ignou
THE PEOPLE'S
UNIVERSITY

Indira Gandhi National
Open University
School of Sciences

BMTC-132
DIFFERENTIAL EQUATIONS

Block

2

ORDINARY DIFFERENTIAL EQUATIONS OF FIRST ORDER

Course Design Committee*

Prof. Amber Habib
Shiv Nadar University
Gautam Buddha Nagar

Dr, Amit Kulshreshtha
IISER, Mohali

Dr. Aparna Mehra
IIT, Delhi

Prof. Meena Sahai
University of Lucknow

Prof. Rahul Roy
Indian Statistical Institute, Delhi

Prof. Rashmi Bhardwaj
G.G.S. Indraprastha University, Delhi

Prof. S. A. Katre
University of Pune

Dr. Sachi Srivastava
University of Delhi

Dr. Sunita Gupta
University of Delhi

Prof. V. Krishna Kumar
NISER, Bhubaneswar

Faculty Members
SOS, IGNOU

Prof. M. S. Nathawat (Director)
Dr. Deepika
Mr. Pawan Kumar
Prof. Poornima Mital
Prof. Parvin Sinclair
Prof. Sujatha Varma
Dr. S. Venkataraman

* The course design is based on the recommendations of the Programme Expert Committee and UGC-CBCS template

Block Preparation Team

Prof. O. P. Bhutani (*Editor*)
I.I.T. Delhi (Retd.)
IIT, Delhi

Prof. Poornima Mital
School of Sciences
IGNOU

Course Coordinator: Prof. Poornima Mital and Prof. Sujatha Varma

Acknowledgement:

- To Prof. Sujatha Varma and Dr. Deepika for their comments on the units.
- To Sh. S. S. Chauhan for the word processing, and the preparation of the CRC, of the block.

PRINT PRODUCTION TEAM

Sunil Kumar
A.R.(P) IGNOU,
Maidan Garhi Delhi-110068

December, 2018

ISBN : 978-93-88498-45-6

Further information on the Indira Gandhi National Open University courses, may be obtained from the University's office at Maidan Garhi, New Delhi-110 068 or ignou website www.ignou.ac.in

Printed and published on behalf of the Indira Gandhi National Open University, New Delhi by Prof. M. S. Nathawat, Director, School of Sciences.

Printed by : Akashdeep Printers, 20-Ansari Road, Daryaganj, New Delhi-110002

BLOCK 2 ORDINARY DIFFERENTIAL EQUATION OF FIRST ORDER

This is the second of the four blocks which you will be studying in the course on differential equations. It consists of four units in which we shall be dealing with ordinary differential equations of the first order and any degree.

Unit 6, which is the first unit of the block begins with the basic definitions related to the study of differential equations. Here, we classify the differential equations into ordinary, partial and total differential equations. We make a distinction between linear and non-linear equations. We have stated the conditions on the nature of solutions and conditions under which a unique solution of first order differential equation exists. We have also formulated in this unit some of the problems of physical and engineering interest in terms of first order linear differential equations.

In Unit 7, we give some methods of solving first order first degree differential equations of the following types: separable equations, homogeneous equations or equations reducible to homogeneous form, exact equations or equations which can be made exact with the help of an integrating factor.

Linear differential equations are discussed in Unit 8. We distinguish between homogeneous and non-homogeneous linear equations. We have also discussed some important properties involving the solutions of homogeneous linear differential equations and discussed the methods of undetermined coefficients and variation of parameters for solving such equations. In this unit, apart from solving the physical problems formulated in Unit 6, we have also solved the Bernoulli's equation.

Unit 9, which is the last unit of this block, deals with differential equations of first order, but of degree greater than one. In particular, we take up the cases when equations of the form $f(x, y, p) = 0$, where $p = \frac{dy}{dx}$, are solvable for x or y or p. In this unit we also discuss some special equations such as Clairaut's and Riccati's.

NOTATIONS AND SYMBOLS

$\mathbf{R}$	:	the set of real numbers
$\mathbf{C}$	:	the set of Complex numbers
$\Rightarrow$	:	implies
$\subseteq$	:	is contained in
$\in$	:	belongs to
$<(\leq)$	:	is less than (is less than or equal to)
$>(\geq)$	:	is greater than (is greater than or equal to)
$\therefore$	:	therefore
$\because$	:	because
i.e.	:	that is
$\forall$	:	for all
w.r.t.	:	with respect to
$\frac{d^n y}{dx^n}, y^{(n)}$	:	nth order derivative of y w.r.t. x
$\frac{\partial^n y}{\partial x^n}$	:	nth order partial derivative of y w.r.t. x
$\propto$	:	is proportional to
] , [	:	open interval
[,]	:	closed interval
[, [	:	semi open or semi closed interval
$\|x\|$	:	absolute value of a number x
IVP	:	initial value problem
I.F.	:	integrating factor

Greek Alphabets

α	:	Alpha
β	:	Beta
γ	:	Gamma
θ	:	Theta
ν	:	Nu
ψ	:	Psi
ϕ	:	Phi
μ	:	Mu
Σ	:	Capital sigma
ι	:	Iota

UNIT 6

INTRODUCING DIFFERENTIAL EQUATIONS

Structure **Page No.**

6.1 INTRODUCTION

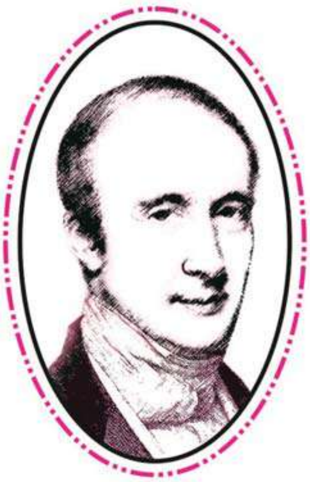
In your calculus course you have learnt that if $y = f(x)$ is a given function then its derivative $\frac{dy}{dx}$ can be interpreted as the rate of change of y with respect to x. But the problem we are going to tackle in this course is given an equation such as $\frac{dy}{dx} = xy$, to find a function $y = f(x)$ that satisfies the equation. Such equations are called **differential** equations. The primary purpose of differential equations is to serve as a tool for studying changes in the physical world.

Sir Isaac Newton (1642-1727), the English mathematician, observed that certain important laws of natural sciences can be formulated in terms of equations involving rates of change. The most famous example of such a natural law is Newton's second law

$$m\frac{d^2u(t)}{dt^2} = F\left[t, u(t), \frac{du(t)}{dt}\right] \qquad (1)$$

for the position $u(t)$ of a particle acted on by a force F, which may be a function of time t, the position $u(t)$, and the velocity $\frac{du(t)}{dt}$. To determine the motion of the particle acted on by a given force F, we need to find a function u satisfying Eqn. (1). Eqn. (1) is an example of differential equation.

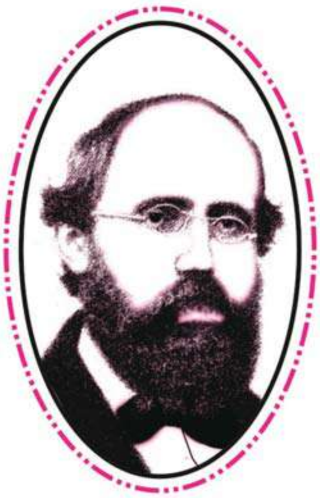
In 1676, Gottfried Leibniz, the German mathematician, introduced the term differential equation to denote a relationship between the differential dx and dy of two variables x and y. This came in connection with the study of inverse tangent problem in geometry, i.e., finding a curve whose tangent satisfies certain conditions. In 1690, the brothers James and John Bernoulli, who were the followers of Leibniz, played a significant part in the development of the theory of differential equations and the use of such equations in the solution of physical problems.



Cauchy (1789-1857)

In the eighteenth century the mathematicians Leonhard Euler, Daniel Bernoulli, Joseph Lagrange and others contributed generously to the development of the subject. The pioneering work that led to the development of ordinary differential equations as a branch of modern mathematics is due to Cauchy, Riemann, Picard, Poincare, Lyapunov, Birkhoff and others.

In many physical situations originating in science and technology and in such varied areas as economics, social sciences etc., differential equations have played an important role in building mathematical models to represent such situations. Besides its uses, the theory of differential equations involving the interplay of functions and their derivatives, is interesting in itself. In this unit, in Sec.6.2 we introduce the basic concepts and definitions related to differential equations. What we mean by the solution of an ordinary differential equation and under what conditions does the solution of a given differential equation exists is discussed in Sec.6.3. In Sec.6.4 we shall see how differential equations arise from family of curves. We shall give the methods of solving first order differential equations of various types in Units 7 and 8. In this unit we shall formulate in Sec.6.5 some of the problems of physical and engineering interest in terms of first order differential equations and solve these problems in Unit 8 after we have learnt the various methods of solving the first order equations.



Riemann (1826-1866)

Objectives

After having gone through this unit, you should be able to

- distinguish between ordinary and partial differential equations;
- define homogeneous, non-homogeneous linear and non-linear ordinary differential equations;
- distinguish between the order and degree of a differential equation;
- define the solution of an ordinary differential equation;
- identify an initial value problem;
- state, and use, the conditions for the existence and uniqueness of solutions of first order initial value problems; and
- derive differential equations for some physical situations.

A mathematical model is a formulation of a problem/system using mathematical concepts and language.

6.2 BASIC CONCEPTS

We shall start the discussion by defining and explaining the basic concepts in the theory of differential equations and illustrate them through examples.

You know from your knowledge of calculus that if a relation $y = f(x)$ involving two variables x and y exist in which the value of y is dependent upon the value of x we call x the **independent variable** and y the **dependent variable**.

Similarly, if you look at Eqn. (1), you may notice that it gives a relation between the independent variable t and dependent variable u and its derivatives $\frac{du}{dt}$ and $\frac{d^2u}{dt^2}$.

Any such equation which gives the relation between the independent and dependent variables and the derivatives of dependent variables is called a **differential equation**.

Some of these equations are as follows:

$$\frac{dy}{dx} = \cos x + \sin x \tag{2}$$

$$m\frac{d^2y}{dt^2} = -ky \tag{3}$$

$$\frac{dy}{dx} + 2xy = e^{-x^2} \tag{4}$$

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - p^2)\ y = 0 \tag{5}$$

$$\frac{d^2x}{dt^2} + \frac{dy}{dt} + x = y \tag{6}$$

The dependent variable in each of the Eqns. (2) to (5) is y, and the independent variable is either t or x. The letters k, m and p represent constants. In Eqn. (6) there are two dependent variables x and y and independent variable is t. There can also be equations involving partial derivatives, which you have studied in Unit 3 of Block-1, e.g.,

$$y^2\frac{\partial z}{\partial x} + xy\frac{\partial z}{\partial y} = nzx \tag{7}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{8}$$

Here $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial x}$ are partial derivatives with z, u and v being dependent variables and x and y as independent variables. Eqns. (2) to (8) are all examples of differential equations.

Formally, we can now have the following definition.

Definition: An equation involving one (or more) dependent variable and its derivatives with respect to one or more independent variables is called a differential equation.

Note that equations of the type

$$\frac{d}{dx}(xy) = y + x\frac{dy}{dx}$$

are not differential equations. In this equation if you expand the left hand side then you will find that left hand side is the same as the right hand side. Such equations are called **identities**.

Identity is an equality between functions that are differently defined.

As you have seen above a differential equation may involve more than one dependent variable as is the case with Eqns. (6) and (8). Based on the nature of the dependent variable and its derivative (or derivatives) in the equation differential equations are classified into various types. Accordingly, we give the following definitions.

Definition: A differential equation involving only ordinary derivatives (that is, derivatives with respect to a single independent variable) is called an **ordinary differential equation** (abbreviated as ODE).

The equations,

$$\frac{d^2y}{dx^2} + y = x^2, \tag{9}$$

$$\left(\frac{dy}{dx}\right)^2 = [\sin(xy) + 2]^2, \text{ and} \tag{10}$$

$$y = x\frac{dy}{dx} + r\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \tag{11}$$

$$\frac{dx}{dt} - \frac{dy}{dt} = \sin t \tag{12}$$

are all ordinary differential equations. Eqns. (9) to (11) involve ordinary derivatives $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ of first and second order, respectively, of the dependent variable y w.r.t. single independent variable x. Eqn. (12) involves ordinary derivatives of two dependent variables x and y, w.r.t. independent variable t.

A typical form of such equations is

$$g\left[x,\ y(x), \frac{dy(x)}{dx}, \frac{d^2y(x)}{dx^2}, \dots, \frac{d^ny(x)}{dx^n}\right] = 0 \tag{13}$$

Note that in Eqn. (13) we assume that g is a known real-valued function and the unknown to be determined is y. Secondly, in an ordinary differential equation, y and its derivatives are evaluated at x.

It may be noted that the equation

$$\left(\frac{dy}{dx}\right)_x = (y)_{x+1}$$

is not a differential equation. This is because y is evaluated at $(x+1)$ whereas $\frac{dy}{dx}$ is evaluated at x.

Further, the equation

$$\frac{dy(x)}{dx} = \int_0^x e^{xs} y(s)\ ds$$

is not a differential equation since the unknown y is appearing inside an integral. Also, in this case the values of y on the right hand side of the equation depend on the interval 0 to x, whereas, in a differential equation, the **unknown y has to be evaluated only at x**.

Let us now define a partial differential equation.

Definition: A differential equation containing partial derivatives, that is, the derivatives of one (or more) dependent variable with respect to two or more independent variables is called a **partial differential equation** (abbreviated as PDE).

The equations

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}=0 \tag{14}$$

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}-z=0, \tag{15}$$

$$\frac{\partial^3 u}{\partial x^3}+\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial t^2}+xtu=0. \tag{16}$$

are all examples of partial differential equations.

You may also **note** that Eqns. (1) to (6) are ordinary differential equations, whereas, Eqns. (7) and (8) are partial differential equations.

You may now check your understanding of ODEs and PDEs while doing the following exercise.

E1) Which of the following are differential equations? Which of the differential equations are ordinary and which are partial? Give reasons for your answer.

i) $\left(\frac{d^2 y}{dx^2}\right)^3+x\frac{dy}{dx}+y^3=5x+2$

ii) $\frac{d^2 x}{dt^2}+\frac{d^2 y}{dt^2}=x+yt$

iii) $\frac{dy}{dx}=\int_0^x \sin[xy(s)]\,ds$

iv) $\frac{d^2 y}{dx^2}+\frac{dy}{dx}+y=\int_0^1 \cos y(s)\,ds$

v) $\frac{\partial^2 w}{\partial t^2}=a^2\frac{\partial^2 w}{\partial x^2}$, a being a constant.

Besides ordinary and partial differential equations, there are a third type of equations, namely, total differential equations. Before giving you the definition of total differential equations, we assign a meaning to the symbols dx and dy which allows us to obtain the derivative $\frac{dy}{dx}$. Symbols dx and dy are variables which take finite real values and are called the **differentials.** For a given function $y=f(x)$, the differential dy is defined by $dy=f'(x)\,dx$, where $f'(x)$ is the derivative of f w.r.t. x. Here dy is a function of x and dx.

In Unit 5 of Block-1 you have learnt that if $u = f(x, y)$ be a function of two independent variables x and y having continuous first order partial derivatives in a region R of the xy-plane, then its total differential du is defined as

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

or, $du = u_x\ dx + u_y\ dy$.

For instance, if $u = x^2 y - 3y$ then

$$du = 2xy\ dx + (x^2 - 3)\ dy.$$

Similarly, for a function $u(x, y, z)$ of three variables x, y, z the total differential du is given by

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz.$$

Now if $u(x, y, z) = c$ where c is a constant then $du = 0$.

$$\Rightarrow\ \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = 0$$

Here, $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$ are known functions of x, y and z, and therefore the above equation can be put in the form

$$P\ dx + Q\ dy + R\ dz = 0$$

which is called the **total differential equation** in three variables. Here P, Q and R are functions of x, y and z. In this equation any one of the variables x, y, z can be treated as an independent variable and the remaining two are then the dependent variables.

Similarly, if $u = u(x, y, z, t)$ then the corresponding total differential equation will be of the form

$$P\ dx + Q\ dy + R\ dz + T\ dt = 0.$$

Remember that a total differential equation always involves **three or more variables**.

We now give the following definition.

Definition: A total differential equation is an equation which contains two or more dependent variables together with their differentials with respect to a single independent variable which may, or may not, appear explicitly in the equation.

For example, equations

$$yz(1 + 4xz)\ dx - xz(1 + 2xz)\ dy - xydz = 0, \text{ and}$$

$$\frac{xdx + ydy + zdz}{\sqrt{x^2 + y^2 + z^2}} + \frac{zdx - xdz}{x^2 + z^2} + 3ax^2dx + 3by^2dy + 3cz^2dz = 0,$$

are total differential equations.

In this course we shall be dealing with ordinary differential equations in Blocks 2 and 3 and Block 4 is devoted to the study of total and partial differential equations.

We next consider the concepts of order and degree of a differential equation on the basis of which differential equations can be further classified.

You may recall from your knowledge of Calculus course that the **nth derivative** of a dependent variable with respect to one or more independent variables is called a derivative of order n, or simply an nth order derivative .

For example, $\frac{d^2y}{dx^2}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x\,\partial y}$ are second order derivatives and $\frac{d^3z}{dx^3}, \frac{\partial^3 z}{\partial x^2 \partial y}$ are third order derivatives. Based on the order of the derivatives of the dependent variable(s) present in the equation we have the following definition.

Definition: The **order** of a differential equation is the order of the highest order derivative appearing in the equation.

For instance, $$\frac{d^2y}{dx^2} + y = x^3, \tag{17}$$

is a **second** order ordinary differential equation (because the highest order derivative is $\frac{d^2y}{dx^2}$, which is of second order), whereas,

$$(x+y)\,(y')^2 = 1, \tag{18}$$

is a **first** order ordinary differential equation (highest order derivative being $y' = \frac{dy}{dx}$).

Similarly,

$$\left[\frac{d^3y}{dx^3}\right]^2 + 2\frac{d^2y}{dx^2} - \frac{dy}{dx} + x^2\left[\frac{dy}{dx}\right]^4 = 0 \tag{19}$$

is a **third** order ordinary differential equation

whereas, $$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial x} = 0 \tag{20}$$

is a **second** order partial differential equation.

Note that the order of a differential equation is a positive integer.

Also, if the order of a differential equation is n then it is not necessary that the equation contains some or all lower order derivatives or independent variables explicitly. For instance, equation $y^{iv} + 2y = 0$, is a fourth order ordinary differential equation.

Now once again look at Eqn. (19) which is of third order as the highest order derivative occurring in the equation is of order 3. Further, you may note that this highest order derivative has an exponent 2. This exponent defines the degree of the differential equation. We say that Eqn. (19) is a third order ODE of degree 2. Accordingly, we have the following definition.

Definition: The **degree** of a differential equation is the highest exponent of the highest order derivative appearing in the equation.

Eqns. (17) and (20) are of first degree, whereas, Eqns. (18) and (19) are of degree two.

However, not every differential equation has a degree. If the derivatives occur in fractions or within radicals the equation have a degree only if it can be rationalized and expressed in the form free from radicals and fractional/negative powers with regard to all the derivatives present in the equation.
Consider for instance the equation

$$y - x\frac{dy}{dx} = r\sqrt{1+\left(\frac{dy}{dx}\right)^3}\,. \tag{21}$$

What is the degree of this equation?
In order to find the degree of this equation we have to make it free from radicals. We thus need to square both the sides of the equation. Squaring Eqn. (21) we obtain

$$\left[y - x\frac{dy}{dx}\right]^2 = r^2\left[1+\left(\frac{dy}{dx}\right)^3\right].$$

You may now **notice** that the highest exponent of the highest derivative, that is, $\frac{dy}{dx}$, in this equation is three. Therefore, the degree of the equation is **three**.
Similarly, the equation

$$y = x\frac{dy}{dx} + \frac{b}{dy/dx} \text{ is of degree two.}$$

This is because we multiply throughout by $\frac{dy}{dx}$ to remove the negative power of $\frac{dy}{dx}$ and get $y\frac{dy}{dx} = x\left[\frac{dy}{dx}\right]^2 + b$.

Can you find the degree of the equation $\cos\left(\frac{dy}{dx}\right) + xy = 3$? No, because cosine function when expanded gives an infinite series. The degree of this equation is not defined.

You may now check your understanding of the order and the degree of a differential equation through the following exercise.

E2) Find the order and degree of the following differential equations.

i) $\left(\frac{d^2y}{dx^2}\right)^2 = \left[1+2\left(\frac{dy}{dx}\right)^2\right]^3$.

ii) $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 + y^2 = x$.

iii) $\sin\left(\frac{d^2y}{dx^2}\right) + x^2y^2 = 0$.

iv) $dy + y^3dx = 0$.

v) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2} = r\frac{d^2y}{dx^2}$.

vi) $\frac{\partial^4 z}{\partial x^4}+\left(\frac{\partial^2 z}{\partial x\,\partial y}\right)^2 = x$.

vii) $x^2(dx)^2+2xy\,dx\,dy+y^2(dy)^2-z^2(dz)^2=0$.

Depending upon the degree of dependent variables and their derivatives present in the equation we further classify the differential equations into two classes, namely, linear and non-linear.

Definition: When, in an ordinary or partial differential equation, the dependent variable and all its derivatives are of the first degree only, and not as higher powers or products, the equation is called **linear**.

The coefficients of a linear equation are therefore either constants or functions of the independent variable or variables. If a differential equation is not linear, we call it **non-linear**.

For example, the ordinary differential equation

$$\frac{d^2y}{dx^2}+y=x^2$$

is linear, whereas equations $(x+y)^2\frac{dy}{dx}=1$ and $\frac{d^2y}{dx^2}+x=y^2$ are non-linear. Non-linearity in the equations is because of the presence of the terms like $y^2\frac{dy}{dx}$, $2xy\frac{dy}{dx}$ and y^2.

Similarly, the equation $\frac{\partial^2 z}{\partial x^2}+(x+y)\frac{\partial^2 z}{\partial y^2}=0$ is a linear partial differential equation whereas,

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}-\left(\frac{\partial^2 z}{\partial x\,\partial y}\right)^2=0$$

is a non-linear partial differential equation, because of the presence of the third term on the left hand side of the equation.

Further, if a partial differential equation is not linear, it can be **quasi-linear, semi-linear** or **non-linear**. We will discuss conditions for these classifications later in Unit 16 of Block 4.

You may now try the following exercise.

E3) Classify the following differential equations into linear and non-linear giving reasons for the same.

i) $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}-z=0$.

ii) $\frac{\partial^2 z}{\partial x^2}\cdot\frac{\partial^2 z}{\partial y^2}-\left(\frac{\partial^2 z}{\partial x\,\partial y}\right)^2=0$.

iii) $\frac{dy}{dx} = (x+y)^2$.

iv) $(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n\,(n+1)\; y = 0,\ n$ a constant.

v) $(x^2+y^2)^{3/2}\frac{d^2y}{dx^2} + \mu x = 0,\ \mu$ a constant.

Normally when we encounter a differential equation, our natural curiosity is to enquire about its solution. It is natural for you to ask as to what exactly is the meaning of its solution. In the following section besides finding an answer to this question we shall also seek answers to questions such as:

i) Does every ordinary differential equation have a solution?

ii) Under what conditions does the solution of a given ordinary differential equation exist?

iii) If the solution exists, then is it a unique solution?

Let us try to find answers to these questions.

6.3 SOLUTION OF A DIFFERENTIAL EQUATION

You have seen that the general ordinary differential equation of the nth order as given by Eqn. (13) is

$$g\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \cdots, \frac{d^ny}{dx^n}\right) = 0.$$

Using the prime notation for derivatives $\left(y' = \frac{dy}{dx},\ y'' = \frac{d^2y}{dx^2},\ \cdots,\ y^{(n)} = \frac{d^ny}{dx^n}\right)$,

we can rewrite Eqn. (13) in the form

$$g(x,\ y,\ y',\ y'',\ldots,\ y^{(n)}) = 0 \tag{22}$$

Let us assume that Eqn. (22) is solvable for $y^{(n)}$, that is, Eqn. (22) can be written in the form

$$y^{(n)} = f(x,\ y,\ y',\ y'',\ldots,\ y^{(n-1)}). \tag{23}$$

It is normally a simple task to verify that a given function ϕ satisfies an equation like (22) or (23). All that is necessary is to compute the derivatives of ϕ and show that ϕ and its derivatives, when substituted in the equation, reduce it to an identity in x. If such a function ϕ exists, we call it a solution of the Eqn. (22) or (23).

However, we assume that

i) ϕ is defined on some interval I;

ii) ϕ is n times differentiable on I;

iii) ϕ can be a real valued function or a complex valued (range is a subset of $\mathbf{C}$) function of x.

Depending on the context, I could represent any interval $[a,\ b],\ ]a,\ b[,\ ]0,\ \infty[$ or $]-\infty,\ \infty[$ and so on.

Formally we have the following definition.

Definition: A real or complex valued function ϕ defined on an interval I is called a **solution** (or an **integral)** of the differential equation $g(x, y, y', \ldots, y^{(n)}) = 0$ if ϕ is n times differentiable and if $x, \phi(x). \phi'(x), \ldots, \phi^{(n)}(x)$ satisfy this equation for all x in I i.e., $g(x, \phi(x), \phi'(x), \ldots, \phi^{(n)}(x)) = 0$ for all x in I.

For example, the first order differential equation

$$\frac{dy}{dx} = 2y - 4x \tag{24}$$

has a solution $y = 2x + 1$ in the interval $\mathrm{I} = \{x : -\infty < x < \infty\}$ i.e., on whole of $\boldsymbol{R}$. This can be checked by substituting the value of y and its derivative $(y' = 2)$ in Eqn. (24). We have

r.h.s of Eqn. (24) $= 2y - 4x = 2(1 + 2x) - 4x = 2 = y' =$ l.h.s. of Eqn. (24).

The function $y = x^4/16$ is a solution of the non-linear equation $\dfrac{dy}{dx} - x\,y^{1/2} = 0$ on $]-\infty, \infty[$. We have $\dfrac{dy}{dx} = \dfrac{4x^3}{16} = \dfrac{x^3}{4}$ and

l.h.s. $= \dfrac{dy}{dx} - x\,y^{1/2} = \dfrac{x^3}{4} - x\left(\dfrac{x^4}{16}\right)^{1/2} = \dfrac{x^3}{4} - \dfrac{x^3}{4} = 0 =$ r.h.s. for every real number x.

In the same way you can check that $y = 1 + 2x + ce^{2x}, -\infty < x < \infty$, is also a solution of Eqn. (24) for any constant c.

In the definition above you might have noticed that a solution y of Eqn. (22) can be real valued or complex valued. In case y is real valued it is called a **real solution**. If y is complex valued, it is called a **complex solution**. We are usually interested in real solutions of Eqn. (22). To help you clarify what we have just said let us consider some more examples.

Example 1: Show that for any constant c, the function $y(x) = ce^x, x \in \boldsymbol{R}$ is a solution of

$$\frac{dy}{dx} = y. \tag{25}$$

Solution: Here I is $\boldsymbol{R}$ itself. For any $x \in \boldsymbol{R}$, we know that

$$\frac{dy}{dx} = \frac{d}{dx}(ce^x) = c\frac{d}{dx}(e^x) = ce^x = y$$

which shows that y satisfies Eqn. (25).

***

Example 2: Show that for real constants a and b, the functions $y(x) = a\cos 2x$ and $z(x) = b\sin 2x$ are solutions of the equation

$$\frac{d^2y}{dx^2} + 4y = 0, \; x \in \boldsymbol{R} \tag{26}$$

Solution: We will first show that $z(x), x \in \boldsymbol{R}$ is a solution of Eqn. (26).

$$\frac{d}{dx}[z(x)] = \frac{d}{dx}(b\sin 2x) = 2b\cos 2x.$$

$$\therefore \frac{d^2}{dx^2}[z(x)] = \frac{d}{dx}(2b\cos 2x) = -4b\sin 2x = -4z(x),$$

so that

$$\frac{d^2 z}{dx^2} + 4z(x) = -4z(x) + 4z(x) = 0,\ x \in \boldsymbol{R}.$$

That is, $z(x)$ satisfies Eqn. (26) for every x in $\boldsymbol{R}$.

Similarly, we have $\frac{d^2 y}{dx^2} = \frac{d}{dx}(-2a\sin 2x) = -4a\cos 2x = -4y(x)$ that is, $y(x) = a\cos 2x$ is also a solution of Eqn. (26) for all x in $\boldsymbol{R}$. You may check here that the sum $y(x) + z(x)$, that is $a\cos 2x + b\sin 2x$ is again a solution of Eqn. (26).

***

Example 3: Show that $y(x) = e^{ix},\ x \in \boldsymbol{R}$ is a solution of $\frac{d^2 y}{dx^2} + y = 0$.

Solution: We have,

$$\frac{dy}{dx} = \frac{d}{dx}(e^{ix}) = ie^{ix}$$

and

$i = \sqrt{-1}$

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}(ie^{ix}) = i^2 e^{ix} = -e^{ix} = -y(x).$$

Thus for $y(x) = e^{ix}$, $\frac{d^2 y}{dx^2} + y = 0$ for every x in $\boldsymbol{R}$.

***

In the examples taken so far you have seen that the solution/solutions of given differential equations exist. In Examples 1 and 2 the solutions were real valued whereas, the solution in Example 3 was a complex valued function. You may also notice that in Examples 1, 2 and 3 the constant function $y = 0$, for all x in $\boldsymbol{R}$, also satisfies the given differential equation. Such a solution is referred to as **trivial solution**. It is not that every differential equation has a trivial solution or necessarily has a solution. Suppose that we are looking for the real roots of the equation $x^2 + 1 = 0$. We know that it does not exist. Likewise, the differential equations $\left|\frac{dy}{dx}\right| + y^2 + 1 = 0$ and $\left(\frac{dy}{dx}\right)^2 + 1 = 0$ possess no real solutions.

Similarly, the differential equation $\sin\left(\frac{dy}{dx}\right) = 2$ does not admit a real solution, because the real values of the sine of a real function lie between -1 and $+1$. Further, the solutions of differential equations can be distinguished as **explicit** or **implicit** solutions. In Example 1, $y(x) = ce^x$ is an explicit solution of the equation $\frac{dy}{dx} = y$ for every x in $\boldsymbol{R}$. Similarly, the solutions in Examples 2 and 3 are the explicit solutions of the given problems. In each of these examples y is given explicitly as a function of x.

Whereas, if you consider the differential equation

$$(1-y^2)\,\frac{dy}{dx}-x^3=0 \qquad (27)$$

then you can check that function

$$3y-x^3-y^3=c \qquad (28)$$

where c is an arbitrary constant satisfies Eqn. (27) and hence is a solution of Eqn. (27). Eqn. (28) gives the solution of Eqn. (27) in an implicit form. We say that an equation

$$F(x,\ y)=0 \qquad (29)$$

involving x and y, defines a solution of a differential equation implicitly on an interval I, if for each x we can solve Eqn. (29) to find the corresponding value of y. If Eqn. (29) is simple, it may be possible to solve it for y and obtain an explicit formula for the solution. However, often it may not be easy to solve Eqn. (29) explicitly for y in terms of x as is the case with Eqn. (28). In such cases, it is better to leave the solution in implicit form.

You may now try the following exercises.

E4) Verify that $y=\cos^{-1}\left(-\frac{x^2}{2}\right)$ is a solution of the equation $\sin y\frac{dy}{dx}=x$. Also state the interval on which y is defined.

E5) Verify that $y=\frac{1}{x}(\ln y+c)$ is a solution of the equation $\frac{dy}{dx}=\frac{y^2}{1-xy}$ for every value of the constant c.

> $\ln y$ is the natural logarithm of y, i.e., $\log_e y$.

E6) Verify that $y=e^{2x}$ and $y=e^{3x}$ are both solutions of the second order equation $y''-5y'+6y=0$. Can you find any other solution of this equation?

E7) Verify that the equation $y^2-2y=x^3+2x^2+2x+c$, where c is an arbitrary constant, gives an implicit solution of the differential equation $\frac{dy}{dx}=\frac{3x^2+4x+2}{2(y-1)}$, $y\neq 1$. Obtain the explicit solution/(s) of the equation, if any.

In the discussion above, you must have **noticed** that a differential equation may have no solution, may have more than one solution. It may even have infinitely many solutions. For instance, each of the functions

$y=\sin x,\ y=\sin x+3,\ y=\sin x-\frac{4}{5}$ is a solution of the differential equation $y'=\cos x$. From your knowledge of calculus you also know that any solution of the equation is of the form

$$y=\sin x+c \qquad (30)$$

where c is a constant. As c can take infinitely many values in $\boldsymbol{R}$, we regard c as arbitrary. The relation (30) then represents the totality of all solutions of the equation. Thus, we can represent even the infinitely many solutions of a differential equation by a simple formula involving arbitrary constants also called parameters. Depending on the number of arbitrary constants involved, we classify the solutions of an ordinary differential equation into various types.

In Example 2, the solution $y = a\cos 2x + b\sin 2x$ of the second order equation $\frac{d^2y}{dx^2} + 4y = 0$, contains two arbitrary constants which is equal to the order of the differential equation. Such a solution is called the general solution or complete solution of the differential equation. Further, if these arbitrary constants are assigned particular values say, $a = 1$ and $b = 3$, then $y = 3\sin 2x + \cos 2x$ will be a particular solution of the equation.

Accordingly, we give the following definitions:

Definition: The solution of an nth order differential equation which contains n arbitrary constants is called its **general** or **complete solution**.

Definition: Any solution which is obtained from the general solution by giving particular values to the arbitrary constants is called a **particular solution**. It is free of arbitrary constants or parameters.

In some cases there may be a further solution of a given equation which cannot be obtained by assigning definite values to the arbitrary constants in its general solution. Such a solution is called a **singular solution** of the differential equation. For example, the equation

$$y'^2 - xy' + y = 0 \tag{31}$$

has the general solution $y = cx - c^2$, c a constant. A further solution of Eqn. (31) is $y = \frac{x^2}{4}$. Since this solution cannot be obtained by assigning a definite value to c in the general solution, it is a singular solution of Eqn. (31).

Thus, we have seen the various types of solutions of an ordinary differential equation. We have also seen that a solution of a differential equation may or may not exist. Even if a solution exists, it may or may not be unique.

We now try to find the conditions under which the solution of a given ordinary differential equation exists and is unique. Here, we shall confine our attention to the first order ordinary differential equations only.

6.3.1 Conditions for the Existence and Uniqueness of Solution

Let us consider the general first order equation.

$$\frac{dy}{dx} = f(x, y) \tag{32}$$

In Eqn. (32) we assume that the function f of x and y is known to us. You may be surprised to know that, though this equation looks simple, it is very difficult to get its explicit solution. To be more specific, we consider the following examples.

Example 4: Does the solution $y(x)$ of an ordinary differential equation

$$\frac{dy}{dx} = f(x), \text{ where } f(x) = \begin{cases} 0 \text{ for } x < 0 \\ 1 \text{ for } x \geq 0 \end{cases}$$

exists $\forall x \in \boldsymbol{R}$?

Solution: Consider the function defined by

$$y(x) = \begin{cases} c_1 & \text{for } x < 0 \\ x + c_2 & \text{for } x \geq 0 \end{cases}$$

where c_1 and c_2 are constants. You can check that this function satisfies the given equation if $x \neq 0$. At the same time this function has no derivative at $x = 0$, because of the discontinuity of $y(x)$ at $x = 0$.
Thus, this differential equation has no valid solution for $x = 0$.
However, $y(x)$ defined above is the solution of the given differential equation at all points other than $x = 0$.

***

Let us look at another example.

Example 5: Does the equation $\dfrac{dy}{dx} = e^{-y}x, \ \forall\ x \in \boldsymbol{R}$ have a unique solution?

Solution: It can be easily verified that $y = \ln\left(\dfrac{x^2}{2} + A\right)$, where A is an arbitrary constant, satisfies the given differential equation.

You know that $\ln x$ is defined for positive values of x only. So, the solution of the given differential equation will exist as long as $\left(\dfrac{x^2}{2} + A\right) > 0$. Clearly, this would imply that for $\forall x \in \boldsymbol{R},\ A > 0$. Also, for different values of A we get different solutions. Thus, the solution of a given differential equation is not unique.

***

It is obvious from the discussion above that the cause for the non-uniqueness of the solution is the arbitrariness of A (but for $A > 0$). Thus, we would like to impose some condition on the solution which might determine A uniquely. One such condition is to specify the value of y at some point x_0 where x_0 is in the interval of existence of y. Such a condition is called an **initial condition** and the problem of solving a differential equation together with an initial condition is called an **initial value problem (IVP)**. In other words, an initial value problem is the problem in which we look for the solution of a given differential equation which satisfies certain conditions at a single value of the independent variable. Thus, the first order initial value problem is

$$\left.\frac{dy}{dx} = f(x,\ y),\ y(x_0) = y_0\right\} \tag{33}$$

where $x_0,\ y_0$ are some fixed values.

Two fundamental questions arise in considering an initial value problem as given by Eqn. (33).

i) Does a solution of the problem exist?

ii) If a solution exists, is it unique, or the only solution of the problem?

Geometrically we are asking in the second question: Of all the solutions of Eqn. (33) that exist on an interval I, is there only one whose graph passes through $(x_0,\ y_0)$?

The above questions are answered by a theorem known as **Existence Uniqueness Theorem**, due to Picard (1856-1941), a French mathematician. We shall now state this theorem for the first order initial value problem (33).

Theorem 1: (Existence-Uniqueness):

Let D be the rectangular domain in the xy-plane defined by

$$D: |x - x_0| < a, |y - y_0| < b$$

that contains the point (x_0, y_0) in its interior (see Fig. 1). If function f is continuous and bounded in D, say

$|f(x, y)| \le k \ \forall \ (x, y)$ in D, where a, b and k are constants. (34)

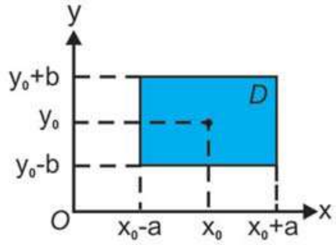


Fig. 1

then the IVP (33) has **at least one solution** $y(x)$ defined for all x in the interval $|x - x_0| < h$, where $h = \min\left(a, \dfrac{b}{k}\right)$.

Further, if $\dfrac{\partial f}{\partial y}$ is continuous and bounded inside D, i.e.,

$$\left|\frac{\partial f}{\partial y}\right| \le L, \forall (x, y) \text{ in } D, L \text{ being a positive constant,} \quad (35)$$

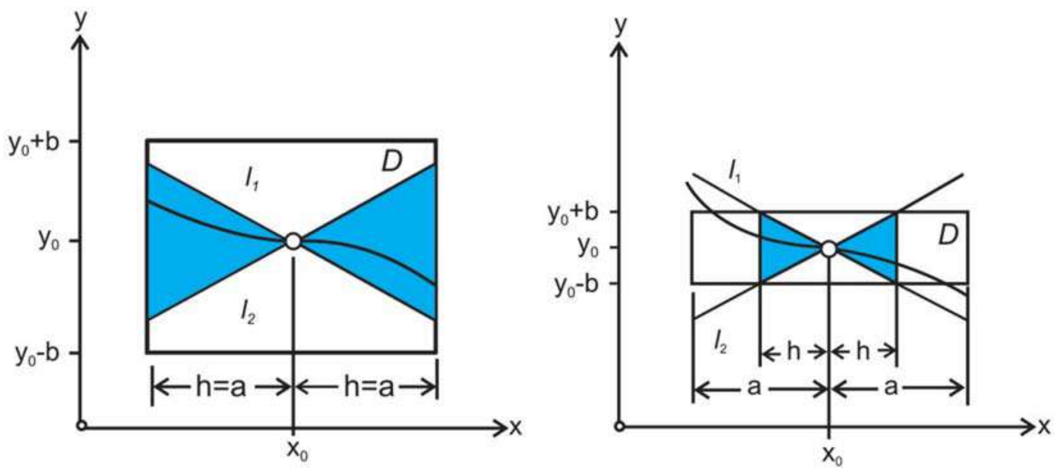
then the solution $y(x)$ is the **unique solution** for all x in the interval $|x - x_0| < h$.

– ■ –

A function $f(x, y)$ is said to be **bounded** when (x, y) varies in a region in the xy-plane and if there is a number k such that $|f| \le k$ when (x, y) is in that region. For example, $f = x^2 + y^2$ is bounded, with $k = 2$ if $|x| < 1$ and $|y| < 1$.

Since the proof of this theorem requires familiarity with many other concepts which are beyond the scope of this course we shall therefore, not be proving this theorem. However, we shall illustrate the theorem with the help of a few examples. The following remark will help you understand what the theorem states.

Remark: Since $y' = f(x, y)$, the condition (34) implies that $|y'| \le k$, that is, the slope of any solution curve $y(x)$ in D is at least $-k$ and at most k. Hence a solution curve which passes through the point (x_0, y_0) must lie in the shaded region shown in Fig. 2 bounded by the lines ℓ_1 and ℓ_2 whose slopes are $-k$ and k, respectively.



(a) (b)

Fig. 2

Determination of a value of h may be difficult. Depending on the form of D, two different cases may arise.

i) We may have $\frac{b}{k} \geq a$. In that case $h = a$, and the solution will exist for all x between $x_0 - a$ and $x_0 + a$ (see Fig. 2(a)).

ii) We may have $\frac{b}{k} < a$. Therefore, $h = \frac{b}{k}$, and the solution will exist for all x between $x_0 - \frac{b}{k}$ and $x_0 + \frac{b}{k}$. In this case the solution curve may leave the rectangle D (see Fig. 2(b)) for larger or smaller values of x and nothing can be concluded about the solution corresponding to those values of x.

The conditions stated in Theorem 1 are **sufficient** but not necessary and can be relaxed. For example, by the mean value theorem of differential calculus, we have

$$f(x, y_2) - f(x, y_1) = (y_2 - y_1)\frac{\partial f}{\partial y}(x, y^*)$$

where (x, y_1) and (x, y_2) are assumed to be distinct points in D with the same *x*-coordinate and y^* being a suitable value between y_1 and y_2. From condition (35) then it follows that

$$|f(x, y_2) - f(x, y_1)| \leq L|(y_2 - y_1)| \tag{36}$$

Condition (35) may be replaced by the weaker condition (36) which is known as a **Lipschitz condition**, named after the German mathematician, Rodolf Lipschitz (1831-1903). The constant L is called the **Lipschitz constant**.

Thus, we can say that for the existence of the solution of the IVP (33), we must have

i) f continuous in D.

ii) f bounded in D.

Further, the solution is unique if in addition to i) and ii) above, we have

iii) $\frac{\partial f}{\partial y}$ continuous in D.

iv) $\frac{\partial f}{\partial y}$ bounded in D (or, Lipschitz condition is satisfied).

However, if the conditions above are not satisfied, then the IVP (33) may still have either (a) no solution, (b) more than one solution, (c) a unique solution. This is because **Theorem 1 provides only sufficient conditions and not necessary conditions**.

For instance, consider $\frac{dy}{dx} = 3y^{2/3}$, $y(0) = 0$, in a domain $D: |x| < a, |y| < b$ where a, b are constants.

Here, $f(x, y) = 3y^{2/3}$. Being a polynomial function f is continuous and it is also bounded in D. Thus, the solution of the IVP exists.

Further, $\frac{\partial f}{\partial y} = \frac{3 \times 2}{3} y^{-1/3} = \frac{2}{y^{1/3}}$ shows that $\frac{\partial f}{\partial y}$ does not exist at $y = 0$.

So $\frac{\partial f}{\partial y}$ is not bounded in any region containing part of the x-axis. Therefore, in view of condition iv) above, we cannot conclude the uniqueness of the solution in any domain containing part of the x-axis. (You can check here that the solutions $y = x^3$ and $y = 0$ of the IVP exist.) But if (x_0, y_0) is any point not on the x-axis, then there is a unique solution of the differential equation $y' = 3y^{2/3}$ passing through (x_0, y_0). Further, for the IVP you may also check that the function $3y^{2/3}$ violates even the Lipschitz condition at $y = 0$. However, in practice there are functions, for instance, $f(x, y) = |y|$ which is not differentiable at $y = 0$ but satisfies the Lipschitz condition (36) with Lipschitz constant $L = 1$.

E8) Check that the function $f(x, y) = 3y^{2/3}$ does not satisfy the Lipschitz condition at $y = 0$.

We now illustrate the theorem through some more examples.

Example 6: Examine $\frac{dy}{dx} = y$ with $y(0) = 1$ for existence and uniqueness of the solution in $D: |x| < a, |y - 1| < b$ where a, b, are constants.

Solution: Here $f(x, y) = y, f_y(x, y) = 1$. Also $x_0 = 0$ and $y_0 = 1$.
In D the function f is continuous and bounded. Hence the solution exists. Further f_y is also continuous and bounded in D. Therefore, the solution is unique.

You may verify that $y = e^x$ is a solution of the given equation satisfying the initial condition $y(0) = 1$. Hence, it is the unique solution.
However, if the initial condition is changed to $y(0) = 0$ then domain D will be of the form

$$D: |x - 0| < a, |y - 0| < b$$

and in that case $y = 0$ will be the unique solution for all x and y in any domain D containing $(0, 0)$.

***

Example 7: Examine $\frac{dy}{dx} = \sqrt{y}$ when $y(0) = 0$, for existence and uniqueness of solution in $D: |x| < a, |y| < b$ for a, b constants.

Solution: Here $f(x, y) = \sqrt{y}, x_0 = 0$ and $y_0 = 0$. Function f is continuous and bounded in D containing the point $(0, 0)$. Hence the solution exists.

In order to test the uniqueness of the solution, consider the Lipschitz condition.

$$\frac{|f(x, y_2) - f(x, y_1)|}{|y_2 - y_1|} = \frac{|\sqrt{y_2} - \sqrt{y_1}|}{|y_2 - y_1|}.$$

For any region containing the line $y = 0$, Lipschitz condition is violated.

Because for $y_1 = 0$ and $y_2 > 0$, we have

$$\frac{|f(x,\ y_2) - f(x,\ y_1)|}{|y_2 - y_1|} = \frac{\sqrt{y_2}}{y_2} = \frac{1}{\sqrt{y_2}},\ \left(\sqrt{y_2} > 0\right)$$

and this can be made as large as we please by choosing y_2 sufficiently small, whereas condition (35) requires that the quotient on the left-hand side of Eqn. (35) does not exceed a fixed constant L.

Therefore, we cannot conclude the uniqueness of the solution. Further, it can be checked that the given problem has the following solutions

i) $y(x) = 0 \ \forall\ x \in D$

ii) $y(x) = \frac{1}{4}x^2 \ \forall\ x \in D$. (37)

***

Example 8: Examine $\frac{dy}{dx} = f(x,\ y)$ where $f(x,\ y) = \begin{cases} y(1-2x) & x > 0 \\ y(2x-1) \text{ for} & x < 0 \end{cases}$ with $y(1) = 1$ for existence and uniqueness of solution in $D: |x-1| < a, |y-1| < b$ for a, b constants.

Solution: Here $x_0 = 1$ and $y_0 = 1$. You may note that the function is not defined at $x = 0$. It is discontinuous at $x = 0$. Thus, at $x = 0$ the solution does not exist. At all other points the polynomial function

$$f(x,\ y) = \begin{cases} y(1-2x) \text{ for } x > 0 \\ y(2x-1) \text{ for } x < 0 \end{cases}$$

and $\frac{\partial f}{\partial y}$ is continuous and bounded in D with $y(1) = 1$. Hence, the solution exists and is unique for all x other than $x = 0$. Further, you may verify that

$$y(x) = \begin{cases} e^{x-x^2} \text{ for } x > 0 \\ e^{x^2-x} \text{ for } x < 0 \end{cases} \qquad (38)$$

is the unique solution of the given problem for all x other than $x = 0$.

***

Note that the way function y is obtained in Eqns. (37) or (38) is just by simple integration of the given differential equation. The details of this would become clear to you after you have learnt the methods of solving first order differential equations in Unit 7.

You may now try the following exercises.

E9) For the following problems determine a region of the xy-plane for which the given differential equation will have a unique solution through a point $(x_0,\ y_0)$ in the region.

i) $x\frac{dy}{dx} = y$

ii) $(4-y^2)\dfrac{dy}{dx}=x^2$

iii) $(x^2+y^2)\dfrac{dy}{dx}=y^2$

iv) $\dfrac{dy}{dx}=x^3\cos y$

E10) Examine $\dfrac{dy}{dx}=f(x,\ y)$ where

$$f(x,\ y)=\begin{cases}\dfrac{4x^3y}{(x^4+y^2)}, & \text{when } x \text{ and } y \text{ are not both zero} \\ 0 & \text{, when } x=y=0\end{cases} \quad \text{with } y(0)=0$$

for existence and uniqueness of the solution in $D:|x|<a, |y|<b$, where a, b are constants.

From the discussion given in Sec. 6.3, you may have realised that the general solution of a first order differential equation is a curve whose equation normally contains one arbitrary constant or a **parameter**. For instance, curve $y=y(x,\ c)$, where c is a parameter. For different choices of this parameter c we get different curves in the family. Each of these curves is a particular solution, or integral curve, of the given differential equation, and all of them together constitute its general solution. In this way, the integral curves of the first order differential equation form a one-parameter family of curves. On the other hand, we expect that the one parameter family of curves $y=y(x,\ c)$ where c is a parameter should yield a first order differential equation. In general we may pose a problem: given an n-parameter family of curves, say, $y=y(x,\ c_1,\ c_2,\ \ldots,\ c_n)$ for $c_1,\ c_2,\ \ldots,\ c_n$ being n-parameters, can we find an associated nth-order differential equation that represents the given family and is entirely free of arbitrary parameters? In most of the cases the answer is yes. We can thus say that differential equations arise from family of curves. In the next section we shall take up this.

6.4 FAMILY OF CURVES AND DIFFERENTIAL EQUATIONS

Let us consider a family of straight lines

$$y=mx+c \tag{39}$$

which is a two-parameter family of curves, the parameters being m and c. It is clear from Eqn. (39) that y can be treated as a function of x, for $x\in \boldsymbol{R}$. Differentiating Eqn. (39) w.r.t. x, we get

$$y'=m \tag{40}$$

w.r.t. represents with respect to.

Differentiating the above equation again w.r.t. x, we get

$$y''=0. \tag{41}$$

Eqns. (40) and (41) are differential equations of order one and two, respectively. The way in which we have arrived at Eqn. (40) or (41) it is clear that we have actually tried to eliminate the parameters, or constants c and m.

In general, we represent one-parameter family of curves by an equation

$$f(x, y, a) = 0 \tag{42}$$

where a is a constant, or a parameter.

In Eqn. (42), let us regard y as a function of x and differentiate it w.r.t. x. Suppose we get

$$g(x, y, y', a) = 0. \tag{43}$$

In case, we are able to eliminate the constant a between Eqns. (42) and (43), then we get a relation connecting x, y and y', say

$$h(x, y, y') = 0. \tag{44}$$

Eqn. (44) is an ODE of order one. In particular, if Eqn. (42) has the form

$$\psi(x, y) = a \tag{45}$$

then the elimination of the constant a from Eqn. (45) leads us to the differential equation

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} y' = 0 \tag{46}$$

For example, $x^2 + y^2 = a^2$ (47)

is the equation of the family of all **concentric circles** with centre at the origin (see Fig. 3).

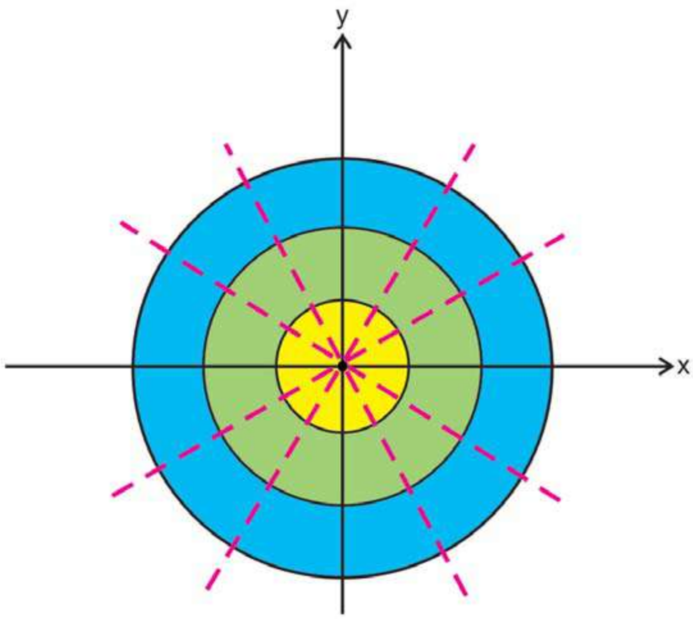


Fig. 3

For different values of a, we get different circles of the family. Differentiating Eqn. (47) with respect to x, we get

$$x + y\frac{dy}{dx} = 0 \tag{48}$$

as the differential equation of the given family of circles.

Continuing with equation $y = mx + c$, if we regard only c as an arbitrary constant to be eliminated, then $y' = m$, represents the required differential equation. Geometrically, for a fixed m, $y' = m$ represents a family of **straight**

lines (in the plane) **whose slope is m** (see Fig. 4).

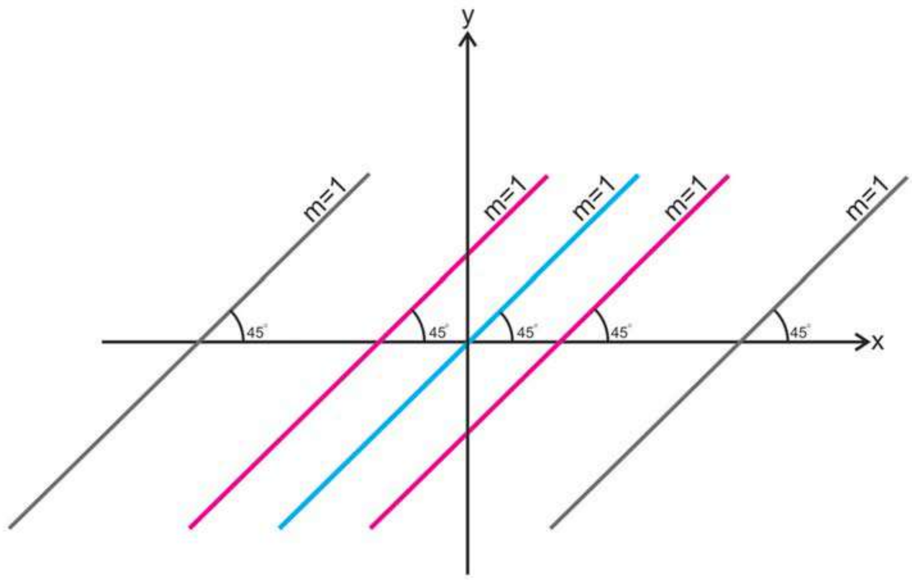


Fig. 4

On the other hand, if we assume that in equation $y = mx + c$ both m and c are constants to be eliminated, then equation $y'' = 0$ represents the required differential equation. Geometrically, it is the family of all **straight lines in the plane**. Some of these lines are shown in Fig. 5

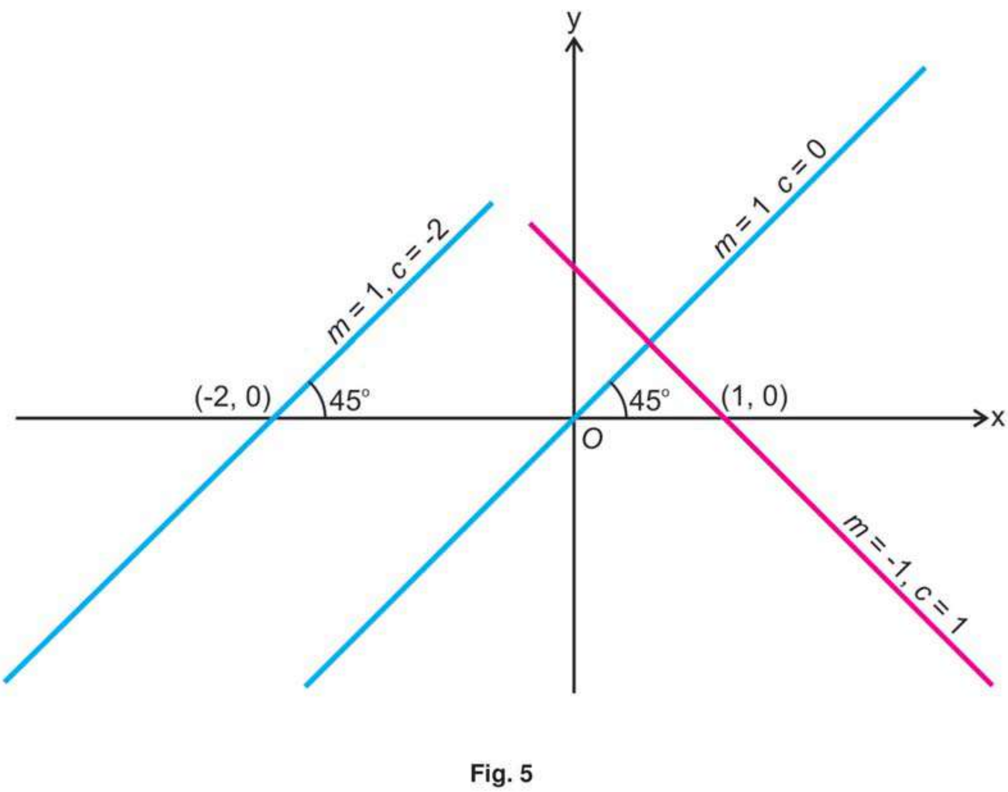


Fig. 5

Let us consider the following examples.

Example 9: Find the differential equation arising from the family of curves $y = cx^3$, c being an arbitrary constant.

Solution: Differentiating $y = cx^3$, w.r.t. x, we obtain

$$\frac{dy}{dx} = 3cx^2, \tag{49}$$

However, we have $c = y/x^3$ from the given equation. Thus Eqn. (49) reduces to

$$\frac{dy}{dx} = 3\left(\frac{y}{x^3}\right)x^2 = 3y/x.$$

Thus the first order linear equation

$$x\frac{dy}{dx} - 3y = 0$$

is the required differential equation.

***

Example 10: Find the differential equation arising from the family of circles passing through the origin with centre on the y-axis.

Solution: You know from your knowledge of analytical geometry that the equation

$$x^2 + y^2 = cy \tag{50}$$

where c is an arbitrary constant, gives the family of circles through the origin with centre on the y-axis. For different values of c different members of the family are obtained.

To find the differential equation of the family given by Eqn. (50), we differentiate it w.r.t. x and obtain

$$2x + 2y\frac{dy}{dx} = c\frac{dy}{dx}$$

Substituting $c = \dfrac{x^2 + y^2}{y}$ in the equation above, we get

$$2x + 2y\frac{dy}{dx} = \left(\frac{x^2 + y^2}{y}\right)\frac{dy}{dx}$$

$$\Rightarrow \quad 2xy + 2y^2\frac{dy}{dx} = (x^2 + y^2)\frac{dy}{dx}$$

$$\Rightarrow \quad (x^2 - y^2)\frac{dy}{dx} = 2xy$$

which is the required differential equation.

***

Example 11: Find the differential equation of the family of parabolas $y = (x+c)^2$, c is an arbitrary constant.

Solution: Differentiating the given equation of the parabola w.r.t. x, we obtain

$$\frac{dy}{dx} = 2(x+c) \tag{51}$$

From the given equation we have $x + c = \pm y^{1/2}$. Substituting this in Eqn. (51), the differential equation representing the family is

$$\frac{dy}{dx} = \pm 2y^{1/2} \text{ or, } \left(\frac{dy}{dx}\right)^2 = 4y$$

You may **observe** here that equation $\dfrac{dy}{dx} = 2y^{1/2}$ does not describe the complete family. It would give the slope only of a right-hand branch $(x > -c)$ of any particular parabola (see Fig. 6) Fig. 6 illustrate the case when $c = 0$. In this case we could say that $y = x^2$ is a solution of $\frac{dy}{dx} = 2y^{1/2}$ on the interval $[0, \infty[$.

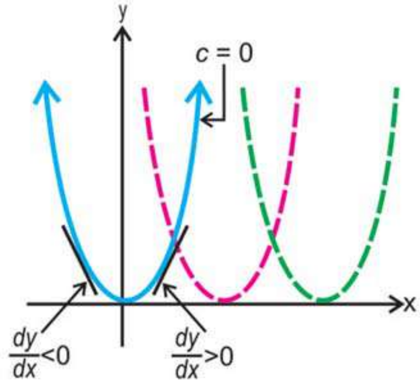


Fig. 6

***

As an interesting application of the method above we consider the problem of finding orthogonal trajectories of a given family of curves.

You may **observe** here that the family of concentric circles represented by Eqn. (47) and the family $y = mx$ of straight lines through the origin (the dotted lines in Fig. 3) have the property that each curve in either family is perpendicular/orthogonal to every curve in the other family. Each family is then said to be a family of orthogonal trajectories of the other. Formally, we have the following definition:

Definition: When all the curves of one family of curves $f(x, y, c_1) = 0$ intersect orthogonally with all the curves of another family $g(x, y, c_2) = 0$, then the families are said to be **orthogonal trajectories** of each other.
In other words, an orthogonal trajectory is any one curve that intersects every curve of another family at right angles. Orthogonal trajectories are of interest in applied mathematics and geometry of plane curves. For instance, if an electric current is flowing in a plane sheet of conducting material, then the lines of equal potential are the orthogonal trajectories of the lines of current flow.

In the case of concentric circles with centre at the origin, it is geometrically obvious that the orthogonal trajectories are the straight lines through the origin. But this may not be the case always. For more complicated situations, we thus need to develop an analytical method of finding orthogonal trajectories.

From your study of analytical geometry you may recall that two lines L_1 and L_2, which are not parallel to the coordinate axes, are perpendicular if and only if the product of their respective slopes say, m_1 and m_2 is –1, i.e., $m_1 m_2 = -1$. In the same way, suppose

$$\frac{dy}{dx} = f(x, y) \tag{52}$$

is the differential equation of the family of curves shown in Fig. 7. The slope at any point (x, y) on each of these curves is given by $f(x, y)$. Then the slope on the orthogonal trajectory through the same point (x, y), shown by the dotted lines, has to be $-1/f(x, y)$, being orthogonal to the curve.

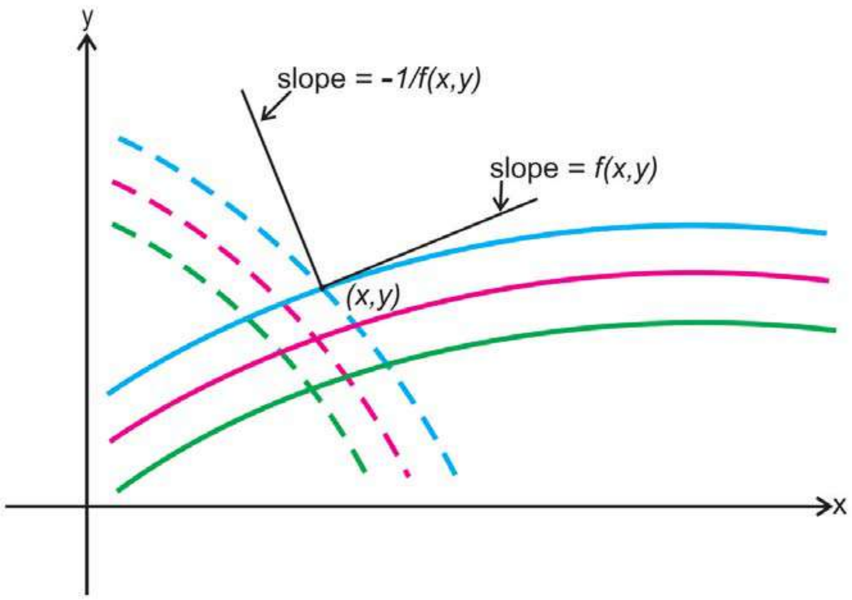


Fig. 7

Thus on the orthogonal trajectory, we have

$$\frac{dy}{dx} = -1/f(x,\ y)$$

or,

$$\frac{1}{dy/dx} = -f(x,\ y) \tag{53}$$

We can say that to find the orthogonal trajectories of a given family of curves, first find the differential equation of the family and then replace $\frac{dy}{dx}$ by $-\frac{1}{dy/dx}$ to obtain the differential equation of the orthogonal trajectories. For instance, in the differential Eqn. (48) of the family of concentric circles if we replace $\frac{dy}{dx}$ by $-\frac{1}{dy/dx}$ then we obtain

$$x + y\left(-\frac{1}{dy/dx}\right) = 0$$

or,

$$\frac{dy}{dx} = \frac{y}{x} \tag{54}$$

as the differential equation of the orthogonal trajectories. On integrating Eqn. (54), we get

$y = cx$, where c is a constant. (55)

as the equation of the orthogonal trajectories to the family of concentric circles. You may **note** that Eqn. (55) represents the family of straight lines through the origin.

Let us take up the following examples to illustrate the procedure.

Example 12: Find the orthogonal trajectories of the family of rectangular hyperbolas $y = \frac{c}{x}$, c being a constant.

Solution: Differentiating $y = \frac{c}{x}$ w.r.t. x, we get

$$\frac{dy}{dx} = \frac{-c}{x^2} \tag{56}$$

Replacing c by $c = xy$ in Eqn. (56) we get the differential equation of the given family as

$$\frac{dy}{dx} = \frac{-y}{x}.$$

The differential equation of the orthogonal family is then given by

$$\frac{dy}{dx} = \frac{-1}{(-y/x)} = \frac{x}{y} \tag{57}$$

Integrating Eqn. (57) we obtain

$y^2 - x^2 = c_1$, where c_1 is a constant.

which is the equation of the family of hyperbolas as the orthogonal trajectories to the family of rectangular hyperbolas.

***

A family of curves is self-orthogonal if a member of orthogonal trajectories is also a member of the original family.

Example 13: Show that the family of parabolas $y^2 = 2cx + c^2$, c being a constant, is self orthogonal.

Solution: Differentiating the given family of parabolas w.r.t. x, we obtain

$$2y\frac{dy}{dx} = 2c \Rightarrow c = yy'$$

The differential equation of the given family is

$$y^2 = 2xy\,y' + y^2 y'^2 \tag{58}$$

In Eqn. (58) replacing y' by $\frac{-1}{y'}$, the differential equation of the family of orthogonal trajectories is

$$y^2 = \frac{-2xy}{y'} + \frac{y^2}{y'^2}$$

or, $$y'^2 y^2 + 2xyy' = y^2 \tag{59}$$

Eqn. (59) is the same as Eqn. (58) and so the given family of parabolas are self orthogonal.

***

You may now try the following exercises.

E11) Assuming y to be a function of x, determine the differential equation by eliminating the arbitrary constant (or constants) indicated in the following problems.

i) $xy = c$ (arbitrary constant is c).

ii) $y = \cos(ax)$ (arbitrary constant is a).

iii) $xy = ae^x + be^{-x} + x^2$ (arbitrary constants are a and b).

E12) Find the orthogonal trajectories of the following families of curves, c being an arbitrary constant:

i) $y = \frac{cx}{1+x}$

ii) $y = cx^2$

iii) $y = ce^x$.

E13) Show that the family of circles $x^2 + y^2 = px$ intersect the family of circles $x^2 + y^2 = qy$ at right angles.

In the discussion above you saw that the general solution of the first order differential equation form a one-parameter family of curves. On the other hand, you also saw how from a given one-parameter (or, n-parameter) family of curves, an associated first-order (or nth order) differential equation is obtained. In the introduction to this unit we mentioned that there are many problems of physical and engineering interest which give rise to differential equations. Alternatively, we can say that many real world problems have representations in the form of differential equations. In the next section we shall take up some of these problems which can be formulated in terms of the first order ordinary differential equations.

6.5 DIFFERENTIAL EQUATIONS ARISING FROM PHYSICAL SITUATIONS

In this section we shall show that differential equations arise not only out of consideration of families of geometric curves, but also as an attempt to describe physical situations, in mathematical terms.
The initial-value problem

$$\left.\begin{aligned} \frac{dy}{dx} &= ky \\ y(t_0) &= y_0 \end{aligned}\right\} \tag{60}$$

where k is a constant of proportionality, occurs in many physical theories involving either **growth** or **decay**. For example, in biology it is often observed that the rate at which certain bacteria grow is proportional to the number of bacteria present at any time. In physics an IVP such as Eqn. (60) provides a model for approximating the remaining amount of a substance that is disintegrating, or decaying, through radioactivity. Eqn. (60) also describes the amount of a substance remaining during certain reactions and could determine the temperature of a cooling body.

Let us now examine the formulation of some of these problems.

I: Population Model

Let $N(t)$ denote the number or amount of a certain species or a substance at time t. Then the rate of growth of $N(t)$ is given by its derivative $\frac{d}{dt}N(t)$.

Thus, if $N(t)$ is growing at a constant rate then $\frac{d}{dt}N(t) = \beta$, a constant. It is sometimes more appropriate to consider the relative rate of growth defined as follows:

Relative rate of growth of $N(t) = \dfrac{\text{actual rate of growth of } N(t)}{\text{size of } N(t)}$

$$= \frac{N'(t)}{N(t)} = \frac{dN(t)/dt}{N(t)}.$$

The relative rate of growth indicates the percentage increase in $N(t)$ or decrease in $N(t)$. For example, an increase of 100 individuals for a species with a population size of 500 would probably have a significant impact being an increase of 20 percent. On the other hand, if the population were 1,000,000 then the addition of 100 would hardly be noticed, being an increase of 0.01 percent. If we assume that the rate of change of N at time t is proportional to the population $N(t)$, present at the time t then,

$$\frac{d}{dt} N(t) \propto N(t)$$

which is written as

$$\frac{d}{dt} N(t) = k\, N(t), \tag{61}$$

where k is a constant of proportionality.

If N increase with t, then $k > 0$ in Eqn. (61).

If N decreases with t, then $k \leq 0$ in Eqn. (61).

Normally, we have the knowledge of the population, say N_0, at some initial time t_0. Along with Eqn. (61) we can thus have the initial condition

$$N(t_0) = N_0. \tag{62}$$

Thus, the population $N(t)$ at time t can be found by solving Eqn. (61) along with the condition (62).

With little modification, the model above can be used to formulate the spread of contagious diseases. For example, in the spread of a virus, it is reasonable to assume that the rate, $\frac{dx}{dt}$, at which the disease spreads is proportional not only to the number of infected people, $x(t)$, who have contracted (developed) the disease, but also to the number of susceptibles $y(t)$, who have yet not been exposed to the disease. Now due to infection the number of susceptibles decreases and the number of infected persons increases. The average number of new cases (i.e., new infectives) of the disease at any time would depend on the rate of the contact between the infectives and the susceptibles and can be assumed to be proportional to i) the number of infectives $x(t)$, and ii) the number of susceptibles $y(t)$, present at that time. Hence we may take the number of new infectives at any time to be proportional to $x(t)y(t)$. That is

Infected are those who have already developed the disease and are capable of transmitting it to others.

$$\frac{dx}{dt} \propto xy$$

or, $$\frac{dx}{dt} = kxy \tag{63}$$

where k is the constant of proportionality called the **contact rate**. If we assume that in a population of n susceptibles, at time $t = 0$ one infected person is introduced then $x(0) = 1$ and x and y are related by the relation

$$x + y = n + 1 \tag{64}$$

Susceptibles are those individuals who are not yet infected by the diseases, but are capable of contracting (developing) the disease and become infective at any time.

Using Eqn. (64) in Eqn. (63), we can write

$$\frac{dx}{dt} = kx(n+1-x) \tag{65}$$

Eqn. (65) along with the initial condition $x(0)=1$ can then be solved to find $x(t)$ at any time t.

Remark: You may have noted that Eqn. (65) is non-linear. It is a special case of a more general equation

$$\frac{dP}{dt} = P(a-bP),\ a \text{ and } b \text{ constants}$$

which is known as the **logistic equation**. The equation is very important in ecological studies. We shall be discussing this equation in detail in another course on mathematical modelling.

II: Newton's Law of Cooling

Here we deal with the temperature variations of a hot object kept in a surrounding which is kept at a constant temperature, say T_0. Under certain conditions, a good approximation to the temperature of an object can be obtained by using Newton's law of cooling. Let the temperature of the object at any time t be T. If $T \geq T_0$, we know that the object radiates heat to the surrounding resulting in the reduction of its (object's) temperature. Newton's law of cooling states that the rate at which the temperature $T(t)$ changes in a cooling body is proportional to the difference between the temperature of the body and the constant temperature T_0 of the surrounding medium. That is,

$$\frac{d}{dt}T(t) \propto (T(t)-T_0)$$

or $$\frac{d}{dt}T(t) = k(T(t)-T_0) \tag{66}$$

where k is a constant of proportionality.

Constant $k<0$, because the temperature of the body is reducing (we have assumed that $T(t) \geq T_0$). We observe that Eqn. (66) is a differential equation of order one and it describes the temperature $T(t)$ of the object at any time t.

III: Radioactive Decay

Many substances are radioactive. This means that the atoms of such a substance break down into atoms of other substances. Physicists have noticed that radioactive material, at time t, decays at a rate proportional to its amount $y(t)$. In other words,

$$\frac{d}{dt}y(t) = ky(t) \tag{67}$$

where $k<0$, is a constant. If the mass of the substance at some initial time, say $t=0$, is A, then $y(t)$ also satisfies the initial condition

$$y(0) = A.$$

Thus, the physical problem of radioactive decay is modelled by the IVP.

$$\frac{d}{dt}y(t) = ky(t),\ y(0) = A \tag{68}$$

where k is a constant and $k<0$.

Remark: I, II and III above indicate situations where differential equations occur naturally. In Unit 8 we shall give the methods of solving these equations and also discuss some more physical situations.

Before we ask you to check your ability to formulate some physical situations we consider an example.

Example 14: A drug is infused into a patient's blood stream at a constant rate r ns/sec. . Simultaneously, the drug is removed at a rate proportional to the amount $x(t)$ of the drug present at any time t. Find the differential equation describing the amount $x(t)$ of the drug in the patient's blood stream at any time t.

Solution: The rate at which the drug is infused into a patients blood $= r$.
The rate at which the drug is removed from a patient blood $\propto x(t) = kx(t)$ where k is the constant of proportionality.
$\therefore$ The rate of change of amount of drug in a patient's blood is

$$\frac{dx}{dt} = r - kx,\ k > 0. \qquad (69)$$

The minus sign in the second term on the r.h.s. of Eqn. (69) shows decrease in the amount of drug in the patient's bloodstream.

Eqn. (69) describes the amount $x(t)$ of the drug in a patient's body at any time t. If it is assumed that no drug was present in a patient's blood at time $t = 0$, then initial condition

$$x(0) = 0$$

can be used to solve Eqn. (69) and obtain $x(t)$ at any time t.

***

You may now try the following exercise.

E14) In the following problems, formulate the differential equation describing the given physical situations.

i) A culture initially has p_0 number of bacteria. Growth of the bacteria is proportional to the number of bacteria present. Find the number p of bacteria present in the culture at any time t.

ii) A quantity of a radioactive substance originally weighing x_0 gms decomposes at a rate proportional to the amount present. If half the original quantity is left after 2 years, then find the amount x of the substance remaining after t years.

We now conclude this unit by giving a summary of what we have covered in it.

6.6 SUMMARY

In this unit, we have covered the following points:

1. An equation involving one (or more) dependent variables and its derivatives w.r.to one or more independent variables is called a **differential equation**.

2. A differential equation involving only ordinary derivatives is called an **ordinary differential equations (ODE).**

3. A differential equation involving partial derivatives is called a **partial differential equation (PDE)**.

4. The **order** of a differential equation is the order of the highest order derivative appearing in the equation.

5. The **degree** of a differential equation is the highest exponent of the highest order derivative appearing in it after the equation has been expressed in the form free from radicals and fractions of the derivatives.

6. In a differential equation, when the dependent variable and its derivatives occur in the first degree only, and not as higher powers or products, the equation is said to be **linear**.

7. If an ordinary differential equation is not linear, it is said to be **non-linear**.

8. A real or complex valued function ϕ defined on an interval I is called a **solution** of the equation

   $g(x,\ y,\ y',\ y'',\ \ldots,\ y^{(n)})=0$

   if ϕ is differentiable n times and if $\phi(x),\ \phi'(x),\ \phi''(x),\ \ldots,\ \phi^{(n)}(x)$ satisfy the above equation for all x in I.

9. The solution of an nth order differential equation which contains n arbitrary constants is called its **general solution.**

10. Any solution which is obtained from the general solution by giving particular values to the arbitrary constants is called a **particular solution** of the differential equation.

11. A solution of a differential equation, which cannot be obtained by assigning definite values to the arbitrary constants in the general solution is called its **singular solution**.

12. Conditions on the value of the dependent variable and its derivatives, at a single value of the independent variable in the interval of existence of the solution, are called **initial conditions**.
13. The problem of solving a differential equation together with the initial conditions is called an **initial value problem**.

14. The **sufficient** conditions for the **existence of solution** of the first order equation $\frac{dy}{dx}=f(x,\ y)$, with $y(x_0)=y_0$, in a region D defined by $|x-x_0|<a$ and $|y-y_0|<b$ are

    i) f is continuous in D
    and

    ii) f is bounded in D.

    Further if the solution exists, then it is **unique** if, in addition to (i) and (ii), we have

    iii) $\frac{\partial f}{\partial y}$ is continuous in D.

iv) $\frac{\partial f}{\partial y}$ is bounded in D (or, Lipschitz condition is satisfied).

15. The general solution of a first order (nth order) differential equation represents **one-parameter** (n-parameter) family of curves.

16. Many physical situations originating in population dynamics, temperature variations in a cooling body, radioactive decay can be represented by first order differential equations.

6.7 SOLUTIONS/ANSWERS

E1) i) ODE

ii) ODE

iii) It is not a differential equation since the unknown y is appearing inside the integral on the right hand side and is to be evaluated in the interval 0 to x.

iv) It is not a differential equation as in iii) above.

v) PDE

E2) i) Order 2, degree 2.

ii) Order 2, degree 1.

iii) Since expansion of sine function gives an infinite series therefore Order 2, degree not defined.

iv) Order 1, degree 1.

v) Rationalising the exponent, we get

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = r^2\left(\frac{d^2y}{dx^2}\right)^2$$

$\therefore$ Order 2, degree 2.

vi) Order 4, degree 1.

vii) Order 1, degree 2.

E3) i) Linear

ii) Non-linear because of the presence of $\left(\frac{\partial^2 z}{\partial x\,\partial y}\right)^2$.

iii) Non-linear, since we have y^2 on the right hand side.

iv) Linear

v) Non-linear, since rationalising exponent, we get

$(x^2+y^2)^3\left(\frac{d^2y}{dx^2}\right)^2 = \mu^2x^2$, and there is a product of y terms with

$\frac{d^2y}{dx^2}$.

E4) Here $2\cos y = -x^2$

Differentiating w.r. t. x, we get $\sin y \frac{dy}{dx} = x$.

Hence $y = \cos^{-1}\left(-\frac{x^2}{2}\right)$ is a solution of $\sin y \frac{dy}{dx} = x$.

Since $\cos y = -\frac{x^2}{2}$, y exists as long as $|x| \le \sqrt{2}$, i.e., the interval of existence of y is $-\sqrt{2} \le x \le \sqrt{2}$.

E5) Differentiating $xy = \ln y + c$ with respect to x, we get $y + x\frac{dy}{dx} = \frac{1}{y}\frac{dy}{dx}$ and hence the result.

E6) $y = c_1 e^{2x} + c_2 e^{3x}$ for all constant values of c_1 and c_2 are solutions of the given equation.

E7) Differentiating $y^2 - 2y = x^3 + 2x^2 + 2x + c$ w.r.t. x, we get

$$(2y-2)\frac{dy}{dx} = 3x^2 + 4x + 2$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)}$$

On solving the quadratic equation

$$y^2 - 2y = x^3 + 2x^2 + 2x + c$$

for y, we get the two explicit solutions as

$$y = 1 \pm \sqrt{x^3 + 2x^2 + 2x + c + 1}.$$

E8) Consider the Lipschitz condition

$$\frac{|f(x, y_2) - f(x, y_1)|}{|y_2 - y_1|} = \frac{|3y_2^{2/3} - 3y_1^{2/3}|}{|y_2 - y_1|}$$

For any domain containing the line $y = 0$, Lipschitz condition is violated. Because for $y_1 = 0$ and $y_2 > 0$, we have

$$\frac{|f(x, y_2) - f(x, y_1)|}{|y_2 - y_1|} = \frac{3}{y_2^{1/2}} \quad (y_2 > 0)$$

which can be made as large as we please by choosing y_2 sufficiently small.

E9) i) $\frac{dy}{dx} = \frac{y}{x} = f(x, y)$

f is continuous and bounded for $x \ne 0$.

Also $f_y = \frac{1}{x}$ is not bounded for $x = 0$.

$\therefore$ Unique solution exits in either of the half plane $x > 0$ or $x < 0$.

ii) $\frac{dy}{dx} = \frac{x^2}{4-y^2} = \frac{x^2}{(2-y)(2+y)} = f(x, y)$

f is continuous for either $y < 2$, $y < -2$ or $-2 < y < 2$.

$$f_y = \frac{2x^2 y}{(4-y^2)^2} = \frac{2x^2 y}{(2-y)^2(2+y)^2}$$

$\therefore$ Unique solution exists for either $y > 2,\ y < -2$ or $-2 < y < 2$.

iii) $f(x,\ y) = \dfrac{y^2}{x^2+y},\ f_y = \dfrac{2yx^2}{(x^2+y^2)^2}$

Unique solution in any region not containing $(0,\ 0)$.

iv) $f(x,\ y) = x^3 \cos y$ and $f_y = -x^2 \sin y$

$\therefore$ Unique solution everywhere in the xy-region

E10) Here $f(x,\ y) = \dfrac{4x^3 y}{x^4+y^2}$ is bounded and continuous in any rectangle containing the point $(0,\ 0)$. Hence the solution exists.

To examine the uniqueness of the solution, let us consider the Lipschitz condition. Here

$$|\,f(x,\ y_1) - f(x,\ y_2)\,| = \left| \frac{4x^3 y_1}{x^4+y_1^2} - \frac{4x^3 y_2}{x^4+y_2^2} \right|$$

$$= \left| \frac{4x^3 y_1(x^4+y_2^2) - 4x^3 y_2(x^4+y_1^2)}{(x^4+y_1^2)\,(x^4+y_2^2)} \right|$$

$$= \left| \frac{4x^3(y_1-y_2)\,(x^4-y_1y_2)}{(x^4+y_1^2)\,(x^4+y_2^2)} \right|$$

$$= \left| \frac{4\left(1-\dfrac{y_1y_2}{x^4}\right)(y_1-y_2)\dfrac{1}{x}}{\left(1+\dfrac{y_1^2}{x^4}\right)\left(1+\dfrac{y_2^2}{x^4}\right)} \right|$$

$$= \frac{4\left|1-\dfrac{y_1y_2}{x^4}\right| \dfrac{|\,y_1-y_2\,|}{|\,x\,|}}{\left|\left(1+\dfrac{y_1^2}{x^4}\right)\left(1+\dfrac{y_2^2}{x^4}\right)\right|}$$

and therefore the Lipschitz condition is not satisfied in any rectangle containing the origin and we cannot conclude the uniqueness of the solution. Further, it can be verified that the equation is satisfied by $y = c^2 - \sqrt{x^4+c^4}$, c is an arbitrary constant, i.e., there is infinity of solution satisfying the initial condition.

E11) i) Here $xy = c$

Differentiating w.r. to x, we get

$$x\frac{dy}{dx}+y=0,$$

which is the required differential equation.

ii) $y=\cos(ax)$ (44)

Differentiating w.r. to x, we get

$y'=-a\sin ax$ (45)

From Eqn. (44), $ax=\cos^{-1}y$ and $\sin(ax)=\sqrt{1-y^2}$

Substituting these in Eqn. (45), we get

$$y'=-\frac{1}{x}(\cos^{-1}y)\sqrt{1-y^2}$$

which is the required differential equation.

iii) $xy=ae^x+be^{-x}+x^2$ (46)

Differentiating w.r. to x, we get

$xy'+y=ae^x-be^{-x}+2x$ (47)

Differentiating Eqn. (47) again w.r.t. x, we get

$$xy''+2y'=ae^x+be^{-x}+2$$

$$=(xy-x^2)+2 \qquad \text{[using Eqn. (46)]}$$

$$\Rightarrow \quad xy''+2y'-xy+x^2-2=0,$$

which is the required differential equation.

E12) i) We have $\frac{dy}{dx}=\frac{c}{(1+x)^2}=\frac{y}{x(1+x)}$

The differential equation of the orthogonal trajectories is then

$$\frac{dy}{dx}=\frac{-x(1+x)}{y}$$

which on integration yields

$$3y^2+3x^2+2x^3=c_1.$$

ii) We have $\frac{dy}{dx}=2cx=\frac{2y}{x}$

The DE of the orthogonal trajectories is then

$$\frac{dy}{dx}=\frac{-x}{2y}$$

which on integration yields the family of ellipses

$$2y^2+x^2=c_1.$$

iii) $y^2+2x=c_1.$

E13) For the given families $y' = (y^2 - x^2)/2xy$ and $y' = 2xy/(x^2 - y^2)$, respectively.

E14) i) $\dfrac{dp}{dt} = kp$

subject to $p(0) = p_0$ where $k > 0$ is a constant.

ii) $\dfrac{dx}{dt} = kx,\ x(0) = x_0$ and $x(2) = \dfrac{x_0}{2}$

where $k < 0$, is a constant.

– x –

UNIT 7

SOLVING FIRST ORDER AND FIRST DEGREE EQUATIONS

Structure **Page No.**

7.1 INTRODUCTION

In Unit 6, we introduced the basic concepts and definitions involved in the study of differential equations. We discussed various types of solutions of an ordinary differential equation. We also stated the conditions for the existence and uniqueness of the solution of the first order initial value problem. However, we do not seem to have paid any attention to the methods of finding these solutions. Accordingly, in this unit we shall confine our attention to this important aspect of differential equations.

In general, it may not be feasible to solve even an apparently simple differential equation $\frac{dy}{dx} = f(x, y)$ or $g\left(x, y, \frac{dy}{dx}\right) = 0$ where f and g are arbitrary functions. This is because no systematic procedure exists for obtaining its solution for arbitrary forms of f and g. However, there are certain standard types of first order equations for which methods of solution are available. In this unit in Secs.72 to 7.5 we shall discuss a few of them with special reference to their applications. A piece of advise – In solving differential equations you will often have to utilize, say, integration by parts, by substitution, or partial fractions. It is therefore worth spending some of your time reviewing the techniques of integration, which you have studied in your calculus course, before you start solving differential equations.

Objectives

After having gone through this unit, you should be able to:

- define separable equations and solve them;
- define homogeneous equations and solve them;
- obtain the solution of equations which are reducible to homogeneous equations;
- identify exact equations and solve them;
- obtain an integrating factor which may reduce a given differential equation into an exact one and eventually provide its solution.

7.2 SEPARATION OF VARIABLES

In our study of first order differential equations

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

we now turn our attention to equations that may be non-linear. For example, consider the equation

$$\frac{dy}{dx} = \frac{x^2}{1-y^2} \tag{2}$$

Here $f(x, y) = \frac{x^2}{1-y^2}$ depends in a non-linear way on the dependent variable y. But it is possible to write Eqn. (2) in the form

$$M(x)dx = N(y)dy \tag{3}$$

where $M(x) = x^2$ and $N(y) = (1-y^2)$. Such an Eqn. (3) is said to be in **variable separable** form, because each side of the equation depends on only one of the variables. As you may know that the problem of finding the tangent to a given curve at a given point was solved by Leibniz (1646-1716), the German mathematician. The search for the solution to the inverse problem of tangents, that is, given the equation of the tangent to a curve at any point to find the equation of the curve led Leibniz to many important developments. A particular mention may be made of the method of separation of variables which was discovered by Leibniz in 1691 by proving that a differential equation of the form

$$\frac{dy}{dx} = M(x)N(y)$$

is integrable by quadratures. However, it is John Bernoulli (1694) who is credited with the introduction of the terminology and the process of separation of variables. In short, it is a method for solving a class of differential equations that arises quite frequently and is defined as follows:

> The process of finding the area of a plane region is called 'quadrature'.

Definition: An equation

$$\frac{dy}{dx} = f(x, y)$$

is called a **separable equation** or **equation in variable separable form** if $f(x, y)$ can be written in the form

$$f(x, y) = M(x)N(y), \tag{4}$$

where M and N are given functions of x and y, respectively.

In other words, Eqn. (1) is a separable equation if f is a product of two functions, one of which is a function of x and the other is a function of y. Here M and N are real valued function of x and y, respectively.

For instance, equation $\frac{dy}{dx} = e^{x+y}$ is a separable equation, since $e^{x+y} = e^x.e^y$ (here $M(x) = e^x$ and $N(y) = e^y$). The equation $\frac{dy}{dx} = x^2(y^2 + y^3)$ is also a separable equation. But the equation $\frac{dy}{dx} = e^{xy}$ is not a separable equation, because it is not possible to express e^{xy} as a product of two functions in which one is a function of x only and the other is a function of y only. Similarly, equation $\frac{dy}{dx} = x + y$ is not a separable equation.

From Eqns. (1) and (4) we can thus write

$$\frac{dy}{dx} = M(x)\,N(y)$$

or, $$\frac{1}{N(y)}\frac{dy}{dx} = M(x)$$

Thus Eqn. (1), when in variable separable form, can be written as

$$N_1(y)\frac{dy}{dx} + M_1(x) = 0 \qquad (5)$$

for some N_1 and M_1, each being function of one variable only.

To solve Eqn. (5) let us assume that there exist functions A and B such that $A'(y) = N_1(y)$ and $B'(x) = M_1(x)$. With this hypothesis, Eqn. (5) can be rewritten as

$$\frac{d}{dx}A(y(x)) + B'(x) = 0 \qquad (6)$$

$$\left[\text{By chain rule } \frac{d}{dx}A(y(x)) = A'(y(x))\frac{dy}{dx} = N_1(y(x))\frac{dy}{dx}\right]$$

Integrating Eqn. (6) with respect to x, we get

$$A(y(x)) + B(x) = c \qquad (7)$$

where c is a constant.

Thus, any solution $y(x)$ of Eqn. (5) is implicitly given by Eqn. (7).

We now take up a few examples to illustrate this method.

Example 1: Solve $\frac{dy}{dx} = e^{x-y}$.

Solution: This equation can be written as

$$\frac{dy}{dx} = e^x\ e^{-y}$$

or $$e^y\frac{dy}{dx} = e^x$$

or $$\frac{d}{dx}(e^y) = e^x$$

which, on integration, gives $e^y = e^x + c$, where c is a constant.
In case $e^x + c \geq 0$, then $y(x) = \ln(e^x + c)$.

***

Example 2: Solve the initial value problem $(1+y^2)\, dx + (1+x^2)\, dy = 0$ with $y(0) = -1$.

Solution: The given equation can be rewritten in variable separable form as

$$\frac{dx}{1+x^2} + \frac{dy}{1+y^2} = 0.$$

Integrating, we get

$$\tan^{-1} x + \tan^{-1} y = c.$$

The initial condition that $y = -1$ when $x = 0$ permits us to determine the value of c and hence obtain the desired particular solution. Since $\tan^{-1} 0 = 0$ and $\tan^{-1}(-1) = -\frac{\pi}{4}$, $c = 0 - \frac{\pi}{4}$. Thus, the solution of the initial value problem is

$$\tan^{-1} x + \tan^{-1} y = -\frac{\pi}{4}. \tag{8}$$

***

You may **observe** that in Example 1 we could solve for y explicitly as a function of x whereas in Example 2 the solution is given by Eqn. (8) in an implicit form. Determining an explicit formula for the solution requires that Eqn. (8) be solved for y as a function of x which may be a formidable task. As we have already mentioned in Unit 6, unless it is important or convenient, you need not try to simplify such an expression to obtain a family of solution for y explicitly in terms of x.

Let us look at another example to illustrate the point made above.

Example 3: Solve the initial value problem

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)}, \ y \neq 1 \text{ with } y(0) = -1. \tag{9}$$

Solution: The given equation can be rewritten in variable separable form as

$$2(y-1)dy = (3x^2 + 4x + 2)dx.$$

Integrating the above equation we get

$$y^2 - 2y = x^3 + 2x^2 + 2x + c \tag{10}$$

where c is an arbitrary constant.
The initial condition that $y = -1$ when $x = 0$ gives, $c = 3$. Substituting $c = 3$ in Eqn. (10) the solution of the IVP is obtained as

$$y^2 - 2y = x^3 + 2x^2 + 2x + 3 \tag{11}$$

Eqn. (11) gives the solution of Eqn. (9) implicitly. In order to obtain the solution of IVP explicitly we must solve Eqn. (11) for y in terms of x which is possible in this case. Since Eqn. (11) is quadratic in y, we obtain

$$y = 1 \pm \sqrt{x^3 + 2x^2 + 2x + 4} \tag{12}$$

You may **observe** that Eqn. (12) gives two solutions of the IVP, viz.,

$$y = 1 + \sqrt{x^3 + 2x^2 + 2x + 4} \text{ and} \tag{13}$$

$$y = 1 - \sqrt{x^3 + 2x^2 + 2x + 4} \tag{14}$$

Out of the two solutions above, only the solution given by Eqn. (14) satisfies the given initial condition. Thus we obtain

$$y = 1 - \sqrt{x^3 + 2x^2 + 2x + 4} \tag{15}$$

as the solution of the IVP (9). Let us now determine the interval in which the solution (15) is valid. The term $\sqrt{x^3 + 2x^2 + 2x + 4}$ in Eqn. (15) is positive for $x > -2$. Thus the solution (15) is valid for $x > -2$. Eqn. (15) gives the explicit solution of Eqn. (9) valid in the region $x > -2$.

***

You may now try the following exercises:

E1) Solve the following differential equations.

i) $(1-x)\, dy - (1+y)\, dx = 0, \; x \neq 1$

ii) $\dfrac{dy}{dx} + \sqrt{\dfrac{1-y^2}{1-x^2}} = 0, \; x \neq 1$

iii) $y - x\dfrac{dy}{dx} = a\left(y^2 + \dfrac{dy}{dx}\right), \; x > 0, \; y > 0$ and a being positive constant

iv) $3e^x \tan y\, dx + (1 - e^x) \sec^2 y\, dy = 0, \; x > 0, \; y > 0$

v) $\dfrac{dy}{dx} = e^{x-y} + x^2 e^{-y}$

E2) Solve the following differential equations satisfying initial conditions indicated alongside.

i) $2xy\dfrac{dy}{dx} = 1 + y^2, \; y(2) = 3 \; \forall x, \; y > 0$

ii) $\dfrac{dy}{dx} = -4xy, \; y(0) = y_0 \; \forall \; y > 0$

iii) $\dfrac{dy}{dx} = xe^{y-x^2}, \; y(0) = 0$

iv) $y\dfrac{dy}{dx} = g, \; y(x_0) = y_0$ where g is a real constant

---

There are many differential equations like $\dfrac{dy}{dx} = \dfrac{y^2 + 2xy}{x^2}, \; x \neq 0$ that are not separable but can be reduced to the separable form by a simple algebraic substitution. In the next section we shall study one such class of equations, known as homogeneous equations. These are the equations of the form $\dfrac{dy}{dx} = f(x, y)$ where the function f does not depend on x and y separately, but only on their ratio y/x or x/y. The method of solving homogeneous differential equations of the first order was given by Leibniz in 1692.

7.3 HOMOGENEOUS EQUATIONS

Let us start by defining a homogeneous function.

Definition: A real-valued function h of two variables x and y is said to be a homogeneous function of degree n, where n is a real number, provided we can write

$$h(\lambda x,\ \lambda y) = \lambda^n h(x,\ y)$$

for all $x,\ y$ and $\lambda > 0$.

For example, $h(x,\ y) = x^3 + 2x^2 y + 3xy^2 + 4y^3$ is a homogeneous function of degree three because $h(\lambda x,\ \lambda y) = \lambda^3 h(x,\ y)$.

Function, $h(x,\ y) = x^2 \cos\left(\frac{y}{x}\right) + (\ln|x| - \ln|y|)\ xy$ is a homogeneous function of degree 2. Here $\frac{y}{x}$ has degree 0 and so

$\cos\left(\frac{y}{x}\right) = 1 + \frac{y}{x} + \left(\frac{y}{x}\right)^2 + \cdots$ has degree 0. The constant 1 also has degree zero. Also, $\ln|x| - \ln|y| = \ln\left|\frac{y}{x}\right|$ is of degree zero as well.

Function $\dfrac{x^2}{x^2 + 2xy + y^2}$ is homogeneous of degree 0. But, the function $h(x,\ y) = x^2 + 2xy + 4$ is not homogeneous because $h(\lambda x,\ \lambda y) \neq \lambda^n (x^2 + xy + 4)$ for any value of n.

We shall be particularly interested in the case where $h(x,\ y)$ is a **homogeneous function of degree** 0 that is, when $h(\lambda x,\ \lambda y) = \lambda^0 h(x,\ y) = h(x,\ y)$. Accordingly, we have the following definition.

Definition: A differential equation

$$y' = f(x,\ y) \qquad (16)$$

is called a **homogeneous differential equation** when f is a homogeneous function of degree 0.

For instance, the following equations are homogeneous differential equations:

i) $\dfrac{dy}{dx} = \dfrac{2y}{x}$,

ii) $\dfrac{dy}{dx} = \dfrac{2x + 3y}{4x} = \dfrac{2 + 3(y/x)}{4}$

iii) $\dfrac{dy}{dx} = \dfrac{x^3 + x^2 y + y^3}{3x^2 y + y^3} = \dfrac{1 + (y/x) + (y/x)^3}{3(y/x) + (y/x)^3}$

In the above examples, you may **notice** that these equations can be put in the form

$$\frac{dy}{dx} = f(x, y) = \frac{M(x, y)}{N(x, y)} \tag{17}$$

where M and N are homogeneous functions of the same degree in x and y, and f is a homogeneous function of degree 0.

Note that if f is a homogeneous function of degree n, then we can write

$$f(x, y) = x^n f(1, y/x) \text{ or } f(x, y) = y^n f(x/y, 1)$$

where $f(1, y/x)$ and $f(x/y, 1)$ are both homogeneous of degree zero.

For instance, $f(x, y) = x^2 + 3xy + y^2$ is a homogeneous function of degree 2, which can be written as

$$f(x, y) = x^2\left[1 + 3\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2\right] = x^2 f(1, y/x)$$

or $$f(x, y) = y^2\left[\left(\frac{x}{y}\right)^2 + 3\left(\frac{x}{y}\right) + 1\right] = y^2 f\left(\frac{x}{y}, 1\right).$$

Further, it follows from Eqn. (17) that when M and N are both homogeneous functions of degree n then we can write

$$\frac{dy}{dx} = f(x, y) = \frac{x^n M(1, y/x)}{x^n N(1, y/x)} = F(y/x).$$

Thus, a homogeneous differential equation can always be expressed as

$$\frac{dy}{dx} = F(y/x). \tag{18}$$

This suggests making the substitution $v = \frac{y}{x}$ to solve this equation.

For instance, consider the homogeneous differential equation

$$\frac{dy}{dx} = \frac{2x + 3y}{4x} = \frac{2 + 3(y/x)}{4} \tag{19}$$

If we substitute $v = y/x$ in Eqn. (19) then since $y(x) = xv(x)$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}.$$

Substituting from above in Eqn. (19) it reduces to

$$v + x\frac{dv}{dx} = \frac{1}{2} + \frac{3}{4}v$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{2 - v}{4}$$

$$\Rightarrow \quad \frac{dv}{dx} = \frac{2 - v}{4x}$$

$$\Rightarrow \quad \frac{dv}{2 - v} = \frac{dx}{4x} \tag{20}$$

Eqn. (20) is a variable separable equation in v and x and can be solved by the method discussed in Sec. 7.2.

In general, the substitution $v(x) = y/x$ reduces Eqn. (18) to the form

$$v + x\frac{dv}{dx} = F(v)$$

or $$\frac{dv}{dx} = \frac{F(v) - v}{x} \tag{21}$$

Eqn. (21) being in variable separable form can be solved for v in terms of x and the solution of Eqn. (18) is then given by $y = vx$.

We may mention here that instead of using the substitution $v = y/x$, you can also use $v = x/y$. At times, it could happen that after using one of these substitutions, you may come across an integral that is difficult or impossible to evaluate in closed form, switching the substitution may lead to an easier problem. In practice whenever the function $M(x, y)$ is simpler than $N(x, y)$, the substitution $x = vy$ works better. However, which substitution is to be used when, comes by practice only.

We now illustrate this method with the help of the following examples.

Example 4: Solve $\frac{dy}{dx} = \frac{2y^2 + 3xy}{x^2}$, $x > 0$.

Solution: You can easily check that the given equation is homogeneous of degree 0. It can be rewritten as

$$\frac{dy}{dx} = 2\left(\frac{y}{x}\right)^2 + 3\left(\frac{y}{x}\right) \tag{22}$$

By making the substitution, $v = \frac{y}{x}$, Eqn. (22) reduces to

$$x\frac{dv}{dx} + v = 2v^2 + 3v$$

or $$x\frac{dv}{dx} = 2v^2 + 2v = 2v(v+1)$$

or $$\frac{dv}{v(v+1)} = \frac{2dx}{x}$$

which is in variable separable form.

Resolving $\frac{1}{v(v+1)}$ into partial fractions, we have

$$\left(\frac{1}{v} - \frac{1}{v+1}\right) dv = \frac{2}{x}\, dx$$

which on integration gives

$$\ln|v| - \ln|v+1| = \ln x^2 + \ln|c| \tag{23}$$

where c is an arbitrary constant.

From Eqn. (23), we have

$$\frac{v}{v+1} = cx^2$$

Replacing v by $\frac{y}{x}$, we get

$$cx^2 = \frac{y/x}{(y/x)+1} = \frac{y}{x+y}$$

or $\quad y = \dfrac{cx^3}{1-cx^2}$,

which is the general solution of the given equation.

***

Example 5: Solve $\dfrac{dy}{dx} = \dfrac{y^3}{x^3} + \dfrac{y}{x}, \quad x > 0$.

Solution: With the substitution $y = vx$, we have

$$v + x\frac{dv}{dx} = v^3 + v$$

or $$\frac{1}{v^3}\frac{dv}{dx} = \frac{1}{x} \tag{24}$$

Integration of Eqn. (24) yields

$$-\frac{1}{2v^2} = \ln x + \ln |c|,$$

where c is a real constant. On substituting $v = \dfrac{y}{x}$ in the equation above, the general solution of the given equation can be expressed as

$$y^2 = -\frac{x^2}{2[\ln x + \ln |c|]} \text{ or } y^2 = -\frac{x^2}{2}\frac{1}{\ln(x|c|)}.$$

***

Let us consider another example.

Example 6: Solve $2x^3 y\, dx + (x^4 + y^4)\, dy = 0,\ x > 0,\ y > 0$.

Solution: The coefficient of dx here is slightly simpler than the coefficient of dy so we try the substitution $x = vy$.

After substitution, the equation reduces to

$$2v^3 y^4 [v\, dy + y\, dv] + (v^4 y^4 + y^4)\, dy = 0$$

$$\Rightarrow \quad 2v^3 y^5 dv + (3v^4 y^4 + y^4) dy = 0$$

or $$\frac{2v^3 dv}{3v^4 + 1} + \frac{dy}{y} = 0 \tag{25}$$

Integrating the above equation we get

$$\frac{1}{6}\ln\,(3v^4 + 1) + \ln\,|\,y\,| = \ln\,|\,c\,|$$

or $$\ln(3v^4 + 1)\, y^6 = 6\ln |c|$$

replacing $v = x/y$, the solution can be expressed as

$3x^4 y^2 + y^6 = c_1$, where $c_1 = 6\ln|c|$ is an arbitrary constant.

You may **note** here that the above solution is in implicit form.

Had the substitution $v = y/x$ been used in the given equation, we would have obtained $\dfrac{dx}{x} + \dfrac{v^4 + 1}{v^5 + 3v} dv = 0$. Although, evaluating the second integral in this equation is not impossible but it appears to be difficult than the one involved in Eqn. (25).

***

How about trying some exercises now?

E3) Solve the following differential equations.

i) $\frac{dy}{dx} = \frac{y}{x}$ for $x \in \,]\, 0, \infty \,[$ and for $x \in \,]\, -\infty, 0 \,[$

ii) $\frac{dy}{dx} = \frac{2x+y}{3x+2y}, \; x > 0, \; y > 0$

iii) $\left(x \sin \frac{y}{x} \right) dy - \left(y \sin \frac{y}{x} - x \right) dx = 0, \; x > 0, \; y > 0$

iv) $x \frac{dy}{dx} = y(\ln y - \ln x + 1), \; x > 0, \; y > 0$

v) $x \, dy - y \, dx = \sqrt{x^2 - y^2} \; dx$

E4) Solve the following differential equations for $x > 0, \; y > 0$, subject to the indicated initial conditions.

i) $2x^2 \frac{dy}{dx} = 3xy + y^2, \; y(1) = -2$

ii) $(x + ye^{y/x}) \, dx - xe^{y/x} \; dy = 0, \; y(1) = 0$

iii) $(y^2 + 3xy) \, dx = (4x^2 + xy) \; dy, \; y(1) = 1$

iv) $y^2 dx + (x^2 + xy + y^2) dy = 0, \; y(0) = 1$

Sometimes it may happen that a given equation is not homogeneous but can be reduced to a homogeneous form by considering a transformation of the variables. We now consider such equations.

Equations reducible to homogeneous form

Let us start by considering a differential equation

$$\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3} \tag{26}$$

You can see that Eqn. (26) is not a homogeneous equation. Let us consider the transformation of the variables x and y of the form:

$$x = X + 1, \; y = Y + 1 \tag{27}$$

with the above transformation, Eqn. (26) reduces to

$$\frac{dy}{dx} = \frac{dY}{dX} = \frac{X+1+2(Y+1)-3}{2(X+1)+Y+1-3}, \quad [\because dx = dX \text{ and } dy = dY]$$

$$\Rightarrow \quad \frac{dY}{dX} = \frac{X+2Y}{2X+Y} \tag{28}$$

which is a homogeneous equation in variables X and Y. Thus Eqn. (26) has been reduced to a homogeneous Eqn. (28) using the transformation (27) of variables x and y in terms of X and Y, respectively. But how do we arrive at a transformation given by Eqn. (27)? However, there is a well defined procedure for getting such transformation which we shall discuss now.

A differential equation of the form

$$\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'}, \; ab'-ba' \neq 0 \tag{29}$$

where a, b, c, a', b' and c' are all constants, can always be reduced to a homogeneous equation by means of the substitution

$x = X + h$ and $y = Y + k$,

Here h and k are constants which are so chosen as to make the given equation homogeneous. In terms of these new variables, Eqn. (29) becomes

$$\frac{dy}{dx} = \frac{dY}{dX} = \frac{aX+bY+(ah+bk+c)}{a'X+b'Y+(a'h+b'k+c')}. \tag{30}$$

Eqn. (30) shall reduce to homogeneous form provided h and k satisfy the following equations:

$$\left.\begin{aligned} ah+bk+c=0 \\ a'h+b'k+c'=0 \end{aligned}\right\} \tag{31}$$

Consequently, Eqn. (30) reduces to

$$\frac{dY}{dX} = \frac{aX+bY}{a'X+b'Y} \tag{32}$$

which can be solved by means of the substitution $Y = vX$.
If the solution of the Eqn. (32) is of the form

$g(X, Y) = 0$,

then the solution of Eqn. (29) is

$g(x-h, \; y-k) = 0$,

where h and k are obtained by solving the simultaneous Eqns. (31).
Solving Eqns. (31) for h and k, we get,

$$h = \frac{bc'-b'c}{ab'-a'b}, \; k = \frac{a'c-ac'}{ab'-a'b}$$

which are defined except when

$ab'-a'b=0$ that is, when $\frac{a}{a'} = \frac{b}{b'}$.

If $\frac{a}{a'} = \frac{b}{b'}$, then h and k have either infinite values or are indeterminate.

Consequently, the question arises is: **what happens if $\frac{a}{a'} = \frac{b}{b'}$?**

In such cases, we let $\frac{a}{a'} = \frac{b}{b'} = \frac{1}{m}$ (say), $m \neq 0$.
Eqn. (29) can then be written as

$$\frac{dy}{dx} = \frac{ax+by+c}{m\,(ax+by)+c'}. \tag{33}$$

On putting $ax+by=v$, Eqn. (33) reduces to

$$\frac{1}{b}\left[\frac{dv}{dx}-a\right]=\frac{v+c}{mv+c'}$$

so that the variables are separated and hence the equation can be solved by the method given in Sec. 7.2.

We now take up some examples to illustrate the above discussion.

Example 7: Solve $\frac{dy}{dx}=\frac{y-x+1}{y+x+5}$. (34)

Solution: Comparing the given equation with Eqn. (29), we have

$$a=1,\ b=-1,\ a'=1,\ b'=1.$$

$\therefore\quad \frac{a}{a'}=1,\ \frac{b}{b'}=-1$ and $\frac{a}{a'}\neq\frac{b}{b'}$

Putting $x=X+h$ and $y=Y+k$ in Eqn. (34), we get

$$\frac{dY}{dX}=\frac{Y-X+k-h+1}{Y+X+k+h+5} \tag{35}$$

We choose h and k such that

$$\left.\begin{aligned}k-h+1=0\\k+h+5=0\end{aligned}\right\} \tag{36}$$

On solving Eqns. (36), we get $h=-2$ and $k=-3$. With these values of h and k, Eqn. (35) reduces to

$$\frac{dY}{dX}=\frac{Y-X}{Y+X} \tag{37}$$

which is a homogeneous equation.
On putting $Y=vX$ in Eqn. (37) and simplifying the resulting equation, we get

$$-\frac{1+v}{1+v^2}\frac{dv}{dX}=\frac{1}{X}.$$

or $$\left(\frac{1}{1+v^2}+\frac{v}{1+v^2}\right)\frac{dv}{dX}=-\frac{1}{X} \tag{38}$$

Integration of Eqn. (38) yields

$\tan^{-1}v+\frac{1}{2}\ln\,(1+v^2)=-\ln|X|+c$, where c is a constant.

or $$\frac{1}{2}\ln(1+v^2)X^2+\tan^{-1}v=c.$$

Replacing v by $\frac{Y}{X}$, we have

$$\frac{1}{2}\ln(X^2+Y^2)+\tan^{-1}\frac{Y}{X}=c.$$

Substituting $X=x+2$ and $Y=y+3$ in the equation above, solution of Eqn. (34) is given by

$$\frac{1}{2}\ln\,[(x+2)^2+(y+3)^2]+\tan^{-1}\left(\frac{y+3}{x+2}\right)=c\,.$$

***

Example 8: Solve the differential equation $(4x+6y+5)\ dy=(3y+2x+5)\,dx$.

Solution: The given equation can be written as

$$\frac{dy}{dx}=\frac{3y+2x+5}{4x+6y+5}$$

$$=\frac{(2x+3y)+5}{2(2x+3y)+5} \qquad (39)$$

In this case $a=2,\ b=3,\ a'=4,\ b'=6$. Thus, $\frac{a}{a'}=\frac{b}{b'}$. Therefore, we put $2x+3y=v$, and Eqn. (39) reduces to,

$$\frac{1}{3}\left(\frac{dv}{dx}-2\right)=\frac{v+5}{2v+5}$$

or $$\frac{dv}{dx}=\frac{3(v+5)}{2v+5}+2=\frac{3v+15+4v+10}{2v+5}=\frac{7v+25}{2v+5}.$$

Now variables are separated and we get

$$\frac{2v+5}{7v+25}\ \frac{dv}{dx}=1$$

or $$\left[\frac{2}{7}-\frac{15}{7(7v+25)}\right]\frac{dv}{dx}=1.$$

Integrating, we get

$$\frac{2}{7}v-\frac{15}{49}\ln\left|\left(v+\frac{25}{7}\right)\right|=x+c$$, where c is a constant of integration.

Substituting $v=2x+3y$, we get the required solution as

$$\frac{2}{7}(2x+3y)-\frac{15}{49}\ \ln\ \left|\left(2x+3y+\frac{25}{7}\right)\right|=x+c\,,$$

or $$14(2x+3y)-15\ln\left|\left(2x+3y+\frac{25}{7}\right)\right|=49(x+c)$$

or $$42y-21x-15\ln|(14x+21y+25)|=49c-15\ln\ 7=c_1$$, say.

***

You may now try the exercise given below. In each of the equations in this exercise you should first check whether $\frac{a}{a'}=\frac{b}{b'}$ and then decide on the substitution.

E5) Solve the following differential equations for $x,\ y>0$.

i) $$\frac{dy}{dx}=\frac{2y-x-4}{y-3x+3}$$

ii) $$(7y-3x+3)\ \frac{dy}{dx}+(3y-7x+7)=0$$

iii) $(2x+y+1)\,dx+(4x+2y-1)\,dy=0$

iv) $(x+y)\,dx+(3x+3y-4)\,dy=0$.

In Unit 6, we defined the total differential of a given function. In the next section we shall make use of this to define and solve exact differential equations.

7.4 EXACT EQUATIONS

Let us start with a very simple equation $x\,dy+y\,dx=0$. You know that this equation is not only separable and homogeneous but it is also equivalent to the differential of the product of x and y. That is $y\,dx+x\,dy=d(xy)=0$. Integrating this equation we obtain a one parameter family of solution $xy=c$. Conversely, the differential equation corresponding to this one parameter family of solutions is given by $d(xy)=0$ or $x\,dy+y\,dx=0$.

In general, if we consider a family of curves $h(x, y)=c$, then its differential equation can be written in terms of its total differential as

$$dh=0,$$

or $$\frac{\partial h}{\partial x}dx+\frac{\partial h}{\partial y}dy=0.$$

In the converse situation let us begin with the differential equation

$$M(x, y)dx+N(x, y)dy=0. \tag{40}$$

If there exists a function h of variables x and y such that

$$\frac{\partial h}{\partial x}=M(x, y) \text{ and } \frac{\partial h}{\partial y}=N(x, y),$$

then Eqn. (40) can be written in the form

$$\frac{\partial h}{\partial x}dx+\frac{\partial h}{\partial y}dy=0 \text{ or } dh=0.$$

That is, $h=$ constant, represents a solution of Eqn. (40).

We then call the expression $M(x, y)dx+N(x, y)dy$ an **exact differential** and Eqn. (40) is called an **exact differential equation**. For instance, equation $x^2y^3dx+x^3y^2dy=0$ is exact, since we have $d\left(\frac{1}{3}x^3y^3\right)=x^2y^3dx+x^3y^2dy$.

Formally, we give the following definitions:

Definition: An expression $M(x, y)\,dx+N(x, y)\,dy$ is an **exact differential** in a region D of the xy-plane if it corresponds to the total differential of some function h, i.e., $dh=M(x, y)dx+N(x, y)dy$.

Definition: An equation

$$M(x, y)dx+N(x, y)dy=0$$

is said to be an **exact differential equation** if the expression $M(x, y)dx+N(x, y)dy$ is an exact differential.

It is sometimes possible to determine exactness and find the function h by mere inspection. Consider, for example, the equations

$$3x^2y^4dx + 4x^3y^3dy = 0$$

and $\quad xe^{xy}dy + (ye^{xy} - 2x)\,dx = 0$.

The first equation can be straight way written as $d(x^3y^4) = 0$ and hence $x^3y^4 = c$, a constant, is its general solution. The second equation can be alternatively written as $(xe^{xy}dy + ye^{xy}dx) - 2x\,dx = 0$ or, $d(e^{xy} - x^2) = 0$. Its general solution would then be $e^{xy} - x^2 = c$ or $e^{xy} = x^2 + c$, where c is a constant.

However, except for some simple cases, the technique of "solution by inspection" does not work. Consequently, we seek an answer to the following question: When does a function h exist such that Eqn. (40) is exact? An answer to this question is given by the following theorem.

Theorem 1: Let M and N be continuous functions of x, and y having continuous first order partial derivatives in a region $D: a < x < b,\ c < y < d$. Then a **necessary** and **sufficient** condition that

$$M(x,\ y)dx + N(x,\ y)dy$$

be an exact differential is

$$\frac{\partial}{\partial y}M(x,\ y) = \frac{\partial}{\partial x}N(x,\ y) \text{ at each point of } D. \qquad (41)$$

We now give the proof of Theorem 1.

Proof: The condition is necessary

Let us assume that the expression

$$M(x,\ y)dx + N(x,\ y)dy$$

is exact. Then there exists a function h of variables x and y such that

$$dh = M(x,\ y)\ dx + N(x,\ y)\ dy.$$

But $\quad dh = \dfrac{\partial h}{\partial x}dx + \dfrac{\partial h}{\partial y}dy$.

$$\therefore\quad \frac{\partial h}{\partial x}\,dx + \frac{\partial h}{\partial y}dy = M(x,\ y)dx + N(x,\ y)\ dy$$

Thus, necessarily,

$$\frac{\partial h}{\partial x} = M(x,\ y) \text{ and } \frac{\partial h}{\partial y} = N(x,\ y) \qquad (42)$$

Since M and N have continuous first order partial derivatives and $M = \dfrac{\partial h}{\partial x}$ and $N = \dfrac{\partial h}{\partial y}$, h possesses continuous second order partial derivatives namely; $\dfrac{\partial^2 h}{\partial y\,\partial x}$ and $\dfrac{\partial^2 h}{\partial x\,\partial y}$.

Now,

$$\frac{\partial}{\partial y}\left(\frac{\partial h}{\partial x}\right) = \frac{\partial^2 h}{\partial y\,\partial x} = \frac{\partial M(x,\ y)}{\partial y} \qquad (43)$$

and $\dfrac{\partial}{\partial x}\left(\dfrac{\partial h}{\partial y}\right)=\dfrac{\partial^2 h}{\partial x\,\partial y}=\dfrac{\partial N(x,\ y)}{\partial x}$ (44)

Since $\dfrac{\partial^2 h}{\partial y\,\partial x}$ and $\dfrac{\partial^2 h}{\partial x\,\partial y}$ are continuous we have $\dfrac{\partial^2 h}{\partial y\,\partial x}=\dfrac{\partial^2 h}{\partial x\,\partial y}$. (ref. Unit 4, Block-1).

Therefore, from Eqns. (43) and (44), we get

$$\frac{\partial N(x,\ y)}{\partial x}=\frac{\partial M(x,\ y)}{\partial y}.$$

The condition is sufficient: The sufficiency part of Theorem consists of showing that there exists a function h for which $\dfrac{\partial h}{\partial x}=M(x,\ y)$ and $\dfrac{\partial h}{\partial y}=N(x,\ y)$ whenever condition (41) holds.

Now let us start with an assumption that

$$\frac{\partial M(x,\ y)}{\partial y}=\frac{\partial N(x,\ y)}{\partial x}. \tag{45}$$

The existence of a function h then amounts to showing that $M(x,\ y)dx+N(x,\ y)dy$ is an exact differential.

Let $\int M(x,\ y)\ dx=h(x,\ y)$, then $\dfrac{\partial h}{\partial x}=M(x,\ y)$

and $\dfrac{\partial^2 h}{\partial y\,\partial x}=\dfrac{\partial M(x,\ y)}{\partial y}$

$=\dfrac{\partial N(x,\ y)}{\partial x}$ (using Condition (45))

$$\therefore\quad \frac{\partial N(x,\ y)}{\partial x}=\frac{\partial}{\partial x}\left(\frac{\partial h}{\partial y}\right) \tag{46}$$

Integration of Eqn. (46) with respect to x, holding y fixed, yields

$$N(x,\ y)=\frac{\partial h}{\partial y}+\phi(y)$$

where ϕ, a function of y only, is a constant of integration. Thus,

$$M(x,\ y)dx+N(x,\ y)dy=\frac{\partial h}{\partial x}dx+\left(\frac{\partial h}{\partial y}+\phi(y)\right)dy$$

$$=d\left[h(x,\ y)+\int_0^y \phi(t)\ dt\right]$$

which establishes that $M(x,y)\ dx+N(x,\ y)dy$ is an exact differential. This completes the proof of Theorem 1.

– ■ –

We now take up a few examples to illustrate the above theorem.

Example 9: Solve the differential equation $\sin y+x\cos y\ y'=0$.

Solution: For the case under consideration, $N(x, y) = x\cos y$ and $M(x, y) = \sin y$. Also, $\frac{\partial}{\partial x}N(x, y) = \cos y = \frac{\partial}{\partial y}M(x, y)$ which shows that the given equation is an exact equation.

Therefore, there exists a function h such that, $\frac{\partial h}{\partial x} = M(x, y)$ and $\frac{\partial h}{\partial y} = N(x, y)$. We thus have

$$\frac{\partial h}{\partial x} = \sin y \tag{47}$$

and

$$\frac{\partial h}{\partial y} = x\cos y. \tag{48}$$

Integrating Eqn. (47) with respect to x, treating y as a constant, we get

$$h(x, y) = x\sin y + \phi(y) \tag{49}$$

where $\phi(y)$ is a constant of integration. Differentiating Eqn. (49) partially w.r.t. y, we get

$$\frac{\partial}{\partial y}h(x, y) = x\cos y + \phi'(y). \tag{50}$$

From Eqns. (48) and (50), we get

$$x\cos y = x\cos y + \phi'(y)$$

which shows that $\phi'(y) = 0 \Rightarrow \phi(y) =$ constant $= c_1$ (say). (51)

From Eqn. (49), we can then write $h(x, y) = x\sin y + c_1$.

Hence the required solution is $x\sin y + c_1 = 0$.

Note that the solution of the equation is not just $h(x, y)$. Rather it is $h(x, y) = c$ or $h(x, y) = 0$ if a constant has already been used in the integration of $\phi'(y)$ as in Eqn. (51).

***

Example 10: Solve $e^x \sin y + e^x \cos y\ y' + 2x = 0$.

Solution: Comparing the given equation with Eqn. (40), we have $N(x, y) = e^x \cos y$ and $M(x, y) = e^x \sin y + 2x$. Therefore,

$$\frac{\partial}{\partial x}N(x, y) = e^x \cos y$$

and $\frac{\partial}{\partial y} M(x, y) = e^x \cos y = \frac{\partial}{\partial x} N(x, y)$.

Hence the given equation is exact and can be written in the form $dh(x, y) = 0$ for some function h where

$$\frac{\partial}{\partial x} h(x, y) = e^x \sin y + 2x \tag{52}$$

and

$$\frac{\partial}{\partial y} h(x, y) = e^x \cos y. \tag{53}$$

Integrating Eqn. (52) w.r.t. x, we get

$$h(x, y) = e^x \sin y + x^2 + \phi(y) \tag{54}$$

where ϕ, a function of y only, is a constant of integration.
From Eqns. (53) and (54), we get

$$\frac{\partial}{\partial y} h(x, y) = e^x \cos y + \phi'(y) = e^x \cos y$$

So we have $\phi'(y) = 0$ or $\phi(y) = c_1$ where c_1 is a constant.
Hence from Eqn. (54) we have the required solution as

$$h(x, y) = e^x \sin y + x^2 + c_1 = 0$$

***

You may **observe** here that Example 9 could also be solved by the separation of variables. But in Example 10, the variables cannot be separated.

On the basis of Theorem 1 and Examples (9) and (10) we can sum up the various steps involved in solving an exact differential equation $M(x, y)dx + N(x, y)dy = 0$ as follows:

Step 1: Integrate $M(x, y)$ w.r.t. x, treating y as a constant.

Step 2: Integrate, with respect to y, those terms in $N(x, y)$ which do not involve x.

Step 3: The sum of the two expressions obtained in Steps 1 and 2 equated to a constant is the required solution.

We now illustrate these steps with the help of another example.

Example 11: Solve $(x^2 - 4xy - 2y^2)\, dx + (y^2 - 4xy - 2x^2)\, dy = 0$. (55)

Solution: Here $N(x, y) = y^2 - 4xy - 2x^2$ and $M(x, y) = x^2 - 4xy - 2y^2$

$\therefore \quad \frac{\partial N}{\partial x} = -4y - 4x$ and $\frac{\partial M}{\partial y} = -4x - 4y$

$\therefore \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$; hence Eqn. (55) is an exact equation.

Step 1: Integrating $M(x, y)$ w.r.t. x, treating y as a constant, we have

$$\int (x^2 - 4xy - 2y^2)\, dx = \frac{x^3}{3} - 2x^2 y - 2xy^2.$$

Step 2: We integrate those terms in $N(x, y)$ w.r.t. y which do not involves x. There is only one such term namely, y^2

$$\therefore \quad \int y^2 dy = \frac{y^3}{3}.$$

Step 3: The required solution is the sum of expressions obtained from Steps 1 and 2 equated to a constant, that is

$$\frac{x^3}{3} - 2x^2 y - 2xy^2 + \frac{y^3}{3} = c_1,$$

or $x^3 - 6x^2y - 6xy^2 + y^3 = c$,

where c and c_1 are constants.

***

Note that Eqn. (55) is a homogeneous differential equation and hence could also have been solved by using the substitution $v = y/x$. Further, note that examining a given equation for exactness and the general procedure for finding its solution can sometimes be simplified. We can pick out those terms of $M(x, y)\,dx + N(x, y)\,dy = 0$ that obviously form an exact differential or can take the form $f(u)\,du$. The remaining expression which is less cumbersome than the original can then be tested and integrated. This procedure is illustrated through the following example.

Example 12: Solve $xdx + ydy + \dfrac{xdy - ydx}{x^2 + y^2} = 0$.

Solution: Note that the first two terms on the left hand side of the given equation are exact differential and hence need not be touched. Dividing the numerator and denominator of the last term by x^2, we get

$$xdx + ydy + \frac{d(y/x)}{1 + (y/x)^2} = 0.$$

Now each term of the above equation is an exact differential. Integrating, we get

$$\frac{x^2}{2} + \frac{y^2}{2} + \tan^{-1}\frac{y}{x} = c$$

as the required solution with c as a constant.

***

You may now try the following exercises.

E6) Show that the following equations are exact and hence solve them.

i) $y\cos x + 2x\,e^y + (\sin x + x^2e^y + 2)y' = 0$.

ii) $y' = -\dfrac{ax + by}{bx + cy}$ (a, b, c, d are given real constants).

iii) $(6x + y/x) + (\ln x + y)\ y' = 0,\ x \geq 1$.

iv) $\left(\dfrac{1}{x} + \dfrac{1}{x^2} - \dfrac{y}{x^2 + y^2}\right)dx + \left(ye^y + \dfrac{x}{x^2 + y^2}\right)dy = 0$.

E7) Determine the values of k for which the equations given below are exact and find the solution for these values of k.

i) $x + kyy' = 0\ (k \neq 0)$.

ii) $y + kxy' = 0\ (k \neq 0)$.

iii) $(2ye^{2xy} + 2x) + k\ x\ e^{2xy}y' = 0$.

In practice the differential equations of the form $M(x, y)dx + N(x, y)dy = 0$ are rarely exact as the condition in Theorem 1 requires a precise balance of the functions M and N. But they can often be transformed into exact equations on multiplication by a suitable function $F \neq 0$. This function is then called an **integrating factor** of the differential equation. The question we, now, must ask is: if

$$M(x, y)dx + N(x, y)dy = 0$$

is not exact, then how to find a function

$F \neq 0$ so that

$$F[N\ dy + M\ dx] = 0$$

is exact? In the next section we shall give an answer to this question.

7.5 INTEGRATING FACTOR

Let us begin with a very simple equation, namely,

$$y' + y = 0 \tag{56}$$

In this case $N(x, y) = 1$ and $M(x, y) = y$. Here $\frac{\partial}{\partial x} N(x, y) = 0$ and $\frac{\partial}{\partial y} M(x, y) = 1$ and hence the given equation is not exact. Let us multiply Eqn. (56) by e^x to get

$$e^x y' + e^x y = 0 \tag{57}$$

You may now check that Eqn. (57) is an exact equation. Thus Eqn. (56) is not exact whereas when we multiply Eqn. (56) by e^x the resulting equation becomes an exact equation. Here e^x is termed as an **integrating factor** for Eqn. (56). How did we think of this integrating factor e^x?. It is not just by hit and trial. There are ways of finding integrating factors for given differential equations which we shall be discussing now.

We first give the following definition.

Definition: A non-zero function, which makes a non-exact differential equation exact once multiplied with it, is known as an integrating factor (abbreviated as I.F.) of the differential equation.

The term I.F., to solve a differential equation, was first introduced by Fatio de Dullier a Swiss Mathematician, in 1687.
For a given equation, there may not be a unique integrating factor.
Consider, for example, the equation

$$ydx - xdy = 0,\ x \neq 0,\ y \neq 0. \tag{58}$$

You can check that Eqn. (58) is not exact, but when multiplied by $\frac{1}{y^2}$, it becomes $\frac{ydx - xdy}{y^2} = 0$, which is exact. This can now be written as $d\left(\frac{x}{y}\right) = 0$ and thus has the solution $\frac{x}{y} = c$ with c being an arbitrary constant.

Further, when Eqn. (58) is multiplied by $\frac{1}{xy}$, it becomes

$$\frac{dx}{x}-\frac{dy}{y}=0,$$

which is again exact and has its solution as $\ln|x|-\ln|y|=c$.

You may **notice** that this solution can be transformed to the solution obtained earlier.

Further, Eqn. (58) when multiplied by $\frac{1}{x^2}$ reduces to an exact equation

$\frac{y}{x^2}dx-\frac{dy}{x}=0$ or, $-d\left(\frac{y}{x}\right)=0$, with $-\frac{y}{x}=c$ as its solution.

Thus, we have seen that some of the integrating factors for Eqn. (58) are $\frac{1}{y^2}, \frac{1}{xy}$ and $\frac{1}{x^2}$.

Now the question arises: Is this the case only with Eqn. (58) or, does an equation of the form $M(x, y)dx+N(x, y)dy=0$ in general, have infinitely many integrating factors?

An answer to this question is given in Theorem 2. Before we discuss this theorem, here is an exercise for you.

E8) In each of the following equations check whether the function F, whose value at (x, y) is indicated alongside a differential equation, is an I.F. of the equation.

i) $6xy\ dx+(4y+9x^2)dy=0;\ F(x, y)=y^2$

ii) $-y^2dx+(x^2+xy)\ dy=0;\ F(x, y)=\frac{1}{xy},\ x\neq 0,\ y\neq 0$

iii) $(-xy\sin x+2y\cos x)\ dx+2x\cos x\ dy=0;\ F(x, y)=xy$

Let us now take up Theorem 2.

Theorem 2: The number of integrating factors for the equation

$$M(x, y)dx+N(x, y)dy=0$$

is infinite.

Proof: Let $g(x, y)$ be an integrating factor of the given equation. Then, by definition

$$g(x, y)[M(x, y)dx+N(x, y)dy]=0 \qquad (59)$$

is an exact differential equation.

Therefore, there exists a function h such that $dh=g(x, y)[M(x, y)dx+N(x, y)dy]$ and $h(x, y)=$constant, is a solution of the given equation.

Let f be an arbitrary function of $h(x, y)$ (f ultimately is a function of x and y as h itself is a function of (x, y)). Then

$$g(x, y).f(h)[M(x, y)dx+N(x, y)dy]=f(h)\ dh=d\left[\int_0^h f(t)dt\right] \qquad (60)$$

Since the term on the right hand side of Eqn. (60) is an exact differential, the term in the left must also be an exact differential. Therefore, $g(x, y).f(h)$ is an integrating factor of the given differential equation.

Since f is an arbitrary function of h, hence the number of integrating factors for equation $M(x, y)dx + N(x, y)dy = 0$ is infinite.

– ■ –

It may be **remarked** here that this fact is of no special assistance in finding an integrating factor of the given equation. Ordinarily, finding an integrating factor of a given equation is as difficult as solving the original equation. Therefore, while in principle integrating factors are powerful tools for solving differential equations, in practice they can usually be found only in special cases. However, rules for finding an integrating factor do exist. We shall now take up these rules one by one.

Rules for finding integrating factors

Rule I: Integrating factors obtainable by inspection: Sometimes integrating factors of a differential equation can be seen at a glance, as is the case with Eqn. (58) above. We give below some more examples in this regard.

Example 13: Solve $(1+xy)\, y\ dx + (1-xy)x\ dy = 0,\ x > 0,\ y > 0$. (61)

Solution: Rearranging the terms of Eqn. (61), we get

$$ydx + xdy + xy^2dx - x^2ydy = 0$$

$$\Rightarrow\ d(xy) + xy^2dx - x^2ydy = 0 \tag{62}$$

Looking at the form of Eqn. (62), the immediate observation is that integrating factor of the form $\frac{1}{x^2y^2}$ could be used to make the equation exact. We multiply Eqn. (62) by $\frac{1}{x^2y^2}$ and obtain the equation

$$\frac{d(xy)}{x^2y^2} + \frac{dx}{x} - \frac{dy}{y} = 0$$

which is integrable and on integration, yields,

$$-\frac{1}{xy} + \ln\ x - \ln\ y = \ln\ c,$$

or $y = cxe^{-1/xy}$ (where c is a constant) is the required solution.

***

Example 14: Solve $(x^4e^x - 2my^2x)dx + 2mx^2y\ dy = 0,\ x > 0,\ y > 0$.

Solution: We can write the given equation as

$$x^4e^xdx + 2m(x^2ydy - xy^2dx) = 0$$

$$\Rightarrow\ x^4e^xdx + 2mx^3y\ d\left(\frac{y}{x}\right) = 0$$

Dividing by x^4 we get

$$e^xdx + 2m\,\frac{y}{x}\,d\left(\frac{y}{x}\right) = 0$$

$$\Rightarrow d\left[e^x + m\left(\frac{y}{x}\right)^2\right] = 0 \text{, which is an exact differential.}$$

Thus $\frac{1}{x^4}$ has served the role of an integrating factor in this case.

The required solution is, then, given by $e^x + m\left(\frac{y}{x}\right)^2 = c$ with c as an arbitrary constant.

***

We would like to mention here that determination of an integrating factor by inspection is a skill which can be developed through practice only.

You may now test your skill of finding an integrating factor through inspection while doing the following exercise.

E9) Solve the following equations.

i) $y(2yx+e^x)dx - e^x dy = 0,\ x \neq 0,\ y \neq 0$

ii) $ydx - xdy + \ln x\ dx = 0\ \forall\, x,\ y > 0$

iii) $(xy - 2y^2)\,dx - (x^2 - 3xy)\ dy = 0\ \forall\, x,\ y > 0$.

We are now giving a rule which applies to the equation $M(x,\ y)dx + N(x,\ y)dy = 0$ only when it is homogeneous.

Rule II: For a homogeneous equation $M(x,\ y)dx + N(x,\ y)\ dy = 0$, when $Mx + Ny \neq 0$, then $\frac{1}{Mx+Ny}$ is an integrating factor.

Proof: Consider the equation

$$M(x,\ y)dx + N(x,\ y)dy = 0$$

Now $Ndy + Mdx = \frac{1}{2}\left[(Mx+Ny)\left(\frac{dx}{x}+\frac{dy}{y}\right) + (Mx-Ny)\left(\frac{dx}{x}-\frac{dy}{y}\right)\right]$

$$\therefore\ \frac{Ndy+Mdx}{Ny+Mx} = \frac{1}{2}\left[\left(\frac{dx}{x}+\frac{dy}{y}\right) + \frac{Mx-Ny}{Mx+Ny}\left(\frac{dx}{x}-\frac{dy}{y}\right)\right]$$

Since the given equation is homogeneous, M and N are of the same degree in x and y and therefore $\frac{Mx-Ny}{Mx+Ny}$ can be written as a function of $\frac{x}{y}$, $y \neq 0$, say $f\left(\frac{x}{y}\right)$.

$$\therefore\ \frac{Ndy+Mdx}{Ny+Mx} = \frac{1}{2}\left[\left(\frac{dx}{x}+\frac{dy}{y}\right) + f\left(\frac{x}{y}\right)\left(\frac{dx}{x}-\frac{dy}{y}\right)\right]$$

$$= \frac{1}{2}\left[d\ (\ln|xy|) + f\,(e^{\ln|x/y|})\ d\left(\ln\left|\frac{x}{y}\right|\right)\right]$$

$$= \frac{1}{2}\left[d(\ln|xy|) + dF\left(\ln\left|\frac{x}{y}\right|\right)\right] \text{ where } dF\left(\ln\left|\frac{x}{y}\right|\right) = f(e^{\ln|x/y|})\, d\left(\ln\left|\frac{x}{y}\right|\right).$$

$$= d\left[\frac{1}{2}\ln|xy| + \frac{1}{2}F\left(\ln\left|\frac{x}{y}\right|\right)\right] \quad (63)$$

Since the right hand side of Eqn. (63) is an exact differential, it shows that $\frac{1}{Ny+Mx}$ is an integrating factor for the homogeneous equation $M(x, y)dx + N(x, y)dy = 0$.

– ■ –

We take up examples to illustrate this rule.

Example 15: Solve $(x^2y - 2xy^2)\, dx - (x^3 - 3x^2y)\, dy = 0\ \forall\ x, y > 0$

Solution: Here the given equation is homogeneous with

$$N(x, y) = -x^3 + 3x^2y \text{ and } M(x, y) = x^2y - 2xy^2$$

$$\therefore\ Mx + Ny = x(x^2y - 2xy^2) + y(-x^3 + 3x^2y) = x^2y^2 \neq 0.$$

$\therefore\ \frac{1}{x^2y^2}$ is an integrating factor.

Multiplying the given differential equation by $\frac{1}{x^2y^2}$, we get

$$\left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0,$$

or $$\left(\frac{dx}{y} - \frac{x}{y^2}dy\right) + 3\frac{dy}{y} - 2\frac{dx}{x} = 0,$$

or $$d\left(\frac{x}{y}\right) + d(3\ln y - 2\ln x) = 0.$$

Therefore, the solution is

$$\frac{x}{y} + 3\ln y - 2\ln x = c_1,$$

or $y^3 = cx^2e^{-x/y}$ where c_1 and c are constants.

***

Note: In case $Mx + Ny = 0$, then $\frac{M}{N} = -\frac{y}{x}$ and the given equation $Mdx + Ndy = 0$ reduces to $\frac{dy}{dx} = \frac{y}{x}$, whose solution is straightaway obtained as $x = cy$.

Example 16: Solve $y^2 + x^2\frac{dy}{dx} = xy\frac{dy}{dx},\ x > 0,\ y > 0$.

Solution: The given equation can be written as $y^2dx + (x^2 - xy)dy = 0$. It is a homogeneous equation with $N(x, y) = x^2 - xy^2$ and $M(x, y) = y^2$.

$\therefore \quad Mx + Ny = xy^2 + yx^2 - xy^2 = yx^2 \neq 0$

$\therefore \quad \dfrac{1}{yx^2}$ is an integrating factor.

Multiplying the given equation by $\dfrac{1}{yx^2}$ we get

$$\frac{y}{x^2} dx + \left(\frac{1}{y} - \frac{1}{x}\right) dy = 0$$

Now $M(x, y) = \dfrac{y}{x^2}$, $N(x, y) = \dfrac{1}{y} - \dfrac{1}{x}$ and $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ thus the equation is exact.

Step 1: Integrating $M(x, y)$ w.r.t. x, treating y as a constant, we have

$$\int \frac{y}{x^2} dx = -\frac{y}{x}.$$

Step 2: Integrating those terms in $N(x, y)$ w.r.t. y which do not involve x, we get

$$\int \frac{1}{y} dy = \ln\ y$$

Step 3: The required solution is

$$-\frac{y}{x} + \ln\ y = c$$

or $\quad \dfrac{y}{x} = \ln\ (c_1 / y)$, where c and c_1 are constants.

***

You may now try this exercise.

E10) Solve the following equations

i) $\quad (x^4 + y^4)\ dx - xy^3\ dy = 0, \ x, \ y > 0$

ii) $\quad (x^2 + 3xy + y^2)dx - x^2 dy = 0, \quad x, \ y > 0.$

Let us consider another rule for finding I.F.

Rule III: When $Mx - Ny \neq 0$ and the differential equation $N(x, y)dy + M(x, y)dx = 0$ can be written in the form $y\, f_1(xy)dx + x\, f_2(xy)\, dy = 0$, where $f_1(xy)$ and $f_2(xy)$ are functions of xy, then $\dfrac{1}{Mx - Ny}$ is an integrating factor.

Proof: If equation $M(x, y)dx + N(x, y)dy = 0$ can be written in the form $y\, f_1(xy)dx + x\, f_2(xy)dy = 0$ then evidently,

$$\frac{N}{x\, f_2(xy)} = \frac{M}{y\, f_1(xy)} = \lambda, \text{ (say) where } x\, f_2(xy) \neq 0 \text{ and } y\, f_1(xy) \neq 0$$

$\therefore \quad N = \lambda x f_2(xy)$ and $M = \lambda y f_1(xy)$

Also $Ndy + Mdx = \frac{1}{2}\left[(Mx+Ny)\left(\frac{dx}{x}+\frac{dy}{y}\right)+(Mx-Ny)\left(\frac{dx}{x}-\frac{dy}{y}\right)\right]$

$$\therefore \quad \frac{Ndy+Mdx}{Mx-Ny} = \frac{1}{2}\left[\frac{Mx+Ny}{Mx-Ny}\left(\frac{dx}{x}+\frac{dy}{y}\right)+\left(\frac{dx}{x}-\frac{dy}{y}\right)\right]$$

$$= \frac{1}{2}\left[\frac{f_1+f_2}{f_1-f_2} d(\ln|xy|) + d\left(\left|\ln\frac{x}{y}\right|\right)\right]$$

$$= \frac{1}{2}\left[f(xy)\, d(\ln|xy|) + d\left(\left|\ln\frac{x}{y}\right|\right)\right] \text{ where } f(xy) = \frac{f_1+f_2}{f_1-f_2}$$

$$= \frac{1}{2}\left[d\,F(\ln|xy|) + d\left(\ln\left(\frac{x}{y}\right)\right)\right]$$

$$= d\left[\frac{1}{2}\ln\left|\frac{x}{y}\right| + \frac{1}{2}F(\ln|xy|)\right]$$

which is an exact differential where, $dF(\ln|xy|) = f(xy)\, d(\ln|xy|)$.

Hence, $\frac{1}{Mx-Ny}$ is an integrating factor.

– ■ –

We now illustrate this rule through examples.

Example 17: Solve $y(xy+2x^2y^2)\,dx + x(xy-x^2y^2)\,dy = 0, \forall\, x,\ y>0$.

Solution: Here $N = x(xy-x^2y^2)$ and $M = y(xy+2x^2y^2)$

$$\therefore \quad Mx - Ny = xy[xy + x^2y^2 - xy + 2x^2y^2]$$

$$= 3x^3y^3 \neq 0.$$

$\therefore \quad \frac{1}{3x^3y^3}$ is an I.F.

Multiplying the given equation by $\frac{1}{3x^3y^3}$, we get

$$\frac{1}{3x^3y^2}(xy+2x^2y^2)\,dx + \frac{1}{3x^2y^3}(xy-x^2y^2)\,dy = 0$$

or $$\frac{dx}{3x^2y} + \frac{2dx}{3x} + \frac{dy}{3xy^2} - \frac{dy}{3y} = 0$$

or $$\left[\frac{dx}{3x^2y} + \frac{dy}{3xy^2}\right] + \frac{2}{3}\frac{dx}{x} - \frac{1}{3}\frac{dy}{y} = 0$$

or $$d\left(-\frac{1}{3}\frac{1}{xy} + \frac{2}{3}\ln x - \frac{1}{3}\ln y\right) = 0.$$

Therefore, the solution is

$-\frac{1}{3xy}+\frac{2}{3}\ln x-\frac{1}{3}\ln y=c_1$ where c_1 is a constant.

or $-\frac{1}{xy}+\ln x^2-\ln y=3c_1=c$ for c being a constant.

or, $\ln\left(\frac{x^2}{y}\right)=c+\frac{1}{xy}$.

***

Note: If $Mx-Ny=0$, i.e., $\frac{M}{N}=\frac{y}{x}$, then given equation $Mdx+Ndy=0$ will be of the form $\frac{dy}{dx}=-\frac{y}{x}$ and has a solution $xy=c$.

Example 18: Solve
$(xy\sin xy+\cos xy)y\ dx+(xy\sin xy-\cos xy)x\ dy=0,\ x>0,\ y>0$.

Solution: Here $N(x,\ y)=(xy\sin xy-\cos xy)x$ and
$M(x,\ y)=(xy\sin xy+\cos xy)y$

$$\therefore\quad Mx-Ny=xy[xy\sin xy+\cos xy-xy\sin xy+\cos xy]$$
$$=2xy\cos xy\neq 0$$

$\therefore\quad \frac{1}{2xy\cos xy}$ is an I.F.

Multiplying both sides of the given equation by $\frac{1}{2xy\cos xy}$, we get

$$\frac{1}{2}\left(y\tan xy+\frac{1}{x}\right)dx+\frac{1}{2}\left(x\tan xy-\frac{1}{y}\right)dy=0.$$

or $\quad \frac{1}{2}(y\tan xy\ dx+x\tan xy\ dy)+\frac{1}{2x}dx-\frac{1}{2y}dy=0$

or $\quad \frac{1}{2}\mathrm{d}\ [\ln\sec\ (\mathrm{xy})]+\frac{1}{2}\left[\frac{\mathrm{dx}}{\mathrm{x}}-\frac{\mathrm{dy}}{\mathrm{y}}\right]=0$

Integrating the above equation, we get

$$\ln\sec\ (xy)+\ln x-\ln y=\ln c$$

or $\quad \ln\left[\frac{x}{y}\sec(xy)\right]=\ln c$

or $\quad \frac{x}{y}\sec xy=c$ for c being an arbitrary constant as the required solution.

***

Before we go to the next rule here is an exercise for you.

E11) Solve the following equations

i) $y(x^2y^2+2)\ dx+x(2-2x^2y^2)\ dy=0,\ x>0,\ y>0$.

ii) $(x^2y^2+xy+1)y\ dx+(x^2y^2-xy+1)x\ dy=0,\ x>0,\ y>0$.

We now give another rule for finding an integrating factor of a given differential equation.

Rule IV: When $\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)$ is a function of x alone, say $f(x)$, then $e^{\int f(x)\,dx}$ is an I.F. of the equation $Mdx+Ndy=0$.

Proof: Consider the equation $e^{\int f(x)\,dx}(Mdx+Ndy)=0$ (64)

Let $r=M\,e^{\int f(x)dx}$ and $s=N\,e^{\int f(x)dx}$

Then Eqn. (64) reduces to $r\,dx+s\,dy=0$

Now, $\frac{\partial r}{\partial y}=\frac{\partial M}{\partial y}e^{\int f(x)\,dx}$

and $\frac{\partial s}{\partial x}=\frac{\partial N}{\partial x}e^{\int f(x)\,dx}+N\,e^{\int f(x)\,dx}.f(x)$

$$=e^{\int f(x)\,dx}\left[\frac{\partial N}{\partial x}+N\;f(x)\right]$$

$$=e^{\int f(x)\,dx}\left[\frac{\partial N}{\partial x}+\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right] \qquad \left[\text{because } \frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=f(x)\right]$$

$$=\frac{\partial M}{\partial y}e^{\int f(x)\,dx}$$

$$=\frac{\partial r}{\partial y}$$

Therefore, $\frac{\partial s}{\partial x}=\frac{\partial r}{\partial y}$ showing that the equation $r\;dx+s\;dy=0$ i.e., $e^{\int f(x)\,dx}(Mdx+Ndy)=0$, is an exact one.

Hence $e^{\int f(x)\,dx}$ is an I.F. of the equation $Mdx+Ndy=0$.

– ■ –

We illustrate this rule with the help of an example.

Example 19: Solve $(x^2+y^2)\;dx-2xydy=0,\; x>0,\; y>0$.

Solution: Here $N=-2xy,\; M=x^2+y^2$

$\therefore\; \frac{\partial M}{\partial y}=2y$ and $\frac{\partial N}{\partial x}=-2y$. Thus, $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$.

Here $\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{1}{-2xy}(2y+2y)=-\frac{2}{x}=f(x)$, which is a function of x alone.

$\therefore\; I.F.=e^{\int f(x)\,dx}=e^{\int -\frac{2}{x}dx}=e^{-2\ln|x|}=\frac{1}{x^2}$.

Multiplying the given equation by $\frac{1}{x^2}$, we get

$$\frac{1}{x^2}(x^2+y^2)\;dx-\frac{2y}{x}dy=0,$$

i.e., $dx+\frac{y^2}{x^2}\,dx-\frac{2y}{x}dy=0$.

i.e., $dx + d\left(-\dfrac{y^2}{x}\right) = 0$ (65)

Integrating Eqn. (65), the required solution is obtained as

$$x - \frac{y^2}{x} = c \text{ (a constant).}$$

***

You may now try this exercise.

E12) Solve the following differential equations for $x > 0,\ y > 0$.

i) $\left(y + \dfrac{1}{3}y^3 + \dfrac{1}{2}x^2\right)dx + \dfrac{1}{4}(x + xy^2)dy = 0$

ii) $(x^2 + y^2 + x)\,dx + xy\,dy = 0$.

Rule V: When $\dfrac{1}{M}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right)$ is a function of y alone, say $f(y)$, then $e^{-\int f(y)\,dy}$ is an I.F. of the differential equation $Mdx + Ndy = 0$.

The proof of this rule is similar to the proof of Rule IV above and we are leaving this as an exercise for you.

E13) Prove Rule V above.

We however illustrate the use of Rule V through the following examples.

Example 20: Solve $(3x^2y^4 + 2xy)\,dx + (2x^3y^3 - x^2)\,dy = 0$.

Solution: Here $N = 2x^3y^3 - x^2$ and $M = 3x^2y^4 + 2xy$

$$\therefore \quad \frac{\partial M}{\partial y} = 12x^2y^3 + 2x \text{ and } \frac{\partial N}{\partial x} = 6x^2y^3 - 2x.$$

Here $\dfrac{1}{M}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right) = \dfrac{1}{3x^2y^4 + 2xy}(12x^2y^3 + 2x - 6x^2y^3 + 2x)$

$$= \frac{2(3x^2y^3 + 2x)}{y(3x^2y^3 + 2x)} = \frac{2}{y}, \text{ which is a function of } y \text{ alone.}$$

Hence $\text{I.F.} = e^{\int\left(\frac{-2}{y}\right)dy} = e^{-2\ln|y|} = y^{-2} = \dfrac{1}{y^2}$.

Multiplying the given equation by the I.F. $= y^{-2}$, and on rearranging the terms, we get

$$(3x^2y^2dx + 2x^3ydy) + \left(\frac{2x}{y}dx - \frac{x^2}{y^2}dy\right) = 0$$

i.e., $d(x^3y^2) + d\left(\dfrac{x^2}{y}\right) = 0$

Integrating the equation above, we get

$x^3y^2 + \frac{x^2}{y} = c$, where c is a constant of integration.

i.e., $x^3y^3 + x^2 = cy$, which is the required solution.

***

Example 21: Solve

$(xy^2 - x^2)dx + (3x^2y^2 + x^2y - 2x^3 + y^2)dy = 0,\ x \neq 0,\ y \neq 0$.

Solution: Here $M(x,\ y) = xy^2 - x^2, N(x,\ y) = 3x^2y^2 + x^2y - 2x^3 + y^2$

$$\frac{\partial M}{\partial y} = 2xy,\ \frac{\partial N}{\partial x} = 6xy^2 + 2xy - 6x^2$$

$$\therefore\ \frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{-1}{xy^2 - x^2}(6xy^2 + 2xy - 6x^2 - 2xy) = \frac{-6(xy^2 - x^2)}{xy^2 - x^2} = -6$$

$$\therefore\ \text{I.F.} = e^{\int 6dy} = e^{6y}$$

Multiplying the given equation by e^{6y}, we obtain

$$(xy^2 - x^2)e^{6y}dx + (3x^2y^2 + x^2y - 2x^3 + y^2)e^{6y}\ dy = 0$$

which is an exact equation.

Step 1: Integrating $M(x,\ y)$ w.r.t x, treating y as a constant, we get

$$\int (xy^2 - x^2)e^{6y}dx = \left(\frac{1}{2}x^2y^2 - \frac{x^3}{3}\right)e^{6y}.$$

Step 2: Integrating terms of $N(x,\ y)$, which do not involve x, w.r.t y, we get

$$\int y^2e^{6y}dy = \frac{e^{6y}}{6}\left(y^2 - \frac{1}{3}y + \frac{1}{18}\right).$$

Step 3: The required solution is

$$e^{6y}\left[\frac{1}{2}x^2y^2 - \frac{x^3}{3} + \frac{1}{6}\left(y^2 - \frac{1}{3}y + \frac{1}{18}\right)\right] = c, \text{ a constant.}$$

***

And now an exercise for you.

E14) Solve the following differential equations

i) $(2xy^4e^y + 2xy^3 + y)dx + (x^2y^4e^y - x^2y^2 - 3x)\ dy = 0,\ x > 0,\ y > 0$.

ii) $(xy^3 + y)dx + 2(x^2y^2 + x + y^4)dy = 0,\ x > 0,\ y > 0$.

Lastly, consider the following rule for finding an integrating factor of differential equations.

Rule VI: If the differential equation $Mdx + Ndy = 0$ can be put in the form $x^{\alpha}y^{\beta}(mydx + nxdy) = 0$, where α, β, m and n are constants, then $x^{km-1-\alpha}.\ y^{kn-1-\beta}$ is an integrating factor, where k can assume any constant value.

Proof: Multiplying the given equation by $x^{km-1-\alpha}.\ y^{kn-1-\beta}$, we get

$$x^{km-1}y^{kn-1}(mydx+nxdy)=0,$$

or $\quad kmx^{km-1}y^{kn}dx+kn\ x^{km}y^{kn-1}dy=0$, where k is a constant.

or $d(x^{km}y^{kn})=0$, which is an exact differential equation. Hence $x^{km-1-\alpha}y^{kn-1-\beta}$ is an I.F. of the differential equation.

It may be **noted** that if the given differential equation is of the form $x^{\alpha}y^{\beta}(my\,dx+nx\,dy)+x^{\alpha_1}y^{\beta_1}(m_1y\,dx+n_1x\,dy)=0$, where $\alpha, \beta, \alpha_1, \beta_1, m, n, m_1$ and n_1 are constants, then also I.F. can be determined.

By Rule VI, $x^{km-1-\alpha}$, $y^{kn-1-\beta}$ will make the first term exact, while $x^{k_1m_1-1-\alpha_1}$, $y^{k_1n_1-1-\beta_1}$ will make the second term exact, for k and k_1 being constants.

These two factors will be identical if

$$km-1-\alpha=k_1m_1-1-\alpha_1$$

and $\quad kn-1-\beta=k_1n_1-1-\beta_1$.

Values of k and k_1 satisfying these two algebraic equations can be obtainted. Then either factor is an integrating factor of the given equation.

– ■ –

We now consider an example to illustrate this rule.

Example 22: Solve $(y^3-2yx^2)\ dx+(2xy^2-x^3)\ dy=0$.

Solution: On rearranging the terms of the given equation, we can write

$$y^2(ydx+2xdy)-x^2(2ydx+xdy)=0 \tag{66}$$

For the first term, $\alpha=0, \beta=2, m=1$ and $n=2$ and hence its I.F. is $x^{k-1}y^{2k-1-2}$.

For the second term, $\alpha_1=2, \beta_1=0, m_1=2, n_1=1$ and hence for the second term I.F. is $x^{2k_1-1-2}y^{k_1-1}$.

These two integrating factors will be identical if

$$\left.\begin{aligned} k-1&=2k_1-1-2 \\ \text{and}\quad 2k-1-2&=k_1-1 \end{aligned}\right\} \tag{67}$$

Solving the system of Eqns. (67) for k and k_1, we get $k=2$ and $k_1=2$ and, therefore, integrating factor of Eqn. (66) for both the values is same i.e., xy.

Multiplying Eqn. (66) by xy, we get

$$xy^3(ydx+2xdy)-x^3y(2ydx+xdy)=0$$

$$\Rightarrow\ (xy^4dx+2x^2y^3dy)-(2x^3y^2dx+x^4ydy)=0$$

$$\Rightarrow\ \frac{1}{2}(2xy^4dx+4x^2y^3dy)-\frac{1}{2}(4x^3y^2dx+2x^4y^4dy)=0$$

$$\Rightarrow\ \frac{1}{2}d(x^2y^4)-\frac{1}{2}d(x^4y^2)=0 \tag{68}$$

Integrating Eqn. (68), we get the required solution as

$$\frac{x^2y^4 - x^4y^2}{2} = c_1 \text{ (a constant)}$$

or $\quad x^2y^2(y^2 - x^2) = 2c_1 = c$ (a constant).

***

Note that the differential equation in Example 22 is a homogeneous differential equation where $Mx + Ny \neq 0$. Hence Rule II also applies to the equation.

Example 23: Solve $(2y\,dx + 3x\,dy) + 2xy(3y\ dx + 4x\ dy) = 0$.

Solution: The given equation can be rewritten as

$$(2y\ dx + 3x\ dy) + xy(6y\ dx + 8x\ dy) = 0$$

For the first term, $\alpha = 0,\ \beta = 0,\ m = 2,\ n = 3$ and hence integrating factor is $x^{2k-1}\ y^{3k-1}$.

For the second term $\alpha_1 = 1,\ \beta_1 = 1,\ m_1 = 6,\ n_1 = 8$ and the integrating factor is $x^{6k_1-2}\ y^{8k_1-2}$.

The two integrating factors will be identical if

$$\left.\begin{aligned} 2k - 1 &= 6k_1 - 2 \\ \text{and}\quad 3k - 1 &= 8k_1 - 2 \end{aligned}\right\} \tag{69}$$

Solving the system of Eqns. (69) for k and k_1, we get $k = 1$ and $k_1 = 1/2$.

The integrating factor of the given equation is xy^2. Multiplying the given equation by xy^2, we get

$$xy^2(2y\,dx + 3x\ dy) + 2x^2y^3(3y\,dx + 4x\ dy) = 0$$

or, $\quad (2xy^3 + 6x^2y^4)dx + (3x^2y^2 + 8x^3y^3)dy = 0 \quad (70)$

Now you can easily check that Eqn. (70) is exact i.e., $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ where

$M(x,\ y) = 2xy^3 + 6x^2y^4,\ N(x,\ y) = 3x^2y^2 + 8x^3y^3$.

Step 1: Integrating $M(x,\ y)$ w.r.t. x, treating y as a constant, we have

$$\int (2xy^3 + 6x^2y^4)\,dx = x^2y^3 + 2x^3y^4.$$

Step 2: There is no term in $N(x,\ y)$ which do not involve x.

Step 3: The required solution is

$x^2y^3 + 2x^3y^4 = c$, where c is an arbitrary constant.

For a given problem which rule is to be applied comes by doing several problems of different kinds only.

***

You can have some practice while doing the following exercises and check your understanding of whatever you have learnt.

E15) Solve the following differential equations

i) $\quad (x^2 + y^2 + 2x)\ dx + 2y\ dy = 0,\ x > 0,\ y > 0$.

ii) $\quad x^2y\ dx - (x^3 + y^3)\ dy = 0$.

iii) $(y^4+2y)\,dx+(xy^3+2y^4-4x)\,dy=0\ \forall\ x,\ y>0$.

iv) $(y^2+2x^2y)\,dx+(2x^3-xy)\,dy=0$.

v) $(2x^2y-3y^4)\,dx+(3x^3+2xy^3)\,dy=0$.

E16) Solve the following equations.

i) $(x+y)^2\dfrac{dy}{dx}=a^2$, a is a constant.

ii) $\dfrac{x+y-a}{x+y-b}\dfrac{dy}{dx}=\dfrac{x+y+a}{x+y+b}$, a, b are constants.

iii) $1+\left(\dfrac{x}{y}-\sin y\right)\dfrac{dy}{dx}=0$.

iv) $(3y^2+2xy)=(2xy+x^2)\dfrac{dy}{dx}=0\ \forall x>0,\ y>0$.

v) $y+y^2+\left(2xy+\dfrac{y}{1+y}\right)\dfrac{dy}{dx}=0,\ x>0,\ y>0$

vi) $2x^2y^3+3x(1+y^2)\dfrac{dy}{dx}=0,\ x>0,\ y>0$

vii) $(x+y-1)\,dy-(x-y-3)\,dx=0$

We now end this unit by giving a summary of what we have covered in it.

7.6 SUMMARY

In this unit, we have covered the following:

1. An equation $\dfrac{dy}{dx}=f(x,\ y)$ is termed as a **separable equation** or an **equation with separable variables** if $f(x,\ y)=M(x)\,N(y)$. To solve a separable equation, we can write it as

$$N(y)\frac{dy}{dx}+M(x)=0$$

for some $N(y)$ and $M(x)$ which on integration gives the required solution.

2. A function h is said to be a **homogeneous function** of degree n, where n is a real number, if we can write $h(\lambda x,\ \lambda y)=\lambda^n h(x,\ y)$ for all $x,\ y$ and $\lambda>0$.

3. A differential equation

$$\frac{dy}{dx}=f(x,\ y)$$

is a **homogeneous differential equation** when f is a homogeneous function of degree zero.

4. A homogeneous differential equation reduces to separable equation by the substitution $y = vx$, where v is some function of x.

5. Equations of the form $\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'}$ where a, b, c, a', b', c' are constants and $\frac{a}{a'} \neq \frac{b}{b'}$ can be **reduced to homogeneous equations** by the substitution $x = X + h,\ y = Y + k$, where h and k are such that $ah + bk + c = 0$ and $a'h + b'k + c' = 0$.
   In case $\frac{a}{a'} = \frac{b}{b'} = \frac{1}{m}$, say, then substitution $ax + by = v$ reduces the equation to separable equation.

6. An exact differential equation is formed by equating an exact differential to zero.

7. The differential equation

   $$M(x,\ y)dx + N(x,\ y)dy = 0$$

   is **exact** if and only if

   $$\frac{\partial}{\partial y} M(x,\ y) = \frac{\partial}{\partial x} N(x,\ y),$$

   provided M, N, $\frac{\partial N}{\partial x}$ and $\frac{\partial M}{\partial y}$ are continuous functions of x and y.

8. A non-zero function, which makes a non-exact differential equation exact when multiplied with it, is known as an **integrating factor (I.F.) of the equation.**

9. The number of integrating factors for the equation $M(x, y)dx + N(x,\ y)dy = 0$ is infinite.

10. For a homogeneous equation $M(x,\ y)dx + N(x,\ y)dy = 0$, when $Mx + Ny \neq 0$, then $\frac{1}{Mx + Ny}$ is an I.F. of the equation.

11. If a differential equation $Mdx + Ndy = 0$ can be written in the form $y\, f_1(xy)dx + x\, f_2(xy)dy = 0$, then $\frac{1}{Mx - Ny}$ is an I.F. of the equation provided $Mx - Ny \neq 0$.

12. When $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is a function of x alone, say $f(x)$, then $e^{\int f(x)\, dx}$ is an I.F. of the equation $Mdx + Ndy = 0$.

13. When $\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is a function of y alone, say $f(y)$, then $e^{-\int f(y)\, dy}$ is an I.F. of the equation $Mdx + Ndy = 0$.

14. If a differential equation is of the form $x^{\alpha} y^{\beta}(my\ dx + nx\ dy) = 0$, for α, β, m, n any real constants, then $x^{km-1-\alpha}\ y^{kn-1-\beta}$ is an I.F. of the equation where k can take any constant value.

7.7 SOLUTIONS/ANSWERS

E1) i) $(1-x)dy - (1+y)dx = 0$

$\Rightarrow\ dy - (x\ dy + y\ dx) - dx = 0$

$\Rightarrow\ dy - d(xy) - dx = 0$

Integrating, we obtain

$y - xy - x = c$, or $y = \dfrac{c+x}{1-x},\ x \neq 1$.

ii) $\sin^{-1} y + \sin^{-1} x = c$.

iii) Given equation can be written as

$$\frac{dx}{a+x} = \frac{dy}{y - ay^2}.$$

$$\Rightarrow\ \frac{dx}{a+x} = \left(\frac{1}{y} + \frac{a}{1-ay}\right)dy$$

Integrating we obtain

$y = c(a+x)\ (1-ay)$.

iv) $\tan y = c\ (1-e^x)^3$.

v) $e^y = e^x + \dfrac{x^3}{3} + c$.

E2) i) Given equation can be written as

$$\frac{2y}{1+y^2}\ \frac{dy}{dx} - \frac{1}{x} = 0$$

Integrating, we get

$\ln\ (1+y^2) - \ln\ x = \ln\ c$, since $x > 0$

or $\dfrac{1+y^2}{x} = c$. Given that $y(2) = 3$

we obtain $c = 5$ and hence $\dfrac{1+y^2}{x} = 5$.

ii) $y = y_0 e^{-2x^2}$.

iii) Given equation can be written as

$$e^{-y} dy = xe^{-x^2}\ dx$$

Integrating we obtain

$$e^{-y} = \frac{e^{-x^2}}{2} + c$$

Using I.C. we obtain $c=\dfrac{1}{2}$

Thus $e^{-y}=\dfrac{e^{-x^2}+1}{2}$

or $e^{y}=\dfrac{2}{1+e^{-x^2}}$

or $y=\ln\left|\dfrac{2}{1+e^{-x^2}}\right|$.

iv) $y^2-y_0^2=2g\,(x-x_0)$.

E3) i) $y=cx$.

ii) Given equation can be written as

$$\frac{dy}{dx}=\frac{2+(y/x)}{3+2(y/x)}.$$

Substituting $(y/x)=v$, in the above equation we obtain

$$x\frac{dv}{dx}+v=\frac{2+v}{3+2v}$$

$$\Rightarrow\quad x\frac{dv}{dx}=\frac{2(1-v-v^2)}{3+2v}$$

$$\Rightarrow\quad \frac{(3+2v)}{1-v-v^2}\,dv=2\frac{dx}{x}$$

$$\Rightarrow\quad \frac{1+2v}{1-v-v^2}\,dv+\frac{2}{1-v-v^2}\,dv=\frac{2}{x}\,dx$$

$$\Rightarrow\quad \frac{1+2v}{1-v-v^2}\,dv-\frac{2dv}{\left(v+\frac{1}{2}\right)^2-\left(\frac{\sqrt{5}}{2}\right)^2}=\frac{2}{x}\,dx$$

Integrating the above equation we obtain

$$-\ln|1-v-v^2|-\frac{2}{\sqrt{5}}\ln\left|\frac{v+\frac{1}{2}+\frac{\sqrt{5}}{2}}{v+\frac{1}{2}-\frac{\sqrt{5}}{2}}\right|=2\ln|x|+\ln c$$

Substituting $v=y/x$ in the above equation and simplifying, we obtain

$$(y^2+xy-x^2).\left[\frac{2y+(1+\sqrt{5})x}{2y+(1-\sqrt{5})x}\right]^{\frac{2}{\sqrt{5}}}=c.$$

iii) Given equations can be written as

$$\frac{dy}{dx}=\frac{(y/x)\sin(y/x)-1}{\sin(y/x)}$$

Putting $(y/x)=v$ and proceeding as in (ii) above, we get

$\ln|x|=\cos(y/x)+c$.

iv) Given equation can be written as

$$\frac{dy}{dx} = (y/x)\,[\ln(y/x) + 1]$$

Substituting $(y/x) = v$ the above equation reduces to

$$\frac{dv}{v \ln v} = \frac{dx}{x}$$

Integrating this equation we get

$$\ln v = cx$$

Substituting $v = y/x$ the solution is obtained as

$$y = xe^{cx}.$$

v) $cx = \exp \sin^{-1}\left(\frac{y}{x}\right).$

E4) i) Dividing throughout by x^2 the given equation reduces to

$$2\frac{dy}{dx} = 3\frac{y}{x} + \left(\frac{y}{x}\right)^2$$

Putting $y/x = v$, we get

$$2x\frac{dv}{dx} = v + v^2$$

$$\Rightarrow \quad 2\frac{dv}{v^2 + v} = \frac{dx}{x}$$

$$\Rightarrow \quad 2\left(\frac{1}{v} - \frac{1}{v+1}\right)dv = \frac{dx}{x}$$

Integrating the above equation we obtain $\left(\frac{v}{v+1}\right)^2 = cx$ which on replacing v by y/x reduces to

$$y^2 = cx(y+x)^2$$

Using the initial condition $y = -2$ at $x = 1$, we obtain $c = 4$ and hence $y^2 = 4x(y+x)^2$ is the required solution.

ii) Given equation can be written as

$$[1 + (y/x)e^{(y/x)}] - e^{(y/x)}\frac{dy}{dx} = 0$$

Putting $(y/x) = v$, the variables are separated. Integrating the resulting equation and evaluating the constant of integration by using initial condition $y = 0$ at $x = 1$, we get

$$\ln|x| = e^{(y/x)} - 1.$$

iii) $4x \ln|(y/x)| + x \ln|x| + y - x = 0.$

iv) Dividing the given equation by y^2 and substituting $\frac{x}{y} = v$, we obtain

$$v + y\frac{dv}{dy} + (1 + v + v^2) = 0$$

$$\Rightarrow \quad \frac{dy}{y} + \frac{dv}{(1+v)^2} = 0$$

Integrating and replacing v by x/y, we obtain

$$\ln|y| - \frac{x}{x+y} = c$$

Using the initial condition $y(0) = 1$, we obtain $c = 0$.

Thus, $(x+y)\ln|y| - x = 0$.

E5) i) Comparing the given equation with Eqn. (29) it can be checked that in this case $\frac{a}{a'} \neq \frac{b}{b'}$. Thus putting $x = X + h$ and $y = Y + k$ the equation reduces to a homogeneous equation $\frac{dY}{dX} = \frac{2Y - X}{Y - 3X}$, where h and k satisfying the equations $2k - h = 4$ and $k - 3h = -3$ are obtained as $h = 2$ and $k = 3$.

Substituting $Y = vX$ the above homogeneous equation reduces to

$$\frac{v-3}{v^2-5v+1}dv = -\frac{dX}{X}$$

$$\Rightarrow \quad \frac{1}{2}\left[\frac{2v-5-1}{v^2-5v+1}\right]dv = -2\frac{dX}{X}$$

$$\Rightarrow \quad \frac{2v-5}{v^2-5v+1}dv - \frac{dv}{\left(v-\frac{5}{2}\right)^2 - \left(\frac{\sqrt{21}}{2}\right)^2} = -2\frac{dX}{X}$$

Integrating the above equation we obtain

$$\ln(v^2-5v+1) - \frac{1}{\sqrt{21}}\ln\left|\frac{v-\frac{5}{2}-\frac{\sqrt{21}}{2}}{v-\frac{5}{2}+\frac{\sqrt{21}}{2}}\right| = -2\ln|X| + \ln|c|$$

$$\Rightarrow \quad \left(\frac{v-\frac{5}{2}-\frac{\sqrt{21}}{2}}{v-\frac{5}{2}+\frac{\sqrt{21}}{2}}\right)^{\frac{1}{\sqrt{21}}} = c\,(v^2-5v+1)\,X^2$$

Substituting $v = \frac{Y}{X} = \frac{y-3}{x-2}$ in the above equation and simplifying, we get

$$\left[\frac{2y-(5+\sqrt{21})x+2(2+\sqrt{21})}{2y-(5-\sqrt{21})x+2(2-\sqrt{21})}\right]^{1/\sqrt{21}}$$
$$= c(y^2-5xy+x^2+11x+4y-17).$$

ii) The substitution $x = X + 1,\ y = Y$ reduces the given equation to the homogeneous form

$$\frac{dY}{dX} = \frac{7X - 3Y}{7Y - 3X}$$

Substituting $Y = vX$ it reduces to variable separable form

$$\frac{7v-3}{7(1-v^2)}dv = \frac{dX}{X}$$

Integration yields

$$(1-v^2)\left(\frac{1+v}{1-v}\right)^{3/7} X^2 = c_1$$

Substituting $v = \frac{Y}{X} = \frac{y}{x-1}$ we get

$$(x-1-y)\ (x-1+y)\left(\frac{x-1+y}{x-1-y}\right)^{3/7} = c_1$$

or $(y-x+1)^2\ (y+x-1)^5 = c$.

iii) The substitution $2x+y=v$ reduces the given equation to the form

$$\frac{dv}{dx}-2=-\frac{v+1}{2v-1}$$

or $$\frac{dv}{dx}=\frac{4v-2-v-1}{2v-1}=\frac{3v-3}{2v-1}$$

or $$\frac{2v-1}{v-1}\frac{dv}{dx}=3 \text{ or } \left(2+\frac{1}{v-1}\right)\frac{dv}{dx}=3$$

Integrating, we get

$$2v+\ln|v-1| = 3x+\ln\ c$$

or $$\ln\frac{|v-1|}{c}=(3x-2v)$$

or $$|v-1| = c\ e^{3x-2v}$$

or $$|(2x+y-1)|\,e^{2y+x}=c \text{ (Putting } v=2x+y)$$

iv) $(x+3y)+2\ln|(2-x-y)|=c$ (Hint: Put $x+y=v$)

E6) i) The given equation can be written as

$$(y\cos x+\sin x.y')+2xe^y+x^2e^y.y'+2y'=0$$

or, $$\frac{d}{dx}(y\sin x)+\frac{d}{dx}(x^2e^y)+\frac{d}{dx}(2y)=0$$

or, $$\frac{d}{dx}(y\sin x+x^2e^y+2y)=0$$

Hence the given equation is exact. On integration, its solution can be expressed as

$$y\sin x+x^2e^y+2y=c$$

where c is an arbitrary constant.

ii) The given equation can be written as

$$(bx+cy)\frac{dy}{dx}+(ax+by)=0$$

You can check that the equation is exact.
Its solution is

$$bxy + c\frac{y^2}{2} + a\frac{x^2}{2} = c_1, \text{ where } c_1 \text{ is an arbitrary constant.}$$

iii) The given equation can be written as

$$6x + \left(\frac{y}{x} + \ln x\frac{dy}{dx}\right) + y\frac{dy}{dx} = 0$$

or $$\frac{d}{dx}\left(3x^2 + y\ln x + \frac{1}{2}y^2\right) = 0$$

Thus, the equation is exact and the required solution is given by the implicit equation $3x^2 + y\ln x + \frac{1}{2}y^2 = c$.

iv) You can easily check that the given equation is exact. The given equation can be written as

$$\left(\frac{1}{x} + \frac{1}{x^2}\right)dx + y\,e^y dy + \frac{xdy - ydx}{x^2 + y^2} = 0$$

Integrating, the solution is obtained as

$$\ln x - \frac{1}{x} + e^y(y-1) + \tan^{-1} y/x = c.$$

E7) i) Here $N(x, y) = ky,\ M(x, y) = x$

$$\frac{\partial N}{\partial x} = 0, \frac{\partial M}{\partial y} = 0$$

$$\therefore \frac{\partial N}{\partial x} = 0 = \frac{\partial M}{\partial y}$$

Hence the given equation is exact for all values of $k(\neq 0)$. The required solution is given by

$$x^2 + ky^2 = c, \quad k \neq 0.$$

ii) Here $N(x, y) = kx,\ M(x, y) = y$

$$\therefore \frac{\partial N}{\partial x} = k, \frac{\partial M}{\partial y} = 1$$

For the given equation to be exact, we need $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$,

$\Rightarrow\ k = 1$. For $k = 1$, the given equation becomes
$y + xy' = 0$

or, $$\frac{d}{dx}(xy) = 0$$

Hence its solution for $k = 1$, can be expressed as

$$xy = c.$$

iii) $k = 2,\ e^{2xy} + x^2 = c$.

E8) i) If $F(x, y) = y^2$ is an I.F. then $y^2[6xydx + (4y + 9x^2)dy] = 0$ must be an exact equation. Here,

$$N(x,\ y) = 4y^3 + 9x^2y^2$$

$$M(x,\ y) = 6xy^3$$

$$\frac{\partial N}{\partial x} = 18xy^2 = \frac{\partial M}{\partial y}$$

Thus $F(x,\ y) = y^2$ is an I.F.

ii) $F(x,\ y)\,[-y^2dx + (x^2 + xy)dy] = 0$

$$\Rightarrow\ N(x,\ y) = \frac{x}{y} + 1,\ M(x,\ y) = \frac{-y}{x}$$

$$\frac{\partial N}{\partial x} = \frac{1}{y} \text{ and } \frac{\partial M}{\partial y} = -\frac{1}{x}$$

$\therefore\ \dfrac{\partial N}{\partial x} \neq \dfrac{\partial M}{\partial y}$ and $F(x,\ y)$ is not an I.F. of the given equation.

iii) $F(x,\ y) = xy$ is an I.F. of the given equation.

E9) i) I.F. $= \dfrac{1}{y^2}$; solution is $x^2 + \dfrac{1}{y}e^x = c$

ii) I.F. $= \dfrac{1}{x^2}$; solution is $y + \ln\ x + cx + 1 = 0$

iii) I.F. $= \dfrac{1}{xy^2}$; solution is $\dfrac{x}{y} + \ln\left(\dfrac{y^3}{x^2}\right) = c$

E10) i) Here $M(x,\ y) = -xy^3,\ N(x,\ y) = x^4 + y^4$

$$Mx + Ny = x^5 \neq 0$$

$\therefore\ \dfrac{1}{x^5}$ is an I.F.

Multiplying the given equation by $\dfrac{1}{x^5}$ and simplifying, it can be put in the form

$$\frac{dy}{dx} = \frac{x^4 + y^4}{xy^3}$$

Substituting $y = vx$ we obtain

$$v^3dv = \frac{dx}{x}$$

Integrating the above equation we obtain

$$\frac{v^4}{4} = \ln|x| + c$$

$\Rightarrow\ y^4 = 4x^4\ln|x| + 4cx^4$ (replacing v by y/x)

which is the required solution.

ii) $M(x,\ y) = x^2 + 3xy + y^2,\ N(x,\ y) = -x^2$

$$Mx + Ny = x(x+y)^2 \neq 0$$

I.F. is $\dfrac{1}{x(x+y)^2}$

Multiplying by I.F. the given equation reduces to

$$\left(\frac{1}{x}+\frac{y}{(x+y)^2}\right)dx-\frac{x}{(x+y)^2}dy=0$$

Integrating the above equation the required solution is obtained as

$$\ln|x|-\frac{y}{x+y}=c\,.$$

E11) i) Here $N(x,\ y)=2x-2x^3y^2,\ M(x,\ y)=x^2y^3+2y$

$$Mx-Ny=3x^3y^3\neq 0$$

$\therefore$ I.F. $=\dfrac{1}{Mx-Ny}=\dfrac{1}{3x^3y^3}$

Multiplying the given equation by the I.F. we obtain

$$\frac{dx}{3x}+\frac{2}{3}\left(\frac{dx}{x^3y^2}+\frac{dy}{x^2y^3}\right)-\frac{2}{3y}dy=0$$

$$\Rightarrow\quad \frac{dx}{x}-d\left(\frac{1}{x^2y^2}\right)-\frac{2}{y}dy=0$$

$\Rightarrow\quad \ln|x|-\dfrac{1}{x^2y^2}-2\ln|y|=\ln c$ or, $\ln\dfrac{x}{cy^2}=\dfrac{1}{x^2y^2}$

or $x=cy^2e^{1/x^2y^2}$ is the required solution.

ii) $M(x,\ y)=(x^2y^2+2y+1)y,\ N(x,\ y)=(x^2y^2-xy+1)x$

$Mx-Ny=2x^2y^2\neq 0$

$\therefore$ I.F. $=\dfrac{1}{2x^2y^2}$

Multiplying the given equation by this factor, we get

$$(y\,dx+x\,dy)+\left(\frac{dx}{x}-\frac{dy}{y}\right)+\left(\frac{1}{x^2y}dx+\frac{1}{xy^2}dy\right)=0$$

or, $\quad d(xy)+\dfrac{dx}{x}-\dfrac{dy}{y}+\dfrac{d(xy)}{x^2y^2}=0$

Integrating, we obtain $xy+\ln x-\ln y-1/xy=c$, a constant which is the required solution.

E12) i) $M(x,\ y)=y+\dfrac{y^3}{3}+\dfrac{x^2}{2},\ N(x,y)=\dfrac{x+xy^2}{4}$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{3}{x}$$

Thus I.F. $-e^{\int\frac{3}{x}dx}=e^{3\ell n\,x}=x^3$

Multiplying the given equation by x^3, we get

$$2x^5dx+(x^4dy+4x^3y\,dx)+\frac{1}{3}(x^4 3y^2dy+4x^3y^3dx)=0$$

or $2x^5dx+d(x^4y)+\dfrac{1}{3}d(x^4y^3)=0$

Integrating, the required solution is

$$x^6 + 3x^4 y + x^4 y^3 = c.$$

ii) I.F. $= x$, solution is $4x^3 + 3x^4 + 6x^2 y^2 = c$.

E13) Consider the equation $e^{-\int f(y)\,dy}\,[Mdx + Ndy] = 0$

Let $r = Me^{-\int f(y)\,dy}$ and $s = Ne^{-\int f(y)\,dy}$ then $\dfrac{\partial s}{\partial y} = \dfrac{\partial N}{\partial x} e^{-\int f(y)dy}$

and $\dfrac{\partial r}{\partial y} = \dfrac{\partial M}{\partial y} e^{-\int f(y)dy} - Me^{-\int f(y)dy} f(y)$

$$= e^{-\int f(y)dy}\left[\frac{\partial M}{\partial y} - Mf(y)\right]$$

$$= e^{-\int f(y)dy}\left[\frac{\partial M}{\partial y} - \frac{\partial M}{\partial y} + \frac{\partial N}{\partial x}\right]\left[\text{because } \frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(y)\right]$$

$$= \frac{\partial N}{\partial x} e^{-\int f(y)dy} = \frac{\partial s}{\partial x}$$

Therefore the equation $r\,dx + s\,dy = 0$ is exact and hence the required result.

E14) i) I.F. $= \dfrac{1}{y^4}$; solution is $x^2 e^y + \dfrac{x^2}{y} + \dfrac{x}{y^3} = c$.

ii) $\dfrac{1}{M}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right) = \dfrac{1}{x}$.

I.F. $= e^{\int \frac{1}{y} dy} = e^{\ln y} = y$

Multiplying the given equation by y, we get

$$(xy^4 + y^2)dx + 2(x^2 y^3 + xy + y^5)dy = 0$$

which is an exact equation.

Integrating, the required solution is

$$3x^2 y^4 + 6xy^2 + 2y^6 = c$$

E15) i) Here $N = 2y,\ M = x^2 + y^2 + 2x$

$\therefore\ \dfrac{1}{N}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right) = \dfrac{1}{2y}(2y - 0) = 1 = f(x)$, say

$\therefore\ \text{I.F.} = e^{\int f(x)\,dx} = e^{\int dx} = e^x$ (using Rule IV)

Solution is $e^x(x^2 + y^2) = c$

ii) Using Rule II, we get

$y = c\,e^{x^3/3y^3}$.

iii) Here $M = y^4 + 2y;\ N = xy^3 + 2y^4 - 4x$

$$\therefore\ \frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{y^2 + 2y}(4y^3 + 2 - y^3 + 4) = \frac{3y^3 + 6}{y(y^3 + 2)} = \frac{3}{y}$$

On using Rule V, I.F. $= e^{-\int 3/y\, dy} = e^{-3\ell n|y|} = y^{-3} = \dfrac{1}{y^3}$

Multiplying the given equation by I.F. $= \dfrac{1}{y^3}$, we get

$$\left(y+\frac{2}{y^2}\right)+\left(x+2y-\frac{4x}{y^3}\right)\frac{dy}{dx}=0$$

or $$\left(y+x\frac{dy}{dx}\right)+2y\frac{dy}{dx}+2\left(\frac{1}{y^2}-\frac{2x}{y^3}\right)\frac{dy}{dx}=0$$

or $$\frac{d}{dx}(xy)+\frac{d}{dx}(y^2)+\frac{d}{dx}\left(\frac{2x}{y^2}\right)=0$$

Hence the solution is

$$xy+y^2+\frac{2x}{y^2}=c$$

iv) I.F. is $x^{-5/2}y^{-1/2}$, solution is

$6\sqrt{xy}-x^{-3/2}y^{3/2}=c$ (Hint: use Rule VI)

v) I.F. is $x^{-\frac{49}{13}}y^{-\frac{28}{13}}$, solution is

$5\left[\dfrac{y^2}{x^3}\right]^{12/13}-12[(x^2y^3)^{-5/13}]=c$ (Hint: use Rule VI)

E16) i) The given equation is

$$(x+y)^2\frac{dy}{dx}=a^2$$

Putting $x+y=v$, the given equation reduces to

$$v^2\left(\frac{dv}{dx}-1\right)=a^2$$

$$\Rightarrow\left(1-\frac{a^2}{a^2+v^2}\right)\frac{dv}{dx}=1$$

Integrating, we get

$$v-\frac{a^2}{a}\tan^{-1}\frac{v}{a}=x+c$$

or $(x+y)-a\tan^{-1}\left(\dfrac{x+y}{a}\right)=x+c$, where c is an arbitrary constant.

ii) Put $x+y=v$

$$\frac{dv}{dx}=\frac{2(v^2-ab)}{(v-a)(v+b)}$$

Integrating

$$v+\frac{b-a}{2}\ln\,(v^2-ab)=2x+c$$

or $\quad (b-a)\ \ln\ \{(x+y)^2-ab\}=2(x-y)+c$.

iii) By inspection, $\text{I.F.}=y$.
Multiplying given equation by I.F., we get

$$y+x\frac{dy}{dx}-y\sin y\frac{dy}{dx}=0$$

or, $\quad \frac{d}{dx}(xy)+\frac{d}{dx}(y\cos y-\sin y)=0$

Integrating, the solution is

$$xy+y\cos y-\sin y=c$$

iv) The given equation can be written as

$$\frac{dy}{dx}=\frac{3y^2+2xy}{2xy+x^2}=\frac{3(y/x)^2+2(y/x)}{2(y/x)+1}$$

Putting $y=vx$, we get

$$v+x\frac{dv}{dx}=\frac{3v^2+2v}{2v+1}$$

or $\quad \frac{2v+1}{v(v+1)}\frac{dv}{dx}=\frac{1}{x}$

or $\quad \left(\frac{1}{v+1}+\frac{1}{v}\right)\frac{dv}{dx}=\frac{1}{x}$

Integrating, we get
$\ln\ (v+1)+\ln\ v=\ln\ x+\ln\ c$, c being an arbitrary constant and $v>0$ since $x>0,\ y>0$.

$\therefore \quad v(v+1)=cx$

or $\quad y(y+x)=cx^3$.

v) Here $N=2xy+\frac{y}{1+y}$, $M=y+y^2$

$\frac{1}{M}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{1}{y+y^2}(1+2y-2y)=\frac{1}{y+y^2}$, which is a function of y alone.

By Rule V, $\text{I.F.}=e^{-\int\frac{1}{y(y+1)}dy}=e^{\int\left(\frac{1}{y+1}-\frac{1}{y}\right)dy}=\frac{y+1}{y}$

Multiplying the given equation by $\text{I.F.}=\frac{y+1}{y}$, we get

$$(y+1)^2+[(2x)\ (y+1)+1]\ \frac{dy}{dx}=0$$

$\Rightarrow \quad \frac{d}{dx}[x(y+1)^2]+\frac{d}{dx}(y)=0$

Hence the solution is

$$x(y+1)^2+y=c.$$

vi) I.F. $= \frac{1}{xy^3}$; solution is $x^2 - \frac{9}{y^2} + 3\ln|y| = c$

Alternatively, variables are separable.

vii) $(y+1)^2 + 2(y+1)(x-2) - (x-2)^2 = c$.

(Hint: put $x = X + 2,\ y = Y - 1$).

– x –

UNIT 8

LINEAR DIFFERENTIAL EQUATIONS

Structure **Page No.**

8.1 INTRODUCTION

In Unit 7, we discussed methods of solving some first order first degree differential equations. We solved mainly the following types of equations:

i) differential equations which could be integrated directly i.e., separable and exact differential equations,

ii) equations which could be reduced to these forms when direct integration is not possible. These include homogeneous equations, equations reducible to homogeneous form and equations that become exact when multiplied by an I.F.

In this unit, we focus our attention to another important type of first order, first degree differential equations known as **linear differential equations**, in which the derivative of highest order is a linear function of the lower order derivatives. These equations are important because of their wide range of applications, for example, the physical situations we considered in Sec. 6.5 of Unit 6, are all governed by linear differential equations. In this unit, we shall solve those and some more physical problems governed by linear differential equations in Sec.8.5.

We shall clasify linear differential equation into homogeneous and non homogeneous equations in Sec.8.2 and discuss the methods of finding particular integral of non-homogeneous linear differential equations in Sec.8.3.

The problem of integrating a linear differential equation was reduced to quadrature by Leibniz in 1692. In December 1695, James Bernoulli proposed a solution of a non-linear differential equation of the first order, now known as Bernoulli's equation. In 1696, Leibniz pointed out that Bernoulli's equation may be reduced to a linear differential equation by changing the dependent variable. We shall discuss this equation in Sec.8.4 of this unit along with some other equations, which may not be of first order or first degree but which can be reduced to linear differential equations.

Objectives

After studying this unit, you should be able to

- identify a linear differential equation;
- distinguish between homogeneous and non-homogeneous linear differential equations;
- obtain the general solution of a linear differential equation;
- obtain a particular integral of a linear equation by the methods of undetermined coefficients and variation of parameters;
- obtain the solution of Bernoulli's equation;
- obtain the solutions of linear equations modelled for certain physical situations.

8.2 CLASSIFICATION OF FIRST ORDER DIFFERENTIAL EQUATIONS

You may recall that in Unit 6 we defined the general form of first order differential equation as

$$g\left(x,\ y,\ \frac{dy}{dx}\right)=0$$

and if the equation is of first degree, then it can be expressed as

$$\frac{dy}{dx}=f(x,\ y).$$

Further, you may recall that a differential equation is **linear** if the dependent variable and all its derivatives appear in the first degree only and also there is no term in the equation involving the product of the derivatives or any derivative and the dependent variable. If a differential equation is not linear we call it **non-linear**

For example, equations $\frac{dy}{dx}+\frac{2y}{x}=x^3$ and $\frac{d^2y}{dx^2}+\frac{dy}{dx}=x\sin x$ are linear differential equations. However, equations $y\frac{dy}{dx}+x^2=10$ and $\frac{dy}{dx}+y^2=x^2$ are both non-linear because of the presence of the terms $y\frac{dy}{dx}$ in the first and y^2 in the second equation.

Accordingly, if the function $f(x,\ y)$ in the above equation is linear then it is a first order linear differential equation.

A **linear differential equation of the first order** can be written in the form

$$p(x)\frac{dy}{dx} = q(x)\,y + r(x), \quad p(x) \neq 0 \tag{1}$$

where p, q, r are continuous real valued functions in some interval $I \subseteq \boldsymbol{R}$. If $r(x)$ is identically zero, then Eqn. (1) reduces to

$$p(x)\frac{dy}{dx} = q(x)\,y\,. \tag{2}$$

You may note that the word homogeneous as it is used here has a very different meaning from that used in Sec. 7.3 of Unit 7.

Eqn. (2) is called a first order linear **homogeneous** differential equation. When $r(x)$ is not zero, Eqn. (1) is called a **non-homogeneous** (or **inhomogeneous**) linear differential equation of the first order.

Equation $\frac{dy}{dx} = y$ is a linear homogeneous equation. Here $p(x) = 1$ and $q(x) = 1$. Similarly, $\frac{dy}{dx} = 0,\ \frac{dy}{dx} = e^x y$ are also linear homogeneous equations. However, equation $\frac{dy}{dx} = e^x y + x$ is a linear non-homogeneous equation with $p(x) = 1,\ q(x) = e^x$ and $r(x) = x$.

Next consider the differential equation $\frac{dy}{dx} = x(y^2+1)$. It is non-linear because of the presence of the term y^2. Similarly, equation $\frac{dy}{dx} = e^y x + x^2$ and $\frac{dy}{dx} = \cos y$ are non-linear because of the presence of the terms like e^y and $\cos y$ (as each can be expressed as an infinite series in powers of y).

You may now try the following exercise to check your understanding of linear/non-linear differential equations.

E1) From the following differential equations, classify which are linear and which are non-linear. Also state the dependent variable in each case.

i) $\frac{dy}{dx} - y = xy^2$

ii) $\alpha(x)\frac{dy}{dx} + \beta(x)\,y = \gamma(x)\,y$

iii) $\frac{di}{dt} - 6i = 10\sin 2t$

iv) $y^2\frac{dx}{dy} = x\cos y$

v) $y\,dx + (xy + x - 3y^2)\,dy = 0$.

vi) $(2s - e^{2t})\,ds = 2(se^{2t} - \cos 2t)\,dt$.

You will realise the need for classifiction of linear differential equations into homogeneous and non-homogeneous equations when we discuss in Sec.10.3, Unit 10 of Block-3, some properties involving the solutions of linear homogeneous/non-homogeneous differential equations. Let us now talk about the general solution of linear non-homogeneous Eqn. (1).

8.3 GENERAL SOLUTION OF LINEAR NON-HOMOGENEOUS EQUATION

Let us start by considering an example.

Example 1: Solve the differential equation

$$\frac{dy}{dx} + y = x + 1, \ x > 0. \tag{3}$$

Solution: You can see that linear non-homogeneous Eqn. (3) is neither in variable separable form nor it is an exact equation. Let us now see if we can find an integrating factor which makes this equation an exact differential equation.
Let us write Eqn. (3) in the form

$$dy + (y - x - 1)dx = 0 \tag{4}$$

If $\mu(x)$ is an I.F. of Eqn. (4) then equation

$$\mu(x)dy + \mu(x)\,(y - x - 1)dx = 0 \tag{5}$$

will be an exact differential equation. Eqn. (5) will be an exact differential if

$$\frac{\partial}{\partial x}(\mu(x)) = \frac{\partial}{\partial y}\{\mu(x)\,(y - x - 1)\}$$

or $$\frac{\partial \mu(x)}{\partial x} = \mu(x)$$

or $$\frac{d\mu}{dx} = \mu$$

This is a separable equation and we can solve it to determine an I.F. μ of the given Eqn. (3) as

$$\ln|\mu| = x \text{ or } \mu = e^x$$

Multiplying Eqn. (3) with the I.F., we obtain

$$e^x\left(\frac{dy}{dx} + y\right) = (x+1)e^x$$

$$\Rightarrow \frac{d}{dx}(ye^x) = (x+1)e^x.$$

Integrating the above equation we obtain

$$ye^x = \int (x+1)e^x dx + C$$, where C is a constant.

$$\therefore \ y = e^{-x}\int (x+1)e^x dx + Ce^{-x}$$

$$= e^{-x}(xe^x - e^x + e^x) + Ce^{-x}$$

$$= x + Ce^{-x}$$

which is the required general solution of Eqn. (3).

***

Let us now see how this method can be used to find the solution of general non-homogeneous linear differential Eqn. (1).

On dividing Eqn. (1) by $p(x)$ it can be put in form

$$\frac{dy}{dx} + P(x)\,y = Q(x), \tag{6}$$

where P and Q are functions of x alone.

In the discussion that follows, we assume that Eqn. (6) has a solution. We look for the solution of Eqn. (6) in an interval $I \subseteq \boldsymbol{R}$ in which P and Q are continuous. You know that, in general, Eqn. (6) may not be exact. But we will show that we can always find an integrating factor $\mu(x)$, which makes this equation exact – a useful property of linear equations.

Let us write Eqn. (6) in the differential form:

$$dy + [P(x)\,y - Q(x)]\,dx = 0. \tag{7}$$

If $\mu(x)$ is an I.F. of Eqn. (7) then equation

$$\mu(x)dy + \mu(x)\,[P(x)\,y - Q(x)]\,dx = 0 \tag{8}$$

will be an exact differential equation. By Theorem 1 of Unit 7, we know that Eqn. (8) will be an exact differential if

$$\frac{\partial}{\partial x}[\mu(x)] = \frac{\partial}{\partial y}\{\mu(x)\,[P(x)\,y - Q(x)]\} \tag{9}$$

or $\quad \dfrac{\partial}{\partial x}\mu(x) = \mu(x)\,P(x) \qquad \left[\because \dfrac{\partial}{\partial y}\mu(x)Q(x) = 0\right]$

or $\quad \dfrac{d\mu}{dx} = \mu P(x)$.

This is a separable equation from which we can determine $\mu(x)$. We have

$$\frac{d\mu}{\mu} = P(x)\,dx$$

or $\quad \ln|\mu| = \int P(x)\,dx$.

Thus, an integrating factor of Eqn. (7) is given by

$$\mu(x) = e^{\int P(x)\,dx}. \tag{10}$$

Note that we need not use a constant of integration in Eqn. (10). Why? This is because Eqn. (8) is unaffected by a constant multiple.

You may also observe here that Eqn. (8) remains an exact differential equation even when $Q(x) = 0$. In fact $Q(x)$ plays no part in determining $\mu(x)$ since we see from Eqn. (9), that $\dfrac{\partial}{\partial y}\mu(x)\,Q(x) = 0$. Thus both

$$e^{\int P(x)dx}dy + e^{\int P(x)dx}\,[P(x)\,y - Q(x)]\,dx \text{ and}$$

$$e^{\int P(x)dx}dy + e^{\int P(x)dx}\,P(x)\,y\,dx$$

are exact differentials. We now multiply Eqn. (6) with the integrating factor obtained and write it in the form

$$e^{\int P\,dx}\left(\frac{dy}{dx} + Py\right) = Qe^{\int Pdx}.$$

This can also be written as

$$\frac{d}{dx}(ye^{\int P dx}) = Qe^{\int P\,dx}.$$

Integrating the above equation, we get

$ye^{\int Pdx} = \int Qe^{\int Pdx} dx + \alpha$, where α is a constant of integration

or $$y = e^{-\int Pdx} \int Qe^{\int Pdx} dx + \alpha e^{-\int Pdx}. \tag{11}$$

Since y represents any solution of Eqn. (6) we conclude that every solution of Eqn. (6) is included in the expression on the right hand side of Eqn. (11). Eqn. (11) gives the **general solution** of Eqn. (6) and can be used as a formula for obtaining the solutions of equations of the form (6). We advise you not to memorise the formula (11) and apply it mechanically for solving linear equations. Instead, use the procedure by which (11) is derived: **multiply by $e^{\int P\,dx}$ and integrate.**

Note that in order to find the solution given by Eqn. (11) **two integrations** are required, **one to obtain $\mu(x)$** from Eqn. (10) and the **other to determine y** from Eqn. (11). Care should be taken while calculating the integrating factor $\mu(x)$ from Eqn. (10). You must ensure that the differential equation is exactly in the form of Eqn. (6), specially, the coefficient of $\frac{dy}{dx}$ must be one. Otherwise, the $P(x)$ used in calculating $\mu(x)$ will be incorrect. Geometrically, Eqn. (11) is an infinite family of curves, one for each value of α. These curves are called **integral curves**. One particular member of the family of integral curves is obtained by using initial condition at a particular point (x_0, y_0) through which the graph of the solution is required to pass.

In case of **linear homogeneous equation, the general solution** can be obtained by putting $Q = 0$ in Eqn. (11) as

$$y = \alpha\, e^{-\int P\,dx} \tag{12}$$

Note that the first term on the right hand side of Eqn. (11) is due to non-homogeneous term Q of Eqn. (6). It is termed as a **particular integral** (P.I) of Eqn. (6). Thus, we have from Eqn. (11)

$$y = \mathbf{P.I} + \alpha e^{-\int Pdx} \tag{13}$$

where $$\mathbf{P.I} = e^{-\int P\,dx} \int Q e^{\int P\,dx} dx. \tag{14}$$

A particular integral does not contain any arbitrary constant.

We now take up some examples and illustrate the method discussed above.

Example 2: Solve $x\frac{dy}{dx} - ay = x + 1$, a is a positive constant and $x > 0$.

Solution: Clearly, the given equation is linear and can be written in the form

$$\frac{dy}{dx} - \frac{a}{x} y = \frac{x+1}{x} \tag{15}$$

$\therefore$ I.F. $= e^{\int (-a/x)\,dx} = e^{-a\ln x} = e^{\ln x^{-a}} = \dfrac{1}{x^a}$.

Multiplying Eqn. (15) by $\dfrac{1}{x^a}$, we get

$$\frac{1}{x^a}\frac{dy}{dx} - \frac{a}{x^{a+1}}\,y = \frac{x+1}{x^{a+1}}$$

i.e., $\dfrac{d}{dx}\left(\dfrac{y}{x^a}\right) = \dfrac{x+1}{x^{a+1}}$.

Integrating the above equation w.r.t. x, we get

$$\frac{y}{x^a} = \int \frac{x+1}{x^{a+1}}\,dx + c \quad (c \text{ is a constant})$$

$$= \frac{x^{-a+1}}{-a+1} + \frac{x^{-a}}{-a} + c.$$

Thus, $y = \dfrac{x}{1-a} - \dfrac{1}{a} + cx^a$ is the required solution.

***

Let us look at another example in which the role of x and y has been interchanged.

Example 3: Solve $y\ln y\dfrac{dx}{dy} + x - \ln y = 0,\ x > 0,\ y > 0$.

Solution: This equation is of first degree in x and $\dfrac{dx}{dy}$. Hence it is a linear equation with y as independent variable and x as dependent variable.
The given equation can be written as

$$\frac{dx}{dy} + \frac{x}{y\ln y} = \frac{1}{y}. \qquad (16)$$

$\therefore$ I.F. $= e^{\int 1/(y\ln y)\,dy} = e^{\int \frac{1/y}{\ln y}dy} = e^{\ln(\ln y)} = \ln y$

Multiplying Eqn. (16) by $\ln y$, we get

$$\ln y\frac{dx}{dy} + \frac{1}{y}x = \frac{1}{y}\ln y,$$

i.e., $\dfrac{d}{dy}(x\ln y) = \dfrac{1}{y}\ln y$,

Integrating the above equation w.r.t. y, we get

$$x\ln y = \int \frac{1}{y}\ln y\,dy + c$$

$$= \frac{(\ln y)^2}{2} + c,\ c \text{ is a constant.}$$

or $2x\ln y = (\ln y)^2 + c_1$ is the required solution where $c_1 = 2c$.

***

Let us consider another example.

Example 4: Solve the equation $ydx+(3x-xy+2)\,dy=0,\ x\neq 0,\ y\neq 0$.

Solution: Since the product $y\,dy$ occurs here, the equation is not linear in dependent variable y. It is, however, linear if we treat variable y as independent variable and x as dependent variable. Therefore, we arrange the terms as

$$y\,dx+(3-y)\,x\,dy=-2\,dy,$$

and write it in the standard form

$$\frac{dx}{dy}+\left(\frac{3}{y}-1\right)x=-\frac{2}{y},\ \text{for } y\neq 0 \qquad (17)$$

Now, $\int\left(\frac{3}{y}-1\right)dy=3\ln|y|-y$,

so that an integrating factor for Eqn. (17) is

$$\begin{aligned}e^{(3\ln|y|-y)}&=e^{-y}e^{3\ln|y|}\\&=e^{\ln|y|^3}e^{-y}\\&=|y|^3e^{-y}\end{aligned}$$

It follows that for $y>0,\ y^3e^{-y}$ is an integrating factor and for $y<0,\ -y^3e^{-y}$ serves as an integrating factor for the given equation. In either case, we obtain the exact differential equation

$$y^3e^{-y}dx+y^2(3-y)e^{-y}x\,dy=-2y^2e^{-y}dy,$$

i.e., $d(xy^3e^{-y})=-2y^2e^{-y}dy$.

Integrating the above equation w.r.t. y, we get

$$\begin{aligned}xy^3e^{-y}&=-2\int y^2e^{-y}\,dy\\&=2y^2e^{-y}-4\int e^{-y}y\,dy \quad \text{(Integrating by parts)}\\&=2y^2e^{-y}+4ye^{-y}+4e^{-y}+c.\end{aligned}$$

Thus, we can express the required solution as
$xy^3=2y^2+4y+4+ce^y$, where c is an arbitrary constant.

***

You may now try the following exercises.

E2) Solve the following equations:

i) $(x^2+1)\frac{dy}{dx}+2xy=4x^2$.

ii) $\frac{dy}{dx}+\frac{2}{x}y=\sin x,\ x\neq 0$.

iii) $\sec x\frac{dy}{dx}+y=\sin x$.

iv) $(1+y^2)\,dx = (\tan^{-1} y - x)\,dy$.

v) $(2x - 10y^3)\dfrac{dy}{dx} + y = 0$.

vi) $\dfrac{dy}{dx} - 2x\,|\,y\,| = 1$

E3) Solve the following equations.

i) $y' = y + \dfrac{e^x}{x}, \quad x \in [1, \infty[$.

ii) $y' = y + x + x^3 + x^5$.

ii) $y' = y + x\sin x\, e^x + x^5$.

iv) $y' + 3y = |\,x\,|, \; y(0) = 1$.

We have seen that the general solution of a linear non-homogeneous differential Eqn. (6) as given by Eqn. (11) involves integrals. Sometime it becomes impossible to evaluate these integrals in terms of known functions. This you must have realised while solving the exercise E2) vi). Further, the complicacy in the evaluation of a particular integral $e^{-\int P\,dx} \int Q(x)\, e^{\int P\,dx}\, dx$ in Eqn. (6) will depend on the form of $Q(x)$. This evaluation may sometimes turn out to be a tedious task. However, in some cases, there are methods available for finding a particular integral without rigorous integration. We shall now briefly discuss these methods of finding a particular integral of Eqn. (6). As these methods are more helpful for higher order differential equations, we shall discuss them in greater detail in Block 3.

8.3.1 Method of Undetermined Coefficients

The method of undetermined coefficients is applicable when in Eqn. (6), i.e.,

$$\frac{dy}{dx} + P(x)\,y = Q(x),$$

$P(x)$ is a constant and $Q(x)$ is any of the following forms:

i) an exponential function

ii) a polynomial in x

iii) of the form $\cos\beta x$ or $\sin\beta x$, β a constant.

iv) a linear combination of i), ii) and iii) above.

The general procedure is to assume the form of a particular solution involving arbitrary or unknown constants and then substitute the assumed expression into given equation to determine the constants satisfying that equation.

We know that on integrating/differentiating functions such as $e^{\alpha x}$ (α constant), x^r ($r > 0$ an integer), $\sin\beta x$ or $\cos\beta x$ (β constant), we again obtain an exponential function, a polynomial or a function which is a linear combination of sine or cosine functions. Hence if the non-homogeneous term $Q(x)$ in Eqn. (6) is in any of the forms i)-iv) above, then we can choose a particular integral accordingly as a suitable combination of the terms in i)-iv) above.

We now take up different cases according to the forms of $Q(x)$.

Case I: $Q(x) = k\,e^{mx}$, k and m are real numbers.

In this case if we take $P(x) = a$ (a constant) then Eqn. (6) reduces to

$$\frac{dy}{dx} + ay = ke^{mx}. \qquad (18)$$

Since $Q(x)$ is an exponential function, we assume a particular solution $y_p(x)$ of Eqn. (18) as well in the exponential form as $y_p(x) = re^{mx}$ where r is a constant to be determined.

Now if $y_p(x)$ is a solution of Eqn. (18) then it must satisfy it. Thus, we get

$$rm\,e^{mx} + ar\,e^{mx} = ke^{mx}$$

or $\quad r = \dfrac{k}{a+m}$ if $m \neq -a$.

Therefore, $y_p(x) = \dfrac{k}{a+m} e^{mx}$ if $m \neq -a$

In case $m + a = 0$, i.e., $m = -a$, then you may verify that $y_p(x) = kx\,e^{mx}$ satisfies Eqn. (18). Thus, we have the following result:

If a, k and m are real constants, then a particular solution of the equation $\dfrac{dy}{dx} + ay = k\,e^{mx}$, is given by

$$y_p(x) = \begin{cases} \dfrac{k}{(a+m)} e^{mx}, & \text{if } m \neq -a \\ kxe^{mx}, & \text{if } m = -a \end{cases}.$$

We illustrate this case by the following example.

Example 5: Solve $y' - y = 2e^x$.

Solution: On comparing the given equation with Eqn. (18), we find that $a = -1, k = 2$, and $m = 1$.

Also, $m + a = -1 + 1 = 0 \Rightarrow m = -a$

$\therefore$ From the results above, a particular integral is $2xe^x$.

Further, using relation (10) I.F. $= e^{\int P\,dx} = e^{-\int dx} = e^{-x}$. $(\because P = -1)$

Therefore, required solution, following relation (13) is

$$y = \text{P.I} + c\,e^x,$$

i.e., $y = 2xe^x + c\,e^x$ where c is a constant.

***

Example 6: Solve $y' + 4y = 2e^{2x},\ y(0) = 1$.

Solution: On comparing the given equation with Eqn. (18), we get

$$a = 4,\ k = 2,\ m = 2$$

Therefore, a particular integral $y_p(x) = \dfrac{k}{a+m} e^{2x} = \dfrac{1}{3} e^{2x}$.

Further, using relation (10) I.F. $= e^{\int 4\, dx} = e^{4x}$ $(\because P = 4)$

The general solution, following relation (13) is

$$y = \frac{1}{3} e^{2x} + c e^{-4x}$$

where c is an arbitrary constant.

Using the given initial condition $y(0) = 1$, we get

$$1 = \frac{1}{3} + c \text{ or } c = \frac{2}{3}.$$

Thus, $y = \dfrac{1}{3} e^{2x} + \dfrac{2}{3} e^{-4x}$ is the required solution.

***

You may now try this exercise.

E4) Solve the following differential equations for given constant b

i) $\dfrac{dy}{dx} + y = 2be^{x}$.

ii) $2\dfrac{dy}{dx} - 6y = be^{3x}$.

Let use now consider the case when $Q(x)$ is a polynomial.

Case II: $\boldsymbol{Q(x) = \sum_{i=0}^{n} a_i x^i}$, where a_i's are real numbers,

that is, $Q(x)$ is a polynomial of degree n. In this case Eqn. (6) for $P(a) = a$ (a being constant) reduces to

$$\frac{dy}{dx} + ay = \sum_{i=0}^{n} a_i x^i \tag{19}$$

If $a = 0$ in Eqn. (19), then a particular solution is

$y_p(x) = \sum_{i=0}^{n} \dfrac{a_i}{i+1} x^{i+1}$, which follows by direct integration.

If in Eqn. (19), $a \neq 0$, then we assume a particular solution in the form

$y_p(x) = \sum_{i=0}^{n} b_i x^i$ ($Q(x)$ being a polynomial in this case), and determine real numbers $b_0, b_1, \ldots, b_n$ so that a particular solution $y_p(x)$ satisfies Eqn. (19).

Substituting this value of $y_p(x)$ in Eqn. (19) (with y replaced by $y_p(x)$), we have

$$\sum_{i=1}^{n} i\, b_i x^{i-1} + \sum_{i=0}^{n} a\, b_i\, x^i = \sum_{i=0}^{n} a_i x^i \ (a \neq 0) \tag{20}$$

Equating the coefficients of like powers of x on both the sides of Eqn. (20), we get

$$\left.\begin{array}{l}\text{coeff. of } x^i : (i+1)b_{i+1} + ab_i = a_i \text{ for } i = 0, 1, 2, \ldots, (n-1) \\ \text{coeff. of } x^n : ab_n = a_n\end{array}\right\} \quad (21)$$

Since $Q(x)$ is a polynomial of degree n, we have $a_n \neq 0$ and we can solve Eqn. (21) for $b_0, b_1, \ldots, b_n$. From Eqn. (21), we get

$$b_n = a_n / a$$

$$b_{n-1} = \left(a_{n-1} - \frac{na_n}{a}\right)\frac{1}{a},$$

$$b_{n-2} = a_{n-2} - \frac{n-1}{a}\left(a_{n-1} - \frac{n}{a}a_n\right)\frac{1}{a}, \text{ and so on.}$$

Once we have obtained $b_0, b_1, b_2, \ldots, y_p(x) = \sum_{i=0}^{n} b_i x^i$ is a particular solution of Eqn. (19).

We illustrate this method with the help of examples.

Example 7: Find a particular solution of $\frac{dy}{dx} + 2y = 2x^2 + 3$.

Solution: In this case $Q(x)$ is a polynomial of degree 2. We thus assume a particular solution of the form:

$$y_p(x) = \sum_{i=0}^{2} b_i x^i = b_0 + b_1 x + b_2 x^2.$$

Substitution of $y_p(x)$ in the given equation yields

$$(b_1 + 2b_2 x) + 2(b_0 + b_1 x + b_2 x^2) = 2x^2 + 3 \quad (22)$$

Equating the coefficients of like powers of x on both the sides of Eqn. (22), we get

Coeff. of x^2 $: 2b_2 = 2$ or $b_2 = 1$.

Coeff. of x $: 2b_2 + 2b_1 = 0$ or $b_1 = -1$

Coeff. of x^0 $: b_1 + 2b_0 = 3$ or $b_0 = 2$

Hence, the required particular solution is

$$y_p(x) = x^2 - x + 2.$$

***

Let us look at another example.

Example 8: Solve $y' - 4y = x^3 + 2x^2 - 3$.

Solution: Here $Q(x)$ is a polynomial of degree 3. We thus assume a particular solution of the form

$$y_p(x) = \sum_{i=0}^{3} b_i x^i = b_0 + b_1 x + b_2 x^2 + b_3 x^3$$

Replacing y' and y in the given equation by $y_p'(x)$ and $y_p(x)$ respectively, we get

$$(b_1 + 2b_2 x + 3b_3 x^2) - 4(b_0 + b_1 x + b_2 x^2 + b_3 x^3) = x^3 + 2x^2 - 3.$$

Equating the coefficients of like powers of x, we have

Coeff. of x^3 $: -4b_3 = 1$ or $b_3 = \frac{-1}{4}$

Coeff. of x^2 $: 3b_3 - 4b_2 = 2$ or $b_2 = \frac{-11}{16}$

Coeff. of x $: 2b_2 - 4b_1 = 0$ or $b_1 = \frac{-11}{32}$

Coeff. of x^0 $: b_1 - 4b_0 = -3$ or $b_0 = \frac{85}{128}$

Hence a particular solution is

$$y_p(x) = \frac{85}{128} - \frac{11}{32}x - \frac{11}{16}x^2 - \frac{1}{4}x^3.$$

Also, I.F. $= e^{-\int 4dx} = e^{-4x}$

Thus the required solution is $y = y_p(x) + c\, e^{4x}$, where c is an arbitrary constant.

***

You may **observe** here that although the method is simple and straight forward in application, the calculations at times, may turn out to be laborious.

And now an exercise for you.

E5) Solve the following differential equation.

i) $\frac{dy}{dx} = y + x^2$.

ii) $\frac{dy}{dx} - 2y = (x+1)^2$.

We now consider the case when $Q(x)$ is a trigonometric function.

Case III: $Q(x) = \sin \beta x$ or $\cos \beta x$ or $a \sin \beta x + b \cos \beta x$,
where β, a and b are real constants.
In all these cases, we assume a particular solution of the form

$$c \sin \beta x + d \cos \beta x$$

where c and d are constants to be determined.

On substituting this solution in the given equation and equating the coefficients of $\sin \beta x$ and $\cos \beta x$ on both the sides of the equation, we determine the constants c and d.

Let us illustrate this case by an example.

Example 9: Find a particular solution of

$$\frac{dy}{dx} + y = \cos 3x.$$

Solution: Here $Q(x) = \cos 3x$. Hence, any particular solution of the given differential equation must be a combination of $\sin 3x$ and $\cos 3x$. Let a particular solution be of the form:

$$y_p(x) = c\cos 3x + d\sin 3x.$$

On substituting this value of $y_p(x)$ in the given equation, we get

$$(-3c\sin 3x + 3d\cos 3x) + (c\cos 3x + d\sin 3x) = \cos 3x \tag{23}$$

Comparing the coefficients of $\cos 3x$ and $\sin 3x$ on both the sides of Eqn. (23), we get

$$c + 3d = 1 \text{ and } d - 3c = 0$$

or, $\quad c = \dfrac{1}{10}$ and $d = \dfrac{3}{10}$.

Hence, a particular solution is

$$y_p(x) = \frac{1}{10}(3\sin 3x + \cos 3x).$$

***

We now take up an example which is a combination of all the three cases discussed above.

Example 10: Find the general solution of the differential equation

$$\frac{dy}{dx} + y = e^x + x + \sin x,\ x > 0,\ y > 0. \tag{24}$$

Solution: Here $Q(x) = Q_1(x) + Q_2(x) + Q_3(x)$,

where $Q_1(x) = e^x$, $Q_2(x) = x$ and $Q_3(x) = \sin x$.

By splitting the right hand side of Eqn. (24), we obtain the three equations

$$\frac{dy}{dx} + y = e^x \tag{25}$$

$$\frac{dy}{dx} + y = x \tag{26}$$

and

$$\frac{dy}{dx} + y = \sin x. \tag{27}$$

If y_1, y_2, y_3 are a particular solutions of Eqns. (25), (26) and (27), respectively, then $y_p = y_1 + y_2 + y_3$ is a particular solution of the given Eqn. (24). First consider Eqn. (25). Let its particular solution be of the form

$y_1 = r\,e^x$, where r is a constant to be determined.

Substituting this in Eqn. (25), we get

$$re^x + re^x = e^x \Rightarrow r = \frac{1}{2}$$

$$\therefore\ y_1 = \frac{1}{2}e^x. \tag{28}$$

For Eqn. (26) we assume a particular solution of the form

$y_2 = a_1 x + a_0$ and obtain the values of the constants a_0 and a_1.

Substituting this in Eqn. (26), we get

$$a_1 + a_1 x + a_0 = x \tag{29}$$

Comparing the coefficients of like powers of x on both the sides of Eqn. (29), we get

$$\left.\begin{aligned} a_0 + a_1 &= 0 \\ a_1 &= 1, \end{aligned}\right\} \Rightarrow a_0 = -1,\ a_1 = 1$$

Hence, $y_2 = x - 1$. (30)

In the case of Eqn. (27), assume a particular solution of the form

$$y_3 = c\sin x + d\cos x\,.$$

Substituting this in Eqn. (27), we get

$$c\cos x - d\sin x + c\sin x + d\cos x = \sin x \tag{31}$$

On equating the coefficients of $\sin x$ and $\cos x$ on both the sides of Eqn. (31), we get

$$\left.\begin{aligned} c - d &= 1 \\ c + d &= 0 \end{aligned}\right\} \Rightarrow c = \frac{1}{2} \text{ and } d = -\frac{1}{2}$$

$$\therefore\quad y_3 = \frac{1}{2}(\sin x - \cos x) \tag{32}$$

Hence, a particular solution of Eqn. (24) can be obtained from Eqns. (28), (30) and (32) as

$$y_p(x) = y_1 + y_2 + y_3 = \frac{1}{2}e^x + x - 1 + \frac{1}{2}(\sin x - \cos x)\,.$$

The solution of the homogeneous part of Eqn. (24), i.e.,

$$\frac{dy}{dx} + y = 0$$

is given by

$$\frac{1}{y}\frac{dy}{dx} + 1 = 0,\ y > 0\,.$$

Integrating the above equation, we get

$\ln\ y + x = \ln\ \alpha$, for some constant α

i.e., $\dfrac{y}{\alpha} = e^{-x}$

or, $y = \alpha e^{-x}$.

Hence, the general solution of Eqn. (24) is given by

$$y = \alpha e^{-x} + \frac{1}{2}e^x + x - 1 + \frac{1}{2}(\sin x - \cos x)\,.$$

***

Note that the solution of the homogeneous equation in Example 10 could also be obtained by finding an integrating factor as we have done in other examples.

You may now try the following exercise.

E6) Solve the following differential equations:

i) $\frac{dy}{dx} - y = 6\cos 2x$.

ii) $\frac{dy}{dx} + 3y = x^2 + 3e^{2x} + 4\sin x$.

You have thus seen that the method of undetermined coefficients for finding a particular integral of the non-homogeneous linear differential Eqn. (6) has certain limitations. The method is applicable only for a certain class of differential equations of the form (6) for which $P(x)$ is a constant and $Q(x)$ assumes either of the forms $e^{\alpha x}, x^r, \sin\beta x$ or $\cos\beta x$, or their combinations. We shall, now, study a method that carries no such restrictions. Such a method is due to Joseph Louis Lagrange (1736-1813) and is called the method of variation of parameters.

8.3.2 Method of Variation of Parameters

Consider the non-homogeneous linear Eqn. (6), namely,

$$\frac{dy}{dx} + P(x)\ y(x) = Q(x)$$

where P, Q are functions of x.
The homogeneous equation corresponding to the above linear equation is

$$\frac{dy}{dx} + P(x)\ y(x) = 0.$$

Further, we know from Eqn. (12), that the solution $y_h(x)$, of the homogeneous linear equation, is given by

$$y_h(x) = \alpha\, e^{-\int P(x)\,dx}, \tag{33}$$

where α is a constant.
The idea associated with the method of variation of parameters is to replace the constant α in Eqn. (33) by a function of x. That is, we vary α with x and assume that the resulting function

$$y(x) = \alpha(x)e^{-\int P(x)\,dx} \tag{34}$$

is again a solution of Eqn. (6). That is, we try to determine $\alpha(x)$ such that y given by Eqn. (34) satisfies Eqn. (6). In other words, we determine a necessary condition on $\alpha(x)$ so that y, defined by relation (34) satisfy Eqn. (6).

On combining Eqns. (6) and (34), we get

$$\frac{d}{dx}\left[\alpha(x)e^{-\int P(x)\,dx}\right] + P(x)\left[\alpha(x)e^{-\int P(x)\,dx}\right] = Q(x)$$

i.e., $\alpha(x)\left[-P(x)e^{-\int P(x)\,dx}\right] + \alpha'(x)e^{-\int P(x)dx} + P(x)\ \alpha(x)e^{-\int P(x)dx} = Q(x)$,

i.e., $\alpha'(x) = Q(x)e^{\int P(x)\,dx}$

Integrating w.r.t. x, we get

$$\alpha(x) = \beta + \int Q(x) e^{\int P(x)\,dx} dx \qquad (35)$$

where β is a constant of integration.

Substituting the value of $\alpha(x)$ from Eqn. (35) in relation (34), the solution of Eqn. (6) can be expressed as

$$y(x) = \beta e^{-\int P(x)\,dx} + e^{-\int P(x)\,dx} \int Q(x)\; e^{\int P(x)\,dx}\; dx$$

You may **note** here that the solution obtained above is same as the one given by Eqn. (11) which we obtained earlier. Further, the method of variation of parameters neither simplifies any integration/solution nor provides any other form of the solution for first order, first degree differential equation. It only provided an alternative approach to arrive at the general solution in this case. However, as we shall see later in Block 3, this method turns out to be quite powerful in solving equations of higher order.

Let us now consider an example to illustrate the method.

Example 11: Find the general solution of the equation

$$x\frac{dy}{dx} - 4y = x^6 e^x .$$

Solution: The equation can be written as

$$\frac{dy}{dx} - \frac{4}{x} y = x^5 e^x .$$

Homogeneous equation corresponding to the above equation is

$$\frac{dy}{dx} - \frac{4}{x} y = 0$$

and its solution is given by

$$y_h(x) = \alpha e^{-\int -\frac{4}{x} dx} = \alpha e^{\int \frac{4}{x} dx} = \alpha e^{4 \ell n\, x} = \alpha x^4$$

Let $y(x) = \alpha(x) x^4$ be the solution of the given non-homogenenous equation. Then we have

$$\frac{d}{dx}(\alpha(x)x^4) - \frac{4}{x}(\alpha(x)x^4) = x^5 e^x$$

$$\Rightarrow \quad \alpha' x^4 + 4\alpha x^3 - 4\alpha x^3 = x^5 e^x$$

$$\Rightarrow \quad \alpha' = x e^x$$

$$\Rightarrow \quad \alpha = \int x e^x dx$$

Integrating by parts we get

$$\alpha(x) = x e^x - \int e^x dx + c$$

$= x e^x - e^x + c,\ c$ being a constant of integration.

Substituting the value of $\alpha(x)$ in $y(x) = \alpha(x) x^4$, we get

$$y(x) = (x e^x - e^x + c) x^4$$

$$= x^5 e^x - x^4 e^x + c x^4$$

as the required solution.

***

You may now try to solve the following exercise.

E7) Solve the following differential equations:

i) $y' - 2y = \sin \pi x + \cos \pi x,\ y(1) = 1.$

ii) $y' - y = \cos 2x + e^x + e^{2x} + x$.

iii) $y' - 3y = x^2 - \cos 3x + 2$.

iv) $y' + y = -x - x^2,\ y(0) = 0$.

v) $y' - y = e^x,\ y(0) = -3$.

In the next section we shall take up equations which are not linear but can be reduced to the linear form by suitable transformations of the variables.

8.4 EQUATIONS REDUCIBLE TO LINEAR EQUATIONS

Let us consider the differential equation of the form

$$f'(y)\frac{dy}{dx} + P(x)\ f(y) = Q(x) \tag{36}$$

where $f'(y)$ is the differential coefficient of $f(y)$.
You can see that Eqn. (36) is non-linear equation. An interesting feature of this first order non-linear differential Eqn. (36) is that it can be reduced to linear equation by putting $v = f(y)$. Using this substitution in Eqn. (36) we get

$$\frac{dv}{dx} + P(x)v = Q(x), \left(\because \frac{dv}{dx} = f'(y)\,\frac{dy}{dx}\right),$$

which is a linear equation with v as dependent variable and x as independent variable.

Consider for example a non-linear differential equation

$$\frac{dy}{dx} + \frac{1}{x}y = xy^2,\ x > 0,\ y > 0 \tag{37}$$

Dividing by y^2 throughout, Eqn. (37) can be written as

$$\frac{1}{y^2}\,\frac{dy}{dx} + \frac{1}{x}\,\frac{1}{y} = x \tag{38}$$

Substituting $\frac{1}{y} = v$ in Eqn. (38), we obtain

$$-\frac{dv}{dx} + \frac{v}{x} = x \ \left(\because \frac{-1}{y^2}\,\frac{dy}{dx} = \frac{dv}{dx}\right)$$

or $$\frac{dv}{dx} - \frac{v}{x} = -x \tag{39}$$

Eqn. (39) is a linear equation with v as the dependent variable and x as independent variable. It can now be solved by the method discussed in Sec. 8.3. We have,

$$\text{I.F.} = e^{-\int \frac{dx}{x}} = e^{-\ln|x|} = x^{-1}$$

Hence $\frac{d}{dx}(x^{-1}v) = -1$.

Integrating the above equation, we get

$$x^{-1}v = -x + c \text{ or } v = -x^2 + cx$$

where c is an arbitrary constant.

Since $v = y^{-1}$, we obtain

$$y = \frac{1}{-x^2 + cx}$$

which is the general solution of Eqn. (37).

Eqn. (37) is an example of a very important and famous equation known as **Bernoulli's Equation**, named after James Bernoulli, who studied it in 1695 for finding its solution. The general form of the equation is

$$\frac{dy}{dx} + P(x)\,y = Q(x)\,y^n, \tag{40}$$

where P and Q are functions of x alone and n is any real number not equal to zero or one. For $n = 0$ and $n = 1$ Eqn. (40) is linear and can be solved by the method discussed in Sec. 8.3.

For $y \neq 0$ dividing Eqn. (40) by y^n, we get

$$y^{-n}\frac{dy}{dx} + P\;y^{1-n} = Q. \tag{41}$$

In the year 1696, Leibniz pointed out that Eqn. (41) can be reduced to a linear equation by considering y^{1-n} as the new dependent variable.

On putting $v = y^{1-n}$ in Eqn. (41), it reduces to

$$\frac{1}{1-n}\frac{dv}{dx} + P\,v = Q\,, \qquad \left(\because \frac{dv}{dx} = (1-n)\,y^{-n}\frac{dy}{dx}\right) \tag{42}$$

which is a linear differential equation in v and x. Eqn (42) can now be solved by the known methods.

Note that when $n = 0$, Eqn. (40) is a linear non-homogeneous equation and when $n = 1$, Eqn. (40) is a linear homogeneous equation.

Let us take up some more examples to illustrate the method above.

Example 12: Solve $\dfrac{dy}{dx} - \dfrac{\tan y}{1+x} = (1+x)\,e^x \sec y,\ x > 0, y > 0.$ (43)

Solution: Dividing Eqn. (43) by $\sec y$, we get

$$\cos y\frac{dy}{dx} - \frac{\sin y}{1+x} = (1+x)e^x. \tag{44}$$

If we put $\sin y = f(y)$ in Eqn. (44), then $f'(y) = \cos y$ and hence Eqn. (44) reduces to the form

$$f'(y)\frac{dy}{dx} - \frac{1}{1+x}f(y) = (1+x)\,e^x$$

which is of the type of Eqn. (36). To reduce it to linear form, we put

$v = f(y)$ and obtain

$$\frac{dv}{dx} - \frac{1}{1+x}\,v = (1+x)e^x. \tag{45}$$

It is a linear equation with I.F. $= e^{-\int \frac{1}{1+x}dx} = e^{-\ln\,(1+x)} = \dfrac{1}{1+x}.$

Multiplying Eqn. (45) by I.F., we get

$$\frac{d}{dx}\left(v\frac{1}{1+x}\right)=\frac{1}{1+x}\,(1+x)\;e^x=e^x$$

Integrating the above equation w.r.t. x, we have

$v\dfrac{1}{1+x}=e^x+c,\ c$ being a constant

i.e., $v=(1+x)e^x+c(1+x)$

Substituting $\sin y$ for v, the required solution of the given Eqn. (43) is

$$\sin y=(1+x)\;e^x+c\,(1+x).$$

***

Let us look at another example.

Example 13: Solve $y\,(axy+e^x)\,dx-e^x\;dy=0,\ x>0,\ y>0$.

Solution: The given equation can be rearranged as

$$e^x\frac{dy}{dx}=e^x y+axy^2\,,$$

i.e., $\dfrac{dy}{dx}-y=axe^{-x}y^2$. (46)

It is a Bernoulli's equation with $n=2$.

To solve it, let $y^{1-2}=v$, i.e., $v=\dfrac{1}{y}$. Then

$$\frac{dv}{dx}=-\frac{1}{y^2}\frac{dy}{dx}=-v^2\frac{dy}{dx}$$

Consequently, Eqn. (46) reduces to

$$-\frac{dv}{dx}-v=axe^{-x},$$

i.e., $\dfrac{dv}{dx}+v=-axe^{-x}$ (47)

It is a linear equation with $\because\quad =e^{\int 1.dx}=e^x$

Multiplying both sides of Eqn. (47) by I.F., we get

$$\frac{d}{dx}(ve^x)=-ax$$

Integrating w.r.t. x, we get

$$ve^x=-\int ax\;dx+c$$

$$=-\frac{ax^2}{2}+c,\ c \text{ a constant.}$$

Replacing v by $\left(\dfrac{1}{y}\right)$, the required solution can be expressed as

$$y=e^x\left(c-\frac{ax^2}{2}\right)^{-1}.$$

***

Remark: There are many second or higher order linear equations which can be solved easily by reducing them to linear first order equations by making some transformation of the variables. We shall take up such equations later in Block 3 when we discuss second order equations.

You may now try the following exercises.

E8) Solve the following differential equations:

a) $\frac{dy}{dx} = \frac{e^y}{x^2} - \frac{1}{x}, \ x > 0$.

b) $xy(x^2y^2 - 1)dx = dy$.

c) $3e^x \tan y + (1 - e^x)\sec^2 y \frac{dy}{dx} = 0$.

E9) Find the solutions of the following differential equations:

a) $dy = x(x^2 \cos^2 y - \sin 2y)dx$.

b) $\frac{dy}{dx} + y = e^x y^3$.

c) $2x\frac{dy}{dx} + y(6y^2 - x - 1) = 0, \ x > 0$.

You may recall that in Unit 6, we discussed some physical situations which when translated into mathematical terms could be expressed in terms of linear differential equations. In the following section we shall find solutions to those problems and also discuss some more physical applications of linear differential equations.

8.5 APPLICATIONS OF LINEAR DIFFERENTIAL EQUATIONS

Let us first consider the population model discussed in Unit 6.

I. Population Model

You may recall that while studying the population problem we had arrived at the initial value problem. (Ref. Eqns. (46) and (47) of Unit 6)

$$\left.\begin{array}{l} \frac{d}{dt}N(t) = k\ N(t) \\ N(t_0) = N_0 \end{array}\right\} \tag{48}$$

Since k is a constant the differential equations in (48) is a linear differential equation of order one. Using the method discussed in Sec. 8.3 solution can be expressed as

$$N(t) = N(t_0)\exp\,[k(t - t_0)]. \tag{49}$$

In Eqn. (49), we normally assume that $N(t_0)$, the population at some initial time t_0, is specified. If k is known then we can find the solution using initial condition given in Eqn. (48). In a particular case, we can actually find the exact value of k, (which gives the rate of growth) if we know the value of N

at $t_1 (t_1 \neq t_0)$. We shall now take up an example to illustrate the procedure for doing so.

Example 14: Assuming that the rate of growth of a species is proportional to the amount $N(t)$ present at time t, find the value of $N(t)$. It is given that $N(0) = 100$ and after one unit of time, the size of the species has grown to 200.

Solution: In this case $t_0 = 0,\ N(0) = 100$. The solution of the problem is given by

$$N(t) = 100\ \exp(kt),\ t \geq 0,\ k > 0$$

We determine k from the additional condition $N(1) = 200$ ($N(1) =$ size of the population at time $t = 1$).
Thus $200 = 100\ \exp(k) \Rightarrow k = \ln 2$.
Hence the solution is

$$N(t) = 100\ \exp(t \ln 2) = 100\ \exp(\ln 2^t)$$

or $\quad N(t) = (100)\, 2^t$, which gives the size of the species at any time t.

***

Example 15: In a culture of yeast, the amount A of active yeast grows at a rate proportional to the amount present. If the original amount A_0 doubles in 2 hours, how long does it take for the original amount to triple?

Solution: The equation governing the growth of yeast is

$A(t) = A_0 e^{kt}$, where the constant $k > 0$.

It is given that $A = 2A_0$ when $t = 2$, thus we get $2A_0 = A_0 e^{2k} \Rightarrow 2 = e^{2k}$. Thus at any time t, we have

$$A(t) = A_0 (2)^{t/2}.$$

To find t when $A = 3A_0$, we get from the above equation

$$3A_0 = A_0 (2)^{t/2}$$

$\Rightarrow \quad 3 = (2)^{t/2}$ or $\ln 3 = \dfrac{t}{2} \ln 2$

$\Rightarrow \quad t = \dfrac{2 \ln 3}{\ln 2} \approx 3.17$ hrs.

Hence it will take approximately 3.17 hrs. i.e., 3 hrs. 10 minutes 12 seconds, for the original amount A_0 of yeast to triple.

***

You can now try to do the following exercises by using the data given in the problems.

E10) A culture initially has N_0 number of bacteria. At $t = 1$ hour, the number of bacteria is measured to be $\left(\dfrac{3}{2}\right) N_0$. If the rate of growth is proportional to the number of bacteria present, determine the time necessary for the number of bacteria to triple.

E11) The population of a town grows at a rate proportional to the population at any time. Its initial population of 500 increases by 15% in 10 years. What will be the population in 30 years?

Let us now consider another application of linear differential equation and discuss the problem of decay of radioactive material.

II. Radioactive Decay

In Unit 6, (Ref Eqn. (52)), we have seen that equation governing the decay of a given radioactive substance is given by

$$y'(t) = k\ y(t) \tag{50}$$

Half-life is the time needed for the material to reduce itself to half of its original mass.

where $y(t)$ is the mass of the radioactive material at time t and $k<0$ is a real constant. Eqn. (50) can be used to find the half-life of the radioactive material.

The following example illustrates the application of the model.

Example 16: A radioactive substance with a mass of 50 gms. was found to have a mass of 40 gms. after 30 years. Find its half-life.

Solution: The mass $y(t)$ of the material satisfies

$$\left.\begin{array}{r} \dfrac{d}{dt}\ y(t) = k\ y(t) \\ y(0) = 50\,\text{gms.} \\ y(30) = 40\,\text{gms.} \end{array}\right] \tag{51}$$

where $k<0$ is a real constant.
The solution of the first two equations in Eqn. (51) can be expressed as

$$y(t) = 50\ \exp(k\,t)$$

Using the third equation in Eqn. (51), we can write

$$y(30) = 40 = 50\exp\ (30k),$$

or $\exp\ (30k) = 4/5$,

i.e., $k = \dfrac{1}{30}\ ln\left(\dfrac{4}{5}\right)$.

Thus, the mass $y(t)$ satisfies

$$y(t) = 50\ \exp\left(\frac{t}{30}\ \ln\frac{4}{5}\right). \tag{52}$$

Let t_1 be its half-life, i.e., after time t_1 the mass reduces to $\dfrac{50}{2} = 25\,\text{gms}$.

Then $y(t_1) = 25$. (53)

We are required to find t_1. Using condition (53) in Eqn. (52) we get

$$25 = 50\ \exp\left(\frac{t_1}{30}\ln\ \frac{4}{5}\right)$$

or $t_1\ \ln\ (4/5) = 30\ln\ (1/2)$.

i.e., $t_1 = 30\ (\ln\ (1/2))/\ln\ (4/5) = \dfrac{30\times\ln(0.5)}{\ln(0.8)} = \dfrac{-20.7944}{-0.22314} \approx 93$ years. (54)

So approximately after 93 years, the mass of the material will be 25 gms. i.e., half of its original mass.

***

Example 17: It is found that 0.5% of radium disappears in 12 years. Using the law of radioactive decay find (i) what percentage will disappear in 1000 years? (ii) what is the half-life of radium?

Solution: Let after t years, A be the quantity (in gms.) of radium present. Then by the law of radioactive decay we have

$$\frac{dA}{dt} = kA$$

where $k < 0$ since the amount of radium is decreasing. If we assume that A_0 be the amount (in gms.) of radium present initially, then 0.5% of A_0 i.e., $0.005A_0$ gms. disappears in 12 years and hence $0.995A_0$ gms. remains. We thus have the initial conditions:

$A = A_0$ gms. at $t = 0$

and $A = 0.995A_0$ gms. at $t = 12$ (years).

Solving the above equation under these conditions, we get

$$A = A_0 e^{kt}$$

Using $A = 0.995A_0$ at $t = 12$, we get

$0.995A_0 = A_0 e^{12k}$ or $e^{12k} = 0.955$

or $12k = \ln(0.995) \approx -0.005$

thus $k = \dfrac{-0.005}{12} \approx -0.0004$

and $A = A_0 e^{-0.0004t}$.

i) When $t = 1000$, $A = A_0 e^{-0.4} \approx 0.670A_0$.

Thus, approximately, $0.670A_0$ gms. of radium is present after 1000 years.

$\therefore$ Amount disappeard $\approx A_0 - .670A_0 = 0.33A_0$ gms. i.e., approximately 33%.

ii) When $A = \dfrac{A_0}{2}$ then we have $\dfrac{A_0}{2} = A_0 e^{-0.0004t}$,

or $e^{0.0004t} = 2$ or $0.0004t = \ln 2 = 0.693147$

or $t = \dfrac{\ln 2}{0.0004} \approx 1732.87$ years.

Thus, half life of the radium is approximately 1733 years.

***

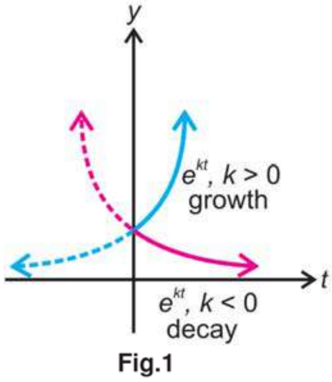


Fig.1

Fig. 1 shows that the exponential function e^{kt} increases as t increases for $k > 0$ and decreases as t increases if $k < 0$. Thus the problems of population growth such as growth of bacteria, species etc. are described by a positive value of k, whereas problems involving decay, will yield a negative value of k.

Let us now deal with the temperature variations of a hot object.

III. Newton's Law of Cooling

The temperature of a hot body kept in a surrounding of constant temperatue T_0 has been discussed in Unit 6. The equation governing the temperature T of the body kept in a surrounding of constant temperature T_0 (Ref. Eqn. (51) of Unit 6) is given by

$$T'(t) = k(T(t) - T_0)\text{, where the constant } k < 0. \tag{55}$$

We now consider an example to illustrate the above situation.

Example 18: A rod of temperature $100^\circ C$ is kept in a surrounding of temperature $20^\circ C$. If the temperature of the rod was found to be $80^\circ C$ after 10 minutes, find the temperature $T(t)$ of the rod.

Solution: We are required to solve

$$\frac{d}{dt}T(t) = k(T(t) - 20), T(0) = 100 \text{ and } T(10) = 80. \tag{56}$$

Let us put $y(t) = T(t) - 20$. Then $y'(t) = T'(t)$ and Eqn. (56) reduces to

$$\frac{d}{dt}y(t) = k\ y(t). \tag{57}$$

Eqn. (56) is not a linear homogeneous equation whereas Eqn. (57) is (which explains the reasons for introducing y). Along with Eqn. (57), we have the following two conditions:

$$\left.\begin{array}{ll} \text{(i)} & y(0) = T(0) - 20 = 100 - 20 = 80^\circ C, \\ \text{(ii)} & y(10) = T(10) - 20 = 80 - 20 = 60^\circ C \end{array}\right] \tag{58}$$

The solution of Eqn. (57), with the condition 58(i), is

$$y(t) = 80\exp(kt).$$

With this value of y and condition 58(ii), we have

$$y(10) = 60 = 80 \exp\ (k.10)$$

or $$k = \frac{1}{10}\ln(6/8) = \frac{1}{10}\ln\ (3/4).$$

Hence the value of y is determined by

$$y(t) = 80\exp\left(\frac{t}{10}\ln\ (0.75)\right),$$

and the temperature T of the rod is given by

$$T(t) = 80\exp\left(\frac{t}{10}\ln(0.75)\right) + 20.$$

***

As another application of the Newton's law of cooling we now take up an example for the determination of the time of death of a human being. The temperature of the dead body taken at two different times gives an estimate of the constant k in Eqn. (55) and then we find the time at which T equals the temperature of a living body. The normal temperature of a human body is considered to be $98.6^\circ F$.

To have a better insight, let us consider the following example.

Example 19: The body of a murder victim was discovered at 11.00 p.m. in a room having constant temperature of $70^\circ F$. The doctor took the temperature of the body at 11.30 p.m., which was $94.6^\circ F$. He again took the temperature after one hour when it showed $93.4^\circ F$. Estimate the time of death.

Solution: The differential equation governing the situation is

$$\frac{d(T(t))}{dt} = k(T - T_0) \tag{59}$$

subject to the conditions

$$T_0 = 70, T(0) = 94.6 \text{ and } T(1) = 93.4. \tag{60}$$

The solution of Eqn. (59) is obtained as

$$T(t) = T_0 + ce^{kt}, \text{ where } c \text{ is a constant.} \tag{61}$$

Using in Eqn. (61) the conditions given in Eqn. (60), we get

$$c = 24.6$$

and $k = \ln\left(\frac{23.4}{24.6}\right) = -0.05$

For the above values of c and k, Eqn. (61) gives

$$T(t) = 70 + 24.6e^{-0.05\,t} \tag{62}$$

We are now required to find the value of t when $T = 98.6$ from Eqn. (62). After simplification we obtain

$$t = -3.0132055 \text{ hrs.}$$

Therefore, the estimated time of death is

$$11.30 - 3.0 = 8.30 \text{ p.m. (approx.).}$$

***

And now exercises for you.

E12) Suppose that a thermometer having a reading of $70^\circ F$ inside a house is placed outside where the air temperature is $10^\circ F$. Three minutes later it is found that the thermometer reading is $25^\circ F$. Find the temperature reading $T(t)$ of the thermometer at time t.

E13) Initally there was 100 gms of a radioactive substance present. After 6 years the mass decreased by 3%. If the rate of decay is proportional to the amount of the substance present at any time, find the half-life of the radioactive substance.

We now take up the problem of mixing of two fluids.

IV. Mixture Problem

The mixing of two fluids sometimes gives rise to a linear first-order differential equation. We explain the formulation of such problems through an example considering the mixture of two salt solutions of different concentrations.

Example 20: A tank containing 300 litres of water has 50 gms of salt dissolved in it. A brine solution is pumped into the tank at a rate of 3 litres per minute, and the well-stirred solution is then pumped out at the same rate (see Fig. 2). If the concentration of the solution entering is 2 gms per litre, determine the amount of salt in the tank at any time. How much salt is present (i) after 50 minutes? (ii) after a long time?

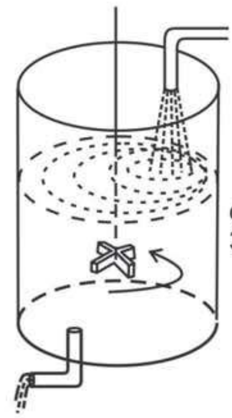


Fig. 2

Solution: Let $P(t)$ be the amount of salt (in gms) in the tank at any time t.

The rate at which $P(t)$ changes is $\frac{dP}{dt}$ and

$$\frac{dP}{dt} = \text{(rate of inflow of salt)} - \text{(rate of outflow of salt)}$$

$$= R_1 - R_2 \tag{63}$$

Eqn. (63) is called the equation of continuity. It states that the mass P is conserved, i.e., no amount of P is created or destroyed in the process.
Now the rate R_1 of inflow of salt in the tank in gms/min. is

$$R_1 = (3\,\text{lit/min}).\,(2\ \text{gms/lit}) = 6\ \text{gms/min.}$$

whereas, the rate R_2 of outflow of salt in the tank is

$$R_2 = (3\ \text{lit/min}).\left(\frac{P}{300}\text{gms/lit}\right) = \frac{P}{100}\text{gms/min.}$$

Note that in the above equation $\frac{P}{300}$ is the concentration of salt in the tank at time t.
Substituting the above values of R_1 and R_2 in Eqn. (63), we obtain the governing equation as

$$\frac{dP}{dt} = 6 - \frac{P}{100}, \tag{64}$$

with the initial condition $P(0) = 50$.
Eqn. (64) is a linear equation with I.F. as $e^{t/100}$. We can thus write Eqn. (64) as

$$\frac{d}{dt}\,[P\,e^{t/100}] = 6e^{t/100}$$

and therefore $P(t) = 600 + c\,e^{-t/100}$, where c is a constant of integration.
Initial condition $P(0) = 50$ gives $c = -550$ and thus

$$P(t) = 600 - 550\,e^{-t/100}. \tag{65}$$

Substituting $t = 50$ in Eqn. (65) we find $P(50) = 266.41$ gms. Also from Eqn. (65) as $t \to \infty$, $P(t) \to 600$ gms. which is the amount of salt expected in the solution over a long period of time.

***

In Example 20 we assumed that the rate at which the solution was pumped in was the same as the rate at which the solution was pumped out. However, this may not be the case always. The solution could be pumped in at a rate faster or slower than the rate at which the solution is pumped out. We are leaving it to you to do the formulation of such a problem while doing the following exercise:

E14) A tank contains 100 litres of fresh water. Two litres of brine, each containing 1 gm of salt, run into the tank per minute. The mixture is kept uniform by stirring and it runs out at the rate of 1 litres per minute. Find the amount of salt present when the tank contains 150 litres of brine.

As a natural extension of the above application, we now take up the absorption problem.

V. Absorption of Drugs in Organs or Cells

For the purpose of studying biological problems mathematically, it is often convenient to consider an organism (such as human, animal or plant) as a collection of individual organs (such as stomach, pancreas, liver or kidney) called ‘compartments’. One such problem is the determination of absorption of chemicals, such as drugs by cells or organs. This has practical applications in the field of medicine. The simplest type of such problems deal only with one compartment. We now illustrate through an example, the formulation of a problem which can arise.

Example 21: A liquid carries a drug into an organ of volume V cm^3 at a rate $a$cm$^3/s$ and leaves at a rate $b$cm$^3/s$. The drug concentration of liquid entering is $c\,g$/cm^3. Find the concentration of the drug in the organ at any time t.

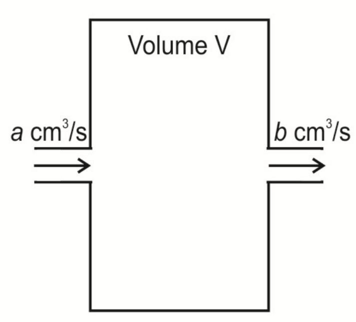


Fig. 3

Solution: Consider a single compartment of volume V together with an inlet and an outlet as shown in Fig. 3. Let $y(t)$ be the concentration of the drug in the organ at any time t.
Then the amount of drug in the organ at any time t is

$$(V\text{cm}^3)\ (y\,g/\text{cm}^3) = Vy\ g. \tag{66}$$

The amount of drug entering the organ at any time t is

$$(a\text{cm}^3/s)\ (c\,g/\text{cm}^3) = ac\,g/s. \tag{67}$$

and the amount leaving the organ is

$$(b\text{cm}^3/s)\ (y\,g/\text{cm}^3) = by\ g/s. \tag{68}$$

From Eqns. (66)-(68), the differential equation governing the given problem is

$$\frac{d}{dt}(yV) = ac - by \tag{69}$$

along with the initial condition that at $t = 0,\ y = y_0$ (say).
Eqn. (69) can be written as

$$\frac{dy}{dt} + \frac{b}{V}y = \frac{ac}{V}$$

which is a linear equation with ∵. $= e^{\int \frac{b}{V}dt} = e^{\frac{b}{V}t}$

$$\therefore\quad ye^{\frac{b}{V}t} = \int \frac{ac}{V}e^{\frac{b}{V}t} + A$$

$$= \frac{ac}{b}e^{\frac{b}{V}t} + A$$

or $$y = \frac{ac}{b} + Ae^{-\frac{bt}{V}} \tag{70}$$

where A is the constant of integration.
Using the initial condition $y = y_0$ at $t = 0$, we obtain

$$A = y_0 - \frac{ac}{b}$$

Substituting the above value of A in Eqn. (70), we obtain

$$y(t) = \frac{ac}{b} + \left(y_0 - \frac{ac}{b}\right)e^{-\frac{bt}{V}} \qquad (71)$$

Which gives the concentration of drug in the organ at any time t.
The following exercise will help you check your understanding of the above application.

E15) A liquid carries a drug into an organ of volume 500cm^3 at a rate of $10\text{cm}^3/\text{s}$ and leaves at the same rate. The concentration of the drug in the entering liquid is $0.08\text{g}/\text{cm}^3$. Assuming that the drug is not present in the organ initially, find

i) the concentration of the drug in the organ after $30s$ and $120s$

ii) the steady-state concentration

iii) how long would it take for the concentration of the drug in the organ to reach $0.04g/\text{cm}^3$ and $0.06g/\text{cm}^3$?

We now end this unit by giving a summary of what we have covered in it.

8.6 SUMMARY

In this unit, we have covered the following points:

1. The general form of the linear differential equation of the first order is $\frac{dy}{dx} + P(x)\,y = Q(x)$, where $P(x)$ and $Q(x)$ are continuous real-valued functions on some interval $I \subseteq \boldsymbol{R}$.

   When $Q(x) = 0$ it is termed as **homogeneous linear differential equation** of order one.

   When $Q(x) \neq 0$, it is called **non-homogeneous** (or **inhomogeneous**) **linear differential equation** of order one.

   I.F. for this equation is $e^{\int P(x)dx}$ and the **general solution** is given by
   $$y = e^{-\int P(x)\,dx}\int Q(x)\,e^{\int P(x)\,dx}dx + c\,e^{-\int P(x)\,dx}$$
   Here, $e^{-\int P(x)\,dx}\int Q(x)\,e^{\int P(x)\,dx}$ is a **particular solution** of the equation.

2. If in the differential equation
   $$\frac{dy}{dx} + P(x)y = Q(x)$$
   $P(x)$ is a constant and $Q(x)$ is any of the forms $e^{\alpha x}$ (α constant),

x^r ($r > 0$, an integer), $\sin \beta x$ or $\cos \beta x$ (β constant) or a linear combination of such functions, then **method of undetermined coefficients** can be applied to find a particular solution of the equation. Particular integral for different forms of $Q(x)$ is given in the following table:

$P(x)$	$Q(x)$	Particular Integral
a (constant)	e^{mx} (m constant)	$\begin{cases} \dfrac{e^{mx}}{m+a} \text{ if } m \neq -a \\ x e^{mx} \text{ if } m = -a \end{cases}$
a (constant)	$\sum_{i=0}^{n} a_i x^i$ ($i > 0$ an integer)	$\begin{cases} \sum_{i=0}^{n} \dfrac{a_i}{i+1} x^{i+1} \; if \; a = 0 \\ \sum_{i=0}^{n} b_i x^i \; if \; a \neq 0 \end{cases}$ with $b_n = \dfrac{a_n}{a}$, $b_{n-1} = \dfrac{1}{a}\left(a_{n-1} - \dfrac{na_n}{a}\right)$, $b_{n-2} = \dfrac{1}{a}\left[a_{n-2} - \dfrac{n-1}{a}\left(a_{n-1} - \dfrac{n}{a} a_n\right)\right]$ and so on.
a (constant)	$\sin \beta x$ $\cos \beta x$ or $A \sin \beta x + B \cos \beta x$ (β, A, B are constants)	A linear combination of $\sin \beta x$ and $\cos \beta x$.

3. **Method of variation of parameters** provides an alternative approach for finding the solution of non-homogeneous linear differential equation
$$\frac{dy}{dx} + P(x) y = Q(x).$$

4. a) **Bernoulli's equation**
$$\frac{dy}{dx} + P(x) y = Q(x)\, y^n,$$
where P and Q are functions of x alone and n is neither zero nor one, reduces, to a linear equation by the substitution $v = y^{1-n}$.

   b) Equations of the type
$$f'(y) \frac{dy}{dx} + P(x)\; f(y) = Q(x)$$
reduce to linear equations by the substitution $f(y) = v$.

5. The differential equations governing physical problems such as population model, radioactive decay, Newton's law of cooling, mixture problem and absorption of drugs in the organ have been solved.

8.7 SOLUTIONS/ANSWERS

E1) i) Nonlinear; y

ii) Linear; y

iii) Linear; i

iv) Linear; x

v) Linear; x, Nonlinear; y

vi) Nonlinear; s, Nonlinear; t

E2) i) Given differential equation can be written as

$$\frac{dy}{dx}+\frac{2x}{x^2+1}y=\frac{4x^2}{x^2+1} \qquad (72)$$

$$\text{I.F.}=e^{\int\frac{2x}{x^2+1}dx}=e^{\ln(x^2+1)}=x^2+1.$$

Multiplying Eqn. (72) by x^2+1 and arranging terms, we get

$$\frac{d}{dx}[y(x^2+1)]=4x^2.$$

Integrating w.r.t. x, we get

$3y(x^2+1)=4x^3+c$, with c as a constant, as the required solution.

ii) $\text{I.F.}=e^{\int\frac{2}{x}dx}=e^{2\ln|x|}=x^2$

Multiplying the given differential equation by x^2, we get

$$\frac{d}{dx}(yx^2)=x^2\sin x$$

Integrating w.r.t. x, we get

$yx^2=c-x^2\cos x+2x\sin x+2\cos x$,

which is the required solution with c as a constant.

iii) Given equation can be written as

$$\frac{dy}{dx}+\cos x\, y=\sin x\cos x$$

$$\text{I.F.}=e^{\int\cos x\,dx}=e^{\sin x}$$

$$\therefore\quad ye^{\sin x}=\int\sin x\cos x e^{\sin x}dx+c$$

$$=\int te^t dt+c \text{ (putting } \sin x=t\text{)}$$

$$=te^t-e^t+c$$

$$=\sin x\, e^{\sin x}-e^{\sin x}+c$$

$\Rightarrow\quad y=c\,e^{-\sin x}+(\sin x-1)$, where c is a constant.

iv) Given equation can be written as

$$\frac{dx}{dy}+\frac{1}{1+y^2}x=\frac{\tan^{-1}y}{1+y^2} \qquad (73)$$

It is linear equation with x as dependent variable, whose I.F. can be written as

$$\text{I.F.}=e^{\int\frac{1}{1+y^2}dy}=e^{\tan^{-1}y}$$

Multiplying Eqn. (73) by $e^{\tan^{-1}y}$ and integrating w.r.t. y, we get

$$x\,e^{\tan^{-1} y} = c + \int \frac{\tan^{-1} y}{1+y^2} e^{\tan^{-1} y}\, dy \text{, } c \text{ a constant.}$$

$$= c + \int t e^t dt \text{ (Put } \tan^{-1} y = t \therefore \frac{1}{1+y^2} dy = dt \text{)}$$

$$= c + te^t - \int e^t\, dt$$

$$= c + te^t - e^t$$

$$= c + (\tan^{-1} y)\, e^{\tan^{-1} y} - e^{\tan^{-1} y} \text{ (Substituting back for } t\text{)}$$

or $x = c\,e^{-\tan^{-1} y} - 1 + \tan^{-1} y$, which is the required solution.

v) $x = 2y^3 + cy^{-2}$. (Hint: Treat x as dependent variable and y as independent variable).

vi) Given equation is linear.

For $y \geq 0$, the given equation is $y' - 2xy = 1$.

Its $\text{I.F.} = e^{\int -2x\, dx} = e^{-x^2}$

Multiplying the given equation by I.F., we get

$$ye^{-x^2} = \int e^{-x^2} dx + c$$

Since $\int e^{-x^2 dx}$ cannot be evaluated in terms of the known functions

$$y = e^{x^2} \int e^{-x^2} dx + c\,e^{x^2}$$

is the general solution of the given equation.

For $y < 0$, the given equation is $y' + 2xy = 1$ and its general solution is $y = e^{-x^2} \int e^{x^2} dx + ce^{-x^2}$.

E3) i) $y = ce^x + e^x \ln x$ for $x \geq 1$.

ii) $y = ce^x - x^5 - 5x^4 - 21x^3 - 63x^2 - 127x - 127$.

iii) $y = ce^x + e^x[-x\cos x + \sin x]$

$$- x^5 - 5x^4 - 20x^3 - 60x^2 - 120x - 120.$$

iv) $\text{I.F.} = e^{3x}$

$\therefore$ Solution is $ye^{3x} = \int |x|\, e^{3x} dx + c$.

If $x \leq 0$, then

$$ye^{3x} = c - \int x\, e^{3x}\, dx \; (|x| = -x \text{ for } x \leq 0)$$

$$= c - x\frac{e^{3x}}{3} + \int \frac{e^{3x}}{3}\, dx \text{ (Integrating by parts)}$$

$$= c - \frac{x}{3} e^{3x} + \frac{1}{9} e^{3x}.$$

$\because$ At $x = 0,\ y = 1$,

$$\therefore 1 = c - 0 + \frac{1}{9},\ i.e.,\ c = \frac{8}{9}.$$

Thus, for $x \leq 0,\ ye^{3x} = \frac{8}{9} - \frac{1}{3} x e^{3x} + \frac{1}{9} e^{3x}$ is the solution.

If $x > 0$, then

$$ye^{3x} = c + \int xe^{3x} dx = c + \frac{x}{3} e^{3x} - \frac{1}{9} e^{3x}$$

Using $y(0)=1$, we get $c=\dfrac{10}{9}$.

$\therefore$ for $x>0,\ ye^{3x}=\dfrac{10}{9}+\dfrac{x}{3}\,e^{3x}-\dfrac{1}{9}e^{3x}$ is the required solution.

E4) i) Comparing the given equation with Eqn. (18)

$a=1,\ k=2b,\ m=1$

Also $m+a=1+1=2\neq 0$

$\therefore\quad y_p(x)=\dfrac{k}{a+m}\,e^{mx}=\dfrac{2b}{1+1}e^{1.x}=be^{x}$

Also in this case, ∴ $=e^{\int P\,dx}=e^{\int dx}=e^{x}\quad (\because P=1)$

$\therefore$ Required solution is

$y=y_p(x)+ce^{-x}$ for some arbitrary constant c.

i.e., $y=be^{x}+ce^{-x}$.

ii) The given equation can be written as

$$\frac{dy}{dx}-3y=\frac{b}{2}e^{3x}$$

Here, $a=-3,\ k=\dfrac{b}{2},\ m=3$

$\therefore\quad m=-a$

A particular solution of the given equation is $y_p(x)=\dfrac{b}{2}xe^{3x}$

I.F. $=e^{-\int 3\,dx}=e^{-3x}$

Therefore, required solution is $y=\dfrac{b}{2}xe^{3x}+c\,e^{3x}$

E5) i) The given equation can be written as

$$\frac{dy}{dx}-y=x^2.$$

Assume a particular solution of the form

$$y_p(x)=\sum_{i=0}^{2}b_i x^i=b_0+b_1x+b_2x^2.$$

Substituting $y_p(x)$ in the given equation, comparing the coefficients of like powers of x on both the sides and solving, we get

$$y_p(x)=-2-2x-x^2.$$

In this case ∴ $=e^{-\int 1.\,dx}=e^{-x}$

$\therefore$ General solution is written as

$y=y_p(x)+ce^{x}$

i.e., $y=ce^{x}-2-2x-x^2$.

ii) Assume $y_p(x)=b_0+b_1x+b_2x^2$.

Substituting $y_p(x)$ and $y_p'(x)$ in the given equation and comparing coefficient of like powers of x, we get

$$b_0=\frac{-5}{4},\ b_1=\frac{-3}{2},\ b_2=\frac{-1}{2}$$

$\therefore \quad y_p(x) = -\frac{5}{4} - \frac{3}{2}x - \frac{1}{2}x^2$

I.F. $= e^{-\int 2\, dx} = e^{-2x}$

Required solution is $y = y_p(x) + c\ e^{2x}$, c a constant.

E6) i) Assume a particular solution as

$y_p(x) = c\sin 2x + d\cos 2x$.

Substituting this value of $y_p(x)$ in the given equation, we get

$(2c\cos 2x - 2d\sin 2x) - (c\sin 2x + d\cos 2x) = 6\cos 2x$

Equating the coefficients of $\sin 2x$ and $\cos 2x$, we have

$$\left.\begin{array}{r} -2d - c = 0 \\ 2c - d = 6 \end{array}\right\} \Rightarrow c = \frac{12}{5} \text{ and } d = -\frac{6}{5}$$

$\therefore \quad y_p(x) = \frac{6}{5}(2\sin 2x - \cos 2x)$.

In this case $\therefore \quad = e^{-\int dx} = e^{-x}$

$\therefore$ General solution is

$y = \alpha e^x + y_p(x)$,

i.e., $y = \alpha e^x + \frac{6}{5}(2\sin 2x - \cos 2x)$.

ii) Particular solution of the given equation is

$y_p(x) = \frac{1}{27}(2 - x + 9x^2) + \frac{3}{5}e^{2x} + \frac{2}{5}(3\sin x - \cos x)$.

The solution of homogeneous part of given equation

i.e., $\frac{dy}{dx} + 3y = 0$ is given by

$y = ce^{-3x}$.

Hence, the general solution is

$y = ce^{-3x} + \frac{1}{27}(2 - 2x + 9x^2) + \frac{3}{5}e^{2x} + \frac{2}{5}(3\sin x - \cos x)$.

E7) i) Assume a particular solution as $y_p(x) = c\cos(\pi x) + d\sin(\pi x)$.

Subsituting $y_p(x)$ in the given equation, we get

$-c\pi\sin(\pi x) + d\pi\cos(\pi x) - 2c\cos(\pi x) - 2d\sin(\pi x) = \sin(\pi x) + \cos(\pi x)$

Equating the coefficients of $\sin(\pi x)$ and $\cos(\pi x)$, we have

$$\left.\begin{array}{r} -c\pi - 2d = 1 \\ d\pi - 2c = 1 \end{array}\right\} \Rightarrow c = -\left(\frac{\pi + 2}{\pi^2 + 4}\right) \text{ and } d = \frac{\pi - 2}{\pi^2 + 4}.$$

Thus, the general solution is

$y(x) = \alpha e^{2x} - \frac{1}{\pi^2 + 4}[(\pi + 2)\cos(\pi x) - (\pi - 2)\sin \pi x]$.

Now, $y(1) = 1$ gives

$1 = \alpha e^2 - \frac{1}{\pi^2 + 4}[(\pi + 2)\cos\pi - (\pi - 2)\sin\pi] = \alpha e^2 + \frac{1}{\pi^2 + 4}(\pi + 2)$

or, $\alpha = \left[1 - \frac{\pi + 2}{\pi^2 + 4}\right]e^{-2} = \frac{(\pi^2 - \pi + 2)}{\pi^2 + 4}\ e^{-2}$.

$\therefore$ The required solution is

$$y(x)=\left(\frac{\pi^2-\pi+2}{\pi^2+4}\right)e^{2x-2}-\frac{1}{\pi^2+4}[(\pi+2)\cos(\pi x)-(\pi-2)\sin(\pi x)]$$

ii) Show that particular solutions of $y'-y=\cos 2x,\ y'-y=e^x$, $y'-y=e^{2x}$ and $y'-y=x$ are respectively, $\frac{1}{5}(2\sin 2x-\cos 2x)$, xe^x, e^{2x} and $(-x-1)$.
The required general solution is given by
$y(x)=ce^x+\frac{1}{5}(2\sin 2x-\cos 2x)+xe^x+e^{2x}-x-1$.

iii) $y(x)=ce^{3x}+\frac{1}{6}(\sin 3x-\cos 3x)+\frac{1}{27}(2+6x+9x^2)-\frac{2}{3}$.

iv) The general solution is
$y(x)=ce^{-x}+x-x^2-1$
$\because\ y(0)=0,\ \therefore\ 0=c+0-0-1\Rightarrow c=1$
$\therefore$ The required solution is
$y(x)=e^{-x}-x^2+x-1$.

v) The general solution is
$y(x)=ce^x+xe^x$
Now $y(0)=-3\Rightarrow -3=c\ or\ c=-3$
$\therefore$ The required solution is $y(x)=(x-3)e^x$.

E8) i) The given equation can be written as
$e^{-y}\frac{dy}{dx}+\frac{1}{x}e^{-y}=\frac{1}{x^2}$
Substituting $v=e^{-y}$ the above equation reduces to $\frac{dv}{dx}-\frac{v}{x}=\frac{-1}{x^2}$.
Solving it for v and replacing v by e^{-y}, we get
$e^{-y}=cx+\frac{1}{2x}$ as the required solution.

ii) The given equation can be written as
$y^{-3}\frac{dy}{dx}+xy^{-2}=x^3$
Substituting $v=y^{-2}$, the given equation reduces to a linear equation $\frac{dv}{dx}-2xv=-2x^3$ whose solution is $v=ce^{x^2}+x^2+1$.
Hence, $\frac{1}{y^2}=ce^{x^2}+x^2+1$ is the required solution.

iii) The given equation can be written as
$\sec^2 y\frac{dy}{dx}+\frac{3e^x}{1-e^x}\tan y=0$
Let $v=\tan y$. Then $\frac{dv}{dx}=\sec^2 y\frac{dy}{dx}$.
Hence, $\frac{dv}{dx}+\frac{3e^x}{1-e^x}v=0$.
It is a linear equation.

The required solution is $\tan y = c(1-e^x)^3$.

E9) i) The given equation can be written as

$$\sec^2 y\frac{dy}{dx} + 2x\tan y = x^3$$

Substituting $v = \tan\ y$ it reduces to

$\dfrac{dv}{dx} + 2xv = x^3$ with ∴. $= e^{x^2}$.

Solving the above equation, we get

$$v = c\,e^{-x^2} + \frac{1}{2}(x^2-1).$$

Hence, $\tan y = c\,e^{-x^2} + \dfrac{1}{2}\,(x^2-1)$ is the required solution.

ii) $\dfrac{1}{y^2} = ce^{2x} + 2e^x$.

iii) Divide the given equation throughout by y^3 and substitute $v = y^{-2}$. The equation reduces to

$$\frac{dv}{dx} + \frac{(x+1)}{x}v = \frac{6}{x}$$

Solving we get

$$v = \frac{6}{x} + \frac{c}{x}e^{-x}.$$

Thus, $\dfrac{1}{y^2} = \dfrac{6}{x} + \dfrac{c}{x}\ e^{-x}$ is the required solution.

E10) Here we have to solve the differential equation

$$\frac{dN}{dt} = kN \qquad (74)$$

subject to $N(0) = N_0$ and $N(1) = \left(\dfrac{3}{2}\right)N_0$.

From Eqn. (74), we have

$N(t) = ce^{kt}$ where c is a constant of integration.

At $t = 0,\ N(0) = N_0$. It follows that $N_0 = ce^0 = c$ so that

$N(t) = N_0 e^{kt}$.

At $t = 1,\ N(1) = \dfrac{3}{2}N_0$, thus

$\dfrac{3}{2}N_0 = N_0\,e^k$ or $e^k = \dfrac{3}{2}$

$\Rightarrow\ k = \ln\left(\dfrac{3}{2}\right)$.

Thus, $N(t) = N_0 e^{t\ln(3/2)}$.

To find the time at which the bacteria have tripled we solve $3N_0 = N_0 e^{t\ln(3/2)}$ for t, which yields

$3 = e^{t\ln(3/2)}$ or $t\ \ln(3/2) = \ln 3$

$$\Rightarrow t = \frac{\ln\ 3}{\ln(3/2)} \approx 2.71 \text{ hours.}$$

E11) Here we have to solve the differential equation

$\frac{dN}{dt} = kN$, $N(0) = 500$ and $N(10) = 500 + 75(15\%$ of $500)$ and obtain the value of $N(30)$.

Solving the given equation under given condition we get

$N(t) = 500e^{kt}$ and $k = \frac{1}{10}\ln(1.15)$.

This gives $N(30) = 500(1.15)^3 = 760$ approximately.

E12) $T(t)$ represent the temperature of the thermometer at time t (in minutes). We are given that when $t = 0, T = 70$ and when $t = 3, T = 25$. Since the thermometer is kept outside the home where the temperature is $10^\circ F$, $T(t)$ satisfies the equation

$$\frac{d}{dt}T(t) \ \alpha \ (T - 10).$$

Since the thermometer temperature is decreasing, it is convenient to choose $(-k)$ as the constant of proportionality. Thus, T is to be determined from the differential equation

$$\frac{dT}{dt} = -k(T - 10) \qquad (75)$$

under the conditions

when $t = 0, T = 70$ (76)

and when $t = 3, T = 25$. (77)

From Eqn. (75), we get

$T(t) = 10 + ce^{-kt}$.

Then condition (76) yields $70 = 10 + c$ or $c = 60$, so that

$T(t) = 10 + 60e^{-kt}$

Using condition (77), we get

$25 = 10 + 60e^{-3k}$

$\Rightarrow e^{-3k} = \frac{1}{4}$ or, $k = \frac{1}{3}\ln 4$

Thus, the temperature $T(t)$ is given by the equation

$$T(t) = 10 + 60\exp\left(\frac{-1}{3}t\ln 4\right).$$

E13) The mass $y(t)$ of the substance at time t (in years) satisfies

$\frac{dy}{dt} = ky(t)$, $y(0) = 100$ gms and $y(6) = 97$ gms.

Solving the above equation under given conditions, we get

$y(t) = 100e^{kt}$ where $k = \frac{1}{6}\ln(0.97) = -0.005076535$.

If t_1 is the half-life of a substance then $y(t_1) = 50$ gms.

This gives $50 = 100e^{kt_1}$ or, $t_1 = \frac{\ln 2}{0.005076535} \approx 136.5$ years.

E14) If $P(t)$ be the amount of salt (in gms) in the tank then

$$\frac{dP}{dt} = \text{(rate of inflow of salt)} - \text{(rate of outflow of salt)}$$
$$= R_1 - R_2.$$

R_1, the rate of inflow of salt in the tank in gms/min. is

$$R_1 = (2 \text{ lit/min}).\ (1 \text{ gm/lit}) = 2 \text{ gms/min}.$$

Since the mixture runs out at a lower rate of 1 lit/min, the mixture is accumulating at a rate of

$$(2-1) \text{ lit/min} = 1 \text{ lit/min}.$$

After t mins. there is $100+t$ litres of brine in the tank. Then the rate of outflow of salt is

$$R_2 = (1 \text{ lit/min}).\left(\frac{P}{100+t}\text{gms/lit}\right) = \frac{P}{100+t} \text{ gms/min}.$$

Hence, $\dfrac{dP}{dt} = 2 - \dfrac{P}{100+t}$ or, $\dfrac{dP}{dt} + \dfrac{P}{100+t} = 2$.

I.F. is $e^{\int \frac{1}{100+t}dt} = e^{\ln(100+t)} = 100+t$

$$\therefore\ P(t)\ (100+t) = \int 2(100+t)\,dt + c$$

$$= 200t + t^2 + c$$

When $t=0,\ P=0$ gives $c=0$, thus

$$P(t) = \frac{200t+t^2}{100+t}$$

Now, if V is the volume of liquid at time t, then $V = 100+t$

$\therefore$ when $V=150$ litres, $t = 150-100 = 50$ minutes and salt content

$$P(t) = \frac{200\times 50 + (50)^2}{150} = 83.3 \text{ gms. (approx.).}$$

E15) Comparing with Eqn. (79) it is given

$$a = b = 10\text{cm}^3/s,\ V = 500\text{cm}^3,\ c = 0.08g/\text{cm}^3,\ y_0 = 0$$

Then the concentration $y(t)$ of drug in the organ at any time t is given by

$$y(t) = 0.08(1-e^{-t/50})$$

i) when $t=30,\ y(t) = 0.08(1-e^{-0.6}) = 0.036g/\text{cm}^3$ (approx.).
 when $t=120,\ y(t) = 0.08(1-e^{-2.4}) = 0.073g/\text{cm}^3$ (approx.).

ii) As $t\to\infty$, the second term of $y(t)$ approaches zero. This term is known as the **transient term;** the remaining term is called the **steady-state** part of the solution.
 Thus the steady-state concentration is $y(t) = 0.08$.

iii) when $y(t) = 0.04$ then

$$1 - e^{-t/50} = \frac{0.04}{0.08} = \frac{1}{2}$$

$$\therefore\ e^{-t/50} = 1 - \frac{1}{2} = \frac{1}{2}$$

or, $t = 50\ln 2 \approx 34.7s$.

Similarly, when $y(t) = 0.06,\ t = 100\ln 2 \approx 69.3s$.

– x –

UNIT 9

FIRST ORDER DIFFERENTIAL EQUATIONS OF DEGREE GREATER THAN ONE

Structure **Page No.**

9.1 INTRODUCTION

In Unit 6, we discussed the nature of differential equations and various types of solutions of differential equations. In Units 7 and 8, we have given you the methods of solving different types of differential equations of first order and first degree. In this unit we shall consider those differential equations which are of first order and of degree greater than one.

If we denote $\frac{dy}{dx}$ by p, then the general form of a differential equation of the first order and nth degree is

$$p^n + P_1 p^{n-1} + P_2 p^{n-2} + \cdots + P_{n-1} p + P_n = 0, \tag{1}$$

where $P_1, P_2, \ldots, P_n$ are functions of x and y.

It is difficult to solve Eqn. (1) in its most general form. Accordingly, in this unit we shall consider only those equations of the form (1) which can be easily solved and discuss the methods of solving such equations. We shall be discussing the equations of the type (1) which can be factorised into linear factors with real coefficients in Sec.9.2 and those which cannot be factorised in Sec.9.3.

It was Isaac Newton (1642-1727), the English mathematician and scientist, who classified differential equations of the first order (then known as fluxional equations) in "Methodus Fluxionum et serierum infinitarum", written around 1671 and published in 1736. Count Jacopo Riccati (1676-1754), an Italian mathematician and philosopher, was mainly responsible for introducing the ideas of Newton to Italy. Riccati was destined to play an impotant part in further advancing the theory of differential equations. In 1712, he reduced an equation of the second order in y to an equation of first order in p. In 1723, he exhibited that under some restricted hypotheses a particular equation, to which the name of Riccati is attached, can be solved.

Clairaut (1713-1765)

Later the French mathematician Alexis Cladue Clairaut (1713-1765) introduced the idea of differentiating the given differential equations in order to solve them. He applied it to the equation that now bear his name and published the method in 1734. He was among the first to discover singular solutions of differential equations. Like many mathematicians of his era, Clairaut was also a physicist and an astronomer. We shall also be discussing the equations introduced by Riccati and Clairaut in Sec.9.3 of the unit.

Objectives

After studying this unit, you should be able to

- find the solution of differential equations which can be resolved into linear factors with real coefficients;
- obtain the solution of differential equations solvable for y, x or p;
- obtain the solution of differential equations in which x or y is absent;
- obtain the solution of differential equations which may be homogeneous in x and y;
- identify the Clairaut's equation and obtain its solution;
- identify the Riccati's equation and obtain its solution.

9.2 EQUATIONS WHICH CAN BE FACTORIZED

Let us consider the general form of differential equation of the first order and nth degree given by Eqn. (1) namely,

$$p^n + P_1 p^{n-1} + P_2 p^{n-2} + \cdots + P_{n-1} p + P_n = 0,$$

where $p = \frac{dy}{dx}$ and $P_1, P_2, \ldots, P_n$ are functions of x and y. This equation can also be written as $f(x, y, p) = 0$.

For this equation the two possibilities may arise:

i) When the left-hand side of Eqn. (1) can be resolved into linear factors with real coefficients.

ii) When the left-hand side of Eqn. (1) cannot be factorized.

In this section we shall take up the first possibility and discuss the second in the next section. We start by taking an example of the equation of form (1) which can be resolved into linear factors with real coefficients.

Example 1: Solve $p^2 + px + py + xy = 0$.

Solution: The given equation can be resolved into linear factors and written in the form

$$(p+x)(p+y) = 0$$

That is, either

$$p + x = 0 \text{ or, } p + y = 0.$$

In other words,

$$\frac{dy}{dx} + x = 0 \text{ or, } \frac{dy}{dx} + y = 0.$$

The above equations are linear first order equations. Their solutions can be written as

$$2y = -x^2 + c_1 \text{ and } \ln|y| = -x + c_2,$$

where c_1 and c_2 are arbitrary constants. We let $c_1 = c_2 = c$ and thus, the general solution of the given equation is

$$(2y + x^2 - c)\,(x + \ln|y| - c) = 0.$$

***

In general when Eqn. (1) can be factorized into linear factors with real coefficients, then it can be written in the form

$$(p - R_1)\,(p - R_2)\ldots(p - R_n) = 0 \qquad (2)$$

for some $R_1, R_2, \ldots, R_n$, which are functions of x and y.
Eqn. (1) will be satisfied by a value of y, and in turn by a value of p that will make any of the factors in Eqn. (2) equal to zero. Hence, to obtain the solution of Eqn. (1), we equate each of the factors in Eqn. (2) equal to zero. Thus, we get

$$p - R_1 = 0,\; p - R_2 = 0, \ldots, \; p - R_n = 0 \qquad (3)$$

These are n equations of first order and first degree. Using the methods given in Units 7 and 8 we can obtain the solutions of the above n equations of first order and first degree.

Let us suppose that the desired solutions for Eqns. (3) are:

$$\left.\begin{array}{l} f_1(x,\, y,\, c_1) = 0 \\ f_2(x,\, y,\, c_2) = 0 \\ \vdots \;\; \vdots \;\; \vdots \;\; \vdots \quad \vdots \\ f_n(x,\, y, c_n) = 0 \end{array}\right\} \qquad (4)$$

where $c_1, c_2, \ldots, c_n$ are the arbitrary constants of integration.
Since each of the constants $c_1, c_2, \ldots, c_n$ can assume an infinite number of values, there is no loss of generality if we take

$$c_1 = c_2 = \cdots = c_n = c, \text{ say.}$$

In that case, the n solutions given by Eqns. (4) will reduce to

$$
\begin{aligned}
&f_1(x,\ y,\ c)=0\\
&f_2(x, y,\ c)=0\\
&f_3(x,\ y, c)=0\\
&\dots\ \dots \quad \dots\\
&\dots\ \dots \quad \dots\\
&f_n(x,\ y,\ c)=0
\end{aligned}
$$

These n solutions can be left distinct or we can combine them into one equation, namely,

$$f_1(x,\ y, c)\ f_2(x,\ y, c)\dots f_n(x, y,\ c)=0.$$

The reason for taking all $c_1, c_2, \dots, c_n$ equal in Eqns. (4) is that the general solution of Eqn. (2), being of first order, can contain only one arbitrary constant.

Let us look at another example to understand the method discussed above.

Example 2: Solve $p^3(x+2y)+3p^2(x+y)+(y+2x)\ p=0$.

Solution: The given equation is equivalent to

$$p\,[p^2(x+2y)+3p(x+y)+(y+2x)]=0$$

$$\Rightarrow\ p\,[p^2(x+2y)+p\{(y+2x)+(x+2y)\}+(y+2x)]=0$$

$$\Rightarrow\ p\,[p^2(x+2y)+p(x+2y)+p(y+2x)+(y+2x)]=0$$

$$\Rightarrow\ p\,[p(p+1)\ (x+2y)+(p+1)(y+2x)]=0$$

$$\Rightarrow\ p(p+1)\,[(x+2y)\ p+(y+2x)]=0$$

Its component equations are

$$p=0,\ p+1=0,\ (x+2y)\ p+(y+2x)=0.$$

Now $p=0 \Rightarrow \dfrac{dy}{dx}=0$, which has the solution

$$y=c. \tag{5}$$

Similarly, $p+1=0 \Rightarrow \dfrac{dy}{dx}+1=0$ or, $dy+dx=0$

which has the solution

$$y+x=c. \tag{6}$$

Further, $(x+2y)p+(y+2x)=0$

$$\Rightarrow\ (x+2y)dy+(y+2x)\ dx=0$$

$$\Rightarrow\ d(xy+x^2+y^2)=0,$$

which has the solution

$$xy+x^2+y^2=c. \tag{7}$$

Therefore, the general solution of the given equation, from Eqns. (5), (6) and (7) is

$$(y-c)(y+x-c)\ (xy+x^2+y^2-c)=0.$$

***

You may now try the following exercise.

E1) Solve the following differential equations:

i) $p^2 y + p(x-y) - x = 0$.

ii) $p^2 - 5p + 6 = 0$.

iii) $4y^2 p^2 + 2pxy(3x+1) + 3x^3 = 0$.

iv) $\left(\frac{dy}{dx}\right)^3 = ax^4$.

v) $x + yp^2 = p(1+xy)$.

As you know from algebra, every equation over $\boldsymbol{R}$ need not have all its roots in $\boldsymbol{R}$, i.e., it need not be factorizable in $\boldsymbol{R}$. We now take up those equations of the form (1) which cannot be factorized into linear factors with real coefficients.

9.3 EQUATIONS WHICH CANNOT BE FACTORIZED

Let us consider Eqn. (1) in the form

$$f(x, y, p) = 0 \tag{8}$$

Since Eqn. (8) cannot be factorized it is not solvable in its most general form.

We shall discuss here only those equations of type (8) which possess one or more of the following properties:

i) It may be solvable for y.

ii) It may be solvable for x.

iii) It may be solvable for p.

iv) Either it may not contain y or it may not conatin x, that is, either x or y is absent from the differential equation.

v) It may be homogeneous in x and y.

vi) It may be of the first degree in x and y.

vii) It may be Riccati's equation.

We now discuss these cases one by one.

9.3.1 Equations Solvable for *y*

Consider an equation

$$xp^2 - yp - y = 0. \tag{9}$$

We can write Eqn. (9) in the form

$$y(p+1) = xp^2$$

$$\Rightarrow \quad y = \frac{xp^2}{p+1}.$$

That is, Eqn. (9) can be solved for y in terms of x and p.

Similarly when Eqn. (8), i.e., $f(x, y, p) = 0$ is solvable for y, then it can be put in the form

$$y = F(x, p). \tag{10}$$

Differentiating Eqn. (10) w.r.t. x, we get an equation of the form

$$p = \phi\left(x,\ p,\ \frac{dp}{dx}\right). \tag{11}$$

Eqn. (11) is in two variables x and p; and we may possibly solve it and get a relation of the type

$$\psi\ (x,\ p, c) = 0 \tag{12}$$

for some constant c.

If we now eliminate p between Eqns. (9) and (12), we get a relation involving x, y and c, which gives the required solution. In the cases when the elimination of p between Eqns. (9) and (12) is not possible, we then obtain the values of x and y in terms of p and these together constitute the required solution.

We now illustrate this method with the help of a few examples.

Example 3: Solve $p^2 - py + x = 0$.

Solution: The given equation is solvable for y.

Solving it for y, we get

$$y = p + \frac{x}{p}. \tag{13}$$

Differentiating Eqn. (13) w.r.t. x, we get

$$p = \frac{dp}{dx} + \frac{1}{p} + x\left(-\frac{1}{p^2}\right)\frac{dp}{dx},$$

i.e, $$\left(p - \frac{1}{p}\right)\frac{dx}{dp} + \frac{1}{p^2}\ x = 1 \tag{14}$$

If we consider p as independent variable and x as dependent variable then Eqn. (14) is a linear equation of the first order.

We can write Eqn. (14) as,

$$\frac{dx}{dp} + \frac{1}{p(p-1)\ (p+1)}\ x = \frac{p}{p^2 - 1}. \tag{15}$$

For Eqn. (15) $e^{\int \frac{1}{p(p^2-1)} dp}$ is an integrating factor.

Now, $e^{\int \frac{1}{p(p^2-1)} dp} = e^{\int \left[\frac{1}{2(p-1)} + \frac{1}{2(p+1)} - \frac{1}{p}\right] dp}$

$$= e^{\ln \frac{(p^2-1)^{1/2}}{p}} = \frac{(p^2-1)^{1/2}}{p}.$$

Thus, the solution of Eqn. (15) is obtained as

$$x\frac{(p^2-1)^{1/2}}{p} = \int \frac{p}{p^2-1}\ \frac{(p^2-1)^{1/2}}{p} dp = \int \frac{1}{\sqrt{p^2-1}} dp = c + \cosh^{-1} p,$$

or, $x = p(c + \cosh^{-1} p)(p^2 - 1)^{-1/2}$. (16)

You may **notice** that elimination of p between Eqns. (13) and (16) is not easy. However, by substituting for x from Eqn. (16) in Eqn. (13), We get

$$y = p + (c + \cosh^{-1} p)\,(p^2 - 1)^{-1/2}. \quad (17)$$

Eqns. (16) and (17) are two equations for x and y in terms of p. These two equations together constitute the general solution of the given differential equation.

***

Let us look at another example.

Example 4: Solve $y = \sin p - p\cos p$.

Solution: The given equation is already solvable for y. Differentiating the given equation w.r.t. x, we get

$$\sin p\, dp = dx \quad (18)$$

Integrating Eqn. (18), we have

$$\cos p = c - x,\ c \text{ is a constant.} \quad (19)$$

Now we have to eliminate p between the given equation and Eqn. (19). From the given equation we have

$$p\cos p = \sin p - y$$

or, $$p = \frac{\sqrt{1-\cos^2 p} - y}{\cos p}.$$

Using Eqn. (19) in the above equation, we get

$$p = \frac{\sqrt{1-(c-x)^2} - y}{c-x}$$

Thus, $$c - x = \cos\left(\frac{\sqrt{1-(c-x)^2} - y}{c-x}\right)$$

which is the required general solution of the given differential equation.

***

We now take up an example in which we also obtain a singular solution of the given equation.

Example 5: Solve $y = 2px + p^4x^2,\ x > 0$.

Solution: The given equation

$$y = 2px + p^4x^2 \quad (20)$$

is in itself solvable for y.

Differentiating it w.r.t. x, we get

$$p = 2p + 2x\frac{dp}{dx} + 2xp^4 + 4x^2p^3\,\frac{dp}{dx},$$

$$\Rightarrow\ p(1+2xp^3) + 2x\frac{dp}{dx}\,(1+2xp^3) = 0,$$

$$\Rightarrow\ (1+2xp^3)\left(p + 2x\frac{dp}{dx}\right) = 0. \quad (21)$$

Eqn. (21) holds when either of the factors $(1+2xp^3)$ or $\left(p+2x\frac{dp}{dx}\right)$ is zero.

First consider the factor

$$p+2x\frac{dp}{dx}=0.$$

$$\Rightarrow \quad \frac{2dp}{pdx}+\frac{1}{x}=0.$$

Integrating the above equation w.r.t. x, we get

$$2\ln|p|+\ln|x| = \text{constant.}$$

$\Rightarrow \quad p^2x=c$, (c an arbitrary constant)

or, $\quad p=\sqrt{\dfrac{c}{x}}$.

Substituting this value of p in the given Eqn. (20), we get

$$y=2\sqrt{cx}+c^2$$

which is the required general solution of Eqn. (20).

If we consider the factor $1+2xp^3=0$ in Eqn. (21), then by elminating p between this factor and the given Eqn. (20), we get another solution. This solution will not contain any arbitrary constant and is a singular solution of the given equation.

***

How about trying an exercise now?

E2) Solve the following differential equations:

i) $y=x+a\tan^{-1}p$.

ii) $x=y+a\ln p,\ p>0$.

iii) $p^3+p=e^y$.

iv) $y=p\tan p+\ln\cos p$.

v) $y=2px+\tan^{-1}(xp^2)$

We next consider the case when Eqn. (8) is solvable for x.

9.3.2 Equations solvable for x

Consider an equation of the form

$$p^3-4xyp+8y^2=0 \tag{22}$$

You may observe that it is difficult to solve Eqn. (22) for y whereas, it is easy to solve it for x as a function of y and p and write

$$x=\frac{p^3+8y^2}{4yp}.$$

In general, Eqn. (8) when solvable for x, can be put in the form

$$x=g(y,p). \tag{23}$$

It can now be solved by differentiating it w.r.t. y and getting an equation of the form

$$\frac{1}{p}=\phi\left(y,\ p,\ \frac{dp}{dy}\right).$$

On solving this equation a relation between p and y can be obtained in the form

$$F(y,\ p,\ c)=0 \tag{24}$$

where c is an arbitrary constant.

If it is possible to eliminate p between Eqns. (23) and (24) then after elimination, we get the required general solution. Otherwise, x and y expressed in terms of p gives the required general solution.

You may **observe** that when Eqn. (8) is solvable for y, we differentiate it w.r.t. x whereas, when it is solvable for x, we differentiate it w.r.t. y.

To have a better understanding let us consider few examples.

Example 6: Solve $p=\tan\left(x-\dfrac{p}{1+p^2}\right)$.

Solution: The given equation can be written as

$$x=\tan^{-1}p+\frac{p}{1+p^2}. \tag{25}$$

Differentiating Eqn. (25) w.r.t. y, we get

$$\frac{1}{p}=\frac{1}{1+p^2}\frac{dp}{dy}+\frac{(1+p^2)-p(2p)}{(1+p^2)^2}\frac{dp}{dy}$$

$$=\frac{1+p^2+1+p^2-2p^2}{(1+p^2)^2}\frac{dp}{dy}$$

$$=\frac{2}{(1+p^2)^2}\frac{dp}{dy}$$

$$\Rightarrow\ dy=\frac{2p}{(1+p^2)^2}\,dp. \tag{26}$$

Note that Eqn. (26) is in variable separable form.
Integrating Eqn. (26), we get

$$y=c-\frac{1}{1+p^2}, \tag{27}$$

c being an arbitrary constant.
It is not possible to eliminate p between Eqns. (25) and (27). Thus, Eqns. (25) and (27) together constitute the general solution of the given equation in terms of p.

***

Let us look at another example.

Example 7: Solve $p^2y+2px=y\ \forall\ x,\ y$ and $p>0$.

Solution: We can write the given equation in the form

$$x = \frac{y}{2p} - \frac{py}{2}. \tag{28}$$

Differentiating Eqn. (28) w.r.t. y, we get

$$\frac{1}{p} = \frac{1}{2p} + \frac{y}{2}\left(-\frac{1}{p^2}\right)\frac{dp}{dy} - \frac{p}{2} - \frac{y}{2}\frac{dp}{dy},$$

$$\Rightarrow \quad \frac{1}{2p} + \frac{p}{2} + \frac{y}{2}\frac{dp}{dy}\left(1 + \frac{1}{p^2}\right) = 0,$$

$$\Rightarrow \quad \frac{1+p^2}{2p} + \frac{y}{2}\left(\frac{1+p^2}{p^2}\right)\frac{dp}{dy} = 0,$$

$$\Rightarrow \quad \frac{1+p^2}{2p}\left(1 + \frac{y}{p}\frac{dp}{dy}\right) = 0 \tag{29}$$

In Eqn. (29), we may have

either $\dfrac{1+p^2}{2p} = 0$

or $\quad 1 + \dfrac{y}{p}\dfrac{dp}{dy} = 0.$

If the first factor equals zero, then $p^2 = -1$, which does not yield the real solutions to the problem.
Thus, the real solution of the given problem is obtained when

$$\frac{1}{y} + \frac{1}{p}\frac{dp}{dy} = 0.$$

Here variables are separable. Integrating, we get

$$\ln y + \ln p = \ln c$$

$$\Rightarrow \quad py = c$$

or, $\quad p = \dfrac{c}{y}. \qquad (30)$

Elminating p between Eqns. (28) and (30), we get

$$x = \frac{y^2}{2c} - \frac{c}{y}\frac{y}{2}$$

or, $\quad x = \dfrac{y^2}{2c} - \dfrac{c}{2},$

which is the required general solution.

***

Note that you could have also solved Example 7 by taking $y = \dfrac{2px}{1-p^2}$ and then proceeding as in Sec. 9.3.1.

You may now try the following exercise.

E3) Solve the following differential equations:

i) $p^2 - py + x = 0.$

ii) $x = y + a \ln p,\ p > 0.$

iii) $x = y + p^2.$

iv) $y^2 \ln y = xyp + p^2, \; x > 0, \; y > 0$.

v) $y = 2px + y^{n-1}p^n$.

As we proceed further you may **note** that we need not consider the case of **equations solvable for p** separately. In this case Eqn. (1) and hence Eqn. (8), which is of nth degree in p, in general, can be reduced to n equations of the first degree in p of the form as given by Eqns. (3). The method of solving these first degree equations in p has already been discussed in Sec. 9.2. Thus, we take up the next case i.e., when Eqn. (8) may not contain either the independent variable x, or, the dependent variable y explicitly.

9.3.3 Equations in which Independent Variable or Dependent Variable is Absent

We shall consider the two cases separately.

Case I: Equations not containing the independent variable:

When Eqn. (8) does not contain the independent variable explicitly then the equation has the form

$$f(y, p) = 0. \tag{31}$$

Consider for instance, the equation

$$y - \frac{1}{\sqrt{1+p^2}} = 0$$

This equation does not contain x explicitly. Also, it is readily solvable for y, since it can be written in the form

$$y = \frac{1}{\sqrt{1+p^2}}. \tag{32}$$

Eqn. (32) can now be solved by the method discussed in Sec. 9.3.1. In case Eqn. (31) is solvable for p, then we can write it in the form

$$p = \frac{dy}{dx} = \phi(y) \tag{33}$$

and then solve it to get the required solution of Eqn. (31).

To be more clear, let us consider the following examples.

Example 8: Solve $y = 2p + 3p^2$.

Solution: We have

$$y = 2p + 3p^2 \tag{34}$$

which is already in the form $y = F(p)$. Following the method discussed in Sec. 9.3.1, we differentiate Eqn. (34) w.r.t. x, so that

$$p = 2\frac{dp}{dx} + 6p\frac{dp}{dx}$$

or, $$\frac{p}{2+6p} = \frac{dp}{dx}.$$

Here variables are separable and we have

$$dx = \left(\frac{2}{p} + 6\right) dp .$$

Integrating, we get

$$x = 6p + 2\ln|p| + c . \qquad (35)$$

c being an arbitrary constant.

Since it is not possible to eliminate p from Eqns. (34) and (35), these equations together yield the required general solution of Eqn. (34) in terms of p .

***

Let us look at another example.

Example 9: Solve $y^2 = a^2(1+p^2)$. (36)

Solution: The given equation is an equation in y and p only. It can be written as

$$p^2 = \frac{y^2}{a^2} - 1$$

Solving for p , we get

$$p = \pm\sqrt{\frac{y^2}{a^2} - 1}$$

$\therefore$ Either $p = \sqrt{\frac{y^2}{a^2} - 1}$ or, $p = -\sqrt{\frac{y^2}{a^2} - 1}$,

Now $p = \sqrt{-1 + \frac{y^2}{a^2}}$ gives

$$\frac{a}{\sqrt{y^2 - a^2}}\, dy = dx .$$

Integrating the above equation, we get

$$a \ln\left|\frac{y + \sqrt{y^2 - a^2}}{a}\right| = x + c ,$$

c being an arbitrary constant.

Similarly, $p = -\sqrt{-1 + \frac{y^2}{a^2}}$, on integration, yields

$$a \ln\left|\frac{y + \sqrt{y^2 - a^2}}{a}\right| = -x + c .$$

Hence, the general solution of the given Eqn. (36) is

$$\left[a \ln\left|\frac{y + \sqrt{y^2 - a^2}}{a}\right| - x - c\right]\left[a \ln\left|\frac{y + \sqrt{y^2 - a^2}}{a}\right| + x - c\right] = 0 .$$

***

Note that we solved Eqn. (36) for p and then integrated it. You could have also integrated it by solving it for y .

We next consider the equations in which the dependent variable is absent.

Case II: Equations not containing the dependent variable:

In this case Eqn. (8) has the form

$$g(x,\ p) = 0\,. \tag{37}$$

As in Case I, Eqn. (37) is either solvable for p or solvable for x. If it is solvable for p, then it can be written as

$$p = \psi(x)$$

which, on integration, gives the solution of Eqn. (37).
If Eqn. (37) is solvable for x, then it corresponds to the case discussed in Sec. 9.3.2 and can be solved by first differentiating it w.r.t. y and then integrating the resulting equation.

We give below examples to illustrate this method.

Example 10: Solve $x(1+p^2)=1$.

Solution: The given equation can be written as

$$x = \frac{1}{1+p^2}\,. \tag{38}$$

Differentiating Eqn. (38) w.r.t. y, we get

$$\frac{1}{p} = \frac{-2p}{(1+p^2)^2}\,\frac{dp}{dy}$$

i.e., $dy = \dfrac{-2p^2}{(1+p^2)^2}\,dp$,

i.e., $dy = 2\left[\dfrac{-1}{1+p^2} + \dfrac{1}{(1+p^2)^2}\right]dp$.

Here variables are separable. Integrating, we get

$$y = -2\tan^{-1} p + 2\int \frac{dp}{(1+p^2)^2} + c\,,\ (c \text{ is a constant})$$

Now $2\displaystyle\int \frac{dp}{(1+p^2)^2} = 2\int \frac{\sec^2\theta}{\sec^4\theta}\,d\theta$ (substituting $p=\tan\theta$)

$$= \int (1+\cos 2\theta)\,d\theta$$

$$= (\theta + \sin\theta\cos\theta)$$

$$= \tan^{-1} p + \frac{p}{p^2+1}$$

$$\therefore\quad y = -\tan^{-1} p + \frac{p}{p^2+1} + c \tag{39}$$

Eqns. (38) and (39) together yield the required general solution in terms of p.

***

Note that problem in Example 10 could have also been done by solving it for p and then integrating it. We illustrate this method in the next example.

Example 11: Solve $p^2 - 2xp + 1 = 0$.

Solution: The given equation is

$$p^2 - 2xp + 1 = 0$$

Solving it for p, we get

$$p = \frac{2x \pm \sqrt{4x^2 - 4}}{2} = x \pm \sqrt{x^2 - 1}$$

$\therefore$ Either $p = x + \sqrt{x^2 - 1}$ or, $p = x - \sqrt{x^2 - 1}$

Now $p = x + \sqrt{x^2 - 1}$ on integration yields

$$y = \frac{x^2}{2} + \frac{x\sqrt{x^2 - 1}}{2} - \frac{1}{2} \ln | x + \sqrt{x^2 - 1} | + c,$$

c being an arbitrary constant.

Similarly, $p = x - \sqrt{x^2 - 1}$ yields

$$y = \frac{x^2}{2} - \frac{1}{2} x\sqrt{x^2 - 1} + \frac{1}{2} \ln | x + \sqrt{x^2 - 1} | + c,$$

Hence, the general solution of the given equation is

$$[x^2 + x\sqrt{x^2 - 1} - \ln | x + \sqrt{x^2 - 1} | - 2y + c_1]$$
$$[x^2 - x\sqrt{x^2 - 1} + \ln | x + \sqrt{x^2 - 1} | - 2y + c_1] = 0$$

where $c_1 = 2c$ is an arbitrary constant.

***

And now some exercises for you.

E4) Solve the following differential equations:

i) $p^2 - 4 = 0$.

ii) $\sin (p) = 0$.

iii) $p^2 + 4p - x^2 = 0, \; x > o$.

E5) Obtain the solution of the following differential equations:

i) $\exp[p + (1 + x^2)] = 1$.

ii) $p^2 - (3x + 2y)p + 6xy = 0$.

iii) $xy^2(p + 2) = 2py^3 + x^3$.

We next discuss the case when Eqn. (8) may be homogeneous in x and y.

9.3.4 Equations Homogeneous in x and y

When Eqn. (8) is homogeneous in x and y it can be expressed in the form

$$\phi\left(p, \frac{y}{x}\right) = 0, \; x > 0. \tag{40}$$

For solving Eqn. (40), we can proceed in two ways. In case Eqn. (40) is solvable for p then it can be expressed as

$$p = \frac{dy}{dx} = f\left(\frac{y}{x}\right) \tag{41}$$

We already know from our knowledge of Unit 7 that equations of the type (41) can be solved by using the substitution $y = vx$.

The second possibility, that is, when Eqn. (40) is solvable for y/x, then it can be put in the form

$$\frac{y}{x} = \psi(p) \text{ or, } y = x\psi(p).$$

In this case we can proceed as in Sec. 9.3.1. Differentiating the above equation w.r.t. x, we get

$$p = \psi(p) + x\psi'(p)\frac{dp}{dx}$$

$$\Rightarrow \quad \frac{dx}{x} = \frac{\psi'(p)\, dp}{p - \psi(p)} \tag{42}$$

Eqn. (42) is in variable separable form. On integration, it yields

$$\ln|x| = c + \int \frac{\psi'(p)}{p - \psi(p)}\, dp$$

$$= c + \phi(p), \text{ say.}$$

The elimination of p between this equation and $y = x\psi(p)$ will give us the required solution. In case it is not possible to eliminate p then the solution can be written in terms of p.

To understand the method, we take an example.

Example 12: Solve $y^2 + xyp - x^2p^2 = 0 \;\forall\; x, y, p > 0$.

Solution: The given equation is homogeneous in x and y and it may be written as

$$p^2 - \left(\frac{y}{x}\right)p - \left(\frac{y}{x}\right)^2 = 0 \tag{43}$$

Solving Eqn. (43) for p, we get

$$p = \frac{(y/x) \pm \sqrt{(y/x)^2 + 4(y/x)^2}}{2} = (y/x)\left(\frac{1 \pm \sqrt{5}}{2}\right)$$

i.e., $\dfrac{dy}{dx} = \dfrac{y}{x}\left(\dfrac{1+\sqrt{5}}{2}\right)$ and $\dfrac{dy}{dx} = \dfrac{y}{x}\left(\dfrac{1-\sqrt{5}}{2}\right)$

Integrating the above equations, we get

$$y = cx^{\frac{1+\sqrt{5}}{2}} \text{ and } y = cx^{\frac{1-\sqrt{5}}{2}}$$

where c is an arbitrary constant.

Hence the general solution of Eqn. (43) is written as

$$[y - cx^{\frac{1+\sqrt{5}}{2}}][y - cx^{\frac{1-\sqrt{5}}{2}}] = 0.$$

***

Now you may try the following exercise.

E6) Solve the following differential equations:

i) $y = yp^2 + 2px.\ x > 0$.

ii) $x^2p^2 + 4xyp - 8y^2 = 0,\ x > 0,\ y > 0$.

We next discuss the case when Eqn. (8) may be of first degree in x and y.

9.3.5 Equation of the First Degree in x and y-Clairaut's Equation

When Eqn. (8) is of first degree in x and y, it can be solved for both x and y and hence can be put in either of the following forms.

$$y = xf_1(p) + f_2(p) \tag{44}$$

or,
$$x = yg_1(p) + g_2(p) \tag{45}$$

Hence, we can use the methods discussed in Sec. 9.3.1 and 9.3.2 to solve equations of the form (44) and (45), respectively.

However, if in Eqn. (44), $f_1(p) = p$, then we get one particular form of this equation known as **Clairaut's Equation** about which we have already mentioned in Sec. 9.1.

The Clairaut's equation is of the form

$$y = px + f(p) \tag{46}$$

In Eqn. (46), $f(p)$ is known function of p which contains neither x nor y explicitly. Depending on the form of $f(p)$, Eqn. (46) can be linear or non-linear. For instance, $y = px + p^2$ and $y = xp + e^p$ are examples of non-linear Clairaut's equation whereas, $y = xp + p$ is a linear Clairaut's equation. You may note that equations $y = xy + p^2$ and $y = xp + yp^2$ are not of the Clairaut's form.

In order to solve Eqn. (46) we follow the procedure discussed in Sec. 9.3.1 and differentiate it w.r.t. x. We obtain

$$p = p + \frac{dp}{dx}x + f'(p)\frac{dp}{dx} \tag{47}$$

$$\Rightarrow \quad \frac{dp}{dx}[x + f'(p)] = 0.$$

Then either $\dfrac{dp}{dx} = 0$ (48)

or, $x + f'(p) = 0$. (49)

The solution of Eqn. (48) is $p = c$, where c is an arbitrary constant. This gives the general solution of Eqn. (46) in the form

$$y = cx + f(c). \tag{50}$$

You may **note** that Eqn. (50) is an equation of a family of straight lines.

Now consider Eqn. (49). Since $f(p)$ and $f'(p)$ are known functions of p, Eqns. (49) and (46) together constitute a set of parametric equations giving x and y in terms of the parameter p.

If it is possible to eliminate p from Eqns. (46) and (49) and if the resulting equation satisfies Eqn. (46), we get yet another solution of Eqn. (46) which does not contain an arbitrary constant and is a singular solution of Eqn. (46).

The following examples will help you understand what we have discussed above.

Example 13: Solve $p^2+4xp-4y=0$.

Solution: The given equation can be re-written as

$$y=px+\frac{1}{4}p^2, \tag{51}$$

which is in the Clairaut's form. Differentiating Eqn. (51) w.r.t. x, we get

$$p=p+\frac{dp}{dx}x+\frac{p}{2}\frac{dp}{dx}$$

$$\Rightarrow \quad \frac{dp}{dx}\left(x+\frac{p}{2}\right)=0.$$

Thus either $\frac{dp}{dx}=0$ which gives $p=c$ (a constant) (52)

or, $\quad x+\frac{p}{2}=0$ (53)

From Eqns. (51) and (52), we obtain

$$y=cx+\frac{c^2}{4}$$

as the general solution of Eqn. (51). Eliminating p from Eqns. (51) and (53), we get

$$y=x(-2x)+\frac{1}{4}(-2x)^2,$$

i.e., $y(x)=-x^2$,

which contains no arbitrary constant. Since this value of y satisfies Eqn. (51), it is a singular solution of Eqn. (51).

***

Let us look at another example.

Example 14: Solve $y=xp+\sqrt{a^2p^2+b^2}$ where a, b are positive constants.

Solution: Differentiating the given equation w.r.t. x, we have

$$\left[x+\frac{a^2p}{\sqrt{a^2p^2+b^2}}\right]\frac{dp}{dx}=0$$

If $\frac{dp}{dx}=0$, then $p=c$ (a constant), gives the general solution as

$$y=cx+\sqrt{a^2c^2+b^2}.$$

If $x+\frac{a^2p}{\sqrt{a^2p^2+b^2}}=0$, we get after simplification

$$p=\frac{bx}{a\sqrt{a^2-x^2}}.$$

Using the above value of p in the given equation, a singular solution is obtained as

$$y^2a^2(a^2-x^2)=b^2(x^2+a^2)^2.$$

***

We shall now illustrate a situation where the given equation is not in the Clairaut's form but it can be reduced to the Clairaut's form by a suitable transformation of the variables.

Example 15: Solve $xyp^2-(x^2+y^2+1)p+xy=0$.

Solution: The given equation is not in Clairaut's form.

Let $x^2=U$ and $y^2=V$

$$\therefore \quad p=\frac{dy}{dx}=\frac{dy}{dV}\frac{dV}{dU}\frac{dU}{dx}$$

$$=\frac{1}{2\sqrt{V}}\frac{dV}{dU}2\sqrt{U}=\sqrt{\frac{U}{V}}\frac{dV}{dU}$$

$\therefore$ The given equation reduces to

$$U\left(\frac{dV}{dU}\right)^2-(U+V-1)\frac{dV}{dU}+V=0$$

or, $$V=UP+\frac{P}{P-1} \text{ where } P=\frac{dV}{dU}. \tag{54}$$

Eqn. (54) is in Clairaut's form. Differentiating it w.r.t U, we get

$$P=P+U\frac{dP}{dU}-\frac{1}{(P-1)^2}\frac{dP}{dU}$$

$$\Rightarrow \quad \frac{dP}{dU}\left[U-\frac{1}{(P-1)^2}\right]=0.$$

If $\frac{dP}{dU}=0$ then $P=c$ (a constant) gives the general solution of Eqn. (54) as

$$V=cU+\frac{c}{c-1}$$

Thus, the general solution of the given equation is

$$y^2=cx^2+\frac{c}{c-1}.$$

If $U-\frac{1}{(P-1)^2}=0$ then $P=1+\frac{1}{\sqrt{U}}$. Putting this value of P in Eqn. (54), we get

$$V=1+U+2\sqrt{U}$$

$$\Rightarrow \quad y^2=1+x^2+2x=(1+x)^2$$

which is a singular solution of the given equation.

***

You may, now try the following exercises.

E7) Solve the following differential equations:

i) $y=xp+\frac{a}{p}$ ($a\neq 0$, is a constant).

ii) $y=xp+p^2$.

iii) $y=xp+p-p^2$.

E8) Solve $e^{4x}(p-1)+e^{2y}p^2=0$.

E9) Solve $y=x^4p^2-px$.

E10) Solve $xy(y-px)=x+py$.

Finally, we take up in the following section, another non-linear equation known as Riccati's equation, which we mentioned in Sec. 9.1.

9.3.6 Riccati's Equation

Originally, this name was given to the first order differential equation

$$\frac{dy}{dx}+by^2=cx^m \tag{55}$$

where b, c and m are constants. This is known as the **special Riccati equation**. Eqn. (55) is solvable in finite terms only if the exponent m is -2 or, of the form $\frac{-4k}{(2k+1)}$ for some integer k. Riccati merely discussed special cases of this equation without offering any solutions. Now a days Riccati's equation is usually understood by an equation of the form

$$y'=a(x)+b(x)y+c(x)y^2, \tag{56}$$

which is a natural extension of the first order linear equation

$$y'=a(x)+b(x)y.$$

Here a, b and c are given functions of x on an interval I (of $\boldsymbol{R}$). Equations $y'=1+xy+e^xy^2$ and $y'=x+x^2y+\sin(x)y^2$ are examples of Riccati's equations whereas, equations $y'=1+y+y^3$, and $y'=1+y+y^4$ are not of Riccati's type.

In general, Riccati's Eqn. (56) cannot be solved by elementary methods. However, if we have the knowledge of a particular solution of Eqn. (56) then its general solution can be obtained. This can be done as follows:

Let y_1 be a particular solution of Eqn. (56). Then we determine a function v so that y defined by the relation

$$y=y_1+\frac{1}{v} \tag{57}$$

satisfy Eqn. (56).
Differentiating Eqn. (57) w.r.t. x, we get

$$y'=y_1'+v'\left(-\frac{1}{v^2}\right).$$

Since y and y_1 satisfy Eqn. (56), we have

$$y_1'=a(x)+b(x)y_1+c(x)y_1^2$$

and $y_1'-\frac{v'}{v^2}=a(x)+b(x)y+c(x)\ y^2$.

Subtracting the second equation from the first, we have

$$v'\left(\frac{1}{v^2}\right)=b(x)\ (y_1-y)+c(x)\ (y_1^2-y^2)$$

or, $\quad v' = b(x)v^2(y_1 - y) + c(x)\, v^2(y_1 - y)\,(y_1 + y) \qquad (58)$

From Eqn. (57), we have

$$(y - y_1)v = 1 \text{ or, } (y_1 - y)\,v = -1. \qquad (59)$$

We can thus write

$(y_1 + y)\, v = (2y_1 + y - y_1)\, v = 2y_1 v + (y - y_1)v = 2y_1 v + 1$ (using Eqn. (59)).

Further, $(y_1^2 - y^2)\, v^2 = (y_1 - y)\, v(y_1 + y)\, v$

$$= (-1)(2y_1 v + 1) = -1 - 2y_1 v \qquad (60)$$

Substituting from Eqn. (60) in Eqn. (58), we get

$$v' = -[b(x) + 2c(x)y_1]v - c(x), \qquad (61)$$

which is a linear (non-homogeneous) equation for determining a function v. The general solution of Eqn. (61) can be obtained and the substitution of this general solution in Eqn. (57) gives us the required solution

Let us now go through some examples to understand the above analysis.

Example 16: Solve $y' = -y + x^2 y^2$.

Solution: On comparing the given equation with Eqn. (56) we find that $a = 0,\, b = -1$ and $c = x^2$. The given equation is a Riccati's equation which has a particular solution $y_1 = 0$. To obtain the general solution y, we consider

$$y = y_1 + \frac{1}{v} = 0 + \frac{1}{v} = \frac{1}{v}$$

and then determine the function v
Substutiting the values of b and c in Eqn. (61), we have

$$\frac{dv}{dx} = v - x^2 \qquad (62)$$

Eqn. (62) is a linear first order equation with
: $\quad = e^{\int -1\, dx} = e^{-x}$
The general solution of Eqn. (62) is then given by

$$v = -e^x \int e^{-x} x^2\, dx + Ae^x$$
$$= (x^2 + 2x + 2) + Ae^x.$$

The general solution y of the given equation then assumes the following form:

$$y = \frac{1}{Ae^x + x^2 + 2x + 2},$$

which contains A as an arbitrary constant of integtration.

***

Let us look at another example.

Example 17: Solve $y' = -1 - x^2 + y^2$.

Solution: By inspection, we see that $y_1(x) = -x$ is a solution of the given Riccati's equation. Comparing the given equation with Eqn. (56), we get $a = -1 - x^2,\, b = 0$ and $c = 1$.

We look for a function v, so that $y = y_1 + \frac{1}{v} = -x + \frac{1}{v}$ satisfy the given equation.
In this case Eqn. (61) reduces to

$$v' = 2xv - 1$$

$$\Rightarrow \frac{dv}{dx} - 2xv = -1$$

The integrating factor for this equation is e^{-x^2}, and so it can be written as

$$\frac{d}{dx}[e^{-x^2} v] = -e^{-x^2}$$

On integrating the above equation, we get

$$e^{-x^2} v = -\int e^{-x^2} dx + c$$

or, $v = e^{x^2} [-\int e^{-x^2} dx + c]$

where c is an arbitrary constant.
Thus, the required general solution is

$$y(x) = -x + \frac{e^{-x^2}}{-\int e^{-x^2} dx + c}.$$

Note that the integral $\int e^{-x^2} dx$ cannot be evaluated in terms of elementary functions. When an initial condition is specified, then integral of the form $\int_{x_0}^{x} e^{-t^2} dt$ can be used.

***

You may now try the following exercises.

E11) State which of the following are Riccati's equation, Clairaut's equation or neither. Give reasons for your answer.

i) $y = 2xp + y^2 p^3$.

ii) $y' = e^x + e^y + y^2$.

iii) $y' = (1 + \sin 2x) + \frac{2}{1+x^2} y + e^x y^2$.

iv) $y = 3px + 6y^2 p^2$.

v) $y' = \sin x + \sin y$.

E12) Given a particular solution $y_1(x)$, find the general solution of the following Riccati's equations:

i) $y' = 1 - xy + y^2 \ (y_1(x) = x)$.

ii) $y' = 2 + 2x + x^2 - y^2 \ (y_1(x) = 1 + x)$.

iii) $y' = 2x - x^2 - x^4 + y + y^2 \ (y_1(x) = x^2)$.

E13) By eliminating arbitrary constant c from the equation

$$y = \frac{cg(x)+G(x)}{cf(x)+F(x)},$$

obtain the Riccati's equation:

$$(gF-Gf)y' = (gG'-g'G)+(Gf'-G'f-gF'+g'F)y+(fF'-f'F)y^2.$$

E14) Show that, when $m=0$, Riccati's equation

$$\frac{dy}{dx}+by^2 = cx^m$$

can be integrated in finite terms.

We end this unit by giving a summary of what we have covered in it.

9.4 SUMMARY

In this unit, we have covered the following:

1. The general differential equation of first order and nth degree is given by Eqn. (1), namely,

$$p^n + P_1 p^{n-1} + P_2 p^{n-2} + \cdots + P_{n-1}p + P_n = 0$$

where $P_1, P_2, \ldots, P_n$ are functions of x, y and $p = \frac{dy}{dx}$.

2. If Eqn. (1) can be resolved into real linear factors, then it assumes the form

$$(p-R_1)\,(p-R_2)\ldots(p-R_n) = 0$$

for some $R_1, R_2, \ldots, R_n$ which are functions of x and y.
Further, if $f_1(x, y, c)=0, f_2(x, y, c)=0, \ldots, f_n(x, y, c)=0$ are the solutions of $p-R_1=0, p-R_2=0, \ldots, p-R_n=0$ respectively, then

$$f_1(x, y, c)\, f_2(x, y, c)\ldots f_n(x, y, c) = 0$$

is the general solution of Eqn. (1).

3. If Eqn. (1) cannot be factorized into real linear factors, then

a) it is said to be solvable for y if we can express it as

$y = F(x, p)$ (see Eqn. (10)).

To solve it, differentiate it with respect to x, and it may be possible to solve the resulting differential equation in x and p. Elimination of p between the solution of the resulting differential equation and equation above gives the required general solution.

b) it is said to be solvable for x if we can express it as

$x = g(y, p)$ (see Eqn. (23)).

To solve it, differentiate it w.r.t. y and it may be possible to solve the resulting differential equation in y and p. Elimination of p

between the solution of the resulting equation and equation above gives the required general solution.

4. If Eqn. (1) does not contain independent variable or dependent variable explicitly and can be put in the form

$f(y, p) = 0$ (see Eqn. (31)).

or, $g(x, p) = 0$ (see Eqn. (37)).

then it may either be possible to factorize it into real linear factors or it may be solvable for y or x, as the case may be.

5. If Eqn. (1) is homogeneous in x and y, then either the substitution $y = vx$ may reduce it to separable equation or it may be put as

$y = x\psi(p)$,

which is solvable for y/x.

6. Clairaut's equation is an equation of first order and of any degree if it can be put in the form

$y = xp + f(p)$ (see Eqn. (46))

This equation is solvable for y and its solution is

$y = cx + f(c)$.

where c is an arbitrary constant.

7. Riccati's equation is an equation of the form

$$\frac{dy}{dx} = a(x) + b(x)y + c(x)y^2 \qquad \text{(see Eqn. (56))}$$

where a, b and c are given functions of x on an interval I of $\boldsymbol{R}$.
The general solution of Eqn. (56) can be obtained if we know a particular solution y_1 of Eqn. (56). We then determine a function v defined by the relation

$$y = y_1 + \frac{1}{v}, \qquad \text{(see Eqn. (57))},$$

so that y satisfies Eqn. (57).

9.5 SOLUTIONS/ANSWERS

E1) i) $(y - x - c)\,(x^2 + y^2 - c) = 0$

ii) $(y - 3x - c)\,(y - 2x - c) = 0$

iii) Given eqnation can be written as

$$\left(p + \frac{x}{2y}\right)\left(p + \frac{3}{2}\frac{x^2}{y}\right) = 0$$

On integrating we get the required solution as

$$(y^2 + x^3 - c)\left(y^2 + \frac{x^2}{2} - c\right) = 0.$$

iv) The given equation is

$$\left(\frac{dy}{dx}\right)^3 = ax^4$$

$$\Rightarrow \quad \frac{dy}{dx} = a^{1/3}x^{4/3}.$$

Integrating, we get

$$c + y = a^{1/3}\frac{x^{7/3}}{7/3}$$

$$\Rightarrow \quad 7(y+c) = a^{1/3}3x^{7/3}$$

$$\Rightarrow \quad 7^3(y+c)^3 = a.\,27x^7 \Rightarrow 343(y+c)^3 = 27ax^7$$

v) $\left(p - \frac{1}{y}\right)(p - x) = 0$

$(y^2 - 2x - c)\,(2y - x^2 - c) = 0$.

E2) i) The given equation is

$$y = x + a\tan^{-1} p \qquad (63)$$

Differentiaing Eqn. (63) w.r.to x, we have

$$p = 1 + \frac{a}{1+p^2}\frac{dp}{dx}$$

$$\Rightarrow \quad dx = \frac{a}{(p-1)(1+p^2)}\,dp$$

$$= \frac{a}{2}\left[\frac{1}{p-1} - \frac{p+1}{p^2+1}\right]dp$$

Integrating the above equation, we get

$$x = \frac{a}{2}\left[\ln(p-1) - \frac{1}{2}\ln(p^2+1)\right] + c$$

$$= \frac{a}{2}\ln\left(\frac{p-1}{\sqrt{p^2+1}}\right) + c \qquad (64)$$

Substituting for x from Eqn. (64) in Eqn. (63), we have

$$y = \frac{a}{2}\ln\left(\frac{p-1}{\sqrt{p^2+1}}\right) + a\tan^{-1} p + c \qquad (65)$$

Eqns. (64) and (65) give us the required solution in terms of p.

ii) $x = -a\,\ln\left(\frac{p-1}{p}\right) + c$

$y = c - a\,\ln(p-1)$

together constitute the required solution.

iii) The given equation is

$$p^3 + p = e^y$$

Taking logrithm of both the sides, we get

$$y = \ln p + \ln(p^2+1) \qquad (66)$$

Differentiating Eqn. (66) w.r.t. x and simplifying, we get

$$dx = \left(\frac{1}{p^2} + \frac{2}{p^2+1}\right)dp \qquad (67)$$

Integrating Eqn. (67), we have

$$x = -\frac{1}{p} + 2\tan^{-1} p + c \tag{68}$$

Eqns. (66) and (68) together consititute the required solution.

iv) $y = p\tan p + \ln\cos p$

$x = \tan p + c$

together constitute the required solution.

v) The given equation is

$$y = 2px + \tan^{-1}(xp^2). \tag{69}$$

Differentiating Eqn. (69) w.r.t. x, we get

$$p = 2p + 2x\frac{dp}{dx} + \frac{1}{1+x^2p^4}\left(p^2 + 2xp\frac{dp}{dx}\right).$$

Simplifying the above equation we get

$$-p(1+x^2p^4+p)dx = 2x(1+x^2p^4+p)dp$$

$$\Rightarrow \frac{dx}{x} + \frac{2dp}{p} = 0$$

Integrating this equation we have

$$xp^2 = c \text{ (a constant).} \tag{70}$$

From Eqns. (69) and (70), the required solution is

$$y = 2c\sqrt{x} + \tan^{-1}(c).$$

E3) i) The given equation can be written as

$$x = py - p^2 \tag{71}$$

Differentiating Eqn. (71) w.r.t. y and simplifying, we get

$$\frac{dy}{dp} - \frac{p}{1-p^2}\, y = -\frac{2p^2}{1-p^2} \tag{72}$$

Eqn. (72) is a linear equation with y as dependent variable

$$\text{I.F.} = e^{-\int \frac{p}{1-p^2}dp} = e^{\frac{1}{2}\ln(1-p^2)} = \sqrt{1-p^2}$$

Multiplying Eqn. (72) by $\sqrt{1-p^2}$ and integrating, we get

$$y\sqrt{1-p^2} = -\int \frac{2p^2}{\sqrt{1-p^2}}\,dp + c \tag{73}$$

Now $\int \frac{2p^2}{\sqrt{1-p^2}}\,dp = \sin^{-1} p - p\sqrt{1-p^2}$ (Putting $p = \sin\theta$) (74)

Hence from Eqns. (73) and (74), we have

$$y = -\frac{\sin^{-1} p}{\sqrt{1-p^2}} + p + \frac{c}{\sqrt{1-p^2}} \tag{75}$$

Thus x and y given by Eqns. (71) and (75) constitute the required solution.

ii) $x = c - a\ln(1-p) + a\ln p$

$y = -a\ln(1-p) + c$

constitute the required solution.

iii) $y = c - [p^2 + 2p + 2\ln(p-1)]$

$x = c - [2p + 2\ln(p-1)]$

together constitute the required solution.

iv) The given equation can be written as

$$x = \frac{y \ln y}{p} - \frac{p}{y}$$

Differentiating the above equation w.r.t. y, we get

$$\frac{\ln y}{p} + \frac{p}{y^2} - \left(\frac{y \ln y}{p^2} + \frac{1}{y}\right)\frac{dp}{dy} = 0$$

$$\Rightarrow \left(\frac{p}{y} - \frac{dp}{dy}\right)\left(\frac{y \ln y}{p^2} + \frac{1}{y}\right) = 0$$

$\therefore$ either $\frac{y \ln y}{p^2} + \frac{1}{y} = 0$ or, $\frac{p}{y} - \frac{dp}{dy} = 0$

If $\frac{y \ln y}{p^2} + \frac{1}{y} = 0$ then eliminating p between this and the given equation we get a singular solution.

And $\frac{p}{y} - \frac{dp}{dy} = 0 \Rightarrow \frac{dp}{p} = \frac{dy}{y} \Rightarrow p = cy$, where c is a constant.

Substituting $p = c\,y$ in the given equation, we get

$\ln\, y = cx + c^2$ as the required solution.

v) The given equation can be written as

$$x = \frac{y}{2p} - \frac{y^{n-1}}{2} p^{n-1}$$

Differentiating the above equation w.r.t. y and simplifying, we get

$$\frac{dp}{p} + \frac{dy}{y} = 0$$

$$\Rightarrow \quad py = c$$

The required solution is

$$y^2 = 2cx + c^n.$$

E4) i) The given equation is

$$p^2 - 4 = 0 \qquad (76)$$

$$\Rightarrow \quad p = \pm 2$$

Now $p = 2$ on integration gives $y = 2x + c$

and $p = -2$ on integration gives $y = -2x + c$

Thus the required solution is

$$(y - 2x - c)(y + 2x - c) = 0.$$

ii) The given equation is

$$\sin p = 0$$

We know that $\sin x = 0$ iff $x = \pm n\pi,\ n = 0, 1, 2, 3, \ldots$.

Thus, the solutions p for the given equation are

$$p = \pm n\pi,\ n = 0, 1, 2, 3, \ldots.$$

$\Rightarrow \quad y(x) = n\pi x + c_n$ or, $y(x) = -n\pi x + c_n$ are the solutions of the given equation, where c_n's are the constants for $n = 0, 1, 2, 3, \ldots$.

iii) $$y(x) = -2x + \frac{1}{2}\left[x\sqrt{4 + x^2} + 4\ln\left|\frac{x + \sqrt{4 + x^2}}{2}\right|\right] + c$$

and $y(x) = -2x - \frac{1}{2}\left[x\sqrt{4+x^2} + 4\ln\left|\frac{x+\sqrt{4+x^2}}{2}\right|\right] + c$

are the solutions of the given equation, where c is an arbitrary constant.

E5) i) We know that exponential function e^x assumes value one only at $x = 0$. Hence the given equation can be written as

$$e^{[p+(1+x^2)]} = 1 = e^o$$

$\Rightarrow \quad p = -1 - x^2$.

Integrating, we have

$$y = -x - \frac{x^3}{3} + c$$

as the required solution, where c is an arbitrary constant.

ii) $p^2 - (3x + 2y)\,p + 6xy = 0$

$\Rightarrow \quad p = 3x$ or, $p = 2y$

Thus, the solution is

$(2y - 3x^2 - 2c)\,(y - e^{2x}c) = 0$.

iii) The given equation can be written as

$$(yp - x)\,[xyp + (x^2 - 2y^2)] = 0$$

If $yp - x = 0$, then integration yields

$$y^2 - x^2 = c$$

If $xyp + x^2 - 2y^2 = 0$, then

$$p = \frac{-x^2 + 2y^2}{xy},$$

which is a homogeneous equation.

Solving the above equation, we get

$$y^2 = cx^4 + x^2$$

Thus the required solution is

$$(y^2 - x^2 - c)\,(y^2 - cx^4 - x^2) = 0$$

E6) i) The given equation can be written as

$$y = \frac{2px}{1-p^2} \qquad (77)$$

Differentiating Eqn. (77) w.r.t. x, we have

$$\frac{2dp}{p(1-p^2)} + \frac{dx}{x} = 0$$

$\Rightarrow \quad \left(\frac{2}{p} + \frac{1}{1-p} - \frac{1}{1+p}\right)dp + \frac{dx}{x} = 0$ (use partial fractions)

Integrating, we get

$$x = c\frac{(1-p^2)}{p^2}. \qquad (78)$$

Eliminating p between Eqns. (77) and (78), we get

$y^2 = 4x\,c + 4c^2$ as the required solution.

ii) The given equation can be written as

$$8(y/x)^2 - 4(y/x)p - p^2 = 0$$

Solving for (y/x), we get

$$\frac{y}{x}=\frac{4p\pm\sqrt{16p^2+32p^2}}{2}=(2\pm2\sqrt{3})\ p$$

$$\Rightarrow\ \frac{dy}{y}=\frac{1}{(2\pm2\sqrt{3})}\ \frac{dx}{x}$$

Integrating, we get

$\ln\ y=\dfrac{1}{2(1+\sqrt{3})}\ln x+\ln\ c$ and $\ln\ y=\dfrac{1}{2(1-\sqrt{3})}\ln\ x+\ln\ c$

or,

$y=cx^{\frac{1}{2(1+\sqrt{3})}}$ and $y=cx^{\frac{1}{2(1-\sqrt{3})}}$

Hence, the general solution is

$$\left[y-c\,x^{\frac{1}{2(1+\sqrt{3})}}\right]\left[y-cx^{\frac{1}{2(1-\sqrt{3})}}\right]=0.$$

E7) i) The given equation is

$$y=xp+\frac{a}{p},\quad a\neq0 \tag{79}$$

It is in Clairaut's form.
Differentiating Eqn. (79) w.r.t. x, we get

$$p=x\frac{dp}{dx}+p-\frac{a}{p^2}\ \frac{dp}{dx}$$

$$\Rightarrow\ x-\frac{a}{p^2}=0 \tag{80}$$

$$\text{or, } \frac{dp}{dx}=0 \tag{81}$$

Eqns. (81) yields, $p=c$, so that the given equation has a solution

$y=cx+\dfrac{a}{c}$, where c is an arbitrary constant.

From Eqn. (80), we have

$$x=a/p^2 \tag{82}$$

Eliminating p between Eqns. (79) and (82), we have

$$y^2=4\,ax,$$

which satisfies the given equation and is its singular solution.

ii) $y=cx+c^2$ is the general solution and
$y=-x^2/4$ is a singular solution.

iii) $y=cx+c-c^2$ is the general solution

and $y=\left[\dfrac{x}{2}(x+1)+\left(\dfrac{x+1}{2}\right)-\left(\dfrac{x+1}{2}\right)^2\right]$ is a singular solution.

E8) Put $e^{2x}=U$ and $e^{2y}=V$

$$\therefore\ p=\frac{dy}{dx}=\frac{dy}{dV}.\frac{dV}{dU}.\frac{dU}{dx}$$

$$=\frac{1}{2}e^{-2y}\frac{dV}{dU}.\ 2e^{2x}$$

$$= \frac{U}{V} \frac{dV}{dU}.$$

The given equation reduces to

$$U^2\left(\frac{U}{V}\frac{dV}{dU} - 1\right) + V\,\frac{U^2}{V^2}\left(\frac{dV}{dU}\right)^2 = 0$$

$\Rightarrow \; V = UP + P^2$ (putting $\frac{dV}{dU} = P$), (83)

which is Clairaut's form.

Differentiating Eqn. (83) w.r.t. U, we have

$$P = P + U\frac{dP}{dU} + 2p\frac{dP}{dU}$$

$$\Rightarrow \; (U + 2P)\frac{dP}{dU} = 0$$

$\Rightarrow$ Either $U + 2P = 0$ or, $\frac{dP}{dU} = 0$

Now $\frac{dP}{dU} = 0$ yields, $P = c$ and the solution of Eqn. (83) is

$$V = Uc + c^2$$

i.e., the general solution of the given equation is

$$e^{2y} = c\;e^{2x} + c^2.$$

Here $U + 2P = 0$ does not give any other solution.

E9) The given equation is

$y = x^4 p^2 - px.$ (84)

Put $x = \frac{1}{t}$

$$\therefore \; p = \frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} = -t^2\frac{dy}{dt}.$$

Let $\frac{dy}{dt} = P$, so that

$p = -t^2 P$

Using this value of p in Eqn. (84), we have

$$y = \frac{1}{t^4}t^4P^2 + t^2P\frac{1}{t}$$

$$\Rightarrow \; y = Pt + P^2.$$

This is in Clairaut's form, which has the solution

$$y = ct + c^2$$

$\therefore$ The general solution of Eqn. (84) is

$$y = c/x + c^2.$$

E10) The given equation is

$xy(y - px) = x + py$ (85)

Put $x^2 = U$ and $y^2 = V$

$$\therefore \; p = \frac{dy}{dx} = \frac{dy}{dV}\,\frac{dV}{dU}\,\frac{dU}{dx} = \sqrt{\frac{U}{V}}\;P$$

where $P = \frac{dV}{dU}$

$\therefore$ Eqn. (85) reduces to

$$\sqrt{UV}\left(\sqrt{V}-\sqrt{\frac{U}{V}}\;P\;\sqrt{U}\right)=\sqrt{U}+\sqrt{\frac{U}{V}}\;P\sqrt{V}$$

$\Rightarrow \quad V = UP + (1+P)$

which is of Clairaut's form and has its solution as

$$V = cU + (1+c).$$

$\therefore$ The solution of Eqn. (85) is

$$y^2 = cx^2 + (1+c).$$

E11) i) Clairaut's (Hint: Substitute $y^2 = v$)

ii) Neither

iii) Riccati's

iv) Clairaut's (Hint: Substitute $y^3 = v$)

v) Neither

E12) i) The given equation is

$$y' = 1 - xy + y^2 \tag{86}$$

Let $y = x + \dfrac{1}{v}$ (given $y_1 = x$)

$$\therefore \quad y' = 1 - \frac{1}{v^2}\frac{dv}{dx}.$$

Then Eqn. (86) reduces to

$$1-\frac{1}{v^2}\frac{dv}{dx} = 1-\left(x^2+\frac{x}{v}\right)+x^2+\frac{1}{v^2}+\frac{2x}{v}$$

$$\Rightarrow \quad \frac{dv}{dx} + xv = -1 \tag{87}$$

Eqn. (87) is a linear equaton with $\because$. $= e^{\int x\,dx} = e^{x^2/2}$

And $v = c\, e^{-x^2/2} - e^{-x^2/2}\int e^{x^2/2}\,dx$.

$\therefore$ The required solution is

$$y = x + 1/\,[c\; e^{x^2/2} - e^{x^2/2}\int e^{x^2/2}\;dx].$$

ii) $$y = 1 + x + \frac{1}{c\,e^{(2x+x^2)} + e^{(2x+x^2)}\int e^{-(2x+x^2)}\,dx}$$

iii) $$y = x^2 + \frac{e^{(x+\frac{2}{3}x^3)}}{c - \int e^{(x+\frac{2}{3}x^3)}\;dx}$$

E13) Here $y = \dfrac{cg(x)+G(x)}{cf(x)+F(x)}$

$$\Rightarrow \quad [cf(x)+F(x)]\;y = cg(x)+G(x) \tag{88}$$

Differentiating w.r.t. x, we get

$$[cf(x)+F(x)]\;y' + (cf' + F')\;y = cg' + G' \tag{89}$$

From Eqn. (88), we get

$$c = [G(x) - yF(x)]/[f(x)y - g(x)] \tag{90}$$

$\therefore$ Substituting for c from Eqn. (90) in Eqn. (89), we get

$$\left[f(x)\frac{(G-Fy)}{(yf-g)}+F\right]y'+\left(f'\frac{(G-yF)}{yf-g}+F'\right)y=G'+g'\left(\frac{G-yF}{yf-g}\right)$$

$\Rightarrow (fG-yfF+yfF-gF)\ y'$

$+(f'G-yf'F+F'yf-gF')\ y$

$=G'yf-G'g+g'G-yg'F$

$\Rightarrow (gF-fG)y'=(gG'-g'G)+(f'G-gF'-fG'+g'F)y+(fF'-f'F)y^2$

which is the required Riccati's equation.

E14) When $m=0$, the given equation becomes

$$\frac{dy}{dx}+b\ y^2=c$$

$$\Rightarrow\ \frac{dy}{dx}=c-by^2$$

$$\Rightarrow\ \left(\frac{1}{c-by^2}\right)\frac{dy}{dx}=1 \qquad (91)$$

If bc is positive then its solution is

$yk=c(Ae^{2xk}-1)/(Ae^{2xk}+1)$, with $k=\sqrt{bc}$

If bc is negative, then solution of Eqn. (91) is

$yk=c\tan\,(A+kx)$, with $k=\sqrt{-bc}$

If $b=0$, then solution of Eqn. (91) is

$y=cx+A$

If $c=0$, then solution of Eqn. (91) is

$y(bx+A)=1$.

– x –

MISCELLANEOUS EXERCISES

1. State whether the following statements are true or false. Justify your answer with the help of a short proof or a counter example.

   i) $x^2(dx)^2 + 2xy\ dx\ dy + y^2(dy)^2 - z^2(dz)^2 = 0$ is a first order, second degree differential equation.

   ii) The integrating factor for the equation $(x^2 + 2xy - y^2)dx + (y^2 + 2xy - x^2)dy = 0$ is $(x+y)^{-2}$.

   iii) If a differential equation of the form $M(x, y)dx + N(x, y)dy = 0$ is separable then it is also exact.

   iv) Differential equation $xy(y')^2 - (x^2 + y^2 + 1)y' + xy = 0$ is of Clairaut's form.

   v) If an initial value problem $y' = f(x, y),\ y(x_0) = y_0$ does not satisfy the conditions for existence and uniqueness of solutions on some domain D then it does not have any solution in D.

2. Examine the following functions for Lipschitz condition on the given domain D. Also find the Lipschitz constant L wherever possible.

   i) $f(x, y) = |xy|,\ D: |x| \le a,\ |y| < \infty$.

   ii) $f(x, y) = x^2|y|,\ D: |x| \le 1,\ |y| \le 1$.

   iii) $f(x, y) = x - y,\ D: |x| \le a,\ |y| \le b$.

   iv) $f(x, y) = \begin{cases} \dfrac{\sin y}{x}, & x \ne 0, \quad D: |x| \le 1, |y| < \infty \\ 0, & x = 0 \end{cases}$.

   v) $f(x, y) = xy^2,\ D: a \le x \le b,\ -\infty < y < \infty$.

3. Solve the following differential equations:

   i) $x\dfrac{dy}{dx} = y(\ln y - \ln x + 1),\ x > 0$.

   ii) $y - x\dfrac{dy}{dx} = a\left(y^2 + \dfrac{dy}{dx}\right)$, a is a constant and $x > 0,\ y > 0$.

   iii) $\dfrac{dy}{dx} = \sin(x+y) + \cos(x+y)$.

   iv) $(x^2y - 2xy^2)dx = (x^3 - 3x^2y)\,dy,\ x > 0,\ y > 0$.

4. Solve the following differential equations:

   i) $\dfrac{dy}{dx} = \dfrac{x - y + 3}{2x - 2y + 5}$.

   ii) $\dfrac{dy}{dx} + y = y^{-2}$.

   iii) $\dfrac{dy}{dx} - y = xy^5$.

iv) $2x(y+1)\,dx - y\,dy = 0$ where $y(0) = -2$.

v) $(1-x^2)\dfrac{dy}{dx} + 2xy = x(1-x^2)^{1/2}$.

5. Solve the following differential equations:

i) $(xy^2 - x^2)dx + (3x^2y^2 + x^2y - 2x^3 + y^2)\,dy = 0$.

ii) $\dfrac{dy}{dx} = \cos x - y\sin x + y^2$.

iii) $2x\dfrac{dy}{dx} + y(6y^2 - x - 1) = 0$.

6. The population of a certain country is known to increase at a rate proportional to the number of people presently living in the country. If after 2 years the population has doubled, and after 3 years the population is 20, 000, find the number of people initially living in the country.

7. Suppose a student carrying a flu virus returns to a college campus of 1000 students. Assuming that the rate at which the virus spreads is proportional to the product of the number x of infected students and the number of students not infected, determine the number of infected students after 6 days. It is given that $x(4) = 50$ and that no student leaves the campus throughout the duration of the disease.

8. i) Solve: $p^2 + 2py\cot x = y^2,\ p = \dfrac{dy}{dx}$.

ii) Solve: $p^3 - 4xyp + 8y^2 = 0,\ p = \dfrac{dy}{dx}$.

iii) Reduce the equation $xp^2 - 2yp + x + 2y = 0$ to Clairaut's form and hence solve it.

9. Solve the following differential equations where $p = \dfrac{dy}{dx}$.

i) $y = -px + x^4p^2$.

ii) $p^2(x^2 - a^2) - 2pxy + y^2 - b^2 = 0,\ a$ and b being constants.

iii) $\sin px\cos y = \cos px\sin y + p$.

10. Solve the following differential equations:

i) $\dfrac{dy}{dx} = 2 + \dfrac{1}{2}\left(x - \dfrac{1}{x}\right)y - \dfrac{1}{2}y^2,\ y_1(x) = x + \dfrac{1}{x}$.

ii) $x(1-x^3)\dfrac{dy}{dx} = x^2 + y - 2xy^2,\ y_1(x) = x^2$.

iii) $\dfrac{dy}{dx} = e^{2x} - ye^x + e^{-x}y^2,\ y_1(x) = e^x$.

11. i) Find the orthogonal trajectories of the system of curves

$$\left(\frac{dy}{dx}\right)^2 = a/x.$$

ii) Find the orthogonal trajectories of the family of coaxial circles $x^2 + y^2 + 2gx + c = 0$, where g is a parameter.

iii) Show that the system of confocal and coaxial parabolas $y^2 = 4a(x+a)$ is self orthogonal.

12. Water at temperature $10^\circ C$ takes 5 min to warm to $20^\circ C$ at a room temperature of $40^\circ C$. Find the temperature of the water after (i) 20 min. and (ii) 30 min. After how much time will the temperature be $25^\circ C$?

13. A tank contains 100 litres of an aqueous solution containing 10 kg. of salt. Water is entering the tank at the rate of 3 litres per minute and the well stirred mixture runs out at 2 litres per minute. How much salt will the tank contain at the end of one hour? After what time will the amount of salt in the tank be 625 gm?

14. Suppose that a room containing $1200\,\text{cm}^3$ of air is originally free of carbon monoxide. Beginning at time $t = 0$ cigarette smoke, containing 4% carbon monoxide, is introduced into the room at a rate of $0.1\text{cm}^3/\text{min}$, and the well-circulated mixture is allowed to leave the room at the same rate.

i) Find an expression for the concentration $x(t)$ of carbon monoxide in the room at any time $t > 0$.

ii) Extended exposure to a carbon monoxide concentration as low as $0.00012\,\text{g/cm}^3$ is harmful to the human body. Find the time T at which this concentration is reached.

– x –

SOLUTIONS/ANSWERS TO MISCELLANEOUS EXERCISES

1. i) **False:** It is first order second degree equation.

   ii) **True:** $\frac{\partial}{\partial y}\left(\frac{x^2+2xy-y^2}{x^2+y^2}\right)=\frac{\partial}{\partial x}\left(\frac{y^2+2xy-x^2}{x^2+y^2}\right)=\frac{-4xy}{(x+y)^3}$.

   iii) **True:** If the equation $M(x, y)dx + N(x, y)dy = 0$ is separable then we can write it in the form $M_1(x)+N_1(y)y'=0$, which is exact since $\frac{\partial}{\partial y}(M_1(x))=\frac{\partial}{\partial x}(N_1(y))=0$.

   iv) **True:** Substitution $x^2=u$, $y^2=v$ reduces the given equation to $v=uP+\frac{P}{P-1}$, where $P=\frac{dv}{du}$, which is Clairaut's form.

   v) **False:** Since conditions for existence and uniqueness of IVP are only sufficient not necessary. Give a counter example.

2. i) $|f(x,y_2)-f(x,y_1)|=\big||xy_2|-|xy_1|\big|\le |x||y_2-y_1|\le a|y_2-y_1|, L=a$

   ii) $|f(x,y_2)-f(x,y_1)|=|x^2|\big||y_1|-|y_2|\big|\le |x^2||y_1-y_2|\le |y_1-y_2|, L=1$

   iii) $|f(x, y_2)-f(x, y_1)|=|y_2-y_1|,\ L=1$

   iv) $|f(x, y)-f(x, 0)|=|x^{-1}\sin y|\le L|y|$ is not possible on the given region, Hence Lipschitz condition is not statisfied.

   v) $|f(x,y_2)-f(x,y_1)|=|x(y_2-y_1)(y_2+y_1)|\le L|y_2-y_1|$ is not possible for $a\le x\le b, -\infty<y<\infty$. Hence Lipschitz condition is not satisfied.

3. i) $x\frac{dy}{dx}=y(\ln y-\ln x+1)$

   Equation is homogeneous. Substituting $y=vx$, we get

   $$\frac{dv}{v\ln v}=\frac{dx}{x}$$

   Integrating $\ln v=cx$ or, $v=e^{cx}$

   Therefore, $y/x=e^{cx}$ is the required solution.

   ii) $y-x\frac{dy}{dx}=a\left(y^2+\frac{dy}{dx}\right)$

   or, $\frac{dy}{y(1-ay)}=\frac{dx}{x+a}$

   or, $\left(\frac{a}{1-ay}+\frac{1}{y}\right)dy=\frac{dx}{x+a}$

   Integrating we get $\frac{y}{1-ay}=c(x+a)$.

   iii) $\frac{dy}{dx}=\sin(x+y)+\cos(x+y)$

Substituting $x+y=v$, we get

$$\frac{dv}{dx}=1+\sin v+\cos v$$

$$=2\cos^2 v/2[1+\tan v/2]$$

or, $\dfrac{\frac{1}{2}\sec^2 v/2dv}{1+\tan v/2}=dx$

Integrating $\ln(1+\tan v/2)=x+c$

or, $\ln\left(1+\tan\dfrac{x+y}{2}\right)=x+c$.

iv) $(x^2y-2xy^2)dx=(x^3-3x^2y)\,dy$.

Substituting $y=vx$, we get

$$\frac{1-3v}{v^2}dv=\frac{dx}{x}.$$

Integrating and simplifying

$$x^2=cy^3e^{x/y}.$$

4. i) $\dfrac{dy}{dx}=\dfrac{x-y+3}{2x-2y+5}$

Substitute $x-y+3=v$ and obtain

$$\frac{2v-1}{v-1}dv=dx$$

which on integration yields

$$(x-2y)+\ln(x-y+2)=c.$$

ii) $\dfrac{dy}{dx}+y=y^{-2}$

Substituting $v=y^3$, the equation reduces to

$$\frac{dv}{dx}+3v=3$$

Integrating and simplifying, we get

$$y^3=1+ce^{-3x}.$$

iii) $y^{-4}=-x+\dfrac{1}{4}+ce^{-4x}$.

iv) $x^2=y-\ln(1+y)+2$.

v) The given equation can be written as

$$\frac{dy}{dx}+\frac{2x}{1-x^2}y=\frac{x}{\sqrt{1-x^2}}$$

I.F. $=e^{\int\frac{2x}{1-x^2}dx}=e^{-\ln(1-x^2)}=\dfrac{1}{1-x^2}$

Therefore, $\dfrac{y}{1-x^2}=\displaystyle\int\frac{x}{(1-x^2)^{3/2}}\,dx+c$

or, $y=\sqrt{1-x^2}+c(1-x^2)$.

5. i) Given equation is not exact. Here

$M = (xy^2 - x^2),\ N = (3x^2y^2 + x^2y - 2x^3 + y^2)$

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = 6\frac{(xy^2 - x^2)}{(xy^2 - x^2)} = 6.$$

$\therefore$ I.F. $= e^{\int 6dy} = e^{6y}$

Multiply the given equation with e^{6y} and integrate to obtain

$$e^{6y}\left[\frac{1}{2}x^2y^2 - \frac{1}{3}x^3 + \frac{1}{6}\left(y^2 - \frac{1}{3}y + \frac{1}{18}\right)\right] = c.$$

ii) Given equation $y' = \cos x - y\sin x + y^2$ is a Riccati equation.

One solution is $y_1 = \sin x$.

Let $y = \frac{1}{v} + \sin x$ then $\frac{dv}{dx} + (\sin x)v = -1$

I.F. is $e^{-\cos x}$ and its solution is $ve^{-\cos x} = c - \int e^{-\cos x}dx$

Since $v = \frac{1}{y - \sin x}$, therefore $\frac{e^{-\cos x}}{y - \sin x} = c - \int e^{-\cos x}dx$.

iii) $2x\frac{dy}{dx} + y(6y^2 - x - 1) = 0$ is Bernoulli's type

Substitute $y^{-2} = z$ to obtain $\frac{dz}{dx} + 2\frac{(x+1)}{2x}z = \frac{6}{x}$

Solving $z = \frac{6}{x} + \frac{c}{x}e^{-x}$ or, $\frac{1}{y^2} = \frac{6}{x} + \frac{c}{x}e^{-x}$.

6. Let $N(t)$ is the population at any time t. We have to solve $\frac{dN}{dt} = kN$ given $N(0) = N_0, N(2) = 2N_0$ and $N(3) = 20,\ 000$ where k is the constant of proportionality.

Solving we get $N = N_0e^{kt}$.

At $t = 2, N = 2N_0 \Rightarrow k = \frac{1}{2}\ln 2$.

At $t = 3,\ N = 20{,}000 \Rightarrow N_0 = \frac{20000}{2^{3/2}} = \frac{10{,}000}{\sqrt{2}} = 7071$ (approximately).

7. Let $x(t)$ be the number of infected students at any time t.

Governing differential equation is

$$\frac{dx}{dt} = kx(1000 - x)$$

where $x(0) = 1,\ x(4) = 50$

Solving by variable separable and evaluating the constant of integration

$$x(t) = \frac{1000}{1 + 999e^{-1000kt}}$$

$x(4) = 50$ gives $k = -\frac{1}{4000}\ln\frac{19}{999}$

$$\therefore\ x(6) = \frac{1000}{1 + 999e^{-1000(k6)}}.$$

8. i) $p^2 + 2py\cot x = y^2$

$\Rightarrow \; p = -y\cot x \pm y\operatorname{cosec} x$

$\Rightarrow \; \dfrac{dy}{dx} = \dfrac{1-\cos x}{\sin x} y$ or, $\dfrac{dy}{dx} = -\dfrac{1+\cos x}{\sin x} y$

Solving $\dfrac{dy}{dx} = \dfrac{1-\cos x}{\sin x} y$, we obtain $y = c\sec^2 x/2$

Solving $\dfrac{dy}{dx} = -\dfrac{1+\cos x}{\sin x} y$, we obtain $y = c\operatorname{cosec}^2 x/2$

Hence, the general solution is

$$(y - c\sec^2 x/2)\,(y - c\operatorname{cosec}^2 x/2) = 0.$$

ii) $p^3 - 4xyp + 8y^2 = 0$

Solving for x, $x = \dfrac{2y}{p} + \dfrac{p^2}{4y}$

Differentiate w.r.t. y and obtain

$$\left(1 - \frac{p^3}{4y^2}\right)\left(\frac{2y}{p^2}\frac{dp}{dy} - \frac{1}{p}\right) = 0$$

Either $\left(1 - \dfrac{p^3}{4y^2}\right) = 0$ or, $\left(\dfrac{2y}{p^2}\dfrac{dp}{dy} - \dfrac{1}{p}\right) = 0$

$\dfrac{2y}{p^2}\dfrac{dp}{dy} - \dfrac{1}{p} = 0$ gives the general solution $c(c-x)^2 = y$

$1 - \dfrac{p^3}{4y^2} = 0$ gives $p^3 = 4y^2$. Elimination of p between this equation and the given equation gives a singular solution.

iii) $xp^2 - 2yp + x + 2y = 0$

Using the substitution $y - x = v$ and $x^2 = u$, the given equation reduces to the Clairaut's form

$$v = uP + 1\frac{1}{2P} \text{ where } P = \frac{dv}{du}.$$

General solution is $v = uc + \dfrac{1}{2c}$

or, $2c^2x^2 - 2c(y-x) + 1 = 0$.

9. i) $y = -px + x^4p^2$

Differentiate w.r.t. x and obtain

$$(2x^3p - 1)\left(x\frac{dp}{dx} + 2p\right) = 0$$

$x\dfrac{dp}{dx} + 2p = 0$ gives $p = c/x^2$. Substituting the value of p in the given equation, we obtain the general solution

$$y = -c/x + c^2.$$

$(2x^3p - 1) = 0$ gives $p = 1/2x^3$. Eliminating p between this equation and the given equation we get a singular solution.

ii) Given equation can be written as

$$(y - px)^2 = a^2p^2 + b^2$$

or, $y = px \pm \sqrt{a^2p^2 + b^2}$ which is Clairaut's form.

General solution is $(y - cx)^2 = a^2c^2 + b^2$.

iii) $\sin px \cos y - \cos px \sin y = p$

$\Rightarrow \sin(px - y) = p$

$\therefore px - y = \sin^{-1} p$ or $y = px - \sin^{-1} p$ which is Clairaut's form.

$\therefore$ General solution is $y = cx - \sin^{-1} c$.

10. i) $\frac{dy}{dx} = 2 + \frac{1}{2}\left(x - \frac{1}{x}\right)y - \frac{1}{2}y^2$ is a Riccati equation.

Let $y = y_1 + \frac{1}{v} = x + \frac{1}{x} + \frac{1}{v}$

$\therefore \frac{dy}{dx} = 1 - \frac{1}{x^2} - \frac{1}{v^2}\frac{dv}{dx}$

Substituting these values of y and $\frac{dy}{dx}$ in the given equation, we obtain

$$1 - \frac{1}{x^2} - \frac{1}{v^2}\frac{dv}{dx} = 2 + \frac{1}{2}\left(x^2 - \frac{1}{x^2}\right) + \frac{1}{2v}\left(x - \frac{1}{x}\right)$$
$$- \frac{1}{2}\left[\left(x + \frac{1}{x}\right)^2 + \frac{2}{v}\left(x + \frac{1}{x}\right) + \frac{1}{v^2}\right]$$

which on simplification reduces to

$$\frac{dv}{dx} - \left(\frac{x}{2} + \frac{3}{2x}\right)v = \frac{1}{2}.$$

$\text{I.F.} = e^{-\int\left(\frac{x}{2}+\frac{3}{2x}\right)dx} = x^{-3/2}e^{-x^2/4}$

$\therefore v x^{-3/2} e^{-x^2/4} = c + \frac{1}{2}\int x^{-3/2}e^{-x^2/4}\,dx.$

Since $v = \frac{x}{xy - x^2 - 1}$, the required general solution is

$$\frac{x^{-1/2}e^{-x^2/4}}{xy - x^2 - 1} = c + \frac{1}{2}\int x^{-3/2}e^{-x^2/4}dx.$$

ii) $x(1 - x^3)\frac{dy}{dx} = x^2 + y - 2xy^2$

Let $y = x^2 + \frac{1}{v}$ then $y' = 2x - \frac{1}{v^2}\frac{dv}{dx}$ and the given equation reduces to

$$\frac{dv}{dx} = -\frac{(1 - 4x^3)}{x(1 - x^3)}v + \frac{2x}{x(1 - x^3)}.$$

Its solution is $v = \frac{x}{1 - x^3} + \frac{c}{x(1 - x^3)}$ and the required general solution is

$\frac{x(1 - x^3)}{y - x^2} = x^2 + c$, c being a constant.

iii) $\dfrac{dy}{dx} = e^{2x} - y e^{x} + e^{-x} y^{2}$.

Taking $y = e^{x} + \dfrac{1}{v}$, we get the equation

$$\frac{dv}{dx} + (2 - e^{x})\, v = -e^{-x}.$$

Solving the above equation, we obtain

$$v(e^{2x-e^{x}}) = -\int e^{-x} e^{2x-e^{x}}\, dx + c.$$

The required general solution is then

$$y = \frac{e^{x} - (e^{2x-e^{x}})}{[c + \int e^{x-e^{x}}\, dx]}.$$

11. i) Differential equation of the given system of curves is

$$p^{2} = a/x.$$

Replacing p by $\left(-\dfrac{1}{p}\right)$ in the above equation we obtain the differential equation of the orthogonal trajectories as

$$p^{2} = \frac{x}{a}$$

$\Rightarrow \ p \pm \sqrt{\dfrac{x}{a}} = 0$ or, $\dfrac{dy}{dx} \pm \sqrt{\dfrac{x}{a}} = 0$

Solving, we obtain

$3\sqrt{a}\,(c + y) = \pm 2x^{3/2}$, c being a constant.

or, $9a\,(c + y)^{2} = 4x^{3}$ is the required system of orthogonal trajectories.

ii) $x^{2} + y^{2} + 2gx + c = 0$

Differentiating the above equation w.r.t. x, we get

$$x + y\frac{dy}{dx} + g = 0 \Rightarrow g = -(x + yp)$$

Putting the value of g in the given equation we get the differential equation of the family of coaxial circles as

$$x^{2} + y^{2} - 2x(x + yp) + c = 0$$

Replacing p by $\dfrac{-1}{p}$, the differential equation of the orthogonal trajectories is

$$2xy\frac{dx}{dy} - x^{2} = -c - y^{2}$$

Putting $x^{2} = v$, the above equation reduces to

$\dfrac{dv}{dy} - \dfrac{v}{y} = -\dfrac{c}{y} - y$, which is linear in v and y and its solution is

$v = c - y^{2} + c_1 y$, c_1 being a constant.

Thus, $x^{2} + y^{2} - c_1 y - c = 0$ is the required family of orthogonal trajectories.

iii) $y^{2} = 4a(x + a)$

Differentiating w.r.t. x, we obtain

$$2yp = 4a \Rightarrow a = \frac{yp}{2}$$

The differential equation of the given family is

$$y^2 = 2xpy + y^2p^2 \qquad \text{(i)}$$

Putting $\frac{-1}{p}$ for p in (i) above, the differential equaton of the required family of orthogonal trajectories is

$$y^2 = 2xy\left(\frac{-1}{p}\right) + y^2\left(\frac{-1}{p}\right)^2$$

or, $y^2 = 2xyp + y^2p^2$ (ii)

which is the same as (i) and so the given system of parabolas are self orthogonal.

12. Let $T(t)$ be the temperature of the water at any time t.
Then the equation governing the given situation is

$$\frac{dT}{dt} = k(T-40) \qquad \text{(i)}$$

where $T(0)=10$ and $T(5)=20$ (ii)

Solving Eqn. (i) we get $T(t) = 40 + Ce^{kt}$, C a constant.

$T(0)=10$ gives $C=-30$ and thus $T(t)=40-30e^{kt}$

$T(5)=20 \Rightarrow 20 = 40-30e^{5k} \Rightarrow k = \frac{1}{5}\ln\frac{2}{3} = -0.08$ (approx).

$\therefore\ T(t) = 40-30e^{-0.08t}$

i) $T(20) = 40-30e^{-(0.08)20} = 40-30e^{-1.6} = 34^\circ C$ (approx).

ii) $T(30) = 40-30e^{-2.4} = 37^\circ C$ (approx).

To find t when $T = 25^\circ C$, we have

$$25 = 40 - 30^{-0.08t} \Rightarrow -0.08t = \ln 1/2 .$$

$$\Rightarrow\ t = \frac{\ln 2}{0.08} = \frac{0.69}{0.08} = 8.6 \text{ min.}$$

13. Let $P(t)$ be the amount of salt in the tank at any time t. Then

$$\frac{dP}{dt} = \text{(rate of inflow of salt)} - \text{(rate of outflow of salt)}$$

$$= R_1 - R_2$$

R_1, the rate of inflow of salt in the tank is

$$R_1 = (3 \text{ lit/min}).\ (0 \text{ g/lit}) = 0 \text{ g/min.}$$

Since the mixture runs out a lower rate of 1 lit/min. the mixture is accumulating at a rate of $(2-1)$ lit/min$=1$ lit/min. After t mins there is $100+t$ lit of salt in the tank.

The rate R_2 of outflow of salt is

$$R_2 = (2 \text{ lit/min}).\left(\frac{P}{100+t}\text{g/lit}\right) = \frac{2P}{100+t}\text{g/min.}$$

$$\therefore\ \frac{dP}{dt} = \frac{-2P}{100+t} \Rightarrow P = \frac{A}{(100+t)^2}$$

$P(0) = 10 \Rightarrow A = 100000$

$\therefore \quad P = \dfrac{100000}{(100+t)^2}$

at $t = 60,\ P = \dfrac{100000}{(160)^2} = 3.9$ kg

when $P = \dfrac{625}{1000}$ kg then $t = 300$ min $= 5$ hrs.

14. Let $x(t)$ be the concentration of carbon monoxide in the room at any time $t > 0$. Then $x(t)$ is given by

$$x(t) = \frac{ac}{b} + \left(y_0 - \frac{ac}{b} \right) e^{-\frac{bt}{V}} \quad \text{(Ref. Eqn. (78))}$$

where $V = 1200\,\text{cm}^3,\ a = 0.1\,\text{cm}^3/\text{min} = b,\ c = .04\,\text{g}/\text{cm}^3$.

$\therefore \quad x(t) = .04(1 - e^{-t/12000})$

when $x(t) = 0.00012\,\text{g}/\text{cm}^3$, then

$$1 - e^{-t/12000} = \frac{3}{1000}$$

$$\Rightarrow \quad e^{-t/12000} = 1 - \frac{3}{1000} = \frac{997}{1000}$$

$$\Rightarrow \quad t = 12000 \ln \frac{1000}{997} \approx 36 \text{ min.}$$

Thus, after 36 min. the concentration of carbon monoxide in the room will be harmful to the human body.

– x –