Indira Gandhi National
Open University
School of Sciences

BPHET-141
ELEMENTS OF MODERN PHYSICS

Block

1

THE SPECIAL THEORY OF RELATIVITY

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ELEMENTS OF MODERN PHYSICS : COURSE INTRODUCTION

So far your study of physics has been restricted to the classical domain – the laws of nature that you have studied are the laws of classical physics. This means that you have studied physical phenomena in the macroscopic world, and in particular, their gross features. For example, using these laws, the motion of a macroscopic system consisting of pulleys, flywheels, levers, etc. can be described if the relevant parameters, viz., the density and modulus of elasticity of the material are given.

However, if you ask why the densities and elastic constants have the values they have, the laws of classical physics are silent. Similarly, if you wish to know why sodium vapour emits yellow light, what makes the sun shine, why the uranium nucleus disintegrates spontaneously or what happens when objects travel at speeds close to the speed of light, classical physics does not provide us the answers. We have to turn to new areas of knowledge, namely, the special theory of relativity, quantum mechanics and nuclear physics. These areas of physics embody concepts which are not a part of our everyday experience. They are the most remarkable intellectual creations of the early twentieth century physics.

Through this course on **Elements of Modern Physics** we extend you an invitation to join us in an adventure of the mind. We promise to provide you a fascinating exposure to these areas, which can be regarded as the fundamental conceptual structures of twentieth century physics. The going may be little tough but the rewards will be rich – intellectually stimulating and deeply satisfying!

In **Block 1**, entitled **The Special Theory of Relativity**, we discuss the theory, which may be regarded as one of the most significant achievements of modern physics. We begin the block by explaining the postulates of special theory of relativity and briefly explain the background in which these were proposed by Einstein. Then we explain the consequences of the special theory of relativity. You will learn how the concepts of Newtonian mechanics of the macroscopic universe get altered in the relativistic world in which objects travel at speeds close to the speed of light. We focus on the core concepts of relativistic kinematics and dynamics. Specifically, we discuss the notions of relativistic space and time, velocity, linear momentum, force and energy. In this block, you will learn about one of the most path breaking equations of physics, namely, $E = mc^2$, and how Einstein arrived at it. It is perhaps the most awe inspiring and beautiful equations of physics of all times!

The special theory of relativity applies to all objects in the universe moving at speeds close to the speed of light. But it still falls in the domain of classical physics. In Blocks 2 and 3, we introduce you to the concepts of quantum theory, the other revolutionary development in physics in the early twentieth century. It is the theory that best explains the behaviour of the microscopic constituents of matter – atoms, and the electrons, protons and neutrons that make up these atoms.

During the late nineteenth century and the first quarter of the twentieth century, several experiments were performed that required new explanations. The experimental results could not be explained by the established classical theories of the time, namely, Newtonian mechanics and Maxwell's electromagnetic theory. You will learn about some of these experiments and the quantum hypotheses put forward by Max Planck and Albert Einstein to explain their results. In the years between 1924 and 1927, a galaxy of physicists (de Broglie, Heisenberg, Schrödinger, Bohr and Born, to name a few) developed the fundamental conceptual framework of a new mechanics that governed the behaviour of all microscopic particles. It was called **quantum mechanics**.

In **Block 2** entitled **Introduction to Quantum Mechanics**, we describe briefly the main experiments that the theories of classical physics could not explain. Then we present the fundamental concepts underlying quantum mechanics, namely, de Broglie hypothesis, wave-particle duality, matter waves and Heisenberg's uncertainty principle. The de Broglie hypothesis ascribed wave-particle duality (till then considered a characteristic of light and all electromagnetic radiation) to all matter. This meant that all matter could behave either as a wave or as a particle and possessed a sort of inherent "duality". He introduced the concept of matter wave. Matter waves were represented by a mathematical function called the wave function. As in Newtonian mechanics, in the new mechanics too, an equation that described the evolution of the wave function in time and space was needed. Schrödinger discovered the differential equation for the wave function. And Born gave the probabilistic interpretation of the wave function to give it a physical meaning. Around the same time, Heisenberg had developed an alternative (matrix) formulation of quantum mechanics involving operators associated with physical observables. These two formulations of quantum mechanics (given by Schrödinger and Heisenberg) were later shown to be equivalent by Dirac. The concepts that you will learn in this block are some of the most intellectually fascinating creations of twentieth century modern physics.

In **Block 3** entitled **Applications of Quantum Mechanics to Simple Systems**, you will learn how to apply these concepts to simple physical systems such as the potential well, step potential and potential barrier, for which it is possible to solve the Schrödinger equation. You will also learn about some of their applications in modern technology, which works at scales at which quantum effects are important.

Finally, in **Block 4** entitled **Nuclear Physics**, we take you into the world of the atomic nucleus. We begin the block with a description of radioactivity, which was a chance discovery that led to fundamental developments in nuclear physics. The alpha particles from radioactive nuclei were used by Rutherford in his classic scattering experiments to 'see' the atomic structure, which led to the discovery of the nuclear model of atom. This opened the flood gates of investigations into the atomic nucleus – its composition and structure, its properties and the forces of interaction between its constituents about which you will learn in this block. Science and its applications are bound together as the fruit to the tree which bears it. This statement of Louis Pasteur beautifully captures the spirit in which we teach physics in our courses. Therefore, we end the block and the course with some major applications of nuclear physics. You will learn about nuclear fission and fusion, and how nuclear energy is produced in nuclear reactors providing us with a viable alternative to fossil fuels.

In the end, we have a word of advice about how to study the course. You should not read this course passively like a story but write down all concepts and work out all mathematical steps given in the course on your own. We advise you to follow the study guide given in each unit. Attempt all self-assessment questions (SAQs) and Terminal Questions (TQs) given in the units. Do not skip any of those as they are designed to assess your understanding of the subject. For any assistance with the course, you can contact us at: slamba@ignou.ac.in; vijayashri@ignou.ac.in and mbnewmai@ignou.ac.in.

We hope you enjoy studying the course. Our best wishes are with you!

BLOCK 1 : THE SPECIAL THEORY OF RELATIVITY

What is it that you think of when you hear or see the word **relativity**? The name of Albert Einstein or the equation $E = mc^2$? This signifies the enormous intellectual impact (more than one hundred years later) of Einstein's special theory of relativity. The development of this theory is rightly regarded as one of the greatest intellectual accomplishments ever made in our understanding of the physical world.

Let us give you a glimpse of the intellectual 'feast' awaiting you by recounting here an adventure of Mr. Tompkins, the hero of George Gamow's book entitled Mr. Tompkins in Wonderland. George Gamow is a well-known physicist of this century. He was a proponent of the Big Bang theory of the origin of the universe.

Mr. Cyril George Henry Tompkins, after listening to a popular lecture on the theory of relativity, dreams of a visit to a fantastic city in which the speed of light is only $25\,\text{kmh}^{-1}$.

What does he observe there? At first, nothing unusual seems to happen around him. A policeman standing on the corner looks as policemen usually do! The streets are nearly empty. But when a cyclist coming down the street approaches Mr. Tompkins, he is absolutely astonished. For, the bicycle and the man on it appear unbelievably flattened to him. When the clock strikes twelve, the cyclist who seems to be in a hurry pedals harder. Though he does not gain much in speed, he appears flattened even more. Mr. Tompkins decides to overtake the cyclist and ask him about it. He borrows a bicycle and pedals on it hoping to get flattened. But amazingly, nothing happens to him. Instead, the picture around him changes completely. The streets grow shorter, the windows of the shops begin to look like narrow slits and the policeman on the corner becomes the thinnest man he had ever seen!

What a topsy-turvy world Mr. Tompkins visits in his dreams and what weird experiences he has. We are sure you will be interested in finding an explanation for these happenings in Mr. Tompkins' dream world. This block will help you do so.

Do you realise that the basic concept of relativity is as old as the mechanics of Galileo and Newton? So what did Einstein do to make his name almost synonymous with relativity?

At the beginning of the twentieth century, two great and beautiful theories were known in the physical sciences – Newtonian mechanics and Maxwell's electromagnetic theory. Both of them gave unified explanations of countless physical phenomena. These theories were expressed in concise mathematical language within a certain conceptual framework. You have learnt both these theories in Blocks 2 and 3 of the course BPHCT-131 entitled Mechanics and Block 4 of the course BPHET-141 entitled Electricity and Magnetism, respectively. You have also learnt numerous applications of these theories. Both theories have been confirmed many a times; and they have been extraordinarily successful in their predictions. And yet these two theories were conceptually in contradiction with one another!

What was this contradiction? In **Unit 1** of this block entitled **Postulates of Special Relativity**, we briefly explain the contradiction. How was this contradiction resolved? Its solution was provided by none other than Albert Einstein. He resolved the contradictions as he saw them and formulated a new theory based on two new postulates. These two postulates led Einstein to a new view of **space** and **time** about which you will study in **Unit 2** entitled **Relativistic Kinematics**. Naturally, a radically new approach to the notions of space and time resulted in changes in well established areas of physics. In **Unit 3** entitled **Relativistic Dynamics**, we discuss the new mechanics that replaced Newtonian mechanics as a result of these changes.

What we had, in effect, was an Einsteinian revolution. Its impact far exceeds that of the Copernican revolution and has rarely been equalled in the history of physics. To quote Roger Penrose, a renowned English physicist, mathematician and philosopher of science,

"In the twentieth century, we have been greatly privileged to witness two major revolutions in our physical picture of the world…We have come to use the term 'relativity' to encompass the first of these revolutions and 'quantum theory' to encompass the second…It is particularly remarkable that a single physicist – Albert Einstein – had such deep perceptions of the workings of Nature that he laid foundation stones for both of these twentieth century revolutions in the single year of 1905.''

Studying this block will require a great deal of imagination and deep thinking. Be prepared to challenge your everyday notions of space, time and mechanics. Pay attention to the phrases and sentences in bold and the summaries of concepts in each section. Try to solve all Examples, SAQs and Terminal Questions on your own for a better understanding of the special theory of relativity. The mathematics of this block is quite simple but the ideas and concepts are profound and need careful reading.

We have tried our best to bring to you the beauty and logic of the special theory of relativity. We hope that you will appreciate and enjoy studying it as much as we did presenting it.

We wish you success!

In 1905, Albert Einstein presented the special theory of relativity that changed our understanding of the physical universe dramatically! The speed of light had an important role in it about which you will learn in this unit. Source of picture: Wikimedia Commons

UNIT 1

POSTULATES OF SPECIAL RELATIVITY

Structure

1.1 Introduction
  Expected Learning Outcomes
1.2 Constancy of the Speed of Light
  Galilean Coordinate Transformation
  Galilean Principle of Relativity
1.3 Postulates of the Special Theory of Relativity
  The Principle of Relativity
  The Principle of Constancy of Speed of Light
  The Nature of Time in Special Relativity
1.4 Summary
1.5 Terminal Questions
1.6 Solutions and Answers

STUDY GUIDE

In this unit, we explain the postulates of the special theory of relativity and briefly discuss how it emerged. We have tried to keep the explanation as simple as possible. However, we will be using many concepts and ideas from our earlier physics courses. We advise you to revise Block 2 of BPHCT-131 (Mechanics) and Block 4 of BPHET-141 (Electricity and Magnetism).

There is hardly any use of mathematics in this unit and the ideas presented in this unit are simple and known to you. But it would require some deep reflection on your part to internalise them. The presentation of some of these ideas may be new for you and you may like to read Secs. 1.2 and 1.3 more than once for a better understanding. Do make sure that you have grasped the concepts of this unit well before you study Unit 2.

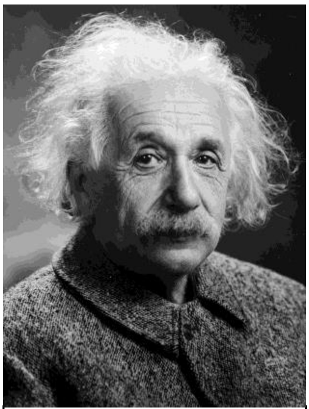
"Most of the fundamental ideas of science are essentially simple, and may, as a rule, be expressed in a language comprehensible to everyone."

Albert Einstein

1.1 INTRODUCTION

You have studied Newtonian mechanics in your school physics courses and in the first semester physics course entitled 'Mechanics' (BPHCT-131). You have learnt the concepts of kinematics and Newton's laws of motion right from your school physics. So, you know what an **inertial frame of reference** is.

In your school physics courses, you have learnt that Newton's laws of motion are the same for all inertial observers. Recall your experiences from everyday life. For example, suppose you throw a ball up in the air or drop a coin while standing on the ground. And at some other time, you travel in a train or bus moving at a constant velocity (inertial frame) and repeat the same activities. What do you observe? The ball and the coin move in the bus/train exactly as they did on the ground: if you throw them vertically upwards, these fall down with acceleration due to gravity. So, the bodies are falling under the force of gravity and obey Newton's laws of motion both on the Earth and in any inertial frame of reference moving with respect to it.



Albert Einstein (1877 – 1955) is regarded as one of the greatest physicists of all time. In a single year, 1905, a year sometimes described as his *annus mirabilis* ('miracle year'), Einstein published four path breaking papers on photoelectric effect, Brownian motion, special relativity, and mass-energy equivalence. His mass-energy equivalence equation $E = mc^2$, has been called "the world's most famous equation". He received the Nobel Prize in Physics in 1921 "for his services to theoretical physics, and especially for his discovery of the law of the photoelectric effect."
Source of picture: commons.wikimedia.org

Both Galileo and Newton were deeply aware of this principle that the laws of mechanics are the same in all inertial frames of reference. This is known as the **classical** or the **Galilean principle of relativity**. So the classical notion of relativity is not new to you although you may not have come across this terminology before.

Now, we ask: Why are we discussing the constancy of the speed of light along with the classical or Galilean principle of relativity in Sec. 1.2 to get to Einstein's special theory of relativity?

To answer this question, we explain briefly the Galilean coordinate transformation in Sec. 1.2.1. It relates the positions (and velocities and accelerations) of objects in two inertial frames moving uniformly with respect to each other. As you will see in this section, you have used these equations many times while solving problems on relative motion in school physics.

Then we ask: What happens when we apply the Galilean coordinate transformation to a flash or pulse of light (generated by, say, bulb or a laser pointer) observed in two inertial frames of reference? What velocities of the pulse of light are measured in these frames moving at a constant velocity with respect to each other? This is how the speed of light enters our discussion on special theory of relativity. You will also learn about the problem that arises in this process.

We also describe how Albert Einstein resolved the problem, in 1905, by a giant leap of imagination, and presented the special theory of relativity. This theory gave us a radically different way of understanding the physical world. And Einstein was not even aware of the experimental results that established that the speed of light was constant when he proposed his special theory of relativity! In Sec. 1.2.2, we revisit the Galilean principle of relativity to set the background for Sec. 1.3.

In Sec. 1.3, we present the two postulates of special theory of relativity proposed by Albert Einstein – the constancy of speed of light and the principle of relativity. You will learn that Einstein was not the first to introduce the relativistic notions in physics. What he did was to generalise the Galilean principle of relativity (applicable only to mechanics) to **all physical laws**. In this way, he resolved the contradictions arising from the constancy of speed of light. You will learn about this generalisation, called the **principle of relativity**, in Sec. 1.3.1. It is the first postulate of the special theory of relativity.

In Sec. 1.3.2, you will learn about the second postulate of the special theory of relativity – the **principle of constancy of speed of light**. In Sec. 1.3.3, we discuss how the postulates of the special theory of relativity changed our understanding of the nature of time. You will learn the concept of **relativity of simultaneity**. In the next unit, you will learn some more consequences of the special theory of relativity. In particular, you will learn about the phenomena of time dilation and length contraction observed in inertial frames moving uniformly at speeds close to the speed of light.

Expected Learning Outcomes

After studying this unit, you should be able to:

- use the Galilean coordinate transformation to describe events in different inertial frames of reference moving uniformly with respect to each other;
- state the Galilean principle of relativity and explain why it became necessary to generalise it;
- state the postulates of special theory of relativity; and
- explain the concept of relativity of simultaneity.

1.2 CONSTANCY OF THE SPEED OF LIGHT

Let us begin our study by revising the concepts you are familiar with. In school physics, you have learnt that the speed of light is constant and has the value $3.0\times10^{8}\,\text{m}\,\text{s}^{-1}$. The question before us is: **How did recognising this simple fact lead to a revolutionary change in our understanding of the physical world**?

In order to answer this question, let us briefly revisit the concept of relative motion. Let us ask: How do we determine the speed of an object with respect to another object in an inertial frame of reference? You have solved many such problems on relative motion in your senior school physics courses.

Do you recall the concept of an inertial frame of reference? You have learnt it in your school physics and in the first semester course on Mechanics (BPHCT-131). Recall that an **inertial frame of reference is the frame of reference in which Newton's first law holds**. Thus, it is a frame of reference that is **at rest** or **moving with a constant velocity**. Now, try to work out the problems given below. Remember, you have already solved many such problems on inertial frames of reference and relative motion in your senior school physics.

PRE-TEST

1. State, giving reasons, which of the following frames of reference are inertial. The frames attached to

   a) a car in circular motion,

   b) spaceships cruising uniformly in space,

   c) an electron accelerating in an electric field in space,

   d) a boat moving at a constant speed in a river flowing uniformly,

   e) a book at rest on a fixed table in your room.

2. A train is moving past a platform at a constant speed of $60\,\text{km}\,\text{h}^{-1}$. A passenger in the train throws a ball horizontally along the length of the compartment (in the train's direction of motion) at a speed of $5\,\text{km}\,\text{h}^{-1}$. What is the speed of the ball measured by a child standing on the platform?

If you have solved these two problems correctly, then you know the basic concepts about relative motion of objects in inertial frames. You should be able to follow the discussion in Secs. 1.2.1 to 1.2.3. Otherwise, you should study these sections thoroughly and solve the pre-test problems and the SAQ given at the end of Sec. 1.2.

1.2.1 Galilean Coordinate Transformation

You have studied the concept of an inertial frame of reference in Unit 5 of Block 2, BPHCT-131 (Mechanics), and that of relative motion in school physics. You are familiar with the **Galilean coordinate transformation**, which gives the relations between the positions, velocities and accelerations of an object measured with respect to two inertial frames in uniform relative motion. Let us revise them quickly with an example similar to problem 2 in the pre-test.

Imagine that you are **standing** on a railway platform. Suppose a train goes past you at a constant speed of $5\,\text{km}\,\text{h}^{-1}$ (Fig. 1.1).

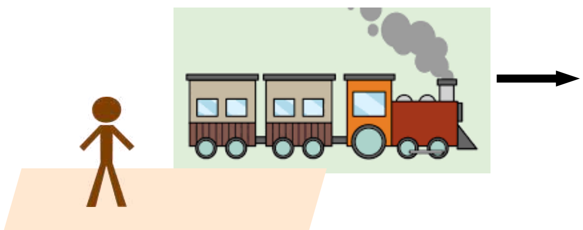


Fig. 1.1: You (the observer) standing on the platform, and a train, moving at a constant speed in a straight line with respect to you.

Suppose you see a child sitting inside a compartment throwing a ball horizontally in the direction of the train's motion. Let the speed of the ball with respect to the child be $1\,\text{km}\,\text{h}^{-1}$. Now, we ask you: What is the speed of the ball with respect to you? If you have solved pre-test problem 2 above, you will say, it is $6\,\text{km}\,\text{h}^{-1}$.

But if you have not solved the problem, we will briefly explain how to calculate the speed of the ball.

To keep the discussion simple, we take all motion to be in a straight line, i.e., we consider one-dimensional motion along the *x*-axis. Suppose the inertial frame attached to you (the observer) is at rest. Let us call it S (the lower line in Fig. 1.2). Let the inertial frame attached to the train/child be S'. Now frame S' is moving in a straight line at a constant speed, say, V, along the *x*-axis with respect to you, that is, frame S. Then the velocity of the frame S' with respect to the frame S is $\vec{\mathbf{V}} = V\hat{\mathbf{i}}$, where $\hat{\mathbf{i}}$ is the unit vector along the *x*-axis. Now refer to Fig. 1.2 again, which shows the two frames of reference at some instant of time t.

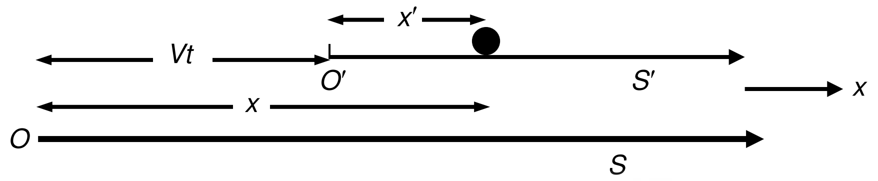


Fig. 1.2: The frames of reference S and S', and their origins at some instant of time t.

Let us take both one-dimensional frames to be along the *x*-axis for this example of straight-line motion. In Fig. 1.2, note that at the instant t, the origins of the frames S and S' are at O and O', respectively. Given that the frame S' is moving at constant speed V along the *x*-axis, what is the distance between the points O and O' at the instant t? From Fig. 1.2, you can see that it is $OO' = Vt$.

Now suppose the positions of the ball at the instant t as observed in the frames S and S' are x and x', respectively (Fig. 1.2). Study Fig. 1.2 and answer the question: What is the relation between x and x'? From Fig. 1.2, you can see that

$$x = x' + Vt \tag{1.1a}$$

or

$$x' = x - Vt \tag{1.1b}$$

Eqs. (1.1a and b) give the **Galilean coordinate transformation** (in one dimension) for two inertial frames, one of which is moving at constant velocity $\vec{\mathbf{V}} = V\hat{\mathbf{i}}$ with respect to the other. We can obtain the speed of the ball by differentiating Eq. (1.1a) with respect to time:

$$v = v' + V \tag{1.2a}$$

or

$$v' = v - V \tag{1.2b}$$

where v and v' are the respective speeds of the ball at the instant t as observed in the frames S and S'. Eqs. (1.2a and b) give the Galilean transformation for the velocity of an object in the two inertial frames. So, from Eq. (1.2a), the speed v of the ball with respect to you is $6\,\text{kmh}^{-1}$.

You may now be wondering: What has the speed of light got to do with this? To answer this question, let us now consider the following situation: Instead of a child throwing a ball in the train, she switches on a torch in the compartment and quickly switches it off (Fig. 1.3). So, a pulse of light travels in the compartment in the direction of the train's motion.

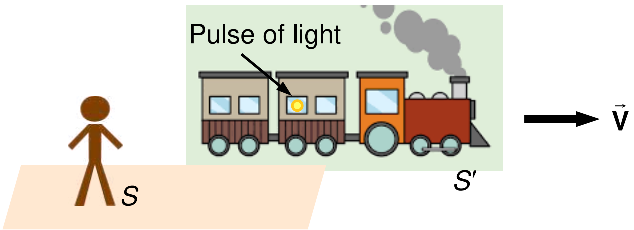


Fig. 1.3: A pulse of light is observed in the frames of reference *S* and *S′* at an instant.

If we apply Eq. (1.2b) to obtain the speed of light observed in the train, we get

$$c' = c - V$$

If we accept this result, then the speed of light would be different in both frames. This contradicts the fact that the speed of light is constant. So, we need to replace the Galilean coordinate transformation with some transformation that is consistent with the constancy of speed of light. We will do that in the next unit.

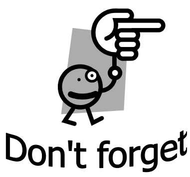
Now we ask: What is the speed of light as measured in the two frames *S* and *S′*? According to the Galilean coordinate transformation, it is c in the frame S and $(c - V)$ in the frame *S′* (read the margin remark). But you know from school physics that the speed of light in vacuum (denoted by c) does not depend on the speed of the source of light. So, the speed of light with respect to an observer in the train moving with a constant velocity and the observer at rest (you) will be the **same**.

You may also know that it has been established in many experiments that the **speed of light in vacuum is constant in the universe. It does not depend on the speed of the source of light. We know that there has been no violation of this fact so far.**

The fact that the speed of light is constant was stated by Einstein in his paper in 1905. We give his statement here:

"**Light always propagates in empty space with a definite velocity *V* that is independent of the state of motion of the emitting body**."

This statement is remarkable because Einstein made it without knowing the experimental results that established that the speed of light in vacuum was constant. He arrived at this statement purely on the basis of theoretical considerations. The above statement of Einstein is a postulate of the special theory of relativity. So, always remember:



The speed of light $(c = 3.0 \times 10^{8}\ \text{m s}^{-1})$ **in vacuum is constant.**

In Sec. 1.3.1, we will discuss the principle of relativity, which is the other postulate of the special theory of relativity. It is an extension of the Galilean principle of relativity. So, you should learn that first.

1.2.2 Galilean Principle of Relativity

In this section, we will briefly explain the Galilean principle of relativity using terms and concepts you already know from school physics and BPHCT-131 (Mechanics). We will be using the term '**physical event**/**event**' quite often in this block. Let us now formally state what we mean by a 'physical event/event' taking place in three-dimensional space.

An idealized version of a physical event is that it is something that happens **<u>at a point in space</u>** and **<u>at an instant in time</u>**. While discussing the theory of relativity, Einstein often used this dramatic example of an event – lightning strikes the ground. A small explosion is an equally dramatic event. In Sec. 1.2.1, a ball being thrown horizontally in a train moving with a constant velocity is an example of a physical event. You can think of several other examples of a physical event.

There are two basic questions that we need to answer in order to describe any physical event.

<u>Where</u> did it take place?

<u>When</u> did it take place?

How do we answer these questions?

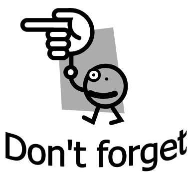
As you know very well, for motion in space, we can describe an event by **four measurements** in a particular frame of reference – three for the position (in three-dimensional space) and one for the instant of time t.

We usually use the Cartesian coordinates (x, y, z) to describe the **position** of any event. You have used the Cartesian coordinate system in all the physics courses that you have studied so far. So, you know it very well.

As an example, consider the event of the collision of two balls. Let us say that the event occurs at the position $x = 1\text{m}$, $y = 2\text{m}$ and $z = 3\text{m}$ and at the instant of time $t = 4\text{s}$ in a frame of reference attached to the Earth such as a laboratory. Then the four numbers (1, 2, 3, 4) specify the event in that reference frame; the first three numbers specify its position and the fourth, the time at which the collision occurred.

So, in order to accurately describe where and when an event happens, we have to use a frame of reference. You also know that for describing an event we are free to use any frame of reference we wish. In this block we shall restrict our study to **inertial frames of reference**. In Sec. 1.2.1 of this unit, you have described one-dimensional motion in an inertial frame of reference. So, you know that

An inertial frame is a frame of reference in which Newton's first law holds true.



You know that in an inertial frame of reference, objects at rest remain at rest and objects moving with constant speed in a straight line continue to do so, unless a net external force is exerted on them. You should now test whether you understand the concept of an inertial frame of reference before studying further. Try the pre-test question 1.

From the above statement, you can also conclude that

Any frame that moves with constant velocity relative to an inertial frame is also an inertial frame.

In Sec. 1.2.1, you have learnt the one-dimensional Galilean coordinate transformation for describing an event in two inertial frames of reference. One of these, say S, is at rest and the other, say S', is moving with a **constant velocity with respect to** S. In such cases, we say that **the two inertial frames of reference are in uniform relative motion**.

The statement that 'two objects/frames are in *uniform relative motion*' means that the objects/frames are moving with constant velocity with respect to each other. Here 'uniform' refers to constant velocity.

Let us now consider a physical event E that occurs at point P in three-dimensional space. For example, E could be a ball thrown vertically upwards. The question now is: How do we describe the event E (motion of the ball in this case) in two inertial frames of reference in **uniform** relative motion?

Suppose that E is being observed from the inertial frames of reference S and S'. Let the frame S' move with a **constant velocity** $\vec{\mathbf{V}}$ with respect to S, along their common xx'-axes. We assume that the metre sticks and clocks of the observers in the two frames have been **jointly** calibrated. Let the other axes in the two frames be parallel to each other: y-axis is parallel to the y'-axis and z-axis is parallel to z'-axis (see Fig. 1.4).

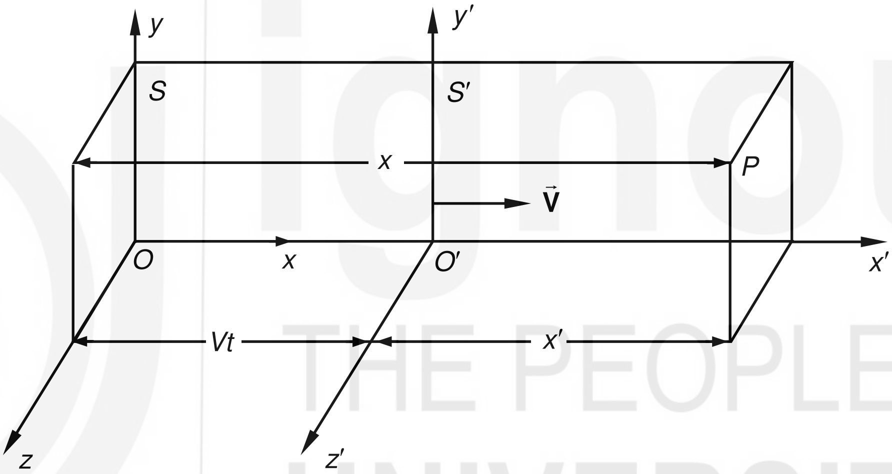
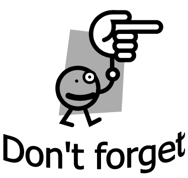


Fig. 1.4: **Two inertial frames of reference S and S'. The frame S' is moving with constant velocity $\vec{\mathbf{V}}$ with respect to S so that the x-x' axes is common and the y-y' and z-z' axes are parallel.**

Also, let $t = 0$ in frame S and $t' = 0$ in frame S' be the instant when the observers start their clocks (start measuring time) and let the origins of the two frames coincide at that instant.

We need to spell out these conditions clearly as these are important in any discussion on relativity. **You must always remember the following initial conditions for describing a physical event in two inertial frames S and S' moving uniformly with respect to each other**.



- **The metre sticks and clocks in the two inertial frames of reference, in which an event is being observed, are calibrated jointly.**
- **The clocks of the observers in the frames S and S' read zero $(t = 0,\ t' = 0)$ at the instant that the origins of these frames coincide.**

Now, let the coordinates of the event E at some instant of time be measured as

(x, y, z, t) in the frame S

And (x', y', z', t') in the frame S'

This means that the coordinates (x, y, z) describe the position of the point P relative to the origin O of the frame S and t is the time at which the event E occurs as recorded by the clock of S. The coordinates (x', y', z') describe the position of P with respect to the origin O' of the frame S' and t' is the time at which E occurs according to the clock of S'. And as you have studied above, the clocks in each frame read zero at the instant that the origins O and O' of the frames S and S' coincide.

Now we ask: What is the relationship between the coordinates (x, y, z, t) and (x', y', z', t')? Since the frame S' is moving along the x-axis, we can use Eq. (1.1b) to relate x and x'. And the other coordinates of the event remain the same. So, the Galilean coordinate transformation equations that relate the coordinates of the event E in three-dimensional inertial frames of reference S and S' are as follows:

$$x' = x - Vt \tag{1.3a}$$

$$y' = y \tag{1.3b}$$

$$z' = z \tag{1.3c}$$

$$t' = t \tag{1.3d}$$

Eq. (1.3d) tells us that the Galilean transformation does not change time. So, the time (or time interval) measured by the both observers (or the two clocks) in the frames S and S' is the **same**. Now in BPHCT-131, you have learnt that the three equations (1.3a to c) can be written as one equation using the following vector notation:

$$\vec{\mathbf{r}}' = \vec{\mathbf{r}} - \vec{\mathbf{V}}t \tag{1.4a}$$

and
$$t' = t \tag{1.4b}$$

where $\vec{\mathbf{r}}$ and $\vec{\mathbf{r}}'$ are the position vectors of the point P (at which the event E occurs) in the frames S and S', respectively. We have taken $\vec{\mathbf{V}} = V\hat{\mathbf{i}}$ and you can see that Eq. (1.4a) is equivalent to Eqs. (1.3a to c). Now, differentiating Eq. (1.4a) with respect to t, we get:

$$\frac{d\vec{\mathbf{r}}'}{dt} = \frac{d\vec{\mathbf{r}}}{dt} - \vec{\mathbf{V}}$$

or
$$\frac{d\vec{\mathbf{r}}'}{dt} = \vec{\mathbf{v}} - \vec{\mathbf{V}} \tag{1.5a}$$

Since $t' = t$, we write the left-hand side of Eq. (1.5a) as $\frac{d\vec{\mathbf{r}}'}{dt} = \frac{d\vec{\mathbf{r}}'}{dt'} = \vec{\mathbf{v}}'$.

Thus,
$$\vec{\mathbf{v}}' = \vec{\mathbf{v}} - \vec{\mathbf{V}} \tag{1.5b}$$

We differentiate Eq. (1.5b) again with respect to t, and use Eq. (1.4b) to get:

$$\vec{\mathbf{a}}' = \vec{\mathbf{a}}, \text{ since } \vec{\mathbf{V}} \text{ is constant.} \tag{1.6}$$

The equation of motion is then given as:

$$m\vec{\mathbf{a}}' = m\vec{\mathbf{a}} = \vec{\mathbf{F}} \tag{1.7}$$

What does Eq. (1.7) tell us? It tells us that the **equation of motion (that is, Newton's second law of motion for constant mass) is the <u>same</u>** in the frames S and S'. So, **the equation of motion, and all laws of mechanics that can be deduced from it do not change under the Galilean coordinate transformation**. We say that **the equation of motion and the laws of mechanics remain <u>invariant</u> under the Galilean coordinate transformation**. This is known as the **classical** or **Galilean principle of relativity**. Let us state it formally.

GALILEAN PRINCIPLE OF RELATIVITY

The laws of mechanics are the same in all inertial frames. If they hold in one inertial frame, they will hold in all inertial frames.

Note that the Galilean principle of relativity is a limited principle that applies only to the laws of mechanics.

Let us understand with the help of a simple example what the Galilean principle of relativity means. Suppose you are in the compartment of a train moving at constant velocity and **you cannot look out**. Now, if you throw a ball vertically upwards, you will observe that it falls vertically along the same path. To you, the motion of the ball would be the same as if you were standing on the ground.

So, all mechanical experiments performed and all mechanical phenomena occurring in the compartment will appear the same to you as if the train were not moving (provided, of course, that you cannot look out). Suppose you could not look out of the train. Then, no mechanical experiment performed in the train could help you determine whether it was moving uniformly or was at rest (provided, of course, **you did not look out**).

So, we hope that you are clear what is meant when we say that if the laws of mechanics are true in one inertial frame of reference, they will be true and be of the same form in any other inertial frame as well.

Thus, as far as mechanics is concerned **there is no preferred inertial frame of reference in which the classical laws have the most basic form.** In mechanics, **all inertial frames are equivalent and there is no absolute frame of reference**.

You may like to pause for a while and find out whether you have understood these ideas. Try the following SAQ.

SAQ 1 - Galilean principle of relativity

From Eq. (1.7), can you conclude that all inertial observers will measure the same values for the position, time and velocity corresponding to an event?

So far, you have learnt that the speed of light is constant. You have also learnt the Galilean principle of relativity. We will now present the postulates of the special theory of relativity as given by Einstein.

1.3 POSTULATES OF THE SPECIAL THEORY OF RELATIVITY

You have learnt in Sec. 1.2.1 that the constancy of the speed of light in all inertial frames is in contradiction with the Galilean coordinate transformation.

In 1905, Albert Einstein applied Galilean coordinate transformation to the laws of electricity and magnetism (Maxwell's equations). He found that the form of these laws **changed** in inertial frames of reference in uniform relative motion. This was a contradiction.

But he also found that Maxwell's equations would remain the same in all inertial frames on the following condition:

The speed of light in vacuum should not depend on the speed of the source of light.

Einstein chose **not to modify** Maxwell's equations of electromagnetic theory. Rather he presented a revolutionary proposal which resolved this contradiction between Newtonian mechanics and the electromagnetic theory.

He **generalised** the Galilean principle of relativity. You have learnt in Sec. 1.2.2 that the Galilean principle of relativity applies to only the *laws of mechanics*. Einstein generalised it to include **all laws of physics**.

In 1905, in his paper "On the Electrodynamics of Moving Bodies", Einstein formulated the following two postulates of the special theory of relativity:

POSTULATES OF THE SPECIAL THEORY OF RELATIVITY

Postulate 1: The principle of relativity

The laws of physics are the same in all inertial frames of reference.

Postulate 2: The principle of constancy of speed of light

The speed of light (in vacuum) has the same constant value in all inertial frames of reference.

The word postulate means a suggestion or an assumption about the existence of a fact, or truth or something, which forms the basis for reasoning, discussion, or belief.

These two assumptions led Einstein to a new theory of physics which is now known as **the special theory of relativity**. It is called 'special' theory because it only deals with observations made in inertial frames.

For example, it does not say anything about the relationship between physical quantities observed in two frames undergoing **relative acceleration**. Non-inertial frames are the subject matter of another one of Einstein's theories – the general theory of relativity.

We now briefly explain both these postulates.

1.3.1 The Principle of Relativity

Postulate 1 generalises the limited Galilean principle of relativity to all laws of physics. It tells us that

Any law of physics that is true in one inertial frame will also be true in all other inertial frames.

This means that laws of nature do not depend upon the choice of an inertial frame of reference or the position or motion of an observer – the laws will always retain their form in any inertial frame of reference.

The measurements of various quantities, like the positions, times, velocities, energies, momenta, etc. may be different in different inertial frames. But the relationships between these quantities **as governed by various laws** would remain the same in all inertial frames.

Actually, the principle of relativity tells us about the **objective** character of the laws of nature; **these laws are independent of the inertial observers.** The laws remain the same regardless of who the inertial observer is.

Always remember: The principle of relativity should never be misrepresented by saying incorrectly that *knowledge is relative* and depends on the observer.

The principle of relativity is also stated in the following form, which you will often encounter:

The laws of physics do not allow us to distinguish between different inertial frames.

In other words, **you will not be able to distinguish through any experiment whether you are at rest or in a state of uniform motion**.

Suppose that such an experiment could be performed. It would mean that the laws of physics depended in some way on your velocity; these would be different when you were at rest. This is **incorrect** according to the principle of relativity.

You should understand that **the principle of relativity does not claim that all inertial frames are the same in all respects**.

To appreciate this point, consider two different spacecrafts, each travelling with a different constant velocity with respect to an inertial frame of reference *S*.

The principle of relativity tells us **that the inertial frames attached to the two spacecrafts cannot be distinguished as far as the laws of physics are concerned**.

However, if one could look outside each spacecraft through a window, it would be easy to know that they are moving at different velocities, relative to *S*. Does this contradict the principle of relativity?

No, because the *velocity* of the spacecraft relative to *S* does not figure in any **law** of physics.

Besides, **in formulating this form of the principle of relativity**, **the essential condition was that the inertial frames (the spacecrafts in this example) were completely isolated**. The observers inside the spacecrafts could not look outside them.

The ideas presented here are worthy of careful thought and you may need to go through them more than once. You may now like to attempt the following SAQ to know whether you have grasped the principle of relativity or not.

SAQ 2 - The principle of relativity

Suppose an observer in an inertial frame finds that a certain equation giving a force law is supported by experiment. Does it automatically follow that the equation represents a law of physics?

Let us now study the second postulate of the special theory of relativity.

1.3.2 The Principle of Constancy of Speed of Light

The second postulate about the constancy of speed of light is crucial in leading to the special theory of relativity. It is very important because it radically alters the classical notions of absolute space and time, as you will learn in Sec. 1.3.3 and the next unit.

Refer to Fig. 1.5. Consider two inertial frames of reference S and S' shown in the figure. The frame S is at rest and the frame S' is moving with constant speed V along the positive x-axis. We say that the two frames are in uniform relative motion along the positive x-axis. Now let there be two inertial observers A and B such that A is at rest with respect to the frame S and B is at rest with respect to the frame S'.

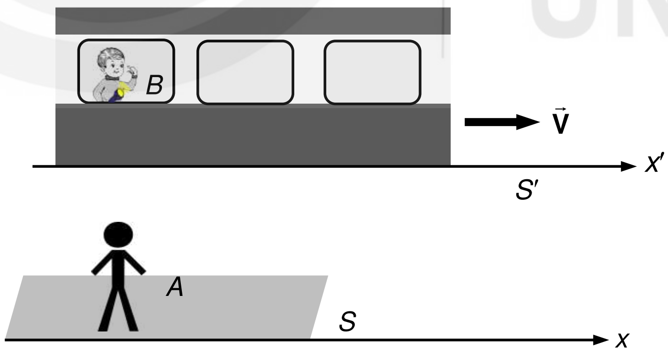


Fig. 1.5: **Observer A is at rest on the platform and the one-dimensional inertial frame of reference S is attached to A. Observer B is inside a train moving with a constant speed V along the positive x-axis. Also, B is at rest with respect to the frame S' attached to the train and him.**

So, according to A, B is moving with constant speed V along the positive x-axis and according to B, A is moving with the same speed V along the negative x-axis. The observer A could be you standing on a platform and observer B could be a child sitting in a train moving uniformly past you. In Fig. 1.5, B holds a source of light such as a torch. According to B, the speed of light is c. To the observer A, the torch is moving with constant speed V in the positive *x*-direction. Now, according to the second postulate given by Einstein, *A* must also measure the same speed of light that is measured by *B*.

You may wonder: Does it really matter if both observers measure the same speed of light or not? It does matter because the fact that the speed of light is constant, radically changes our ideas about the nature of time and space. In the next section, we briefly explain how our understanding of the nature of time changes due to the constancy of speed of light.

1.3.3 The Nature of Time in Special Relativity

The second postulate of the special theory of relativity radically alters the classical notions of absolute space and time. Let us explain how.

You should understand that the basic premise of Newtonian mechanics was that the **same time scale applied to all inertial frames of reference**. So, it is a universal time scale. While recording the occurrence of any event, the clocks of all inertial observers would always show the same values of time. This is reflected in the equation $t' = t$ in Galilean coordinate transformation.

Now, using this universal time scale, we must be able to give meaning to statements such as "**Events** *A* **and** *B* **occurred at the same time**," without referring to any inertial frame of reference.

Let us use the example given by Einstein to explain what we mean. When we say that a train arrives at 7 o'clock, what we mean is this: The pointing of the clock hand to 7 and the arrival of the train are two events that take place **at the same time**. So, these are **simultaneous** events. Thus, the assigning of a given time to events involves **judging whether they are simultaneous or not**.

So, suppose all observers, independent of their position and velocity, agreed that any two events (e. g., the arrival of the train at the station and the pointing of the clock hand to 7) are simultaneous. Then we could certainly say that the *absolute Newtonian time scale existed*.

If you find the discussion of this section difficult to grasp, you must read it again till you feel confident that you have understood the point being made here.

However, **if different inertial observers did not agree that two events were simultaneous, we would certainly not have an absolute time scale**. This would happen if one inertial observer said that two events occurred at the same time and another inertial observer said that they did not. Then we would not be able to talk about an absolute time scale.

This is precisely what happens if we uphold the constancy of the speed of light. Consider an event similar to the one shown in Fig. 1.5 to grasp this idea. Suppose the train compartment is travelling at a **very high** constant speed *V* to the right of an observer *S* at rest on the platform (see Fig. 1.6). An observer *S′* is sitting in the compartment. Consider a high speed flashbulb situated at the exact centre (*C*) of the compartment. It sends out two light pulses to the right and left, respectively, when it flashes. Suppose that the positions of observers *S* and *S′* coincide with that of the bulb when it flashes (Fig. 1.6a). Let there be detectors (photocells) at each end of the compartment, which register a tick when a pulse of light strikes them.

Now let us find out: What does the observer S' sitting in the compartment observe after the flashbulb emits the pulses? Note that the flashbulb is at rest with respect to S' and is at the centre of the compartment. So, when it flashes, **two light pulses travel equal distances to the two ends of the compartment in equal times**. Hence, S' observes that the light pulses hit the two detectors at the ends of the compartment **at the same time**.

Now, suppose by using some smart device, the observer S on the platform can also observe the two pulses. What does S observe? Can you take a guess? Refer to Figs. 1.6b and c. Consider the light pulses travelling to right and left in the frame of reference of S. Note that the compartment is moving to the right with respect to S. So, **as time passes, in the frame S, the distance between the point at which S observes the pulse and the left end of the compartment is decreasing**. Let us explain this point further.

Let CA be the distance between the centre of the compartment and its left end at the instant when the clocks of the observers start, i.e., at the instant $t = 0$ in S (Fig. 1.6a). Let the pulse reach the point P with respect to S at some given instant t (Fig. 1.6b) and the point Q at the instant $(t + \Delta t)$, that is, after some more time passes (Fig. 1.6c). Study Figs. 1.6b and c carefully. Note that at the instant t, the distance between the point P at which S observes the pulse and the left end of the compartment is PA. It is shorter than CA: $PA < CA$. At a later instant, say, $t + \Delta t$, we have $QA < PA < CA$.

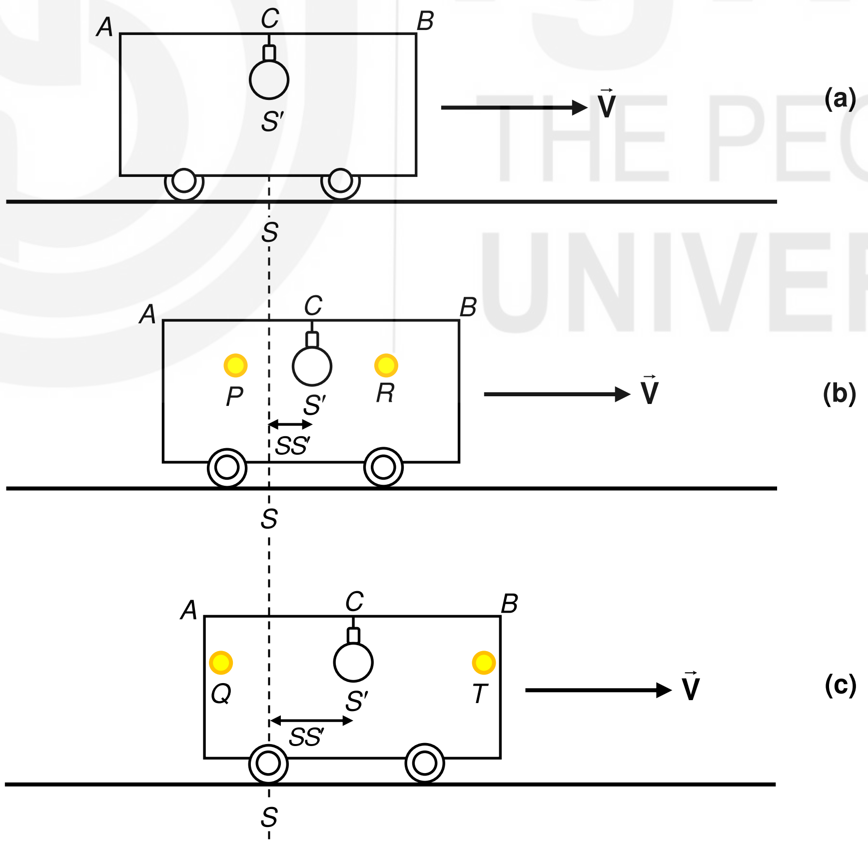


Fig. 1.6: a) The inertial observer S' in the moving train compartment observes that the two pulses of light emitted by the flashbulb reach the two ends of the compartment at the same time; b) and c) unlike S', the stationary observer S on the ground observes that the light pulses do not strike the ends of the compartment simultaneously. The figures are drawn with respect to the inertial observer S.

So, the distance travelled by the left pulse (i.e., the pulse travelling to the left) to the left end *A* of the compartment is **decreasing** with respect to *S*. When the left pulse reaches *A*, then we can infer from Fig. 1.6c that $AS < AS'$.

Now, what about the distance travelled by the right pulse between the centre *C* and the right end *B* of the compartment with respect to *S*?

You can see from Figs. 1.6b and c that, with respect to frame *S*, the distance travelled by the pulse to the right end *B* of the compartment is **increasing**. When the pulse reaches the right end of the compartment, then we can infer from Fig. 1.6c that $BS > BS'$.

Now consider the instant at which the left and right pulses hit the detectors placed at the left and right ends (*A* and *B*) of the compartment, respectively.

With respect to *S*, the distance travelled by the left pulse to the end *A* of the compartment is **shorter** as compared to the distance travelled by the right pulse to the end *B*. This is what you have just learnt.

Thus, as observed by *S*, the left pulse will reach the end *A* in **lesser** time and the right pulse will take **more** time to reach the end *B*. This is because with respect to *S*, the left pulse travels lesser distance as compared with the right pulse and *both pulses travel with the same constant speed of light*.

What is the result? Observer *S* observes that the pulse moving to the left reaches the end *A* of the compartment **<u>before</u>** the pulse moving to the right reaches the end *B*.

This means that for the observer *S*, the light pulses **do <u>not</u> reach the two ends of the compartment at the same time**.

So, **the two events are <u>not</u> taking place at the same time for the inertial observer** *S*. We say that these events are **<u>not simultaneous</u>** for observer *S*.

This result arises because of the **constancy of the speed of light**. And also because we have taken the speed *V* to be very high, close to the speed of light. You will appreciate this point better in the next unit.

Now, suppose we applied Galilean transformation used in Newtonian mechanics to this example.

Then the observer *S* would measure a lower speed $(c - V)$ of the pulse travelling *smaller* distance to the left (opposite to the direction of the train's motion). Observer *S* would also measure a *greater* speed of light $(c + V)$ of the pulse travelling greater distance to the right.

So, in Newtonian mechanics the time taken by each pulse to travel from the centre to the ends of the compartment would be measured by *S* to be the same. Thus, *S* would conclude that the two pulses reach the respective ends of the compartment at the same time.

But, **as per the special theory of relativity, the speed of light is constant**. Therefore, in the frame *S*, the two events (the two light pulses reaching the two respective ends of the compartment) **<u>do not</u>** occur simultaneously.

Do you now see that this is a major break from the older ideas of absolute time?

This is because, now, **different observers do not agree on what the same time is**.

Of course, remember that this result is arrived at for events occurring at different locations (the two ends of the compartment for instance). In the next unit we shall come back to this discussion and also consider events occurring simultaneously **at the same point in space**.

To sum up, we can conclude that **the notion of absolute time is contradicted by the second postulate because**

Events (occurring at different point in space) that are simultaneous in one inertial frame of reference are not simultaneous in another frame of reference in uniform relative motion.

This conclusion is called the **relativity of simultaneity**.

This is the fundamental difference between Newtonian mechanics and the special theory of relativity regarding the nature of time. **The idea of absolute time as existed in Newtonian mechanics gives way to the idea of relativity of simultaneity**.

In Newtonian mechanics, **observers in any two inertial frames in relative motion always agree everywhere about the simultaneity of events,** that is, they would always agree about the events occurring at the **same time**.

But in special theory of relativity, **different inertial observers travelling close to the speed of light would not agree on what the same time is**.

Events occurring at the same time in one inertial frame may not be occurring at the same time (i.e., may not be simultaneous) in another inertial frame.

We will explore this notion again in detail in the next unit.

This is also the origin of other features of space and time that follow from the special theory of relativity, namely, the phenomena of length contraction, time dilation, etc., which you will learn in the next unit.

Let us now summarise what you have learnt in this unit.

1.4 SUMMARY

Concept	Description
Physical event	■ A **physical event** is an occurrence that happens at a point in space and at an instant in time. In a Cartesian coordinate system, a physical event is described by (x, y, z, t) where coordinates (x, y, z) describe the position of the point and t is the time at which the event occurs.

Galilean coordinate transformation

- The **Galilean coordinate transformation** relates the coordinates of the event E in three-dimensional frames of reference S and S' as follows:

$$x' = x - Vt$$

$$y' = y$$

$$z' = z$$

$$t' = t$$

or $\vec{\mathbf{r}}' = \vec{\mathbf{r}} - \vec{\mathbf{V}}t$

whence $\vec{\mathbf{v}}' = \vec{\mathbf{v}} - \vec{\mathbf{V}}$

and $\vec{\mathbf{a}}' = \vec{\mathbf{a}}$

Galilean principle of relativity

- The **Galilean** or **classical principle of relativity** states that the **laws of mechanics are the same in all inertial frames of reference**. If they hold in one frame of reference, they also hold in all other inertial frames.

Postulates of special theory of relativity

- The **postulates of special theory of relativity** are as follows:

Postulate 1: The principle of relativity

The laws of physics are the same in all inertial frames of reference.

Postulate 2: The principle of constancy of speed of light

The speed of light (in vacuum) has the same constant value in all inertial frames.

Relativity of simultaneity

- Events that occur at different positions in space and are simultaneous in one inertial frame of reference are not simultaneous in another frame of reference in uniform relative motion. This concept is termed as **relativity of simultaneity.**

This is the fundamental difference between the notion of the nature of time in Newtonian mechanics and the special theory of relativity. *In Newtonian mechanics,* **observers in any two inertial frames in relative motion always agree everywhere about simultaneity of events**. But in special theory of relativity, **different inertial observers travelling at speeds close to the speed of light do not always agree on what the same time is**.

1.5 TERMINAL QUESTIONS

1. Are the laws of conservation of linear momentum and energy invariant under Galilean coordinate transformation?

2. Linear momentum is conserved in an elastic collision. Using this fact and the Galilean coordinate transformation, show that if linear momentum is

conserved in an elastic collision in one inertial frame, then it is conserved in all inertial frames.

3. What is the difference between the Galilean principle of relativity and the principle of relativity stated by Einstein?

4. Suppose you observe the motion of particle *A* in an inertial frame. You find that the *x*-component of the acceleration of the particle is described by the following equation:

$$\frac{d^2x}{dt^2} = k_1\frac{dx}{dt} + k_2[(x-X)^2 + (y-Y)^2 + (z-Z)^2]$$

where k_1 and k_2 are constants and (X, Y, Z) are the coordinates of a second particle *B* in your frame. If this equation is to be regarded as a possible law in physics, what relationship must an observer in a different inertial frame find to be experimentally valid? Write the equation.

5. The speed of light is $3.0 \times 10^8\ \mathrm{m\,s^{-1}}$. You travel towards a source of light at a constant speed of $2.0 \times 10^8\ \mathrm{m\,s^{-1}}$. Suppose that both speeds are measured with respect to an inertial frame at rest. What is the speed of light with respect to you?

6. An inertial observer moves towards a star at a constant speed *V*, and another inertial observer moves away from it at the same speed. Which amongst the three quantities, the speeds, frequencies and wavelengths of the starlight, would these observers measure to be the same, and which to be different?

7. Suppose the compartment of the train in Fig. 1.6 shrinks so that the distance between its two ends approaches zero. Can you give a simple argument to show that two events happening at the same position in space are simultaneous for all inertial observers?

8. In an inertial frame *S*, one of Maxwell's equations is written as $\vec{\nabla}\times\vec{\mathbf{E}} = -\frac{\partial\Phi_B}{\partial t}$. Use the principle of relativity to write it in another inertial frame S' moving uniformly with respect to *S*.

9. What is meant by the term "simultaneous events"? Specify the simultaneous events in each of the following statements:

   a) The arrival time of an aircraft at an airport is 5 pm.

   b) Two trains arrive at a time interval of half an hour at a platform with the earlier one arriving at 4 am.

   c) An athlete runs 100 m race in 10 s.

10. Explain how the nature of time is different in classical mechanics and the special theory of relativity.

1.6 SOLUTIONS AND ANSWERS

Pre-test

1. The frames in b), d) and e) are inertial as these are moving with constant velocity with respect to the Earth. The frames in a) and c) are non-inertial as these are accelerating.

2. The child on the platform will measure the speed of the ball to be $65\,\text{km h}^{-1}$.

Self-Assessment Questions

1. No. Different inertial observers can measure different values of these physical quantities but the laws giving the relationship between them will remain the same. So, the positions, times, velocities, etc. may be measured as having different values in different inertial frames but the equation of motion relating force with mass and acceleration (Newton's second law of motion) and other laws of mechanics will remain the same in both inertial frames.

2. No. The equation governing the force law has to satisfy the principle of relativity also, that is, it has to be invariant or hold in all inertial frames of reference to be called a law of physics.

Terminal Questions

1. Yes, since these are the laws of Newtonian mechanics.

2. Suppose in an elastic collision, an object having mass m and velocity $\vec{\mathbf{v}}_1$ collides with another object having mass M and velocity $\vec{\mathbf{V}}_1$ as observed in frame S. Let their velocities after collision be $\vec{\mathbf{v}}_2$ and $\vec{\mathbf{V}}_2$ in frame S. Then from conservation of linear momentum in elastic collision, we have

$$m\vec{\mathbf{v}}_1 + M\vec{\mathbf{V}}_1 = m\vec{\mathbf{v}}_2 + M\vec{\mathbf{V}}_2 \qquad \text{(i)}$$

Now suppose frame S' moves with constant velocity $\vec{\mathbf{V}}$ with respect to the frame S. Let the velocities of the two objects of masses m and M measured in the frame S' before and after collision be $\vec{\mathbf{v}}'_1, \vec{\mathbf{V}}'_1, \vec{\mathbf{v}}'_2$ and $\vec{\mathbf{V}}'_2$, respectively. Then using the Galilean velocity transformation [Eq. (1.5b)], we can write:

$$\vec{\mathbf{v}}'_1 = \vec{\mathbf{v}}_1 - \vec{\mathbf{V}}, \quad \vec{\mathbf{V}}'_1 = \vec{\mathbf{V}}_1 - \vec{\mathbf{V}}, \quad \vec{\mathbf{v}}'_2 = \vec{\mathbf{v}}_2 - \vec{\mathbf{V}}, \quad \vec{\mathbf{V}}'_2 = \vec{\mathbf{V}}_2 - \vec{\mathbf{V}} \qquad \text{(ii)}$$

Substituting the values of $\vec{\mathbf{v}}_1, \vec{\mathbf{V}}_1, \vec{\mathbf{v}}_2$ and $\vec{\mathbf{V}}_2$ from Eq. (ii) in Eq. (i), we get:

$$m(\vec{\mathbf{v}}'_1 + \vec{\mathbf{V}}) + M(\vec{\mathbf{V}}'_1 + \vec{\mathbf{V}}) = m(\vec{\mathbf{v}}'_2 + \vec{\mathbf{V}}) + M(\vec{\mathbf{V}}'_2 + \vec{\mathbf{V}})$$

or $$m\vec{\mathbf{v}}'_1 + M\vec{\mathbf{V}}'_1 = m\vec{\mathbf{v}}'_2 + M\vec{\mathbf{V}}'_2 \qquad \text{(iii)}$$

Eq. (iii) tells us that linear momentum is conserved in frame S'. So, the linear momentum will be conserved in all inertial frames of reference.

3. The difference in Galilean principle of relativity and the principle of relativity stated by Einstein is as follows:

   According to the Galilean principle of relativity, the **laws of mechanics** hold in all inertial frames of reference.

   According to the principle of relativity stated by Einstein, **all laws of physics** hold in all inertial frames of reference.

4. The relationship in another inertial frame of reference S' should be of the form:

$$\frac{d^2x'}{dt^2} = k_1' \frac{dx'}{dt} + k_2' [(x' - X')^2 + (y' - Y')^2 + (z' - Z')^2]$$

   where k_1' and k_2' are constants and (X', Y', Z') are the coordinates of the particle B in the frame S' and (x', y', z', t) are the space-time coordinates of the particle A in the frame S'.

5. It is $3.0 \times 10^8 \text{ m s}^{-1}$ since the speed of light is constant everywhere.

6. The speed of light will be measured to be the same by the two observers. The frequencies and wavelengths will be measured to be different by the two observers.

7. The loss of simultaneity in the frame S is due to the finite length of the compartment. This results in the observer S measuring the distance to the left to be smaller.

   If the compartment length is shrunk to zero, the two pulses would strike its ends simultaneously for all inertial frames. Thus, two events occurring at the same position would occur at the same time for all inertial observers.

8. The form of Maxwell's equation in the frame S' moving at a constant velocity with respect to frame S will be:

$$\vec{\nabla}' \times \vec{\mathbf{E}}' = -\frac{\partial \Phi'_B}{\partial t'}$$

9. The term "simultaneous events" means that the events take place **at the same time**. Simultaneous events in each case are as follows:

   a) The pointing of the clock hand to 5 and the arrival of the aircraft are two events that take place **at the same time**.

   b) The pointing of the clock hand to 4 and the arrival of the earlier train are two events that take place **at the same time**.

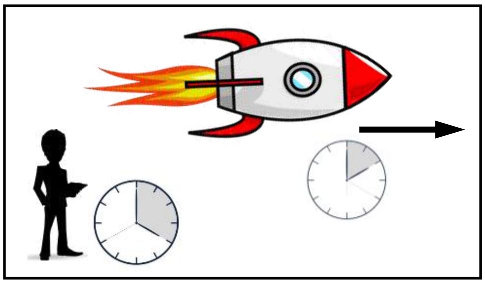
      The pointing of the clock hand to 30 minutes past 4 and the arrival of the latter train are two events that take place **at the same time**.

c) The pointing of the stop watch hand to 0 s and the athlete starting the race are two events that take place **at the same time**.

The pointing of the same stop watch kept on during the race to 10 s and the athlete completing the race at 100 m are two events that take place **at the same time**.

10. In classical mechanics, observers in different inertial frames will always agree about the time at which an event occurs. If two events occur at the same time for one inertial observer, then according to classical mechanics, these two events will be simultaneous for all other inertial observers.

    This need not be so according to the special theory of relativity. Thus, in special relativity, two events need not be simultaneous for two inertial observers in uniform relative motion. So, **if two events occur at different positions in space**, these observers do not measure the times at which two events occur in their respective frames **to be the same**. That is, the events will occur at different times for them and will **not** be simultaneous for them.



One way of expressing time dilation in the special theory of relativity is to say that 'moving clocks run slow'. In this unit, you will learn about time dilation and many other consequences of this theory.

UNIT 2

RELATIVISTIC KINEMATICS

Structure

2.1 Introduction
 Expected Learning Outcomes
2.2 Lorentz Transformation
2.3 Relativity of Simultaneity
2.4 Time Dilation and Length Contraction
 Time Dilation
 Length Contraction
2.5 Relativistic Transformation of Velocities
2.6 Relativistic Doppler Effect
2.7 Summary
2.8 Terminal Questions
2.9 Solutions and Answers

STUDY GUIDE

In this unit, we explain how the notions of space (lengths) and time intervals were modified in the special theory of relativity. Since length, time and velocity are kinematical variables in mechanics, this unit is called Relativistic Kinematics. The concepts presented in this unit are entirely new for you. You will need to study them carefully and more than once to understand them and solve problems based on them. Once again, the use of mathematics in this unit is simple. You should read the text carefully, make notes and ask your Counsellor to help you with your difficulties. You should study Secs. 2.3 and 2.4 very carefully and make sure you understand the concepts of simultaneity of relativity, time dilation and length contraction thoroughly. You will understand Secs. 2.5 and 2.6 better, once you have learnt the concepts in Secs. 2.2 to 2.4 well. You should also solve the SAQs and Terminal Questions on your own to confirm if you have grasped what you have studied in the unit.

"One thing I have learned in a long life: that all our science, measured against reality, is primitive and childlike – and yet it is the most precious thing we have".

Albert Einstein

2.1 INTRODUCTION

In Unit 1, you have learnt the postulates of the special theory of relativity. You have also learnt briefly what led to them. Essentially, you have learnt that according to Galilean velocity transformation, the speed of light turns out to be different in different inertial frames of reference. However, experiments show that the speed of light in vacuum is a universal constant. So, you have seen that there is a need for a new coordinate transformation.

We begin this unit by introducing the new coordinate transformation, the **Lorentz transformation**, in Sec.2.2. Then we discuss the implications of the postulates of special relativity for particle kinematics. In Sec. 1.3.3 of Unit 1, you have learnt, in brief, how the special theory of relativity alters the classical notion of time. In Sec. 2.3 of this unit, we examine the consequences of the special theory of relativity in somewhat greater detail. You will learn how the established ideas of the nature of time are revised radically in this theory.

Next, we revisit the Newtonian concept that the measurements of distances between two points (lengths) and the time intervals are the ***same*** for all observers. You have learnt in Sec. 1.3.3 of Unit 1 that this is not so for time. It does not hold for length measurements too. This will become clearer to you when you study the phenomena of time dilation and length contraction in Secs. 2.4.2 and 2.4.3. In Sec. 2.5, you will learn how the relativistic velocity of an object transforms in two inertial frames in uniform relative motion. Finally, in Sec. 2.6, we discuss the relativistic Doppler effect as an application of special relativity to optics.

In the next unit, we turn our attention to relativistic dynamics. You will learn the modifications carried out in Newtonian mechanics to make it compatible with the special theory of relativity.

Expected Learning Outcomes

After studying this unit, you should be able to:

- ❖ write and apply the Lorentz transformation;
- ❖ explain the phenomena of time dilation and length contraction;
- ❖ transform relativistic velocities in two inertial frames of reference in uniform relative motion and perform relativistic addition of velocities;
- ❖ calculate the relativistic Doppler shift; and
- ❖ solve numerical problems based on the special theory of relativity.

2.2 LORENTZ TRANSFORMATION

Do you recall the Galilean coordinate transformation, [Eqs. (1.1a, b) and Eqs. (1.2a, b)], from Unit 1? What happens when we apply them to the following situation: An inertial observer A (in frame S') is travelling at the constant speed V with respect to another inertial observer B at rest (in frame

S). What is the speed of light measured by observer A? From Eq. (1.2a), we get the result that $c' = c - V$.

But you know that the speed of light is constant. So, what does this imply? This means that Galilean transformation cannot be applied to this situation. We have to find a new transformation to relate the space and time coordinates of a given event in the inertial frames S and S' in uniform relative motion. This new transformation should be consistent with the postulates of special relativity. Thus, it should give the result that the speed of light has the same constant value for both observers and the laws of physics have the same form in both inertial frames.

Let us suppose that the frame of reference S' is moving with speed V with respect to S in the positive x-direction (see Fig. 2.1). Both frames are rectangular and their respective x, y and z-axes are parallel. We also define the origins of time $t = 0$ in S and $t' = 0$ in S' to be that instant of time when the origins of S and S' coincide.

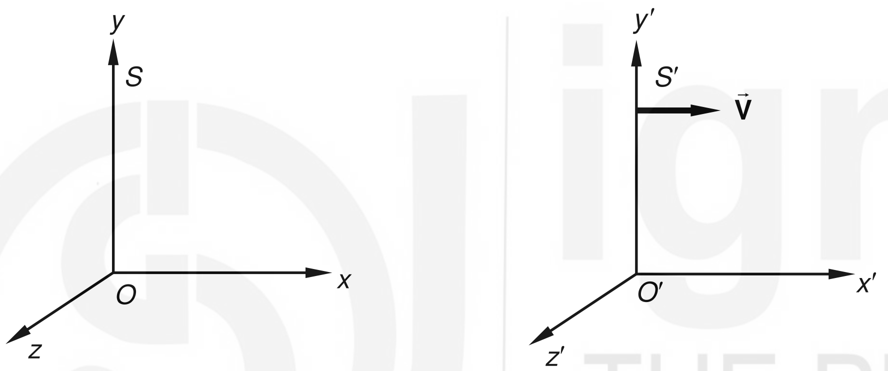


Fig. 2.1: **Inertial frames of reference S and S' with respective parallel x, y and z-axes. Their origins $O\,(x = 0)$ and $O'\,(x' = 0)$ coincide at $t = 0$ in S and $t' = 0$ in S'.**

Before you learn about this new coordinate transformation, **you should revise the method of assigning coordinates to an event in an inertial frame** S. **Remember**: We assume that *every observer is equipped with a jointly calibrated standard clock and standard of length, e. g., a metre stick*. The observer can then assign Cartesian right-handed rectangular space coordinates (x, y, z) to any event in S. Since we know the distance of the event and the time at which the observer receives the light signal from it, we can assign a time coordinate t to the event. Given this information, we now write the new coordinate transformation – the **Lorentz transformation**.

LORENTZ TRANSFORMATION

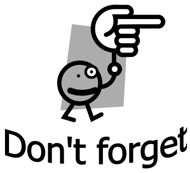
$$x' = \frac{x - Vt}{\sqrt{1 - V^2/c^2}} \tag{2.1a}$$

$$y' = y \tag{2.1b}$$

$$z' = z \tag{2.1c}$$

$$t' = \frac{t - (V/c^2)x}{\sqrt{1 - V^2/c^2}} \tag{2.1d}$$

Eqs. (2.1a to d) can be deduced from the postulates of special relativity. But here we will not go into those details. We can write Lorentz transformation in the following compact form, which **you must always remember**. We will be using them very often.



$$x' = \gamma(x - Vt) = \gamma(x - \beta ct) \quad (2.2a)$$

$$y' = y \quad (2.2b)$$

$$z' = z \quad (2.2c)$$

$$t' = \gamma\left(t - Vx/c^2\right) = \gamma(t - \beta x/c) \quad (2.2d)$$

where $$\gamma = \frac{1}{\sqrt{1 - V^2/c^2}} \quad \text{and} \quad \beta = V/c \quad (2.2e)$$

The factor γ is known as the **Lorentz factor**. Note that Lorentz transformation is **linear** in the variables x and t.

What do Eqs. (2.2a to d) tell us? Note from Eqs. (2.2a and d) that V **can never be greater than** c. For $V > c$, the space and time coordinates become imaginary, which is physically impossible. Thus, we arrive at the following conclusion, which **you should always remember**:

We cannot measure speeds greater than the speed of light. $c\,(=3.0\times10^8\,\mathrm{m\,s^{-1}})$ **is the limiting speed in the universe. No object can travel faster than light.**

It is interesting to know what happens to the Lorentz transformation when the speed V is much less than the speed of light c, i.e., $V/c \ll 1$. You can work this out yourself. Try SAQ 1.

SAQ 1 - Lorentz transformation for $V/c \ll 1$

When $V/c \ll 1$ in the Lorentz transformation, what equations do you get? Complete the following equations:

$x' =$

$y' =$

$z' =$

$t' =$

Now that you have solved SAQ 1, you know that **Lorentz transformation reduces to the Galilean transformation for speeds much smaller compared to the speed of light**. Thus, although the **<u>special theory of relativity applies to all objects, it is more relevant for objects travelling at speeds close to the speed of light</u>**.

You may like to know: **How does the factor γ change for various values of** *V*? You can work out an SAQ to obtain the results.

SAQ 2 – Factor γ for various values of V

Show that

a) $\gamma \geq 1$ always.

b) $\gamma \to 1$ as $V \to 0$.

c) $\gamma \to \infty$ as $V \to c$.

You may like to know: What is the inverse relationship between the position and time coordinates of Eqs. (2.2a to d)? It is important for you to learn the **inverse Lorentz transformation** as these too will be used very often henceforth. Let us write them here.

INVERSE LORENTZ TRANSFORMATION

$$x = \gamma(x' + Vt') = \gamma(x' + \beta ct') \tag{2.3a}$$

$$y = y' \tag{2.3b}$$

$$z = z' \tag{2.3c}$$

$$t = \gamma(t' + Vx'/c^2) = \gamma(t' + \beta x'/c) \tag{2.3d}$$

In fact, we are setting the proof of inverse Lorentz transformation as an exercise for you. Solve SAQ 3.

SAQ 3 – Inverse Lorentz transformation

Verify Eqs. (2.3a to d).

With this information, you are ready to learn about the implications of the postulates of the special theory of relativity for length and time measurements. Let us ask: What time intervals are measured by two inertial observers in uniform relative motion? The answer to this question leads us to the concept of relativity of simultaneity that you have studied briefly in Sec. 1.3.3 of Unit 1. We now discuss the idea further.

2.3 RELATIVITY OF SIMULTANEITY

In Sec. 1.3.3 of Unit 1, you have studied the concept of **relativity of simultaneity**. What you have learnt is this:

According to the special theory of relativity, two events observed to occur simultaneously in one inertial frame are **not** necessarily simultaneous in another inertial frame moving at a constant speed (close to the speed of light) with respect to the first. We illustrated the relativity of simultaneity using a simple example of one-dimensional motion in Sec. 1.3.3 of Unit 1.

We had considered two simultaneous events occurring at two different points in the reference frame S'. Recall that the high speed flashbulb of Fig. 1.6 (Unit 1) emits two pulses of light at the same time with respect to the frame attached to the moving train. These pulses were detected by two detectors situated at different points in the train's compartment.

You have learnt in Sec. 1.3.3 that these two events (the two pulses of light striking the respective detectors) were simultaneous in the inertial frame S'. However, these events were not simultaneous in the inertial frame S attached to an observer standing on the platform.

We now use Lorentz transformation to show that, in general, those events which are simultaneous in one inertial frame are not simultaneous in another inertial frame that is in uniform relative motion.

As an example, consider an atom (P) at rest with respect to an observer B in the frame S' (Fig. 2.2). The observer B and the atom are at rest with respect to the frame S' attached to the train. The frame S' and observer B are moving at a constant speed V with respect to the frame S attached to an observer A standing on the platform.

Now suppose the atom emits two photons in opposite directions at the same instant of time with respect to observer B in the frame S'. Let these photons be detected by two detectors D_1 and D_2 kept at the same distance R from the atom on the opposite sides to it. So, the two events are: One photon each is being detected by each one of the detectors D_1 and D_2.

Since the two detectors are at the same opposite distances from the atom, the time taken by both photons (travelling with speed c) to strike their respective detectors is equal: $t = R/c$. So, the two photons are detected at the same time (simultaneously) in the frame S'. And these two events are simultaneous in the frame S'.

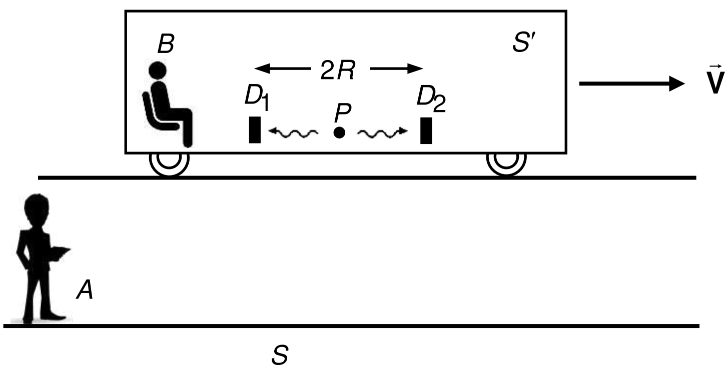


Fig. 2.2: With respect to the observer *B* or the frame S', the atom (*P*) is at rest and emits two photons at the same instant of time in opposite directions. The photons are detected at the same instant by the two detectors D_1 and D_2 in frame S' since the detectors are kept at equal distance *R* from the atom on both sides. But these two events, i.e., detection of photons by the respective detectors are not simultaneous for observer *A* standing on the platform (frame *S*).

However, you have learnt in Sec. 1.3.3 of Unit 1 that the two events need not necessarily be simultaneous for observer *A* in the frame *S*. Let us use Lorentz transformation and find out what result we get.

All conditions for the reference frames are as stated in Sec. 2.2. Since the frame S' moves in the positive *x*-direction, the *y* and *z* coordinates of any event in both frames will be equal. So, to keep the algebra simple, we will use only the transformation equations for the *x* and *t* coordinates. Let the two events occur at (x_1, t_1) and (x_2, t_2), respectively, in the frame *S* and at (x'_1, t'_1) and (x'_2, t'_2), respectively, in the frame S'. So, from Lorentz transformation [Eqs. (2.2a and d)], we can write:

$$x'_1 = \gamma(x_1 - Vt_1) \quad \text{and} \quad x'_2 = \gamma(x_2 - Vt_2) \tag{2.4a}$$

or
$$x'_2 - x'_1 = \gamma(x_2 - x_1) - \gamma V(t_2 - t_1) \tag{2.4b}$$

If we write $\Delta x' = (x'_2 - x'_1)$, $\Delta x = (x_2 - x_1)$ and $\Delta t = (t_2 - t_1)$, we can express Eq. (2.4b) as:

$$\Delta x' = \gamma(\Delta x - V\Delta t) \tag{2.5}$$

Similarly, we can obtain the expression for time intervals in both frames in terms of $\Delta x'$. For this, we use Eq. (2.3d) and write:

$$t_1 = \gamma(t'_1 + Vx'_1/c^2) \quad \text{and} \quad t_2 = \gamma(t'_2 + Vx'_2/c^2) \tag{2.6a}$$

so that
$$t_2 - t_1 = \gamma(t'_2 - t'_1) + \gamma V(x'_2 - x'_1)/c^2 \tag{2.6b}$$

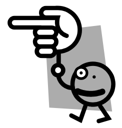
or
$$\Delta t = \gamma(\Delta t' + V\Delta x'/c^2) \tag{2.7}$$

Now, in the frame S', the events are simultaneous and $\Delta x' = 2R$. So, in Eq. (2.7), we put $\Delta t' = 0$ and $\Delta x' = 2R$ to obtain:

$$\Delta t = 2\gamma VR/c^2 \tag{2.8a}$$

What does Eq. (2.8a) tell us? It tells us that $\Delta t \neq 0$. This means that **two events that are simultaneous in the reference frame S' are not simultaneous in the reference frame *S*.** This is the result we had obtained in Sec. 1.3.3 of Unit 1. Note that these events are occurring at *different positions* in the frame S' and so, we can conclude that

> **Simultaneous events that occur at <u>different positions</u> with respect to one inertial frame are not simultaneous in another inertial frame in uniform relative motion.**



Don't forget

Let us now ask: What happens if the two events occur **at the <u>same time</u> and at the <u>same</u> <u>position</u>** in the frame S', i.e., at $\Delta t' = 0$ and $\Delta x' = 0$? Then Eq. (2.7) tells us that:

$$\Delta t = 0 \quad \text{or} \quad t_1 = t_2 \tag{2.8b}$$

This means that the two events are also simultaneous in the frame *S*. Also, if we substitute $\Delta x' = 0$ and $\Delta t = 0$ in Eq. (2.5), we get that

$$\Delta x = 0 \quad \text{or} \quad x_1 = x_2 \tag{2.8c}$$

Don't forget

> If two events occur **simultaneously** and **at the same position** in one inertial frame, then they are **simultaneous and occur at the same position** in every other inertial frame of reference.

To sum up, observers in two inertial frames S and S' in uniform relative motion **agree on the simultaneity of events occurring at the same point in space**. But, they will disagree on simultaneity of events occurring at different positions in space: If two events occurring at different positions in space are simultaneous in an inertial frame, they will **not be simultaneous** in any other inertial frame in uniform relative motion.

Similarly, we can show that events occurring at the same point in space but at different times in the frame S' will appear to be occurring at different points in the frame S. We have given this as an exercise for you in SAQ 4.

SAQ 4 – Relativity of simultaneity

Show that events occurring at the same position in the inertial frame S' at different instants of time do not occur at the same points in the inertial frame S, as long as $V \neq 0$.

To sum up, in the above discussion, we have illustrated the fundamental difference between classical/Galilean relativity and the special theory of relativity. In classical relativity, observers in the inertial frames S and S' **always agree about simultaneity at all points in space**. You can see that for $V << c$ or $\gamma \rightarrow 1$, $t' = t$ everywhere.

This is not so in the special theory of relativity. Let us revise what you have learnt about the concept of simultaneity in special relativity.

Recap

SIMULTANEITY IN SPECIAL RELATIVITY

- **Events** that occur at **different positions** but **at the same instant of time with respect to one inertial frame are not simultaneous in another inertial frame in uniform relative motion.** So,

  if $x_1' \neq x_2'$ and $t_2' = t_1'$ in S' then $t_2 \neq t_1$ in S

  or if $x_1 \neq x_2$ and $t_2 = t_1$ in S then $t_2' \neq t_1'$ in S'

- Events that occur at the **same position in space** and **at the same instant of time** are **simultaneous** in two inertial frames in uniform relative motion. Observers in these frames **always agree on the simultaneity of these events.** So,

  if $x_1' = x_2'$ and $t_2' = t_1'$ in S' then $t_2 = t_1$ in S

  or if $x_1 = x_2$ and $t_2 = t_1$ in S then $t_2' = t_1'$ in S'

Having solved SAQ 4, you have also seen that:

If an inertial observer observes that events occur at the same position but at different instants of time, then another inertial observer in uniform relative motion (non-zero V) will observe that the events occur at two different positions in her/his frame of reference. So,

if $x'_1 = x'_2$ and $t'_2 \neq t'_1$ in S' then $x_2 \neq x_1$ in S as long as $V \neq 0$

or if $x_1 = x_2$ and $t_2 \neq t_1$ in S then $x'_2 \neq x'_1$ in S' as long as $V \neq 0$

You must understand thoroughly the relativity of simultaneity at all points in space-time. It is the origin of the phenomena of length contraction and time dilation, i.e., the relativity of length and time-interval measurements. You will study these phenomena in the next section. But **always remember, relativity of simultaneity is experimentally observed only at speeds close to the speed of light**.

We now discuss the phenomena of time dilation and length contraction.

2.4 TIME DILATION AND LENGTH CONTRACTION

In this section, we discuss two major implications of the special theory of relativity that result from the relativity of simultaneity observed at speeds close to the speed of light. These are the phenomena of **time dilation** and **length contraction**. These may take more than one reading of the text to understand. But these are interesting phenomena that have been observed in experiments. So, do read the following sections carefully.

2.4.1 Time Dilation

Refer to Fig. 2.3. Note that an observer B is at rest inside a train (frame S') moving with constant speed V with respect to an observer A standing on the platform (frame S).

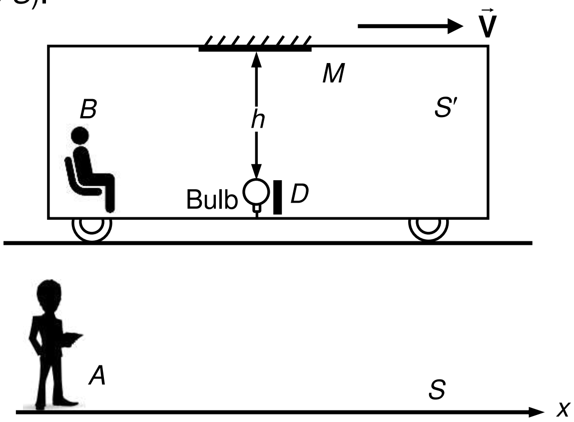


Fig. 2.3: Two observers A and B at rest in the inertial frames S and S', respectively. The frames are in uniform motion with respect to each other. The train and the frame S' attached to it are moving at constant speed V in the positive x-direction with respect to the frame S. In the frame S' attached to the train, the observer B, the bulb, the mirror M and the detector D are at rest.

Notice also that there is a bulb on the floor of the compartment and a mirror *M* right above the bulb on the ceiling at a height *h*. Note that a detector *D* is placed on the floor and is extremely close to the bulb, which is at a distance *h* from the mirror. The bulb, the detector and the mirror are at rest with respect to *B*, i.e., the frame S'.

Now refer to Fig. 2.4a. Suppose observer *B* switches on the bulb and quickly turns it off to produce a pulse of light. The pulse is reflected by the mirror *M* and the reflected pulse is detected by detector *D*. So, we have two events:

Event 1: Emission of light pulse by the bulb

Event 2: Detection of light pulse by the detector

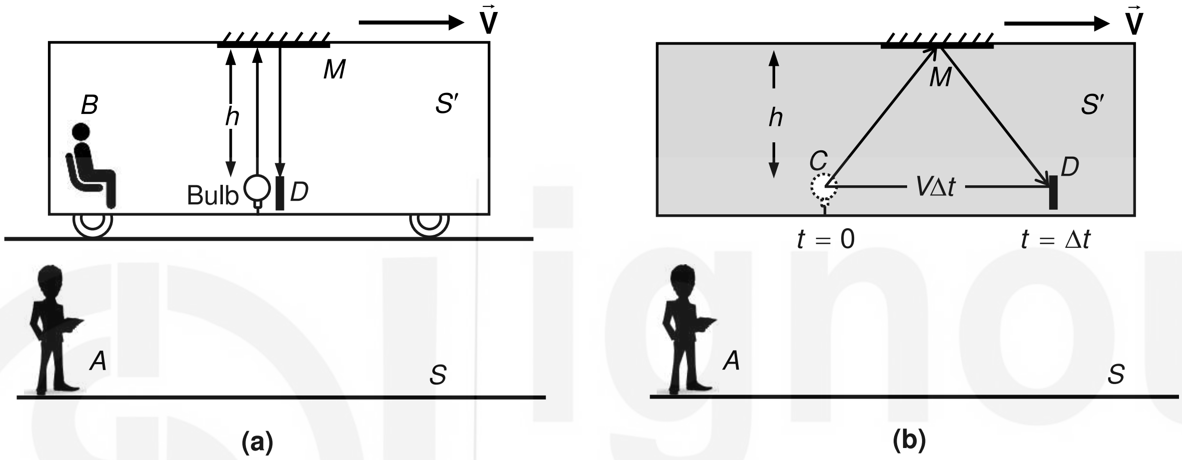


Fig. 2.4: The same events as observed in two inertial frames in uniform motion with respect to each other. a) In the frame S' attached to the train, the observer *B*, the bulb, the mirror *M* and the detector *D* are at rest. Frame S' is moving at constant speed *V* with respect to frame *S*. In the frame S', the total distance travelled by the pulse from the bulb to the mirror and back is $2h$; b) the observer *A* (who is at rest in the frame *S*), observes that when the pulse of light reaches the detector *D*, the detector has moved by a distance $V\Delta t$. In the figure, *C* shows the point when the flash was emitted with respect to observer *A* at the instant $t = 0$.

Let $\Delta t'$ be the time interval measured by observer *B* between the two events in the frame S'. Note from Fig. 2.4a that the distance travelled by the pulse with speed *c*, during this time interval is $2h$. Therefore,

$$\Delta t' = \frac{2h}{c} \tag{2.9}$$

Now we ask: What is the time interval Δt between these two events as measured by observer *A* in the frame *S*? Note that this is the time interval between the two events as observed by *A*: **Event 1** – *emission of light pulse by the bulb* and **Event 2** – *detection of light pulse by the detector*.

For observer *A*, the train is moving with constant speed *V* in the positive *x*-direction. So, with respect to *A*, the detector *D* moves a distance $V\Delta t$ (see Fig. 2.4b). So, *A* would observe that the light pulse takes a longer path than the one observed by *B*. The pulse travels the distance *CM*, strikes the mirror and after reflection, it travels the distance *MD* when it is detected. So, the total distance travelled by the light pulse is $CM + MD$. Since the speed of light is the same for the two observers, the time interval measured by *A* would be:

$$\Delta t = \frac{CM + MD}{c} \tag{2.10a}$$

Now, we use basic trigonometry to obtain the distances *CM* and *MD*. Refer to Fig. 2.5, which is a part of Fig. 2.4b repeated here in the margin. Let *O* be the midpoint of *CD*. Then, from the right-angled triangles *COM* and *MOD* in Fig. 2.5, you can see that

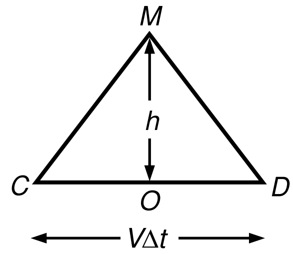


Fig. 2.5: Obtaining distances *CM* and *MD* from basic trigonometry.

$$(CM)^2 = h^2 + (CO)^2 = h^2 + \left(\frac{V\Delta t}{2}\right)^2 = (MD)^2$$

or $$CM = MD = \sqrt{h^2 + \left(\frac{V\Delta t}{2}\right)^2} \tag{2.10b}$$

Substituting Eq. (2.10b) in Eq. (2.10a), we get:

$$\Delta t = \frac{2}{c}\sqrt{h^2 + \left(\frac{V\Delta t}{2}\right)^2} \tag{2.10c}$$

Squaring Eq. (2.10c), we can write:

$$(\Delta t)^2 = \frac{4h^2}{c^2} + \left(\frac{V\Delta t}{c}\right)^2 \tag{2.10d}$$

From Eq. (2.9), you can see that the first term of the right-hand side of Eq. (2.10d) is just $(\Delta t')^2 = (2h/c)^2$. Therefore, we can write Eq. (2.10d) as:

$$(\Delta t)^2 = (\Delta t')^2 + \left(\frac{V\Delta t}{c}\right)^2$$

or $$(\Delta t)^2\left(1 - \frac{V^2}{c^2}\right) = (\Delta t')^2$$

or $$\Delta t' = (\Delta t)\sqrt{\left(1 - \frac{V^2}{c^2}\right)} = \frac{\Delta t}{\gamma} \tag{2.11a}$$

where as you know from Eq. (2.2e), that

$$\gamma = \frac{1}{\sqrt{\left(1 - \frac{V^2}{c^2}\right)}} \tag{2.11b}$$

$\therefore$ $$\boxed{\Delta t = \gamma \Delta t'} \tag{2.12}$$

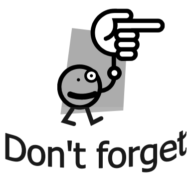
We can arrive at the same result using inverse Lorentz transformation. Recall Eq. (2.7). We write it again for your convenience: $\Delta t = \gamma(\Delta t' + V\Delta x'/c^2)$

Since the events are occurring at the same position in the frame S', we put $\Delta x' = 0$ in Eq. (2.7) and so it gives us Eq. (2.12). Let us now interpret what Eq. (2.12) tells us.

Recall that with respect to the observer B, the bulb, the mirror and the detector are at rest. And the detector is extremely close to the bulb so that the two events (switching on the bulb and detection of the pulse) occur at the **same position** (same coordinate on the x-axis).

The time interval between two events occurring at the same position is known as the proper time interval or proper time.

Now, you can see that $\Delta t'$ is the time interval between two events occurring **at the same position**. So, $\Delta t'$ is the **proper time** between the two events. **While solving problems on time dilation in special relativity, you will need to identify proper time.** You should always remember the following about proper time:



> The **time interval** measured between two events in the frame of reference in which the events occur at the **same position** is called the **proper time interval** or just **proper time**. The time interval measured in the frame of reference in which the two events occur at **different positions** is called **improper time**.

So, in this situation, the time interval measured by observer B in the frame S' is proper time whereas the time interval measured by observer A in frame S is improper. Remember that we are interested in relativistic speeds for which γ is greater than 1. For such speeds, Eq. (2.12) tells us that the improper time interval measured by an observer in the frame S is **longer** than the proper time interval measured by the observer in the frame S'. This phenomenon is called **time dilation**. For example,

$$\text{if } V = c/2, \quad \text{then } \gamma = \frac{1}{\sqrt{1-1/4}} = \frac{2}{\sqrt{3}} = 1.15 \quad \text{and} \quad \Delta t = 1.15\Delta t'$$

So, a time interval of 10 s measured in the frame S' will be measured as 11.5 s in the frame S. We say that: The time interval measured in any inertial frame (which is in uniform motion with respect to an inertial frame in which **the events occur at the same position**) gets '**dilated**'.

Here we would like to point out that a statement like 'moving clocks run slow' can be misunderstood. What this statement actually means is that a clock moving at a constant velocity relative to an inertial frame S goes **slower** when timed by the clocks in S.

Note that the time interval in S' is **smaller** than the one in S. This means that to the observer in frame S, the moving clock in frame S' runs **slower** by the Lorentz factor γ. You should now be able to explain what is meant when it is said that "moving clocks run slower." In brief, it means the following:

A clock at rest with respect to the frame S' measures the proper time interval, that is, it measures the time interval between events occurring at a fixed position in the frame. When S' (the clock) moves uniformly with a velocity V relative to an inertial frame, say S, the clock at rest in S records a **longer** (**improper**) time interval between the two events than the moving clock.

Thus, the **clock in** S', which is **moving** with respect to S, records a shorter time interval, i.e., it goes **slow** by a factor γ relative to the **stationary clock** in S. You can see this effect in Fig. 2.6. The clock in the moving rocket records proper (shorter) time than the clocks on the Earth, i.e., it runs slow. Remember that **both clocks are at rest with respect to their respective frames**.

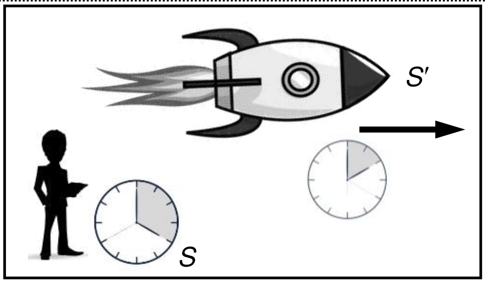


Fig. 2.6: The proper time interval between two events recorded by a clock in the rocket (frame S') moving uniformly with respect to the observer on the Earth (frame S) is shorter than the improper time interval recorded by a clock in the stationary frame S. The clock in S' goes slower by a factor γ relative to the clock in S: 'moving clocks run slow'.

You can argue that motion is relative and to an observer in S', the clock in S is moving. Hence, it should also slow down as compared to the stationary clocks in S'. This would be correct provided that the time interval between events is measured by the clock in S **at the same position** ($x_1 = x_2$), i.e., it is **proper time.** Then using Lorentz transformation equation (2.2d), we would get the result:

$$\Delta t' = \gamma \Delta t \tag{2.13}$$

where Δt is the **proper time interval** in S (read the margin remark). In fact, why don't you do it as an exercise? Solve SAQ 5.

You should memorise that

Improper time interval $= \gamma$ (proper time interval)

Also **proper time interval** is the time interval between two events occurring at the **same position** in a given frame.

SAQ 5 – Time dilation

Verify Eq. (2.13) when the proper time interval is measured in S.

The important point to note is that **the clock measuring the proper time interval of two events occurring at the same position records the shortest time interval**. In the second case, it is the clock in S. In this case, to the observer in S', the clock in S is moving with a speed V. So, now the proper time interval (Δt) in S will be recorded as the improper (dilated) time interval $\gamma\Delta t$ by a clock in S'. Let us revisit what you have learnt about time dilation in this section.

Recap

TIME DILATION

- Consider two inertial reference frames in uniform relative motion. Each frame has observers and clocks held at rest in that frame.
- Suppose two events occur at a fixed position in the frame S', separated by a **proper time interval** $\Delta t'$ as measured by a clock at rest in S'. Then the **improper time interval** Δt between these events measured by a clock at rest in the frame S with respect to which S' is moving, will be

$$\Delta t = \gamma \Delta t' \tag{2.12}$$

It will be longer than $\Delta t'$, the proper time. Thus, the clocks in S' moving with respect to a stationary clock in S, will run slower. So, we say that "moving clocks run slow".

Recap

TIME DILATION

- Alternatively, suppose two events occur at a fixed position in the frame S, separated by the proper time interval Δt as measured by a clock at rest in S. Then the improper time interval $\Delta t'$ between these events measured by a clock at rest in the frame S', with respect to which S is moving will be longer than Δt, and be given by

$$\Delta t' = \gamma \Delta t \tag{2.13}$$

With respect to a stationary clock in S', the moving clock in S will run slower.

Remember, in all cases we are referring to **measurements** and do not confuse the term observer with one who *sees*. An observer is one who **measures** physical quantities.

We would like to put in a **word of caution** here: It is not easy to understand time dilation in an intuitive way and you may take some time to adjust to this notion. To enable you to understand time dilation further, let us compute some actual time intervals in the following example.

EXAMPLE 2.1 : TIME DILATION

Consider two events observed from two inertial frames S and S'. The frame S' is moving at constant speed V with respect to S along the positive x-direction. Use either Eq. (2.12) or Eq. (2.13) giving reasons and do the following calculations:

a) The clock in S' measures the time interval between two events occurring at the same position as 2.0 s. What is the time interval measured by a clock in S for $V = 0.75c$ and $V = 0.90c$?

b) The clock in S measures the time interval between two events occurring at the same position as 2.0 s. What is the time interval measured by a clock in S' for $V = 0.60c$ and $V = 0.99c$?

c) The proper time interval measured by an inertial observer A is 10 s. What is the speed of the inertial frame (with respect to A) in which the time interval is measured to be 16 s?

d) The mean life time of a free neutron at rest is measured to be 900 s. If you measure the mean life time of the neutron as 2700 s, what is the speed of the neutron with respect to you?

SOLUTION ■ We use either Eq. (2.12) or Eq. (2.13) as follows:

a) Since the clock in frame S' measures the time interval at the same position, it measures the proper time. So we use Eq. (2.12) and substitute $\Delta t' = 2.0\,\text{s}$ and the respective values of V.

For $V = 0.75c$, $\gamma = \dfrac{1}{\sqrt{1 - 9/16}} = \dfrac{4}{\sqrt{7}} = 1.5$

and $\Delta t = 1.5 \times 2.0\,\text{s} = 3.0\,\text{s}$

For $V = 0.90c$, $\gamma = \dfrac{1}{\sqrt{1-0.81}} = \dfrac{1}{\sqrt{0.19}} = 2.3$

and $\Delta t = 2.3 \times 2.0\,\text{s} = 4.6\,\text{s}$

b) In this case, the clock in frame S measures the proper time interval. So, we use Eq. (2.13) and substitute $\Delta t = 2.0\,\text{s}$ and the respective values of V.

For $V = 0.60c$, $\gamma = \dfrac{1}{\sqrt{1-0.36}} = \dfrac{1}{\sqrt{0.64}} = 1.25$ and

$\Delta t = 1.25 \times 2.0\,\text{s} = 2.5\,\text{s}$

For $V = 0.99c$, $\gamma = \dfrac{1}{\sqrt{1-0.98}} = \dfrac{1}{\sqrt{0.02}} = 7.1$ and

$\Delta t = 7.1 \times 2.0\,\text{s} = 14.2\,\text{s}$

c) We are given the proper and improper time intervals and we have to determine the speed of the moving frame. We can use either Eq. (2.12) or Eq. (2.13) with the correct substitutions of proper and improper time intervals. In Eq. (2.12), we substitute $\Delta t' = 10\,\text{s}$ and $\Delta t = 16\,\text{s}$. So, we have

$$16\,\text{s} = \gamma \times 10\,\text{s} \quad \text{or} \quad \gamma = \frac{1}{\sqrt{\left(1-\dfrac{V^2}{c^2}\right)}} = 1.6$$

$$\text{or} \quad \left(1-\frac{V^2}{c^2}\right) = \frac{1}{(1.6)^2} \Rightarrow \frac{V^2}{c^2} = 1 - 0.39 = 0.61$$

$$\text{or} \quad V = 0.78c = 2.3\times 10^8\,\text{m}\,\text{s}^{-1}$$

Alternatively, had we used Eq. (2.13), we would have taken $\Delta t = 10\,\text{s}$ and $\Delta t' = 16\,\text{s}$. The result would be the same.

d) The mean life time of a free neutron at rest is the proper mean life time, which is 900 s. The mean life time of the neutron measured by you is 2700 s, which is the improper time interval. So, we have

$$2700\,\text{s} = \gamma \times 900\,\text{s} \quad \text{or} \quad \gamma = \frac{1}{\sqrt{\left(1-\dfrac{V^2}{c^2}\right)}} = 3$$

$$\text{or} \quad \left(1-\frac{V^2}{c^2}\right) = \frac{1}{9} \Rightarrow \frac{V^2}{c^2} = 1 - 0.11 = 0.89$$

$$\text{or} \quad V = 0.94c = 2.8\times 10^8\,\text{m}\,\text{s}^{-1}$$

So, the speed of the neutron with respect to you is $2.8\times 10^8\,\text{ms}^{-1}$.

You may now like to solve an SAQ based on time dilation.

SAQ 6 - Time dilation

When a pion is at rest, its mean life time is $2.6\times10^{-8}\,\mathrm{m\,s}^{-1}$. What is its mean life time measured by an observer at rest when the pion is moving relative to the observer at a speed of 0.98*c*? What would an observer moving with the pion measure its mean life time to be?

We end this section by presenting the experimental evidence for time dilation.

Experimental Evidence for Time Dilation

Time dilation was first confirmed experimentally by observations on elementary particles called muons (μ-mesons) by B. Rossi and D. B. Hall in 1941. When cosmic rays interact with gaseous particles in the upper layers of the Earth's atmosphere, at a height of about 10 km from the Earth's surface, muons are produced in large numbers. These muons decay very quickly; as their mean life time is 2.2 μs, which is very small. It is so small that even if muons travelled at the speed of light from the top of the atmosphere where they are created, they would not reach the surface of the Earth (a distance of about 10 km). This is because the distance travelled by them in 2.2 μs would be only

$$(2.2\times10^{-6}\,\mathrm{s})\times(3.0\times10^{8}\,\mathrm{ms}^{-1}) = 660\,\mathrm{m},$$

which is much smaller than 10 km. Yet some of these muons are detected in laboratories on the Earth in spite of their very small mean life time. How do we explain this?

This can be explained only by the fact that in our frames of reference, the life time of the muon is dilated by a factor of γ, i. e., it is $[\gamma\times(2.2\times10^{-6}\,\mathrm{s})]$.

Now, if $V = 0.998c$, so that $\gamma = 16$, then the life time of the muon would be

$$16 \times (2.2\times10^{-6}\,\mathrm{s}) = 35\,\mu\mathrm{s}$$

Then the muon would travel a distance of

$$(35\times10^{-6}\,\mathrm{s})\times(3.0\times10^{8}\,\mathrm{ms}^{-1}) = 10{,}500\,\mathrm{m} = 10.5\,\mathrm{km}$$

And it would be detected at the surface of the Earth. So, the detection of muons at the surface of the Earth is a direct experimental evidence of time dilation.

Time dilation effects assume great significance for physicists working at high-energy accelerators with beams of particles which decay very fast, i. e., they have very small mean lifetimes. For example, pions and kaons decay spontaneously over 100 times as fast as muons. Were it not for time dilation, they would decay and disappear before they had travelled a few metres even at the speed of light. Due to time dilation, their mean life time increases and their decay is slowed down. Thus, they can be observed hundreds of metres from the point in the accelerator where they are produced. Therefore, they can be used for further experiments. In this manner, time dilation becomes a matter of everyday concern to these physicists.

So far, you have studied about implications of special theory of relativity for the notion of time and its measurement. You will now study its implication for length measurements.

2.4.2 Length Contraction

You have learnt about the relativity of simultaneity in Sec. 1.3.3 of Unit 1 and Sec. 2.2. This has an interesting consequence for the measurement of lengths in two inertial frames in uniform relative motion.

The problem is this: Suppose a body's length is measured to be L_0 in an inertial frame in which it is at rest. Will an inertial observer in relative motion with respect to the body, measure the length of the body to the same, i. e., L_0? Let us find the answer.

We consider two inertial frames of reference S and S' in uniform relative motion along the positive x-axis. Refer to Fig. 2.7. Note that observers A and B are at rest with respect to the frames S and S', respectively. Also note the rigid rod RR' lying along the x-axis and at rest in the frame S. Since the rod is at rest in S, the position coordinates of its ends, say x_1 and x_2, are independent of time as measured in S. So, the rod's length in S is

$$L_0 = x_2 - x_1 = \Delta x \tag{2.14}$$

The **length** L_0 **of the rod in the inertial frame in which the rod is at rest, is known as its rest length** or **proper length**. You should always remember:

> **Proper length is the distance between two points measured by an observer who is at rest relative to both the points.**

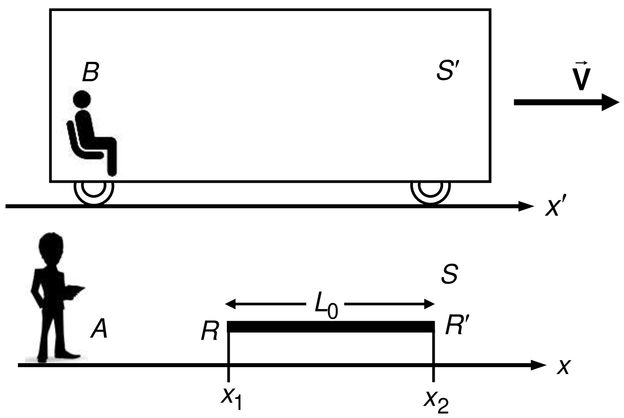


Fig. 2.7: A rod RR' of length L_0 is kept at rest with respect to the observer A in the frame S attached to the platform. The frame S' attached to the train is moving at a constant speed V with respect to S in the positive x-direction.

Now we ask: What is the length of the rod measured by the observer B in the inertial frame S'?

Remember that the frame S' is moving with a constant speed V in the positive x-direction with respect to the frame S. Refer to Figs. 2.8a and b. Note the arrow P in both figures. The two events for the length measurement are:

Event 1: When the arrow P is right above the end R of the rod.

Event 2: When the arrow P is right above the end R' of the rod.

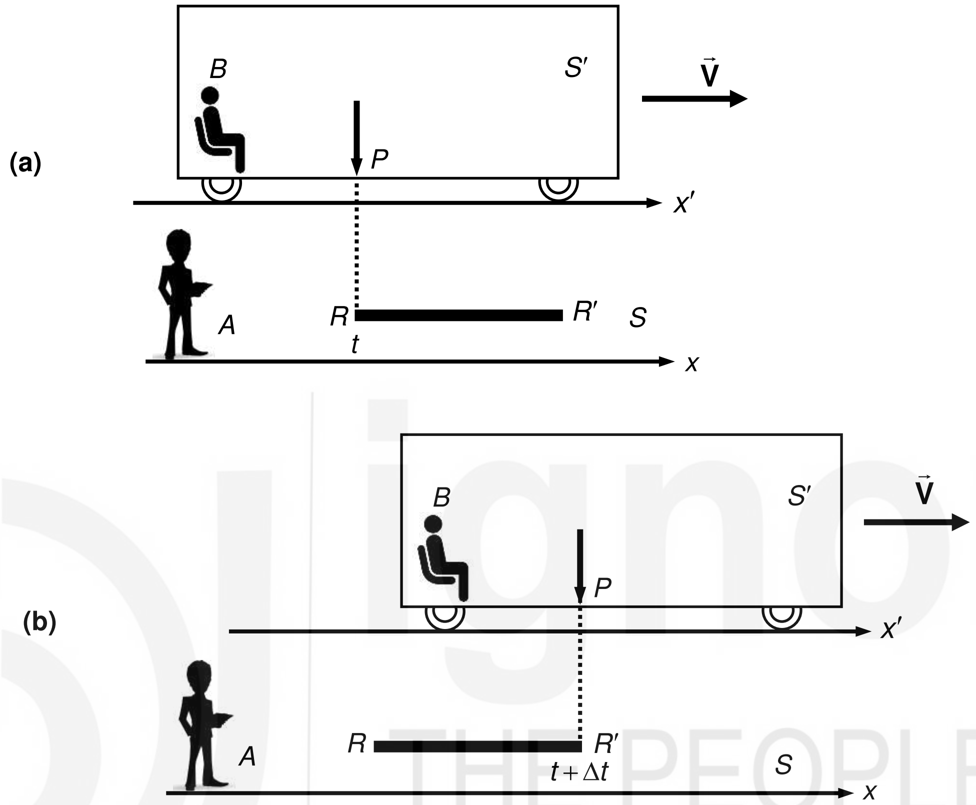


Fig. 2.8: A rod RR' of length L_0 is at rest with respect to observer A in frame S. The frame S' attached to the train is moving at a constant speed V with respect to S in the positive x-direction. The observer B and an arrow P are in the frame S'. a) Event 1 observed by S at the instant t; b) Event 2 observed by S at the instant $t + \Delta t$. Note that with respect to observer A, the frame S' and observer B have moved the distance $RR' = L_0 = V\Delta t$ in the time interval Δt.

With respect to the observer A in the inertial frame S, the arrow moves with constant speed V in the positive x-direction (Fig. 2.8b). So, for observer A, if the time interval between the arrow going from R to R' (time interval between the two events mentioned above) is Δt, then

$$L_0 = V\Delta t \qquad (2.15a)$$

What can we say about the length of the rod measured by observer B?

Refer to Figs. 2.9a and b. Note that with respect to observer B in frame S', the observer A, the frame S and the rod (at rest in the frame S) move with constant speed V in the **negative** x-direction. Now, let $\Delta t'$ be the time interval between the two events as measured in the frame S'. So, it is the difference between the instants when the ends R and R' of the rod are coincident with the arrow.

Then, the length of the rod as measured by observer B in the inertial frame S' is given by

$$L = V\Delta t' \qquad (2.15b)$$

Note from Figs. 2.9a and b that the two events have been observed at the same position P in the frame S'. So, $\Delta t'$ is the **proper time**.

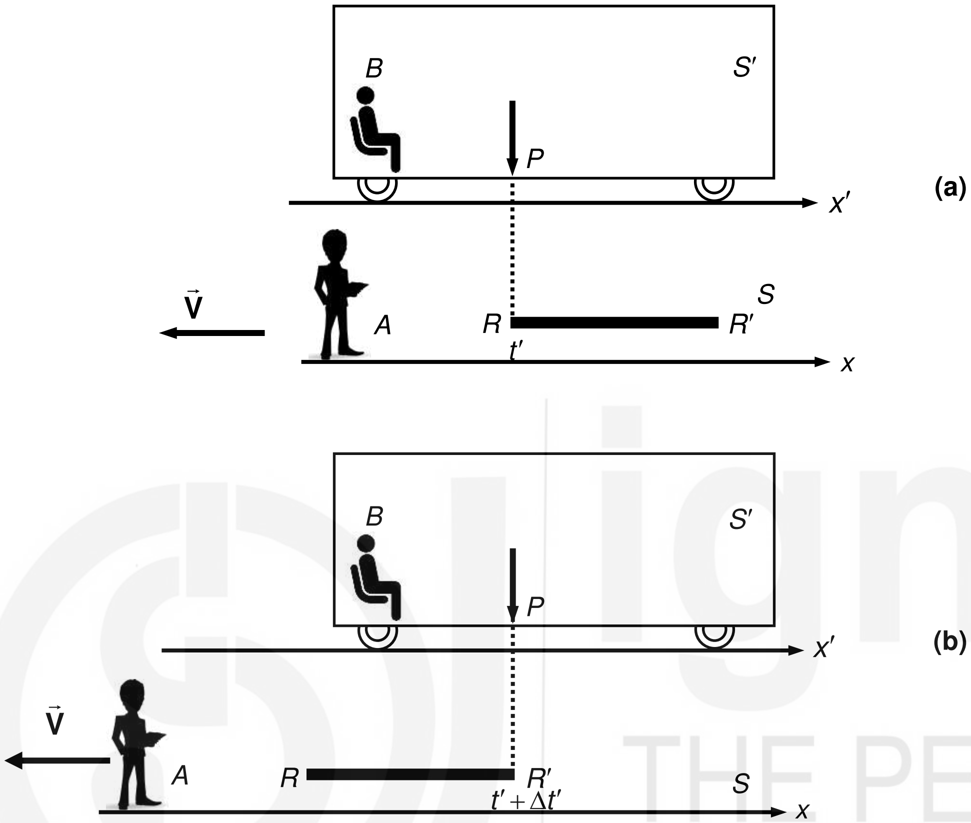


Fig. 2.9: The rod RR' observed in frame S', which moves at a constant speed V with respect to frame S (which is at rest) in the positive x-direction. With respect to the observer B in frame S', the rod moves in the <u>negative</u> x-direction. a) Event 1 as observed by observer B in S' at the instant t'; b) Event 2 as observed by observer B in S' at the instant $t' + \Delta t'$. Note that with respect to observer B, the frame S and observer A have moved the distance $V\Delta t'$ in the proper time interval $\Delta t'$. So, the length of the rod measured by observer B is $L = V\Delta t'$.

Using the result $\Delta t = \gamma \Delta t'$ from Eq. (2.12) in Eq. (2.15b), we can write:

$$L = V\frac{\Delta t}{\gamma} \qquad (2.15c)$$

Using the result $L_0 = V\Delta t$ from Eq. (2.15a) in Eq. (2.15c), we get:

$$\boxed{L = \frac{L_0}{\gamma}} \qquad \text{where } \gamma = \frac{1}{\sqrt{1-\frac{V^2}{c^2}}} \qquad (2.16)$$

So, the observer B measures a shorter length or '**contracted**' length of the rod given by Eq. (2.16). This phenomenon is known as <u>**length contraction**</u>. We say that 'a moving rod contracts.'

We can arrive at this result using the Lorentz transformation also with the help of a bit of algebra. Let us do that. You have just learnt that the two events occur at the same position P in the frame S' and so the time interval $\Delta t'$ is the proper time. Now we use Eq. (2.2d) for the proper time and write:

$$\Delta t' = \gamma(\Delta t - V\Delta x/c^2) \tag{2.17a}$$

Next we use the result $\Delta t = \gamma \Delta t'$ from Eq. (2.12) to write:

$$\Delta t' = \gamma(\gamma\Delta t' - V\Delta x/c^2) \tag{2.17b}$$

From Eq. (2.14), you know that $L_0 = \Delta x$ and so we can recast Eq. (2.17b) as follows (read the margin remark):

Remember, in all cases we are referring to **measurements**. Do not confuse the term observer with one who *sees*. An observer is one who **measures** physical quantities.

$$\Delta t' = \gamma^2 \Delta t' - \gamma V L_0/c^2 \quad \Rightarrow \quad \frac{\gamma V L_0}{c^2} = (\gamma^2 - 1)\Delta t' = \frac{V^2/c^2}{1 - V^2/c^2}\Delta t'$$

or
$$\frac{\gamma V L_0}{c^2} = \gamma^2 \frac{V^2}{c^2}\Delta t' \quad \Rightarrow \quad L_0 = \gamma V \Delta t' \tag{2.17c}$$

Now, from Eq. (2.15b), $L = V\Delta t'$ and so, we get Eq. (2.16):

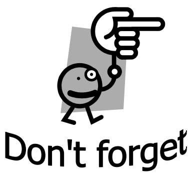
$$L = \frac{L_0}{\gamma}$$

which describes length contraction. Let us write this result in a more general form.

LENGTH CONTRACTION

With respect to two points at a fixed distance L_0 in a stationary frame, an observer in motion along the line joining the two points will measure that distance to be $L = \dfrac{L_0}{\gamma}$, which is shorter than the proper distance L_0.

In this context, you should always remember that



Length contraction takes place only <u>in the direction of motion</u> and <u>not</u> in the direction perpendicular to it.

So, for the rod in our discussion, its width and depth would not change. Only its length would.

The effect of length contraction becomes particularly significant when the velocity of an object approaches the speed of light. You can verify this statement. Take $V = 0.9c$ and calculate the ratio $\dfrac{L}{L_0} = \dfrac{1}{\gamma}$. It is 0.44.

So, the length of the rod measured at this speed is less than half of its proper length. You may be wondering whether this consequence of special theory of relativity has been verified experimentally. The answer is yes. The muon experiment that we described in Sec. 2.4.1 also provides evidence for length contraction. Let us see how.

Experimental Evidence for Length Contraction

Detection of μ-mesons or muons near the Earth's surface is also a direct evidence for length contraction. You know that muons are highly unstable with an average life time of 2.2 μs. Their speeds can be as high as 0.998*c*. So in their life time the muons are able to travel a distance of $(2.2 \times 10^{-6}\,\text{s}) \times (3.0 \times 10^{8}\,\text{m s}^{-1}) \times 0.998 = 658\,\text{m}$.

However, some muons are detected near the Earth's surface after travelling a distance of 10 km. How is this explained? To unfold this puzzle, we use the relation $L = \dfrac{L_0}{\gamma}$, where L is the distance travelled by a muon in its own frame of reference and L_0, the distance measured in the Earth's frame of reference.

For $V = 0.998c$, $\gamma = 16$ and L_0 is given by

$$L_0 = \gamma L = 16 \times 658\,\text{m} = 10.5\,\text{km}$$

Hence, the fact that in spite of their short life time, muons are able to reach the Earth's surface from great altitudes where they are produced provides experimental evidence for length contraction.

Let us apply these ideas about length contraction in an example.

EXAMPLE 2.2 : LENGTH CONTRACTION

a) What is the length of a scale as measured by an observer moving at the speed 0.75*c* with respect to the frame in which the scale is at rest and of length 1.0 m?

b) An observer on the Earth measures the length of a spaceship to be 5.0 m as it moves past her. Its length is measured to be 15 m by an astronaut travelling in it. What is the speed of the spaceship with respect to the observer on the Earth?

c) A particle is travelling through the Earth's atmosphere at a speed of 0.90*c*. An inertial observer at rest on the Earth measures the distance travelled by it as 5.50 km. What distance does the particle travel in its own frame of reference?

SOLUTION ■ We use Eq. (2.16).

a) The proper length of the scale is 1.0 m as it is the length measured in the frame in which it is at rest. So, $L_0 = 1.0\,\text{m}$. For an observer moving with speed 0.75*c* with respect to the scale, $V = 0.75c$, and

$$\gamma = \frac{1}{\sqrt{1 - 9/16}} = \frac{4}{\sqrt{7}} = 1.5$$

Therefore, from Eq. (2.16), $L = \frac{L_0}{\gamma} = \frac{1.0\,\text{m}}{1.5} = 0.67\,\text{m}$

b) The proper length of the spaceship is measured by the astronaut as the spaceship is at rest with respect to the astronaut. Therefore, $L_0 = 15\,\text{m}$ and $L = 5\,\text{m}$. Substituting these values in Eq. (2.16), we get:

$$\gamma = \frac{L_0}{L} = 1\bigg/\sqrt{1-\frac{V^2}{c^2}} = 3 \Rightarrow 1-\frac{V^2}{c^2} = \frac{1}{9} \Rightarrow \frac{V^2}{c^2} = 1 - \frac{1}{9} = \frac{8}{9}$$

or $V = 0.94c$

c) Here $L_0 = 5.50$ km as it is the distance measured by the observer at rest. It is given that $V = 0.90c$. Therefore,

$$\gamma = \frac{1}{\sqrt{1-0.81}} = 2.3$$

From Eq. (2.16), the distance measured in the particle's frame of reference is:

$$L = \frac{L_0}{\gamma} = \frac{5.50\,\text{km}}{2.3} = 2.4\text{ km}$$

You may now like to solve an SAQ based on length contraction.

SAQ 7 - Length contraction

a) A rod of proper length 1.0 m measures 50 cm in an inertial frame that is moving with respect to the rod. What is the speed of the moving frame?

b) A scale is kept perpendicular to the direction of motion of a spaceship travelling at a very high speed in space with respect to the Earth. Will an observer in the spaceship measure any change in its length? What change does an observer on the Earth measure in the scale's length?

So far, you have studied about two consequences of the special theory of relativity, namely, time dilation and length contraction. Apart from these, there are some other consequences of special relativity, which we discuss in the next two sections. One of these is: How do you determine the relative velocities of objects moving towards or away from each other at speeds close to the speed of light?

For example, consider this problem: A spaceship, named Enterprise, moves in one direction at a speed of 0.9*c* while another spaceship, named Endeavour moves in the opposite direction at a speed of 0.9*c* with respect to an observer on the Earth (see Fig. 2.10). What is the relative speed of the ships? From the Galilean principle of relativity, it would be 1.8*c*. So, Endeavour's crew would see the Enterprise moving at a speed faster than of light and vice-versa.

But as per the special theory of relativity, no object can travel at a speed greater than the speed of light. So, what will the speed of these spaceships relative to each other be?

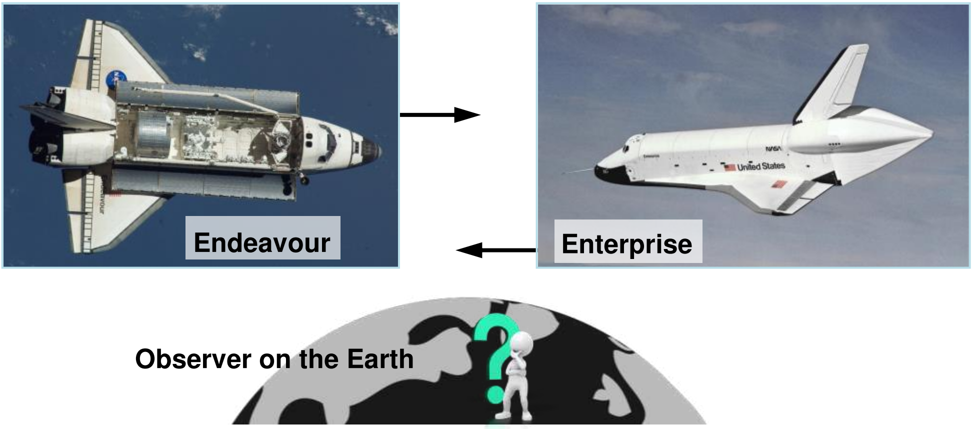
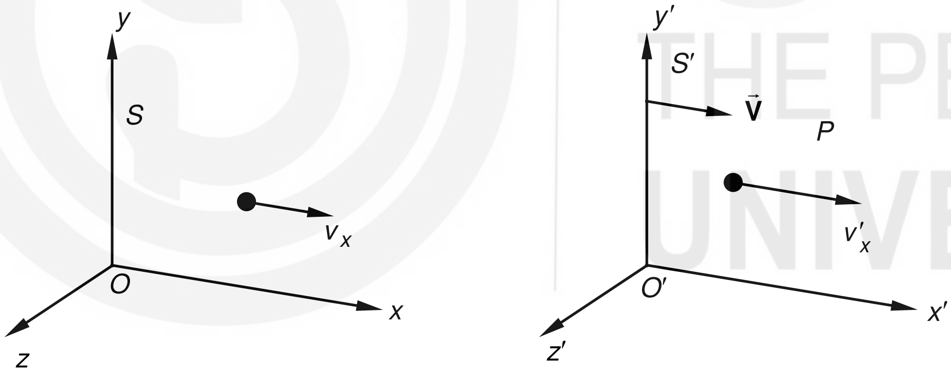


Fig. 2.10: How do we calculate the relative speed of objects moving at high speeds?

In special relativity, we use the Lorentz transformation to obtain relative speeds. This is called the **relativistic transformation of velocities**. Let us see what it is.

2.5 RELATIVISTIC TRANSFORMATION OF VELOCITIES

The problem we have to solve is this: Suppose an object moves with uniform velocity $\vec{\mathbf{v}}$ with respect to an inertial frame *S*, which is at rest. What is the velocity $\vec{\mathbf{v}}'$ of the object relative to the frame *S*′, which moves with respect to *S* with uniform velocity $\vec{\mathbf{V}}$? Let us find out. Let $\vec{\mathbf{V}}$ be in the positive *x*-direction.



(a) **(b)**

Fig. 2.11: a) An object has speed v_x in the frame of reference *S*; b) The frame *S*′ is moving with constant velocity $\vec{\mathbf{V}}$ along the *x*-*x*′ axis with respect to *S* so that the *y*-*y*′ and *z*-*z*′ axes are parallel. The velocity (v'_x) of the object in the *S*′ frame is obtained using the Lorentz transformation.

Refer to Fig. 2.11. Let us consider the *x*-component of the velocity $\vec{\mathbf{v}}$. Recall Eqs. (2.2a and d) of the Lorentz transformation:

$$x' = \gamma(x - Vt) \qquad \text{and} \qquad t' = \gamma\left(t - \frac{Vx}{c^2}\right) \tag{2.18a}$$

We take the differential of all variables in Eq. (2.18a). So, we get:

$$dx' = \gamma(dx - Vdt) \qquad \text{and} \qquad dt' = \gamma\left(dt - \frac{Vdx}{c^2}\right) \tag{2.18b}$$

Then we can obtain the components of the velocity $\vec{\mathbf{v}}'$ in terms of the components of $\vec{\mathbf{v}}$ and $\vec{\mathbf{V}}$ as follows: Using the basic definition of velocity and Eq. (2.18b), we can write the *x*-component of the velocity $\vec{\mathbf{v}}'$ as:

$$v'_x = \frac{dx'}{dt'} = \frac{dx - Vdt}{dt - \dfrac{Vdx}{c^2}} = \frac{\dfrac{dx}{dt} - V}{1 - \dfrac{V}{c^2}\dfrac{dx}{dt}}$$

or $$v'_x = \frac{v_x - V}{1 - \dfrac{v_x V}{c^2}} \qquad (2.19a)$$

What are the *y* and *z* components of $\vec{\mathbf{v}}'$? Since $y' = y$ and $z' = z$, these are, respectively,

$$v'_y = \frac{dy'}{dt'} = \frac{dy}{\gamma\left(dt - \dfrac{Vdx}{c^2}\right)} = \frac{\dfrac{dy}{dt}}{\gamma\left(1 - \dfrac{V}{c^2}\dfrac{dx}{dt}\right)}$$

or $$v'_y = \frac{v_y}{\gamma\left(1 - \dfrac{v_x V}{c^2}\right)} \qquad (2.19b)$$

Similarly, you can show that

$$v'_z = \frac{v_z}{\gamma\left(1 - \dfrac{v_x V}{c^2}\right)} \qquad (2.19c)$$

You should verify Eq. (2.19c) for practice before studying further. You can obtain the inverse velocity transformations from Eqs. (2.2a to d), or by solving Eqs. (2.19a) to (2.19c) for the unprimed velocity components. Solve SAQ 8.

SAQ 8 - Relativistic velocity addition

Prove that $$v_x = \frac{v'_x + V}{1 + \dfrac{v'_x V}{c^2}}, \qquad (2.20a)$$

$$v_y = \frac{v'_y}{\gamma\left(1 + \dfrac{v'_x V}{c^2}\right)} \qquad (2.20b)$$

and $$v_z = \frac{v'_z}{\gamma\left(1 + \dfrac{v'_x V}{c^2}\right)} \qquad (2.20c)$$

Note that Eqs. (2.19a to c) are together known as **relativistic velocity transformation**. The inverse velocity transformation equations (2.20a to c) are also known as **relativistic velocity addition formulae.** These equations can be regarded as giving the resultant of the velocities $\vec{\mathbf{v}}' = (v'_x, v'_y, v'_z)$ and $\vec{\mathbf{V}} = (V, 0, 0)$ to be $\vec{\mathbf{v}} = (v_x, v_y, v_z)$.

Compare Eq. (2.19a) with the result obtained from Galilean transformation: $v'_x = v_x - V$. Note that for $V << c$, Eqs. (2.19a and 2.20a) reduce to the Galilean transformation equations:

$$v'_x = v_x - V \qquad \text{or} \qquad v_x = v'_x + V \tag{2.21}$$

Let us now apply Eqs. (2.19a and 2.20a) to the example of the spaceships Enterprise and Endeavour (Fig. 2.12).

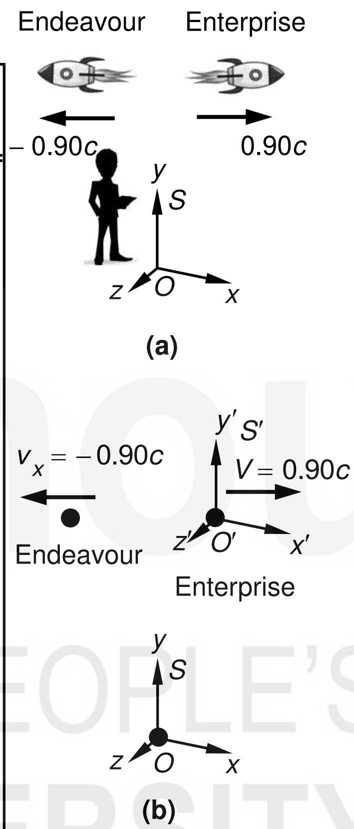


Fig. 2.12: **Relativistic transformation of velocities. a) Spaceships Endeavour and Enterprise are travelling in opposite directions with the same velocities with respect to an observer on the Earth; b) the relative speed of the spaceships is obtained using the relativistic velocity transformation equations and can never be more than the speed of light.**

Example 2.3 : RELATIVISTIC VELOCITY TRANSFORMATION

The speed of the spaceship Enterprise relative to an observer on the Earth is 0.90*c*, and the speed of the spaceship Endeavour relative to the observer $-0.90c$. What is the speed of the spaceships with respect to each other?

SOLUTION ■ We can use either Eq. (2.19a or 2.20a). **The important point is to identify the frames *S* and *S′* correctly**. In order to solve such problems, we attach the frame *S′* to one of the moving objects.

Since the spaceships are moving with respect to the Earth, let us attach the stationary frame *S* to the Earth (Fig. 2.12a). It is given that the speeds of Enterprise and Endeavour relative to *S* are 0.90*c* and $-0.90c$, respectively. So, as observed by an observer in the frame *S*, the spaceships are travelling with speeds $\pm 0.90c$ opposite to each other.

Let us attach the frame *S′* to the spaceship Enterprise (Fig. 2.12b). In Fig. 2.12b, we show the observer and the spaceships by dots to keep the figure simple. Then $V = 0.90c$. We know that the speed of Endeavour with respect to the Earth is $v_x = -0.90c$.

Substituting these values in Eq. (2.19a), we get the speed of Endeavour (v'_x) with respect to Enterprise as:

$$v'_x = \frac{v_x - V}{1 - \dfrac{v_x V}{c^2}} = \frac{-0.90c - 0.90c}{1 - (-0.90c)(0.90c)/c^2} = -\frac{1.8c}{1.81} = -0.99c$$

Thus, the speed of Endeavour with respect to Enterprise is $-0.99c$.

Alternatively, we could attach the frame *S′* to Endeavour.

Then the speed of Enterprise relative to Endeavour would be 0.99*c*, since now $V = -0.90c$ and $v_x = 0.90c$.

Let us verify if the relativistic transformation of velocities and Lorentz transformation are consistent with the special theory of relativity. Suppose that the travelling object is a photon. Let us find out what its speed is in the frames *S* and *S′*, where the frame *S′* is moving with respect to the frame *S* with constant speed *V*. Study Example 2.4.

EXAMPLE 2.4 : RELATIVISTIC VELOCITY TRANSFORMATION

The speed of a photon (speed of light) travelling along the x-axis is measured to be c relative to the frame S.

a) What is the photon's speed with respect to a frame S' moving with speed V in the positive x-direction?

b) If the photon were travelling in the y-direction, what would its speed be as measured in the frame S'?

c) If the photon were travelling with speed c in the frame S' and the frame S' were moving with speed c relative to the frame S in the positive x-direction, what would the speed of the photon be in the frame S?

SOLUTION ■ We can use either Eqs. (2.19a to c) or Eqs. (2.20a to c).

a) It is given that $v_x = c$ in frame S. Then from Eq. (2.19a), we see that

$$v'_x = \frac{v_x - V}{1 - v_x V/c^2} = \frac{c - V}{1 - cV/c^2} = c \tag{2.22a}$$

Thus, the speed of the photon in the frame S' is also c. The Lorentz transformation was designed to be consistent with this result. Is it not reassuring that we obtain c to be the photon speed (speed of light) in both reference frames?

b) Now, it is given that the photon is travelling along the y-axis, that is, $v_y = c$ and $v_x = 0$. So, putting $v_y = c$ and $v_x = 0$ in Eqs. (2.19a and b), we get

$$v'_x = -V \quad \text{and} \quad v'_y = \frac{c}{\gamma\left(1 - \dfrac{v_x V}{c^2}\right)} = \frac{c}{\gamma} = c\left(1 - \frac{V^2}{c^2}\right)^{1/2} \tag{2.22b}$$

Thus, the speed of the photon in the frame S' is:

$$\sqrt{v'^2_x + v'^2_y} = \sqrt{V^2 + c^2 - V^2} = c \tag{2.22c}$$

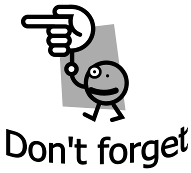
c) Now, it is given that the photon is travelling at speed c in the frame S' and frame S' is travelling relative to the frame S with speed c in the positive x-direction. Will the photon speed be measured as $2c$ in frame S? Let us find out. Applying Eq. (2.20a), we get:

$$v_x = \frac{c + c}{1 + \dfrac{c^2}{c^2}} = c \tag{2.22d}$$

So, the relativistic transformation of velocities implies that the photon speed measured in frame S is c, and not $2c$. Also, note from Eqs. (2.22a to d) that the speed of the photon, i.e., the speed of light, is **not zero** in either frame. In fact, **the speed of light can never be zero in any reference frame.**

So, the relativistic velocity transformation, which we have derived from the Lorentz transformation, is consistent with the fact that there exists an **ultimate speed** *c* in nature. So, always remember the following about the speed of light:

There exists an ultimate speed *c*, the speed of light, in nature, which cannot be exceeded by changing reference frames. Also, there is no frame in which the speed of light is zero or a photon (light quantum) is at rest.



You may now like to solve a problem using Eqs. (2.19a to c).

SAQ 9 - Relativistic velocity transformation

A spacecraft *A* has a speed 0.90*c* with respect to the Earth. If the speed of another spacecraft *B* with respect to spacecraft *A* is 0.50*c*, what is the speed of *B* with respect to the Earth?

Finally, in the last section of this unit, we consider an application of special theory of relativity to optics. You are familiar with the Doppler effect in sound and optics from school physics. Basically, the shift in frequency of sound or light due to the motion of the source or observer is called the **Doppler effect**.

The Doppler shift in sound is a familiar experience: Do you recall hearing a rapid drop in the honk of a car speeding away from you or an increase in the pitch of the whistle of a train approaching you? That is explained by Doppler shift. The frequency shift towards the red end in the spectral lines of distant receding galaxies arises due to Doppler effect in optics.

You know that the Doppler effect in sound gives different results when

(i) the source is at rest and the observer is moving, and

(ii) the source is moving and the observer is at rest.

However, if we applied these results to light waves, as is done in the classical Doppler effect in optics, we would be able to know which of the two inertial systems (source or observer) was at rest. But this would contradict the principle of relativity. To resolve this difficulty, we have to add a relativistic correction to the classic optical Doppler effect. This is precisely what we do in the next section.

2.6 RELATIVISTIC DOPPLER EFFECT

Consider a transmitter of light *T* (say, a torch), which flashes with a period τ in the frame in which it is at rest (its rest frame) *S* (see Fig. 2.13). Let the transmitter be located at $x = 0$, the origin of the reference frame *S*. Suppose the transmitter *T* emits two successive pulses of light at $t = 0$ and $t = \tau$, respectively, in *S*. Let these pulses be detected in the reference frame S' moving with speed *V* with respect to *S* in the positive *x*-direction by a detector kept at its origin O'. Let the respective origins *O* and O' of the frames *S* and S', coincide at the instants $t = 0$ and $t' = 0$, that is, the instants when the measurement begins. So the **initial pulse** emitted at $x = 0$ and $t = 0$ in S is detected at $x' = 0$ and $t' = 0$ in the frame S'. Now, the **second pulse** is emitted at $x = 0$ and $t = \tau$ in the frame S.

What are the coordinates of this event (emission of the second pulse having coordinates $x = 0$ and $t = \tau$ in the frame S), in the moving frame S'?

Let these be X' and t'. These are given by the Lorentz transformation equations (2.2a and d). Since $x = 0$ and $t = \tau$, we get:

$$X' = \gamma(x - Vt) = -\gamma V\tau = \frac{-V\tau}{\sqrt{1 - \dfrac{V^2}{c^2}}} \tag{2.23a}$$

and

$$t' = \gamma\left(t - \frac{Vx}{c^2}\right) = \gamma\tau = \frac{\tau}{\sqrt{1 - \dfrac{V^2}{c^2}}} \tag{2.23b}$$

So the event (emission of the second pulse) has coordinates X' and t' given by Eqs. (2.23a and b) in the frame S'. Note that X' is negative, which means that this coordinate is to the left of O' (see Fig. 2.13).

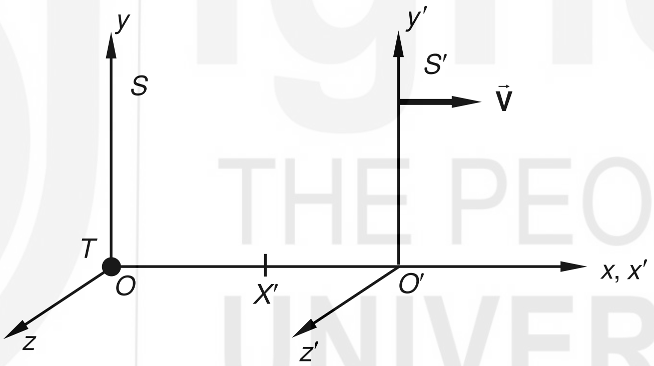


Fig. 2.13: **A transmitter T in frame S is located at its origin $O(x = 0)$ and sends pulses of light with a time period τ. The successive pulses of light are detected in S', which is moving with respect to S with velocity $\vec{\mathbf{V}}$. The initial pulse emitted in S at $O(x = 0)$ is detected in S' at $O'(x' = 0)$ when their origins coincide. The event (emission of the second pulse in S at $x = 0$) has the coordinate $x' = X'$ in S'.**

We have to determine the additional time taken by the second pulse in reaching O' from X' so that it is detected in the frame S'. Let $\Delta t'$ be the time taken by the second pulse to travel in S' from $X' = -\gamma V\tau$ to its origin $x' = 0$, at the speed c. It is given by:

$$\Delta t' = \frac{\Delta x'}{c} = \frac{x' - X'}{c} = \frac{0 - (-\gamma V\tau)}{c} = \frac{\gamma V\tau}{c} = \frac{V\tau/c}{\sqrt{1 - \dfrac{V^2}{c^2}}} \tag{2.24}$$

So the second pulse is detected at the origin O' $(x' = 0)$ of S' at the instant $t' + \Delta t'$. Since the initial pulse was detected in S' at $t' = 0$, the **total time** elapsed between the detection of the two pulses at $x' = 0$ in S' is:

$$t' + \Delta t' = \gamma\tau + \frac{\gamma V\tau}{c} = \gamma\tau\left(1 + \frac{V}{c}\right) = \frac{\tau(1 + V/c)}{\sqrt{1 - \dfrac{V^2}{c^2}}} = \frac{\tau(1 + V/c)}{\sqrt{(1 + V/c)(1 - V/c)}}$$

or $$t' + \Delta t' = \tau\left[\frac{1 + V/c}{1 - V/c}\right]^{1/2} \qquad (2.25)$$

The time interval between two successive pulses emitted from the transmitter in S and received in S' at its origin is also the **time period** of the light wave as measured in S'. From Block 4 of BPHCT-131 (Mechanics), you know that the frequency is the inverse of the time period of the wave. So, using Eq. (2.25), we can write:

$$\nu' = \frac{1}{t' + \Delta t'} = \frac{1}{\tau}\left[\frac{1 - V/c}{1 + V/c}\right]^{1/2}$$

or $$\nu' = \frac{1}{t' + \Delta t'} = \nu\left[\frac{1 - V/c}{1 + V/c}\right]^{1/2} \quad \text{(Source and observer receding)} \qquad (2.26a)$$

Here ν' is the measured frequency of the light wave detected in S' and ν is the measured frequency of the light wave transmitted from S. If the observer is receding from the source, then V is positive and ν' is less than ν.

You should note that even if the source were receding from the observer, we would get the same result. This is unlike the Doppler effect in sound where the two results are different.

If the observer is approaching the source, we take V to be negative and ν' is greater than ν. Again the result would be the same if the source were approaching the observer:

$$\nu' = \nu\left[\frac{1 + V/c}{1 - V/c}\right]^{1/2} \quad \text{(Source and observer approaching)} \qquad (2.26b)$$

Let us write these results in terms of wavelengths, $\lambda = c/\nu$ and $\lambda' = c/\nu'$. We get that

$$\lambda' = \lambda\left[\frac{1 + V/c}{1 - V/c}\right]^{1/2} \quad \text{(Source and observer receding)} \qquad (2.27a)$$

and $$\lambda' = \lambda\left[\frac{1 - V/c}{1 + V/c}\right]^{1/2} \quad \text{(Source and observer approaching)} \qquad (2.27b)$$

Remember that while solving problems using Eqs. (2.26a, b and 2.27a, b), you have to substitute only the magnitude of the velocity V in them as its sign has already been taken into account in these equations.

Eqs. (2.26a, b and 2.27a, b) describe the **relativistic longitudinal Doppler effect** for light waves in vacuum. It expresses the effect when the relative motion of the source and observer is along the same axis. Do you realise that the **Doppler shift is essentially a time dilation effect**? It occurs due to the relative motion of the transmitter/source and the detector/observer.

Note that the time interval between two pulses of light as measured in S' is an improper time interval. The time interval between the two pulses measured in S is the **proper time interval** as the two events occur at the same point $x = 0$.

If we regard the source of light as a clock then to an observer in S', it acts as a moving clock. Thus, an observer in S' measures a longer time interval between the two pulses [given by Eq. (2.25)], as compared to the proper time interval measured by an observer in S. This results in the Doppler shift in frequency/wavelength given by Eqs. (2.26a, b)/(2.27a, b).

With this, we end the discussion on relativistic kinematics.

In this unit, you have learnt about some features of space and time that follow from the special theory of relativity, namely, the phenomena of time dilation, length contraction, relativistic transformation of velocities and the relativistic Doppler effect. These phenomena become appreciable at speeds close to the speed of light. Such speeds are also termed **relativistic speeds**. You have also learnt how Lorentz transformation connects the space and time variables in two inertial frames in uniform relative motion. In the next unit, you will learn about linear momentum and force, and the equation of motion in the special theory of relativity, which constitute relativistic dynamics. Let us now sum up what you have learnt in this unit.

2.7 SUMMARY

Concept | **Description**

Lorentz transformation

- The coordinate transformation consistent with the postulates of special theory of relativity is called the **Lorentz transformation**. It links the coordinates (x, y, z, t) assigned to an event in an inertial frame S, with the coordinates (x', y', z', t') assigned to the same event in another inertial frame S' moving at a constant velocity $\vec{\mathbf{V}}$ with respect to S.

  The **Lorentz transformation** is given by:

$$x' = \gamma(x - Vt), \qquad \gamma = \left(1 - V^2/c^2\right)^{-1/2}$$
$$y' = y$$
$$z' = z$$
$$t' = \gamma(t - Vx/c^2)$$

  The **inverse Lorentz transformation** is given by:

$$x = \gamma(x' + Vt'), \qquad \gamma = \left(1 - V^2/c^2\right)^{-1/2}$$
$$y = y'$$
$$z = z'$$
$$t = \gamma(t' + Vx'/c^2)$$

Relativity of simultaneity

- The postulates of special theory of relativity have implications for the measurement of the kinematical variables of distance (length) and time. The first and foremost is the breakdown in simultaneity for objects/frames travelling at speeds close to the speed of light. This is known as the **relativity of simultaneity:**

  Simultaneity is relative. Events that occur at **different positions** but **at the same time** with respect to one inertial frame are **not simultaneous** in another inertial frame in uniform relative motion. So,

  if $x'_1 \neq x'_2$ and $t'_2 = t'_1$ in S' then $t_2 \neq t_1$ in S

  or if $x_1 \neq x_2$ and $t_2 = t_1$ in S then $t'_2 \neq t'_1$ in S'

  Events that occur at the **same position in space** and **at the same time** are **simultaneous** in two inertial frames in uniform relative motion. Observers in these frames **always agree on the simultaneity of events**. So,

  if $x'_1 = x'_2$ and $t'_2 = t'_1$ in S' then $t_2 = t_1$ in S

  or if $x_1 = x_2$ and $t_2 = t_1$ in S then $t'_2 = t'_1$ in S'

Time dilation

- The measurement of time interval depends on the inertial frame of reference. The time interval measured in the reference frame in which the two events occur at the **same position** is called the **proper time interval** or **proper time**. The time interval measured between the two events occurring at different positions in any reference frame is called **improper time interval** or **improper time**.

  For two events that occur at the same position in the inertial frame S', separated by a **proper time interval** $\Delta t'$, the **improper time interval** Δt between these events measured in S (with respect to which S' is moving at constant velocity $\vec{\mathbf{V}}$) **is longer** and given by:

  $$\Delta t = \gamma \Delta t'$$

  This phenomenon is called **time dilation**.

Length contraction

- The length of an object or the distance between two events depends upon the inertial frame in which it is measured. **Proper length** is the distance between two points measured by an **observer who is at rest relative to both the points**. Suppose L_0 is the proper length or proper distance measured by an observer at rest with respect to the two points at which the events occur. Then an observer in uniform motion along the line joining the two points will measure that length or distance to be:

  $$L = \frac{L_0}{\gamma}$$

  It will be shorter than the proper distance L_0. This phenomenon is called **length contraction. It takes place only in the direction of motion and not in the direction perpendicular to it.**

Relativistic velocity transformation

- The **relativistic velocity transformation** in two frames S and S' in uniform relative motion is given by :

$$v'_x = \frac{v_x - V}{1 - \frac{v_x V}{c^2}}$$

$$v'_y = \frac{v_y}{\gamma\left(1 - \frac{v_x V}{c^2}\right)}$$

$$v'_z = \frac{v_z}{\gamma\left(1 - \frac{v_x V}{c^2}\right)}$$

where (v_x, v_y, v_z) are the velocity components in frame S, and (v'_x, v'_y, v'_z), the velocity components in the frame S', which is moving with speed V relative to S in the positive x-direction. The inverse velocity transformation is also referred to as the **relativistic velocity addition formulae** and is given by:

$$v_x = \frac{v'_x + V}{1 + \frac{v_x V}{c^2}}$$

$$v_y = \frac{v'_y}{\gamma\left(1 + \frac{v'_x V}{c^2}\right)}$$

$$v_z = \frac{v'_z}{\gamma\left(1 + \frac{v'_x V}{c^2}\right)}$$

Relativistic Doppler effect

- The application of special relativity to Doppler effect in optics leads to a correction. The **relativistic Doppler effect** relates the frequency ν of a light wave in a frame S (in which the source of light is at rest) to the frequency ν' measured by an observer in S' moving with a velocity $\vec{\mathbf{V}}$ relative to S in a given direction:

$$\nu' = \nu\left[\frac{1 - V/c}{1 + V/c}\right]^{1/2} \qquad \text{(Source and observer receding)}$$

$$\nu' = \nu\left[\frac{1 + V/c}{1 - V/c}\right]^{1/2} \qquad \text{(Source and observer approaching)}$$

Since wavelengths $\lambda = c/\nu$ and $\lambda' = c/\nu'$, we have

$$\lambda' = \lambda\left[\frac{1 + V/c}{1 - V/c}\right]^{1/2} \qquad \text{(Source and observer receding)}$$

and $$\lambda' = \lambda\left[\frac{1 - V/c}{1 + V/c}\right]^{1/2} \qquad \text{(Source and observer approaching)}$$

2.8 TERMINAL QUESTIONS

1. An observer in frame S assigns the following coordinates to two random events E_1 and E_2 :

   Event E_1 : $x_1 = 1.2\times10^9\,\text{m}$, $y_1 = 0$, $z_1 = 0$, $t_1 = 7.0\,\text{s}$

   Event E_2 : $x_2 = 3.0\times10^9\,\text{m}$, $y_2 = 0$, $z_2 = 0$, $t_2 = 11\,\text{s}$

   Determine the coordinates of E_1 and E_2 as measured by an observer in frame S' moving relative to frame S at speed $0.80c$.

2. Show that two events simultaneous and separated by the distance Δx in the frame S will be separated in a frame S' in both space and time, such that

   $$\Delta x' = \gamma\,\Delta x \qquad \text{and} \qquad \Delta t' = -\frac{\gamma V}{c^2}\Delta x$$

   where S' is moving relative to S at speed V in the positive x-direction.

3. Suppose two events are simultaneous at $t = 0$ and $t' = 0$ and at $x = 0$ and $x' = 0$ of the inertial frames S and S' in uniform relative motion. Show that events occurring at all other points in space $(x \neq 0)$ are not simultaneous in S and S'.

4. Why do we not observe the effect of time dilation in every day phenomena?

5. Calculate the speed of a moving clock that runs at a rate which is one-half the rate of a clock at rest.

6. Light takes 4.2 years to travel from the star system Alpha Centauri to the Earth. If an astronaut travelled from the Earth to Alpha Centauri, at a speed of $0.90c$, how long would it take, according to the astronaut's clock, for him/her to reach there?

7. A spaceship moves relative to an observer at a speed of $0.60c$. The observer measures its length to be 20 m. What is the length of the spaceship when it is at rest relative to the observer?

8. A particle is travelling through the Earth's atmosphere at a speed of $0.80c$. To an Earth-bound observer, the distance it travels is 5.0 km. How far does the particle travel in its own frame of reference?

9. Show that if L_0^3 is the rest volume of a cube, then the volume measured in a frame moving with speed u in a direction parallel to an edge of the cube is $L_0^3(1 - u^2/c^2)^{1/2}$.

10. A rocket is moving with a speed of $0.90c$ with respect to the Earth in a certain direction. It is given an additional speed of $0.40c$ in the same direction. Calculate its final speed with respect to the Earth.

11. From Earth, we observe two galaxies receding in opposite directions at speeds 0.60c. What speed of recession would an observer in one of these galaxies observe for the other galaxy?

12. Two spaceships of proper length L_0 approach the Earth from opposite directions at velocities $\pm c/3$. What is the length of one of the spaceships with respect to the other?

13. A star emits the blue line of hydrogen with wavelength $\lambda = 434\,\text{nm}$. Suppose that in the light from a distant galaxy, the same line has a wavelength of 600 nm on the Earth, which is not blue but red (therefore, the term "reddening"). What is the "receding speed" of that galaxy?

14. A spaceship is receding from the Earth at a speed of 0.20c. Light of wavelength $\lambda = 450\,\text{nm}$ in the spaceship appears blue to a passenger on the spaceship. What colour would it appear to an observer on the Earth?

15. A proton beam is accelerated so that protons in it attain a speed of $2.0\times10^6\,\text{m s}^{-1}$. Afterwards they drift at a constant speed through a region where they are neutralized to hydrogen atoms. In this process light is emitted and observed in a spectrometer. What is the Doppler shift of the wavelength in the light spectrum? The wavelength of light emitted when the atom is at rest is $\lambda = 4861.33\,\text{Å}$.

2.9 SOLUTIONS AND ANSWERS

Self-Assessment Questions

1. When $V/c \ll 1$ in Lorentz transformation, we can neglect $\dfrac{V^2}{c^2}$ in comparison with 1, and get the following equations:

$$x' = \frac{x - Vt}{(1 - V^2/c^2)^{1/2}} \cong x - Vt, \quad y' = y, \quad z' = z$$

In the last equation, we consider the motion of the origin O' given by $x = Vt$ and substitute it in the expression for t' given by Eq. (2.2d). Then

$$t' = \frac{t - V^2 t/c^2}{(1 - V^2/c^2)^{1/2}} = t(1 - V^2/c^2)^{1/2}$$

When we take $V/c \ll 1$ in the above equation, we get $t' = t$

2. From Eq. (2.2e), $\gamma = \dfrac{1}{(1 - V^2/c^2)^{1/2}}$

   a) Since V is finite, γ will always be greater than or equal to 1 as the denominator will be less than or equal to 1.

   b) As $V \to 0$, the denominator tends to 1, and $\gamma \to 1$.

   c) As $V \to c$, the denominator tends to zero, $\gamma \to \infty$.

3. Substituting $\gamma t = (t' + \gamma Vx/c^2)$ from Eq. (2.2d) in Eq. (2.2a), we get:

$$x' = \gamma x - (Vt' + \gamma V^2 x/c^2)$$

or $\quad x' + Vt' = \gamma x(1 - V^2/c^2) = x(1 - V^2/c^2)^{1/2} = \dfrac{x}{\gamma}$

or $\quad x = \gamma(x' + Vt')$

This is Eq. (2.3a). Eqs. (2.3b and c) remain the same. Now we substitute $\gamma x = (x' + \gamma Vt)$ from Eq. (2.2a) in Eq. (2.2d) and get:

$$t' = \gamma t - \frac{V}{c^2}(x' + \gamma Vt) = \gamma t(1 - \frac{V^2}{c^2}) - \frac{V}{c^2}x' = \frac{t}{\gamma} - \frac{V}{c^2}x'$$

or $\quad t = \gamma(t' + \dfrac{V}{c^2}x')$, which is Eq. (2.3d).

4. Using Eq. (2.3a), we can write:

$$x_2 - x_1 = \gamma(x'_2 - x'_1) + \gamma V(t'_2 - t'_1)$$

Since it is given that $x'_2 = x'_1$, but $t'_2 \neq t'_1$, we have

$$x_2 - x_1 = \gamma V(t'_2 - t'_1) \neq 0 \quad \text{as long as } V \neq 0.$$

So events occurring at the same position in S' $(x'_2 = x'_1)$, but at different times $t'_2 \neq t'_1$, do not occur at the same points in S as long as $V \neq 0$.

5. We use Lorentz transformation [Eq. (2.2d)] and write:

$$t'_1 = \gamma(t_1 - Vx_1/c^2) \quad \text{and} \quad t'_2 = \gamma(t_2 - Vx_2/c^2)$$

So, $\quad t'_2 - t'_1 = \gamma(t_2 - t_1) \qquad (\because\ x_2 = x_1)$

or $\quad \Delta t' = \gamma \Delta t$

where Δt is the proper time interval measured in S at the same position.

6. The proper mean life time of the pion is 2.6×10^{-8} s. It is given that $V = 0.98c$ and so,

$$\gamma = \frac{1}{(1 - V^2/c^2)^{1/2}} = \frac{1}{[1 - (0.98)^2]^{1/2}} = 5$$

Therefore, the mean life time of the pion when it is moving with a speed of 0.98c with respect to the observer at rest is given by Eq. (2.12) and is equal to:

$$5 \times 2.6 \times 10^{-8}\text{ s} = 1.3 \times 10^{-7}\text{ s}$$

To an observer moving with the pion, the pion would be at rest and the observer will measure the proper mean life time of the charged pion, which is 2.6×10^{-8} s.

7. a) It is given that $L_0 = 1.0$ m and $L = 50$ cm $= 0.50$ m. Therefore,

$$\gamma = \frac{L_0}{L} = 2 \quad \text{or} \quad \frac{1}{(1 - V^2/c^2)^{1/2}} = 2 \quad \Rightarrow \quad \left(1 - \frac{V^2}{c^2}\right)^{1/2} = \frac{1}{2}$$

or

$$\left(1 - \frac{V^2}{c^2}\right) = \frac{1}{4} \quad \Rightarrow \quad \frac{V^2}{c^2} = \frac{3}{4} \quad \Rightarrow \quad V = \frac{\sqrt{3}}{2}c = 2.6 \times 10^8\text{ ms}^{-1}$$

b) Since the scale is at rest with respect to the spaceship, its length will remain unchanged to the observer at rest in the spaceship. Since the scale is kept perpendicular to the direction of motion of the spaceship, to the observer on the Earth too, the length of the scale will remain unchanged. The observer on the Earth will measure the same length of the scale as the observer in the spaceship. Recall that the phenomenon of length contraction is observed only in the direction of motion and not in the direction perpendicular to it.

8. From Eqs. (2.3a and d), we have:

$$x = \gamma(x' + Vt') \qquad \text{and} \qquad t = \gamma\left(t' + Vx'/c^2\right)$$

$$\therefore \qquad dx = \gamma(dx' + Vdt') \qquad \text{and} \qquad dt = \gamma\left(dt' + Vdx'/c^2\right)$$

Thus, $$v_x = \frac{dx}{dt} = \frac{(dx' + Vdt')}{(dt' + Vdx'/c^2)} = \frac{\frac{dx'}{dt'} + V}{1 + \frac{V}{c^2}\frac{dx'}{dt'}} = \frac{v'_x + V}{1 + \frac{v'_x V}{c^2}}$$

which is Eq. (2.20a). Again, using Eq. (2.3b), we can write:

$$v_y = \frac{dy}{dt} = \frac{(dy')}{\gamma(dt' + Vdx'/c^2)} = \frac{\frac{dy'}{dt'}}{\gamma\left(1 + \frac{V}{c^2}\frac{dx'}{dt'}\right)} = \frac{v'_y}{\gamma\left(1 + \frac{v'_x V}{c^2}\right)}$$

which is Eq. (2.20b). Similarly, using Eq. (2.3c), we get:

$$v_z = \frac{dz}{dt} = \frac{(dz')}{\gamma(dt' + Vdx'/c^2)} = \frac{\frac{dz'}{dt'}}{\gamma\left(1 + \frac{V}{c^2}\frac{dx'}{dt'}\right)} = \frac{v'_z}{\gamma\left(1 + \frac{v'_x V}{c^2}\right)}$$

which is Eq. (2.20c).

9. We apply the relativistic velocity transformation [Eq. (2.19a)]. Let S be an inertial frame fixed on the Earth and let S' be the inertial frame attached to the spacecraft B. With this choice, v_x is the speed of the spacecraft A with respect to the Earth and it is given that $v_x = 0.90c$. The speed of B with respect to A is given to be $0.50c$. We have to determine the speed of B with respect to the Earth, i.e., the speed V in Eq. (2.19a).Since the speed of B, i.e., S' with respect to A is $0.50c$, the speed of A with respect to S' is $v'_x = -0.50c$. Hence, from Eq. (2.19a), we have:

$$v'_x = \frac{v_x - V}{1 - \frac{v_x V}{c^2}}$$

or $$-0.50c = \frac{0.90c - V}{1 - \frac{0.90cV}{c^2}} \qquad \text{or} \qquad -0.50c + 0.45V = 0.90c - V$$

or $$1.45V = 1.40c \qquad \Rightarrow \qquad V = \frac{1.40}{1.45}c = 0.97c$$

Terminal Questions

1. The space-time coordinates of the events E_1 and E_2 in the inertial frame S' can be obtained from the Lorentz transformation [Eqs. (2.2a to d)].

   Now,

$$\gamma = \frac{1}{(1 - V^2/c^2)^{1/2}} = \frac{1}{[1 - (0.80)^2]^{1/2}} = \frac{1}{[1 - 0.64]^{1/2}} = \frac{1}{[0.36]^{1/2}} = \frac{5}{3}$$

For event E_1: $x_1' = \frac{5}{3}\left(1.2\times10^9\,\text{m} - 0.80c\times7.0\,\text{s}\right) = 2.0\times10^9\,\text{m} - 2.8\times10^9\,\text{m}$

or $x_1' = -8.0\times10^8\,\text{m}$, $y_1' = y_1 = 0$, $z_1' = z_1 = 0$,

$$t_1' = \frac{5}{3}\times\left(7.0\,\text{s} - 0.80c\times1.2\times10^9\,\text{m}/c^2\right) = \frac{35}{3}\,\text{s} - \frac{16}{3}\,\text{s} = 6.3\,\text{s}$$

For event E_2: $x_2' = \frac{5}{3}\left(3.0\times10^9\,\text{m} - 0.80c\times11\,\text{s}\right) = 5.0\times10^9\,\text{m} - 4.4\times10^9\,\text{m}$

or $x_2' = 6.0\times10^8\,\text{m}$, $y_2' = y_2 = 0$, $z_2' = z_2 = 0$,

$$t_2' = \frac{5}{3}\times\left(11\,\text{s} - 0.80c\times3.0\times10^9\,\text{m}/c^2\right) = 5\,\text{s}$$

2. It is given that $x_2 \neq x_1$ and $t_2 = t_1$ in the frame S. We use the Lorentz transformation [Eqs. (2.2a and d)] since the events are simultaneous in S and obtain:

$$x_1' = \gamma(x_1 - Vt_1) \quad \text{and} \quad x_2' = \gamma(x_2 - Vt_2)$$

whence $x_2' - x_1' = \gamma(x_2 - x_1) - \gamma V(t_2 - t_1)$ or $\Delta x' = \gamma\Delta x$ since $t_2 = t_1$

Similarly, $t_1' = \gamma(t_1 - \frac{V}{c^2}x_1)$ and $t_2' = \gamma(t_2 - \frac{V}{c^2}x_2)$ whence

$$t_2' - t_1' = \gamma(t_2 - t_1) - \gamma\frac{V}{c^2}(x_2 - x_1) \quad \text{or} \quad \Delta t' = -\frac{\gamma V}{c^2}\Delta x \text{ since } t_2 = t_1$$

3. It is given that two events are simultaneous at $t = 0$ and $t' = 0$ and at the positions $x = 0$ and $x' = 0$ of the inertial frames S and S' in uniform relative motion. For all other points $(x \neq 0)$, Eq. (2.2d) gives:

$$t' = \gamma\left(t - \frac{Vx}{c^2}\right)$$

   So, if $x \neq 0$, then $t' \neq t$ in the above equation. Thus, events occurring at all other points do not occur at the same time, i.e., these are not simultaneous in S and S'.

4. We do not observe the effect of time dilation in every day phenomena because the speeds we encounter in daily life are much smaller than c, the speed of light. So, the factor γ is approximately equal to 1. Thus, $t' = t$ and we do not observe the effects of time dilation in every day phenomena around us.

5. In this case, the proper time (i.e., time measured at a fixed position) is being measured in the moving clock's frame. The clock at rest measures the improper time (that is, it measures events at two different positions). So, we use Eq. (2.12). It is given that $\Delta t' = \frac{1}{2}\Delta t$. Thus, from Eq. (2.12),

$$\Delta t = 2\Delta t' \quad \Rightarrow \quad \gamma = 2 \ \text{ or } \ \frac{1}{(1 - V^2/c^2)^{1/2}} = 2$$

$$\text{or } \ 1 - \frac{V^2}{c^2} = \frac{1}{4} \quad \Rightarrow \quad \frac{V}{c} = \frac{\sqrt{3}}{2} \quad \Rightarrow \quad V = \frac{\sqrt{3}}{2}c = 2.6 \times 10^8 \,\text{ms}^{-1}$$

6. It is given that light takes 4.2 years to travel from Alpha Centauri to the Earth. So, the distance between Alpha Centauri and the Earth is (4.2 years)c, where c is the speed of light. Now, the astronaut travels at the speed $0.90c$ from the Earth to Alpha Centauri. So, in a frame attached to the Earth, the time taken by the astronaut to complete the journey is:

$$\Delta t = \frac{4.2\,c}{0.90c}\,\text{years} = 4.7 \text{ years}$$

Since the astronaut's clock is at rest with respect to her, it measures the proper time interval $\Delta t'$ and the clock on the Earth measures the improper time interval Δt. These are related by:

$$\Delta t = \gamma\,\Delta t' \qquad \Rightarrow \qquad \Delta t' = \frac{4.7}{\gamma}\,\text{years}$$

$$\text{For } V = 0.90c,\ \gamma = \frac{1}{\left(1 - \frac{(0.90)^2 c^2}{c^2}\right)^{1/2}} = \frac{1}{(1 - 0.81)^{1/2}} = \frac{1}{0.43} = 2.3$$

$$\therefore \qquad \Delta t' = \frac{4.7}{2.3}\,\text{years} = 2\,\text{years}$$

Thus, in the astronaut's frame of reference, the time elapsed is 2 years. As per the astronaut's clock, it has taken 2 years for her to reach Alpha Centauri from the Earth.

7. The length of the moving spaceship measured by the observer is its improper length as the observer is not at rest with respect to its ends. It is given to be 20 m when the speed of the spaceship is $0.60c$. The length of the spaceship when it is at rest relative to the observer is its proper length and is given by Eq. (2.16). Using $L = 20$ m and $V = 0.60c$ in Eq. (2.16), we get:

$$L_0 = L\gamma = \frac{L}{(1 - V^2/c^2)^{1/2}} = \frac{20\,\text{m}}{(1 - 0.36)^{1/2}} = \frac{20\,\text{m}}{0.8} = 25\,\text{m}$$

8. In the frame of an Earth-bound observer at rest, the distance travelled by the particle is the proper length, since the Earth-bound observer is at rest with respect to the particle's initial and final positions. It is given to be 5.0 km. The distance travelled by the particle in its own frame of reference is the improper length as the particle is not at rest with respect to its initial and final positions. So, we use $L_0 = 5.0$ km and $V = 0.80c$ in Eq. (2.16) to get:

$$L = L_0/\gamma = L_0\,(1 - V^2/c^2)^{1/2} = 5.0\,\text{km} \times \left(1 - (0.80)^2\right)^{1/2} = 3.0\,\text{km}$$

So, the particle travels 3.0 km in its own frame of reference.

9. Suppose the frame S' moves with a velocity u in a direction parallel to the x-edge of the cube. Then the length of only that edge of the cube is observed to be contracted and is given by $L = L_0/\gamma$. The lengths of the other two edges (along y and z-axes) remain the same as those lengths are in directions perpendicular to the line of motion. Therefore, the volume of the cube measured in the frame S' will be

$$V = \frac{L_0}{\gamma} \times L_0 \times L_0 = L_0^3 (1 - u^2/c^2)^{1/2}$$

10. We use Eq. (2.20a) giving the relativistic velocity addition to solve this problem. Let the inertial frame S' be moving with a speed of $0.90c$ with respect to the inertial frame S attached to the Earth. So, $V = 0.90c$. And the rocket is initially at rest with respect to the frame S'. Then the rocket is given an additional speed of $0.40c$ in the same direction as the direction of its initial motion. This means that its speed becomes $0.40c$ in the frame S', that is, $v'_x = 0.40c$. So, its final speed (v_x) with respect to the Earth (frame S) is given by Eq. (2.20a) as:

One way to check your reasoning and answers in all such problems is to always remember that no object can have speed greater than the speed of light.

$$v_x = \frac{v'_x + V}{1 + \dfrac{Vv'_x}{c^2}} = \frac{0.40c + 0.90c}{\left(1 + \dfrac{0.40 \times 0.90\, c^2}{c^2}\right)} = \frac{1.3}{1.36} c = 0.96c$$

11. We follow the steps given in Example 2.3. We can use either Eq. (2.19a) or Eq. (2.20a). Since the galaxies are moving with respect to us, let us attach the frame S to ourselves on the Earth. So, as observed by us in frame S, the galaxies (say, 1 and 2) are receding at speeds $\pm\, 0.60c$ opposite to each other. Let us attach the frame S' to the galaxy 1 moving with the speed $+\,0.60c$ with respect to us, i.e., the Earth. Then $V = 0.60c$. Now the speed of the other galaxy 2 with respect to the Earth is $v_x = -0.60c$. We have to calculate v'_x, which is the speed of the galaxy 2 with respect to the frame S'. Substituting the values of V and v_x in Eq. (2.19a), we get the value of v'_x as:

$$v'_x = \frac{v_x - V}{1 - \dfrac{Vv_x}{c^2}} = \frac{-0.60c - 0.60c}{1 - \dfrac{(-0.60c)\times(0.60c)}{c^2}} = -\frac{1.2c}{1.36c} = -0.88c$$

12. Let us attach the frame S to the Earth. So, in the frame S, the two spaceships have the speeds $\pm\, c/3$. Let us attach the frame S' to the spaceship (say 1) moving with speed $+\, c/3$. Then $V = c/3$ and the speed of the spaceship (say 2) with respect to the Earth is $v_x = -c/3$. From Eq. (2.19a), the speed of the spaceship 2 moving in the opposite direction with respect to the spaceship 1 is given as:

$$v'_x = \frac{v_x - V}{1 - \dfrac{Vv_x}{c^2}} = \frac{-c/3 - c/3}{1 - \dfrac{(-c/3)\times(c/3)}{c^2}} = -\frac{2c/3}{10c/9} = -\frac{3}{5}c$$

So, spaceship 2 moves with a speed of $-3c/5$ with respect to spaceship 1 (frame S'). So, its contracted length with respect to spaceship 1 is given by:

$$L = \frac{L_0}{\gamma} = L_0\sqrt{1-(3/5)^2} = \frac{4L_0}{5}$$

13. We use Eq. (2.27a) for the source and observer receding. In this case, the source is the star and it emits the blue line of hydrogen with wavelength $\lambda = 434\,\text{nm}$. The same line has wavelength of $\lambda' = 600\,\text{nm}$ when observed on the Earth. So, substituting these values in Eq. (2.27a), we get:

$$600\,\text{nm} = 434\,\text{nm}\left[\frac{1+V/c}{1-V/c}\right]^{1/2} \quad\Rightarrow\quad \frac{1+V/c}{1-V/c} = \left(\frac{600}{434}\right)^2 = 1.9$$

or $$c + V = 1.9\times(c - V) \quad\Rightarrow\quad V = 0.31\,c$$

So, the "receding speed" of the galaxy is $0.31c$.

14. Since the source of light is at rest in the spaceship, we attach the frame S to it. So substituting $\lambda = 450$ nm and $V = 0.20c$ in Eq. (2.27a), we get the wavelength of light in the frame attached to the Earth as:

$$\lambda' = 450\,\text{nm}\left[\frac{1+0.20}{1-0.20}\right]^{1/2} \quad\Rightarrow\quad \lambda' = 450\times 1.22\text{ nm} = 551\text{ nm}$$

Light of this wavelength would appear green to the observer on the Earth.

15. It is given that the rest wavelength of light is $\lambda = 4861.33\,\text{Å}$ ($1\,\text{Å} = 10^{-10}$ m). Since the protons are approaching the region where they are neutralized to hydrogen atoms, we use Eq. (2.27b) to calculate the Doppler-shifted wavelength:

$$\lambda' = \lambda\left[\frac{1-V/c}{1+V/c}\right]^{1/2}$$

It is given that the protons are moving with a speed of $2.0\times10^6\,\text{m s}^{-1}$. Substituting the values of λ and V in the above equation, we get:

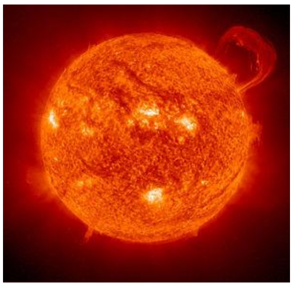
$$\lambda' = (4861.33)\times\left[\frac{1-2.0\times10^6\,\text{ms}^{-1}/3.0\times10^8\,\text{ms}^{-1}}{1+2.0\times10^6\,\text{ms}^{-1}/3.0\times10^8\,\text{ms}^{-1}}\right]^{1/2}\text{Å}$$

$$= (4861.33)\times\left[\frac{1-2.0/300}{1+2.0/300}\right]^{1/2}\text{Å} = (4861.33)\times\left[\frac{298}{302}\right]^{1/2}\text{Å}$$

or $$\lambda' = 4829.03\,\text{Å}$$

The Doppler shift of the wavelength in the light spectrum is

$$4861.33\,\text{Å} - 4829.03\,\text{Å} = 32.3\,\text{Å}$$



The cover of the Time Magazine (July, 1946) showed the connection between the mass-energy equivalence $E = mc^2$ **and the atom bomb represented by the mushroom cloud. This equation also explains the source of energy released in nuclear fusion in stars.** Source of pictures: Wikimedia Commons

UNIT 3

RELATIVISTIC DYNAMICS

Structure

3.1 Introduction
Expected Learning Outcomes
3.2 Dynamics of a Single Particle
Relativistic Linear Momentum
Relativistic Equation of Motion
3.3 Mass-Energy Equivalence
Relativistic Kinetic Energy
Relativistic Energy and Linear Momentum of a Free Particle
3.4 Summary
3.5 Terminal Questions
3.6 Solutions and Answers

STUDY GUIDE

In this unit, we explain the basic concepts of relativistic dynamics such as the relativistic linear momentum and force, the equation of motion of relativistic particles, and their relativistic mass. In order to learn the concepts of this unit better, you should revise the basic concepts of Newtonian mechanics explained in Blocks 2 and 3 of the first semester course BPHCT-131 (Mechanics). So, you must revise the classical concepts of dynamics of motion (the equation of motion, the force law and linear momentum) explained in Units 5, 6 and 8 of Block 2, BPHCT-131 (Mechanics). You must also revise the concepts of work and energy, and the work-energy theorem explained in Unit 9 of BPHCT-131 because these will be used in Sec. 3.3 on relativistic mass-energy equivalence. We have made use of a great deal of algebra in this unit. You should try to work out all steps yourself. So, keep a pen and paper with you while studying it. You should read the text carefully, make notes and ask your Counsellor to help you with your difficulties. You should also solve the SAQs and Terminal Questions on your own to confirm if you have grasped what you have studied in the unit.

"Look deep into nature and then you will understand everything better."	***Albert Einstein***

3.1 INTRODUCTION

In Unit 1, you have learnt the postulates of the special theory of relativity. You have also studied that Galilean coordinate transformation is incompatible with the special theory of relativity. In Unit 2, we obtained the Lorentz transformation and examined the implications of special relativity for length and time measurements. You have also learnt about relativistic transformation of velocities. So, you have learnt the basic concepts of relativistic kinematics in Unit 2. Now we would like you to learn the basic concepts of relativistic dynamics, namely, linear momentum, equation of motion and energy of a relativistic particle.

You may ask: Why?

You may find it interesting to know that experiments performed in high energy particle accelerators reveal that Newton's laws of motion do not apply to the motion of relativistic particles. As you may recall, these are particles moving at high speeds close to the speed of light. Therefore, we need to suitably modify the Newtonian concepts of linear momentum and force so that these are consistent with the special theory of relativity.

In Sec. 3.2, we examine the concepts of linear momentum and inertial mass for the **relativistic motion** of a **single particle**. You will learn that these dynamical physical quantities need to be redefined for relativistic particles. Using the expressions for relativistic linear momentum and relativistic mass, we rewrite the equation of motion (Newton's second law) and apply it to the motion of a particle moving at high speeds.

Recall from Units 9 and 10 of BPHCT-131 (Mechanics), how force is related to work and energy. So, once we recast the equation of motion of mechanics in the special theory of relativity, we have to re-examine the concept of energy. When we revisit the concept of energy in special relativity, we arrive at the famous equation $E = mc^2$ that demonstrates the equivalence of mass and energy. This single equation has, perhaps, transformed our world in unparalleled ways and you will discover its power in Sec. 3.3.

With this unit we end the discussion on the special theory of relativity. In the next block, we introduce yet another area of physics that revolutionised the way we understand the microscopic world, namely, quantum mechanics.

Expected Learning Outcomes

After studying this unit, you should be able to:

- ❖ derive the expression of relativistic linear momentum;
- ❖ derive relativistic equation of motion and apply it to simple situations; and
- ❖ compute the mass, speed, linear momentum and energy of a relativistic particle.

3.2 DYNAMICS OF A SINGLE PARTICLE

In this section our aim is to reformulate Newton's laws of motion so that they are consistent with the special theory of relativity.

You know from school physics and BPHCT-131 that Newton's second law defines force as the rate of change of linear momentum, which is the product of mass and velocity of a body: $\vec{\mathbf{F}} = \frac{d\vec{\mathbf{p}}}{dt} = \frac{d(m\vec{\mathbf{v}})}{dt}$.

Recall that one of the basic assumptions of Newtonian mechanics is that **the mass of an object is independent of its state of motion with respect to the observer.** Thus, equal forces acting on an object would produce equal accelerations. As a result, if a net non-zero force were to be exerted indefinitely on an object, its velocity would go on increasing indefinitely, at a constant rate.

Note that this result contradicts what you have learnt in Secs. 2.2 and 2.5 of Unit 2: That **no object can travel with a speed greater than the speed of light in vacuum.**

Can we, therefore, intuitively conclude that the mass of a body should increase with its speed and tend to infinity as its speed approaches *c*, the speed of light? Then this modified definition of mass would yield a different linear momentum. This is one point of departure of the special theory of relativity from Newton's laws.

Another difficulty with classical mechanics is that it requires action and reaction to be equal and opposite at all instants. When it is applied to **forces in contact (acting at the same point)**, even according to special relativity we can say that the forces act at the same instant [recall Eqs. (2.8b and c) in Sec. 2.3 of Unit 2]. However, you have learnt in Sec. 2.3 of Unit 2 that in special relativity, for forces acting at a distance, the same instant differs from one inertial observer to another. Therefore, we cannot give meaning to action and reaction independent of the observer's frame of reference.

Thus, in modifying Newtonian mechanics we have to exclude the notion of forces 'acting-at-a-distance'. But we can include phenomena involving contact forces or force fields such as gravitational and electromagnetic force fields.

We now explain the concept of relativistic linear momentum.

3.2.1 Relativistic Linear Momentum

From Newton's second law you know that if the net external force on a system is zero, its **linear momentum remains conserved**. Our starting point of this discussion is that we **retain the law of conservation of linear momentum for relativistic particles**. This is because it is a fundamental law of nature. Hence, it is logical to assume that the law of conservation of linear momentum should be valid in relativistic mechanics as well.

So, we should redefine linear momentum in a way that it is consistent with special relativity, that is, **the law of conservation of linear momentum should satisfy the principle of relativity**. Thus, **we should define linear momentum in such a way that conservation of linear momentum holds in all inertial reference frames** (frames moving at constant velocities with respect to each other). The modified linear momentum should remain unchanged under Lorentz transformation. We say that '**linear momentum should be Lorentz invariant**'. This is what we do in this section.

To keep the discussion simple, we consider a system of two interacting particles (say, two balls) P_1 and P_2 in one-dimensional motion along the positive x-axis (see Fig. 3.1). We observe their motion from two inertial frames, S and S'. Note that frame S' is moving with constant speed V in the positive x-direction with respect to frame S.

Let the masses of the particles P_1 and P_2 be m_1 and m_2 in frame S, and m_1' and m_2' in frame S', respectively. **Note** that we are **not** assuming the masses of the particles to be the same in both frames. We will have to find out if this is so. Let the respective velocities of the particles be $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$ in the frame S and $\vec{\mathbf{v}}_1'$ and $\vec{\mathbf{v}}_2'$ in the frame S'.

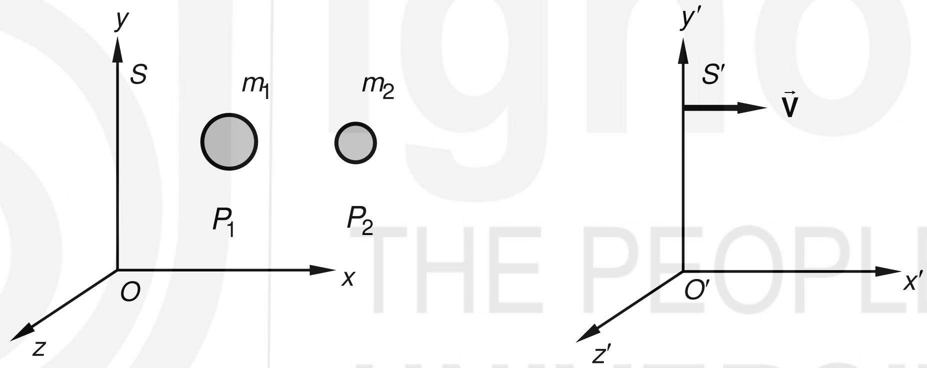


Fig. 3.1: **Two interacting particles P_1 and P_2, having masses m_1 and m_2, respectively, in frame S are being observed from the frames S and S' with parallel x, y and z-axes. The particles are moving with velocities $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$ as measured in S and $\vec{\mathbf{v}}_1'$ and $\vec{\mathbf{v}}_2'$ as measured in S'.**

Remember: These particles are moving subject to the condition that **the total mass and total linear momentum of the system are conserved in frame S**:

$$m_1 + m_2 = \text{constant} \tag{3.1a}$$

and $$m_1 \vec{\mathbf{v}}_1 + m_2 \vec{\mathbf{v}}_2 = \text{constant} \tag{3.1b}$$

We have to find **the conditions under which the total mass and total linear momentum of the system are also conserved in the frame S'**. So, we ask: **Under what conditions do the following equations hold along with Eqs. (3.1a and b)**?

$$m_1' + m_2' = \text{constant} \tag{3.2a}$$

and $$m_1' \vec{\mathbf{v}}_1' + m_2' \vec{\mathbf{v}}_2' = \text{constant} \tag{3.2b}$$

where m_1', m_2' and $\vec{\mathbf{v}}_1'$, $\vec{\mathbf{v}}_2'$ are the respective masses and velocities of the particles in frame S'. So, in a sense, we now have to find whether or not

Eqs. (3.1a, b and 3.2a, b) are compatible. For this, in the remaining part of this section, we will make use of some algebra, which you need not memorise. We are giving it so that you understand how we get the result. We will be using the following relations:

$$\gamma_1 = \frac{1}{\sqrt{1 - v_1^2/c^2}}, \quad \gamma_1' = \frac{1}{\sqrt{1 - v_1'^2/c^2}} \quad \text{and} \quad \gamma = \frac{1}{\sqrt{1 - V^2/c^2}} \tag{3.3}$$

Since the particles are moving in the positive *x*-direction, we use the velocity transformation [Eqs. (2.19a and 2.20a)] and write:

$$v_1' = \frac{v_1 - V}{1 - \frac{v_1 V}{c^2}} \tag{3.4a}$$

and

$$v_1 = \frac{v_1' + V}{1 + \frac{v_1' V}{c^2}} \tag{3.4b}$$

We will now determine a relation between $\gamma_1' v_1'$ and $\gamma_1 v_1$. Using Eq. (3.4a) we can write:

$$\gamma_1' v_1' = \left(\frac{1}{\sqrt{1 - \frac{v_1'^2}{c^2}}}\right) \times \left(\frac{v_1 - V}{1 - \frac{v_1 V}{c^2}}\right)$$

We can write the *denominator* of the second term on the RHS of the above equation as the square root of the square of the same quantity. So, we get:

$$\gamma_1' v_1' = \left(\frac{v_1 - V}{\sqrt{\left(1 - \frac{v_1 V}{c^2}\right)^2 \left(1 - \frac{v_1'^2}{c^2}\right)}}\right) \tag{3.5}$$

From Eq. (3.4a) we can also show that:

$$\left(1 - \frac{v_1 V}{c^2}\right)^2 \left(1 - \frac{v_1'^2}{c^2}\right) = \left(1 - \frac{v_1^2}{c^2}\right)\left(1 - \frac{V^2}{c^2}\right) \tag{3.6}$$

In fact, you should work these steps yourself. Try SAQ 1.

SAQ 1 – Verifying Eq. (3.6)

Verify Eq. (3.6).

Substituting Eq. (3.6) in Eq. (3.5) we get

$$\gamma_1' v_1' = \frac{v_1 - V}{\sqrt{\left(1 - \frac{v_1^2}{c^2}\right)\left(1 - \frac{V^2}{c^2}\right)}}$$

Using the results for γ_1 and γ from Eq. (3.3), we can write

$$\gamma_1' v_1' = \gamma_1 \gamma (v_1 - V) \tag{3.7a}$$

Similarly, we can show that

$$\gamma_2' v_2' = \gamma_2 \gamma (v_2 - V) \tag{3.7b}$$

Prove Eq. (3.7b) as an exercise given in SAQ 2.

SAQ 2 - **Proving Eq. (3.7b)**

Prove Eq. (3.7b).

Now remember that V and γ are constant. So, we can multiply Eq. (3.1a) by γV and the scalar form of Eq. (3.1b) by γ and their right-hand side would still be constant, though of a different value. So, we can write:

$$(m_1 + m_2)\gamma V = \text{constant} \tag{3.8a}$$

and

$$(m_1 v_1 + m_2 v_2)\gamma = \text{constant} \tag{3.8b}$$

Subtracting Eq. (3.8a) from Eq. (3.8b), we get:

$$m_1 \gamma (v_1 - V) + m_2 \gamma (v_2 - V) = \text{constant} \tag{3.9a}$$

In compact summation notation, we can write Eq. (3.9a) as follows:

$$\sum_{i=1}^{2} m_i \gamma (v_i - V) = \text{constant} \tag{3.9b}$$

You could expand Eq. (3.9b) and check that it is the same as Eq. (3.9a) before studying further. Now, from Eqs. (3.7a and b), we have

$$(v_i - V) = \frac{\gamma_i' v_i'}{\gamma_i \gamma}, \qquad (\text{for } i = 1, 2) \tag{3.10}$$

Substituting Eq. (3.10) in Eq. (3.9b), we can write:

$$\sum_{i=1}^{2} \left(m_i \frac{\gamma_i'}{\gamma_i} \right) v_i' = \text{constant} \tag{3.11}$$

Now, compare Eq. (3.11) with the scalar form of Eq. (3.2b). What are the coefficients of v_1' and v_2' in both these equations? Equating them, and writing them in compact notation, we get:

$$m'_i = m_i \frac{\gamma'_i}{\gamma_i} \qquad \text{(for } i = 1, 2\text{)} \tag{3.12a}$$

or

$$\frac{m'_i}{\gamma'_i} = \frac{m_i}{\gamma_i} \tag{3.12b}$$

Thus, **if Eqs. (3.2a and b), which represent mass and linear momentum conservation are to hold in frame** S', then the **masses** of **individual particles should satisfy Eq. (3.12a** or **3.12b)**.

What are the implications of Eq. (3.12a or 3.12b)?

Note that the quantities on the left-hand side of Eq. (3.12b) come from measurements in the frame S' and the quantities on its right-hand side from those in the frame S. These measurements are independent of each other. So, if their respective ratios are equal, then these must be equal to a constant and we can write:

$$\frac{m'_i}{\gamma'_i} = \frac{m_i}{\gamma_i} = \text{constant} = m_0, \text{ say} \tag{3.13}$$

Let us express the result of Eq. (3.13) generally. We say that: If $\vec{\mathbf{v}}$ is the velocity of a relativistic particle of mass m with respect to an inertial frame of reference, then

$$m = \gamma\, m_0 = \frac{m_0}{\sqrt{1 - \dfrac{v^2}{c^2}}}, \qquad \gamma = \frac{1}{\sqrt{1 - \dfrac{v^2}{c^2}}} \tag{3.14}$$

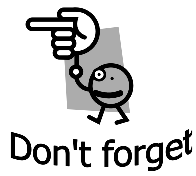
For a **particle at rest** with respect to the inertial frame, $v = 0$ and Eq. (3.14) yields

$$m = m_0 \tag{3.15}$$

The constant m_0 in Eq. (3.14 or 3.15) is termed the <u>**rest mass**</u> of the particle: It is the mass of the particle measured when it is at rest with respect to the reference frame. It is also called the **proper mass** of the particle. Eq. (3.14) tells us that **the relativistic mass of a particle depends on its velocity and increases with an increase in its velocity**.

That is why the speed of a particle having finite mass can never be equal to the speed of light. Why? Because its mass would become infinite when $v = c$ [see Eq. (3.14)]. Thus, **always remember**

Particles having finite mass can never travel with the speed of light.



This is where the special theory of relativity differs from Newtonian mechanics.

Using Eq. (3.14), the **relativistic linear momentum of a particle** is given by

$$\vec{\mathbf{p}} = m\vec{\mathbf{v}} = \frac{m_0 \vec{\mathbf{v}}}{\sqrt{1 - \dfrac{v^2}{c^2}}} \tag{3.16}$$

You may like to take the advice of Richard Feynman, a Nobel Laureate, who wrote the following about the Lorentz factor:

"For those who want to learn just enough about it so that they can solve problems, that is all there is to the theory of relativity – it just changes Newton's laws by introducing a correction factor to the mass."

This should make it easy for you to remember all important equations of relativistic dynamics.

The *x*, *y* and *z*-components of the relativistic linear momentum are:

$$p_x = \frac{m_0 v_x}{\sqrt{1-\dfrac{v^2}{c^2}}}, \qquad p_y = \frac{m_0 v_y}{\sqrt{1-\dfrac{v^2}{c^2}}}, \qquad p_z = \frac{m_0 v_z}{\sqrt{1-\dfrac{v^2}{c^2}}} \qquad (3.17a)$$

Notice that the denominator of each component in Eq. (3.17a) contains the *magnitude v* of the velocity, which is

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2} \qquad (3.17b)$$

Thus, we have found the conditions under which the law of conservation of linear momentum will hold in both inertial frames in relative motion. Under these conditions,

Mass and linear momentum have to be redefined as in Eqs. (3.14 and 3.16). Then Eqs. (3.1b and 3.2b) give the respective relativistic linear momentum conservation laws in the frames *S* and *S′*. This also means that the law of conservation of relativistic linear momentum **remains unchanged under Lorentz transformation**. Note that, now,

The **total mass of the system is conserved** (see Eqs. 3.1a and 3.2a)**. The masses of individual particles are not constant.**

The **relativistic linear momentum conservation law** has been proved to be true by experiments. Let us revise what you have learnt so far.

Recap

RELATIVISTIC LINEAR MOMENTUM CONSERVATION

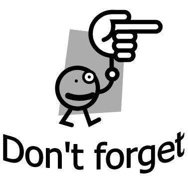
The relativistic law of conservation of linear momentum holds in all inertial frames of reference in relative motion provided mass and linear momentum in these frames are redefined as in Eqs. (3.14, 3.16). So, if the net force on a relativistic two-particle system is zero, its linear momentum is conserved:

$$m_1 \vec{\mathbf{v}}_1 + m_2 \vec{\mathbf{v}}_2 = \text{constant} \qquad (3.1b)$$

and

$$m_1' \vec{\mathbf{v}}_1' + m_2' \vec{\mathbf{v}}_2' = \text{constant} \qquad (3.2b)$$

where $m_1 + m_2 = \text{constant}$ and $m_1' + m_2' = \text{constant}$



Here m_i, m_i' are **relativistic masses**. Also,

Relativistic mass *m*: $m = \gamma m_0 = \dfrac{m_0}{\sqrt{1-\dfrac{v^2}{c^2}}}$ (3.14)

Rest mass m_0 : for $v = 0$ in Eq. (3.14), $m = m_0$ (3.15)

Relativistic linear momentum $\vec{\mathbf{p}} = m\vec{\mathbf{v}} = \dfrac{m_0 \vec{\mathbf{v}}}{\sqrt{1-\dfrac{v^2}{c^2}}}$ (3.16)

In this section, we have arrived at the definitions of relativistic mass and relativistic linear momentum for one-dimensional motion of two particles. But the same results are obtained for three-dimensional motion of a many-particle system. However, that discussion is beyond the scope of this course.

You may now like to calculate the relativistic mass of a particle given its relativistic linear momentum. Try SAQ 3.

SAQ 3 - Relativistic mass and rest mass

Calculate the relativistic mass and the rest mass of a negatively charged particle which has linear momentum of magnitude $1.92 \times 10^{-21}\ \text{kg m s}^{-1}$ at a speed of 0.99*c*.

You can now learn the equation of motion for a particle in relativistic mechanics.

3.2.2 Relativistic Equation of Motion

Recall Newton's second law of motion from the course BPHCT-131. It tells us that for a suitable choice of units, the **rate of change of linear momentum of a body is equal to the net force experienced by the body and is directed in the line of this force.** This law has the following general form, which is consistent with the special theory of relativity.

$$\vec{\mathbf{F}} = \frac{d\vec{\mathbf{p}}}{dt} = \frac{d}{dt}(m\vec{\mathbf{v}}) \tag{3.18a}$$

or

$$\vec{\mathbf{F}} = m\frac{d\vec{\mathbf{v}}}{dt} + \vec{\mathbf{v}}\frac{dm}{dt} \tag{3.18b}$$

Substituting for *m* from Eq. (3.14), we can write Eq. (3.18b) as

$$\vec{\mathbf{F}} = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}\frac{d\vec{\mathbf{v}}}{dt} + \vec{\mathbf{v}}\frac{d}{dt}\left(\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}\right) \tag{3.18c}$$

In the limit $v << c$,

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = 1$$

and since m_0 is constant,

$$\frac{dm_0}{dt} = 0$$

You can verify that in the limit as $v << c$, Eq. (3.18c) reduces to:

$$\vec{\mathbf{F}} = m_0\frac{d\vec{\mathbf{v}}}{dt} \tag{3.19}$$

which is the well known Newton's second law of motion.

Thus, Newtonian mechanics and relativistic mechanics overlap in a large domain of applications in which the speeds of objects are low compared with the speed of light, i. e., when $v << c$ or $\gamma = 1.$ When γ exceeds unity for a particle in motion, Newtonian mechanics is in slight error. But for high values of γ as observed for cosmic ray protons, Newtonian mechanics does not hold.

<u>Always Remember</u>: Within its acceptable slow motion domain, Newtonian mechanics continues to remain valid and useful.

However, there is one fundamental difference between the non-relativistic and relativistic cases as explained below:

In the non-relativistic case, it is possible to write the expression of force exerted on a body of **constant** mass as the product of its mass and its acceleration: $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$. But the relativistic equation of motion [Eq. 3.18b] shows that the force acting upon a body and its acceleration may not even be parallel.

You can see that Eq. (3.18b) is **<u>NOT</u>** equivalent to writing

$$\vec{\mathbf{F}} = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}\frac{d\vec{\mathbf{v}}}{dt} = m\vec{\mathbf{a}}$$

As defined in Eq. (3.18a), the relativistic equation of motion automatically leads to conservation of relativistic linear momentum:

RELATIVISTIC LINEAR MOMENTUM CONSERVATION

If the net external force $\vec{\mathbf{F}}$ is zero [in Eq. (3.18a)], then the relativistic linear momentum

$$\vec{\mathbf{p}} = \frac{m_0\vec{\mathbf{v}}}{\sqrt{1-\frac{v^2}{c^2}}}$$

must be constant. Hence,

In the absence of net external force on an object, its relativistic linear momentum is conserved.

We can also define **impulse** for relativistic particles.

Suppose $\vec{\mathbf{F}}$ [as defined by Eq. (3.18a)] is not zero and is exerted on a body, or a system of particles for some time. Then it changes the relativistic momentum of the body or the system by an amount $\Delta\vec{\mathbf{p}}$, which is equal to the **total impulse** given to the system:

$$\Delta\vec{\mathbf{p}} = \vec{\mathbf{J}} = \int \vec{\mathbf{F}}\,dt \qquad (3.20)$$

We will now consider an application of the relativistic equation of motion [Eqs. (3.18a, b)] for a single particle. Study Example 3.1 and work out all steps in it.

EXAMPLE 3.1 : RELATIVISTIC EQUATION OF MOTION

Let us apply the relativistic equation of motion to a charged particle moving perpendicular to a uniform magnetic field at relativistic speeds in a circular orbit. What is the radius of its orbit?

SOLUTION ■ We use Eq. (3.18b).

Recall from school physics and the course BPHET-141 that the Lorentz force on the charged particle due to the magnetic field is given by

$$\vec{\mathbf{F}} = q\vec{\mathbf{v}} \times \vec{\mathbf{B}} \tag{i}$$

It is given that the particle travels perpendicular to the magnetic field. So, the Lorentz force on the particle is:

$$\vec{\mathbf{F}} = qvB\sin 90^\circ\,\hat{\mathbf{n}} = qvB\hat{\mathbf{n}} \tag{ii}$$

where $\hat{\mathbf{n}}$ is the unit vector perpendicular to both $\vec{\mathbf{v}}$ and $\vec{\mathbf{B}}$ (recall the definition of cross product from Unit 2 of Block 1, BPHCT-131). Also recall the concept of 'no-work' force from Unit 9 of BPHCT-131. Recall that a force acting perpendicular to the particle's displacement does zero work on the particle (read the margin remark). So, from work-energy theorem, it follows that the particle's speed does not change under the action of such a force. Since the speed of the particle remains constant, you can see from Eq. (3.14) that its mass remains constant. Therefore,

Note that it is given in the problem that the particle is moving in a circular orbit perpendicular to the magnetic field. Recall that for circular motion, the velocity and displacement of the particle lie in the same plane. Since the force is perpendicular to the particle's velocity, it is also perpendicular to its displacement.

$$\frac{dm}{dt} = 0 \tag{iii}$$

for the charged particle and its equation of motion reduces to

$$\vec{\mathbf{F}} = m\frac{d\vec{\mathbf{v}}}{dt} = \gamma m_0 \frac{d\vec{\mathbf{v}}}{dt} \tag{iv}$$

Substituting the expression for Lorentz force from Eq. (ii) in Eq. (iv), we get:

$$qvB\hat{\mathbf{n}} = \gamma m_0 \frac{d\vec{\mathbf{v}}}{dt} = \gamma m_0 \vec{\mathbf{a}} \tag{v}$$

Since the particle is moving in a circle of radius, say r, the acceleration is simply the centripetal acceleration and substituting $a = v^2/r$ in Eq. (v), we get:

$$qvB = \frac{\gamma m_0 v^2}{r} \tag{vi}$$

$$\therefore \qquad r = \frac{\gamma m_0 v}{qB} \tag{vii}$$

This is of the same form as the classical expression for the radius of the orbit $\left(\dfrac{m_0 v}{qB}\right)$ except that it is γ times larger in this case. Note further that the charged particle's mass increases with an increase in its speed.

The increase in radius by a factor γ explains why high-energy charged particle accelerators are so huge in size (see Fig. 3.2).

Fig. 3.2: **Aerial view of the particle accelerator Tevatron at Fermilab, USA. Until 2007 it was the most powerful particle accelerator in the world, which accelerated protons to energies of over 1 TeV (tera electron volts $= 10^{12}$ eV). Beams of circulating protons in two circular vacuum chambers in the two (6.28 km) rings shown in the picture collided at their intersection point.**

You may now like to work out an SAQ to consolidate your understanding of the relativistic equation of motion.

SAQ 4 – **Relativistic equation of motion**

Show that the relativistic force required to give acceleration $\vec{\mathbf{a}}$ to a particle in the direction of its motion is given by

$$\vec{\mathbf{F}} = \gamma^3 m_0 \vec{\mathbf{a}}$$

We now discuss the crucial step taken by Einstein in developing the concept of relativistic energy. This greatly extended the impact of the special theory of relativity. In doing so, Einstein arrived at, perhaps, the most famous equation of all times: $E = mc^2$, which demonstrates the equivalence of mass and energy.

3.3 MASS-ENERGY EQUIVALENCE

In this section, we will first derive the mass-energy relationship using a simple argument and then present a more formal approach.

Consider two inertial reference frames S and S' in uniform relative motion (Fig. 3.3). Let frame S be at rest and frame S' move with a constant speed V with respect to S in the positive x-direction. Note that there are two observers P_1 and P_2 in frame S and one observer P in frame S'. Now, consider an atom (A) at rest in the frame S'. Suppose, it emits two photons of the same frequency ν at the same instant but in opposite directions as shown in Fig. 3.3.

Since the photons are moving in **opposite** directions with the same speed (c), their total linear momentum as measured in the frame S' by the observer P will be zero. Therefore, from the law of conservation of linear momentum, the linear momentum of the atom in S' will always be zero.

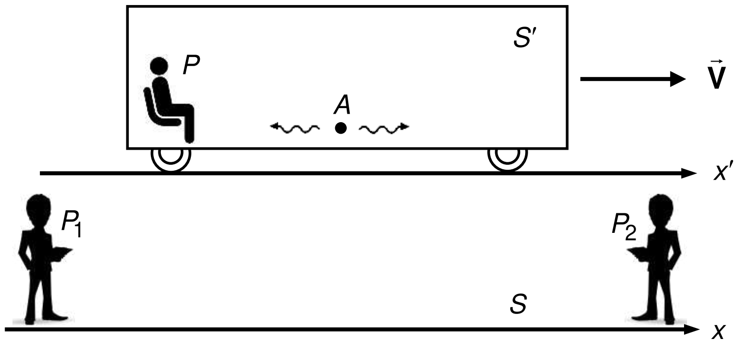


Fig. 3.3: With respect to the observer P (at rest) in the frame S', the atom (A) is at rest and emits two photons of the same frequency in opposite directions at the same instant. Two observers P_1 and P_2 at rest in frame S measure Doppler shifted frequencies of the two photons.

From school physics, you know that the energy of a photon of frequency ν is equal to $h\nu$, where h is Planck's constant. Since the atom emits two photons of energy $h\nu$ each, the change in its energy (in the frame S') is given by

$$(\Delta E)_{S'} = 2h\nu \tag{3.21}$$

Now, what do the two observers in frame S measure the change in the atom's energy as? Let us find out.

Note that with respect to P_1, the source of light (the atom) is moving **away from** the observer and with respect to P_2, the source is moving **towards** the observer. So, the observers will measure Doppler shifts in the respective frequencies of photons reaching them.

What are the Doppler shifted frequencies of the photons measured by the two observers in S? Recall Eqs. (2.26a and b) of Unit 2. From Eq. (2.26a), the frequency of the photon measured by the observer P_1 will be:

$$\nu_1 = \nu\left[\frac{1 - V/c}{1 + V/c}\right]^{1/2} \quad \text{(Source and observer receding)} \tag{3.22a}$$

And the frequency of the photon measured by the observer P_2 will be:

$$\nu_2 = \nu\left[\frac{1 + V/c}{1 - V/c}\right]^{1/2} \quad \text{(Source and observer approaching)} \tag{3.22b}$$

Thus, the **change** in the energy of the atom as measured in frame S (which is the sum of the energies of photons observed in S) will be

$$(\Delta E)_S = h\nu_1 + h\nu_2 = h\nu\left(\left[\frac{1 - V/c}{1 + V/c}\right]^{1/2} + \left[\frac{1 + V/c}{1 - V/c}\right]^{1/2}\right) \tag{3.23}$$

$$\left[\frac{1 - V/c}{1 + V/c}\right]^{1/2} + \left[\frac{1 + V/c}{1 - V/c}\right]^{1/2}$$

$$= \frac{1 - V/c + 1 + V/c}{\sqrt{1 - V^2/c^2}}$$

$$= \frac{2}{\sqrt{1 - V^2/c^2}}$$

Simplifying Eq. (3.23), we get (read the margin remark):

$$(\Delta E)_S = \frac{2h\nu}{\sqrt{1 - \frac{V^2}{c^2}}} = 2h\nu\gamma, \text{ where } \gamma = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}} \tag{3.24a}$$

Using $(\Delta E)_{S'} = 2h\nu$ from Eq. (3.21) in Eq. (3.24), we get:

$$(\Delta E)_S = \gamma (\Delta E)_{S'} \tag{3.24b}$$

Now, let us apply the law of conservation of linear momentum to the system of (atom + two photons) in frame S. The *linear momentum* of the atom must be equal to the sum of the linear momenta of the two photons (taking into account their direction of motion).

Note that the atom is moving with speed V with respect to S. Recall from school physics that the linear momentum of the photon is given by the relation $E = pc$ or $p = \frac{h\nu}{c}$. So, the linear momentum of the two photons will be different as these have different frequencies in the reference frame S.

Since the photons are travelling in different directions, the total linear momentum of the two photons in S is given by:

$$\Delta p = \frac{h\nu_2}{c} - \frac{h\nu_1}{c} \tag{3.24c}$$

$$\frac{h\nu_2}{c} - \frac{h\nu_1}{c}$$

$$= \frac{h\nu}{c}\left(\left[\frac{1 + V/c}{1 - V/c}\right]^{1/2} - \left[\frac{1 - V/c}{1 + V/c}\right]^{1/2}\right)$$

$$= \frac{h\nu}{c}\left(\frac{1 + V/c - 1 + V/c}{\sqrt{1 - V^2/c^2}}\right)$$

$$= \frac{h\nu}{c}\left(\frac{2V/c}{\sqrt{1 - V^2/c^2}}\right)$$

$$= \frac{2h\nu}{\sqrt{1 - V^2/c^2}}\left(\frac{V}{c^2}\right)$$

This should be equal to the linear momentum of the atom (after emitting the photons), which is the product of its mass and velocity in the frame S. Let us denote the atom's final mass in frame S by $(\Delta m)_S$. Then the linear momentum of the atom in frame S is $(\Delta m)_S V$.

From the law of conservation of linear momentum, we can write for frame S:

$$(\Delta m)_S V = \frac{h\nu_2}{c} - \frac{h\nu_1}{c} \tag{3.24d}$$

Using Eqs. (3.22a and b) and simplifying the expression (read the margin remark), we can write Eq. (3.24d) as:

$$(\Delta m)_S V = \frac{2h\nu}{\sqrt{1 - \frac{V^2}{c^2}}}\left(\frac{V}{c^2}\right) \tag{3.24e}$$

Using the result $(\Delta E)_S = 2h\nu\gamma$ from Eq. (3.24a) in Eq. (3.24e), we get:

$$(\Delta m)_S V = (\Delta E)_S \left(\frac{V}{c^2}\right) \tag{3.24f}$$

or
$$(\Delta E)_S = (\Delta m)_S c^2 \tag{3.25}$$

Note that Eq. (3.25) is independent of the value of V. Now, in Eq. (3.25), ΔE and Δm need not be infinitesimal quantities. Then, we can write the **mass-energy relation** as follows:

$$\boxed{E = mc^2} \tag{3.26}$$

What do Eqs. (3.25 and 3.26) tell us? These equations tell us that **whenever the mass of a system changes, the difference in the mass gets converted to energy**. For example, if a hydrogen atom emits a photon while making a transition from an excited state to its ground state, its mass will decrease by a small amount (although it still consists of the proton and electron) and be converted into energy. For the sake of completeness, we now derive this equation formally. We first derive the expression for relativistic kinetic energy using the work-energy theorem and then arrive at mass-energy equivalence given by Eq. (3.26) once again.

3.3.1 Relativistic Kinetic Energy

Recall the work-energy theorem from Unit 9 (Block 2) of BPHCT-131, which relates the work done by the net external force on a particle and the change in its kinetic energy. Suppose a net external force $\vec{\mathbf{F}}$ does work on a particle and it is displaced by $d\vec{\mathbf{l}}$ as it moves from some point A to point B in space. Let the speed of the particle increase from some value v_A to v_B. Then from the work-energy theorem, we can write:

$$T_B - T_A = \int_A^B \vec{\mathbf{F}} . d\vec{\mathbf{l}} \tag{3.27a}$$

where
$$\vec{\mathbf{F}} = \frac{d\vec{\mathbf{p}}}{dt} = \frac{d}{dt}\left[\frac{m_0\,\vec{\mathbf{v}}}{\left(1 - v^2/c^2\right)^{1/2}}\right] \tag{3.27b}$$

Here we have used the expression for linear momentum from Eq. (3.16) to write Eq. (3.27b). As you know, T_A and T_B are the kinetic energies of the particle at points A and B, respectively. Substituting the expression of force from Eq. (3.27b) in the right-hand side of Eq. (3.27a), we get:

$$T_B - T_A = \int_A^B \frac{d\vec{\mathbf{p}}}{dt} . d\vec{\mathbf{l}} \tag{3.27c}$$

Now, you know from calculus that we can write the differential $d\vec{\mathbf{l}}$ as $d\vec{\mathbf{l}} = \frac{d\vec{\mathbf{l}}}{dt}dt = \vec{\mathbf{v}}dt.$ Hence, we can write Eq. (3.27c) as follows:

$$T_B - T_A = \int_A^B \frac{d\vec{\mathbf{p}}}{dt} . \vec{\mathbf{v}}\, dt = \int_A^B \vec{\mathbf{v}} . \frac{d\vec{\mathbf{p}}}{dt} dt \qquad \left(\because\ \vec{\mathbf{a}}.\vec{\mathbf{b}} = \vec{\mathbf{b}}.\vec{\mathbf{a}}\right)$$

or
$$T_B - T_A = \int_A^B \vec{\mathbf{v}} . d\vec{\mathbf{p}} \qquad \left(\because\ \frac{d\vec{\mathbf{p}}}{dt} dt = d\vec{\mathbf{p}}\right) \tag{3.27d}$$

We can rewrite the integrand in Eq. (3.27d) making use of the following relation:

$$d(\vec{\mathbf{v}}.\vec{\mathbf{p}}) = \vec{\mathbf{v}}.d\vec{\mathbf{p}} + \vec{\mathbf{p}}.d\vec{\mathbf{v}} \quad \text{or} \quad \vec{\mathbf{v}}.d\vec{\mathbf{p}} = d(\vec{\mathbf{v}}.\vec{\mathbf{p}}) - \vec{\mathbf{p}}.d\vec{\mathbf{v}}$$

Using this relation, we can write Eq. (3.27d) as follows:

$$T_B - T_A = \int_A^B [d(\vec{\mathbf{v}}.\vec{\mathbf{p}}) - \vec{\mathbf{p}}.d\vec{\mathbf{v}}] \tag{3.27e}$$

Integrating the first term on the RHS of Eq. (3.27e), we get:

$$T_B - T_A = (\vec{\mathbf{v}}.\vec{\mathbf{p}})\Big|_A^B - \int_A^B \vec{\mathbf{p}}.d\vec{\mathbf{v}} \tag{3.28a}$$

We now substitute the expression for relativistic linear momentum from Eq. (3.16) in Eq. (3.28a) and get:

$$T_B - T_A = \frac{m_0 v^2}{\left(1 - v^2/c^2\right)^{1/2}}\Bigg|_A^B - \int_A^B \left[\frac{m_0\, \vec{\mathbf{v}}.d\vec{\mathbf{v}}}{\left(1 - v^2/c^2\right)^{1/2}}\right] \tag{3.28b}$$

Next, we use the following result:

$$d\left(\frac{\vec{\mathbf{v}}.\vec{\mathbf{v}}}{2}\right) = \vec{\mathbf{v}}.\frac{d\vec{\mathbf{v}}}{2} + \frac{d\vec{\mathbf{v}}.\vec{\mathbf{v}}}{2} = \vec{\mathbf{v}}.d\vec{\mathbf{v}}$$

or

$$\vec{\mathbf{v}}.d\vec{\mathbf{v}} = d\left(\frac{\vec{\mathbf{v}}.\vec{\mathbf{v}}}{2}\right) = d\left(\frac{v^2}{2}\right) = \frac{2v}{2}dv = vdv$$

and write Eq. (3.28b) as:

$$T_B - T_A = \frac{m_0 v^2}{\left(1 - v^2/c^2\right)^{1/2}}\Bigg|_A^B - \int_A^B \left[\frac{m_0\, vdv}{\left(1 - v^2/c^2\right)^{1/2}}\right] \tag{3.28c}$$

We can solve the second integral on the right-hand side of Eq. (3.28c) by making the change of variable $\left(1 - v^2/c^2\right) = t$ (read the margin remark). Then we get

With the change of variable $\left(1 - v^2/c^2\right) = t$ we have $\frac{-2vdv}{c^2} = dt$

We can write the integral on the RHS of Eq. (3.28c) as:

$$-\int_A^B \left[\frac{m_0\, vdv}{\left(1 - v^2/c^2\right)^{1/2}}\right] = \frac{m_0 c^2}{2}\int_A^B \frac{dt}{\sqrt{t}} = \frac{m_0 c^2}{2} 2\sqrt{t}\Big|_A^B = m_0 c^2 \sqrt{1 - v^2/c^2}\Big|_A^B$$

$$T_B - T_A = \frac{m_0 v^2}{\left(1 - v^2/c^2\right)^{1/2}}\Bigg|_A^B + m_0 c^2 \left(1 - v^2/c^2\right)^{1/2}\Big|_A^B \tag{3.28d}$$

To simplify Eq. (3.28d), we assume that the particle is at rest at the point A, so that $v_A = 0$ and B is any arbitrary point, at which $v_B = v$.

Then, the **relativistic kinetic energy** of the particle **starting from rest** is

$$T = \frac{m_0 v^2}{\left(1 - v^2/c^2\right)^{1/2}} + m_0 c^2 \left(1 - v^2/c^2\right)^{1/2} - m_0 c^2$$

$$= \frac{m_0 v^2 + m_0 c^2 \left(1 - v^2/c^2\right)}{\left(1 - v^2/c^2\right)^{1/2}} - m_0 c^2$$

or $$T = \frac{m_0 c^2}{\left(1 - v^2/c^2\right)^{1/2}} - m_0 c^2 \qquad (3.29a)$$

or $$T = \gamma m_0 c^2 - m_0 c^2 = m_0 c^2 (\gamma - 1) \qquad (3.29b)$$

Using the expression of relativistic mass from Eq. (3.14) in Eq. (3.29b), we can write the expression for **relativistic kinetic energy** as:

$$T = mc^2 - m_0 c^2 \qquad (3.29c)$$

The term $m_0 c^2$ in Eqs. (3.29a, b, c) is called the **rest mass energy** of the particle.

You can see that in the limit $v << c$, Eq. (3.29c) tends to the classical expression of kinetic energy: $T = \frac{1}{2} m_0 v^2$. Why don't you work it out quickly?

SAQ 5 - **Classical limit of relativistic kinetic energy**

Show that in the limit $v << c$, the expression of relativistic kinetic energy approaches that of the classical kinetic energy.

We will now present the interpretation of Eq. (3.29c) given by Einstein to arrive at **the equivalence of mass and energy**.

The kinetic energy T [Eq. (3.29c)] of a particle arises from the work done on it to bring it to speed v from rest. We now rewrite (Eq. 3.29c) as

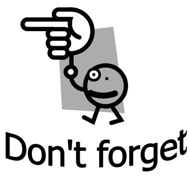
$$mc^2 = T + m_0 c^2 \qquad (3.29d)$$

$$= \text{work done on the particle to change its speed} + m_0 c^2$$

where $m_0 c^2$ in Eq. (3.29d) is the rest mass energy of the particle.

Einstein proposed a bold interpretation for Eq. (3.29d) as follows, which **you should always remember**:

$$\boxed{mc^2 \text{ is the } \textbf{total energy } E \text{ of the particle.}}$$



Let us explain this point further.

The first term on the RHS in Eq. (3.29d) arises from (external) work done on the particle by the net external force. The second term, $m_0 c^2$, represents the "rest" energy the particle possessed by virtue of its mass. Thus, we get:

$$E = mc^2 \qquad (3.26)$$

which is just Eq. (3.26). Let us further understand the meaning of Eq. (3.26).

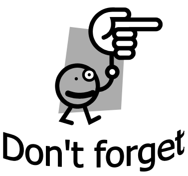
Eq. (3.26) states that **mass and energy are equivalent** – mass and energy are different names for the same quantity, which we can call mass-energy. Anything that has a mass m has an energy $E\ (= mc^2)$ and anything that has an energy E has a mass $(= E/c^2)$. If energy ΔE is added to or taken away from an object, its mass will change by $\Delta m = \Delta E/c^2$, whatever be the form of energy:

$$\Delta m = \Delta E/c^2 \tag{3.30}$$

So, ΔE could represent change in mechanical energy or heat energy, absorption/emission of light, or any other form of energy. Thus, Einstein's generalisation of the conservation of energy goes far beyond the classical conservation law of mechanical energy.

One of the most significant consequences of mass-energy equivalence is that the law of conservation of total mass of a system (discussed in Sec. 3.2) automatically implies the law of conservation of total energy of the system. These two classical laws then merge into **one relativistic law of conservation of mass-energy** given by Eq. (3.26).

Let us write the results obtained in this section at one place before proceeding further. You should always remember these equations as you will be solving problems based on them.



MASS-ENERGY EQUIVALENCE

Mass and energy are equivalent. Anything that has a mass m has an energy $E\ (= mc^2)$ and anything that has an energy E has a mass $(= E/c^2)$:

$$E = mc^2 \tag{3.26}$$

where

$$m = \frac{m_0 c^2}{\left(1 - v^2/c^2\right)^{1/2}} \tag{3.14}$$

If energy ΔE is added to or taken away from an object, its mass will change by

$$\Delta m = \Delta E/c^2 \tag{3.30}$$

The **kinetic energy** of a relativistic particle moving at speed v is given by

$$T = mc^2 - m_0 c^2 \tag{3.29c}$$

The **rest mass energy** of the particle is given by

$$m_0 c^2$$

Eq. (3.26) embodying the equivalence of mass and energy has had a profound influence on our understanding of the universe. Perhaps its most dramatic examples are provided by the promise of unlimited energy supply through nuclear fission and nuclear fusion.

On the other hand, the threat of destruction by nuclear (atomic) bombs has also loomed large over us. The atomic bombs that destroyed Hiroshima and Nagasaki involved the annihilation of just a few grams of matter. In this sense, special theory of relativity becomes not just the concern of scientists alone but of the entire humanity as it may well control our destiny!

We now take up a few interesting examples of mass-energy equivalence to demonstrate the magnitudes of energy involved.

Fig. 3.4: The Kudankulam nuclear power plant in Tamil Nadu has a capacity of 2000 MW.

EXAMPLE 3.2 : MASS-ENERGY EQUIVALENCE

a) Calculate the relativistic kinetic energy of an object of mass 100 kg if it travels with a speed of 0.80*c*.

b) What is the rest mass energy of a proton given that its rest mass is 1.67×10^{-27} kg? Calculate the relativistic kinetic energy of a proton that moves with a speed of 0.99999999*c* in a particle accelerator.

c) In the Sun, about 4.0×10^{9} kg of mass gets converted into energy every second. What is the total energy (per s) released in the process?

SOLUTION ■ We use Eq. (3.29b) or Eq. (3.26/3.30).

a) We use Eq. (3.29b).

$$\gamma = \frac{1}{\left(1 - v^2/c^2\right)^{1/2}} = \frac{1}{\left(1 - (0.80)^2\right)^{1/2}} = \frac{1}{0.6} = \frac{5}{3}$$

and $T = m_0 c^2 (\gamma - 1) = 100\,\text{kg} \times (3.0\times 10^8\,\text{ms}^{-1})^2 \times \frac{2}{3} = 6.0\times10^{18}\,\text{J}$

This is a huge amount of energy. You can get an idea about the magnitude by comparing it with the power output of the largest nuclear power plant, the Kudankulam power plant in Tamil Nadu, which has a capacity of 2000 MW (Fig. 3.4).

How long will a 2000 MW plant take to generate the energy 6.0×10^{18} J?

Suppose it takes *N* years. There are 31536000 s in 1 year. Using the relation, Power $=$ Energy/ time,

or

Energy $=$ Power $\times$ time,

we get,

$(2000\times10^{6}\,\text{W}) \times N \times 31536000\,\text{s} = 6.0\times10^{18}\,\text{J}$

or

$$N = \frac{6.0\times10^{18}\,\text{J}}{2000\times10^{6}\,\text{W}\times31536000\,\text{s}} = 95\,\text{years}$$

So, with 100 kg fuel, the Kudankulam nuclear power plant can supply energy up to 95 years if it operates at full capacity.

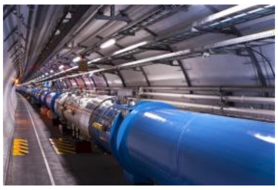
a) Rest mass energy of the proton is:

$$m_0 c^2 = 1.67\times 10^{-27}\,\text{kg} \times (3.0\times 10^8\,\text{ms}^{-1})^2$$

$$= 1.5\times10^{-10}\,\text{J} = 938\,\text{MeV} \quad (\because\ 1\,eV = 1.6\times10^{-19}\,\text{J})$$

The kinetic energy of the proton at a speed of 0.99999999*c* is given by Eq. (3.29b). For this speed:

$$\gamma = \frac{1}{\left(1 - v^2/c^2\right)^{1/2}} = \frac{1}{\left(1 - (0.99999999)^2\right)^{1/2}} = 7071$$



(a)

(b)

(c)

Fig. 3.5: The Large Hadron Collider (LHC) is the world's largest particle collider and the largest machine. It lies in a tunnel 27 km in circumference, beneath the France-Switzerland border near Geneva. It accelerates protons up to 13 TeV total collision energy. The collider has seven detectors. The pictures above show a) a section of the LHC; b) the proton source, red cylinder; c) view inside a detector.

$$\therefore \qquad T = m_0 c^2 (\gamma - 1) = 938\,\text{MeV} \times 7070 = 6631660 \text{ MeV}$$

$$\text{or} \qquad T = 6.6\,\text{TeV} \qquad (\text{where } 1 \text{ TeV} = 10^{12} \text{ eV})$$

Such extremely high energy protons are used in the Large Hadron Collider (LHC) to detect elementary particles and verify theories in particle physics (see Fig. 3.5). If you wish to know more about LHC, you may like to visit the webpage: https://en.wikipedia.org/wiki/Large_Hadron_Collider

c) From Eq. (3.26), the energy released by the Sun per second is:

$$E = mc^2 = 4.0 \times 10^9 \text{ kg} \times (3.0 \times 10^8 \text{ ms}^{-1})^2 = 3.6 \times 10^{26} \text{ J}$$

You should now solve an SAQ to check your understanding.

SAQ 6 - Mass-energy equivalence

Calculate the relativistic mass of a free proton having relativistic kinetic energy 200 MeV given that its rest mass is $938 \text{ MeV}/c^2$.

The experimental evidence for mass-energy equivalence [Eq. (3.26/3.30)] first came from the mass defect of atomic nuclei about which you will learn in Block 4 of this course. Mass defect represents the deficit in mass of a nucleus compared with the sum of the masses of its constituent nucleons. Mass-energy equivalence is evident in the processes of nuclear fusion (fusion of light atomic nuclei) and nuclear fission (breaking up of heavy atomic nuclei).

The basic concept underlying these phenomena is that of **binding energy** of nuclei. In Block 4, you will learn the concept of binding energy and calculate its values for many atomic nuclei using the mass-energy equation. You will also learn about nuclear fission and fusion. Nuclear fusion is the source of energy in all stars.

The first direct experimental verification of mass-energy equivalence was done for the following nuclear fusion reaction:

$${}_3\text{Li}^7 + {}_1\text{H}^1 \rightarrow 2\,{}_2\text{He}^4 + \Delta E$$

With a mass defect of 3.0886×10^{-29} kg, it has an equivalent energy of $\Delta mc^2 = 27.8 \times 10^{-13}$ J or 27.8×10^{-6} erg $(\because 1\text{ J} = 10^7 \text{ erg})$. Measurements of the difference between the total kinetic energy of the α-particles produced and that of the incident protons gave the value $(27.6 \pm 0.05) \times 10^{-6}$ erg. This was in excellent agreement with the theoretical result. Many experiments have now fully confirmed Eq. (3.26). So, we hope that you appreciate the importance of mass-energy equivalence.

In the next section, we will use Eqs. (3.16) and (3.26), and express the total energy of a **free particle** in terms of its linear momentum. In doing so, we will arrive at an interesting result about massless particles.

3.3.2 Relativistic Energy and Linear Momentum of a Free Particle

You have learnt in Unit 9 of the course BPHCT-131 (Mechanics) that the kinetic energy of a free particle is given by:

$$K.E. = \frac{1}{2}mv^2 \tag{3.31a}$$

We can write kinetic energy of a free particle in terms of its linear momentum $(p = mv)$ as

$$K.E. = \frac{1}{2}m\left(\frac{p}{m}\right)^2 = \frac{p^2}{2m} \tag{3.31b}$$

For a free particle moving at relativistic speeds (speeds close to the speed of light), we combine Eqs. (3.16) and (3.26) and arrive at a corresponding relation. Squaring Eq. (3.16) for linear momentum, we get:

$$p^2 = \frac{m_0^2 v^2}{\left(1 - \frac{v^2}{c^2}\right)} \tag{3.32a}$$

We write Eq. (3.32a) as:

$$p^2\left(1 - \frac{v^2}{c^2}\right) = m_0^2 v^2$$

$$= m_0^2 \frac{v^2}{c^2} c^2$$

Thus, we have:

$$\frac{v^2}{c^2}\left(m_0^2 c^2 + p^2\right) = p^2$$

Using some algebra (read the margin remark), we can simplify Eq. (3.32a) to write:

$$\frac{v^2}{c^2} = \frac{p^2}{p^2 + m_0^2 c^2} \tag{3.32b}$$

$$\therefore \qquad \gamma = \frac{1}{\left(1 - v^2/c^2\right)^{1/2}} = \left(1 + \frac{p^2}{m_0^2 c^2}\right)^{1/2} \tag{3.32c}$$

Verify Eq. (3.32c) before studying further. Now, from Eq. (3.26),

$$E = mc^2 = \gamma m_0 c^2 \tag{3.32d}$$

Substituting for γ from Eq. (3.32c) in Eq. (3.32d), we get:

$$E = m_0 c^2 \left(1 + \frac{p^2}{m_0^2 c^2}\right)^{1/2} \tag{3.32e}$$

Squaring both sides of Eq. (3.32e) and rearranging the terms, we get:

$$\boxed{E^2 = p^2c^2 + \left(m_0 c^2\right)^2} \tag{3.33}$$

You should work out the intermediate steps in arriving at Eq. (3.33) from Eq. (3.32e) yourself. Eq. (3.33) is the **relativistic energy-momentum relation for a free particle**.

A consequence of this relation is the possibility of '**massless**' particles – particles which possess energy and momentum but no rest mass. Do you remember encountering an example of such a particle earlier in this unit? It is the photon! Let us briefly discuss this.

<u>Massless Particles</u>

Now there is an interesting additional effect contained in Eq. (3.33). For a massless particle, we put $m_0 = 0$ in it and get:

$$E = pc \tag{3.34}$$

Thus, we conclude that massless particles possess energy given by Eq. (3.34). Hence the momentum of a massless particle such as the photon is given by

$$p = \frac{E}{c} = \frac{h\nu}{c} \tag{3.35}$$

Let us explain this result further.

If a particle has no rest mass $(m_0 = 0)$ **<u>and</u>** it is at rest $(p = 0)$, then from Eq. (3.33), its total energy is zero $(E = 0)$. But an object with zero energy and zero mass is nothing at all. Therefore, **if an object with no mass is to physically exist, it can never be at rest**. It travels with the speed of light. Such is the case with the photon. **All massless particles must travel at speed of light *c* in all reference frames. They can never be at rest.**

Photon is such an object. But photon is not the only massless object. Gluons and the hypothetical gravitons are also massless, and therefore travel at speed *c* in all frames. Conversely,

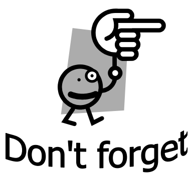
Particles which travel at the speed of light have zero rest mass.

Always remember:

All massless particles travel at the speed of light *c* in all reference frames. Particles which travel at the speed of light have zero rest mass.

We now write the results obtained in this section at one place. You should memorise these equations as you will be solving problems based on them.

RELATIVISTIC FREE PARTICLE



The **relativistic energy-momentum** relation for a free particle is:

$$E^2 = p^2c^2 + \left(m_0 c^2\right)^2 \tag{3.26}$$

The **energy of a massless particle** $(m_0 = 0)$ is related to its **linear momentum** as:

$$E = pc \tag{3.34}$$

whence $$p = \frac{E}{c} = \frac{h\nu}{c} \tag{3.35}$$

Here *h* is Planck's constant and its value is 6.626×10^{-34} Js.

We end this section with an SAQ for you.

SAQ 7 - Massless particles

What is the linear momentum of a photon of wavelength 5.0×10^{-7} m?

Let us now sum up what you have learnt in this unit.

3.4 SUMMARY

Concept	Description
Relativistic linear momentum	■ The **relativistic linear momentum** of a particle of rest mass m_0 moving with velocity $\vec{\mathbf{v}}$ is defined as: $$\vec{\mathbf{p}} = m\vec{\mathbf{v}} = \frac{m_0\,\vec{\mathbf{v}}}{\left(1 - v^2/c^2\right)^{1/2}} = \gamma\, m_0\, \vec{\mathbf{v}}, \quad \gamma = \frac{1}{\left(1 - v^2/c^2\right)^{1/2}}$$ where m is the relativistic mass of the particle defined as: $$m = \frac{m_0}{\left(1 - v^2/c^2\right)^{1/2}} = \gamma\, m_0$$
Relativistic equation of motion	■ The **relativistic equation of motion** is given by $$\vec{\mathbf{F}} = \frac{d\vec{\mathbf{p}}}{dt}$$ where $\vec{\mathbf{p}}$ is the relativistic linear momentum.
Conservation of linear momentum	■ If the net external force being exerted on a system is zero, the relativistic linear momentum is conserved.
Relativistic impulse	■ If a force is exerted on an object for a given time, it produces a change in its relativistic linear momentum, equal to its **relativistic impulse** defined as: $$\Delta\vec{\mathbf{p}} = \vec{\mathbf{J}} = \int \vec{\mathbf{F}}\, dt$$
Relativistic energy and mass-energy equivalence	■ The **total relativistic energy** of a particle is the sum of its **relativistic kinetic energy** and **rest mass energy**: $$E = T + m_0 c^2$$ where the relativistic kinetic energy is given by: $$T = mc^2 - m_0 c^2$$ This results in an equivalence between mass and energy represented by the equations: $$E = mc^2 \quad \text{and} \quad \Delta E = \Delta mc^2$$

Relativistic energy and linear momentum of a free particle

- The **relativistic energy and linear momentum of a free particle** are related by:

$$E^2 = p^2c^2 + \left(m_0 c^2\right)^2$$

The energy of a particle of zero rest mass (massless particle, $m_0 = 0$), is related to its linear momentum as:

$$E = pc$$

whence $p = \frac{E}{c} = \frac{h\nu}{c}$

All massless particles travel at the speed of light in all reference frames.

Conversely, particles that travel at the speed of light have zero

3.5 TERMINAL QUESTIONS

1. What is the mass of an object moving at the speed 0.9999c if its rest mass is 2.0×10^{-28} kg?
2. Calculate the magnitude of the linear momentum of an electron moving at the speed 0.99c given that its rest mass is 9.11×10^{-31} kg.
3. Calculate the rest mass of a particle which has a linear momentum of magnitude 1.0×10^{-20} kg m s^{-1} and is moving at the speed 0.90c.
4. The rest mass of proton is 1.67×10^{-27} kg. Determine the magnitude of the force needed to give it an acceleration of 1.0×10^{15} m s^{-2} in the direction of motion when $v = 0.90c$.
5. A particle of rest mass 5.0 kg has an initial speed of 2.0×10^{8} m s^{-1}. A constant relativistic force of magnitude 1.0×10^{6} N is exerted on the particle in the same direction as the initial relativistic momentum for 1000 s. Calculate the magnitude of the initial relativistic linear momentum of the particle.
6. Calculate the magnitude of the final relativistic linear momentum of the particle in Terminal Question 5.
7. Calculate the final speed of the particle in Terminal Question 5.
8. Calculate the linear momentum, kinetic energy and total energy (in J and MeV) of a proton travelling with speed 0.80c given that its rest energy is 938 MeV and 1 MeV $= 1.6 \times 10^{-13}$ J. Take the rest mass of the proton as $m_p = 1.67 \times 10^{-27}$ kg.
9. A free proton travels at the speed of 0.999c in a laboratory. Calculate its energy and linear momentum in an inertial frame travelling in the same direction as the proton at the speed of 0.990c with respect to the laboratory. Proton rest energy is 938 MeV and 1 MeV $= 1.6 \times 10^{-13}$ J.

10. Calculate the linear momentum of a free proton when its total energy is twice its rest mass energy. It is given that the rest mass of proton is 1.67×10^{-27} kg. Take $c = 3.0\times10^{8}\,\text{ms}^{-1}$.

3.6 SOLUTIONS AND ANSWERS

Self-Assessment Questions

1. We have to show that

$$\left(1-\frac{v_1V}{c^2}\right)^2\left(1-\frac{v_1'^2}{c^2}\right) = \left(1-\frac{v_1^2}{c^2}\right)\left(1-\frac{V^2}{c^2}\right)$$

Using Eq. (3.4a), we can write

$$\left(1-\frac{v_1'^2}{c^2}\right) = 1-\frac{1}{c^2}\left(\frac{v_1-V}{1-\dfrac{v_1V}{c^2}}\right)^2$$

$$= \frac{c^2\left(1-\dfrac{v_1V}{c^2}\right)^2-(v_1-V)^2}{c^2\left(1-\dfrac{v_1V}{c^2}\right)^2}$$

or $$\left(1-\frac{v_1V}{c^2}\right)^2\left(1-\frac{v_1'^2}{c^2}\right) = \left(1-\frac{v_1V}{c^2}\right)^2-\frac{1}{c^2}(v_1-V)^2$$

Simplifying the right-hand side of the above equation, we get:

$$\left(1-\frac{v_1V}{c^2}\right)^2\left(1-\frac{v_1'^2}{c^2}\right) = 1+\frac{v_1^2V^2}{c^4}-2\frac{v_1V}{c^2}-\frac{v_1^2}{c^2}-\frac{V^2}{c^2}+\frac{2v_1V}{c^2}$$

$$= 1+\frac{v_1^2V^2}{c^4}-\frac{v_1^2}{c^2}-\frac{V^2}{c^2}$$

$$= \left(1-\frac{V^2}{c^2}\right)-\frac{v_1^2}{c^2}\left(1-\frac{V^2}{c^2}\right) = \left(1-\frac{v_1^2}{c^2}\right)\left(1-\frac{V^2}{c^2}\right)$$

2. We have to show that

$$\gamma_2'v_2' = \gamma_2\gamma\,(v_2-V)$$

Let us write the expressions for γ_2, γ_2', v_2' and v_2 on the lines of Eqs. (3.3, 3.4a, b) as follows:

$$\gamma_2 = \frac{1}{\sqrt{1-v_2^2/c^2}},\qquad \gamma_2' = \frac{1}{\sqrt{1-v_2'^2/c^2}},\qquad v_2' = \frac{v_2-V}{1-\dfrac{v_2V}{c^2}}$$

and $$v_2 = \frac{v'_2 + V}{1 + \dfrac{v'_2 V}{c^2}} \qquad \text{(i)}$$

Now, we follow the same steps until Eq. (3.7a) to obtain Eq. (3.7b). Substituting the expressions for γ'_2 and v'_2 in the left hand side of Eq. (3.7b), we get

$$\gamma'_2 v'_2 = \left(\frac{1}{\sqrt{1 - \dfrac{v'^2_2}{c^2}}}\right) \times \left(\frac{v_2 - V}{1 - \dfrac{v_2 V}{c^2}}\right)$$

We can write the denominator of the second term in the RHS of the above equation as the square root of the square of the same quantity. So, we get:

$$\gamma'_2 v'_2 = \left(\frac{v_2 - V}{\sqrt{\left(1 - \dfrac{v_2 V}{c^2}\right)^2 \left(1 - \dfrac{v'^2_2}{c^2}\right)}}\right) \qquad \text{(ii)}$$

As you have shown in SAQ 1, you can also show that

$$\left(1 - \frac{v_2 V}{c^2}\right)^2 \left(1 - \frac{v'^2_2}{c^2}\right) = \left(1 - \frac{v_2^2}{c^2}\right)\left(1 - \frac{V^2}{c^2}\right) \qquad \text{(iii)}$$

Substituting Eq. (iii) in Eq. (ii), we get

$$\gamma'_2 v'_2 = \frac{v_2 - V}{\sqrt{\left(1 - \dfrac{v_2^2}{c^2}\right)\left(1 - \dfrac{V^2}{c^2}\right)}}$$

Using the results for γ_2 and γ from Eq. (i), we can write

$$\gamma'_2 v'_2 = \gamma_2 \gamma \,(v_2 - V)$$

3. From Eq. (3.16), we have: $\vec{\mathbf{p}} = m\vec{\mathbf{v}} = \dfrac{m_0 \vec{\mathbf{v}}}{\sqrt{1 - \dfrac{v^2}{c^2}}}$

The magnitude of linear momentum is:

$$p = mv = \frac{m_0 v}{\sqrt{1 - \dfrac{v^2}{c^2}}}$$

where $v = 0.99c$ and $p = 1.92 \times 10^{-21}\ \text{kg m s}^{-1}$.

$$\therefore \quad m = \frac{p}{v} = \frac{1.92 \times 10^{-21}\ \text{kg m s}^{-1}}{0.99 \times 3.0 \times 10^8\ \text{m s}^{-1}} = 6.5 \times 10^{-30}\ \text{kg}$$

and $$m_0 = \frac{p}{v}\sqrt{1-\frac{v^2}{c^2}} = \frac{1.92\times10^{-21}\,\text{kg m s}^{-1}}{0.99\times3.0\times10^{8}\,\text{m s}^{-1}}[1-(0.99)^2]^{1/2}$$

$$= 9.1\times10^{-31}\,\text{kg}$$

4. To determine the expression for force on the proton when its acceleration is in the direction of its motion, we use Eq. (3.18b):

$$\vec{\mathbf{F}} = m\frac{d\vec{\mathbf{v}}}{dt} + \vec{\mathbf{v}}\frac{dm}{dt} = m\vec{\mathbf{a}} + \vec{\mathbf{v}}\frac{dm}{dt}$$

where $$\frac{dm}{dt} = m_0\frac{d}{dt}\left(\frac{1}{(1-v^2/c^2)^{1/2}}\right) = \frac{m_0\,v}{c^2\,(1-v^2/c^2)^{3/2}}\frac{dv}{dt}$$

Thus,

$$\vec{\mathbf{F}} = m\vec{\mathbf{a}} + \vec{\mathbf{v}}\,\frac{m_0 v}{c^2\,(1-v^2/c^2)^{3/2}}\frac{dv}{dt} = m\vec{\mathbf{a}} + v\hat{\mathbf{v}}\,\frac{m_0 v}{c^2\,(1-v^2/c^2)^{3/2}}\frac{dv}{dt}$$

or $$\vec{\mathbf{F}} = m\vec{\mathbf{a}} + \frac{m_0}{(1-v^2/c^2)^{3/2}}\frac{v^2}{c^2}\,\hat{\mathbf{v}}\frac{dv}{dt}$$

Since the acceleration of the proton is in the direction of its velocity, we can write $\vec{\mathbf{a}} = \hat{\mathbf{v}}\dfrac{dv}{dt}$ and, therefore, substituting $m = \dfrac{m_0}{(1-v^2/c^2)^{1/2}}$ and $\hat{\mathbf{v}}\dfrac{dv}{dt} = \vec{\mathbf{a}}$ in the second term in the RHS of the above equation, we get:

$$\vec{\mathbf{F}} = \frac{m_0}{(1-v^2/c^2)^{1/2}}\vec{\mathbf{a}} + \frac{m_0}{(1-v^2/c^2)^{3/2}}\frac{v^2}{c^2}\vec{\mathbf{a}}$$

$$= \frac{m_0}{(1-v^2/c^2)^{1/2}}\left(1 + \frac{v^2/c^2}{(1-v^2/c^2)}\right)\vec{\mathbf{a}} = \frac{m_0}{(1-v^2/c^2)^{3/2}}\vec{\mathbf{a}}$$

or $$\vec{\mathbf{F}} = \gamma^3\, m_0\,\vec{\mathbf{a}}$$

5. In the classical limit $v << c$ or $v/c << 1$, we expand the following expression binomially and neglect the higher order terms to make the approximation

$$\frac{1}{(1-v^2/c^2)^{1/2}} \approx 1 + \frac{1}{2}\left(\frac{v^2}{c^2}\right)$$

Then

$$T = \frac{m_0\,c^2}{(1-v^2/c^2)^{1/2}} - m_0\,c^2 \approx m_0\,c^2\left(1 + \frac{1}{2}\times\frac{v^2}{c^2} - 1\right) = \frac{1}{2}m_0\,v^2$$

6. It is given that the kinetic energy of the proton is 200 MeV and its rest mass is $938\,\text{MeV}/c^2$.

Therefore, its rest mass energy is

$$m_0 c^2 = 938 \text{ MeV} \quad \text{and} \quad T = 200 \text{ MeV}$$

So, its total energy is $E = T + m_0 c^2 = (938 + 200) \text{ MeV} = 1138 \text{ MeV}$

The relativistic mass of the proton is

$$m = \frac{E}{c^2} = \frac{1138 \times 1.6 \times 10^{-13} \text{ J}}{9.0 \times 10^{16} \text{ m}^2\text{s}^{-4}} = 2.02 \times 10^{-27} \text{ kg}$$

7. From Eq. (3.34b), $p = \frac{E}{c} = \frac{h\nu}{c}$ for the photon.

   Since $\lambda = \frac{c}{\nu}$ for the photon,

$$p = \frac{h}{\lambda} = \frac{6.626 \times 10^{-34} \text{ Js}}{5.0 \times 10^{-7} \text{ m}} = 1.33 \times 10^{-27} \text{ kg m s}^{-1}$$

Terminal Questions

1. From Eq. (3.14), we have $m = \dfrac{m_0}{\sqrt{1 - \dfrac{v^2}{c^2}}}$

   Substituting the values $m_0 = 2.0 \times 10^{-28}$ kg and $v = 0.9999c$ in the above expression, we get

$$m = \frac{m_0}{\sqrt{1 - \dfrac{v^2}{c^2}}} = \frac{2.0 \times 10^{-28} \text{ kg}}{\sqrt{1 - (0.9999)^2}} = \frac{2.0 \times 10^{-28} \text{ kg}}{0.0141} = 1.4 \times 10^{-26} \text{ kg}$$

2. From Eq. (3.16), we can write the magnitude of the linear momentum of the electron in terms of its rest mass as:

$$p = \frac{m_0 v}{\sqrt{1 - \dfrac{v^2}{c^2}}}$$

   Substituting the values $m_0 = 9.11 \times 10^{-31}$ kg and $v = 0.99c$, we get

$$p = \frac{9.11 \times 10^{-31} \text{ kg} \times 0.99c}{\sqrt{1 - (0.99)^2}} = \frac{27.1 \times 10^{-23}}{0.141} \text{kg m s}^{-1} = 1.92 \times 10^{-21} \text{ kg m s}^{-1}$$

3. Using Eq. (3.16), we can write the rest mass of the particle in terms of its linear momentum as:

$$m_0 = \frac{p}{v}\sqrt{1 - \frac{v^2}{c^2}}$$

Substituting the values $p = 1.0\times10^{-20}\,\text{kg m s}^{-1}$ and $v = 0.90c,$ we get

$$m_0 = \frac{1.0\times10^{-20}\,\text{kg m s}^{-1}}{0.90c}\sqrt{1-(0.90)^2} = 1.61\times10^{-29}\,\text{kg}$$

4. We use the result obtained in SAQ 4. In this problem, it is given that the rest mass of proton is $m_0 = 1.67\times10^{-27}$ kg, $a = 1.0\times10^{15}\,\text{ms}^{-2}$ and $v = 0.90c.$

$$\gamma = \frac{1}{(1-v^2/c^2)^{1/2}} = \frac{1}{(1-0.81)^{1/2}} = 2.3$$

Substituting these values in the expression $F = \gamma^3 m_0 a,$ we get:

$$F = \gamma^3 m_0 a = (2.3)^3 \times 1.67\times10^{-27}\,\text{kg}\times 1.0\times10^{15}\,\text{ms}^{-2}$$

or $\quad F = 2.03\times10^{-11}\,\text{N}$

5. From Eq. (3.16), the magnitude of the initial relativistic linear momentum is:

$$p_i = \frac{m_0 v_i}{\left(1-v_i^2/c^2\right)^{1/2}} = \frac{5.0\,\text{kg}\times2.0\times10^8\,\text{m s}^{-1}}{(1-4/9)^{1/2}}$$

$$= 1.34\times10^9\,\text{kg m s}^{-1} \approx 1.3\times10^9\,\text{kg m s}^{-1}$$

6. From Eq. (3.20), the change in the particle's relativistic linear momentum when a constant relativistic force of magnitude 1.0×10^6 N is exerted on it, is:

$$\Delta\vec{\mathbf{p}} = \int \vec{\mathbf{F}}\,dt$$

or $\quad \Delta p = p_f - p_i = F\Delta t = 1.0\times10^6\,\text{N}\times10^3\,\text{s} = 1.0\times10^9\,\text{kg m s}^{-1}$

From the answer of TQ 5, $p_i = 1.34\times10^9\,\text{kg m s}^{-1},$ therefore,

$$\therefore\ p_f = p_i + 1.0\times10^9\,\text{kg m s}^{-1} = 1.34\times10^9\,\text{kg m s}^{-1} + 1.0\times10^9\,\text{kg m s}^{-1}$$

$$= 2.34\times10^9\,\text{kg m s}^{-1} \approx 2.3\times10^9\,\text{kg m s}^{-1}$$

7. The final speed of the particle is obtained from Eq. (3.16) as follows:

$$p_f = \frac{m_0 v_f}{\left(1-v_f^2/c^2\right)^{1/2}}$$

or $\quad p_f^2 = \dfrac{(m_0 v_f)^2}{1-v_f^2/c^2}$

or $\quad p_f^2\,[1-v_f^2/c^2] = m_0^2 v_f^2$

or $\quad v_f^2\,[m_0^2 + p_f^2/c^2] = p_f^2 \quad\Rightarrow\quad v_f = \dfrac{p_f}{\left(m_0^2 + p_f^2/c^2\right)^{1/2}}$

$$\therefore\ v_f = \frac{2.34\times10^9\,\text{kg m s}^{-1}}{\left(25\,\text{kg}^2 + 60.8\,\text{kg}^2\right)^{1/2}} = 2.53\times10^8\,\text{m s}^{-1} \approx 2.5\times10^8\,\text{m s}^{-1}$$

8. From Eq. (3.16), the linear momentum of the proton is:

$$p = \frac{m_0 v}{\sqrt{\left(1 - v^2/c^2\right)}} = \frac{1.67 \times 10^{-27}\,\text{kg} \times 0.8c}{\sqrt{1 - 0.64}} = 6.7 \times 10^{-19}\,\text{kg m s}^{-1},$$

The total energy of the proton is:

$$E = mc^2 = \frac{1.67 \times 10^{-27}}{\sqrt{1 - 0.64}} \times 9 \times 10^{16}\,\text{J} = 2.5 \times 10^{-10}\,\text{J}$$

$$= \frac{2.5 \times 10^{-10}}{1.6 \times 10^{-13}}\,\text{MeV} = 1562\,\text{MeV}$$

From Eq. (3.29c), the kinetic energy of the proton (in MeV and J) is:

$$T = mc^2 - m_0 c^2 = (1562 - 938)\,\text{MeV} = 624\,\text{MeV}$$

$$= 624 \times 1.6 \times 10^{-13}\,\text{J} = 9.98 \times 10^{-11}\,\text{J}$$

9. We first calculate the speed of the proton in the frame, say S', travelling at the speed of $0.990c$ with respect to the laboratory. Let v' be the speed of the proton with respect to the frame S'. Let v denote the speed of the proton with respect to the laboratory and V, the speed of the frame S' with respect to the frame S attached to the laboratory, which is at rest. Then from velocity transformation equation [Eq. 2.19a)], we can write:

$$v' = \frac{v - V}{1 - \dfrac{vV}{c^2}}$$

where $v = 0.999c$ and $V = 0.990c$. Thus, we get

$$v' = \frac{0.999c - 0.990c}{1 - \dfrac{(0.999)(0.990)\,c^2}{c^2}} = \frac{0.009c}{1 - 0.989} = \frac{0.009c}{0.011} = 0.82c$$

Therefore, the energy of the proton in the frame S' is:

$$E = mc^2 = \frac{m_0 c^2}{(1 - v'^2/c^2)^{1/2}} = \frac{938\,\text{MeV}}{[1 - (0.82)^2]^{1/2}} = \frac{938\,\text{MeV}}{0.572} = 1640\,\text{MeV}$$

We obtain the linear momentum of the free proton from Eq. (3.33) as follows:

$$E^2 = p^2c^2 + \left(m_0 c^2\right)^2$$

or $$p^2c^2 = E^2 - m_0^2 c^4$$

or $$p = \frac{1}{c}\left(E^2 - m_0^2 c^4\right)^{1/2}$$

$$= \frac{1}{3 \times 10^8}\left((1640)^2 - (938)^2\right)^{1/2} \times \left(1.6 \times 10^{-13}\right)\text{kg m s}^{-1}$$

Note that we have converted the unit of MeV into joules in the above step. Thus,

$$p = 7.2 \times 10^{-19}\,\text{kg m s}^{-1}$$

10. We apply Eq. (3.26). In this problem, it is given that $E = 2\,m_p c^2$, where m_p is the rest mass of the proton. Therefore,

$$p^2c^2 = \left(2\,m_p c^2\right)^2 - \left(m_p c^2\right)^2 = 3m_p^2\,c^4$$

$$\therefore\ p = \sqrt{3}m_p\,c = \sqrt{3}\times 1.67\times 10^{-27}\times 3.0\times 10^{8}\,\text{kg m s}^{-1} = 8.7\times 10^{-19}\,\text{kg m s}^{-1}$$

FURTHER READINGS

1. **Einstein and the Special Theory of Relativity**; **Ajoy Ghatak, Viva Books** (2011).

2. **Introduction to Special Relativity**; R. Resnick, Wiley India, (1st Edition, 2007).

TABLE OF PHYSICAL CONSTANTS

Symbol	Quantity	Value
c	Speed of light in vacuum	$3.00 \times 10^{8}\ \text{ms}^{-1}$
μ_0	Permeability of free space	$1.26 \times 10^{-6}\ \text{NA}^{-2}$
ε_0	Permittivity of free space	$8.85 \times 10^{-12}\ \text{C}^{2}\,\text{N}^{-1}\,\text{m}^{-2}$
$1/4\pi\varepsilon_0$		$8.99 \times 10^{9}\ \text{Nm}^{2}\,\text{C}^{-2}$
e	Charge of the proton	$1.60 \times 10^{-19}\ \text{C}$
$-e$	Charge of the electron	$-1.60 \times 10^{-19}\ \text{C}$
h	Planck's constant	$6.63 \times 10^{-34}\ \text{Js}$
$\hbar$	$h/2\pi$	$1.05 \times 10^{-34}\ \text{Js}$
m_e	Electron rest mass	$9.11 \times 10^{-31}\ \text{kg}$
$-e/m_e$	Electron charge to mass ratio	$-1.76 \times 10^{11}\ \text{Ckg}^{-1}$
m_p	Proton rest mass	$1.67 \times 10^{-27}\ \text{kg}$ (1 amu)
m_n	Neutron rest mass	$1.68 \times 10^{-27}\ \text{kg}$
a_0	Bohr radius	$5.29 \times 10^{-11}\ \text{m}$
N_A	Avogadro constant	$6.02 \times 10^{23}\ \text{mol}^{-1}$
R	Universal gas constant	$8.31\ \text{Jmol}^{-1}\,\text{K}^{-1}$
k_B	Boltzmann constant	$1.38 \times 10^{-23}\ \text{J K}^{-1}$
G	Universal gravitational constant	$6.67 \times 10^{-11}\ \text{Nm}^{2}\,\text{kg}^{-2}$

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Special Theory of Relativity: Constancy of Speed of Light, Postulates of Special Theory of Relativity, Lorentz Transformation, Length Contraction, Time Dilation, Examples, Velocity Addition Theorem, Doppler Effect, Variation of Mass with Velocity, Energy-Mass Equivalence, Relativistic Energy and Momentum.

An Introduction to Quantum Mechanics: Birth of Quantum Physics, Blackbody Radiation, Planck's Quantum Hypothesis, Planck's constant and light as a collection of photons; Photo-electric effect and Compton scattering. Problems with Rutherford model – instability of atoms and observation of discrete atomic spectra; Bohr Model, Bohr's quantization rule and atomic stability; calculation of energy levels for hydrogen like atoms and their spectra.

de-Broglie hypothesis and matter waves; Davisson-Germer experiment, Wave-Particle Duality, Matter waves and wave amplitude, Wave Packet, Group Velocity.

The Uncertainty Principle and its Consequences, Thought Experiments: Position measurement – gamma ray microscope thought experiment; Two slit interference experiment with photons, atoms and particles; Complementarity; Estimating minimum energy of a confined particle using uncertainty principle; Energy-time uncertainty principle.

Postulates of Quantum Mechanics, Time Dependent and Time Independent Schrödinger Equation in One Dimension, Statistical Interpretation of Wave Function, Probability Current Density and Continuity Equation, Normalization of Wave Functions, Wave Function in Momentum Space; Observables and Operators, Linear Momentum and Energy Operators; Parity Operator and its Eigenvalues; Commutation Relations, Expectation Values.

Applications of Quantum Mechanics: One Dimensional Infinitely Rigid Box – energy eigenvalues and eigenfunctions, normalization, quantum dot as an example, Particle in a Box, Free Particle, Step Potential and Rectangular Potential Barrier, Quantum mechanical scattering and Tunneling; One Dimensional Potential Well and barrier.

Nuclear Physics: Radioactivity: Stability of nucleus; Law of Radioactive Decay, Mean life and half-life; Radioactive Equilibria, Natural Radioactive Series, Law of α decay, β decay – energy released, spectrum and Pauli's prediction of neutrino; γ-ray emission. General Properties of Nuclei: Size and structure of atomic nucleus and its relation with atomic weight; Impossibility of an electron being in the nucleus as a consequence of the uncertainty principle. Nature of nuclear force, NZ graph, semi-empirical mass formula and binding energy; Stability of Nuclei: Binding Energy Curve. Fission and fusion – mass defect, generation of energy; Fission – nature of fragments and emission of neutrons. Nuclear reactor: slow neutrons and their interactions with Uranium 235; Fusion and thermonuclear reactions.