions of one variable from Sec. 7.3, Unit 7 the of Calculus course. We shall now study this concept for functions of several variables i.e. for 2 and 3 variables.
Let us recall the definition of the limit, L, of a function f of a single variable x as x tends to real number a :

Definition 1: Let f be a real valued function of single variable x defined on an open interval $]a-h, a+h]$ around the point a, expect possibly at the point a, where $h>0$. We say that the limit of $f(x)$ as x tends to a is equal to a real number L, if given $\varepsilon>0$, there exists a positive real number δ (depending on ε), $\delta<h$, such that $0<|x-a|<\delta \Rightarrow |f(x)-L|<\varepsilon$.

From Definition 1 it is clear that the limit of a function f gives us an idea how the functional values $f(x)$ behaves as x approaches a particular value a. We extend this idea to functions of 2 and 3 variables.

To begin with let us consider two functions f and g of two variables x and y as given below and compare the behavior of $f(x, y)$ and $g(x, y)$ as (x, y) approaches $(0,0)$ i.e. as $x \to 0$ and $y \to 0$ simultaneously.

$$f(x, y)=\frac{\sin(x^2+y^2)}{x^2+y^2}((x, y) \neq (0,0))$$

$$g(x, y)=\frac{x^2-y^2}{x^2+y^2}(x, y) \neq (0,0)$$

Table 1 Values of $f(x, y)$

$x \backslash y$	−1.0	−0.5	−0.2	0	0.2	0.5	1.0
−1.0	0.455	0.759	0.829	0.841	0.829	0.759	0.455
−0.5	0.759	0.959	0.986	0.990	0.986	0.959	0.759
−0.2	0.829	0.986	0.999	1.000	0.999	0.986	0.829
0	0.841	0.990	1.000		1.000	0.990	0.841
0.2	0.829	0.986	0.999	1.000	0.999	0.986	0.829
0.5	0.759	0.959	0.986	0.999	0.986	0.959	0.759
1.0	0.455	0.759	0.829	0.841	0.829	0.759	0.455

Table 2 Values of $g(x, y)$

$x \backslash y$	−1.0	−0.5	−0.2	0	0.2	0.5	1.0
−1.0	0.000	0.600	0.923	1.000	0.923	0.600	0.000
−0.5	−0.600	0.000	0.724	1.000	0.724	0.000	−0.600
−0.2	−0.923	−0.724	0.000	1.000	0.000	−0.724	−0.923
0	−1.000	−1.000	−1.000		−1.000	−1.000	−1.000
0.2	−0.923	−0.724	0.000	1.000	0.000	−0.724	−0.923
0.5	−0.600	0.000	0.724	1.000	0.724	0.000	−0.600
1.0	0.000	0.600	0.923	1.000	0.923	0.600	0.000

Table 1 and 2 show values of $f(x, y)$ and $g(x, y)$ as (x, y) approaches $(0,0)$. Note that neither functions are defined on $(0,0)$. From Table 1 we guess that as (x, y) approaches $(0,0)$, $f(x, y)$ approaches 1 whereas the values of $g(x, y)$ are not approaching any number. Based on these numerical evidences we write that

$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = 1$ and $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ does not exist.

In general if f is a function of 2 or 3 variables, then we use the following notation to indicate that $f(x,y)$ approaches L as (x,y) approaches (a,b) or that the limit of $f(x,y)$ is L as (x,y) tends to (a,b).

$f(x,y) \to L$ as $(x,y) \to (a,b)$ or $\lim_{(x,y)\to(a,b)} f(x,y) = L$

The notation above also means that we can make the values of $f(x,y)$ as close to L as we like by taking the point (x,y) sufficiently close to the point (a,b). That is the distance between $f(x,y)$ and L can be made sufficiently small by making the distance between (x,y) and (a,b) sufficiently small. Recall here that the distance between two points $\boldsymbol{x} = (x_1, x_2)$ and $\boldsymbol{y} = (y_1, y_2)$ is denoted $\|\boldsymbol{x} - \boldsymbol{y}\|$ which is defined by

$$\|\boldsymbol{x} - \boldsymbol{y}\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

This is the Euclidean distance in $\boldsymbol{R}^2$. Similarly we can define the Euclidean distance in $\boldsymbol{R}^3$ also.

The distance between two points $\boldsymbol{x} = (x_1, x_2, x_3)$ and $\boldsymbol{y} = (y_1, y_2, y_3)$ is denoted by $\|\boldsymbol{x} - \boldsymbol{y}\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}$.

With the notion of distance we can define the following subsets of $\boldsymbol{R}^n$ for $n = 2$ or 3 as $B(a,h) = \{\boldsymbol{x} \mid \boldsymbol{x} \in \boldsymbol{R}^n, \|\boldsymbol{x} - a\| < h\}, n = 2$ or 3.

For $n = 2$, the set, $B(\boldsymbol{a},h)$ consists of all the points in the plane, whose distance from $\boldsymbol{a}$ is less than h. In other words, this is the set of points lying inside the circle of radius h with centre at a.

For $n = 3$, the set will consist of all the points in space, lying inside the sphere of radius h with centre at a.

> Note that '$\|.\|$' is used to denote the distance in $\boldsymbol{R}^n$ for $n = 2,3$ and '$|.|$' is used to denote the distance in $\boldsymbol{R}$.

Please note that the set, $B(\boldsymbol{a},h)$ is called a **neighbourhood of the point $\boldsymbol{a}$**. It is also called the **h-neighbourhood of $\boldsymbol{a}$**.

Here we make a note:

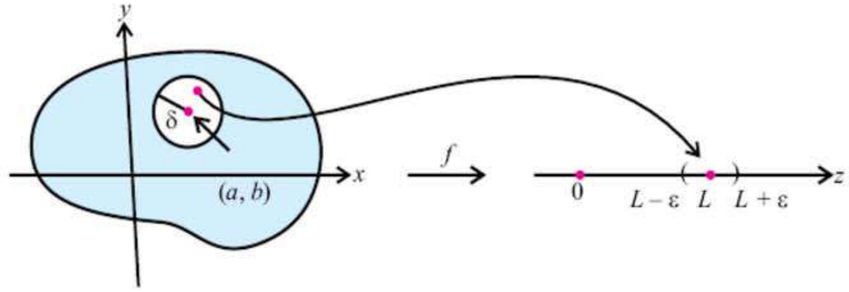
Note: Hence forth we will not specify the domain every time. It is understood that it is the largest possible set for which the expression for f makes sense. Now we formally define the limit.

> Note that the bold letters are used to denote the elements of $\boldsymbol{R}^n$, $n = 2,3$.

Definition 2: Let f be a real-valued function defined in a set $B(\boldsymbol{a},h)$ in $\boldsymbol{R}^n$ ($n = 2$ or 3). We say that the limit of $f(\boldsymbol{x})$ as $\boldsymbol{x}$ tends to $\boldsymbol{a}$ is equal to a real number L, if given $\varepsilon > 0$, there exists a positive real number δ (depending on ε), $\delta < h$, such that

$$0 < \| \boldsymbol{x} - \boldsymbol{a} \| < \delta \Rightarrow | f(\boldsymbol{x}) - L | < \varepsilon. \quad \dots (1)$$

In the definition notice that $|f(x)-L|$ is the distance between the numbers $f(\boldsymbol{x})$ and L in $\boldsymbol{R}$ and $\|\boldsymbol{x}-\boldsymbol{a}\|=\sqrt{(x_1-a_1)^2+(x_2-a_2)^2}$ is the distance between the numbers $\boldsymbol{x}$ and $\boldsymbol{a}$ in $\mathbb{R}^2$ where $\boldsymbol{x}=(x_1,x_2)$ and $\boldsymbol{a}=(a_1,a_2)$. The following figure Fig. 1 illustrates the definition. The figure shows that if any small interval $(L-\varepsilon, L+\varepsilon)$ is given around L, then we can find a disc D_δ with centre $\boldsymbol{a}$ and radius $\delta>0$ such that f maps all the points in D_δ (except possibly at $\boldsymbol{a}$) into the interval $(L-\varepsilon, L+\varepsilon)$.



(a) Shows values of (x, y) in $\boldsymbol{R}^2$ sufficiently close to (a,b).

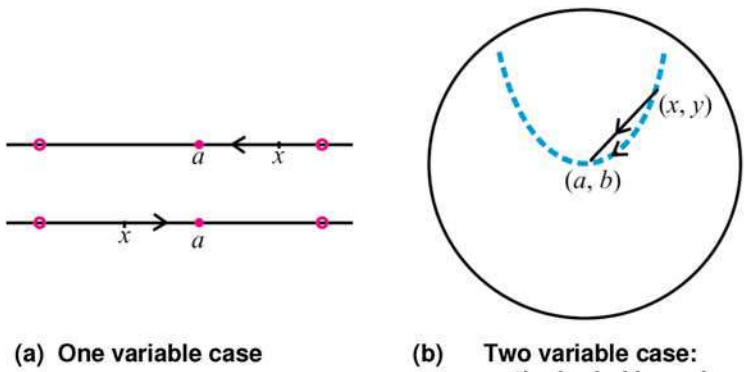
(b) Shows the values of $f(x, y)$ in $\boldsymbol{R}$ close to L

Fig. 1

You might have noticed that this definition is almost similar to the definition of the limit of a function of a single variable. But there are some differences. We explain that in the case of a function of single variable, when x approaches a, there are only two possible directions of approach, from left or from right as shown in Fig. 2 (a) which are denoted by $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$.

From the Calculus course you have learnt that if $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ are not equal, then $\lim_{x\to a} f(x)$ does not exist.

In the case of functions of 2 or 3 variables the situation is not so simple. This is because here (x, y) can approach (a,b) from an infinite number of directions. Fig. 2(b) shows two different paths, a line segment and an arc for $n=2$.



(a) One variable case

(b) Two variable case: path shaded by red path shaded by blue

Fig. 2

Therefore if the limit exists then $f(x, y)$ must approach L in whatever path or directions (x, y) approach (a,b). In other words if $f(x, y)$ approaches two different values along any two different paths, then we can say that the limit does not exist. This fact is very useful in checking the non-existance of a limit. We shall explain more about this little later.

Remark 1:

i) In general, the value of δ will depend on the value of ε.

ii) In Definition 1, the condition, $0 < \| \boldsymbol{x} - \boldsymbol{a} \|$ means that the distance between $\boldsymbol{x}$ and $\boldsymbol{a}$ is greater than zero. That is, the point $\boldsymbol{x}$ is not the same as $\boldsymbol{a}$.

We have already defined a neighbourhood of a point in $\boldsymbol{R}^n$ for $n = 2, 3$. If we remove the point a from a neighbourhood of a, the remaining set is called a **deleted neighbourhood of** a.

Before we discuss examples for computation of limits, we state a theorem about the algebra of limits. This will be very useful for calculating the limits of functions of several variables. You may recall that you had studied and used a similar theorem for the limits of functions of a single variable. We won't give the proof of the theorem here.

Theorem 1 (Algebra of Limits): Let f and g be two real-valued functions defined in a deleted neighbourhood of a point $\boldsymbol{a}$ in $\boldsymbol{R}^n$ where $n = 2$ or 3. If $\lim_{x \to a} f(\boldsymbol{x}) = L$ and $\lim_{x \to a} g(\boldsymbol{x}) = M$, then the following holds:

i) $\lim_{x \to a} (\alpha f)(\boldsymbol{x}) = \alpha \lim_{x \to a} f(\boldsymbol{x}) = \alpha L$ for any $\alpha \in \boldsymbol{R}$

ii) $\lim_{x \to a} (f \pm g)\ (\boldsymbol{x}) = L \pm M$

iii) $\lim_{x \to a} (fg)\ (\boldsymbol{x}) = LM$

iv) $\lim_{x \to a} (f / g)\ (\boldsymbol{x}) = \dfrac{L}{M}$, provided $M \neq 0$.

– ■ –

We shall now prove a simple result which makes the computation of limits easier. Here we take $n = 3$, a similar result holds for $n = 2$ also.

Theorem 2: Let f be a real-valued function defined in a deleted h-neighbourhood of a point $\boldsymbol{a}$ of $\boldsymbol{R}^3$. Then $\lim_{x \to a} f(x) = L$, if and only if given $\varepsilon > 0$, there exist positive real numbers $\delta_1, \delta_2, \delta_3$ (which depend on ε), $\delta_i < h$, $1 \le i \le 3$, such that whenever $0 < |x_i - a_i| < \delta_i, \forall\ i = 1,2,3,$ then $|f(\boldsymbol{x}) - L| < \varepsilon$, where $\boldsymbol{a} = (a_1, a_2, a_3)$ and $\boldsymbol{x} = (x_1, x_2, x_3)$.

Proof: Let $\lim_{x \to a} f(\boldsymbol{x}) = L$. Then given $\varepsilon > 0$ there exists a real number $\delta > 0$, $\delta < h$, such that $0 < \| x - a \| < \delta \Rightarrow | f(x) - L | < \varepsilon$.

Let $\delta_i = \delta / \sqrt{3}, 1 \le i \le 3$.

Now, if for any point $\boldsymbol{x} = (x_1, x_2, x_3)$, we have $|x_i - a_i| < \delta_i = \delta / \sqrt{3}$ for all i, such that $1 \le i \le 3$, then

$$\|\boldsymbol{x} - \boldsymbol{a}\| = \sqrt{(x_1 - a_1)^2 + (x_2 - a_2)^2 + (x_3 - a_3)^2} < \sqrt{\frac{\delta^2}{3} + \frac{\delta^2}{3} + \frac{\delta^2}{3}} = \delta, \text{ and}$$

therefore whenever $0 < |x_i - a_i| < \delta_i, \forall i$ such that $1 \le i \le 3$, then $|f(x) - L| < \varepsilon$.

Conversely, suppose that the given condition is satisfied.
Let $\delta = \min\ \{\delta_1, \delta_2, \delta_3\}$.

Then, $0 < \| x - a \| < \delta \Rightarrow 0 < |x_i - a_i| = \sqrt{(x_i - a_i)^2} \leq$

$\sqrt{(x_1 - a_1)^2 + (x_2 - a_2)^2 + (x_3 - a_3)^2} < \delta \leq \delta_i, \forall i$ such that $1 \leq i \leq 3$.

This means that $\lim\limits_{x \to a} f(x) = L.$

Hence the result.

– ■ –

We now apply Theorems 1 and 2 to calculate the limits in the following example.

Example 1: Show that

i) $\lim\limits_{\substack{x \to 0 \\ y \to 2}} (x^2 + y) = 2$

ii) $\lim\limits_{\substack{x \to 0 \\ y \to 0}} \left(x \sin \frac{1}{y} + y \sin \frac{1}{x} \right) = 0$

iii) $\lim\limits_{(x,y,z) \to (a,b,c)} x = a$

$\lim\limits_{(x,y,z) \to (a,b,c)} y = b$

$\lim\limits_{(x,y,z) \to (a,b,c)} z = c$

iv) $\lim\limits_{\substack{x \to 2 \\ y \to 3}} (x^2 + xy + y^3) = 37.$

v) $\lim\limits_{(x,y,z) \to (a,b,c)} (xy + yz + zx) = ab + bc + ca$

Solution: We'll use Theorem 2 to check the limits in i) and ii).

i) The given function is $f(x, y) = x^2 + y$. Let $0 < \varepsilon < 1$ be given. Then $|f(\mathbf{x}) - L| = |x^2 + y - 2| \leq |x^2| + |y - 2| < \varepsilon$ whenever $|x| < \sqrt{\varepsilon / 2}$ and $|y - 2| < \varepsilon / 2$. Thus, $\lim\limits_{\substack{x \to 0 \\ y \to 2}} (x^2 + y) = 2$, in view of Theorem 2.

Note that here we have taken $\delta_1 = \sqrt{\varepsilon / 2}, \delta_2 = \varepsilon / 2.$

ii) The function f is given by $f(x, y) = x \sin \frac{1}{y} + y \sin \frac{1}{x}$.

Then, we have $\left| x \sin \frac{1}{y} + y \sin \frac{1}{x} \right| \leq |x| + |y| < \varepsilon,$

whenever $|x| < \varepsilon / 2, |y| < \varepsilon / 2.$

Thus, applying Theorem 2 again, we can say that

$$\lim_{\substack{x \to 0 \\ y \to 0}} \left(x \sin \frac{1}{y} + y \sin \frac{1}{x} \right) = 0.$$

We will use the algebra of limits (Theorem 1) to check the limits in iii) and iv).

iii) We first prove that $\lim\limits_{(x,y,z)\to(a,b,c)} x = a$. Observe that $|x-a| \le \|(x,y,z)-(a,b,c)\|$. So, for a given $\varepsilon > 0$, if we take $\delta = \varepsilon$, then a direct use of Definition 1 gives us the required limit.

In a similar manner we can prove that $\lim\limits_{(x,y,z)\to(a,b,c)} y = b$, and $\lim\limits_{(x,y,z)\to(a,b,c)} z = c$.

iv) Using the algebra of limits we get that

$$\lim_{\substack{x\to 2\\ y\to 3}}(x^2+xy+y^3) = \lim_{\substack{x\to 2\\ y\to 3}} x^2 + \lim_{\substack{x\to 2\\ y\to 3}} xy + \lim_{\substack{x\to 2\\ y\to 3}} y^3$$
$$= \lim(x).\lim(x) + \lim(x).\lim(y) + \lim(y).\lim(y).\lim(y) = 4+6+27 = 37.$$

v) Now, using algebra of limits, we get

$$\lim_{(x,y,z)\to(a,b,c)}(xy+yz+zx) = \lim_{(x,y,z)\to(a,b,c)} xy + \lim_{(x,y,z)\to(a,b,c)} yz + \lim_{(x,y,z)\to(a,b,c)} zx$$
$$= \lim x.\lim y + \lim y.\lim z + \lim z.\lim x$$
$$= ab + bc + ca$$

***

Next we shall observe an important fact related to limits. You recall that a single variable function is said to be bounded on a set D of $\boldsymbol{R}$ if there exist real numbers m and M such that $m \le f(x) \le M$ for every $x \in D$. In a similar way a function f of two variables x and y is said to be bounded on a set W in $\boldsymbol{R}^2$, if there exists real numbers m and M such that $m \le f(x,y) \le M$ for all $(x,y) \in W$.

Similarly we can define bounded sets in $\boldsymbol{R}^3$ also.

We shall now prove the following result.

Theorem 3: If $\lim\limits_{x\to a} f(\boldsymbol{x}) = L$, then $f(\boldsymbol{x})$ is bounded in a deleted neighbourhood of a. That is, there exist real numbers m and M such that $m \le f(\boldsymbol{x}) \le M$ for all $\boldsymbol{x}$ in a deleted neighbourhood of $\boldsymbol{a}$.

Proof: We use Definition 2. By the definition $\lim\limits_{x\to a} f(\boldsymbol{x}) = L$ implies that given some $\varepsilon > 0$, $\exists\, \delta > 0$, such that

$$0 < \| \boldsymbol{x} - \boldsymbol{a} \| < \delta \Rightarrow | f(\boldsymbol{x}) - L | < \varepsilon$$
$$\Rightarrow L - \varepsilon < f(\boldsymbol{x}) < L + \varepsilon.$$

Now take $m = L - \varepsilon$ and $M = L + \varepsilon$. Then we get that f is bounded in the neighbourhood $B(a,\delta)$, except possibly at $\boldsymbol{x} = \boldsymbol{a}$ (see Remark 1(ii)). This shows that whenever $\lim\limits_{x\to a} f(x)$ exists, then f is bounded in a deleted neighbourhood of $\boldsymbol{a}$.

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The converse of this theorem is not true. In other words, if a function f is bounded in a deleted neighbourhood of some point a, it is not necessary that the limit of the function exists at $\boldsymbol{a}$. In the following example you will see a function which supports this statement.

Example 2: Show that the following functions are bounded but $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

i) $$f(x,y)=\begin{cases}1, & x\neq 0\\ 0, & x=0\end{cases}$$

ii) $$f(x,y)=\begin{cases}1, & x \text{ irrational}\\ 0, & x \text{ rational}\end{cases}$$

Solution: Let us take (i) and (ii) one by one.

i) We first note that $0\le f(x,y)\le 1$ i.e. f is bounded. Next we prove that the limit does not exist. On the contrary, let us suppose that $\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y)=L.$

Then, for a given ε, such that $0<\varepsilon<1, \exists$ a real number $\delta>0$ such that $0<\|(x,y)\|=\sqrt{x^2+y^2}<\delta \Rightarrow |f(x,y)-L|<\varepsilon/2$.

In particular, if $(x_1,y_1),(x_2,y_2)$ are two points with $\sqrt{x_1^2+y_1^2}<\delta, \sqrt{x_2^2+y_2^2}<\delta,$ then we get

$$\begin{aligned}|f(x_1,y_1)-f(x_2,y_2)| &= |f(x_1,y_1)-L+L-f(x_2,y_2)|\\ &\le |f(x_1,y_1)|-L|+|f(x_2,y_2)-L|\\ &<\varepsilon/2+\varepsilon/2=\varepsilon. \qquad \text{... (2)}\end{aligned}$$

Now consider the points $(x_1,y_1)=\left(\frac{\delta}{2},0\right)$ and $(x_2,y_2)=\left(0,\frac{\delta}{2}\right)$.

Then $f(x_1,y_1)=f\left(\frac{\delta}{2},0\right)=1$, and $f(x_2,y_2)=f\left(0,\frac{\delta}{2}\right)=0$. Thus, $|f(x_1,y_1)-f(x_2,y_2)|=1>\varepsilon$.

This contradicts (2). Therefore our assumption that the limit exists cannot be true. Hence we conclude that $\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y)$ does not exist.

Let us consider the second function now.

ii) Here also we first note that f is bounded and $0\le f(x,y)\le 1$. We start with the assumption that $\lim\limits_{\substack{x\to a\\ y\to b}} f(x,y)$ exists. Then, proceeding as in i), we see that given any ε such that $0<\varepsilon<1$, there exists a real number $\delta>0$ such that, whenever $(x_1,y_1),(x_2,y_2)$ belong to the neighbourhood B_δ of (a, b) with radius δ ,we have

$$|f(x_1,y_1)-f(x_2,y_2)|<\varepsilon \qquad \text{... (3)}$$

Now we choose $(x_1,y_1),(x_2,y_2)$ in this neighbourhood B_δ, so that x_1 is irrational and x_2 is rational. Then $f(x_1,y_1)=1$ and $f(x_2,y_2)=0$. Thus, $|f(x_1,y_1)-f(x_2,y_2)|=1>\varepsilon$.

This contradicts (3). Hence our assumption that the limit exists is not possible. Therefore we conclude that $\lim_{\substack{x\to a\\ y\to b}} f(x,y)$ does not exist.

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As we remarked earlier when we are talking about limits of functions in $\boldsymbol{R}^2$ or $\boldsymbol{R}^3$, there are infinitely many ways in which a point $\boldsymbol{x}$ can approach another point $\boldsymbol{a}$. So, there are infinitely many ways of calculating limits along these different paths. But if the limit of $f(x)$ as x tends to a exists, then all these different limits exist and are all equal.

We prove the following result.

Theorem 4: Let f be a real-valued function of two variables such that $\lim_{(x,y)\to(a,b)} f(x,y) = L.$ If $\phi(x)$ is a real-valued function of a real variable such that $\lim_{x\to a}\phi(x) = b$, then $\lim_{x\to a} f(x,\phi(x)) = L.$

Proof: Let $\varepsilon > 0$. Then there exists a real number $\delta > 0$ such that $f(x,y)$ is defined in the δ-neighbourhood of (a,b), except possibly at (a,b), and

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \Rightarrow |f(x,y) - L| < \varepsilon.$$

Since $\lim_{x\to a}\phi(x) = b$, for this $\delta > 0$, there exists a real number $\delta_1 > 0$, $\delta_1 < \frac{\delta}{\sqrt{2}}$, such that $\phi(x)$ is defined for all x, such that $0 < |x-a| < \delta_1$, and

$$0 < |\mathrm{x} - \mathrm{a}| < \delta_1 \Rightarrow |\phi(\mathrm{x}) - \mathrm{b}| < \frac{\delta}{\sqrt{2}}.$$

Thus,

$$0 < |x-a| < \delta_1 \Rightarrow \sqrt{(x-a)^2 + (\phi(x)-b)^2} < \sqrt{\delta_1^2 + \frac{\delta^2}{2}} < \sqrt{\frac{\delta^2}{2} + \frac{\delta^2}{2}} = \delta, \text{and}$$

therefore, $|f(x,\phi(x)) - L| < \varepsilon$ i.e., $\lim_{x\to a} f(x,\phi(x)) = L.$

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Remark 2: The theorem above says that if $\lim_{(x,y)\to(a,b)} f(x,y)$ exists and is equal to L, then $f(x,y)$ approaches L, as (x, y) approaches (a,b) along any path. That means, if $\lim_{(x,y)\to(a,b)} f(x) = L$ exists, then the limit of f as (x,y) tends to (a,b) along any curve $y = \phi(x)$ is L. In other words, the limit of f is independent of the path along which the point (x,y) approaches the point (a,b). As we have observed earlier this result can also be interpreted as follows:

If $f(x,y)$ tends to two different limits as $(x,y)\to(a,b)$ along two different paths, then $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

This interpretation is very useful in proving the non-existence of certain limits. We'll now state this more precisely as a corollary to Theorem 4.

Corollary 1: Suppose $f(x, y)$ is a real-valued function defined in some deleted neighbourhood of the point (a,b). If there exist real-valued functions $\phi_1(x)$ and $\phi_2(x)$ such that $\lim_{x\to a}\phi_1(x) = b = \lim_{x\to a}\phi_2(x)$, and $\lim_{x\to a} f(x,\phi_1(x)) \neq \lim_{x\to a} f(x,\phi_2(x))$, then $\lim_{(x,y)\to(a,b)} f(x, y)$ does not exist.

We shall illustrate the usefulness of this corollary with the help of an example.

Example 3: Show that the limits of the following functions do not exist as $(x, y) \to (0,0)$

i) $f(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$

ii) $f(x, y) = \begin{cases} \dfrac{x^3 + y^3}{x - y}, & x \neq y \\ 0, & x = y \end{cases}$

Solution: i) Let $y = mx$. Then

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{x\to 0} \frac{x^2 - m^2x^2}{x^2 + m^2x^2} = \frac{1 - m^2}{1 + m^2}$$

Now the values of $\dfrac{1-m^2}{1+m^2}$ are different for different m. You can check by putting $m = 1$ and $m = 2$. This means that $f(x, y)$ approaches different values as $(x, y) \to (0,0)$ along different radial vectors. See Fig. 3. In the margin, (showing the lines for $m = 1$ and $m = 2$).

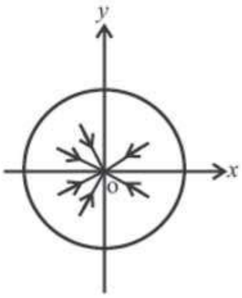

Fig. 3

Thus, the limit of $\dfrac{x^2 - y^2}{x^2 + y^2}$ does not exist as (x, y) tends to $(0,0)$, in view of Corollary1.

(Here we can take $\phi_1(x) = m_1x,\ \phi_2(x) = m_2\, x,\ m_1 \neq \pm m_2$)

ii) Let $\phi_1(x) = x - x^3$, $\phi_2(x) = x - x^2$. Note that $\lim_{x\to 0}\ \phi_1(x) = \lim_{x\to 0}\ \phi_2(x) = 0$.

Then

$$\lim_{x\to 0} f(x,\phi_1(x)) = \lim_{x\to 0} \frac{x^3 + (x - x^3)^3}{x^3} = \lim_{x\to 0} \frac{x^3 + x^3 - 3x^5 + 3x^7 - x^9}{x^3} = 2, \text{ and}$$

$$\lim_{x\to 0} f(x,\phi_2(x)) = \lim_{x\to 0} \frac{x^3 + (x - x^2)^3}{x^2} = 0.$$

Therefore, the limit of $f(x, y)$ does not exist as $(x, y) \to (0,0)$ by Corollary 1 again.

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Example 4: Let $f(x, y) = \dfrac{xy^2}{x^2 + y^4}$. Check whether $\lim_{(x,y)\to(0,0)} f(x, y)$ exist.

Solution: We shall first check the limit when $(x, y) \to (0,0)$ along different paths. Let us consider the path $y = mx$;

$$f(x, y) = f(x, mx) = \frac{m^2x^3}{x^2 + (mx)^4} = \frac{m^2x}{1 + m^4x^2}$$

Then $f(x, y) \to 0$ as $(x, y) \to (0,0)$ along $y = mx$.

This shows that f has the same limiting value along every non vertical line through the origin. But that does not show that the given limit is 0 for if we consider the path along the parabola $x = y^2$, we have

$$f(x, y) = f(y^2, y) = \frac{y^4}{2y^4} = \frac{1}{2}$$

Then $f(x, y) \to \frac{1}{2}$ as $(x, y) \to (0,0)$ along $x = y^2$. Since f takes different limiting values along different paths, we conclude that the $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ does not exist.

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We shall now explain you how the conversion to polar coordinates, i.e., the use of the substitution $x = r\cos\theta$, $y = r\sin\theta$ is useful for evaluation of certain limits.

Consider the following example.

Example 5: Prove that $\lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{x^3 - y^3}{x^2 + y^2} = 0.$

Solution: For this we'll make the substitution $x = r\cos\theta$, $y = r\sin\theta$, so that $x^2 + y^2 = r^2$.

Then

$$\left|\frac{x^3 - y^3}{x^2 + y^2}\right| = \left|\frac{r^3(\cos^3\theta - \sin^3\theta)}{r^2}\right|$$

$$= \left|r(\cos^3\theta - \sin^3\theta)\right| \le r(\left|\cos^3\theta\right| + \left|\sin^3\theta\right|) \le 2r = 2\sqrt{x^2 + y^2} \quad \text{(why ?)}$$

Now, if $|x| < \frac{\varepsilon}{\sqrt{8}}$ and $|y| < \frac{\varepsilon}{\sqrt{8}}$, then $2\sqrt{x^2 + y^2} < \varepsilon$, and therefore

$$\left|\frac{x^3 - y^3}{x^2 + y^2}\right| < \varepsilon.$$

That is, $\lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{x^3 - y^3}{x^2 + y^2} = 0.$

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In the next example we shall illustrate how $\varepsilon - \delta$ definition of limit is used to find the limit.

Example 6: Find the $\lim\limits_{(x,y)\to(0,0)} \frac{5x^2 y}{x^2 + y^2}$.

Solution: We apply Definition 2.

Let $\in > 0$. We want to find $\delta > 0$ such that

$\left|\frac{5x^2 y}{x^2 + y^2} - 0\right| < \varepsilon$ whenever $0 < \sqrt{x^2 + y^2} < \delta.$

That is $\frac{5x^2|y|}{x^2+y^2} - 0 < \varepsilon$ whenever $0 < \sqrt{x^2+y^2} < \delta.$

But $x^2 \le x^2 + y^2$ since $y^2 \ge 0$, so $\frac{x^2}{x^2+y^2} \le 1$ and therefore

$$\frac{5x^2|y|}{x^2+y^2} \le 5|y| = 5\sqrt{y^2} \le 5\sqrt{x^2+y^2}.$$

Now if we choose $\delta = \frac{\varepsilon}{5}$, then whenever $0 \le \sqrt{x^2+y^2} < \delta$, we get

$$\left|\frac{5x^2 y}{x^2+y^2} - 0\right| \le 5\sqrt{x^2+y^2} \le 5\,\delta = 5 \times \frac{\varepsilon}{5} = \varepsilon.$$

Hence by Definition 1, $\lim\limits_{(x,y)\to(0,0)} \frac{5x^2 y}{x^2+y^2} = 0.$

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Now see if you can solve these exercises.

E1) Show that

a) $\lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{xy}{\sqrt{x^2+y^2}} = 0$

b) $\lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{x^2 y^2}{\sqrt{x^2+y^2}} = 0$

by converting to polar coordinates.

E2) Use algebra of limits to show the following:

a) $\lim\limits_{(x,y,z)\to(0,1,2)} \frac{x^2 + 3xyz - 5z^2}{xy^3 + 5z^2 - 3xy + x^3} = -1$

b) $\lim\limits_{(x,y)\to(0,0)} \frac{x \sin y}{2x^2+1} = 0$

E3) Show that the limits of the following functions exist or not as $x \to 0,\ y \to 0$.

a) $f(x,y) = \frac{x^4}{x^4+y^2}$. Try the paths $y = x^2, y = 2x^2$.

b) $f(x,y) = \frac{x^3 y}{x^6+y^2}$. Try the paths $y = x^3, y = 2x^3$.

c) $f(x,y) = \frac{2x^2}{x^2 - y^2 + x}$. Try the paths $y = x, y^2 = x$.

d) $f(x,y) = \frac{x^4 - y^2}{x^4+y^2}$. Try the paths $y = 0, x = 0$, that is along the two axes.

E4) Check whether the limits of the following functions do not exist as $x \to 0,\ y \to 0$.

a) $\dfrac{xy}{x^2+y^2}$

b) $\dfrac{x^2}{x^2+y}$

c) $\dfrac{x^2-y^2}{x^2+y^2}+\dfrac{2xy}{\sqrt{x^2+y^2}}$

E5) If $f(x,y)=xy$ and $\varepsilon=0.0004$, show how close to the origin should we take the point (x,y) to make $|f(x,y)-f(0,0)|<\varepsilon$, for the given ε. In other words, find δ corresponding to $\varepsilon=0.0004$.

E6) Find $\lim\limits_{(x,y)\to(0,0)}\dfrac{2xy^2}{x^2+y^2}$ using $\varepsilon-\delta$ definition.

By now you must have become familiar with the concept of the limit of a function of several variables. We shall now discuss the continuity of these functions.

2.3 CONTINUITY OF REAL-VALUED FUNCTIONS

In Unit 2 of the Calculus course you saw that the knowledge of the limit of a function of one variable is necessary for studying the continuity of these functions.

In the last section we have studied the concept of limit for real-valued functions of several variables. Let us now see how we can use this knowledge to define continuous functions from $\boldsymbol{R}^n$ to $\boldsymbol{R}$ for $n=2$ and 3.

Recall that a function $f:\boldsymbol{R}\to\boldsymbol{R}$ is continuous at a point $a\in\boldsymbol{R}$, if $\lim\limits_{x\to a}f(x)=f(a)$.

Definition 3: Let f be a real-valued function of several variables, defined on $\boldsymbol{R}^n$ $(n=2$ or 3), and $\boldsymbol{a}\in\boldsymbol{R}^n$. We say that the function f is continuous at $\boldsymbol{a}$ if given $\varepsilon>0$, there exists a real number $\delta>0$ (depending upon ε), such that $\|\boldsymbol{x}-\boldsymbol{a}\|<\delta\Rightarrow|f(\boldsymbol{x})-f(\boldsymbol{a})|<\varepsilon$. That means $\lim\limits_{\boldsymbol{x}\to\boldsymbol{a}}f(\boldsymbol{x})=f(\boldsymbol{a})$. In other words a function f is continuous at a point (a,b) if $\lim\limits_{(x,y)\to(a,b)}f(x,y)=f(a,b)$.

The definition above implies that if a function f is continuous at a point (a,b), then (i) and (ii) given below necessarily holds:

i) f is defined at (a,b).

ii) $\lim\limits_{(x,y)\to(a,b)}f(x,y)=f(a,b)$.

Note-1: The condition (ii) indicates not only that the limit exists, but also that the limit is equal to $f(\boldsymbol{a})$. This is the essential difference between the existence of the limit of a function at a point and the continuity of a function at that point.

Note-2: The necessary conditions (i) and (ii) above also implies that if either of the condition (i) or (ii) fails at a point $\boldsymbol{a}$, then we can conclude that f is not continuous at that point. This fact is very important in checking whether a function is continuous or not.

Let us see some examples:

Example 7: Check the continuity of the following functions at $(0,0)$.

i) $f(x,y)=\dfrac{x^2-y^2}{x^2+y^2}$

ii) $f(x,y)=\begin{cases}\dfrac{5x^2y}{x^2+y^2}, & if\ (x,y)\neq(0,0)\\ 0 & , if\ (x,y)=(0,0)\end{cases}$

iii) $f(x,y)=\begin{cases}\dfrac{x^2y}{x^2+y^2}, & if\ (x,y)\neq(0,0)\\ \dfrac{1}{2} & , if\ (x,y)=(0,0)\end{cases}$

Solution: Let us try (i), (ii) and (iii) one by one.

i) We note that the given function is not defined at $(0,0)$. Therefore f is not continuous at $(0,0)$.

ii) We first note that the given function is defined at $(0,0)$ and $f(0,0)=0$. Next we check whether limit of the function exists at $(0,0)$. In Example 6 we have shown that the limit exists and the limit is equal to 0. Thus we have

$$\lim_{(x,y)\to(0,0)} f(x,y)=f(0,0).$$

Hence f is continuous at $(0,0)$.

iii) We note that the function is defined at $(0,0)$. We shall check whether the limit of the function f exists or not. For that we shall let $(x,y)\to(0,0)$ along different lines. We consider the line $y=mx$

$$f(x,y)=f(x,mx)=\frac{m^2x}{1+m^4x^2}$$

Thus $f(x,y)\to 0$ as $(x,y)\to(0,0)$ along the line $y=mx$.
Next we shall let $(x,y)\to(0,0)$ along the parabola $x=y^2$.

$$f(x,y)=f(y^2,y)=\frac{y^4}{2y^4}=\frac{1}{2}.$$

Thus $f(x,y)\to\dfrac{1}{2}$ as $(x,y)\to(0,0)$ along curve $x=y^2$.

Since different paths lead to different limiting values, the given limit does not exist. Hence the function is not continuous at $(0,0)$.

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Here are some definitions.

Definition 4: A real-valued function of several variables is **continuous on a set A** contained in the domain of the function, if the function is continuous at each point of A. A real-valued function of several variables is said to be a **continuous function** if the function is continuous at every point of its domain of definition.

Using the properties of the limits, you can see that sums, differences, products and quotients of continuous functions (whenever defined) are continuous on their domains. We have the following theorem.

Theorem 5 (Algebra of Continuous Functions): Let f and g be two real-valued functions of n variables, which are continuous at a point $\boldsymbol{a} \in \boldsymbol{R}^n, n = 2,3$. Then

i) $f \pm g$ is continuous at $\boldsymbol{a}$.

ii) αf is continuous at $\boldsymbol{a}$ for every $\alpha \in \boldsymbol{R}$.

iii) fg is continuous at $\boldsymbol{a}$.

iv) $\dfrac{f}{g}$ is continuous at $\boldsymbol{a}$, , provided $g(\boldsymbol{a}) \neq 0$.

Proof: i) $\lim\limits_{x \to a} (f \pm g)(\boldsymbol{x}) = \lim\limits_{x \to a} [f(\boldsymbol{x}) \pm g(\boldsymbol{x})]$

$= \lim\limits_{x \to a} f(\boldsymbol{x}) \pm \lim\limits_{x \to a} g(\boldsymbol{x})$, by Theorem 2.

$= f(\boldsymbol{a}) \pm g(\boldsymbol{a})$, since f and g are continuous at $\boldsymbol{a}$.

$= (f \pm g)(\boldsymbol{a})$

ii) $\lim\limits_{x \to a} (\alpha f)(\boldsymbol{x}) = \lim\limits_{x \to a} \alpha f(\boldsymbol{x}) = \alpha \lim\limits_{x \to a} f(\boldsymbol{x}) = \alpha f(\boldsymbol{a}) = (\alpha f)(\boldsymbol{a})$

iii) $\lim\limits_{x \to a} (fg)(\boldsymbol{x}) = \lim\limits_{x \to a} f(\boldsymbol{x})\, g(\boldsymbol{x}) = \lim\limits_{x \to a} f(\boldsymbol{x}). \lim\limits_{x \to a} g(\boldsymbol{x}) = f(\boldsymbol{a}).$
$g(\boldsymbol{a}) = (fg)(\boldsymbol{a}).$

iv) $\lim\limits_{x \to a} \left(\dfrac{f}{g}\right)(\boldsymbol{x}) = \lim\limits_{x \to a} \dfrac{f(\boldsymbol{x})}{g(\boldsymbol{x})} = \dfrac{\lim\limits_{x \to a} f(\boldsymbol{x})}{\lim\limits_{x \to a} g(\boldsymbol{x})} = \dfrac{f(\boldsymbol{a})}{g(\boldsymbol{a})} = \left(\dfrac{f}{g}\right)(\boldsymbol{a}).$

This shows that $(f \pm g)$, αf, fg, $\dfrac{f}{g}$ are all continuous at $\boldsymbol{a}$.

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From Theorem 5 we can conclude that if f is any polynomial in 2 or 3 variables, then f is a continuous function. Note that a polynomial in 2 variables is of the form $c\,x^m y^n$ where c is a constant and m, n are non-negative integers. For instance, the function $f(x, y) = x^4 + 5x^3y^2 + 6xy^4 - 7y + 6$ is a polynomial and therefore is continuous on whole of $\boldsymbol{R}^2$. The function $g(x, y) = \dfrac{3xy + 5}{x^2 + y^2}$ is also continuous at all points of $\boldsymbol{R}^2$ except at $(0,0)$. Note that $g(x, y)$ is a quotient of two polynomial functions. Such functions are called rational functions.

Let us see some examples.

Example 8: Check the limit and continuity of the following functions at the indicated points.

i) $f(x,y)=\dfrac{7xy+1}{x^2+y^2},(x,y)\neq(0,0)$ as $(x,y)\to(1,1)$

ii) $f(x,y)=2x^2y^3-3x^3y^2+4x+2y$ as $(x,y)\to(-2,0)$

iii) $f(x,y,z)=\dfrac{xy+yz^2+xz^2}{x^2+y^2+z^2}$ as $(x,y,z)\to(1,0,1)$

Solution: Let us try one by one.

i) We first note that $f(x,y)=\dfrac{7xy+1}{x^2+y^2},(x,y)\neq(0,0)$ is a rational function.

Then

$$\lim_{(x,y)\to(1,1)}\frac{7xy+1}{x^2+y^2}=\frac{7\lim x\times\lim y+1}{\lim x^2+\lim y^2},(x,y)\neq(0,0)$$
$$=\frac{7\times1\times1+1}{1+1}=\frac{8}{2}=4=f(1,1)$$

Hence the limit exists and is equal to the value of the function. Therefore, the function is continuous at the given point.

ii) $f(x,y)$ is a polynomial and $f(-2,0)=-8.$ Also

$$\lim_{(x,y)\to(-2,0)}f(x,y)=\lim(2x^2y^3)-\lim 3x^3y^2+4\lim x+2\lim y$$
$$=2\times4\times0-3\times(-8)\times0+4\times(-2)+0$$
$$=-8$$
$$=f(-2,0)$$

Since the value of the limit exists and is equal to the value of the function at the given point, the function is continuous.

iii) $f(x,y,z)$ is a rational function in three variables and $f(1,0,1)=\dfrac{1}{2}.$

$$\lim_{(x,y,z)\to(1,0,1)}f(x,y,z)=\frac{\lim(xy)+\lim(yz^2)+\lim(xz^2)}{\lim(x^2+y^2+z^2)},(x,y,z)\neq(0,0,0)$$
$$=\frac{0+0+1}{1+1}=\frac{1}{2}=f(1,0,1).$$

Hence the limit exists and it is equal to the value of the function. Hence it is continuous.

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Next we state and prove a result about the continuity of the composite of two continuous functions. You must have studied a similar result for functions of one variable.

Theorem 6: Let f be a real-valued function of several variables, which is continuous at a point $\boldsymbol{a}\in\boldsymbol{R}^n\,(n=2$ or $3)$, and let g be a real-valued function of a real variable, which is continuous at $f(\boldsymbol{a})$. Then the composite function $g\circ f$ is continuous at $\boldsymbol{a}$.

Proof: Let $\varepsilon>0.$ The continuity of the function g at $f(\boldsymbol{a})$ implies that there exists a real number $\delta>0,$ such that

$$|y-f(\boldsymbol{a})|<\delta\Rightarrow|g(y)-g(f(\boldsymbol{a}))|<\varepsilon \qquad \dots (5)$$

Now, the continuity of the function f at $\boldsymbol{a}$ implies that given $\delta > 0$ there exists a positive number $\eta > 0$, such that

$$\|\boldsymbol{x}-\boldsymbol{a}\| < \eta \Rightarrow |f(\boldsymbol{x}) - f(\boldsymbol{a})| < \delta \qquad \ldots (6)$$

Combining (5) and (6), we see that

$\|\boldsymbol{x}-\boldsymbol{a}\| < \eta \Rightarrow |g(f(\boldsymbol{x})) - g(f(\boldsymbol{a}))| < \varepsilon$, that is,

$\|\boldsymbol{x}-\boldsymbol{a}\| < \eta \Rightarrow |g \circ f(\boldsymbol{x}) - g \circ f(\boldsymbol{a})| < \varepsilon$

i.e., $g \circ f$ is continuous at $\boldsymbol{a}$.

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Example 9: Show that $\lim\limits_{(x,y)\to(0,\,\ln 5)} e^{x+y} = 5.$

Solution: Clearly, the functions $f(x,y) = x + y$ and $g(t) = e^t$ are continuous everywhere on their domains. Therefore, the composite function $g \circ f$ is continuous everywhere, in view of Theorem 6. Consequently,

$$\lim_{(x,y)\to(0,\,\ln 5)} e^{x+y} = e^{0+\ln 5} = 5.$$

***

We conclude this section with the following exercises.

E7) Show that the following functions are not continuous at (0, 0).

a) $f(x,y) = \begin{cases} \dfrac{x^2 y}{x^3 + y^2}, & (x,y) \neq (0,0) \\ 2, & (x,y) = (0,0) \end{cases}$

b) $f(x,y) = \begin{cases} y \sin \dfrac{1}{x} + x \sin \dfrac{1}{y}, & x \neq 0,\ y \neq 0 \\ 1, & \text{otherwise} \end{cases},$

E8) Let p_1 and p_2 be the functions defined on $\boldsymbol{R}^2$ by $p_1(x,y) = x$ and $p_2(x,y) = y$. Prove that π_1 and π_2 are continuous functions on $\boldsymbol{R}^2$. The functions p_1 and p_2 are called projection maps.

E9) Show that the following functions are continuous at the origin. Mention clearly the theorems or known results used.

a) $x \sin y + y \sin z + z \sin x$

b) $e^x \cos y + e^y \cos z + e^z \cos x$

c) $\ln(1 + x^2 + y^2 + z^2)$

d) $|x_1| + |x_2| + |x_3|$

By now you have learnt how the concepts of limit and continuity are extended to functions of several variables. In the next section we shall discuss one more way of defining the limit of a function of several variables.

2.4 REPEATED LIMITS

The definition of limit which you have studied in Sec. 2.2 is a generalization of the definition of limit for functions from $\boldsymbol{R} \to \boldsymbol{R}$. We shall now consider another type of limit, which is peculiar to functions of several variables. For the sake of simplicity we confine ourselves to functions of two variables.

Let f be a real-valued function of two variables, defined on some deleted neighbourhood of a point (a,b).

Suppose we first find the limit of $f(x,y)$ as y tends to b, while x is treated as a constant and then we find the limit of the resulting function, as x tends to a. This limit is known as a repeated limit and express by $\lim\limits_{x\to a}\left(\lim\limits_{y\to b} f(x,y)\right)$.

In a similar way suppose, we find the limit as x tends to a, treating y as a constant, and then find the limit of this function as y tends to b. This is known as another repeated limit and express it as $\lim\limits_{y\to b}\left(\lim\limits_{x\to a} f(x,y)\right)$.

Please note that when we find the repeated limits, we are actually taking two specific paths P_1 and P_2 one after the another as shown in Fig. 4 to approach the point (a,b). P_1 is the path consisting of one horizontal and one vertical line segment corresponding to $\lim\limits_{y\to b}(\lim\limits_{x\to a} f(x,y))$ and P_2 is the path consisting of one vertical dotted and one horizontal dotted line segment corresponding to $\lim\limits_{x\to a}(\lim\limits_{y\to b} f(x,y))$.

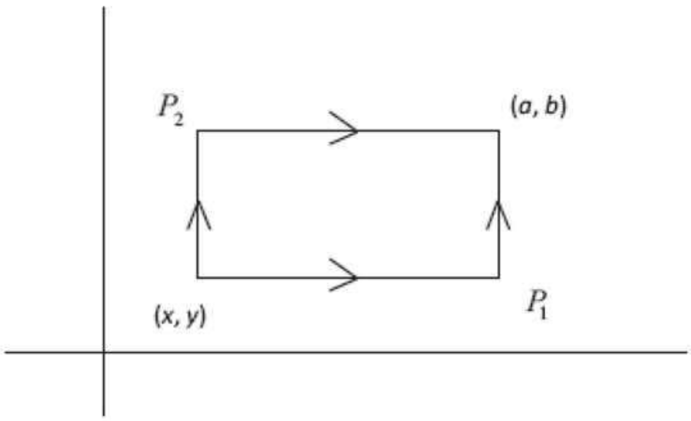


Fig. 4: P_1 is the straight line path joining (x, y) and (a,b) in one direction. P_2 is another straight line path joining (x, y) and (a,b) in the other direction.

These two limits, which are called repeated limits, are independent of the limit, $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ which we defined in the previous section (See Definition 4 in the earlier section). We refer to $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ as the simultaneous limit, as x and y are approaching a and b, respectively, at the same time i.e. simultaneously.

In the following example we illustrate the computation of the two repeated limits and the simultaneous limit.

Example 10: Let $f(x,y)=\dfrac{(x-y)^2}{x^2+y^2}$. Show that the two repeated limits

$\lim\limits_{x\to 0}\left(\lim\limits_{y\to 0} f(x,y)\right)$ and $\lim\limits_{y\to 0}\left(\lim\limits_{x\to 0} f(x,y)\right)$ exist and are equal, but the simultaneous limit $\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y)$ does not exist.

Solution: Clearly $\lim\limits_{x\to 0} f(x,y)=1=\lim\limits_{y\to 0} f(x,y)$, and therefore

$$\lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right)=1=\lim_{y\to 0}(\lim_{x\to 0} f(x,y))$$

Next we shall consider the simultaneous limit. Let $y=mx$. Then along the line $y=mx$, we have

$$f(x,y)=\frac{(1-m)^2x^2}{(1+m^2)x^2}=\frac{(1-m)^2}{1+m^2}.$$

Therefore $f(x,y)$ approaches different values as $(x,y)\to(0,0)$ on different paths.

Thus, the simultaneous limit $\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y)$ does not exist.

***

The example above shows that the existence of the simultaneous limit need not imply the existence of repeated limits and vice-versa.

Let us see another example.

Example 11: Let $f(x,y)=\dfrac{xy}{|y|}$. Show that $\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y)$ exists, but

$\lim\limits_{x\to 0}\left(\lim\limits_{y\to 0} f(x,y)\right)$ does not exist, and $\lim\limits_{y\to 0}\left(\lim\limits_{x\to 0} f(x,y)\right)=0.$

Solution: Since

$$|f(x,y)|=\frac{|xy|}{|y|}=|x|,$$

it follows that $\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y)=0.$

Further, $\lim\limits_{y\to 0^+}\dfrac{xy}{|y|}=x$, and $\lim\limits_{y\to 0^-}\dfrac{xy}{|y|}=-x.$

This shows that $\lim\limits_{y\to 0}\dfrac{xy}{|y|}$ does not exist when $x\neq 0$, and therefore

$\lim\limits_{x\to 0}\left(\lim\limits_{y\to 0}\dfrac{xy}{|y|}\right)$ also does not exist.

Now, $\lim\limits_{x\to 0}\frac{|xy|}{|y|}=\lim\limits_{x\to 0}|x|=0$. Therefore, $\lim\limits_{x\to 0}\frac{xy}{|y|}=0$, and thus,

$$\lim_{y\to 0}\left(\lim_{x\to 0}\frac{xy}{|y|}\right)=0.$$

***

So, does it mean that the simultaneous limit and the repeated limits are totally unrelated? No. The situation is not so bad. In some cases we can relate the two.

Here is a theorem which gives the connection between simultaneous limits and repeated limits.

Theorem 7: Let $f(x, y)$ be a real-valued function such that $\lim\limits_{\substack{x\to a\\ y\to b}} f(x, y)=L.$ If both the repeated limits $\lim\limits_{x\to a}\left(\lim\limits_{y\to b} f(x, y)\right)$ and $\lim\limits_{y\to b}\left(\lim\limits_{x\to a} f(x, y)\right)$ exist, then each one of these limits is equal to L.

We are not giving the proof of this theorem here, as it is beyond the scope of this course.

See if you can solve these exercises now.

E10) For the function $f(x, y)=\dfrac{xy}{x^2+y^2}$, prove that the simultaneous limit does not exist at (0, 0), while the two repeated limits exist and are equal.

E11) For the function $f(x, y)=\dfrac{y-x}{y+x}\dfrac{1+x^2}{1+y^2}$, show that $\lim\limits_{x\to 0}\left(\lim\limits_{y\to 0} f(x, y)\right)=-1$ and $\lim\limits_{y\to 0}\left(\lim\limits_{x\to 0} f(x, y)\right)=1$. Apply Theorem 7 to decide the existence of the simultaneous limit as $(x, y)\to(0,0)$.

E12) Let $f(x, y)=\begin{cases}1+xy, & xy\neq 0\\ 0 & , xy=0\end{cases}$

Then prove that $\lim\limits_{x\to 0}\left[\lim\limits_{y\to 0} f(x, y)\right]=1=\lim\limits_{y\to 0}\left[\lim\limits_{x\to 0} f(x, y)\right]$. But $\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x, y)$ does not exist.

This brings us to the end of the unit. Let us briefly recall what we have covered in it.

2.5 SUMMARY

In this unit we have

1) defined the limit of a function $f: \boldsymbol{R}^n\to\boldsymbol{R}$ $(n=2 \text{ or } 3)$: $\lim\limits_{x\to a} f(x)=L,$ if for every $\varepsilon>0$, there exists a positive real number δ (depending on ε), such that $0<\|x-a\|<\delta \Rightarrow |f(x)-L|<\varepsilon.$

2) defined continuity for real-valued functions of several variables:
A function f is continuous at $\boldsymbol{a}$ if $\lim_{\boldsymbol{x}\to\boldsymbol{a}} f(\boldsymbol{x}) = f(\boldsymbol{a})$, i.e., given $\varepsilon > 0$ there exists a real number $\delta > 0$ (depending upon ε) such that
$\|x-a\| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.$

3) stated and proved results about the algebra of continuous functions and the continuity of the composite of two continuous functions.

4) discussed repeated limits and their connection with the simultaneous limit.

2.6 SOLUTIONS/ANSWERS

E1) a) Put $x = r\cos\theta,\ y = r\sin\theta$. Then

$$\left|\frac{xy}{\sqrt{x^2+y^2}}\right| = \left|\frac{r\cos\theta.\ r\sin\theta}{\sqrt{r^2(\cos^2\theta+\sin^2\theta)}}\right|$$
$$= \left|\frac{r^2\ \cos\theta\ \sin\theta}{r}\right|$$
$$\le r = \sqrt{x^2+y^2} = \|(x,y)\|$$

Therefore, as $\|(x,y)\|$ tends to zero, $\dfrac{xy}{\sqrt{x^2+y^2}}$ also tends to zero.

b) Put $x = r\cos\theta,\ y = r\sin\theta$. Then

$$\left|\frac{x^2y^2}{\sqrt{x^2+y^2}}\right| = \left|\frac{r^2\cos^2\theta.\ r^2\sin^2\theta}{\sqrt{r^2(\cos^2\theta+\sin^2\theta}}\right|$$
$$= \left|\frac{r^4\cos^2\theta.\sin^2\theta}{r}\right|$$
$$\le r^3 = \left(x^2+y^2\right)^{3/2} = \|(x,y)\|^3.$$

Now, as $\|(x,y)\|$ tends to zero, $\|(x,y)\|^3$ also tends to zero, and therefore, $\dfrac{x^2y^2}{\sqrt{x^2+y^2}}$ tends to zero.

E2) a) Using algebra of limits we get

$$\lim_{(x,y,z)\to(0,1,2)} (x^2+3xyz-5z^2) = -20 \text{ and}$$
$$\lim_{(x,y,z)\to(0,1,2)} (xy^3+5z^2-3xy+x^3) = 20,$$

Again, using algebra of limits, we get

$$\lim_{(x,y,z)\to(0,1,2)} \left(\frac{x^2+3xyz-5z^2}{xy^3+5z^2-3xy+x^3}\right)$$
$$= \frac{\lim\limits_{(x,y,z)\to(0,1,2)} (x^2+3xyz-5z^2)}{\lim\limits_{(x,y,z)\to(0,1,2)} (xy^3+5z^2-3xy+x^3)} = \frac{-20}{20} = -1$$

b) Now, $|x\sin y| \le |x|$, and therefore, $\lim\limits_{(x,y)\to(0,0)} x\sin y = 0.$

Using algebra of limits, we get

$$\lim_{(x,y)\to(0,0)} (2x^2+1) = \lim_{(x,y)\to(0,0)} 2x^2 + \lim_{(x,y)\to(0,0)} 1 = 0+1 = 1.$$

Again using algebra of limits, we get

$$\lim_{(x,y)\to(0,0)} \left(\frac{x\sin y}{2x^2+1}\right) = \frac{\lim\limits_{(x,y)\to(0,0)} x\sin y}{\lim\limits_{(x,y)\to(0,0)} (2x^2+1)} = \frac{0}{1} = 0.$$

E3) a) $f(x,y) = \dfrac{x^4}{x^4+y^2}$. Along the path $y = x^2$,

$\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{x\to 0} \dfrac{1}{2} = \dfrac{1}{2}.$

Along the path $y = 2x^2$, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{x\to 0} \dfrac{1}{5} = \dfrac{1}{5}.$

So, the limit does not exist.

b) $f(x,y) = \dfrac{x^3 y}{x^6+y^2}$. Along the path $y = x^3$, the limit is ½, and along the path $y = 2x^3$, the limit is 1/5. So, the limit does not exist.

c) $f(x,y) = \dfrac{2x^2}{x^2-y^2+x}$. Along the path

$y = x$, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{x\to 0} 2x = 0.$

Along the path $y^2 = x$, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{x\to 0} 2 = 2.$

So, the limit does not exist.

d) $f(x,y) = \dfrac{x^4-y^2}{x^4+y^2}$. Along the path $y = 0$, that is along the x-axis, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{x\to 0} 1 = 1$, and along the path $x = 0$, that is along the y-axis, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{x\to 0} -1 = -1$.

So, the limit does not exist.

E4) a) $f(x,y) = \dfrac{xy}{x^2+y^2}$

If we put $y = \phi_1(x) = mx$, then

$$f(x, \phi_1(x)) = \frac{mx^2}{x^2(1+m^2)} = \frac{m}{1+m^2}$$

This value is different for different m, which shows that the function has different limits in different directions. Therefore the limit does not exist.

b) In view of Corollary 1, it is enough to prove that there exist real-valued functions $\lim\limits_{x\to 0} \phi_1(x) = 0 = \lim\limits_{x\to 0} \phi_2(x)$ and

$\lim\limits_{x\to 0} f(x, \phi_1(x)) \neq \lim\limits_{x\to 0} f(x, \phi_2(x))$

Let $\phi_1(x) = x^2$ and $\phi_2(x) = x - x^2$. Then

$\lim\limits_{x\to 0} \phi_1(x) = \lim\limits_{x\to 0} \phi_2(x) = 0$

Also, $\lim\limits_{x\to 0} f(x, \phi_1(x)) = \lim\limits_{x\to 0} \dfrac{x^2}{x^2 + x^2} = \lim\limits_{x\to 0} \dfrac{1}{2} = \dfrac{1}{2}$

and $\lim\limits_{x\to 0} f(x, \phi_2(x)) = \lim\limits_{x\to 0} \dfrac{x^2}{x^2 + x - x^2}$

$= \lim\limits_{x\to 0} x = 0$

Thus, $\lim\limits_{x\to 0} f(x, \phi_1(x)) \neq \lim\limits_{x\to 0} f(x, \phi_2(x))$

Therefore, by Corollary 1, the limit does not exist.

c) Let $\phi_1(x) = x$ and $\phi_2(x) = -x$.

$\lim\limits_{x\to 0} f(x, \phi_1(x)) = \dfrac{x^2 - x^2}{x^2 + x^2} + \dfrac{2.x.x}{x^2 + x^2} = 1$

Similarly $\lim\limits_{x\to 0} f(x, \phi_2(x)) = \dfrac{x^2 - x^2}{x^2 + x^2} - \dfrac{2.x.x}{x^2 + x^2}$

Thus, $\lim\limits_{x\to 0} f(x, \phi_1(x)) \neq \lim\limits_{x\to 0} f(x, \phi_2(x))$. Hence, the limit does not exist.

E5) a) We have to find δ such that, if $|x| < \delta$, $|y| < \delta$ and $|z| < \delta$, then $|x^2 + y^2 + z^2| < 0.01$.

Now, $|x| < \delta$, $|y| < \delta$ and $|z| < \delta \Rightarrow x^2 + y^2 + z^2 < 3\delta^2$.

If we choose δ such that $3\delta^2 < 0.01$, then we are through. So, any positive number, which is less than $\sqrt{\dfrac{0.01}{3}}$ will do. For example, we can take $\delta = 0.05$.

b) We have to find δ such that, if $|x| < \delta$, $|y| < \delta$ then $|xy| < 0.0004$.

If $|x| < \delta$, $|y| < \delta$, then $|xy| < \delta^2$. So we can take any δ such that $\delta^2 < 0.0004$. So, any positive number less than $\sqrt{0.0004}$ will do. For example, $\delta = 0.01$.

E6) We note that

$$\left|\frac{2x^2 y}{x^2 + y^2}\right| \leq \left|\frac{2x^2 y}{x^2 + y^2}\right| = 2|y|$$

Suppose that $\varepsilon > 0$ is given. If we choose $\delta = \dfrac{\varepsilon}{2}$, then we have

$0 < \|(x, y) - (0,0)\| = \sqrt{x^2 + y^2} < \delta$. This implies that $|y| < \sqrt{x^2 + y^2} < \delta$.

This together with the inequality shown above implies that whenever $|y| < \delta = \dfrac{\varepsilon}{2}$ we have

$$\left|\frac{2x^2y}{x^2+y^2}\right| \le 2|y| \le 2\frac{\varepsilon}{2} = \varepsilon.$$

Hence the limit is 0.

E7) a) We first note that to prove that a function is discontinuous at a point, it is enough to show that the limit in any particular direction exists and is not equal to $f(0,0)$.

Let $\phi_1(x) = x$. Then

$$\lim_{x\to 0} f(x, \phi_1(x)) = \lim_{x\to 0}\frac{x^3}{x^3+x^2} = \lim_{x\to 0}\frac{x}{x+1} = 0 \ne 2 = f(0,0).$$

Therefore f is not continuous at (0, 0).

b) You have already seen in Example 1 that

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)}\left(y\sin\frac{1}{x} + x\sin\frac{1}{y}\right) = 0.$$

But $f(0,0) = 1$. Therefore, the function is not continuous at (0, 0).

Caution: Whenever you write solution to this problem you should show that

$\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)}\left(y\sin\frac{1}{x} + x\sin\frac{1}{y}\right) = 0$. Do not just refer to Example 1.

E8) Let $a = (a_1, a_2) \in \boldsymbol{R}^2$. We first show that $\lim\limits_{x\to a} p_1(x) = p_1(a)$.

Note that for any $\boldsymbol{x}, \boldsymbol{a} \in \boldsymbol{R}^2$, we have

$|p_1(\boldsymbol{x}) - p_1(\boldsymbol{a})| = |p_1(x_1, x_2) - p_1(a_1, a_2)| = |x_1 - a_1|$, where
$\boldsymbol{x} = (x_1, x_2)$
$\boldsymbol{a} = (a_1, a_2)$.

Let $a > 0$ be given. Take $\delta_1 = \varepsilon$. Then,
$0 < |x_1 - a_1| < \delta_1 \Rightarrow |\pi_1(x) - \pi_1(a)| = |x_1 - a_1| < \delta_1 = \varepsilon$.

This shows that $\lim\limits_{x\to a} p_1(x) = p_1(a)$.

Therefore p_1 is continuous at a. Since a was an arbitrary point in $\boldsymbol{R}^2$, we conclude that p_1 is continuous on the whole of $\boldsymbol{R}^2$.

Hint: A similar argument shows that the map p_2 is continuous.

E9) a) Let $f(x, y, z) = x\sin y + y\sin z + z\sin x$.

$|x\sin y + y\sin z + z\sin x| \le |x| + |y| + |z|$

Therefore $|x| < \frac{\varepsilon}{3}, |y| < \frac{\varepsilon}{3}$ and $|z| < \frac{\varepsilon}{3}$ implies that
$|x\sin y + y\sin z + z\sin x| < \varepsilon.$

Hence $\lim\limits_{\substack{x\to 0\\ y\to 0\\ z\to 0}} (x\sin y + y\sin z + z\sin x) = 0 = f(0,0,0)$. Since this limit is equal to $f(0,0,0)$, we conclude that the function is continuous at $(0,0,0)$.

b) Let $f(x,y,z) = e^x\cos y + e^y\cos z + e^z\cos x$

Using algebra of limits,

$$\lim_{(x,y,z)\to(0,0,0)} (e^x\cos y + e^y\cos z + e^z\cos x) =$$

$$\lim_{(x,y,z)\to(0,0,0)} e^x . \lim_{(x,y,z)\to(0,0,0)} \cos y + \lim_{(x,y,z)\to(0,0,0)} e^y .$$

$$\lim_{(x,y,z)\to(0,0,0)} \cos z + \lim_{(x,y,z)\to(0,0,0)} e^z . \lim_{(x,y,z)\to(0,0,0)} \cos x = 1+1+1 = 3 = f(0,0,0)$$

(Note that, $\lim\limits_{t\to 0} e^t = 1$, and $\lim\limits_{t\to 0} \cos t = \cos 0 = 1$).

This shows that f is continuous at $(0,0,0)$.

c) Let $f(x,y,z) = \ln(1 + x^2 + y^2 + z^2)$.
Then $f = g\circ h$ where $h(x,y,z) = 1 + x^2 + y^2 + z^2$ and $g(t) = \ln t$. Clearly, the function h is continuous at $(0,0,0)$ and g is continuous at $h(0,0,0) = 1$. Since the composite of two continuous function is continuous, f is continuous at $(0,0,0)$.

d) $f(x_1, x_2, x_3) = |x_1| + |x_2| + |x_3|$

Let $f_i(x_1, x_2, x_3) = |x_i|,\ i = 1,2,3.$ Then $f_i, i = 1,2,3$ is a real-valued function of 3 variables and $f = f_1 + f_2 + f_3$. Also, $f_i = go\pi_i$, where $g(t) = |t|$. For each i, the function π_i is continuous at $(0,0,0)$, and g is continuous at $\pi_i(0,0,0) = 0$. Therefore, f_i is continuous at $(0,0,0)$, being the composite of two continuous functions.

Then by using algebra of limits, we conclude that f is continuous at $(0,0,0)$.

E10) We have already seen in the solution for E5 (a) that the simultaneous limit of the function $f(x,y) = \frac{xy}{x^2 + y^2}$ does not exist at $(0,0)$. The repeated limits $\lim\limits_{x\to 0}\left[\lim\limits_{y\to 0} \frac{xy}{x^2+y^2}\right]$ and $\lim\limits_{y\to 0}\left[\lim\limits_{x\to 0} \frac{xy}{x^2+y^2}\right]$ exist and are equal to 0, since $\lim\limits_{x\to 0} \frac{xy}{x^2+y^2} = 0$ and $\lim\limits_{y\to 0} \frac{xy}{x^2+y^2} = 0$.

E11) $\lim\limits_{x\to 0}\left(\dfrac{y-x}{y+x}\,\dfrac{1+x^2}{1+y^2}\right)=\dfrac{y}{y(1+y^2)}=\dfrac{1}{1+y^2}$

Therefore, $\lim\limits_{y\to 0}\left(\lim\limits_{x\to 0}\dfrac{y-x}{y+x}\,\dfrac{1+x^2}{1+y^2}\right)=\lim\limits_{y\to 0}\dfrac{1}{1+y^2}=1.$

Now,

$\lim\limits_{y\to 0}\left(\dfrac{y-x}{y+x}.\dfrac{1+x^2}{1+y^2}\right)=\dfrac{-x(1+x^2)}{x}=-(1+x^2)$, and therefore,

$$\lim_{x\to 0}\left(\lim_{y\to 0}\left(\frac{y-x}{y+x}.\frac{1+x^2}{1+y^2}\right)\right)=-1$$

This shows that the two repeated limits exist and are not equal. Therefore the simultaneous limit

$$\lim_{\substack{x\to 0\\ y\to 0}}\frac{(y-x)(1+x^2)}{(y+x)(1+y^2)}$$

does not exist (by applying Theorem 8).

E12) If $x\neq 0, f(x,y)=1+xy,\text{and}\lim\limits_{y\to 0} f(x,y)=1$, and then

$$\lim_{x\to 0}\left[\lim_{y\to 0} f(x,y)\right]=1.$$

(Note that if $x=0$, then $f(x,y)=0$, and $\lim\limits_{y\to 0} f(x,y)=0$. But this need not be considered in finding $\lim\limits_{x\to 0}\left[\lim\limits_{y\to 0} f(x,y)\right]=1.$)

Similarly, $\lim\limits_{y\to 0}\left[\lim\limits_{x\to 0} f(x,y)\right]=1.$

Consider $y=\phi_1(x)=x.$ Then we get $\lim\limits_{x\to 0} f(x,\phi_1(x))=\lim\limits_{x\to 0}(1+x^2)=1.$

When we take $y=\phi_2(x)=0, \lim\limits_{x\to 0} f(x,\phi_2(x))=0,$ since

$y=0\Rightarrow xy=0\Rightarrow f(x,y)=0$ for every $x.$

This shows that the simultaneous limit does not exist.

UNIT 3

FIRST ORDER PARTIAL DERIVATIVES AND DIFFERENTIABILITY

Structure ______ Page No.

3.1 INTRODUCTION

You are already familiar with the concept of a derivative of a real-valued function of a real variable from the Calculus course, Block 3, Unit 9. In this unit, we shall study this concept for functions from $\boldsymbol{R}^n \to \boldsymbol{R}$, where $n = 2,3$. You have learnt from the previous units that the notions of limit and continuity can be extended to these functions. But the definition of the derivative of a real-valued function of a real variable cannot be extended, as it is, to a real-valued function f of 2 variables or 3 variables. This is because for any $\boldsymbol{h} \in \boldsymbol{R}^2$, $\boldsymbol{h} \neq 0$, the quotient $\frac{f(x+h)-f(x)}{h}$ does not make sense as the division by a element in $\boldsymbol{R}^2$ or $\boldsymbol{R}^3$ is not defined.

However, if we examine the definition of derivative more closely, we realise that a function of a single variable is differentiable at a point if and only if the two directional derivatives, i.e., the right hand derivative and the left hand derivative, exist and are equal at that point. The only hitch is that in

$R^2 (or\ R^3)$, we have to deal with infinitely many directions which are parallel to the coordinate axes. This leads us to the notion of partial derivatives.

In Sec 3.2 we'll discuss the concept of partial derivatives of a function of more than one variables. You will learn that this notion of partial derivatives does not fully generalise the concept of derivative of a real-valued function of a real-variable. Later in Sec 3.3 we introduce the concept of differentiability for functions on $R^3 (or\ R^2)$, and discuss the relationship between differentiability, continuity and the existence of partial derivatives.

Objectives

After reading this unit, you should be able to:

- define partial derivatives of the first order for a function of or 2 or 3 variables,
- check whether the partial derivative of a given real-valued function of 2 or 3 variables with respect to any particular value of the variable exists or not and compute the partial derivative, if it exists,
- give the geometric interpretation of first order partial derivatives of function of two variables,
- decide whether a given function of two or three variables is differentiable or not,
- give examples to establish relationships between the continuity, differentiability and existence of partial derivatives at a point for a function of two variables.

3.2 FIRST ORDER PARTIAL DERIVATIVES

In this section we shall discuss what the partial derivative of a function at a point means. We already know how to define the derivative of a function of a single variable. We'll use this knowledge in defining the partial derivatives of functions of several variables.

We shall begin with some definitions and examples.

3.2.1 Definitions and Examples

Let us start with a situation.

Situation: The heat index which we denote by I indicates the air temperature when the actual temperature in Fahrenheit is t and the relative humidity is h. So I is a function of two variables, t and h i.e.,

$$I = f(t, h)$$

The following table gives values of I corresponding to different values of t and h.

(This table is part of the table of values compiled by National Weather Services)

Table 1: Heat Index I as a function of temperature and humidity

Relative humidity (%)

Actual Temperature ($^\circ F$) T \ H	50	55	60	65	70	75	80	85	90
90	96	98	100	103	106	109	112	115	119
92	100	103	105	108	112	115	119	123	128
94	104	107	111	114	118	122	127	132	137
96	109	113	116	121	125	130	135	141	146
98	114	118	123	127	133	138	144	150	157
100	119	124	129	135	141	147	154	161	168

By looking closely at the table you can see that the raw against the temperature value indicates the heat index values for the different humidity index, say $h = 104, 107, 111, 118, 122, 127, 132$ and 137 for a fixed temperature value $t = 94$. That means in this situation, the heat index I can be treated as a function of one variable h with a fixed value $t = 94$. Let us denote this one variable function as $g(h) = f(94, h)$. Then $\mathrm{g(h)}$ describes how the heat index I changes as the humidity changes when the temperature is 94.

From the one variable case we know that the rate of change of I with respect to h when $t = 94^\circ$ is known as the derivative of the function g w.r.t. to the variable h which is denoted by $g'(h)$. Also we have for $h = h_0$,

$$g'(h_0) = \lim_{h\to 0} \frac{g(h_0 + h) - g(h_0)}{h} \qquad \dots (1)$$

Then g' is called the partial derivative of f w.r.t. the variable h, keeping the other variable, a constant.

Similarly suppose we fix $h = 75$, say. Then the column against $h = 75$, indicates the I values for the temperatures $t = 109, 115, 122, 130, 138, 147$. Thus if we denote $k(t) = f(t, 75)$, then k is a function of one-variable and its derivative at a point $t = t_0$ is given by

$$k'(t_0) = \lim_{t\to 0} \frac{k(t_0 + t) - k(t_0)}{t} \qquad \dots (2)$$

Then k' is called the partial derivative of f w.r.t the variable t, keeping the other variable, a constant.

The above situation shows that if f is a function of two variables x and y and suppose we allow only x vary, keeping y fixed say $y = b$, then we are actually considering a function of single variable x, namely $g(x) = f(x, b)$. If the function g has a derivative at a point $a \in \boldsymbol{R}$, then the derivative of g w.r.t. x at the point a is called **partial derivative of f w.r.t. x** at the point a. We formally define it now.

Definition 1: If f is a function of two variables x and y, then the **partial derivative of** f with respect to x is the function defined by

$$f_x(x, y) = \lim_{h\to 0} \frac{f(x + h, y) - f(x, y)}{h} \qquad \dots (3)$$

When we have to compute the partial derivative f_x at a particular point (a, b), then we use the following expression:

$$f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h} \quad \ldots (4)$$

Similarly we can define the partial derivative w.r.t. y.

Definition 2: If f is a function of two variables x and y, then the **partial derivative of** f w.r.t. y is the function defined by

$$f_y(x,\ y) = \lim_{k \to 0} \frac{f(x,\ y+k) - f(x,\ y)}{k} \quad \ldots (5)$$

For computing the partial derivative f_y at a particular point (a,b), we use the following expression:

$$f_y(a,b) = \lim_{k \to 0} \frac{f(a,b+k) - f(a,b)}{k} \quad \ldots (6)$$

If you go back to the situation of heat index which we have discussed in the beginning, then we you will observe that the derivative which we have considered in Eqn. (1) is infact the partial derivative of I w.r.t. h at the point $(94,\ h_0)$ denoted by $I_h(94,\ h_0)$. The partial derivative of the heat index I with constant relative humidity $h = 75$, at a point t_0 is $I_t(t_0,\ 75)$.

There are different symbols available in literature to denote the partial derivatives of a given function. However, we shall use only the following

Notation for Partial derivatives

Partial derivative w.r.t. x

$$f_x,\ D_1 f,\ \frac{\partial f}{\partial x}$$

Partial derivative w.r.t. y

$$f_y,\ D_2 f,\ \frac{\partial f}{\partial y}$$

You have seen in the Calculus course that we write $y = f(x)$ to express a real-valued function of a real variable. For functions of two variables it is customary to write $z = f(x, y)$, and the two partial derivatives of f at the point (a,b) are then denoted by $\left.\frac{\partial z}{\partial x}\right|_{(a,\ b)}$ and $\left.\frac{\partial z}{\partial y}\right|_{(a,\ b)}$.

Remark 1: i) Note that the notion of partial derivative of a function (like continuity) is local in character. That means, we check the existence of the partial derivative of a function at a point. Thus, when we say that a function has partial derivatives on a set A, we mean that the function has partial derivatives at each point of A.

ii) It is obvious from the definition of a partial derivative of a function at a point that the function must be defined in a neighbourhood of the point. Also, we can talk about the partial derivatives only at the interior points of the domain D. For example, if D is a disc in $\boldsymbol{R}^2$, then we cannot talk about the partial derivative of a point on the circumference of this disc.

The partial derivatives can be defined for functions of 3-variables in the same way as for 2-variables as given in the following definition.

Definition 3: If f is a function of three variables $x,\ y$ and $z,$ then its partial derivatives w.r.t. $x,\ y,\ z$, respectively are given by

$$f_x(x, y, z) = \lim_{p \to 0} \frac{f(x+p, y, z) - f(x, y, z)}{p} \quad \dots (7)$$

$$f_y(x, y, z) = \lim_{q \to 0} \frac{f(x, y+q, z) - f(x, y, z)}{q} \quad \dots (8)$$

$$f_z(x, y, z) = \lim_{r \to 0} \frac{f(x, y, z+r) - f(x, y, z)}{r} \quad \dots (9)$$

When we have to compute the partial derivatives of a particular point (a,b,c) , then we use the following

$$f_x(a, b, c) = \lim_{p \to 0} \frac{f(a+p, b, c) - f(a, b, c)}{p} \quad \dots (10)$$

$$f_y(a,b,c) = \lim_{q \to 0} \frac{f(a,b+q,c) - f(a,b,c)}{q} \quad \dots (11)$$

$$f_z(a,b,c) = \lim_{r \to 0} \frac{f(a,b,c+r) - f(a,b,c)}{r} \quad \dots (12)$$

We now give some examples to show how to obtain partial derivatives of given function at a given point. If we do not mention a specific point at which partial derivatives are to be calculated, then these are to be calculated at a generic point (x, y) or (x, y, z) using the expressions given in (3), (5), (7), (8) and (9).

Example 1: Let $f : \boldsymbol{R}^2 \to \boldsymbol{R}$ be a function defined by $f(x, y) = x^2 + xy + y^3$. Find $f_x(x, y)$ and $f_y(x, y)$.

Solution: By definition,

$$\begin{aligned}
f_x(x, y) &= \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h} \\
&= \lim_{h \to 0} \frac{(x+h)^2 + (x+h)y + y^3 - x^2 - xy - y^3}{h} \\
&= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + xy + hy + y^3 - x^2 - xy - y^3}{h} \\
&= \lim_{h \to 0} (2x + h + y) \\
&= 2x + y
\end{aligned}$$

Similarly ,

$$\begin{aligned}
f_y(x, y) &= \lim_{k \to 0} \frac{f(x,\ y+k) - f(x, y)}{k} \\
&= \lim_{k \to 0} \frac{x^2 + x(y+k) + (y+k)^3 - x^2 - xy - y^3}{k} \\
&= \lim_{k \to 0} \frac{xk + 3y^2k + 3yk^2 + k^3}{k}
\end{aligned}$$

$$= \lim_{k \to 0} \left(x + 3y^2 + 3yk + k^2\right)$$
$$= x + 3y^2$$

***

When we are considering functions of two variables x and y, then for the increment in x we normally use the letter h and for the increment in y, the letter k. Similarly, when we are dealing with function of three variables x, y and z, we use the letters p, q and r for increments in x, y and z, respectively. This is only a matter of convention, not a rule.

In the next example we consider a function of three variables.

Example 2: Let $f : \boldsymbol{R}^3 \to \boldsymbol{R}$ be a function defined by $f(x, y, z) = xy + yz + zx$. Find the partial derivatives at the point (a, b, c).

Solution: By definition,

$$f_x(a,b,c) = \lim_{p \to 0} \frac{f(a+p, b, c) - f(a,b,c)}{p}$$
$$= \lim_{p \to 0} \frac{(a+p)b + bc + c(a+p) - ab - bc - ca}{p}$$
$$= b + c$$

$$f_y(a,b,c) = \lim_{q \to 0} \frac{f(a, b+q, c) - f(a,b,c)}{q}$$
$$= \lim_{q \to 0} \frac{a(b+q) + (b+q)c + ca - ab - bc - ca}{q}$$
$$= a + c$$

$$f_z(a,b,c) = \lim_{r \to 0} \frac{f(a, b, c+r) - f(a,b,c)}{r}$$

$$= \lim_{r \to 0} \frac{ab + b(c+r) + (c+r)a - ab - bc - ca}{r}$$
$$= b + a$$

***

Example 3: Let $f : \boldsymbol{R}^3 \to \boldsymbol{R}$ be a function defined by $f(x, y, z) = x^2 + y^2 + z^2$. Find the partial derivatives of f with respect to x, y, z at the point $(-1, 0, 2)$.

Solution: Here we use the expressions given in (10), (11) and (12). Therefore we have

$$f_x(-1, 0, 2) = \lim_{p \to 0} \frac{f(-1+p, 0, 2) - f(-1, 0, 2)}{p}$$
$$= \lim_{p \to 0} \frac{(-1+p)^2 + 0^2 + 2^2 - 5}{p}$$

$$= \lim_{p\to 0} \frac{1-2p+p^2+4-5}{p}$$
$$= \lim_{p\to 0} -2+p$$
$$= -2$$
$$f_y(-1, 0, 2) = \lim_{q\to 0} \frac{f(-1, 0+q, 2) - f(-1, 0, 2)}{q}$$
$$= \lim_{q\to 0} \frac{(-1)^2+q^2+4-5}{q}$$
$$= 0$$
$$f_z(-1, 0, 2) = \lim_{r\to 0} \frac{f(-1, 0, 2+r) - f(-1, 0, 2)}{r}$$
$$= \lim_{r\to 0} \frac{4+4r+r^2-5}{r}$$
$$= 4$$

***

You must have come across functions from $\boldsymbol{R} \to \boldsymbol{R}$ which do not possess derivatives at some points. For example, $f(x) = |x|$ cannot be differentiated at $x = 0$. Here are some example of a function of two variables whose partial derivatives fail to exist at some points.

Example 4: Let $f : \boldsymbol{R}^3 \to \boldsymbol{R}$ be defined by $f(x, y, z) = |x| + |y| + |z|$. Show that f does not possess any of the three partial derivatives at (0, 0, 0).

Solution: The partial derivative f_x of f at $(0,0,0)$ is given by

$$f_x(0, 0, 0) = \frac{f(0+p, 0, 0) - f(0,0,0)}{p} = \frac{|p|}{p}.$$

We have $\lim\limits_{p\to 0^+} f_x(0, 0, 0) = \lim\limits_{\substack{p\to 0\\ p>0}} \frac{|p|}{p} = \lim\limits_{\substack{p\to 0\\ p>0}} \frac{p}{p} = 1,$

and $\lim\limits_{p\to 0^-} f_x(0, 0, 0) = \lim\limits_{\substack{p\to 0\\ p<0}} \frac{|p|}{p} = \lim\limits_{\substack{p\to 0\\ p<0}} \frac{-p}{p} = -1.$

Hence, $\lim\limits_{p\to 0} f_x(0,0,0)$ does not exist.

Similarly we can show that $\lim\limits_{q\to 0} f_y(0,0,0)$ and $\lim\limits_{r\to 0} f_z(0,0,0)$ also do not exist.

Thus, f does not possess any of the first order partial derivatives at the point $(0, 0, 0)$.

***

Example 5: If $f : \boldsymbol{R}^2 \to \boldsymbol{R}$ is defined by

$$f(x, y) = \begin{cases} \dfrac{x}{y} + \dfrac{y}{x}, & y \neq 0, x \neq 0 \\ 0 & \text{,otherwise} \end{cases}$$

then show that $f_x(0,1)$ and $f_y(1,0)$ do not exist.

Solution: To prove this, we shall first note the following:

$$\frac{f(0+h,1)-f(0,1)}{h}=\frac{h+\frac{1}{h}-0}{h}=1+\frac{1}{h^2}$$

and $\dfrac{f(1,0+k)-f(1,0)}{k}=\dfrac{\frac{1}{k}+k-0}{k}=\dfrac{1}{k^2}+1.$

Since $\lim\limits_{h\to 0}\dfrac{1}{h^2}=\infty$, neither $\lim\limits_{h\to 0}\dfrac{f(0+h,1)-f(0,1)}{h}$, nor $\lim\limits_{k\to 0}\dfrac{f(1,0+k)-f(1,0)}{k}$ exist. So f_x and f_y do not exist, respectively, at the points $(0,1)$ and $(1,0)$.

***

Try to solve these exercises now.

E1) Let $f:\boldsymbol{R}^2\to\boldsymbol{R}$ be the constant function defined by $f(x,y)=c$ for all (x,y). Show that $f_x(a,b)=0=f_y(a,b)$ for all points (a,b).

E2) Let $f:\boldsymbol{R}^2\to\boldsymbol{R}$ be defined by

$$f(x,y)=\begin{cases}1 & if\ (x,y)\neq(0,0)\\ 0 & if\ (x,y)=(0,0)\end{cases}$$

Show that $f_x(0,0)$ as well as $f_y(0,0)$ do not exist.

E3) Let $f:\boldsymbol{R}^3\to\boldsymbol{R}$ be defined by

$$f(x,y)=\begin{cases}x & if\ (x,\ y,\ z)\neq(0,0,0)\\ 0 & if\ (x,\ y,\ z)=(0,0,0)\end{cases}$$

Show that $f_x(0,\ 0,\ 0)=1$ and $f_y(0,\ 0,\ 0)=0$ and $f_z(0,\ 0,\ 0)=0$.

From these examples and exercises you must have observed that $f_x(x,y)$ is nothing but the derivative of $f(x,y)$ considered as a function of a single variable x, treating y as a constant. Similarly, $f_y(x,y)$ is nothing but the derivative of $f(x,y)$ considering it as a function of the single variable y, and treating x as a constant. Thus, for calculating partial derivatives, we can use our knowledge of calculating derivatives of functions of a single real variable.

Rule for finding partial derivative of a two variable function $f(x,y)$

1. To find f_x, regard y as a constant and differentiate $f(x,y)$ with respect to x.
2. To find f_y, regard x as a constant and differentiate $f(x,y)$ with respect to y.

Rule for finding partial derivation of a three variable function $f(x, y, z)$

1. To find f_x, regard y and z as a constant and differentiate $f(x, y, z)$ with respect to x.
2. To find f_y, regard x and z as a constant and differentiate $f(x, y, z)$ with respect to y.
3. To find f_z, regard x and y as a constant and differentiate $f(x, y, z)$ with respect to z.

Note 1: It is important to check whether the derivative exists, when considered as a function of single variable before computing the partial derivative.

Note 2: If we do not mention a specific point at which partial derivatives are to be calculated, we mean that they are to be calculated at a general point (x, y) or (x, y, z) according as $n=2$ or $n=3$.

Note 3: Using the rules for algebra of limits, we get that the partial derivatives of the sum or difference or product of functions is equal to the sum of the partial derives of each functions.

Here is an example to illustrate this.

Example 6: Let $f: \boldsymbol{R}^2 \to \boldsymbol{R}$ be a function given by $f(x, y) = x^2 + xy + y^3$. Find $f_x(x, y)$ and $f_y(x, y)$.

Solution: Method 1: By the definition,

$$
\begin{aligned}
f_x(x, y) &= \lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^2 + (x+h)y + y^3 - x^2 - xy - y^3}{h} \\
&= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 + xy + hy + y^3 - x^2 - xy - y^3}{h} \\
&= \lim_{h\to 0} (2x + h + y) \\
&= 2x + y
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
f_y(x, y) &= \lim_{k\to 0} \frac{f(x, y+k) - f(x\ y)}{k} \\
&= \lim_{k\to 0} \frac{x^2 + x(y+k) + (y+k)^3 - x^2 - xy - y^3}{k} \\
&= \lim_{k\to 0} \frac{xk + 3y^2k + 3yk^2 + k^3}{k} \\
&= \lim_{k\to 0} (x + 3y^2 + 3yk + k^2) \\
&= x + 3y^2
\end{aligned}
$$

Method 2: We write $f(x, y) = f_1(x, y) + f_2(x, y) + f_3(x, y)$ where

$$f_1(x, y) = x^2,\ f_2(x, y) = xy,\ f_3(x, y) = y^3$$

Suppose we find the partial derivatives w.r.t. the variable x of f_1, f_2 and f_3. For instance f_1 is a function of single variable x i.e., x^2, and its derivative is $2x$.

Whereas f_2 is a function of two variables i.e., xy. For calculating the partial derivative of f_2, we treat y as a constant and therefore by direct differentiation, the derivative of f_2 is y.

The function f_3 is a function of single variable y. For calculating the partial of f_3, we treat y as a constant and therefore by direct differentiation, the derivative of f_3 is 0. Please note that the derivative of a constant function is 0.

Then the sum of the partial derivatives of f_1, f_2 and f_3 with respect to the variable x is $2x+y$. This is nothing but the partial derivative of f which we have calculated by Method 1.

Similarly we find the partial derivatives of f_1, f_2 and f_3 with respect to y, treating x as a constants.

Since f_1 is a function of single variable x, by direct differentiation we get that the partial derivative of f, with respect to y is 0.

Similarly the partial derivative of f_2 w.r.t. y is x. The partial derivative of f_3 w.r.t. y is $3y^2$.

Then the sum of the partial derivative of f_1, f_2 and f_3 w.r.t. y is $x+3y^2$. This is nothing but the partial derivative of f w.r.t. y which we have calculated by Method 1.

***

Here is another example.

Example 7: Find the partial derivatives of the following functions.

i) $z = x^3 - 4x^2y^2 + 8y^2$

ii) $z = x \sin y + y \cos x$

iii) $z = x\, e^y + y\, e^x$.

Solution: We first note that in all the three cases, the single variable functions involved are either polynomials or trigonometric or exponential functions. This ensures that the partial derivatives exist. Then we calculate the partial derivatives by applying the direct differentiation technique for single variable. Therefore we have the following:

i) $\dfrac{\partial z}{\partial x} = 3x^2 - 8xy^2; \dfrac{\partial z}{\partial y} = -8x^2y + 16y$

ii) $\dfrac{\partial z}{\partial x} = \sin y - y \sin x : \dfrac{\partial z}{\partial y} = x \cos y + \cos x$

iii) $\frac{\partial z}{\partial x} = e^y + ye^x$; $\frac{\partial z}{\partial y} = xe^y + e^x$

***

The calculation of partial derivatives is not always as simple as in these examples. In some exceptional cases, we may have to apply the definition directly as in the case of one variable. You will be able to recognize such cases with practice.

Let us consider one such situation.

Example 8: Suppose $f : \boldsymbol{R}^2 \to \boldsymbol{R}$ is defined by

$$f(x,y) = \begin{cases} \dfrac{xy}{x^4 + y^4}, & if\ (x,y) \neq (0,0) \\ 0 & ,\ if\ (x,y) = (0,0) \end{cases}$$

Find the two partial derivatives at the points $(0,0)$, $(a,0)$, $(0,b)$ and (a,b), where $a \neq 0,\ b \neq 0$.

Solution: Here we apply the definition directly.

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0.$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,0+k) - f(0,0)}{k} = \lim_{k\to 0} \frac{0-0}{k} = 0.$$

$$f_x(a,0) = \lim_{h\to 0} \frac{f(a+h,0) - f(a,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0.$$

$$f_y(a,0) = \lim_{k\to 0} \frac{f(a,0+k) - f(a,0)}{k} = \lim_{k\to 0} \frac{\frac{ak}{a^4+k^4}}{k} = \frac{1}{a^3}$$

$$f_x(0,b) = \lim_{h\to 0} \frac{f(0+h,b) - f(0,b)}{h} = \lim_{h\to 0} \frac{\frac{bh}{b^4+h^4}}{h} = \frac{1}{b^3}$$

$$f_y(0,b) = \lim_{k\to 0} \frac{f(0,b+k) - f(0,b)}{k} = \lim_{k\to 0} \frac{0-0}{k} = 0.$$

$$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{(a+h)b}{(a+h)^4 + b^4} - \frac{ab}{a^4+b^4}}{h}$$

$$= \lim_{h\to 0} \frac{(ab+hb)(a^4+b^4) - (ab)(a^4 + 4a^3h + 6a^2h^2 + 4ah^3 + h^4 + b^4)}{h(a^4+b^4)\left[(a+h)^4 + b^4\right]}$$

$$= \lim_{h\to 0} \frac{b(a^4+b^4) - (ab)\left(4a^3 + 6a^2h + 4h^2a + h^3\right)}{(a^4+b^4)\left[(a+h)^4 + b^4\right]}$$

$$= \frac{b^5 - 3a^4 b}{\left(a^4 + b^4\right)^2}$$

$$f_y(a,b) = \lim_{k \to 0} \frac{f(a,b+k) - f(a,b)}{k}$$

$$= \lim_{k \to 0} \frac{\dfrac{a(b+k)}{a^4 + (b+k)^4} - \dfrac{ab}{a^4 + b^4}}{k}$$

$$= \lim_{k \to 0} \frac{(ab+ab)\left(a^4 + b^4\right) - (ab)\left(a^4 + b^4 + 4b^3 k + 6b^2 k^2 + 4bk^3 + k^4\right)}{k\left(a^4 + b^4\right)\left[a^4 + (b+k)^4\right]}$$

$$= \lim_{k \to 0} \frac{a\left(a^4 + b^4\right) - (ab)\left(4b^3 + 6b^2 k + 4bk^2 + k^3\right)}{\left(a^4 + b^4\right)\left[a^4 + (b+k)^4\right]}$$

$$= \frac{a^5 - 3ab^4}{\left(a^4 + b^4\right)^2}.$$

***

You may note here that by direct differentiation, we could have obtained $f_x(a,b)$ and $f_y(a,b)$ for $(a,b) \neq (0,0)$ directly, but not $f_x(0,0)$ or $f_y(0,0)$. Can you say why? Since f is defined as a quotient of two polynomial functions for all $(x, y) \neq (0,0)$ and therefore we can use direct differentiation to calculate partial derivatives at these points. But to calculate $f_x(0,0)$ or $f_y(0,0)$ we need to use $f(0,0)$, which is not defined by the same quotient. Also note that after obtaining $f_x(a,b)$ and $f_y(a,b)$, we could have substituted $a = 0$ or $b = 0$ to get $f_x(0,b)$, $f_y(0,b)$, $f_x(a,0)$ and $f_y(a,0)$.

Why don't you try some exercises now?

E4) If $f(x, y) = 2x^2 - xy + 2y^2$, find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ at the point (1, 2).

E5) Find all the first order partial derivatives of the following functions.

a) $\sin\left(x^2 - y\right)$

b) $\dfrac{1}{\sqrt{x + y^2 + z^2 + 1}}$

c) $y \sin xz$

d) x^y

e) $x^3 y + e^{xy^2}$

E6) Show that the functions $u = e^x \cos y$, $v = e^x \sin y$ satisfy the conditions $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$.

E7) Let $f : \mathbf{R^2} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ be two differentiable functions.
Let $F(x, y) = f(x) + g(y)$ for all x and y. Show that
$F_x(x, y) = f'(x)$ and $F_y(x, y) = g'(y)$.

E8) Let f and g be two real-valued functions for which $f_x(a,b)$ and $g_x(a,b)$ exist. Show that $\frac{\partial(f+g)}{\partial x}$ exists at (a,b) and is equal to $f_x(a,b) + g_x(a,b)$. Is the converse true? Justify your answer.

Through these exercises you must have gained enough practice for calculating the partial derivatives. In the next sub-section we shall try to interpret the notion of partial derivatives geometrically.

3.2.2 Geometric Interpretation

In the case of a real-valued function f of one variable x, you know that the derivative $f'(x)$ gives the slope of the tangent to the curve $y = f(x)$ at a generic point (x, y). We shall now try to visualize the partial derivatives of real-valued function of two variables. Such a function, as you might know, represent a surface in $\mathbf{R}^3$.

To get a geometrical interpretation, let us consider $f(x, y)$ be a real-valued function of two variables and let $S = \{(x, y, z) \mid z = f(x, y)\}$ be the surface represented by the function $f(x, y)$ in $\mathbf{R}^3$. Suppose that $f(x, y)$ has both the partial derivatives at a point (a,b) and $c = f(a,b)$. Then the point $P(a,b,c)$ lies on the surface S. When we fix $y = b$, we get that the plane $y = b$, which is parallel to XOZ plane and passes through P, will intersect the surface in a curve C_1 (see Fig. 1).

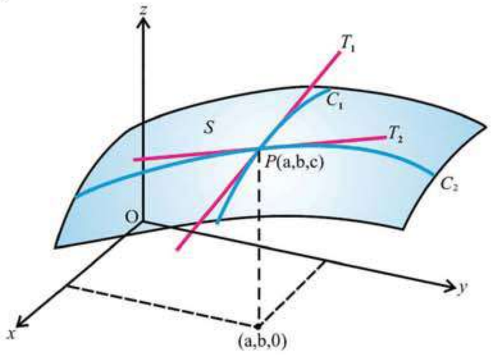


Fig. 1: The curves marked in blue colour are C_1 and C_2 and the curves marked in red colour are the tangents T_1 and T_2.

Likewise when we fix $x = a$, we get that the plane $x = a$ which is parallel to YOZ plane and passes through the point $P(a, b, c)$, will intersect the surface in a curve C_2. Note that both the curves C_1 and C_2 pass through the point $P(a, b, c)$ (see Fig. 1). Notice that the curve C_1 is the graph of the one-variable function $g_1(x) = f(x, b)$. You already know that the derivative g_1' of g_1 at a point (a) gives the slope of the tangent at (a, b) (See Fig. 1 showing tangent T_1). But $g_1'(a) = f_x(a, b)$ i.e., the partial derivative of f w.r.t. x at similarly the curve C_2 is the graph of the function $g_2(y) = f(a, y)$. You already know that the derivative g_2' of g_2 at (a, b) gives the slope of the tangent at (a,b). [See Fig. 1 showing the tangent T_2]. But $g_2'(a) = f_y(a, b)$ - the partial derivative of f w.r.t. y. Thus the partial derivatives $f_x(a, b)$ and $f_y(a, b)$ can be interpreted geometrically as the slopes of the tangent lines to the curves of intersects of the planes $y = b$ and $x = a$, respectively.

We shall illustrate this with an example.

Example 9: Find f_x and f_y for the function $f(x, y) = 4 - x^2 - 2y^2$ at the point $(1, 1, 1)$ and geometrically interpret the values.

Solution: We first note that $f(1, 1) = 1$. Therefore the point $(1, 1, 1)$ lies on the surface given by $z = 4 - x^2 - 2y^2$.

Following the method in Example 6, we get that

$$f_x(x, y) = -2x$$

$$\therefore \quad f_x(1, 1) = -2$$

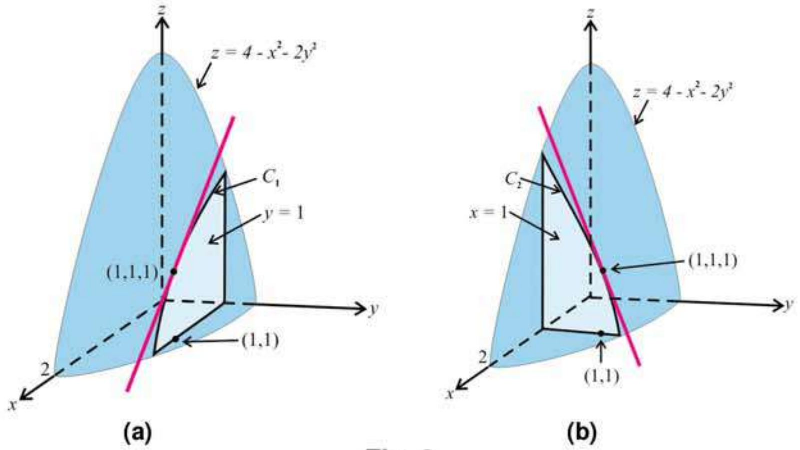
and

$$f_y(x, y) = -4y$$

$$f_y(1, 1) = -4$$

To find the geometric interpretation, we obtain the curves C_1 and C_2.
To obtain C_1, we put $y = 1$ in the expression for f and we get the curve $z = 2 - x^2$, $y = 1$ (See Fig. 2 (a)). You know from the one-variable case that the slope of the curve at $x = 1$ is obtained by finding the derivative at $x = 1$, which is equal to -2. This is the same as $f_x(1, 1) = -2$.

Similarly to obtain C_2, we put $x = 1$ in the expression for f and we get the curve $z = 3 - 2y^2$, $x = 1$ (see Fig. 2(b)) and to obtain the slope of this curve at $y = 1$, we find the derivative of z at $y = 1$ which is -4. This is same as $f_y(1, 1) = -4$. This is illustrated by the following figure.



(a) (b)

Fig. 2

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Note: The geometrical interpretation need to be explained only if it is required. Otherwise it is enough to calculate the partial derivatives at the indicated points.

Example 10: Find the slopes of the tangents to the curves of intersection of the planes $x = 2$ and $y = 3$ and the surface $z = xy + 3x^2$ at the point $(2,3,18)$.

Solution: We know that the slope of the tangent to the curve of intersection of the plane $x = 2$ and the surface $z = xy + 3x^2$ at the point $(2,3,18)$ will be given by $\left(\frac{\partial z}{\partial y}\right)_{(2,3)}$.

Now, $\left(\frac{\partial z}{\partial y}\right)_{(2,3)} = (x)_{(2,3)} = 2.$

Therefore, the slope of the tangent at the point $(2,3,18)$ to the curve of intersection of the plane x = 2 and the surface $z = xy + 3x^2$ is 2.

Similarly, since $\left(\frac{\partial z}{\partial x}\right)_{(2,3)} = (y + 6x)_{(2,3)} = 15,$ therefore, the slope of the tangent at the point $(2,3,18)$ to the curve of intersection of the plane $y = 3$ and the surface $z = xy + 3x^2$ is 15.

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You should be able to do this exercise now.

E9) Find the slope of the tangent at the point $(1,2,14)$ to the curve of intersection of the plane $y = 2$ and the surface $z = 2x^2 + 3y^2$.

You know that if a function $f : \mathbf{R} \to \mathbf{R}$ is differentiable at a point, it is also continuous at that point. In the next sub-section we shall see whether any such link exists between the continuity and the existence of partial derivatives of functions from $\mathbf{R}^2 \to \mathbf{R}$.

3.2.3 Continuity and Partial Derivatives

If the two partial derivatives of a function $f(x, y)$ exist, then what can we infer from this? Let's see.

Suppose $f(x, y)$ is a real-valued function having partial derivatives at a point (a, b). Then for $h \neq 0,$ we have

$$f(a+h,b) - f(a,b) = \frac{f(a+h,b) - f(a,b)}{h} \times h$$

and therefore,

$$\begin{aligned}\lim_{h\to 0}\,[f(a+h,b) - f(a,b)] &= \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h} \lim_{h\to 0} h \\ &= f_x(a,b).0 \\ &= 0.\end{aligned}$$

Therefore, we can say that $f(a+h,b) \to f(a,b)$ as $h \to 0$, i.e., $f(x,y) \to f(a,b)$ as (x,y) approaches (a,b) along a line parallel to the x-axis. Similarly, the existence of the other partial derivative shows that $f(x,y) \to f(a,b)$ as (x,y) approaches (a,b) along a line parallel to the y-axis. The existence of $f_x(a,b)$ and $f_y(a,b)$ of f does not give us any further information. So we do not know whether the limit of $f(x,y)$ exists or not if $(x,y) \to (a,b)$ along any other path. But you have learnt in Unit 2 that $f(x,y)$ would be continuous at (a,b) if $f(x,y) \to f(a,b)$ as $(x,y) \to (a,b)$ along any path (which need not even be a straight line).

Thus, it is clear from the above discussion that the mere existence of partial derivatives need not ensure the continuity of the function at the point. This, in fact, is the case shown in the following example. Later on we shall see that if the partial derivatives satisfy some additional requirements, then their existence does imply continuity.

Example 11: Let the function $f : \boldsymbol{R}^2 \to \boldsymbol{R}$ be defined by

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$$

Find the partial derivative of f at (0, 0), this is, $f_x(0,0)$ and $f_y(0,0)$. Check whether f is continuous at (0, 0).

Solution: We have,

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0, \text{ and}$$
$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,0+k) - f(0,0)}{k} = \lim_{k\to 0} \frac{0-0}{k} = 0.$$

So, f possesses both the first order partial derivatives at (0, 0) and they are equal to 0 at (0, 0). However, this function is not continuous at the point (0, 0) since $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist. (See Unit 2, E2).

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We know that a real-valued continuous function of a real variable need not be differentiable. The same is true for functions of several variables. This means a function of several variables which is continuous at a point need not have any of the partial derivatives at the point.

If you have understood the above examples, you should be able to solve the following exercises.

E10) Let $f(x,y) = \sqrt{x^2+y^2}$ for all $(x,y) \in \boldsymbol{R}^2$. Show that f is continuous at (0, 0). but $f_x(0,0)$ as well as $f_y(0,0)$ do not exist.

E11) Let $f(x,y)=\begin{cases} x\sin\dfrac{1}{x}+y\sin\dfrac{1}{y}, & xy\neq 0 \\ 0 & , xy=0 \end{cases}$.

Show that $f_x(0,0)$ as well as $f_y(0,0)$ exist. Also show that f is continuous at (0, 0).

E12) Show that the function $f:\boldsymbol{R}^2\to\boldsymbol{R}$ defined by

$$f(x,y)=\begin{cases} \dfrac{x^2y}{x^4+y^2}, & \text{if } x^4+y^2\neq 0 \\ 0 & , \text{if } x=0=y \end{cases}$$

possess first partial derivatives everywhere including the origin but the function is discontinuous at the origin.

E13) Let $f(x,y)=\begin{cases} \dfrac{xy}{|y|} & , \text{if } y\neq 0 \\ 0 & , \text{if } y=0 \end{cases}$

a) Prove that f is continuous at (0, 0), and both $f_x(0,0)$ and $f_y(0,0)$ exist.
b) Show that $f_x(1,0)$ exists, but $f_y(1,0)$ does not.

In the examples and exercises above we have observed that the existence of partial derivatives does not imply continuity. However, if the partial derivatives satisfy some more conditions, then we can ensure continuity. You will study this in Theorem 1. We have omitted the proof of the theorem as it is beyond the scope of this course.

We now state Theorem.

Theorem 1: Let f be a real-valued function of two variables defined in a neighbourhood N of a point (a,b) such that one of the first order partial derivatives exists at all points $(x,y)\in N$ and is bounded on N, whereas the other partial derivative exists at the point (a,b). Then the function f is continuous at the point (a,b).

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Corollary 1: Let f be real-valued function of two variables defined in a neighbourhood N of a point (a,b), such that both the partial derivatives of f exists at all points of N and one of them is bounded on N. Then the function f is continuous everywhere on N.

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Note that the condition given in Theorem 1 are only sufficient and are not necessary. We have already seen in Example 11 that a function may be continuous even when none of the partial derivatives exist.
In the next example we explain how to use Corollary 1 to prove the continuity of the given function.

Example 12: Apply corollary 1 and check whether the function $f : \boldsymbol{R}^2 \to \boldsymbol{R}$ given by $f(x, y) = ye^x$ is continuous everywhere.

Solution: We first note that $f_x(x, y) = ye^x$ and $f_y(x, y) = e^x$. Let (a,b) be any point $\boldsymbol{R}^2$, and consider a neighbourhood.

$N = \left\{(x, y) \middle| \sqrt{(x-a)^2 + (y-b)^2} < 1\right\}$ of (a,b). Now, $f_x(x, y)$ and $f_y(x, y)$ exist at all points of N. Further, since $|x-a| \le \sqrt{(x-a)^2 + (y-b)^2}$, for all $(x, y) \in N$ we have $|x-a| < 1$, i.e., $a-1 < x < a+1$, i.e., $e^{a-1} < e^x < e^{a+1}$.

So f_y is bounded on N. Therefore, in view of Corollary 1, f is continuous at the point (a,b). Since (a,b) was any arbitrary point of $\boldsymbol{R}^2$, we can say that f is continuous everywhere on $\boldsymbol{R}^2$.

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Try this exercise now.

E14) Use Corollary 1 to show that the following functions are continuous everywhere on $\boldsymbol{R}^2$.

a) $f(x, y) = xe^y$

b) $f(x, y) = 3xy$

In this section we have seen that the mere existence of partial derivatives does not imply continuity. This show that the concept of partial derivatives does not generalize the concept of differentiation of functions from $\boldsymbol{R} \to \boldsymbol{R}$. In the next section we'll introduce a concept which is a generalization of the concept of differentiation of real-valued functions of a single variable.

3.3 DIFFERENTIABILITY OF FUNCTIONS FROM R^2 TO R

We begin with the definition of derivative of a single variable function. We say that a function f from $\boldsymbol{R}$ to $\boldsymbol{R}$ is differentiable at a point c if the following limit exists

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = A \qquad \ldots (13)$$

Then A is called the derivative of f at c and is denoted by $f'(c)$.

We put $x = c + h$. Then the equation Eqn. (13) can be written as

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c) \qquad \ldots (14)$$

Then, it is the same as writing

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} - f'(c) = 0$$

i.e. $$\lim_{x \to c} \frac{f(x) - f(c) - f'(c)(x - c)}{x - c} = 0$$

i.e. $$\lim_{h \to 0} \frac{f(c + h) - f(c) - f'(c)h}{h} = 0.$$

This is the same as writing

$$f(c + h) - f(c) = f'(c)h + h\eta(h)$$

where $\eta(h) \to 0$

we know that if the derivative exists at c, then $f'(c) = A$, a constant.
Therefore we have

$$f(c + h) - f(c) = Ah + h\eta(h) \qquad \ldots (15)$$

where $\eta(h) \to 0$ as $h \to 0$.

This definition of differentiability of a function given in (15) of a single variable can be generalized in a natural way for functions of several variables. In this section we shall study the differentiability of real-valued functions of two variables. Here is it's definition.

Definition 2: Let f be a real-valued function defined in a neighbourhood N of a point (a, b). We say that the function f is differentiable at (a, b), if the following holds:

$f(a + h, b + k) - f(a, b) = Ah + Bk + h\phi(h, k) + k\psi(h, k)$, where

- h and k are real numbers such that $(a + h, b + k) \in N$,
- A and B are constants independent of h and k but dependent on the function f and the point (a, b).
- ϕ and ψ are two functions tending to zero as $(h, k) \to (0,0)$.

We would like to make an important remark here.

Remark 2: (i) In the literature you may find another definition of differentiability which we give below.

Let f be a real-valued function defined in a neighbourhood N of a point (a, b). Then the function f is said to be differentiable at the point (a, b), if there exist two constants A and B (depending on f and the point (a, b) only) such that $f(a + h, b + k) - f(a, b) = Ah + Bk + \sqrt{h^2 + k^2}\,\phi(h, k)$ where $\phi(h, k)$ is a real-valued function such that $\phi(h, k) \to 0$ as $(h, k) \to (0,0)$.
The equivalence of these two definitions can be established easily by using the identity $\sqrt{h^2 + k^2} = h\left(\frac{h}{\sqrt{h^2 + k^2}}\right) + k\left(\frac{k}{\sqrt{h^2 + k^2}}\right)$.

ii) We, make another observation here For a function f from $\mathbf{R}$ to $\mathbf{R}$, if $f'(x_0)$ exists, then we can approximate $f(x)-f(x_0)$ by the linear function $(x-x_0)f'(x_0)$ near x_0. Similarly, if a two variable function $g(x,y)$ is differentiable at (a,b), then $g(x,y)-g(a,b)$ can be approximated by the linear function $(x-a)A+(y-b)B$ in a neighbourhood of the point (a,b). We now illustrate the definition of differentiability with the help of a few examples.

Example 13: Let $f(x,y)=x^2+y^2$. Show that f is differentiable at any point (a,b).

Solution: Let (a, b) be an arbitrary point.

For any two real numbers h and k, we have

$$\begin{aligned} f(a+h,b+k)-f(a,b) &= (a+h)^2+(b+k)^2-(a^2+b^2) \\ &= 2ah+2bk+hh+kk \end{aligned}$$

If we set $A=2a,\ B=2b,\ \phi(h,k)=h,\ \psi(h,k)=k$, then $f(a+h,b+k)-f(a,b)=Ah+Bk+h\phi(h,k)+k\psi(h,k)$, where A and B are constants independent of h and k, $\phi(h,k)\to 0$ and $\psi(h,k)\to 0$ as $(h,k)\to(0,0)$. Thus f is differentiable at the point (a,b).

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Example 14: Let $f(x,y)=\frac{x}{y}$. Show that f is differentiable at all points (a,b) in the domain of definition of the function.

Solution: Since f is not defined for $y=0$, we take $b\neq 0$. Let h and k be two real numbers such that $(a+h,b+k)$ is a point in a neighbourhood N of (a,b), which is contained in the domain of f. Then $b+k\neq 0$, and

$$\begin{aligned} f(a+h,b+k)-f(a,b) &= \frac{a+h}{b+k}-\frac{a}{b} \\ &= \frac{a}{b+k}-\frac{a}{b}+\frac{h}{b+k} \\ &= -\frac{ak}{b(b+k)}+\frac{h}{b+k} \\ &= -\frac{a}{b^2}\left[1-\frac{k}{b+k}\right]k+\frac{h}{b}\left[1-\frac{k}{b+k}\right] \\ &= \frac{1}{b}h-\frac{a}{b^2}k+h\left(\frac{-k}{b(b+k)}\right)+k\left(\frac{ak}{b^2(b+k)}\right) \end{aligned}$$

Set $A=\frac{1}{b},\ B=-\frac{a}{b^2},\ \phi(h,k)=\frac{-k}{b(b+k)},\ \psi(h,k)=\frac{ak}{b^2(b+k)}$.

Then $f(a+h,b+k)-f(a,b)=Ah+Bk+h\phi(h,k)+k\psi(h,k)$, where A and B are constants independent of h and k. Also $\phi(h,k)\to 0$ and $\psi(h,k)\to 0$ as $(h,k)\to(0,0)$. Hence, f is differentiable at the point (a,b).

***

Here is an example of a function which is not differentiable.

Example 15: We will prove that the function given by $f(x, y) = |x| + |y|$ is not differentiable at $(0,0)$.

Solution: Suppose, if possible, that f is differentiable at $(0,0)$. Then $f(0+h,\, 0+k) - f(0,0) = Ah + Bk + h\phi(h,k) + k\psi(h,k)$ where A and B are constants, $\phi(h,k) \to 0$ and $\psi(h,k) \to 0$ as $(h,k) \to (0,0)$.

Therefore, $|h| + |k| = Ah + Bk + h\phi(h,k) + k\psi(h,k)$.

Let $h = 0$ and $k > 0$. Then $k = Bk + k\psi(0,k)$.

Taking limits on both sides as $(h,k) \to (0,0)$, we get $B = 1,$ because $\psi(0,k) \to 0$.

Now let $h = 0$ and $k < 0$. Then
$$-k = Bk + k\psi(0,k)$$
$$-1 = B + \psi(0,k)$$

Taking limits on both sides as $(h,k) \to (0,0)$, we get $B = -1$, because $\psi(0,k) \to 0.$

Thus the assumption that the given function is differentiable at $(0,0)$ leads us to the contradiction $B = 1 = -1.$ Hence $|x| + |y|$ is not differentiable at $(0,0)$.

***

Now see if you can do these exercises.

E15) We have listed some results about the differentiability of real-valued functions of a real variable in the first column of the following table. Write analogous statements for real-valued functions of two variables in the second column, and check whether each of them is true.

One Variable	Two variables
a) A constant function is differentiable b) If f is differentiable at $a \in \boldsymbol{R}$, then $cf\,(c \in \boldsymbol{R})$ is also differentiable at a. c) If f and g are differentiable at $a \in \boldsymbol{R}$, then $f \pm g$ is also differentiable at a. d) If f, g are differentiable at $a \in \boldsymbol{R}$ then fg is also differentiable at a.	

E16) Show that the function $x^2 + y + xy$ is differentiable at $(0,0)$.

E17) Show that $\cos(x+y)$ is differentiable at the point $\left(\frac{\pi}{4},\frac{\pi}{4}\right)$.

E18) Show that the following function f is not differentiable at $(0,0)$.

$$f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0)\\ 0 & ,(x,y)=(0,0)\end{cases}$$

In the case of a real-valued function of real variable, continuity does not imply differentiability. The same is true or real-valued functions of two variables. Consider the function $f(x,y)=|x|+|y|$ of Example 15. According to E9) d) of Unit 2, it is continuous at (0, 0). In Example 15 we have seen that it is not differentiable at (0, 0). So continuity at a point does not imply differentiability at that point. However, every function which is differentiable at a point, is also continuous at that point. This is proved in the theorem that follows.

Theorem 2: Let f be a real-valued function defined in a neighbourhood N of a point (a,b). If f is differentiable at (a,b), then f is continuous at (a,b).

Proof: Let h and k be two real numbers such that $(a+h,b+k)\in N$. Then differentiability of f at (a,b) implies that there exist two constants A, B and two functions $\phi(h,k)$, $\psi(h,k)$ such that

$$f(a+h,b+k)-f(a,b)=Ah+Bk+h\phi(h,k)+k\psi(h,k) \qquad \ldots (16)$$

where $\phi(h,k)\to 0,\ \psi(h,k)\to 0$ as $(h,k)\to(0,0)$.

Now taking the limit on both sides of the Eqn. (16) as $(h,k)\to(0,0)$, we get

$\lim\limits_{(h,k)\to(0,0)}\left(f(a+h,b+k)-f(a,b)\right)=0$, or

$\lim\limits_{(h,k)\to(0,0)} f(a+h,b+k)=f(a,b)$

– ■ –

This shows that the function f is continuous at (a,b).

We can use this result to establish the non-differentiability of a function at a given point. For instance, we have seen in Unit 2. that the function

$$f(x,y)=\begin{cases}\dfrac{x^2-y^2}{x^2+y^2}, & if\ \ x^2+y^2\neq 0\\ 0 & , if\ \ x=y=0\end{cases}$$

is not continuous at (0, 0). Thus, in view of Theorem 4, we can conclude that this function is not differentiable at (0, 0).

You can now use Theorem 2 to solve this exercise.

E19) Show that the following functions are not differentiable at $(0,0)$ by showing that they are discontinuous at $(0,0)$.

a) $f(x,y)=\begin{cases}\dfrac{x^2-y^2}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

b) $f(x,y)=\begin{cases}\dfrac{2xy}{x^2+y^2}, & (x,y)\neq(0,0)\\ 2, & (x,y)=(0,0)\end{cases}$

c) $f(x,y)=\begin{cases}\dfrac{x^2+y^2}{x-y}, & x\neq y\\ 0, & x=y\end{cases}$

d) $f(x,y)=\begin{cases}\dfrac{x^5}{x^4+y^4}, & (x,y)\neq(0,0)\\ 3, & (x,y)=(0,0)\end{cases}$

So far we have not said anything about the values of the constants occurring in the definition of differentiability of a function at a point. We shall now show (Theorem 3) that the constant A, B mentioned in Definition 2 are nothing but the two partial derivatives of the function under consideration at a point. This, in particular, would show that if a function is differentiable at a point, then it has both the partial derivatives at that point. But is the converse true? That is, if a function has both the partial derivatives at a point, can we conclude that it is differentiable there? No. The existence of partial derivatives, as we have seen in Example 11, does not guarantee even continuity. So, obviously, it cannot guarantee differentiability.

Look at the following theorem now.

Theorem 3: Let f be a real-valued function defined in a neighbourhood N of the point (a,b). If f is differentiable at (a,b), then f possesses both the partial derivatives at (a,b).

Proof: Let h,k be real numbers such that $(a+h,b+k)\in N$. Since f is differentiable at the point (a,b), it follows that

$$f(a+h,b+k)-f(a,b)=Ah+Bk+h\phi(h,k)+k\psi(h,k)$$

where A and B are constants $\phi(h,k)\to 0$, $\psi(h,k)\to 0$ as $(h,k)\to(0,0)$. You can see from Fig. 3, that if $(a+h,b+k)\in N$, then $(a+h,b)$ and $(a,b+k)$ also belong to N . so if we let $k=0$ in the above equation, then we get
$f(a+h,b)-f(a,b)=Ah+h\phi(h,0)$

i.e., $\dfrac{f(a+h,b)-f(a,b)}{h}=A+\phi(h,0)$ for $h\neq 0$.

Therefore, $\lim\limits_{h\to 0}\dfrac{f(a+h,b)-f(a,b)}{h}=A,$

i.e., $f_x(a,b)=A.$

In the same way we can show that $f_y(a,\, b)$ exists and $f_y(a,b)=B$

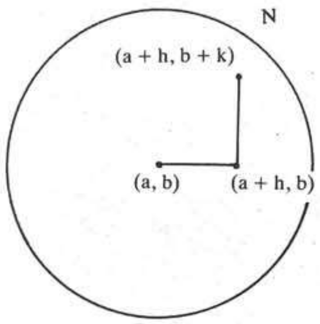


Fig. 3

$\therefore f(a+h,b+k)-f(a,b)=hf_x(a,b)+kf_y(a,b)+h\phi(h,k)+k\psi(h,k)$
where $\phi(h,k)\to 0$ and $\psi(h,k)\to 0$ as $(h,k)\to(0,0)$.

– ■ –

From the arguement given above we see that for small values of h and k we can approximate $f(a+h,b+k)-f(a,b)$ by the expression $hf_x(a,b)+kf_y(a,b)$. This expression is given a special name, as you will now see.

Definition 3: Let $f(x,y)$ be a real-valued function defined in a neighbourhood of the point (a,b). If $f(x,y)$ is differentiable at (a,b), then the linear function $T:\boldsymbol{R}^2\to\boldsymbol{R}$ defined by

$$T(h,k)=hf_x(a,b)+kf_y(a,b)$$

is called the differential of f at (a,b). It will be denoted by $df(a,b)$.

We will now give an example to show that a function may posses both the partial derivatives and still not be differentiable.

Example 16: If $f(x,y)=\begin{cases}\dfrac{x^3-y^3}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0 & ,\ (x,y)=(0,0)\end{cases}$

then show that f possesses both the first order partial derivatives at the point (0, 0), but is not differentiable at (0, 0).

Solution: Now, $\lim\limits_{h\to 0}\dfrac{f(0+h,0)-f(0,0)}{h}=\lim\limits_{h\to 0}\dfrac{h-0}{h}=1$

and $\lim\limits_{k\to 0}\dfrac{f(0,0+k)-f(0,0)}{k}=\lim\limits_{k\to 0}\dfrac{-k-0}{k}=-1$

Therefore, both the first order partial derivatives exist at the point (0, 0) and $f_x(0,0)=1,\ f_y(0,0)=-1$.

Suppose, if possible, that f is differentiable at (0, 0). Then, by Remark 2 (i),
$f(0+h,0+k)-f(0,0)=hf_x(0,0)+kf_y(0,0)+\sqrt{h^2+k^2}\ \phi(h,k)$
where $\phi(h,k)\to 0$ as $(h,k)\to(0,0)$

Now, $\phi(h,k)=\dfrac{f(h,k)-h+k}{\sqrt{h^2+k^2}}$

This means that $\lim\limits_{(h,k)\to(0,0)}\dfrac{f(h,k)-h+k}{\sqrt{h^2+k^2}}=0$

Now, if $h=r\cos\theta,\ k=r\sin\theta$, then

$$\frac{f(h,k)-h+k}{\sqrt{h^2+k^2}}=\cos^3\theta-\sin^3\theta-\cos\theta+\sin\theta.$$

Therefore,

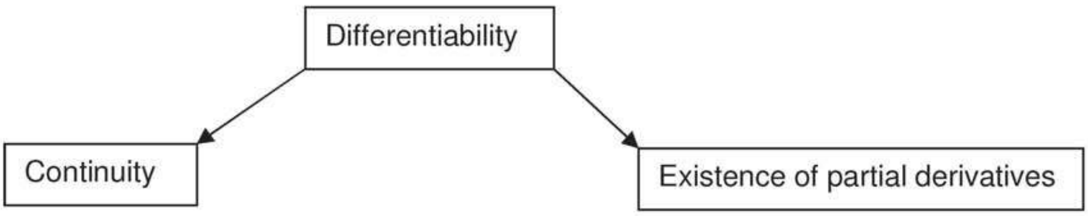
$$0=\lim_{(h,k)\to(0,0)}\frac{f(h,k)-h+k}{\sqrt{h^2+k^2}}=\lim_{r\to 0}\left(\cos^3\theta-\sin^3\theta-\cos\theta+\sin\theta\right) \qquad \ldots (17)$$

Now since the expression $\cos^3\theta-\sin^3\theta-\cos\theta+\sin\theta$ is independent of r, Eqn. (17) implies that $\cos^3\theta-\sin^3\theta-\cos\theta+\sin\theta=0$ for all θ.

But this is not possible. So we have arrived at a contradiction, proving that the given function is not differentiable at (0, 0).

***

We now present the result of this section in the following chart.



Note that the arrows in this chart cannot be reversed. We shall now give (without proof) a sufficient set of conditions, which would ensure the differentiability of the function under consideration.

Theorem 4: If f is a real-valued function defined in a neighbourhood of (a,b) such that

i) f_x is continuous at (a,b), and
ii) f_y exists at (a,b).

then f is differentiable at the point (a,b).

– ■ –

Similarly, the statement that f is differentiable at (a,b) if f_x exists at (a,b) and f_y is continuous at (a,b) is true. Thus, the continuity of one of the partial derivative and the existence of the other guarantees the differentiability of the function under consideration.

Now a function, both of whose partial derivatives are continuous, is given a special name. Here is the precise definition.

Definition 4: A real-valued function f of two variables is said to be **continuously differentiable** at a point (a,b) if both the first order partial derivatives exist in a neighbourhood of (a,b) and are continuous at the point (a,b).

Note that the above definition requires that a neighbourhood of (a,b) should be contained in the domain D of the given function.

An immediate consequence of Definition 4 and Theorem 4 is the following.

Theorem 5: A function, which is continuously differentiable at a point is differentiable at that point.

– ■ –

Now we shall illustrate the above discussion with some examples.

Example 17: Let $f: \boldsymbol{R}^2 \to \boldsymbol{R}$ be a function defined by

$$f(x,y)=\begin{cases} xy\,\dfrac{x^2-y^2}{x^2+y^2}, & if \quad x^2+y^2 \neq 0 \\ 0 \quad , & if \quad x=0=y \end{cases}$$

Prove that f is differentiable at (0, 0).

Solution: We shall prove the result using Theorem 6.

Now $f_x(0,0)=\lim\limits_{h\to 0} \dfrac{f(h,0)-f(0,0)}{h}$

$=0$

Similarly, $f_y(0,0)=0$ and for $(x,y)\neq(0,0)$,

$$f_y(x,y)=\frac{x^4y+4x^2y^3-y^5}{\left(x^2+y^2\right)^2}$$

Using polar coordinates, $x=r\cos\theta,\ y=r\sin\theta$, we get

$\left|f_x(x,y)\right|=r\left|\left(\cos^4\theta\sin\theta+4\cos^2\theta\sin^3\theta-\sin^5\theta\right)\right|$

$\leq 6\sqrt{x^2+y^2}$, since $\sin\theta\leq 1$ and $\cos\theta\leq 1$.

This can be made less than a given ε if we take $|x|<\dfrac{\varepsilon}{\sqrt{72}}$ and $|y|<\dfrac{\varepsilon}{\sqrt{72}}$.

This means that $\lim\limits_{(x,y)\to(0,0)} f_x(x,y)=f_x(0,0)$. Therefore, f_x is continuous at $(0,0)$. Thus f satisfies the conditions (i) and (ii) in Theorem 6. Consequently, f is differentiable at $(0,0)$.

***

Example 18: Show that the function $f(x,y)=e^{x+y}\sin x+x^2+2xy$ is continuously differentiable.

Solution: We first note that we compute $f_x(x,y)$ and $f_y(x,y)$. We have

$f_x(x,y)=e^{x+y}\sin x+e^{x+y}\cos x+2x+2y$

and

$f_y(x,y)=e^{x+y}\sin x+2x$

Since both f_x and f_y are continuous everywhere. It follows that f is continuously differentiable.

***

The following example shows that the condition stated in Theorem 6 are sufficient but not necessary. That is, a function can be differentiable at a point even when none of the partial derivatives is continuous at that point.

Example 19: Consider the function $f : \boldsymbol{R}^2 \to \boldsymbol{R}$ given by

$$f(x,y)=\begin{cases} x^2\sin\dfrac{1}{x}+y^2\sin\dfrac{1}{y}, & \text{if } xy\neq 0 \\ x^2\sin\dfrac{1}{x}, & \text{if } x\neq 0 \text{ and } y=0 \\ y^2\sin\dfrac{1}{y}, & \text{if } x=0 \text{ and } y\neq 0 \\ 0 \quad , & \text{if } x=0=y \end{cases}$$

Prove that f is differentiable at (0, 0), but neither f_x nor f_y is continuous at (0, 0).

Solution: Here $f_x(x,y)=\begin{cases} 2x\sin\dfrac{1}{x}-\cos\dfrac{1}{x} & \text{if } x\neq 0 \\ 0 & \text{if } x=0 \end{cases}$

and $f_y(x,y)=\begin{cases} 2y\sin\dfrac{1}{y}-\cos\dfrac{1}{y} & \text{if } y\neq 0 \\ 0 & \text{if } y=0 \end{cases}$

Since $\lim\limits_{t\to 0}\cos\dfrac{1}{t}$ does not exist, and $\lim\limits_{t\to 0} t\sin\dfrac{1}{t}=0,$ it follows that $\lim\limits_{(x,y)\to(0,0)} f_x(x,y)$ and $\lim\limits_{(x,y)\to(0,0)} f_y(x,y)$ do not exist.

This means that f_x and f_y are discontinuous at (0, 0).

However, $f(h,k)-f(0,0)=0.h+0.k+h\left(h\sin\dfrac{1}{h}\right)+k\left(k\sin\dfrac{1}{k}\right),$

where $\lim\limits_{(h,k)\to(0,0)} h\sin\dfrac{1}{h}=0=\lim\limits_{(h,k)\to(0,0)} k\sin\dfrac{1}{k}.$

Therefore, f is differentiable at (0, 0).

***

See if you can solve these exercises now.

E20) Show that the function $f : \boldsymbol{R}^2 \to \boldsymbol{R}$ defined by

$$f(x,y)=\begin{cases} x\sin\frac{1}{x}+y\sin\frac{1}{y}, & xy\neq 0 \\ x\sin\frac{1}{x}, & y=0\ ,\ x\neq 0 \\ y\sin\frac{1}{y}, & x=0\ ,\ y\neq 0 \\ 0\ , & x=0=y \end{cases}$$

is continuous but not differentiable at the origin.

E21) Check the following functions for continuity and differentiability at the origin.

a) $$f(x,y)=\begin{cases} \frac{xy^3}{x^2+y^2} & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } (x,y)=(0,0) \end{cases}$$

b) $$f(x,y)=\begin{cases} y\sin\frac{1}{x} & \text{if } x\neq 0 \\ 0 & \text{if } x=0 \end{cases}$$

E22) Show that the following functions are continuously differentiable everywhere.

a) $f(x,y)=e^{x+y}$

b) $f(x,y)=2\sinh x+3\cosh y$

With this we come to an end of this section and the unit.

3.4 SUMMARY

In this unit we have extended the concept of derivatives to functions of several variables. In the process, we have covered the following points:

1. Defined first order partial derivatives of a function at a point, if $f:\boldsymbol{R}^n\to\boldsymbol{R}$ for $n=2$ and 3.

2. Discussed the methods to prove the existence of first order partial derivatives at a point and to evaluate them.

3. Give examples to establish the point that the existence of first order partial derivatives at a point need not imply the continuity of the function at that point.

4. Defined a differentiable function of a function $f:\boldsymbol{R}^n\to\boldsymbol{R}$, $n=2$ or 3.

5. Brought out the connection between the existence of partial derivatives, differentiability and continuity of a function.

3.5 SOLUTIONS/ANSWERS

E1) $f_x(a,b)=\lim\limits_{h\to 0}\dfrac{f(a+h,b)-f(a,b)}{h}=\lim\limits_{h\to 0}\dfrac{c-c}{h}=0.$

Similarly you can show that $f_y(a, b) = 0$.

E2) By definition $f_x(0, 0) = \lim_{h \to 0} \frac{f(0+h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{1}{h}$

Since $\lim_{h \to 0} \frac{1}{h}$ does not exist, $f_x(0, 0)$ does not exist.

Similarly, $f_y(0, 0) = \lim_{k \to 0} \frac{f(0, k) - f(0, 0)}{k} = \lim_{k \to 0} \frac{1}{k}$ does not exist.

E3) $f_x(0, 0, 0) = \lim_{p \to 0} \frac{f(p, 0, 0) - f(0, 0, 0)}{p} = \lim_{h \to 0} \frac{p-0}{p} = 1$

$f_y(0, 0, 0) = \lim_{q \to 0} \frac{f(0, q, 0) - f(0, 0, 0)}{q} = \lim_{k \to 0} \frac{0-0}{k} = 0$

$f_z(0, 0, 0) = \lim_{r \to 0} \frac{f(0, 0, r) - f(0, 0, 0)}{r} = \lim_{r \to 0} \frac{0-0}{r} = 0.$

E4) The function $f(x, y) = 2x^2 - xy + 2y^2$ is a polynomial in x and y.
Therefore the partial derivatives exist.
We first find $\frac{\partial f}{\partial x}$. Then y is treated as a constant.

Therefore we get

$$\begin{aligned} \frac{\partial f}{\partial x} &= 4x - y + 0 \\ &= 4x - y \end{aligned}$$

Then we find $\frac{\partial f}{\partial y}$. Here x is treated as constant.

Thus

$$\frac{\partial f}{\partial y} = 0 - x + 4y = -x + 4y.$$

E5) a) **Hint:** Since f is a polynomial, by direct differentiation we get,

$$\frac{\partial f}{\partial x} = 2x\cos(x^2 - y); \; \frac{\partial f}{\partial y} = -\cos(x^2 - y).$$

b)

$$\frac{\partial f}{\partial x} = -\frac{1}{2} \frac{1}{(x + y^2 + z^2 + 1)^{3/2}}$$

$$\frac{\partial f}{\partial y} = -\frac{1}{2} \frac{2y}{(x + y^2 + z^2 + 1)^{3/2}} = -\frac{y}{(x + y^2 + z^2 + 1)^{3/2}}$$

$$\frac{\partial f}{\partial z} = -\frac{1}{2} \frac{2z}{(x + y^2 + z^2 + 1)^{3/2}} = -\frac{z}{(x + y^2 + z^2 + 1)^{3/2}}$$

c) **Hint:** Since f is a polynomial, by direct differentiation we get,

$$\frac{\partial f}{\partial x} = yz\cos xz; \; \frac{\partial f}{\partial y} = \sin xz; \; \frac{\partial f}{\partial z} = xy\cos xz$$

d) **Hint:** Direct differentiation gives $\frac{\partial f}{\partial x} = yx^{y-1}$; $\frac{\partial f}{\partial y} = x^y \ln x$.

e) $\frac{\partial f}{\partial x} = 3x^2 y + y^2 e^{xy^2}$; $\frac{\partial f}{\partial y} = x^3 + 2xy\, e^{xy^2}$.

E6) $u = e^x \cos y$, $v = e^x \sin y$. Since the functions involved in u and v are exponential and trigonometric functions, the partial derivatives exist.

Then

$\frac{\partial u}{\partial x} = e^x \cos y$ and $\frac{\partial u}{\partial y} = -e^x \sin y$ and
$\frac{\partial v}{\partial x} = e^x \sin y$ and $\frac{\partial v}{\partial y} = e^x \cos y$. Hence $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

E7)
$$
\begin{aligned}
F_x(x, y) &= \lim_{h\to 0} \frac{F(x+h, y) - F(x, y)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h) + g(y) - (f(x) + g(y))}{h} \\
&= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = f'(x) \\
F_y(x, y) &= \lim_{k\to 0} \frac{F(x, y+k) - F(x, y)}{k} \\
&= \lim_{k\to 0} \frac{f(x) + g(y+k) - (f(x) + g(y))}{k} \\
&= \lim_{k\to 0} \frac{g(y+k) - g(y)}{k} = g'(y).
\end{aligned}
$$

E8) By definition,

$$
\begin{aligned}
\left(\frac{\partial (f+g)}{\partial x}\right)_{(a,\,b)} &= \lim_{h\to 0} \frac{(f+g)\,(a+h, b) - (f+g)\,(a, b)}{h} \\
&= \lim_{h\to 0} \left[\frac{f(a+h, b) - f(a, b)}{h} + \frac{g(a+h, b) - g(a, b)}{h}\right] \\
&= \lim_{h\to 0} \frac{f(a+h, b) - f(a, b)}{h} + \lim_{h\to 0} \frac{g(a+h, b) - g(a, b)}{h} \\
&= \left(\frac{\partial f}{\partial x}\right)_{(a,\,b)} + \left(\frac{\partial g}{\partial x}\right)_{(a\,b)}
\end{aligned}
$$

The converse is not necessarily true.

Consider the following functions f and g.

$$
f(x, y) = \begin{cases} 1 & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}
$$

$$
g(x, y) = \begin{cases} 0 & \text{if } (x, y) \neq (0, 0) \\ 1 & \text{if } (x, y) = (0, 0) \end{cases}
$$

Then, $(f+g)(x, y) = 1$ for all (x, y).

We have shown eariler that $f_x(0, 0)$ does not exist.

Similarly, we can show that $g_x(0, 0)$ also does not exist.

However, $\frac{\partial}{\partial x}(f+g)$ exists at $(0, 0)$ since $f+g$ is a constant function.

E9) The slope of the tangent at the point $(1, 2, 14)$ to the curve of intersection of the plane $y = 2$ and the surface $z = 2x^2 + 3y^2$ is given by $\left(\frac{\partial z}{\partial x}\right)_{1, 2, 14)}$.

Now, $z = 2x^2 + 3y^2$. So $\frac{\partial z}{\partial x} = 4x$ and hence $\left(\frac{\partial z}{\partial x}\right)_{(1, 2, 14)} = 4.$

Thus, the slope is 4.

E10) To show the continuity at $(0, 0)$ of f, let $\varepsilon > 0$ be given a real number.

Choose $\delta = \frac{\varepsilon}{\sqrt{8}}$. Then,

$|x| < \frac{\varepsilon}{\sqrt{8}}, |y| < \frac{\varepsilon}{\sqrt{8}}$ implies that $\sqrt{x^2 + y^2} < \frac{\varepsilon}{2} < \varepsilon.$

Therefore, $\lim\limits_{\substack{x\to 0\\ y\to 0}} \sqrt{x^2 + y^2} = 0 = f(0, 0).$

Hence, f is continuous at $(0, 0)$. Now, by definition

$$f_x(0, 0) = \lim_{h\to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h\to 0} \frac{\sqrt{h^2}}{h} = \lim_{h\to 0} \frac{|h|}{h}$$

Since, $\lim\limits_{h\to 0} \frac{|h|}{h}$ does not exist, $f_x(0, 0)$ does not exist.

Similarly, $f_y(0, 0)$ does not exist.

E11)
$$f_x(0, 0) = \lim_{h\to 0} \frac{f(h, 0) - f(0, 0)}{h}$$
$$= \lim_{h\to 0} \frac{0-0}{h} = 0$$

Similarly, $f_y(0, 0) = 0$. Now, we have to show that f is continuous at $(0, 0)$. We have

$$\left| x\sin\frac{1}{x} + y\sin\frac{1}{y} \right| \le |x| + |y|.$$

This implies that $\lim\limits_{x\to 0}\left(x\sin\frac{1}{x} + y\sin\frac{1}{y} \right) = (0, 0) = f(0, 0).$

Therefore, f is continuous at $(0, 0)$.

E12) $f_x(0, 0) = \lim_{h \to 0} \dfrac{f(0+h, 0) - f(0, 0)}{h}$

$$= \lim_{h \to 0} \frac{f(h, 0) - 0}{h} = 0$$

Similarly, $f_y(0, 0) = 0$

Similarly, for points (x, y) such that $xy = 0$ we get that $f_x(x, y) = f_y(x, y) = 0.$

Now suppose $x \neq 0 \neq y.$ Then both the partial derivatives at (x, y) exist since $f(x, y)$ is a quotient of two differentiable functions in x and $y.$ By direct differentiation

$$f_x(x, y) = \frac{(x^4 + y^2)\,(2xy) - x^2 y(4x^3)}{(x^4 + y^2)^2}$$
$$= \frac{2x^5 y + 2xy^3 - 4x^5 y}{(x^4 + y^2)^2}$$

and

$$f_y(x, y) = \frac{(x^4 + y^2)\,(x^2) - x^2 y(2y)}{(x^4 + y^2)^2}$$
$$= \frac{x^6 - x^2 y^2}{(x^4 + y^2)^2}$$

But $\lim_{\substack{x \to 0 \\ y \to 0}} f(x, y)$ does not exist, because when we put $y = x,$

$\lim_{\substack{x \to 0 \\ y \to 0}} f(x, y) = \lim_{x \to 0} \dfrac{x^3}{x^4 + x^2} = 0,$ and when we put

$y = x^2,$ $\lim_{\substack{x \to 0 \\ y \to 0}} f(x, y) = \lim_{x \to 0} \dfrac{x^4}{x^4 + x^4} = \dfrac{1}{2}.$

Hence f is discontinuous at the origin.

E13) a) Since $\left|\dfrac{xy}{y}\right| = \dfrac{|x||y|}{|y|} = |x|$ for all $y \neq 0,$ $\lim_{\substack{x \to 0 \\ y \to 0}} \dfrac{xy}{|y|} = 0 = f(0, 0).$

Hence f is continuous at $(0, 0).$

Also, $f_x(0, 0) = \lim_{h \to 0} \dfrac{f(0+h, 0) - f(0, 0)}{h}$

$$= \lim_{h \to 0} \frac{0 - 0}{h} = 0.$$

Similarly, $f_y(0, 0) = 0$

b) $f_x(1, 0) = 0$. However, $f_y(1, 0)$ does not exist since

$$\frac{f(1, k) - f(1, 0)}{k} = \frac{\frac{k}{|k|} - 0}{k} = \frac{1}{|k|} \text{ and } \lim_{k \to 0} \frac{1}{|k|} \text{ does not exist.}$$

E14) a) **Hint:** $f(x, y) = xe^y \Rightarrow f_x(x, y) = e^x$ and $f_y(x, y) = xe^y$. We can show that f_x is bounded in any neighbourhood of any point of $\boldsymbol{R}^2$. Therefore, by Corollary 1, f is continuous everywhere.

b) $f(x, y) = 3xy \Rightarrow f_x(x, y) = 3y$ and $f_y(x, y) = 3x$. Both the partial derivatives f_x and f_y satisfy the conditions of Corollary 1. Therefore f is continuous everywhere.

E15) a) A constant function of two variables is differentiable everywhere.

b) If f is differentiable at $(a, b) \in \boldsymbol{R}^2$, then $cf\,(c \in \boldsymbol{R})$ is also differentiable at (a, b).

c) If f and g are differentiable at $(a, b) \in \boldsymbol{R}^2$, then $f \pm g$ is also differentiable at (a, b).

d) If f and g are differentiable at (a, b) then fg is also differentiable at (a, b).

Now we shall check the validity of these statements.

a) Let $f(x, y) = c$ be a given constant function. Let (a, b) be any point of $\boldsymbol{R}^2$. Then

$$\begin{aligned} f(a+h, b+k) - f(a, b) &= c - c = 0 \\ &= 0.h + 0.k + h\phi(h, k) + k\psi(h, k). \end{aligned}$$

where $\phi(h, k) = 0 = \psi(h, k)$ for all h and k.

Since (a, b) was any arbitrary point of $\boldsymbol{R}^2, f$ is differentiable everywhere.

b) Let f be differentiable at (a, b). Then there exist constants A and B and functions ϕ and ψ which tend to zero as $(h, k) \to (0, 0)$ such that

$$f(a+h, b+k) - f(a, b) = Ah + Bk + h\phi(h, k) + k\psi(h, k).$$

If $c \neq 0$, is constant, then multiplying the above equation by c,

we get

$$(cf)\,(a+h, b+k) - (cf)(a, b) = A'h + B'k + h\phi'(h, k) + k\psi'(h, k)$$
where $A' = cA, B' = cB, \phi' = c\phi, \psi' = c\psi$.

From the previous unit, Unit 2, you know that if ϕ and ψ tend to zero as $(h, k) \to (0, 0)$ then ϕ' and ψ' also tend to zero as $(h, k) \to (0, 0)$. Therefore cf is differentiable where c non-zero constant. If c is zero, then cf is a constant function taking every point to zero and hence is differentiable.

c) Since f is differentiable at (a, b) there exist constants A' and B' and functions $\phi'(h, k)$ and $\psi(h, k)$ which tend to zero as $(h, k) \to (0, 0)$ such that

$$f(a+h, b+k) - f(a, b) = A'h + B'k + h\phi'(h,k) + k\psi'(h,k).$$

Similarly, since g is differentiable at (a, b) there exist constants A'' and B'' and functions $\phi''(h, k)$ and $\psi''(h, k)$ which tend to zero as $(h, k) \to (0, 0)$ such that

$$h(a+h, b+k) - g(a, b) = A''h + B''k + h\phi''(h,k) + k\psi''(h,k)$$

Then, $(f \pm g)\ (a+h, b+k) - (f \pm g)\ (a, b)$
$= (A' \pm A'')h + (B' \pm B'')k + h[\phi'(h, k) \pm \phi''(h, k)]$
$\quad + k[\psi'(h, k) \pm \psi''(h, k)]$
$= Ah + Bk + h\phi(h, k) + k\psi(h, k)$

where $A = A' \pm A''$, $B = B' \pm B''$, $\phi(h, k) = \phi'(h, k) \pm \phi''(h, k)$ and $\psi(h, k) = \psi'(h, k) \pm \psi''(h, k)$.

Now since A and B are constants and functions ϕ and ψ tend to zero as $(h, k) \to (0, 0)$, $f + g$ as well as $f - g$ is differentiable at (a, b).

d) Proceed as in c).

Then

$$\begin{aligned} fg(a+h, b+k) - fg(a, b) &= f(a+h, b+k)\ g(a+h, b+k) \\ &\quad - f(a+h, b+k)\, g(a, b) + f(a+h, b+k)\, g(a, b) \\ &\quad - f(a, b)\ g(a, b). \end{aligned}$$

$$= f(a+h, b+k)[g(a+h, b+k) - g(a, b)] + g(a, b)$$
$$[f(a+h, b+k) - f(a, b)]$$

$$= [f(a,b) + A'h + B'k + h\phi'(h,k) + k\psi'(h,k)][A''h + B''k$$
$$+ h\phi''(h,k) + k\psi''(h,k)] + g(a,b)[A'h + B'k + h\phi'(h,k) + k\psi'(h,k)]$$

$$= Ah + Bk + h\phi(h, k) + k\psi'(h,k),$$

where $A = A''f(a,b) + A'g(a,b), B = B''f(a,b) + B'g(a,b)$
$\phi(h,k) = [A' + \phi'(h,k)]\,[A''h + B''k + h\phi''(h,k) + k\psi''(h,k)]$
$\qquad + f(a,b)\phi''(h,k) + g(a,b)\phi'(h,k)$

and $\psi(h,k)=[B'+\psi'(h,k)][A''h+B''k+h\phi''(h,k)+k\psi''(h,k)]$
$+f(a,b)\psi''(h,k)+g(a,b)\psi'(h,k).$

E16) Here $f(h,k)-f(0,0)=h^2+k+hk=0.h+1.k+h\phi(h,k)+k\psi(h,k)$ where $\phi(h,k)=h=\psi(h,k)$ for all h, k and therefore $\phi(h,k)\to 0$ and $\psi(h,k)\to 0$ as $(h,k)\to(0,0)$. Thus, f is differentiable at $(0, 0)$.

E17) Let $f(x,y)=\cos(x+y)$. Consider

$$
\begin{aligned}
&f\left(\frac{\pi}{4}+h, \frac{\pi}{4}+k\right)-f\left(\frac{\pi}{4},\frac{\pi}{4}\right)\\
&=\cos\left(\frac{\pi}{4}+h+\frac{\pi}{4}+k\right)-\cos\left(\frac{\pi}{4}+\frac{\pi}{4}\right)\\
&=\cos\left(\frac{\pi}{2}+h+k\right)-\cos\left(\frac{\pi}{2}\right)\\
&=-\sin(h+k)\\
&=-h-k+h\left[1-\frac{\sin(h+k)}{h+k}\right]+k\left[1-\frac{\sin(h+k)}{h+k}\right]\\
&=Ah+Bk+h\phi(h,k)+k\psi(h,k),
\end{aligned}
$$

where $A=-1,\ B=-1,\ \phi(h,k)=\psi(h,k)=1-\dfrac{\sin(h+k}{(h+k)}$

Now, $\lim\limits_{(h,k)\to(0,0)}\phi(h,k)$

$$
\begin{aligned}
&=\lim_{(h,k)\to(0,0)}\psi(h,k)=1-\lim_{(h,k)\to(0,0)}\frac{\sin(h+k)}{h+k}\\
&=1-1=0,
\end{aligned}
$$

since $\lim\limits_{(h,k)\to(0,0)}\dfrac{\sin(h+k)}{h+k}=\lim\limits_{t\to 0}\dfrac{\sin t}{t}=1$ where $t=h+k$.

Therefore, f is differentiable at $\left(\dfrac{\pi}{4},\dfrac{\pi}{4}\right)$.

E18) Suppose, if possible, that f is differentiable at $(0, 0)$. Then there exist constants A and B and functions ϕ and ψ which tend to zero as $(h,k)\to(0,0)$, such that

$$f(0+h, 0+k)-f(0,0)=Ah+Bk+h\phi(h\ k)+k\psi(h,k),$$

i.e., $\dfrac{hk}{\sqrt{h^2+k^2}}=Ah+Bk+h\phi(h,k)+k\psi(h,k).$

Put $h=0, k\neq 0$. Then $0=Bk+k\psi(0,k)$

i.e., $B+\psi(0,k)=0.$

Taking the limit as $k\to 0$, we get $B=0$.

Similarly, by putting $h\neq 0,\ k=0$, we get $A=0$

Now if we let $h=k\neq 0$, then we get

$$\frac{h}{\sqrt{2}} = Ah + Bh + h\phi(h, h) + h\psi(h, h)$$

i.e., $\frac{1}{\sqrt{2}} = A + B + \phi(h, h) + \psi(h, h)$

and so taking limit as $h \to 0$ we get $A + B = \frac{1}{\sqrt{2}}$ which is impossible as $A = 0 = B.$

Thus we arrive at a contradiction and so f cannot be differentiable at $(0, 0).$

E19) a) You have already seen in Example 3, Unit 2 that $\lim\limits_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$ does not exist at $(0, 0)$. This shows that the function is discontinuous at $(0, 0)$, and therefore not differentiable at $(0, 0)$.

b) If we put $y = mx,\ f(x, y) = \frac{2}{1+m^2}$. This means that the limit of $f(x, y)$ as $(x, y) \to (0, 0)$ along $y = mx$ is $\frac{2}{1+m^2}$, and this is different for different values of m.

Therefore, $\lim\limits_{(x, y)\to(0, 0)} \frac{2xy}{x^2 + y^2}$ does not exist and hence f is not continuous at $(0, 0)$.

c) When we put $y = mx, m \neq 1$, then $\lim\limits_{(x, y)\to(0, 0)} \frac{x^2 + y^2}{x - y}$

$= \lim\limits_{x\to 0} \frac{x(1+m^2)}{1-m} = 0.$ But $f(0, 0) = 1.$

Therefore, f is discontinuous at $(0, 0)$.

d) $x^4 + y^4 \le x^4$

$\Rightarrow\ | f(x, y) | \le \left| \frac{x^5}{x^4} \right| = | x |$

This shows that $\lim\limits_{(x, y)\to(0, 0)} f(x, y) = 0,$ which is different from $f(0, 0) = 3.$ Thus f is not continuous at $(0, 0)$.

E20) Now $| f(x, y) | \le | x | + | y |$ in all cases.

$\lim\limits_{(x, y)\to(0, 0)} f(x, y) = f(0, 0)$ and hence f is continuous at $(0, 0)$. Now, we have to show that f is not differentiable at $(0, 0)$.

In view of Theorem 5, it is enough to show that either f_x or f_y does not exist.

Now $f_x(0, 0)$ does not exist, since

$$\frac{f(h, 0)-f(0, 0)}{h}=\frac{h\sin\frac{1}{h}-0}{h}=\sin\frac{1}{h}$$

and $\lim\limits_{h\to 0}\sin\frac{1}{h}$ does not exist. Therefore f is not differentiable as $(0, 0)$.

E21) a) Now $f_x(0, 0)=0=f_y(0, 0)$.

At $(x, y)\neq(0, 0)$, $f_x(x, y)=\dfrac{y^5-x^2y^3}{(x^2+y^2)^2}$ and

$f_y(x, y)=\dfrac{3x^3y^2+xy^4}{(x^2+y^2)^2}$.

Putting $x=r\cos\theta,\ y=r\sin\theta$, we find that

$$|f_x(x, y)|=r\,|\sin^5\theta-\cos^2\theta\sin^3\theta|\leq 2r=2\sqrt{x^2+y^2}.$$

This implies that $\lim\limits_{(x, y)\to(0, 0)} f_x(x, y)=f_x(0, 0)$. Hence f_x is continuous at $(0, 0)$ and $f_y(0, 0)$ exists. Thus f_x and f_y satisfy all conditions of Theorem 6. Therefore, f is differentiable at $(0, 0)$, and hence f is also continuous at $(0, 0)$.

b) Since $|f(x, y)|\leq|y|$, it can be easily shown that f is continuous at $(0, 0)$. Now $f_x(0, 0)=0$ and $f_y(0, 0)=1$.

Suppose, if possible, that f is differentiable at $(0, 0)$.

Then there exist functions ϕ and ψ, which tend to zero as $(h, k)\to(0, 0)$, such that
$f(h, k)-f(0, 0)=0.h+1.k+h\phi(h, k)+k\psi(h, k)$.

Let $h=k\neq 0$.

Then

$$h\sin\frac{1}{h}=h+h\phi(h, h)+h\psi(h, h)$$

i.e., $\sin\dfrac{1}{h}=1+\phi(h, h)+\psi(h, h)$

Therefore, $\lim\limits_{h\to 0}\sin\frac{1}{h}=1$, which contradicts the fact that $\lim\limits_{h\to 0}\sin\frac{1}{h}$ does not exist.

E22) a) Since $f_x(x, y)=e^{x+y}=f_y(x, y)$ it can be easily seen that f_x and f_y are continuous everywhere. Therefore, f is continuously differentiable everywhere.

b) $f_x(x,\ y) = 2\cosh x$ and $f_y(x,\ y) = 3\sinh y$.

So f_x and f_y are continuous everywhere and hence f is continuously differentiable everywhere.

UNIT 4

HIGHER ORDER PARTIAL DERIVATIVES

Structure

4.1 INTRODUCTION

In the last unit you studied the new concept of partial derivatives for functions of several variables. The partial derivatives that we computed for the various functions in the examples there, are called partial derivatives of first order. You must have seen that partial derivatives of first order, again, are functions. For instance, if

$f(x, y) = 3x^3 + 2xy^2 + 5y^2 + 6$, then $f_x(x, y) = 9x^2 + 2y^2$ and $f_y(x, y) = 4xy + 10y$

are again real-valued functions of two variables with domain $\boldsymbol{R}^2$. Thus, we can talk of partial derivatives of these new functions. These new derivatives are called second order partial derivatives of the original function.

The second order partial derivatives are important as they occur in many physical applications such as heat equation describing heat flow and wave equation in the study of sound and water wave and so on.

If we consider a function of two variables, there are two first order partial derivatives, which may give rise to four second order partial derivatives, which might again turn out to be functions. If this chain continues, then we obtain third order, fourth order, and so on, partial derivatives. In this unit we shall study such higher order partial derivatives. The importance of these partial derivatives will become clear to you in the coming blocks.

In this unit you will also study Euler's, Schwarz's and Young's theorems, which give some sets of conditions under which the mixed partial derivatives become equal.

Objectives

After reading this unit, you should be able to

- define and evaluate higher order partial derivatives,
- state Euler's theorem, Schwarz's and Young's theorems,
- decide about the commutativity of the operations of taking partial derivatives with respect to different variables.

4.2 HIGHER ORDER PARTIAL DERIVATIVES

In this section we discuss higher order partial derivatives mainly second order partial derivatives. In the introduction you have seen that the partial derivative f_x of the function $f(x, y)$ is again a function of x and y. For instance, the partial derivative f_x of the function $f(x, y) = 3x^3 + 2xy^2 + 5y^2 + 6$ is $f_x(x, y) = 9x^2 + 2y^2$, which is a function of x and y.

In general, let $D \subset \boldsymbol{R}^2$, and let $f : D \to \boldsymbol{R}$ have a first order partial derivative f_x at every point of D. Then we get a new function, say $g = f_x$, which is defined on D. This new function g may or may not possess first order partial derivatives. In case it does, then g_x and g_y are called the second order partial derivatives of *f* and are denoted by f_{xx} and f_{xy}, respectively. Similarly, if the function *f* has a first order partial derivative f_y at every point of D, then f_y defines a new function. And if this new function has first order partial derivatives, then we get two more second order partial derivatives, namely, f_{yx} and f_{yy}. Thus, if $f(x, y)$ is a real-valued function defined in a neighbourhood of (a,b) having both the partial derivatives at all the points of the neighbourhood, then

$$f_{xx}(a,b) = \lim_{h\to 0} \frac{f_x(a+h,b) - f_x(a,b)}{h}$$
$$f_{xy}(a,b) = \lim_{k\to 0} \frac{f_x(a,b+k) - f_x(a,b)}{k}$$
$$f_{yx}(a,b) = \lim_{h\to 0} \frac{f_y(a+h,b) - f_y(a,b)}{h}$$
$$f_{yy}(a,b) = \lim_{k\to 0} \frac{f_y(a,b+k) - f_y(a,b)}{k},$$

provided each one of these limits exists.

This means that

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \; f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right), f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right).$$

We also denote the second order partial derivatives of f by

$$f_{xx} = \frac{\partial^2 f}{\partial x^2}; \quad f_{xy} = \frac{\partial^2 f}{\partial y\, \partial x};$$

$$f_{yx} = \frac{\partial^2 f}{\partial x\, \partial y}; \; f_{yy} = \frac{\partial^2 f}{\partial y^2}$$

If we want to indicate the particular point at which the second order partial derivatives are taken, then we write

$$\left(\frac{\partial^2 f}{\partial x^2}\right)_{(a,b)}, \frac{\partial^2 f(a,b)}{\partial x^2}, f_{xx}(a,b), \left(\frac{\partial^2 f}{\partial x\, \partial y}\right)_{(a,b)}$$

$\dfrac{\partial^2 f(a,b)}{\partial x\, \partial y}$, $f_{xy}(a,b)$, and so on.

Note that the second order partial derivatives f_{xy} and f_{yx} are called **mixed second order partial derivatives or simply mixed partial derivatives**.

In a similar manner partial derivatives of order higher than two are defined. For example,

$$\frac{\partial^3 f}{\partial x\, \partial x\, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x \partial y}\right) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\right]$$

i.e., $\dfrac{\partial^3 f}{\partial x\, \partial x\, \partial y}$ stands for the partial derivative of $\dfrac{\partial^2 f}{\partial x\, \partial y}$ with respect to x and is written as $\dfrac{\partial^3 f}{\partial^2 x\, \partial y}$.

Note: The higher order partial derivatives for a function of three variables can also be defined in a similar way.

In the following examples, we show how to calculate these higher order partial derivatives.

Example 1: Find all the second order partial derivatives of the function $U(x, y) = x^3 + y^3 + 3axy$, where a is constant.

Solution: Let's consider the function U given by

$$U(x, y) = x^3 + y^3 + 3axy,$$

We first find the first order partial derivatives.

$\dfrac{\partial U}{\partial x} = 3x^2 + 3ay$ and $\dfrac{\partial U}{\partial y} = 3y^2 + 3ax.$

Then we calculate the second order partial derivatives.

$$\frac{\partial^2 U}{\partial x^2} = \frac{\partial}{\partial x}\left(3x^2 + 3ay\right) = 6x,$$

$$\frac{\partial^2 U}{\partial y\, \partial x} = \frac{\partial}{\partial y}\left(3x^2 + 3ay\right) = 3a = \frac{\partial^2 U}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left(3y^2 + 3ax\right) \text{ and}$$

$$\frac{\partial^2 U}{\partial y^2} = \frac{\partial}{\partial y}\left(3y^2 + 3ax\right) = 6y.$$

***

Example 2: If $f(x, y) = x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y},\ x \neq 0, y \neq 0,$

prove that $\dfrac{\partial^2 f}{\partial x\, \partial y} = \dfrac{x^2 - y^2}{x^2 + y^2}.$

Solution: We first note that $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right)$. Here,

$$\begin{aligned}\frac{\partial f}{\partial y} &= x^2 . \frac{1}{1 + y^2/x^2} . \frac{1}{x} - 2y \tan^{-1}\frac{x}{y} - y^2 . \frac{1}{1 + x^2/y^2}\left(-\frac{x}{y^2}\right) \\ &= \frac{x^3}{x^2 + y^2} + \frac{xy^2}{x^2 + y^2} - 2y \tan^{-1}\frac{x}{y} \\ &= x - 2y \tan^{-1}\frac{x}{y}\end{aligned}$$

And therefore,

$$\begin{aligned}\frac{\partial^2 f}{\partial x\, \partial y} &= \frac{\partial}{\partial x}\left(x - 2y \tan^{-1}\frac{x}{y}\right) \\ &= 1 - 2y . \frac{1}{1 + x^2/y^2} . \frac{1}{y} \\ &= 1 - \frac{2y^2}{x^2 + y^2} \\ &= \frac{x^2 - y^2}{x^2 + y^2}\end{aligned}$$

***

In the next example we go a step further and calculate a third order partial derivative.

Example 3: If $u(x, y) = e^{xy}$, show that

$$\frac{\partial^3 u}{\partial x\, \partial x\, \partial y} = \frac{\partial^3 u}{\partial y\, \partial x\, \partial x}.$$

Solution: Now $u(x, y) = e^{xy}$. Then,

$$\frac{\partial u}{\partial x} = ye^{xy}, \frac{\partial u}{\partial y} = xe^{xy}$$

So, $\dfrac{\partial^2 u}{\partial x\,\partial y} = \dfrac{\partial}{\partial x}(xe^{xy}) = e^{xy} + xy\,e^{xy} = (1+xy)e^{xy}$

$$\Rightarrow \frac{\partial^3 u}{\partial x\,\partial x\,\partial y} = \frac{\partial}{\partial x}\left\{(1+xy)e^{xy}\right\} = (1+xy)\,y\,e^{xy} + ye^{xy} = (2+xy)\,y\,e^{xy}$$

Similarly $\dfrac{\partial u}{\partial x} = y\,e^{xy}$

$$\Rightarrow \frac{\partial^2 u}{\partial x \partial x} = \frac{\partial}{\partial x}\left(ye^{xy}\right) = y^2 e^{xy}$$

$$\Rightarrow \frac{\partial^3 u}{\partial y\,\partial x\,\partial x} = \frac{\partial}{\partial y}\left(y^2\,e^{xy}\right) = 2y\,e^{xy} + y^2 x e^{xy}$$

$$= y\,e^{xy}(2+yx)$$

Hence the result.

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We are sure you will be able to solve these exercises now.

E1) Find all the second order partial derivatives of the following functions.

a) $f(x,y) = \cos\dfrac{y}{x}; \quad (x \neq 0)$

b) $f(x,y) = x^5 + y^4 \sin\, x^6$

E2) Does the function $f(x,y) = x^3 - 3xy^2$ satisfy the Laplace equation $f_{xx} + f_{yy} = 0$? Justify your answer.

E3) Verify that $\dfrac{\partial^2 f}{\partial x\,\partial y} = \dfrac{\partial^2 f}{\partial y\,\partial x}$ for each of the following functions.

a) $f(x,y) = x^3 y + e^{xy^2}$

b) $f(x,y) = \tan\left(xy^3\right)$

Remark 1: In Unit 3 you have seen that it is not always possible to find first order partial derivatives by direct differentiation (See Example 5 of Unit 3). The same is true for higher order partial derivatives of some functions. This is illustrated by the following examples.

Example 4: Evaluate the second order partial derivatives of f at $(0,0)$, where f is the function given by

$$f(x,y) = \begin{cases} \dfrac{xy\left(x^2 - y^2\right)}{x^2 + y^2}, & (x,y) \neq (0,0) \\ 0 & , \ (x,y) = (0,0) \end{cases}$$

Solution: Since $f_{xx}(0,0)=\lim\limits_{h\to 0}\dfrac{f_x(h,0)-f_x(0,0)}{h}$, we have to first evaluate $f_x(h,0)$ and $f_x(0,0)$.

$$f_x(0,0)=\lim_{t\to 0}\frac{f(t,0)-f(0,0)}{t}=\lim_{t\to 0}\frac{0-0}{t}=0.$$

$$f_x(h,0)=\lim_{t\to 0}\frac{f(h+t,0)-f(h,0)}{t}=\lim_{t\to 0}\frac{0-0}{t}=0.$$

Therefore,

$$\boxed{f_{xx}(0,0)=\lim_{h\to 0}\frac{0-0}{h}=0}.$$

Since $f_{xy}(0,0)=\lim\limits_{k\to 0}\dfrac{f_x(0,k)-f_x(0,0)}{k}$, we must first evaluate $f_x(0,k)$.

Now, $f_x(0,k)=\lim\limits_{t\to 0}\dfrac{f(t,k)-f(0,k)}{t}$

$$=\lim_{t\to 0}\frac{\dfrac{tk(t^2-k^2)}{t^2+k^2}-0}{t}$$
$$=\lim_{t\to 0}\frac{k(t^2-k^2)}{t^2+k^2}$$
$$=-\frac{k^3}{k^2}$$
$$=-k.$$

Therefore, $f_{xy}(0,0)=\lim\limits_{k\to 0}\dfrac{-k-0}{k}=-1.$

Since $f_{yx}(0,0)=\lim\limits_{h\to 0}\dfrac{f_y(h,0)-f_y(0,0)}{h}$, we first evaluate $f_y(h,0)$ and $f_y(0,0)$.

Now, $f_y(0,0)=\lim\limits_{s\to 0}\dfrac{f(0,s)-f(0,0)}{s}=\lim\limits_{s\to 0}\dfrac{0-0}{s}=0.$

$$f_y(h,0)=\lim_{s\to 0}\frac{f(h,s)-f(h,0)}{s}$$
$$=\lim_{s\to 0}\frac{\dfrac{hs(h^2-s^2)}{h^2+s^2}-0}{s}$$
$$=\lim_{s\to 0}\frac{h(h^2-s^2)}{h^2+s^2}$$
$$=\frac{h^3}{h^2}$$
$$=h$$

Therefore, $f_{yx}(0,0) = \lim\limits_{h \to 0} \dfrac{h-0}{h} = 1$

Since $f_{yy}(0,0) = \lim\limits_{k \to 0} \dfrac{f_y(0,k) - f_y(0,0)}{k}$, we first evaluate $f_y(0,k)$.

Now $f_y(0,k) = \lim\limits_{s \to 0} \dfrac{f(0,k+s) - f(0,k)}{k} = \lim\limits_{s \to 0} \dfrac{0-0}{k} = 0$.

Therefore, $f_{yy}(0,0) = \lim\limits_{k \to 0} \dfrac{0-0}{k} = 0$

***

Remark 2: i) Thus, we have observed that to evaluate the partial derivatives of a function, sometimes we have to resort to the definition of partial derivatives, and direct differentiation is possible.

ii) The example above also shows that $f_{xy}(0,0)$ and $f_{yx}(0,0)$ exist, but they are not equal.

In the next example we take up a function which is slightly more complicated.

Example 5: Evaluate $f_{xy}(0,0)$ and $f_{yx}(0,0)$, for the function f given by

$$f(x,y) = \begin{cases} \left(x^4 + y^4\right)\tan^{-1}\left(y^2/x^2\right), & x \neq 0 \\ \dfrac{\pi y^4}{2}, & x = 0 \end{cases}$$

Solution: We first note that

$$f_x(0,0) = \lim_{t \to 0} \frac{f(t,0) - f(0,0)}{t} = \lim_{t \to 0} \frac{0-0}{t} = 0, \text{ and}$$

$$f_x(0,k) = \lim_{t \to 0} \frac{f(t,k) - f(0,k)}{t}.$$

$$= \lim_{t \to 0} \frac{\left(t^4 + k^4\right)\tan^{-1}\left(k^2/t^2\right) - \pi k^4/2}{t}$$

By L'Hospital's rule (Refer Unit, Calculus course) we have

$$f_x(0,k) = \lim_{t \to 0} \frac{4t^3 \tan^{-1}\dfrac{k^2}{t^2} + \left(k^4 + t^4\right)\cdot\dfrac{1}{1+\left(k^4/t^4\right)}\left(-\dfrac{2k^2}{t^3}\right)}{1}$$

$$= \lim_{t \to 0}\left[4t^3 \tan^{-1} k^2/t^2 - 2k^2 t\right]$$

$$= 0$$

Therefore, $f_{xy}(0,0) = \lim\limits_{k \to 0} \dfrac{0-0}{k} = 0$

Now, $f_y(0,0) = \lim\limits_{s\to 0} \dfrac{f(0,s) - f(0,0)}{s}$

$$= \lim_{s\to 0} \frac{(\pi s^4/2) - 0}{s}$$

$$= 0$$

Further, $f_y(h,0) = \lim\limits_{s\to 0} \dfrac{f(h,s) - f(h,0)}{s}$

$$= \lim_{s\to 0} \frac{(h^4 + s^4)\tan^{-1}(s^2/h^2) - 0}{s}$$

$$= \lim_{s\to 0} \frac{4s^3 \tan^{-1}(s^2/h^2) + (h^4 + s^4).\left(\dfrac{1}{1 + s^4/h^4}\right)(2s/h^2)}{1}$$

$$= \lim_{s\to 0} \left[4s^3 \tan^{-1} s^2/h^2 + 2sh^2\right]$$

$$= 0$$

Consequently, $f_{yx}(0,0) = \lim\limits_{h\to 0} \dfrac{f_y(h,0) - f_y(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{0-0}{h} = 0.$

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In Unit 3 you have seen examples of some functions, whose first order partial derivatives do not exist. See Example 3 of Unit 3.

Here we will give you an example of a function whose first order partial derivatives exist, but higher order ones do not exist. From this example you will also see that the existence of a partial derivative of a particular order does not imply the existence of partial derivatives of the higher order.

Example 6: Examine whether the second order partial derivatives of *f* at $(0,0)$ exist or not, if $f : \boldsymbol{R}^2 \to \boldsymbol{R}$ is defined by

$$f(x,y) = \begin{cases} \dfrac{xy^2}{\sqrt{x^2 + y^2}}, & xy \neq 0 \\ 0 & , xy = 0 \end{cases}$$

Solution: Now, $f_x(0,0) = \lim\limits_{t\to 0} \dfrac{f_y(t,0) - f_y(0,0)}{t}$

$$= \lim_{t\to 0} \frac{0-0}{t} = 0.$$

Similarly, for $h \neq 0$, $f_x(h,0) = \lim\limits_{t\to 0} \dfrac{f(h+t,0) - f(h,0)}{t}$

$$= \lim_{t\to 0} \frac{0-0}{t} = 0.$$

Therefore, $\boxed{f_{xx}(0,0) = \lim\limits_{h\to 0} \dfrac{f_x(h,0) - f_x(0,0)}{h} = 0}$

Now to check the existence of f_{xy}, we will have to see whether

$$\lim_{h\to 0}\frac{f_x(0,k)-f_x(0,0)}{k} \text{ exists or not.}$$

Therefore, let us find $f_x(0,k)$, for $k \neq 0$.

For $k \neq 0$, $f_x(0,k) = \lim\limits_{t\to 0}\dfrac{f(t,k)-f(0,k)}{t}$

$$= \lim_{t\to 0}\frac{1}{t}\left[\frac{tk^2}{\sqrt{t^2+k^2}}\right]$$

$$= \lim_{t\to 0}\frac{k^2}{\sqrt{t^2+k^2}}$$

$$= \frac{k^2}{\sqrt{k^2}}$$

$$= |k|. \qquad \text{(Remember, } \sqrt{k^2} = |k|\text{)}$$

Now, $\lim\limits_{k\to 0}\dfrac{f_x(0,k)-f_x(0,0)}{k}$

$$= \lim_{k\to 0}\frac{|k|}{k},$$

which does not exist, showing that $\boxed{f_{xy} \text{ does not exist at } (0,0)}$.

Now

$$f_y(0,0) = \lim_{s\to 0}\frac{f(0,s)-f(0,0)}{s} = \lim_{s\to 0}\frac{0-0}{s} = 0,$$

and for $h \neq 0$, $f_y(h,0) = \lim\limits_{s\to 0}\dfrac{f(h,s)-f(h,0)}{s}$

$$= \lim_{s\to 0}\frac{\dfrac{hs^2}{\sqrt{h^2+s^2}}-0}{s}$$

$$= \lim_{s\to 0}\frac{hs}{\sqrt{h^2+s^2}} = 0$$

Therefore, $\boxed{f_{yx}(0,0) = \lim\limits_{h\to 0}\dfrac{f_y(h,0)-f_y(0,0)}{h} = \lim\limits_{h\to 0}\dfrac{0-0}{h} = 0}$

Again, for $k \neq 0$, $f_y(0,k) = \lim\limits_{s\to 0}\dfrac{f(0,k+s)-f(0,k)}{s} = \lim\limits_{s\to 0}\dfrac{0-0}{s} = 0$

Therefore, $\boxed{f_{yy}(0,0) = \lim\limits_{k\to 0}\dfrac{f_y(0,k)-f_y(0,0)}{k} = \lim\limits_{k\to 0}\dfrac{0-0}{k} = 0}$.

Thus, f_{xx}, f_{yy} and f_{yx} exist at $(0,0)$ and are equal to 0, while $f_{xy}(0,0)$ does not exist.

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See if you can solve these exercises now.

E4) Show that $f_{xy}(0,0) \neq f_{yx}(0,0)$ for the function $f : \boldsymbol{R}^2 \to \boldsymbol{R}$ defined by

$$f(x,y) = \begin{cases} \dfrac{xy^5}{x^2+y^4}, & \text{if } (x,y) \neq (0,0) \\ 0 & , \text{if } (x,y) = (0,0) \end{cases}$$

E5) Examine the following functions for equality of f_{xy} and f_{yx} at (0,0).

a) $$f(x,y) = \begin{cases} \dfrac{x^2y^2}{\sqrt{x^4+y^4}}, & \text{if } (x,y) \neq (0,0) \\ 0 & , \text{if } (x,y) = (0,0) \end{cases}$$

b) $$f(x,y) = \begin{cases} \dfrac{xy^3}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0 & , \text{if } (x,y) = (0,0) \end{cases}$$

E6) Show that $f_{xy}(0,0) \neq f_{yx}(0,0)$ for the function f defined by

$$f(x,y) = \begin{cases} xy, & \text{if } |y| \leq |x| \\ -xy, & \text{if } |y| > |x| \end{cases}$$

The study of the above examples and exercises might have convinced you that we have to be careful about the order of variables with respect to which higher order derivatives are taken. For instance, from Example 4 it is clear that f_{xy} need not be equal to f_{yx}. Example 6 goes a step further, where f_{xy} exists at $(0,0)$, while f_{yx} does not, showing that the question of their equality does not arise at all. If you look at the definitions of f_{xy} and f_{yx} at a point (a,b) more carefully, you would see why the expectation of the equality $f_{xy}(a,b) = f_{yx}(a,b)$ is farfetched. By definition

$$f_{xy}(a,b) = \lim_{k\to 0} \frac{f_x(a,b+k) - f_x(a,b)}{k}$$

$$= \lim_{k\to 0}\left[\frac{1}{k}\left\{\lim_{h\to 0}\frac{f(a+h,b+k)-f(a,b+k)}{h} - \lim_{h\to 0}\frac{f(a+h,b)-f(a,b)}{h}\right\}\right]$$

$$= \lim_{k\to 0}\left[\frac{1}{k}\left\{\lim_{h\to 0}\frac{f(a+h,b+k)-f(a+h,b)+f(a,b+k)-f(a,b)}{hk}\right\}\right],$$

Similarly,

$$f_{yx}(a,b) = \lim_{h\to 0}\left[\lim_{k\to 0}\left\{\frac{f(a+h,b+k)-f(a+h,b)-f(a,b+k)-f(a,b)}{hk}\right\}\right]$$

and we have already seen in Unit 2, that repeated limits are not equal, in general.

In the next section we will study the conditions under which these mixed partial derivatives become equal.

4.3 EQUALITY OF MIXED PARTIAL DERIVATIVES

So far we have seen how to compute the second order partial derivatives. We have also discussed the equality of mixed partial derivatives f_{xy} and f_{yx} of functions. The second order partial derivatives are very important in the study of many physical phenomenon such as heat and wave propagation and gravitational force etc. Given below are some of the well-known equations.

i) Heat Equation – Let $T(x, y, z, t)$ denotes the temperature of the body at the point (x, y, z) at time t. Then T satisfy the **heat equation:**

$$k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)=\frac{\partial T}{\partial t}$$

where k is a constant whose value depends on the conductivity of the material comprising the body.

ii) One – dimensional wave equation.

$$\frac{\partial^2 f}{\partial x^2}=c^2\frac{\partial^2 f}{\partial t^2}$$

The equations involving partial derivatives are called **partial differential equations**. These equations are studied in detail in Block 4.

We shall now give a set of sufficient conditions which would ensure that the order of the variables with respect to which higher order partial derivatives are taken is immaterial. In other words, if a function f satisfies these conditions, then its mixed partial derivatives will be equal.

We shall state three theorems and illustrate them with examples.

Theorem 1: Let $f(x, y)$ be a real-valued function such that the two second order partial derivatives f_{xy} and f_{yx} are continuous at a point (a,b). Then

$$f_{xy}(a,b)=f_{yx}(a,b)$$

This result was proved by Leonard Euler around 1734, when he was working on some problems in hydrodynamics. Later the German mathematician Hermann Amandus Schwarz (1843-1921) proved another theorem about the equality of mixed partial derivatives. The conditions in Schwarz's theorem are less restrictive than those in Euler's theorem (Theorem 1). We give only the statement of Schwarz's theorem here.

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Theorem 2: (Schwarz's Theorem): Let $f(x, y)$ be a real-valued function defined in a neighbourhood of (a,b) such that

i) f_y exists on a certain neighbourhood of (a,b).
ii) f_{xy} is continuous at (a,b).

Then f_{yx} exists at (a,b) and $f_{yx}(a,b) = f_{xy}(a,b)$.

– ■ –

We now give examples to illustrate the theorems.

Example 7: Check the conditions of the Schwarz's theorem for the function f defined by $f(x, y) = x^4 + x^2 y^2 + y^6$. What can you conclude about the equality of the mixed partial derivatives? Jusitfy your answer.

Solution: Here we shall first find f_{xy}, and then use Schwarz's theorem to evaluate f_{yx} at the point (x, y).

By direct differentiation you can see that $f_y(x, y) = 2x^2 y + 6y^5$. Therefore, $f_{xy}(x, y) = 4xy.$

Since $4xy$ is a polynomial, f_{xy} is a continuous function.

Hence f satisfies the conditions of Schwarz's theorem. So we can conclude that $f_{yx}(x, y) = f_{xy}(x, y) = 4xy.$

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Example 8: Verify Euler's theorem for the function $f(x, y) = e^x \cos y - e^y \sin x.$

Solution: $f(x, y) = e^x \cos y - e^y \sin x$

$\therefore f_y(x, y) = -e^x \sin y - e^y \sin x$ and $f_x(x, y) = e^x \cos y - e^y \cos x.$

By looking at the expressions for f_x and f_y we can observe that the derivatives of these functions exist and are also continuous. This means that both f_{xy} and f_{yx} exist everywhere and are continuous everywhere. Therefore by Euler's theorem we conclude that both the mixed partial derivatives are equal at all points of $\boldsymbol{R}^2$.

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Note: Note that for the example above we can directly compute that

$f_{xy}(x,y) = -e^x \sin y - e^y \cos x$ and $f_{yx}(x,y) = -e^x \sin y - e^y \cos x$.

Therefore, it is easy to see that both f_{xy} and f_{yx} are equal and continuous everywhere. You don't require Euler's theorem to conclude about the equality of mixed partial derivatives. But in real life situations mostly the functions are very complicated and may not easy to compute the second partial derivatives. In such case Euler's theorem is applied to conclude the equality of mixed partial derivatives.

You can try this exercise now.

E7) Evaluate f_{xy} at a point (x, y) for each of the following functions.

a) $f(x,y) = x^2 + xy + y^2$

b) $f(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2}$, $x \neq 0$, $y \neq 0$

Verify that each of these functions satisfies the requirements of Schwarz's theorem and hence evaluate $f_{yx}(x, y)$.

In Euler's Theorem we assume that both the mixed partial derivatives are continuous, whereas in Schwarz's theorem we assume that only one of them, say f_{xy} is continuous, and that f_y exists. But even though the conditions of Schwarz's theorem are less strict, these are still not necessary for the equality of mixed partial derivatives. In other words, we can have functions whose mixed partial derivatives at some point are equal, but which do not satisfy the requirements of Schwarz's theorem. We give one such function in the following example.

Example 9: Consider the function f defined by

$$f(x,y) = \begin{cases} \dfrac{x^2 y^2}{x^2 + y^2}, & (x,y) \neq (0,0) \\ 0, & x = 0 = y \end{cases}$$

Show that $f_{xy}(0,0) = f_{yx}(0,0)$, even though f does not fulfill the requirements of Schwarz's theorem.

Solution: Now, $f_x(0,0) = \lim\limits_{h \to 0} \dfrac{f(h,0) - f(0,0)}{h}$

$$= \lim_{h \to 0} \frac{0 - 0}{h} = 0.$$

Also, for $y \neq 0$,

$$f_x(0,y) = \lim_{h \to 0} \frac{f(h,y) - f(0,y)}{h}$$

$$= \lim_{h \to 0} \frac{h^2 y^2}{h^2 + y^2} \cdot \frac{1}{h}$$

$$= \lim_{h \to 0} \frac{hy^2}{h^2 + y^2}$$

$$= 0.$$

Therefore, $f_{xy}(0,0) = \lim_{k \to 0} \dfrac{f_x(0,k) - f_x(0,0)}{k}$

$$= 0.$$

Similarly, you can check that

$f_y(0,0) = 0$ and for $x \neq 0$, we have

$$f_y(x,0) = \lim_{k \to 0} \frac{f(x,k) - f(x,0)}{k}$$

$$= \lim_{k \to 0} \frac{x^2 k^2}{x^2 + k^2} \cdot \frac{1}{k}$$

$$= 0.$$

From this we get

$$f_{yx}(0,0) = \lim_{h \to 0} \frac{f_y(x,0) - f_y(0,0)}{h}$$

$$= 0.$$

Hence, we have shown that $f_{xy}(0,0) = f_{yx}(0,0)$.

We'll now show that the conditions of Schwarz's theorem are not satisfied. Now, for $x \neq 0, y \neq 0$, we can find the partial derivatives of *f* at (x, y) by differentiating directly. Thus,

$$f_x(x, y) = \frac{\partial}{\partial x}\left[\frac{x^2 y^2}{x^2 + y^2}\right]$$

$$= \frac{2(x^2 + y^2)xy^2 - 2x^3 y^2}{(x^2 + y^2)^2}$$

$$= \frac{2xy^4}{(x^2 + y^2)^2}$$

Further, $f_{xy}(x, y) = \dfrac{\partial}{\partial y}\left[\dfrac{2xy^4}{(x^2 + y^2)^2}\right]$

$$= \frac{8x(x^2 + y^2)^2 y^3 - 8xy^5(x^2 + y^2)}{(x^2 + y^2)^4}$$

$$= \frac{8xy^3(x^2+y^2)\left[x^2+y^2-y^2\right]}{\left(x^2+y^2\right)^4}$$

$$= \frac{8x^3y^3}{\left(x^2+y^2\right)^3}$$

Now, $\lim\limits_{(x,y)\to(0,0)} \dfrac{8x^3y^3}{\left(x^2+y^2\right)^3}$ does not exist. For this we put $y = mx$ in $\dfrac{8x^3y^3}{\left(x^2+y^2\right)^3}$, and take the limit as $x \to 0$. Then we find that the limit is different for different values of m. This means that $\lim\limits_{(x,y)\to(0,0)} f_{xy}(x,y)$ does not exist, which implies that f_{xy} is not continuous at $(0,0)$.

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There is another criterion which tells us when f_{xy} equals f_{yx} at a particular point. We state this also without proof.

Theorem 3 (Young's Theorem): Let $f(x,y)$ be a real-valued function defined in a neighbourhood of a point (a,b) such that both the first order partial derivatives f_x and f_y are differentiable at (a,b). Then $f_{xy}(a,b) = f_{yx}(a,b)$.

As in the case of Schwarz's theorem the conditions stated in Young's theorem are less strict than in Theorem 1(why?). However, these are also not necessary for the equality of mixed partial derivatives.

For most of the functions that we come across, all the partial derivatives are continuous, and therefore the value of the mixed partial derivatives do not change when there is a change in the order of variables with respect to which the partial derivatives are taken. Let us look at few more examples.

Example 10: Consider the function $f : \boldsymbol{R}^3 \to \boldsymbol{R}$ defined by

$$f(x,y,z) = \begin{cases} \dfrac{x^3y - xy^3 + yz^2}{x^2+y^2+z^2}, & (x,y,z) \neq (0,0,0) \\ 0, & (x,y,z) = (0,0,0) \end{cases}$$

Show that $f_{xy}(0,0,0) \neq f_{yx}(0,0,0)$, whereas $f_{xz}(0,0,0) = f_{zx}(0,0,0)$.

Solution: Let us first calculate $f_{xy}(0,0,0)$. For this we need to evaluate $f_x(0,0,0)$ and $f_x(0,k,0)$. Now

$$f_x(0,0,0) = \lim_{p\to 0} \frac{f(p,0,0) - f(0,0,0)}{p} = \lim_{p\to 0} \frac{0-0}{p} = 0 \text{ and}$$

$$f_x(0,k,0) = \lim_{p\to 0} \frac{f(p,k,0) - f(0,k,0)}{p}$$

$$= \lim_{p \to 0} \frac{\frac{p^3 k - pk^3}{p^2 + k^2} - 0}{p}$$

$$= \lim_{p \to 0} \frac{p^2 k - k^3}{p^2 + k^2} = -k \, .$$

Therefore, $f_{xy}(0,0,0) = \lim\limits_{k \to 0} \dfrac{f_x(0,k,0) - f_x(0,0,0)}{k}$

$$= \lim_{k \to 0} \frac{-k - 0}{k}$$

$$= -1.$$

Next, we'll evaluate $f_{yx}(0,0,0)$. For this we need $f_y(h,0,0)$ and $f_y(0,0,0)$.

Now, $f_y(0,0,0) = \lim\limits_{q \to 0} \dfrac{f(0,q,0) - f(0,0,0)}{q} = 0$, and

$$f_y(h,0,0) = \lim_{q \to 0} \frac{f(h,q,0) - f(h,0,0)}{q} \, .$$

$$= \lim_{q \to 0} \frac{\frac{h^3 q - hq^3}{h^2 + q^2} - 0}{q}$$

$$= \lim_{q \to 0} \frac{h^3 - hq^2}{h^2 + q^2}$$

$$= h.$$

Therefore,

$$f_{yx}(0,0,0) = \lim_{h \to 0} \frac{f_y(h,0,0) - f_y(0,0,0)}{h}$$

$$= \lim_{h \to 0} \frac{h - 0}{h} = 1$$

So $f_{xy}(0,0,0) \neq f_{yx}(0,0,0)$

Now, $f_z(0,0,0) = \lim\limits_{r \to 0} \dfrac{f(0,0,r) - f(0,0,0)}{r} = 0$, and

$$f_z(h,0,0) = \lim_{r \to 0} \frac{f(h,0,r) - f(h,0,0)}{r} = 0$$

Therefore, $f_{zx}(0,0,0) = \lim\limits_{h \to 0} \dfrac{f_z(h,0,0) - f_z(0,0,0)}{h} = 0$

Also $f_x(0,0,r) = \lim\limits_{p \to 0} \dfrac{f(p,0,r) - f(0,0,r)}{p} = 0$

This means, $f_{xz}(0,0,0) = \lim_{r \to 0} \dfrac{f_x(0,0,r) - f(0,0,0)}{r} = 0$

Hence $f_{xz}(0,0,0) = f_{zx}(0,0,0)$.

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You can try this exercise now.

E8) Let $f : \boldsymbol{R}^3 \to \boldsymbol{R}$ be defined by

$$f(x,y,z) = \begin{cases} \dfrac{x}{y} + \dfrac{y}{z}, & y \neq 0,\ z \neq 0 \\ 0 & , \text{otherwise} \end{cases}$$

Show that at the origin f_{xy}, f_{yx}, f_{xz} and f_{zx} all exist, but neither f_{zy} nor f_{yz} exists.

With this we come to an end of this unit.

Now let us briefly recall what we have covered in this unit.

4.4 SUMMARY

In this unit we have covered the following:

1) Introduced the partial derivatives of order greater than one
2) Evaluated higher order partial derivatives for some functions
3) Studied examples of functions which illustrate that mixed partial derivatives need not be equal
4) Stated three theorems:- Euler's theorem, Schwarz's theorem and Young's theorem, which give sufficient conditions that ensure the equality of f_{xy} and f_{yx}.

   Euler's Theorem:

   If f_{xy} and f_{yx} are both continuous at (a,b), then $f_{xy}(a,b) = f_{xy}(a,b)$

   Schwarz's Theorem:

   If f_{xy} is continuous at (a,b), and if f_y exists at (a,b), then $f_{xy} = f_{yx}$ at (a,b).

   Young's Theorem:

   If f_x and f_y are differentiable at (a,b), then $f_{xy} = f_{yx}$ at (a,b).

5) Explained through examples that Schwarz's and Young's theorems give sets of only sufficient conditions for the equality of f_{xy} and f_{yx}. The conditions are not necessary.

4.5 SOLUTIONS/ANSWERS

E1) a) $f(x, y) = \cos\frac{y}{x}$. Then

$$f_x = -\sin\left(\frac{y}{x}\right).\left(-\frac{y}{x^2}\right) = \frac{y}{x^2}\sin\frac{y}{x}$$

$$f_y = -\sin\left(\frac{y}{x}\right).\left(\frac{1}{x}\right) = -\frac{1}{x}\sin\frac{y}{x}$$

$$f_{xx} = \frac{y}{x^2}\cos\left(\frac{y}{x}\right).\left(-\frac{y}{x^2}\right) + \left(-\frac{2y}{x^3}\right)\sin\frac{y}{x}$$

$$= -\frac{y^2}{x^4}\cos\frac{y}{x} - \frac{2y}{x^3}\sin\frac{y}{x}$$

$$f_{yx} = \frac{1}{x^2}\sin\frac{y}{x} + \frac{y}{x^2}\left(\cos\frac{y}{x}\right)\left(\frac{1}{x}\right)$$

$$= \frac{1}{x^2}\sin\frac{y}{x} + \frac{y}{x^3}\cos\frac{y}{x}$$

$$f_{xy} = \frac{1}{x^2}\sin\frac{y}{x} - \frac{1}{x}\left(\cos\frac{y}{x}\right)\left(-\frac{y}{x^2}\right)$$

$$= \frac{1}{x^2}\sin\frac{y}{x} + \frac{y}{x^3}\cos\frac{y}{x}$$

$$f_{yy} = -\frac{1}{x^2}\cos\frac{y}{x}$$

b) $f(x, y) = x^5 + y^4\sin(x^6)$

$$\therefore f_x = 5x^4 + 6x^5y^4\cos(x^6)$$

$$f_y = 4y^3\sin x^6$$

$$f_{xx} = 20x^3 + 30x^4y^4\cos(x^6) - 36x^{10}y^4\sin(x^6)$$

$$f_{yx} = 24x^5y^3\cos(x^6) = f_{xy}$$

$$f_{yy} = 12y^2\sin(x^6)$$

E2) $f(x, y) = x^3 - 3xy^2$

$$\frac{\partial f}{\partial x} = 3x^2 - 3y^2 \qquad \therefore \frac{\partial^2 f}{\partial x^2} = 6x$$

$$\frac{\partial f}{\partial y} = -6xy \qquad \therefore \frac{\partial^2 f}{\partial y^2} = -6x.$$

Thus $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0$.

Therefore the answer is "yes", that is, the given function satisfies the Laplace equation.

E3) a) $f(x,y)=x^3y+e^{xy^2}$

$$\frac{\partial f}{\partial x}=3x^2y+y^2e^{xy^2}$$

$$\frac{\partial^2 f}{\partial y\partial x}=3x^2+2y\,e^{xy^2}+2xy^3\,e^{xy^2}$$

$$\frac{\partial f}{\partial y}=x^3+2xy\,e^{xy^2}$$

$$\frac{\partial^2 f}{\partial x\partial y}=3x^2+2y\,e^{xy^2}+2xy^3\,e^{xy^2}$$

So, $\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial y\partial x}$

b) $f(x,y)=\tan(xy^3)$

$$\therefore \frac{\partial f}{\partial x}=y^3\sec^2(xy^3)$$

$$\frac{\partial^2 f}{\partial y\partial x}=3y^2\sec^2(xy)^3+6xy^5\sec^2(xy^3)\tan(xy^3)$$

$$\frac{\partial f}{\partial y}=3xy^2\sec^2(xy^3)$$

$$\frac{\partial^2 f}{\partial x\partial y}=3y^2\sec^2(xy^3)+6xy^5\sec^2(xy^3)\tan(xy^3)$$

So, $\frac{\partial^2 f}{\partial y\partial x}=\frac{\partial^2 f}{\partial x\partial y}$

E4) $f_x(0,0)=\lim_{t\to 0}\frac{f(t,0)-f(0,0)}{t}=0$. Similarly, $f_y(0,0)=0$,

and for $k\neq 0, f_x(0,k)=\lim_{h\to 0}\frac{f(h,k)-f(0,k)}{k}=1$

So, $f_{yx}(0,0)=\lim_{k\to 0}\frac{f_x(0,k)-f_x(0,0)}{k}=1$.

Also, for $h \neq 0, f_y(h,0) = \lim_{k\to 0} \frac{f(h,k) - f(h,0)}{k}$

$$= \lim_{k\to 0} \frac{hk^4}{h^2 + k^4} = 0$$

So, $f_{xy}(0,0) = \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h} = 0.$

Hence, $f_{xy}(0,0) \neq f_{yx}(0,0)$.

E5) a) $f_x(0,0) = 0$.

For $k \neq 0, f_x(0,k) = \lim_{h\to 0} \frac{f(h,k) - f(0,k)}{h} = \lim_{h\to 0} \frac{hk^2}{\sqrt{h^4 + k^4}} = 0.$

So, $f_{yx}(0,0) = \lim_{k\to 0} \frac{f_x(0,k) - f_x(0,0)}{k} = 0$

Similarly, $f_{xy}(0,0) = 0$.

Hence, $f_{xy}(0,0) = f_{yx}(0,0)$

b) $f_x(0,0) = 0 = f_y(0,0)$

For $k \neq 0, f_x(0,k) = \lim_{h\to 0} \frac{f(h,k) - f(0,k)}{h} = \lim_{h\to 0} \frac{k^3}{\sqrt{h^2 + k^4}} = k.$

So, $f_{yx}(0,0) = \lim_{k\to 0} \frac{f_x(0,k) - f_x(0,0)}{k} = 1$

For $h \neq 0, f_y(h,0) = \lim_{k\to 0} \frac{f(h,k) - f(h,0)}{k} = \lim_{k\to 0} \frac{hk^2}{\sqrt{h^2 + k^4}} = 0$

So, $f_{xy}(0,0) = \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h} = 0$

Consequently, $f_{xy}(0,0) \neq f_{yx}(0,0)$.

E6) $f_x(0,0) = 0 = f_y(0,0)$

For $k \neq 0, f_x(0,k) = \lim_{h\to 0} \frac{f(h,k) - f(0,k)}{h}$

$= \lim_{h\to 0} \frac{-hk - 0}{h}$ (since k is fixed we can suppose that $|h| < |k|$)

$= -k$

For $h \neq 0, f_y(h,0) = \lim_{k\to 0} \frac{f(h,k) - f(h,0)}{k}$

$= \lim_{k\to 0} \frac{hk - 0}{k}$ (since k is fixed we can suppose that $|k| < |h|$)

$= h$.

Therefore, $f_{yx}(0,0) = \lim_{k \to 0} \frac{f_x(0,k) - f_x(0,0)}{k} = \lim_{k \to 0} \frac{-k-0}{k} = -1$

and $f_{xy}(0,0) = \lim_{h \to 0} \frac{f_y(h,0) - f_y(0,0)}{h} = \lim_{h \to 0} \frac{h-0}{h} = 1$

Hence, $f_{xy}(0,0) \neq f_{yx}(0,0)$.

E7) a) $f(x,y) = x^2 + xy + y^2$. Then

$f_y(x,y) = x + 2y$

$f_{xy}(x,y) = 1$

Clearly f_y exists everywhere and f_{xy} is continuous being a constant function. This shows that f satisfies the requirements of Schwarz's theorem. Hence, by Schwarz's theorem, f_{yx} exists and $f_{yx} = f_{xy} = 1$.

b) $f(x,y) = \frac{x^2 - y^2}{x^2 + y^2}, x \neq 0, y \neq 0$.

$$f_y(x,y) = \frac{-4x^2 y}{(x^2 + y^2)^2}$$

$$f_{xy}(x,y) = \frac{8xy(x^2 - y^2)}{(x^2 + y^2)^3}$$

Since f_y exists and f_{xy} is continuous at all points (x,y), where $x \neq 0$, $y \neq 0$, by Schwarz's theorem, we have

$$f_{yx}(x,y) = f_{xy}(x,y) = \frac{8xy(x^2 - y^2)}{(x^2 + y^2)^3}$$

E8) Now $f_x(0,0,0) = f_y(0,0,0) = f_z(0,0,0) = 0$.

For $k \neq 0, f_x(0,k,0) = \lim_{h \to 0} \frac{f(h,k,0) - f(0,k,0)}{h} = 0$

Therefore, $f_{yx}(0,0,0) = \lim_{k \to 0} \frac{f_x(0,k,0) - f_x(0,0,0)}{k} = 0$

For, $r \neq 0, f_y(0,0,r) = \lim_{k \to 0} \frac{f(0,k,r) - f(0,0,r)}{k}$

$$= \lim_{k \to 0} \frac{k/r}{k}$$

$$= \lim_{k \to 0} \frac{1}{r} = \frac{1}{r}$$

So, $f_{zy}(0,0,0) = \lim_{r \to 0} \frac{f_y(0,0,r) - f_y(0,0,0)}{r} = \lim_{r \to 0} \frac{1}{r^2}$

But $\lim\limits_{r\to 0}\frac{1}{r^2}$ does not exist.

Hence, $f_{zy}(0,0,0)$ does not exist.

Since $f_z(0,k,0)=\lim\limits_{r\to 0}\frac{f(0,k,r)-f(0,k,0)}{r}=k\lim\limits_{r\to 0}\frac{1}{r^2}$

therefore, $f_z(0,k,0)$ does not exist as $\lim\limits_{r\to 0}\frac{1}{r^2}$ does not exist.

Hence $f_{yz}(0,0,0)=\lim\limits_{k\to 0}\frac{f_z(0,k,0)-f_z(0,0,0)}{k}$ does not exist.

UNIT 5

CHAIN RULE AND HOMOGENEOUS FUNCTIONS

Structure — Page No.

5.1 INTRODUCTION

You are already familiar with the chain rule which is used for evaluating the derivative of a function of a function (Calculus, Unit 3). In this unit we shall study the chain rule to evaluate the partial derivatives of functions of 2 or 3 variables where each variable itself is a function of 2 or 3 independent variables. For instance if $\mathrm{u}, \mathrm{v}, \mathrm{w}$ are functions of a single variable t, then the function $\mathrm{f}(\mathrm{u}, \mathrm{v}, \mathrm{w})$ which is a function of three variables, is a function $F(t)$ of the single variable t. We shall describe how to find the 'total derivative' $F'(t)$. In this unit we shall also discuss homogeneous functions and an important theorem known as Euler's Theorem which characterizes these function.

Objectives

After reading this unit, you should be able to:

- define and evaluate the total derivative of a function using the chain rule,
- use the various forms of the chain rule to solve problems,
- define and identify homogeneous functions,
- state and use Euler's theorem for homogeneous functions.

5.2 CHAIN RULE

In this section, we describe the chain rule which enables us to calculate the derivatives of functions of several variables where each variable itself is a function of an independent variable. Recall that you have learnt the chain rule for function of one variable in your calculus course. The rule says that, suppose we have a function $y = f(x)$, where x is a function of t, say $x = g(t)$, then y also may be regarded as a function of t, say $y = F(t)$, and we have

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

or, in other words, $\boxed{F'(t) = f'(x)g'(t) = f'(g(t))g'(t)}$.

Now we extend the chain rule for functions of one variable to functions of 2-variables. Recall that we have defined the composite of functions of 2 or 3 variables in Unit 1. There you have seen that there are several ways of forming a composite function. For instance,

Case 1: Let $f(x, y) = x^2 + xy + y^2$ be a function from $\mathbf{R}^2 \to \mathbf{R}$, $g(t) = \sin t$ be a function from $\mathbf{R} \to \mathbf{R}$. Then the composite function $g \circ f$ defined by $g \circ f(x, y) = g(f(x, y)) = \sin(x^2 + xy + y^2)$ is a function from $\mathbf{R}^2 \to \mathbf{R}$.

Case 2: Consider the function $\phi(x, y) = x^y + y^x$ from $\mathbf{R}^2 \to \mathbf{R}$ and the functions $f(t) = \sin t$ and $g(t) = \tan t$ from $\mathbf{R} \to \mathbf{R}$. Then the function F defined by $F(t) = \phi(\sin t, \tan t) = (\sin t)^{\tan t} + (\tan t)^{\sin t}$ is a function from $\mathbf{R} \to \mathbf{R}$.

Since there are many ways of forming composite functions, we will have to derive chain rule separately for each type. In Theorem 1 below, we shall derive the chin rule for finding the derivatives of composite functions as discussed Case 1. Later in Theorem 2, we derive chain rule for the case discussed in Case 2. Let us now state Theorem 1, proof of which is omitted.

Theorem 1: Let $f(x, y)$ be a real-valued function having continuous first order partial derivatives at a point (a, b), and let g be a real-valued function of a real variable which is differentiable at the point $f(a, b)$. Then the composite function $\phi = g \circ f$ from $\mathbf{R}^2 \to \mathbf{R}$ has first order partial derivatives at (a, b) and

$$\phi_x(a,b) = g'(f(a,b))\, f_x(a,b)$$
$$\phi_y(a,b) = g'(f(a,b))\, f_y(a,b).$$

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We illustrate this theorem with an example.

Example 1: Consider the composite function $\phi(x, y) = \sin(x^2 + xy + y^2)$, given in Case 1. Calculate $\frac{\partial \phi}{\partial x}(a, b)$ and $\frac{\partial \phi}{\partial y}(a, b)$.

Solution: Here $\phi = g \circ f$ where $f(x, y) = x^2 + xy + y^2$ and $g(t) = \sin t$. Both functions f and g satisfy the requirements of Theorem 1.

So by Theorem 1, $\frac{\partial \phi}{\partial x}(a,b) = g'(f(a,b)).\frac{\partial \phi}{\partial x}(a,b)$

$$= \cos(a^2 + ab + b^2).(2a + b)$$
$$= (2a + b)\cos(a^2 + ab + b^2)$$

$$\frac{\partial \phi}{\partial y}(a,b) = g'(f(a,b))\,\frac{\partial \phi}{\partial y}(a,b)$$
$$= \cos(a^2 + ab + b^2).(a + 2b)$$
$$= (a + 2b)\cos(a^2 + ab + b^2)$$

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You can try this exercise now.

E1) Let $f(x, y) = x^2 + 3xy + y^2$ and $g(t) = \cos t$. Find the partial derivatives of $\phi = g \circ f$ at $\left(\sqrt{\frac{\pi}{2}}, \sqrt{\frac{\pi}{2}}\right)$

In the following theorem we state the chain rule for the functions discussed in Case 2. The proof of the theorem is beyond the scope of this course.

Theorem 2: If $f(t)$ and $g(t)$ are two real-valued functions which are differentiable at a point t_0 and if $\phi(x, y)$ is a real-valued function of two variables, which is differentiable at the point $(f(t_0), g(t_0))$, then the function $F(t) = \phi(f(t),\ g(t))$ is differentiable at t_0 and
$F'(t) = f'(t_0)\phi_x(f(t_0), g(t_0)) + g'(t_0)\phi_y(f(t_0),\ g(t_0))$.

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Fig. 1 gives an illustration of the functions considered in Theorem 2.

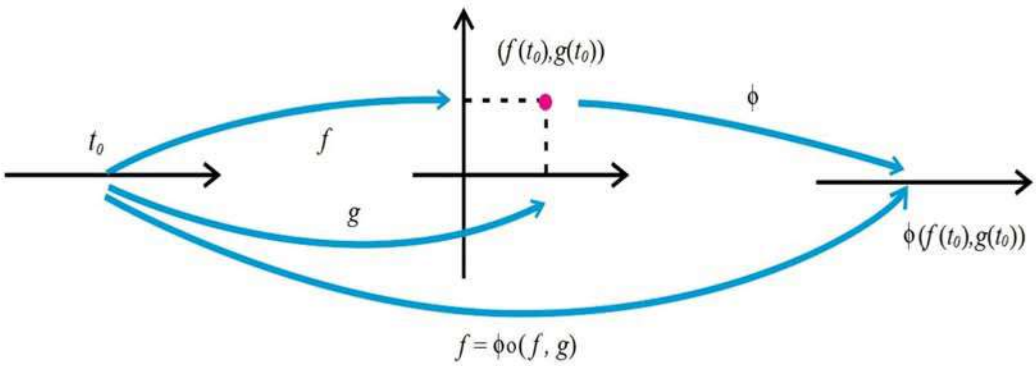

Fig. 1

If we write $x = f(t),\ y = g(t),\ z = F(t) = \phi(f(t, g(t))) = \phi(x, y)$, then the result of the above theorem can be written as

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{\partial t} + \frac{\partial z}{\partial y}\frac{dy}{dt}}$$

Remark: This is known as the **chain rule for partial derivatives**. The derivative $\frac{dz}{dt}$ is also known as the **total derivative of z**.
Let us look at some examples.

Example 2: Find the total derivative of the function

$f(x, y) = x^2 y - 2x + 3y - 4$, where $x = t - 2$ and $y = t^2$.

Solution: You can easily verify that all the requirements of Theorem 2 are satisfied.

Therefore, we have

$$\begin{aligned}\frac{df}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \\ &= (2xy - 2)(1) + (x^2 + 3)(2t) \\ &= [2(t-2)t^2 - 2] + [(t-2)^2 + 3]\,(2t) \\ &= 2t^3 - 4t^2 - 2 + 2t^3 - 8t^2 + 14t \\ &= 4t^3 - 12t^2 + 14t - 2.\end{aligned}$$

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Example 3: Consider the function $f : \boldsymbol{R}^2 \to \boldsymbol{R}$ defined by $f(x, y) = xy + yx$ where $x = t, y = e^t$. Find its total derivative.

Solution: Using the chain rule, we get

$$\begin{aligned}\frac{df}{dt} &= \frac{\partial f}{\partial x}.\frac{dx}{dt} + \frac{\partial f}{\partial y}.\frac{\partial y}{dt} \\ &= y + xe^t \\ &= e^t + te^t \\ &= e^t(1+t).\end{aligned}$$

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Example 4: Find the total derivative of the function $z = xy$ where $x = \cos t,\ y = t^2$.

Solution: By the chain rule we get

$$\frac{dz}{dt} = y(-\sin t) + x.2t$$

Note that we can also write

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial y}\frac{dy}{dt} + \frac{\partial z}{\partial x}\frac{dx}{dt} \\ &= x\frac{dy}{dt} + y\frac{dx}{dt}\end{aligned}$$

This formula is familiar to you. This is nothing but the product rule of differentiation for functions of one-variable.

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Note that in Examples 3 and 4, instead of using the chain rule, we could have first substituted for x and y in terms of t, and then differentiated the resulting function w.r.t. t. Thus, for the function in Example 3, namely $f(x, y) = xy$, where $x = \cos t, y = t^2$, we can write $f(t) = t^2 \cos t$.

And therefore, $f'(t) = 2t\cos t - t^2 \sin t$.

You can see that this is the same as the total derivative which we have calculated in Example 3 by the chain rule.

You might be wondering why we have done so much work to discover the additional (complicated) method for finding $\frac{dz}{dt}$. There are several reasons.

- First of all, it may not be always possible to express x or y explicitly in terms of t.
- Secondly, substituting for x and y could make the expression of z very complicated.

Thus, the evaluation of $\frac{dz}{dt}$ may become very lengthy and tedious. In our formula, we are carrying out the calculations in bits which are usually simpler than the calculations involved in the evaluation of $\frac{dz}{dt}$ after z has been expressed as a function of t.

We illustrate this by means of an example.

Example 5: Find the derivative of $(\sin t)^{\tan t} + (\tan t)^{\sin t}$.

Solution: Let $F(t) = (\sin t)^{\tan t} + (\tan t)^{\sin t}$

We rewrite $F(t)$ as $F(t) = \phi(f(t), g(t))$ where

$\phi(x, y) = x^y + y^x$, $x = f(t) = \sin t$ and $y = g(t) = \tan t$.

Then, ϕ, f and g satisfy the conditions of Theorem 2. Therefore, by chain rule

$$\begin{aligned}
\frac{dF}{dt} &= \frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt} \\
&= \left[yx^{y-1} + (\ln y)y^x\right](\cos t) + \left[(\ln x)x^y + xy^{x-1}\right]\sec^2 t \\
&= (\tan t)\frac{(\sin t)^{\tan t}}{\sin t}\cos t + (\ln \tan t)(\tan t)^{\sin t}(\cos t) \\
&\quad + (\ln \sin t)(\sin t)^{\tan t}(\sec^2 t) + \sin t\frac{(\tan t)^{\sin t}}{\tan t}(\sec^2 t) \\
&= \left[1 + \sec^2 t \ \ln\ \sin t\right](\sin t)^{\tan t} + \left[\cos t \ln \tan t + \sec t\right](\tan t)^{\sin t}
\end{aligned}$$

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Can you imagine how long it would have taken had we done this without writing F as a composite function?

Now if you have gone through the examples carefully, you should be able to do these exercises.

E2) Find the total derivative with respect to t in each of the following cases:

a) $z = x^2 + .3xy + y^2$ if $x = 2\cos\frac{\pi t}{8},\ y = 3 + \sin\frac{\pi t}{8}.$

b) $z = \frac{2x+3}{3y-2}$ if $x = e^t + t,\ y = e^{-t} - t.$

E3) Find $\frac{dz}{dt}$ in each of the following cases.

a) $z = \ln\left(x^2 + 3xy\right),\ x = e^t,\ y = e^{-t}$

b) $z = \tan^{-1}\frac{y}{x},\ x = \ln t,\ y = e^t$

E4) Find the derivatives of the following functions using the concept of total derivative.

a) $t^{3\sin t} + (\sin t)^{t^3}$

b) $t^{2t} + (t+1)^{t^2}$

c) $e^{t^4} + t^{4\cos t}$

In the next section we shall discuss an application of the chain rule.

5.3 HOMOGENEOUS FUNCTIONS

In this section we shall introduce you to a class of functions of several variables called homogeneous functions. We shall mainly describe a theorem known as Euler's theorem, which characterizes homogeneous functions using the techniques of the chain rule.

What is a homogeneous function? Let us see.

You have come across polynomials of the type $ax + by,\ 2x^2 + 3xy + 5y^2$ in various contexts. Note that each term in $ax + by$ is of degree 1, while each term in $2x^2 + 3xy + 5y^2$ is of degree 2. Suppose we replace x by $t\,x$ and y by $t\,y$ in the first polynomial. Then we get $atx + bty = t(ax + by)$. Similarly replacing x by $t\,x$ and y by $t\,y$ in the second polynomial, we get $2t^2x^2 + 3txty + 5t^2y^2 = t^2\left(2x^2 + 3xy + 5y^2\right).$

These are the simplest examples of homogeneous polynomials of degree 1 and 2, respectively. More generally, a polynomial with real coefficients in two variables x and y is called a homogeneous polynomial of degree h, if each term in the polynomial is of degree h. The most general polynomial of degree h in x, y is

$$p(x, y) = \sum_{\substack{\lambda+\mu=h \\ \lambda\geq 0,\ \mu\geq 0}} a_{\lambda\mu}\, x^{\lambda} y^{\mu} \qquad \ldots (1)$$

In the expression (1) we note that if we replace x by tx and y by ty, then we get $p(tx,ty)=t^h p(x,y)$. Thus, if $p(x,y)$ is a homogeneous polynomial of degree h, then $p(x,y)$ gets multiplied by t^h if x is replaced by tx and y is replaced by ty for any real number t. This phenomenon makes sense for functions other than polynomials too.

For instance, if $f(x,y)=\sqrt{x^2+y^2}$, then $f(tx,ty)=tf(x,y)$ for all $t>0$.

We call $\sqrt{x^2+y^2}$ a homogeneous function of degree 1. More formally, we have the following definition.

Definition 1: Let D be a subset of $\boldsymbol{R}^2$ such that if $(x,y)\in D$, then $(tx,ty)\in D$ for all $t>0$. A function $f: \mathrm{D}\to \boldsymbol{R}$ is said to be a homogeneous function of degree h, h being a real number, if $f(tx,ty)=t^h f(x,y)$ for all points $(x,y)\in D$ and for all $t>0$.

In the same way we can define homogeneous functions of 3 variables and we check the expression $f(tx,ty,tz)=t^h f(x,y,z)$.

Let us look at some examples.

Example 6: Show that the following functions are homogeneous functions.

i) $f(x,y)=\tan\dfrac{y}{x}$

ii) $f(x,y)=(x^4+3y^4)^{1/3}$

iii) $f(x,y)=\dfrac{\sin\left(\dfrac{x^2y}{x^3+y^3}\right)}{\ln\left(\dfrac{x+y}{x}\right)}$

iv) $f(x,y,z)=\dfrac{xy^2+yz^2+zx^2}{x+y+z}$

Solution: Let us take these one by one.

i) Replacing x by tx and y by ty when t is a positive real number, we get

$$f(tx,ty)=\tan\frac{ty}{tx}=\tan\frac{y}{x}=t^0 f(x,y).$$

Thus, $f(x,y)$ is a homogeneous function of two variables of degree zero.

ii)
$$\begin{aligned} f(tx,ty) &= \sqrt[3]{t^4x^4+3t^4y^4} \\ &= \left(t^4\right)^{1/3} 3\sqrt{x^4+3y^4} \\ &= t^{4/3} f(x,y) \end{aligned}$$

Thus, $f(x,y)$ is a homogenous function of two variables of degree 4/3.

iii) $$f(tx,ty)=\frac{\sin\left(\dfrac{t^2x^2.ty}{t^3x^3+t^3y^3}\right)}{\ln\left(\dfrac{tx+ty}{tx}\right)},t>0$$

$$=\frac{\sin\left(\dfrac{t^3x^2y}{t^3\left(x^3+y^3\right)}\right)}{\ln\left(\dfrac{x+y}{x}\right)}=\frac{\sin\left(\dfrac{x^2y}{x^3+y^3}\right)}{\ln\left(\dfrac{x+y}{x}\right)}$$

$$=t^0f(x,y).$$

Thus $f(x,y)$ is a homogenous function of degree zero.

iv) $$f(tx,ty,tz)=\frac{tx.t^2y^2+ty.t^2z^2+tz.t^2x^2}{tx+ty+tz}\quad t>0$$

$$=\frac{t^3\left(xy^2+yz^2+zx^2\right)}{t(x+y+z)}$$

$$=t^2f(x,y,z).$$

Thus $f(x,y,z)$ is a homogeneous function of three variables of degree two. Did you notice that the degree of homogeneity for the functions in i), iii) and iv) is an integer, whereas the degree of the function in ii) is not an integer? You should be able to do these exercises now.

E5) Which of the following functions are homogeneous? If a function is homogenous, determine the degree of homogeneity.

a) $f(x,y)=\max\left\{\dfrac{x}{y},y\right\}$

b) $f(x,y)=\dfrac{\sin x}{\sin y}$

c) $f(x,y)=x^{1/3}.y^{-5/3}$

d) $f(x,y)=3x^2y+xy^2-\pi y^3$

e) $f(x,y)=x^2y+2xy^2+xy+4y^3$

We shall now state Euler's theorem, which gives a beautiful characterization of homogeneous functions using the chain rule.

Theorem 3 (Euler's Theorem): Let D be a subset of $\boldsymbol{R}^2$ such that

i) for any $(x,y)\in D$, there exists an open disc of radius r with centre (x,y) contained in D, and

ii) for any point $(x,y)\in D$, the point $(tx,ty)\in D$ for all $t>0$.

Let $f : D \to \boldsymbol{R}$ be a function having continuous partial derivatives of first order at all points of D. Then $f(x, y)$ is a homogeneous function of degree h if and only if $af_x(a,b)+bf_y(a,b)=hf(a,b)$ for any point (a,b) in D.

Proof: Suppose $f(x, y)$ is a homogeneous function of degree h. Define the function $F :]0,\infty[\to \boldsymbol{R}$ by $F(t)=f(at,bt)=f(u(t),v(t))$, where (a,b) is any point of D and $u(t)=$ at and $v(t)=bt$.

Regarding F as a function of two variables, x and y, where $x=u(t)=at$, $y=v(t)=bt$, you can check that F satisfies all the conditions of Theorem 2.

Thus, we get

$$\begin{aligned}F'(t)&=u'(t)\, f_x(u(t),v(t))+v'(t)f_y(u(t),v(t))\\&=af_x(u(t),v(t))+bf_y(u(t),v(t)),\text{ since } u'(t)=a \text{ and } v'(t)=b.\\&=af_x(at,bt)+bf_y(at,bt) \qquad \ldots (2)\end{aligned}$$

But $f(x, y)$ is a homogeneous function of degree h. Therefore
$F(t)=f(at,bt)=t^h f(a,b)$
and $F'(t)=ht^{h-1}f(a,b)$. ... (3)

Equations (2) and (3), we get

$$af_x(at,bt)+bf_y(at,bt)=ht^{h-1}f(a,b) \text{ for all } t>0. \qquad \ldots (4)$$

Putting $t=1$ in (4) we get

$$af_x(a,b)+bf_y(a,b)=hf(a,b).$$

Conversely, suppose that the function $f(x, y)$ satisfies the relation

$$af_x(a,b)+bf_y(a,b)=hf(a,b) \text{ for all } (a,b) \text{ in } D. \qquad \ldots (5)$$

For the function $F(t)$ defined above, we have $F'(t)=af_x(at,bt)+bf_y(at,bt)$, where (a,b) is any point of D. Since $(a,b)\in D$ implies that $(at,bt)\in D$, it follows from (3) and (4) that $atf_x(at,bt)+btf_y(at,bt)=f(at,bt)$.

Consequently, $tF'(t)=hf(at,bt)=hF(t)$
or $F'(t)=\dfrac{h}{t}F(t)$
Consider the function $\phi(t)=t^{-h}F(t)$ for $t>0$.

$$\begin{aligned}\text{Clearly } \phi'(t)&=t^{-h}F'(t)-ht^{-h-1}F(t)\\&=t^{-h}\left[F'(t)-\frac{h}{t}F(t)\right]\\&=0 \text{ for all } t>0.\end{aligned}$$

Therefore, $\phi(t)$ is a constant function for all $t > 0$.

But $\phi(1) = F(1) = f(a,b)$. Therefore, $\phi(t) = f(a,b) \forall t > 0$
i.e., $t^{-h} F(t) = f(a,b)$ for all $t > 0$
i.e., $F(t) = t^h f(a,b)$.

Thus, $f(at,bt) = t^h f(a,b)$ for any point $(a,b) \in D$. This means that f is a homogeneous function of degree h.

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Remark 1: If we write $z = f(x, y)$, then by Euler's theorem $f(x, y)$ is a homogeneous function of degree n if and only if

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz \qquad \dots (6)$$

This relation is known as **Euler's relation**.

Try this exercise before proceeding further.

E6) If D is a subset of $\boldsymbol{R}^2$ which satisfies (i) and (ii) of Theorem 3, and if $f : D \to \boldsymbol{R}$ is a homogeneous function of degree n, which has continuous second order partial derivatives, then $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are homogenous.

This exercise leads us to the following result.

Corollary 1: Let $f : D \to \boldsymbol{R}$ where D is a subset of $\boldsymbol{R}^2$ as mentioned in the statement of Theorem 3. If f is a homogeneous function of degree n and if f has continuous partial derivatives of second order on all points of D, then

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = n(n-1)z$$

for all points $(x, y) \in D$, where $z = f(x, y)$.

Proof: Since z is a homogeneous function of degree n and has continuous second order partial derivatives of all points of D, it follows that both $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are homogeneous function of degree $(n-1)$ (see E6), and have continuous partial derivatives of first order at al points of D. Thus, applying Euler's theorem to the functions $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, we obtain

$$x\frac{\partial^2 z}{\partial x^2}+y\frac{\partial^2 z}{\partial y\,\partial x}=(n-1)\frac{\partial z}{\partial x} \qquad \dots (7)$$

and

$$x\frac{\partial^2 z}{\partial x\,\partial y}+y\frac{\partial^2 z}{\partial y^2}=(n-1)\frac{\partial z}{\partial y} \qquad \dots (8)$$

But $\dfrac{\partial^2 z}{\partial x\,\partial y}=\dfrac{\partial^2 z}{\partial y\,\partial x}$, in view of Schwarz's theorem (Theorem 2, Unit 4),

Therefore, multiplying (7) by x and (8) by y and adding we get

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\,\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=(n-1)\left(x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}\right)$$

Therefore,

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\,\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=n(n-1)z \qquad \text{(By Euler's relation)}$$

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We shall now illustrate Euler's theorem with some examples.

Example 7: Show that the function $\dfrac{xy}{x+y}$, $x>0,\ y>0$ satisfies the requirements of Euler's theorem, and then verify Euler's relation by direct computation.

Solution: Let $D=\{(x,y)\mid x>0,\ y>0\}$, and $f:D\to \boldsymbol{R}$ be defined by $f(x,y)=\dfrac{xy}{x+y}$.

Then

i) $(x,y)\in D \Rightarrow (tx,ty)\in D$ for all $t>0$

ii) if $(a,b)\in D,$ then the disc of radius $r=\dfrac{1}{2}\min\{a,b\}$ with centre (a,b) is contained in D. See Fig. 2.

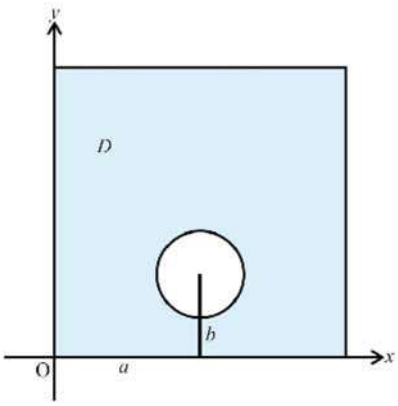

Fig. 2

Now the given function is a homogeneous function of degree 1, because
$f(tx,ty)=\frac{t^2xy}{t(x+y)}=tf(x,y).$

Further, for any point $(a,b)\in D,$ a simple calculation shows that
$f_x(x,y)=\frac{y^2}{(x+y)^2}$ and $f_y(x,y)=\frac{x^2}{(x+y)^2}$ and these are clearly continuous on $D.$ Thus, all the requirements of Euler's theorem are satisfied.

To verify Euler's relation we have to prove that $xf_x(x,y)+yf_y(x,y)=1.f(x,y)$

Now,

$$\begin{aligned} xf_x(x,y)+yf_y(x,y) &= x\frac{y^2}{(x+y)^2}+y\frac{x^2}{(x+y)^2} \\ &= xy\left[\frac{x+y}{(x+y)^2}\right] \\ &= \frac{xy}{(x+y)} \\ &= 1.\quad f(x,y) \end{aligned}$$

This proves Euler's relation.

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In the next two examples we consider inverse trigonometric functions.

Example 8: For the function $z=\sin^{-1}\frac{x}{y}+\tan^{-1}\frac{y}{x}$ defined on
$D=\{(x,y)\,|0<x<y\}$, prove that
$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=0.$$

Solution: We first note that D satisfies conditions (i) and (ii) of Euler's theorem. Let $f(x,y)=\sin^{-1}\frac{x}{y}+\tan^{-1}\frac{y}{x}.$

Then since

(i) $(tx,ty)\in D$ for all $(x,y)\in D,\ t>0$, and

(ii) $f(tx,ty)=t^0 f(x,y)$ for all $(x,y)\in D,$ we can say that $z=f(x,y)$ is a homogeneous function of degree 0.

Further, $f_x(x,y)=\frac{\partial z}{\partial x}=\frac{1}{y}\left(\frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\right)-\frac{y}{x^2}\left(\frac{1}{1+\frac{y^2}{x^2}}\right)$

Since $(x,y)\in D$ is such that $0<x<y, f_x$ is defined and continuous for all points of $D.$ Similarly

$$f_y(x,y)=\frac{\partial z}{\partial x}=\left(\frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\right)\left(-\frac{x}{y^2}\right)+\left(\frac{1}{1+\frac{y^2}{x^2}}\right)\left(\frac{1}{x}\right)$$

$$=-\frac{x}{y^2}\left(\frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\right)+\frac{1}{x}\left(\frac{1}{\sqrt{1+\frac{y^2}{x^2}}}\right)$$

is defined and continuous for all points of D. Thus by Euler's theorem,
$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=0.$

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Example 9: If $u=\sin^{-1}\frac{x^2+y^2}{x+y}$, $0<x<1,\ 0<y<1$, prove that

$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\tan u.$

Solution: Let $D=\{(x,y)\,|x>0,\ y>0\}$ and $f:D\to \boldsymbol{R}$ be defined by

$$f(x,y)=\frac{x^2+y^2}{x+y}.$$

Then the function $z=f(x,y)$ is a homogeneous function of degree 1 and satisfies the requirements of Euler's theorem. Therefore,

$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z$ for all $(x,y)\in D$. … (9)

Now take $D'=\{(x,y)\,|0<x<1,\ 0<y<1\}$. Then $D'\subset D$ and hence equation (8) is true in particular for all $(x,y)\in D'$. Also, for all $(x,y)\in D'$, we have

$$\sin u=\frac{x^2+y^2}{x+y}=z.$$

Therefore, $\frac{\partial z}{\partial x}=\cos u\frac{\partial u}{\partial x}$ and $\frac{\partial z}{\partial y}=\cos u\frac{\partial u}{\partial y}$. Consequently, substituting these values in (9) we get

$$\left(x\,\frac{\partial u}{\partial x}+y\,\frac{\partial u}{\partial y}\right)\cos u=\sin u$$

or $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\tan u.$

***

Why don't you try some exercises now?

E7) If $z = \frac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$, show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{1}{20}z.$

E8) Verify Euler's relation for the function

a) $u = \frac{x^3 + y^3}{x + y}$

b) $u = \tan^{-1}\frac{y}{x}$

by direct calculation.

E9) If $z = \sin^{-1}\frac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}$, then show that

$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0.$

E10) If $z = \tan^{-1}\frac{x^3 + y^3}{x + y}$, then show that

$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \sin 2x.$

Now let us briefly recall the main points discussed in this unit.

5.4 SUMMARY

In this unit, we have:

1. Discussed the following form of chain rule to differentiate composite functions:

   If $f : \boldsymbol{R} \to \boldsymbol{R}, g : \boldsymbol{R^2} \to \boldsymbol{R}$ and $\phi = f \circ g : \boldsymbol{R^2} \to \boldsymbol{R}$, other partial derivatives of the composite function $\phi = fog$ are given by $\phi_x(a,b) = g'(f(a,b))f_x(a,b)$ and $\phi_y(a,b) = g'(f(a,b))f_y(a,b)$.

2. Explained the notion of the total derivative of functions of 2 variables x and y where each x and y is a function of t. The total derivative of z is

   $$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

3. Defined homogeneous functions of 2 or 3 variables : f is a homogeneous function of degree h if

   $f(tx,ty) = t^h f(x,y), \quad \forall\ t > 0.$

   $f(tx,ty,tz) = t^h f(x,y,z), \quad \forall t > 0\,.$

4. Stated and discussed how to use Euler's theorem for homogeneous functions.

5.5 SOLUTIONS/ANSWERS

E1) By applying Theorem 1, we have

$$\frac{\partial\phi}{\partial x}\left(\sqrt{\frac{\pi}{2}},\sqrt{\frac{\pi}{2}}\right)=g'\left[f\left(\sqrt{\frac{\pi}{2}},\sqrt{\frac{\pi}{2}}\right)\right].\frac{\partial f}{\partial x}\left(\sqrt{\frac{\pi}{2}},\sqrt{\frac{\pi}{2}}\right)$$

$$=-\sin\left(\frac{\pi}{2}+3\frac{\pi}{2}+\frac{\pi}{2}\right)\times\left(2\sqrt{\frac{\pi}{2}}+3\sqrt{\frac{\pi}{2}}\right)$$

$$=-5\sqrt{\frac{\pi}{2}}$$

$$\frac{\partial\phi}{\partial y}\left(\sqrt{\frac{\pi}{2}},\sqrt{\frac{\pi}{2}}\right)=g'\left[f\left(\sqrt{\frac{\pi}{2}},\sqrt{\frac{\pi}{2}}\right)\right]\times\frac{\partial f}{\partial y}\left(\sqrt{\frac{\pi}{2}},\sqrt{\frac{\pi}{2}}\right)$$

$$=-\left(3\sqrt{\frac{\pi}{2}}+2\sqrt{\frac{\pi}{2}}\right)$$

$$=-5\sqrt{\frac{\pi}{2}}$$

E2) a) $$\frac{dz}{dt}=\frac{-\pi}{8}(2x+3y)\sin\frac{\pi t}{8}+\frac{\pi}{8}(3x+2y)\cos\frac{\pi t}{8}$$

$$=\frac{\pi}{8}\left[12\cos\frac{\pi t}{8}-13\sin\frac{\pi t}{8}+3\left(\cos^2\frac{\pi t}{8}-\sin^2\frac{\pi t}{8}\right)\right]$$

b) $$\frac{dz}{dt}=\frac{2}{3y-2}\left(e^t+1\right)+(2x+3)\left(-\frac{3}{(3y-2)^2}\right)\left(-e^t-1\right)$$

$$=\frac{\left(2e^t+2\right)\left(3e^{-t}-3t-2\right)+\left(2e^t+2t+3\right)\left(3e^{-t}+3\right)}{\left(3e^{-t}-3t-2\right)^2}$$

$$=\frac{17-6te^t+6te^{-t}+2e^t+15e^{-t}}{\left(3e^{-t}-3t-2\right)^2}$$

E3) a) $$\frac{dz}{dt}=\frac{1}{x^2+3yx}.(2x+3y)e^t+\frac{1}{x^2+3xy}.3x\left(-e^t\right)$$

$$=\frac{(2e^t+3e^{-t})e^t-3e^te^{-t}}{e^{2t}+3e^t.e^{-t}}$$

$$=\frac{2e^{2t}}{e^{2t}+3}$$

b) $$\frac{dz}{dt}=\frac{1}{1+y^2/x^2}\left(-\frac{y}{x^2}\right).\frac{1}{t}+\frac{1}{1+y^2/x^2}\left(\frac{1}{x}\right).e^t$$

$$=\frac{-e^t\dfrac{1}{t}+e^t.\ln t}{(\ln t)^2+e^{2t}}=\frac{e^t\left(t\ln t-1\right)}{t[(\ln t)^2+e^{2t}]}$$

E4) a) Let $f(x,y)=x^y+y^x$, where $x=t^3, y=\sin t$ so that the function f when considered as a function of t gives us the function whose derivative we wish to find out. Thus,

$$\frac{df}{dt}=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$=[yx^{y-1}+\ln x.y^x]3t^2+[\ln y.x^y+xy^{x-1}]\cos t$$

$$=[(\sin t)t^{3(\sin t-1)}+\ln t^3.(\sin t)^{t^3}]3t^2+[\ln\sin t.t^{3\sin t}+t^3(\sin t)^{t^3-1}]$$

b) Let $f(x,y)=x^{y-1}+y^x$, where $x=t^2, y=t+1$

Therefore, $\dfrac{df}{dt}=[(y-1)x^{y-2}+y^x\ln y]2t+[\ln x.x^{y-1}+xy^{x-1}]$

$$=[t.t^{2(t-1)}+(t-1)^{t^2}\ln(t+1)]2t+[\ln t^2.t^{2t}+t^2(t+1)^{t^2-1}]$$

$$=2t^{2t}+2t(t+1)^{t^2}\ln(t+1)+t^{2t}\ln t^2+t^2(t+1)^{t^2-1}$$

c) Let $f(x,y)=e^x+x^y$, where $x=t^4, y=\cos t$.

Then, $\dfrac{df}{dt}=[e^x+yx^{y-1}](4t^3)+(x^y\ln x)(-\sin t)$

$$=4t^3e^{t^4}+4t^3\cos t\,t^{4(\cos t-1)}-\sin t.t^{4\cos t}\ln t^4$$

E5) a) For $t>0$, $f(tx,ty)=\max\left\{\dfrac{tx}{ty},ty\right\}$

$$=\max\left\{\frac{x}{y},ty\right\}$$

which need not be equal to $t\max\left\{\dfrac{x}{y},y\right\}=t\,f(x,y)$.

check with $x=2, y=1$ and $t=2$,

$\therefore f$ is not homogeneous.

b) Since $f(tx,ty)=\dfrac{\sin tx}{\sin ty}\neq t^n\dfrac{\sin x}{\sin y}=t^nf(x,y)$, for any n,

f is not homogeneous.

Check with $x=\pi, y=\dfrac{\pi}{2}, t=\dfrac{1}{2}$.

c) $f(tx,ty)=(tx)^{1/3}(ty)^{-5/3}$

$$=t^{-4/3}x^{1/3}y^{-5/3}$$

$$=t^{-4/3}f(x,y)$$

Thus, f is a homogeneous function of degree $-4/3$.

d) $\mathrm{f(tx,ty)=t^3(3x^2y+xy^2-\pi y^3)}$

$$=t^3f(x,y)$$

Thus, f is a homogeneous function of degree 3.

e) f is not homogeneous.

E6) Since f is a homogeneous function of degree n, and has continuous first order partial derivatives, we can apply Euler's theorem to f.

Therefore, we get

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf$$

Differentiating the above equation with respect to x, we get

$$x\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)+\frac{\partial f}{\partial x}+y\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=n\frac{\partial f}{\partial x}$$

Since f admits continuous partial derivatives of second order, in view of Schwarz's theorem (see Unit 4), we get

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

Consequently, we have

$$x\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)+y\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=(n-1)\frac{\partial f}{\partial x}$$

Therefore, by using Euler's theorem we get that $\frac{\partial f}{\partial x}$ is a homogeneous function of degree $n-1$. Similarly, we can show that $\frac{\partial f}{\partial y}$ is also a homogeneous function of degree $n-1$.

E7) Note that z is a homogeneous function of degree $\frac{1}{20}$ and so Euler's theorem gives $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=\frac{1}{20}z.$

E8) a) Note that u is a homogeneous function of degree 2. We have to verify Euler's relation $x\cdot\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2u.$ Now,

i) $$\frac{\partial u}{\partial x}=\frac{3x^2(x+y)-(x^3+y^3)}{(x+y)^2}=\frac{2x^3+3x^2y-y^3}{(x+y)^2}$$

ii) $$\frac{\partial u}{\partial y}=\frac{3y^2(x+y)-(x^3+y^3)}{(x+y)^2}=\frac{-x^3+3xy^2+2y^3}{(x+y)^2}$$

$$\therefore x\cdot\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{2x^4+2x^3y+2xy^3+2y^4}{(x+y)^2}$$

$$= 2 \cdot \frac{x^3 + y^3}{(x + y)} = 2u .$$

b) u is a homogeneous function of degree 0. Now,

$\frac{\partial u}{\partial x} = \frac{1}{1 + y^2 / x^2} \left(-\frac{y}{x^2} \right) = -\frac{y}{x^2 + y^2}$ and

$\frac{\partial u}{\partial y} = \frac{1}{1 + y^2 / x^2} \left(\frac{1}{x} \right) = \frac{x}{x^2 + y^2}$. Thus

$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0.u$. This proves Euler's relation.

E9) Let $D = [(x, y) \mid x > 0, y > 0\}$

Then D satisfies the requirements (i) and (ii) of Euler's theorem. Moreover the function

$$z(x, y) = \sin^{-1} \left(\frac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}} \right)$$

is a homogeneous function of degree 0 and has continuous partial derivatives on D . Therefore by applying Euler's theorem,

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 0.$$

E10) Let $D = \{(x, y) \mid x \neq y\}$ and $f : D \to \boldsymbol{R}$ such that

$$f(x, y) = \frac{x^3 + y^3}{x - y}$$

Then f is a homogeneous function of degree 2 and has continuous partial derivatives of first order.

Therefore, in view of Euler's Theorem, $x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = 2f$

Now we have $\tan z = f$. Then $\frac{\partial f}{\partial x} = \sec^2 z . \frac{\partial z}{\partial x}, \frac{\partial f}{\partial y} = \sec^2 z \frac{\partial z}{\partial y}$

Substituting for $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ in Euler's relation, we get

$$x \sec^2 z \frac{\partial z}{\partial x} + y \sec^2 z \frac{\partial z}{\partial y} = 2 \tan z$$

i.e., $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = \sin 2z.$

MISCELLANEOUS EXAMPLES AND EXERCISES

The examples and exercises given below cover the concepts and processes you have studied in this block. Doing them will give you a better understanding of the concepts concerned, as well as practice in solving such problems.

We shall first consider miscellaneous examples. You can solve the exercises in the similar way. We advise you not to look at the solutions of the exercises given at the end unless you have tried to solve on your own.

Miscellaneous Examples

Example 1: State whether the following statements are true or false. Justify your answer with the help of a short proof or a counter example.

i) The function $f(x, y) = \sin\sqrt{x^2 + y^2}$ is not a homogenous function.

ii) If $P(x, y, z)$ is a point in $\boldsymbol{R}^3$ such that $|x| < \frac{1}{\sqrt{3}}, |y| < \frac{1}{\sqrt{3}}$ and $|z| \le \frac{1}{\sqrt{3}}$, then P lies inside the unit sphere in $\boldsymbol{R}^3$.

iii) If f and g are two real-valued functions defined on $\boldsymbol{R}^2$ such that $\frac{\partial(f+g)}{\partial x}$ exists at a point (a,b), then both $f_x(a,b)$ and $g_x(a,b)$ exist.

iv) $\lim\limits_{(x,y)\to(0,1)} e^{xy}\cos(\pi xy) = \pi$

v) If $f(x, y) = x\ln y$, then $f_{xy} = f_{yx}$ for $y \neq 0$.

Solution: Before we discuss the solution we make an important note.

[Note: Suppose the given statements is $\lim\limits_{(x,y)\to(0,0)} (x + y) = 2$. Simply saying that this is false is not enough. You have to give reasons for saying so. The reasons may be a proof or an example. In this case you should show that the limit is not 2 it is zero. Similarly, if you believe that a particular statement is true, then you have to give a reason for saying so. For example, the statement is like the one is given below.

"A function of two variables which is continuous at a point need not have any of the partial derivatives at that point".

This statement is true. Here you can give an example of a function f which is continuous at some point and which does not have any of the partial derivatives at that point.]

Now we shall discuss the solution of i), ii) , iii), iv) and v) one by one.

i) We replace x by tx and y by ty in the expression for $f(x, y)$ given by $f(x, y) = \sin\sqrt{x^2 + y^2}$.

Then $f(tx, ty) = \sin\sqrt{t^2x^2 + t^2y^2}$
$= \sin t\sqrt{x^2 + y^2}$

$$\neq t^n \sin\sqrt{x^2+y^2} = t^n f(x,y), \text{for any } n.$$

Therefore the statement is **true**.

ii) To check this we calculate the distance OP where O is the origin $(0,0,0)$.

Then $OP = \sqrt{x^2+y^2+z^2} < \sqrt{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = 1$

Since OP is less than radius of the unit sphere which is 1, we get that P lies inside the sphere.

Hence the statement is true.

iii) Let $f(x,y) = |x|$

$g(x,y) = -|x|$

Then $f(x,y) + g(x,y) = 0$

$\therefore \frac{\partial(f+g)}{\partial x}(x,y)$ exists at all points.

But $\frac{\partial f}{\partial x}$ does not exist at $(0,0)$. Why?

This shows that the given statement is **false.**

iv) The given function $f(x,y)$ is a product of two functions $f(x,y) = e^{xy}$ and $g(x,y) = \cos(\pi xy)$. Both are continuous functions. Therefore,

$\lim_{(x,y)\to(0,1)} f(x,y) = e^0 = 1$ and

$\lim_{(x,y)\to(0,1)} \cos(\pi xy) = \cos(0) = 1$

$\therefore$ Thus $\lim_{(x,y)\to(0,1)} e^{xy}\cos(\pi xy) = 1 \neq \pi$.

Hence the statement is **false**.

v) We have $f_x(x,y) = \ln y$ and $f_y(x,y) = \frac{x}{y}, y \neq 0$

$f_{yx} = \frac{1}{y}; f_{xy} = \frac{1}{y}, y \neq 0$.

Then both functions are not defined for $y = 0$ and so we can assume that the result is true for all points of their domains.

Hence the given statement is true.

***

Example 2: Find the following:

i) the rectangular coordinates of the point with cylindrical coordinates $(r,\theta,z) = (4,\pi/3,-3)$.

ii) the rectangular coordinates of the point with spherical coordinates $(\rho, \theta, \phi) = (4, \pi/3, \pi/4)$.

iii) the spherical coordinates of the point that has rectangular coordinates $(x, y, z) = (4, -4, 4\sqrt{6})$.

Solution: i) Applying the cylindrical-to-rectangular conversion formulas, we get $x = r\cos\theta = 4\cos\dfrac{\pi}{3} = 2,\ y = r\sin\theta = 4\sin\dfrac{\pi}{3} = 2\sqrt{3},\ z = -3$

Thus, the rectangular coordinates of the point are $(x, y, z) = (2, 2\sqrt{3}, -3)$

ii) Applying the spherical-to-rectangular conversion formula, we get

$$x = \rho\sin\phi\cos\theta = 4\sin\frac{\pi}{4}\cos\frac{\pi}{3} = \sqrt{2}$$

$$y = \rho\sin\phi\sin\theta = 4\sin\frac{\pi}{4}\sin\frac{\pi}{3} = \sqrt{6}$$

$$z = \rho\cos\phi = 4\cos\frac{\pi}{4} = 2\sqrt{2}$$

Hence the rectangular coordinates of the point are $(x, y, z) = (\sqrt{2}, \sqrt{6}, 2\sqrt{2})$

iii) From the rectangular-to-spherical conversion formulas, we obtain

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{16+16+96} = \sqrt{128} = 8\sqrt{2}$$

$$\tan\theta = \frac{y}{x} = -1$$

$$\cos\phi = \frac{z}{\sqrt{x^2 + y^2 + z^2}} = \frac{4\sqrt{6}}{8\sqrt{2}} = \frac{\sqrt{3}}{2}$$

From the restriction $0 \le \theta < 2\pi$ and the computed value of $\tan\theta$, the possibilities for θ are $\theta = 3\pi/4$ and $\theta = 7\pi/4$. However, the given point has a negative $y-$coordinate, so we must have $\theta = 7\pi/4$. Moreover, from the restriction $0 \le \phi \le \pi$ and the computed value of $\cos\phi$, the only possibility for ϕ is $\phi = \pi/6$. Thus, the spherical coordinates of the point are $(\rho, \theta, \phi) = (8\sqrt{2}, 7\pi/4, \pi/6)$.

***

Example 3: Find equations of the paraboloid $z = x^2 + y^2$ in cylindrical and spherical coordinates.

Solution: The rectangular-to-cylindrical conversion formula, gives

$$z = r^2 \qquad \text{... (1)}$$

which is the equation in cylindrical coordinates.

The conversion formula from rectangular to spherical coordinates gives

$\rho\cos\phi = \rho^2 \sin^2 \phi$

Which we can rewrite as $\rho = \cos\phi \operatorname{cosec}^2\phi.$.

This gives the equation in spherical coordinates.

***

Example 4: Let $f(x, y, z) = \sqrt{1 - x^2 - y^2 - z^2}$. Find $f\left(0, \frac{1}{2}, -\frac{1}{2}\right)$ and the domain of f.

Solution: By substitution, $f\left(0, \frac{1}{2}, -\frac{1}{2}\right) = \sqrt{1 - (0)^2 - \left(\frac{1}{2}\right)^2 - \left(-\frac{1}{2}\right)^2} = \sqrt{\frac{1}{2}}$

Because of the square root sign, we must have $0 \le 1 - x^2 - y^2 - z^2$ in order to have a real value for $f(x, y, z)$. Rewriting this inequality in the form $x^2 + y^2 + z^2 \le 1$, we see that the domain of f consists of all points on or within the sphere $x^2 + y^2 + z^2 = 1$.

***

Example 5: Find the following:

i) The level curves of the surface $f(x, y) = x^2 - y^2$ for the values $c = 1, 0, 4, -1$.

ii) The level surfaces given by $f(x, y, z) = x^2 + y^2$ for $c = 1, 0, 9$.

Solution: i) The given surface is $z = x^2 - y^2$. For $c = 1$, the level curves is $x^2 - y^2 = 1$ which is a hyperbola that passes vertically through the x-axis at the point $(\pm 1, 0)$. For $c = 0$, the level curves are straight lines. For $c = 4$, the level curves is a hyperbola through x-axis at the point $(\pm 2, 0)$. For $c = -1$, the level curve is the hyperbola passes through y-axis at the points $(0, \pm 1)$.

ii) The surface is given $f(x, y) = x^2 + y^2, z$. For $c = 1$, the level surface represents a cylinder of radius 1. For $c = 0$, the level surface is z-axis. For $c = 9$, the level surface is a cylinder of radius 3.

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Example 6: Find the limit of $f(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$ as $(x, y) \to (0, 0)$ along the lines

i) $y = 3x$

ii) $y = 5x$

What can you conclude about $\lim\limits_{(x,y)\to(0,0)} f(x, y)$? Justify your answer.

Solution: i) Along the line $y=3x, f(x,y)=\dfrac{x^2-9x^2}{x^2+9x^2}=\dfrac{-8}{10}=\dfrac{-4}{5}$ when $x\neq 0$.

Therefore $f(x,y)\to -\dfrac{4}{5}$ as $(x,y)\to(0,0)$ along $y=3x$.

ii) Along $y=5x, f(x,y)\to\dfrac{-12}{13}$ as $(x,y)\to(0,0)$.

Since $f(x,y)$ takes different limits in different directions, $\lim f(x,y)$ does not exist as $(x,y)\to(0,0)$.

***

Example 7: Verify Schwarz theorem for the following function at the point $(0,1)$. What can you conclude about f_{xy} and f_{yx}?

$$f(x,y)=2e^{3y}\cos x+e^x\sin 2y.$$

Solution: We have $f(x,y)=2e^{3y}\cos x+e^x\sin 2y$. Then $f_y(x,y)=6e^{3y}\cos x+2e^x\cos 2y$ and $f_{xy}(x,y)=-6e^{3y}\sin x+2e^x\cos 2y$ at $(0,1)$.

Then both by f_y and f_{xy} are continuous being the sum of the continuous functions. Also $f_{xy}(0,1)=2\cos 2$.

Therefore the function satisfies all the conditions of Schwarz's theorem and hence by the theorem, we have

$$f_{yx}(0,1)=f_{xy}(0,1)=2\cos 2.$$

***

Example 8: Let $f(x,y)=\dfrac{\cos x+e^{xy}}{x^2+y^2}$. Show that f is differentiable at all points $(x,y)\neq(0,0)$.

Solution: We calculate the partial derivatives

$$f_x=\frac{(x^2+y^2)(ye^{xy}-\sin x)-2x(\cos x+e^{xy})}{(x^2+y^2)^2}$$
$$f_y=\frac{(x^2+y^2)\,xe^{xy}-2y(\cos x+e^{xy})}{(x^2+y^2)^2}$$

Since both f_x and f_y are continuous at $(x,y)\neq(0,0)$, by Euler's theorem, f is differentiable at $(x,y)\neq(0,0)$.

***

Example 9: If $z=x^2y+3xy^4$ where $x=\sin 2t$ and $y=\cos t$, find $\dfrac{dz}{dt}$ when $t=0$ by i) chain rule and ii) by direct substitution.

Solution: i) The chain rule gives $\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dt}{dt}$

We have $z=x^2y+3xy^4$

Therefore $\frac{\partial z}{\partial x}=2xy+3y^4$

$\frac{\partial z}{\partial y}=x^2+12xy^3$

$\frac{dx}{dt}=2\cos 2t$

$\frac{dy}{dt}=-\sin t$

Thus $\frac{dz}{dt}=(2xy+3y^4)(2\cos 2t)+(x^2+12xy^3)(-\sin t)$

$=6$, since when $t=0, x=0$ and $y=1.$

ii) Direct substitution gives

$$z=\sin^2 2t\cos t+3\sin 2t\cos^4 t$$

$$\frac{dz}{dt}=2\sin 2t\cos t+-\sin t\sin^2 2+6\cos 2t\cos^4(t)+12\sin 2t\cos^3 t$$

$=6$ when $t=0.$

***

Example 10: Let $f(x,y)=x^{-1/3}y^{5/3}$. Show that $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=\frac{4}{3}z.$

Solution: Let $D=\{(x,y):x>0,y>0\}$

Then,

i) $(x,y)\in D\Rightarrow(tx,ty)\in D$ for all $t>0$.

ii) $(x,y)\in D$, then disc of radius $r=\frac{1}{2}\min\{x,y\}$ with centre (x,y) is contained in D.

Also by replacing x by tx and y by ty, we get

$$f(tx,ty)=(tx)^{-1/3}(ty)^{5/3}$$
$$=t^{4/3}x^{-1/3}y^{5/3}$$
$$=t^{4/3}f(x,y).$$

Therefore f is a homogenous function of degree $4/3$. Further, for any point $(x,y)\in D$ the partial derivatives f_x and f_y are given by

$$f_x(x,y)=\left(-\frac{1}{3}\right)x^{-4/3}y^{5/3}$$

$$f_y(x,y)=\frac{5}{3}x^{-1/3}y^{2/3}$$

Both f_x and f_y are continuous on D. Therefore f satisfies all the conditions of Euler's theorem and by Euler's relation we have

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=\frac{4}{3}z.$$

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Example 11: Find the slope of the tangent at $(3,1,1)$ to the curve of intersection of $x=3$ and $z=x^2y-3xy^2+1$.

Solution: The slope is given by $\frac{\partial z}{\partial y}$ at $(3,1,1)$.

$$\begin{aligned}\frac{\partial z}{\partial y}&=x^2-6xy\\&=9-18 \text{ at } (3,1,1)\\&=-9\end{aligned}$$

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Miscellaneous Exercises

E1) State whether the following statements are true or false. Justify your answer with the help of a short proof or a counter example.

i) The domain of ***f/g***, where $f(x,y)=2\sin x+\sin y$ and $g(x,y)=\frac{1}{x^2}\cos y$ is $\boldsymbol{R}^2-\left\{\left(0,\frac{\pi}{2}\right)\right\}$.

ii) The function $F(x,y)=\ln\left(\frac{x+y}{y}\right)$ is not a homogeneous function.

iii) The set $S=\{(x,y,z):|x|<1,|y|<1,|z|<1\}$ is an open cube with one corner having coordinates $(-1,-1,-1)$.

iv) A real valued function of three variables, which is continuous every where is differentiable.

v) $z=f(x^2y)$ where f is differentiable satisfies $x\left(\frac{\partial z}{\partial x}\right)=2y\left(\frac{\partial z}{\partial y}\right)$.

E2) Find the repeat limits of the following functions at $(0,0)$ and check whether they are equal or not in each case:

i) $f(x,y)=\frac{(1+y)(1+x^2)}{(2+x)(1+y^3)}$

ii) $f(x,y)=\frac{(y-3x)}{(2y+x)}\frac{(2+x^3)}{(1+y^3)}$

E3) Show that $f_{yz}(0,0,0)$ as well as $f_{zy}(0,0,0)$ do not exists for the function f defined by

$$f(x,y,z)=\begin{cases}\frac{2x}{y}+\frac{y}{3z}, & y\neq 0, z\neq 0\\ 0 & , \text{ otherwise}\end{cases}$$

E4) If $z = \tan^{-1}\left(\frac{x^2 + y^2}{x + y}\right)$, then show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{1}{2}\sin 2z$.

E5) Using Cauchy's inequality show that $|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$, for all x, y in $\mathbf{R^2}$.

E6) Find the level surface of the following for the value given against it.

i) $f(x, y, z) = x^2 + y^2 + z^2 - 24$ for $k = 1$

ii) $f(x, y, z) = x + y - z + 3$, for $k = 2$

E7) Determine $f_x(1,3)$ and $f_y(1,3)$ for the function given by

$$f(x, y) = 2x^2y^2 + 2y + 4x.$$

E8) Let $f(x, y) = x^2y + 5y^3$. Find the slopes of the tangents to the curves of intersection of the planes $x = 1$ and $y = 2$ and the surface $z = f(x, y)$ at the point $(1, 2, 42)$.

E9) If $f(x, y, z) = x^3y^2z^4 + 2xy + z$, then find all the 1st order partial derivatives of f .

E10) Find all the second-order partial derivatives of $f(x, y) = x^2y^3 + x^4y$.

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HINTS/SOLUTIONS TO MISCELLANEOUS EXERCISES

E1) Hints:

i) $$\frac{f(x, y)}{g(x, y)} = \frac{x^2(\sin x + \sin y)}{\cos y}$$

At point $\left(1, \frac{\pi}{2}\right)$, $g(x, y)$ is not defined (There are other points also).

Therefore the statement is false.

ii) Hint: False – Show that it is homogenous.

iii) Hint: False – The point $(-1, -1, -1)$ does not lie on the cube.

iv) Hint: False – $f(x, y, z) = |x| + |y| + |z|$ is continuous at $(0,0,0)$ but not differentiable.

v) Put $u = x^2y$. Then $z = f(u)$.

$$\therefore \frac{\partial z}{\partial x} = f'(u)2xy$$

$$\frac{\partial z}{\partial y} = f'(u)x^2$$

$$\therefore x\frac{\partial z}{\partial x} = f'(u)2x^2 y$$

$$=2y\, f'(u)2x^2$$

$$=2y\frac{\partial z}{\partial y}$$

Hence the statement is true.

E2) i) $$f(x, y) = \frac{(1+y)(1+x^3)}{(2+x)(1+y^3)}$$

$$\lim_{x\to 0}\lim_{y\to 0} f(x, y) = \lim_{x\to 0}\left\{\lim_{y\to 0}\frac{(1+y)(1+x^3)}{(2+x)(1+y^3)}\right\} = \lim_{x\to 0}\frac{(1+x^3)}{(2+x)} = \frac{1}{2}$$

$$\lim_{y\to 0}\lim_{x\to 0} f(x, y) = \lim_{y\to 0}\left\{\lim_{x\to 0}\frac{(1+y)(1+x^3)}{(1+y^3)(2+x)}\right\} = \lim_{y\to 0}\frac{1+y}{2(1+y^3)} = \frac{1}{2}.$$

These limits are equal.

ii) $$f(x, y) = \frac{(y-3x)(2+x^2)}{(2y+x)(1+y^2)}$$

$$\lim_{x\to 0}\lim_{y\to 0} f(x, y) = \lim_{x\to 0}\left\{\lim_{y\to 0}\frac{(y-3x)(2+x^2)}{(2y+x)(1+y^2)}\right\}$$

$$= \lim_{x\to 0}\left\{\left(\frac{-3x}{x}\right)(2+x^2)\right\} = \lim_{x\to 0} -3(2+x^2) = -6$$

$$\lim_{y\to 0}\lim_{x\to 0} f(x, y) = \lim_{y\to 0}\left(\lim_{x\to 0}\frac{(y-3x)(2+x^2)}{(2y+x)(1+y^2)}\right)$$

$$= \lim_{y\to 0}\left\{\frac{2y}{2y}\times\frac{1}{(1+y^2)}\right\} = \lim_{y\to 0}\frac{1}{1+y^2} = 1$$

These limits are not equal.

E3) $$f_y(0,0,0) = \lim_{k\to 0}\frac{f(0,k,0)-f(0,0,0)}{k} = 0$$

$$f_y(0,0,t) = \lim_{k\to 0}\frac{f(0,k,t)-f(0,0,t)}{k}$$

$$= \lim_{k\to 0}\frac{\frac{k}{3t}-0}{k} = \frac{1}{3t}$$

$$f_{zy}(0,0,t) = \lim_{t\to 0}\frac{f_y(0,0,t)-f_y(0,0,0)}{t}$$

$$= \lim_{t\to 0}\frac{1}{3t^2}$$

The limit on the R.H.S does not exists and therefore f_{zy} does not exist.

$$f_z(0,0,0)=0$$

$$f_z(0,k,0)=\lim_{t\to 0}\frac{f(0,k,t)-f(0,k,0)}{t}$$

$$=\lim_{t\to 0}\frac{\frac{k}{3t}}{t}$$

$$=\lim_{t\to 0}\frac{k}{3t^2}$$

Since the limit on the R.H.S does not exist, f_z does not exists and consequently f_{yz} do not exist.

E4) We check whether the given function satisfies the conditions of Euler's theorem. For this, let $D=\{x,y\}:x>0,y>0\}$ and let $f(x,y)=\dfrac{x^2+y^2}{x+y}$.

Then f is a function defined on $D\to \boldsymbol{R}$. We also have

i) $(x,y)\in D \Rightarrow (tx,ty)\in D$ for $t>0$.

ii) If $(x,y)\in D$, then the disc of radius $r=\dfrac{1}{2}\min\{a,b\}$ with centre (a,b) is contained in D.

Now we replace x by tx and y by ty in the expression for f. Then we have $f(tx,ty)=t\,f(x,y)$.

Therefore f is a homogenous function of degree 1. Also the partial derivatives of f exists on D. Therefore by Euler's theorem.

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=f(x,y) \qquad \text{... (2)}$$

Now we have $\tan z=f(x,y)$. Then

$$\frac{\partial f}{\partial x}=\sec^2 z\frac{\partial z}{\partial x},$$

$$\frac{\partial f}{\partial y}=\sec^2 z\frac{\partial z}{\partial y}$$

Substituing for $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ in 2, we get

$$x\sec^2 z\frac{\partial z}{\partial x}+y\sec^2 z\frac{\partial z}{\partial y}=\tan z$$

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=\frac{\tan z}{\sec^2 z}$$

$$=\frac{\sin z}{\cos z}\times\cos^2 z$$

$$=\sin 2z$$

This is the required equation.

E5) We have to prove that $|\boldsymbol{x}+\boldsymbol{y}| \le |\boldsymbol{x}|+|\boldsymbol{y}|$, for any two points $\boldsymbol{x}, \boldsymbol{y}$ in $\boldsymbol{R}^2$.

Let $\boldsymbol{x}=(x_1, x_2)$ and $\boldsymbol{y}=(y_1, y_2)$ in $\boldsymbol{R}^2$. Then

$$\begin{aligned}|\boldsymbol{x}+\boldsymbol{y}|^2 &= \sum_{i=1}^{2}(x_i+y_i)^2 \\ &= \sum_{i=1}^{2} x_i^2 + 2\sum_{i=1}^{2} x_i y_i + \sum_{i=1}^{2} y_i^2 \\ &\le \sum_{i=1}^{2} x_i^2 + 2\sqrt{\sum_{i=1}^{2} x_i^2}\sqrt{\sum_{i=1}^{2} y_i^2} + \sum_{i=1}^{2} y_i^2\end{aligned}$$

(in view of Cauchy inequality)

Consequently,

$$\|\boldsymbol{x}+\boldsymbol{y}\|^2 \le \|\boldsymbol{x}\|^2 + 2\|\boldsymbol{x}\|\,\|\boldsymbol{y}\| + \|\boldsymbol{y}\|^2$$

or $$\|(\boldsymbol{x}+\boldsymbol{y})\|^2 \le (\|\boldsymbol{x}\|+\|\boldsymbol{y}\|)^2$$

or $$\|(\boldsymbol{x}+\boldsymbol{y})\| \le \|\boldsymbol{x}\|+\|\boldsymbol{y}\|.$$

E6) i) The level surfaces have equations of the form $x^2+y^2+z^2-24=1$. This represents a sphere of radius 5.

ii) Hint: It is a plane.

E7) Since $f_x(x,3)=\dfrac{d}{dx}[f(x,3)]=\dfrac{d}{dx}[18x^3+4x+16]=54x^2+4$,

we get that $f_x(1,3)=54+4=58$

since $f_y(1,y)=\dfrac{d}{dx}[f(1,y)]=\dfrac{d}{dy}[2y^2+2y+4]=4y+2,$

we get that $f_y(1,3)=4\times 3+2=14$.

E8) The slope corresponding to $x=1$ is given by $f_y(x,y)=x^2+15y^2$ at $(1,2)$ i.e. $f_y(1,2)=61$.

The slope corresponding to $y=2$ is given by $f_x(x,y)=2xy$ at $(1,2)$. i.e. $f_x(1,2)=4$.

E9) $$\begin{aligned}f_x(x,y,z) &= 3x^2y^2z^4+2y \\ f_y(x,y,z) &= 2x^3yz^4+2x \\ f_z(x,y,z) &= 4x^3y^2z^3+1\end{aligned}$$

E10) We have $\dfrac{\partial f}{\partial x}=2xy^3+4x^3y$ and $\dfrac{\partial f}{\partial y}=3x^2y^2+x^4$

So that $\dfrac{\partial^2 f}{\partial x^2}=\dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right)=\dfrac{\partial}{\partial x}(2xy^3+4x^3y)=2y^3+12x^2y$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(3x^2y^2 + x^4) = 6x^2y$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(3x^2y^2 + x^4) = 6xy^2 + 4x^3$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(2xy^3 + 4x^3y) = 6xy^2 + 4x^3$$

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