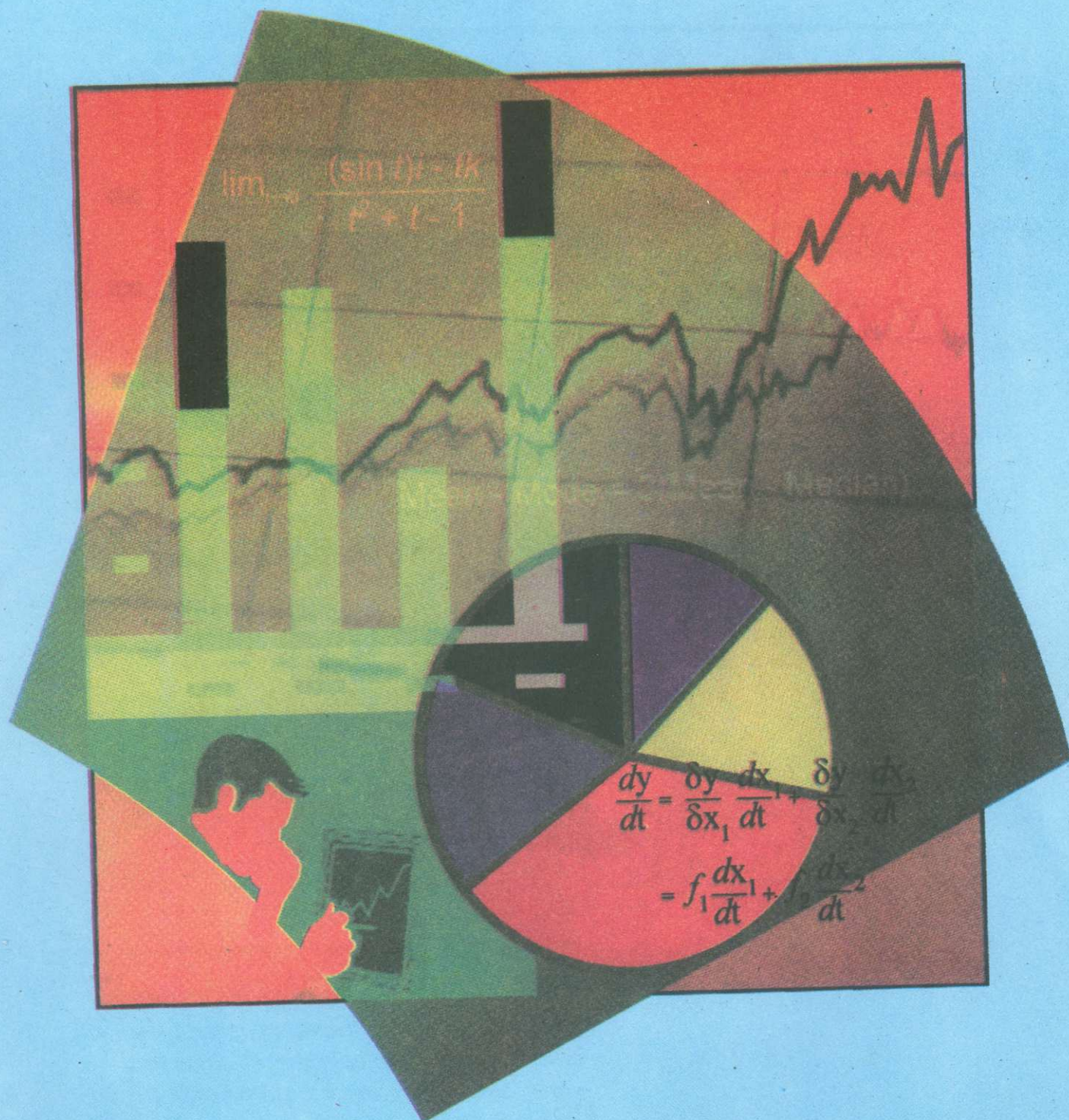




“शिक्षा मानव को बन्धनों से मुक्त करती है और आज के युग में तो यह लोकतंत्रा की भावना का आधार भी है। जन्म तथा अन्य कारणों से उत्पन्न जाति एवं वर्गगत विषमताओं को दूर करते हुए मनुष्य को इन सबसे ऊपर उठाती है।”

- इन्दिरा गाँधी

“Education is a liberating force, and in our age it is also a democratising force, cutting across the barriers of caste and class, smoothing out inequalities imposed by birth and other circumstances.”

- Indira Gandhi

Block

3

INTEGRAL CALCULUS AND ECONOMIC DYNAMICS

UNIT 7**Integration Techniques** **5**

UNIT 8**Integration and Economic Dynamics** **44**

UNIT 9**Difference Equations and Applications in Economic Dynamics** **72**

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BLOCK 3 INTEGRAL CALCULUS AND ECONOMIC DYNAMICS

Introduction

This block deals with the technique of finding the effects of a change in variables over a period. For that purpose, **Unit 7** deals with integration as a prelude to their application in economic scenarios while **Unit 8** takes up differential equations and their applications in Economics. Time is taken as the independent variable and the economic variables as dependent to capture the underlying process of dynamism. In **Unit 9** applications of difference equations in Economic Dynamics is covered. It will be useful to note that this block uses the materials of Block 3 EEC-5 of IGNOU BDP course after suitable modifications and revision.



UNIT 7 INTEGRATION TECHNIQUES

- 7.0 Objectives
- 7.1 Introduction
- 7.2 Indefinite Integrals
 - 7.2.1 The Integral Sign
- 7.3 Properties of Indefinite Integrals
 - 7.3.1 Integral of a Power of x
 - 7.3.2 Power Formula for Integration
 - 7.3.3 Method of Substitution
- 7.4 Integration of Trigonometric Functions
- 7.5 Integration by Parts
- 7.6 Some Useful Formulae
- 7.7 Determination of the Constant of Integration
- 7.8 Definite Integrals
 - 7.8.1 Finding Area
 - 7.8.2 Definition of Definite Integral
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- 7.9 The Fundamental Theorem of Calculus
 - 7.9.1 Evaluation of Definite Integrals
- 7.10 Integration with Chain Rule
 - 7.10.1 Integration with Chain Rule and Substitution
- 7.11 Improper Integrals
 - 7.11.1 Convergent and Divergent Integrals
- 7.12 Integration with Partial Fractions
- 7.13 Let Us Sum Up
- 7.14 Key Words
- 7.15 Useful Books
- 7.16 Answer or Hints to Check Your Progress
- 7.17 Exercises and Answer or Hints

7.0 OBJECTIVES

After going through this unit, you will be able to:

- understand integration as a mathematical process for problem solving;

- integrate various types functions using the different integration methods: substitution rule, integration by part, trigonometric substitution, partial fractions, rational substitution; and
- evaluate improper integrals and use these definitions to compute an improper integral if it is convergent, or to show that it is divergent.

7.1 INTRODUCTION

This unit is devoted to finding and computing integrals. There are two types of integrals that we'll be looking at in this unit, viz., Indefinite Integrals and Definite Integrals. The first half of the unit is devoted to indefinite integrals and the last half is to definite integrals. As we will see in the last half of the unit if we don't know indefinite integrals we will not be able to do definite integrals. It may be useful to point out that in developing the unit the study materials available in the internet sources in the public domain have been utilised often. Credit goes to authors for their contributions to an increased accessibility of learning.

7.2 INDEFINITE INTEGRALS

Indefinite integral is reverse differentiation. The differential calculus is used to derive the rate of change (or, derivative) of a given function whereas indefinite integration reverses such a process and finds the unknown function whose rate of change (derivative) is given. Thus, differentiation is just like finding square root of a number while integration is the process of finding the square.

If the function $f(x)$ is written symbolically as $\int f(x)dx$, which is read as "the integral of $f(x)$ with respect to x " and $f(x)$ is the 'integrand'. We can write $\int f(x)dx = F(x)$, provided $F(x)$ is the derivative of $f(x)$ i.e., $f(x) = \frac{d}{dx}F(x)$. It is immediately clear that the function $F(x)$ is not unique, because if $F(x)$ has the derivative $f(x)$, then so has $F(x) + C$ where C is any arbitrary constant. That is why the indefinite integral is always written with an arbitrary constant, called 'the constant of integration'. As we wish to perform the **opposite process** to differentiation it is called "anti-differentiation" and we will call it "integration".

Example 1: If we know that $\frac{dy}{dx} = 3x^2$, and we need to know the function from which this derivative has come from, then we "undo" the differentiation process. It is easy to see that $y = x^3$ is one anti-derivative of $\frac{dy}{dx} = 3x^2$.

Example 2: Find $\int (x^2 - 5)dx$

The anti-derivative of x^2 is $x^3/3$, and the anti-derivative of 5 is $5x$, so we can write:

$$\int (x^2 - 5)dx = x^3/3 - 5x + C.$$

In general, we say $y = x^3 + C$ is the **indefinite integral** of $3x^2$. The number C is called the **constant of integration**. Thus, you write: $\int 3x^2 dx = x^3 + C$ to say in words as "The integral of $3x^2$ with respect to x equals $x^3 + C$."

7.2.1 The Integral Sign

The $\int$ sign is an elongated "S", standing for "sum". This "long s" notation was introduced by Leibniz when he developed the concepts of integration. As we will see below the integral is the sum of the areas of infinitesimally thin rectangles. Sometimes we write a capital letter to signify integration. For example, we write $F(x)$ to mean the integral of $f(x)$. In that usage we have: $F(x) = \int f(x) dx$.

7.3 PROPERTIES OF THE INDEFINITE INTEGRALS

- 1) $\int k f(x) dx = k \int f(x) dx$, where k is any number. So, we can factor multiplicative constants out of indefinite integrals.

Proof: Suppose that $F(x)$ is an anti-derivative of $F'(x) = f(x)$. Then by the basic properties of derivatives we also get $(kF(x))' = kF'(x) = k f(x)$.

So, $kF(x)$ is an anti-derivative of $k f(x)$; i.e., $(kF(x))' = k f(x)$. Thus, $\int k f(x) dx = k F(x) + C = k \int f(x) dx$.

Example: $\int 5x^2 dx = 5 \int x^2 dx = 5 \left(\frac{x^3}{3} + c \right)$

- 2) $\int f(x) \pm g(x) dx = \int f(x) dx \pm \int g(x) dx$

Proof: Suppose that $F(x)$ is an anti-derivative of $f(x)$ and that $G(x)$ is an anti-derivative of $g(x)$. So we have that $F'(x) = f(x)$ and $G'(x) = g(x)$. Basic properties of derivatives also tell us that $(F(x) \pm G(x))' = (F'(x) \pm G'(x)) = f(x) \pm g(x)$. So $F(x) + G(x)$ is an anti-derivative of $f(x) + g(x)$ and $F(x) - G(x)$ is an anti-derivative of $f(x) - g(x)$. In other words, $\int f(x) \pm g(x) dx = F(x) \pm G(x) + C = \int f(x) dx \pm \int g(x) dx$.

Example: $\int (4x^3 + 7x + 3) dx = \int (4x^3) dx + \int (7x) dx + \int 3 dx$
 $= \frac{4}{5} x^5 + \frac{7}{2} x^2 + 3x + c$

7.3.1 Integral of a Power of x

$x^n dx = \frac{x^{n+1}}{n+1}$ (This is true as long as $n \neq -1$ why?)

For the integral of a power of x we add 1 to the power and divide by the new number.

Example 4: Integrate $\int x^5 dx$

Using the formula obtain: $\int x^5 dx = \frac{x^{5+1}}{5+1} + C = \frac{x^6}{6} + C$

Example 5: Integrate $\int 8x^6 dx$

Ans. $8 \frac{x^{6+1}}{6+1} = 8 \frac{x^7}{7} + C$

See that 8 is a constant, so it is kept out the front.

Example 6: Integrate $dy = (5x^2 - 4x + 3)dx$

Ans.: Write $\int dy = \int (5x^2 - 4x + 3)dx$.

In the left side $\int dy = \int (1)dy = y$. Integration of the constant 1 with respect to y yields y

On the right hand side, we integrate each of the terms, one at a time: $\int (5x^2 - 4x + 3)dx = 5 \frac{x^3}{3} - 4 \frac{x^2}{2} + 3x + C$

So putting it all together, we have the solution:

$$y = 5(x^3)/3 - 2x^2 + 3x + C$$

Example 7: $\int (3x^2 + \sqrt{x} - \frac{5}{x^3})dx$

Ans. Re-write the exponents to make it easier to integrate. So $\int (3x^2 + \sqrt{x} - \frac{5}{x^3})dx = \int (3x^2 + x^{\frac{1}{2}} - 5x^{-3})dx = x^3 + \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - 5 \frac{x^{-2}}{-2} + C = x^3 + \frac{2x^{\frac{3}{2}}}{3} + 5 \frac{x^{-2}}{2} + C$

Example 8: A particular curve has its derivative given by $\frac{dy}{dx} = 3x^2 - 2x$.

We are told that the curve passes through the point (2, 5). Find the equation of the curve.

Answer: The first step for this problem is to **integrate** the expression (i.e. find the **anti-derivative**). This will give us the expression for y : $\int (3x^2 - 2x)dx = x^3 - x^2 + C$

So we have $y = x^3 - x^2 + C$

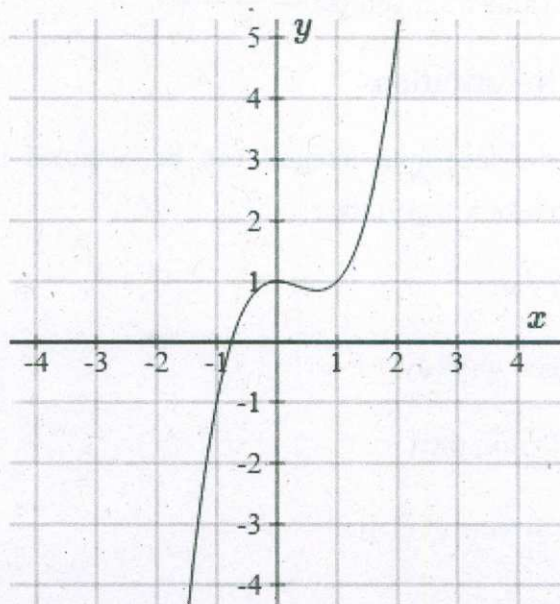
This represents a **family of curves**, and depends on the value of C for the y -intercept.

We must now find the value of C from the information given in the question.

Since the curve passes through (2, 5), we substitute these values into $y = x^3 - x^2 + C$

to give: $5 = (2)^3 - (2)^2 + C$ or, $5 = 8 - 4 + C$. So $C = 1$, giving the required curve is $y = x^3 - x^2 + 1$

Here is a graph of the curve we found in Example 10:



Notice the curve passes through the point (2,5)

Example 9: Consider the integration- $\int (2x^4 - 5)^6 x^3 dx$

An examination of the expression indicate that we cannot integrate it using the rules we have learned so far.

So we have to use the reverse of the Chain Rule, which we have seen in the section on differentiation.

We introduce such a new rule for integrating to deal with cases like these.

7.3.2 Power Formula for Integration

If $\int u^n du = \frac{u^{n+1}}{n+1} + C$, it is true if $n \neq -1$.

We need a substitution step, where $u(x)$ is some function of x . Now back to the problem to see how to apply this formula.

Integrate $(2x^4 - 5)^6 x^3 dx$.

We use, as a starting point, the substitution

$$u = 2x^4 - 5.$$

Why? Because $2x^4 - 5$ is the expression in brackets in the question. Differentiating u gives: $du = 8x^3 dx$

On dividing both sides by 8, we get $\frac{1}{8} du = x^3 dx$.

Rewriting the integration as $\int (2x^4 - 5)^6 x^3 dx = \frac{1}{8} \int u^6 du$, we integrate with respect to u we get $\frac{1}{8} \times \frac{u^7}{7} + C = \frac{u^7}{56} + C$.

Substituting back the value of x , you get $\frac{(2x^4 - 5)^7}{56} + C$.

7.3.3 Method of Substitution

If $F(x) = \int f(x) dx$, the indefinite integral can be obtained by resorting to transformation. If we take $x = g(y)$ then

$$\int f(x) dx = \int [g(y)] g'(y) dy.$$

See that $x = g(y) \Rightarrow dx = g'(y) dy$.

If $\phi(y) = \int f[g(y)] g'(y) dy$, then

$F(x) = \phi[g^{-1}(\cdot)]$ is the inverse of $g(\cdot)$.

Example: (i) Solve $F(x) = \int (1 + 5x)^{\frac{1}{2}} dx$.

Let $y = 1 + 5x$

$$dy = 5dx, \text{ or } dx = \frac{1}{5} dy$$

$$\therefore F(x) = \frac{1}{5} \int y^{\frac{1}{2}} dy = \frac{1}{5} \cdot \frac{2}{3} y^{\frac{3}{2}} + c$$

$$= \frac{2}{15} \cdot y^{\frac{3}{2}} + c = \frac{2}{15} (1 + 5x)^{\frac{3}{2}} + c$$

Example: (ii) Solve $F(x) = \int \frac{dx}{(3x-1)^2}$

Let $y = 3x - 1$

$$\text{So, } dy = 3 dx, \text{ or, } dx = \frac{1}{3} dy.$$

$$\therefore F(x) = \frac{1}{3} \int \frac{dy}{y^2} = \frac{1}{3} \left(-\frac{1}{y} \right) + c = \frac{1}{3(3x-1)} + c$$

Example: (iii) Evaluate $\int e^x e^x dx$ using substitution.

Since $e^x e^x = e^{2x}$, we have $\int e^{2x} dx$.

Let $u = 2x$, so $du = 2dx$ and $dx = \frac{1}{2} du$. Thus, we get $\frac{1}{2} \int e^u du$. Integration gives us $\frac{1}{2} \int e^u du = \frac{1}{2} e^u = \frac{e^{2x}}{2}$.

Example: iv Find $\int 2x \sqrt{x^2 + 1} dx$

Here we let $g(x) = x^2 + 1$. We have $\int 2x \sqrt{x^2 + 1} dx = \int \sqrt{g(x)} g'(x) dx$

Now we let

$u = g(x) = x^2 + 1$, giving us that

$\frac{du}{dx} = 2x$ giving us that $du = 2x dx$. Therefore, we have

$$\int 2x \sqrt{x^2 + 1} dx = \int \sqrt{u} du = \frac{2u^{\frac{3}{2}}}{\frac{3}{2}} + C.$$

We convert our answer back to an answer in terms of the variable

x , to get $\int 2x \sqrt{x^2 + 1} dx = \frac{2u^{\frac{3}{2}}}{\frac{3}{2}} + C$

Check Your Progress 1

- 1) What do you understand by indefinite integral?

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- 2) Find $\int 4 dx$

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- 3) $\int (x^2 + 3) dx$

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- 4) $\int 9e^{-3x} dx$

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5) $\int \frac{1}{x+2} dx$

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7.4 INTEGRATION OF TRIGONOMETRIC FUNCTIONS

Recall the basic trigonometric functions before entering into their integration.

i) Definitions:

$$\tan x = \frac{\sin x}{\cos x}, \sec x = \frac{1}{\cos x}; \cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}; \operatorname{cosec} x = \frac{1}{\sin x}.$$

ii) Important identities are:

$$\sin^2 A + \cos^2 A = 1$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2 \cos^2 A - 1 \text{ or, } \cos^2 A = \frac{1 + \cos 2A}{2}$$

$$\cos 2A = 1 - 2 \sin^2 A \text{ or, } \sin^2 A = \frac{1 - \cos 2A}{2}$$

$$1 + \tan^2 A = \sec^2 A$$

$$1 + \cot^2 A = \operatorname{cosec}^2 A$$

iii) Differentiation rules for trigonometry functions are:

$$D(\sin A) = \cos A$$

$$D(\cos A) = -\sin A$$

$$D(\tan A) = \sec^2 A$$

$$D(\cot A) = \operatorname{cosec}^2 A$$

$$D(\sec A) = \sec A \tan A$$

$$D(\operatorname{cosec} A) = -\operatorname{cosec} A \cot A$$

iv) Angle-Sum and -Difference Identities are:

$$2 \sin A \cos B = \sin(A + B) + \sin(A - B)$$

$$2 \cos A \cos B = \cos(A - B) + \cos(A + B)$$

$$2 \sin A \sin B = \sin(A + B) - \sin(A - B)$$

$$2 \cos A \sin B = \sin(A + B) - \sin(A - B)$$

$$2 \sin A \sin B = \cos(A - B) - \cos(A + B)$$

v) Some Generalised formulas for Integration:

Integral Rule	General Rule
$\int \cos x dx = \sin x + C$	$\int \cos(ax + b) dx = \frac{1}{a} \sin(ax + b) + C$
$\int \sin x dx = -\cos x + C$	$\int \sin(ax + b) dx = -\frac{1}{a} \cos(ax + b) + C$
$\int \tan x dx = -\ln \cos x + C$	$\int \tan(ax + b) dx = -\frac{1}{a} \ln \cos(ax + b) + C$
$\int \cot x dx = \ln \sin x + C$	$\int \cot(ax + b) dx = \frac{1}{a} \ln \sin(ax + b) + C$
$\int \sec x dx = \ln \sec x + \tan x + C$	$\int \sec(ax + b) dx = \frac{1}{a} \ln \sec(ax + b) + \tan(ax + b) + C$
$\int \operatorname{cosec} x dx = -\ln \operatorname{cosec} x + \cot x + C$	$\int \operatorname{cosec}(ax + b) dx = -\frac{1}{a} \ln \operatorname{cosec}(ax + b) + \cot(ax + b) + C$

Example 1: Evaluate the integral $\int \cos(15x) \cos(4x) dx$

Solution: $\int \cos(15x) \cos(4x) dx = \frac{1}{2} \int \cos(11x) + \cos(19x) dx =$
 $\frac{1}{2} \left(\frac{1}{11} \sin(11x) + \frac{1}{19} \sin(19x) \right) + C.$

Example 2: Evaluate $\int \cos x \sin^5 x dx$

Put $u = \sin x$ so that $du = \cos x dx$

$$\text{Thus, } \int \cos x \sin^5 x dx = \frac{1}{6} u^5 du = \frac{1}{6} \sin^6 x + C$$

Example 3: Evaluate the following integral: $\int \sin^5 x dx$

$$\text{Solution: } \int \sin^5 x dx = \int \sin^4 x \sin x dx = \int (\sin^2 x)^2 \sin x dx = \int (1 - \cos^2 x)^2 \sin x dx.$$

$$\text{Put } u = \cos x \text{ to get } \int \sin^5 x dx = -\int (1 - u^2)^2 du = -\int (1 - 2u^2 + u^4) du = -\left(u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right) + C = \left(-\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C\right).$$

Example 4: Evaluate $\int (3\sin x - 4\sec 2x) dx$

$$\text{Solution: } \int (3\sin x - 4\sec 2x) dx = 3 \int \sin x dx - \int 4\sec 2x,$$

$$\text{or, } 3 \int \sin x dx - 4 \int \sec 2x dx = 3 \int \sin x dx - 4 \int \sec^2 x dx = -3\cos x - 4\tan x + C.$$

Example 5: Evaluate $\int \cos(2x - 6) dx$

$$\text{Solution: } \int \cos(2x - 6) dx$$

$$\text{Put } u = 2x - 6 \text{ so that } \frac{du}{dx} = 2 \text{ or, } dx = \frac{1}{2} du$$

$$\text{We now have } \int \cos(2x - 6) dx = \int \cos u \cdot \frac{1}{2} du = \frac{1}{2} \int \cos u du = \frac{\sin u}{2} + C = \frac{\sin(2x - 6)}{2} + C.$$

Example 6: Evaluate $\int \sin x \cos 2x dx$

$$\text{Solution: Put } u = \cos x \text{ so that } \frac{du}{dx} = -\sin x \text{ or, } dx = -\frac{du}{\sin x}$$

$$\text{Now we have } \int \sin x \cos^2 x dx = \int (\sin x) u^2 \left(-\frac{1}{\sin x}\right) du = -\int u^2 du = -\frac{u^3}{3} + C = -\frac{\cos^3 x}{3} + C.$$

Example 7: Evaluate $\int \tan x dx$

$$\text{Solution: Write } \int \tan x dx = \int \frac{\sin x}{\cos x} dx. \text{ Using substitution put } \cos x = u \text{ so that } du = -\sin x dx \text{ or, } dx = -\frac{du}{\sin x}.$$

$$\text{Now } \int \tan x dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{du}{u} = -\ln|u| + C = -\ln|\cos x| + C$$

Example: Evaluate the integral: $\int \cos^2 x dx$

$$\text{Ans.: Take } \sin^2 x = \frac{1 - \cos 2x}{2} \text{ and } \cos^2 x = \frac{1 + \cos 2x}{2}$$

Using the half angle formula for $\cos^2 x$, we have:

$$\int \cos^2 x dx = \int \frac{(1 + \cos 2x)}{2} dx = \frac{x}{2} + \frac{\sin 2x}{4} + C$$

7.5 INTEGRATION BY PARTS

A method of **integration** is often useful when two functions are multiplied together. Let us return to differentiation of product of two functions $u = f(x)$ and $v = g(x)$ which gives,

$$d(uv) = u \frac{dv}{dx} + v \frac{du}{dx}.$$

From this, we can obtain

$$\int (uv)' dx = \int uv' dx + \int vu' dx$$

$$\therefore \int (uv)' dx = uv$$

$$\therefore uv' dx = uv - \int vu' dx$$

$$\text{Solve } F(x) = \int xe^{-x} dx$$

$$\text{Let } u(x) = x, v(x) = -e^{-x}$$

$$\text{So that } u'(x) = 1, v'(x) = e^{-x}$$

$$\therefore f(x) = \int u.v' dx$$

$$= uv - \int vu' dx$$

$$= -xe^{-x} + \int e^{-x} dx$$

$$= -xe^{-x} - e^{-x} + c = -e^{-x}(1+x) + c$$

Evaluate the following integral: $\int (3t + 5) \cos\left(\frac{t}{4}\right) dt$

$$\text{Put } u = 3t + 5 \text{ and } dv = \cos\left(\frac{t}{4}\right) dt$$

$$\text{So that } du = 3t \text{ and } v = 4\sin\left(\frac{t}{4}\right)$$

The integral is then,

$$\int (3t + 5) \cos\left(\frac{t}{4}\right) dt = 4(3t + 5) \sin\left(\frac{t}{4}\right) - 12 \int \sin\left(\frac{t}{4}\right) dt = 4(3t + 5) \sin\left(\frac{t}{4}\right) + 48 \int \cos\left(\frac{t}{4}\right) + C.$$

Example: Integrate $\int x \ln x dx$

$$\text{Solution: Let } u = \ln x \text{ and } dv = x dx \text{ so that } du = \frac{1}{x} dx \text{ and } v = \frac{x^2}{2}.$$

$$\text{Therefore, } \int x \ln x dx = \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \frac{1}{x} dx = \frac{x^2}{2} \ln x - \frac{1}{2} \int x dx =$$

$$\frac{x^2}{2} \ln x - \frac{1}{2} \frac{x^2}{2} + C = \frac{x^2}{2} \ln x - \frac{1}{4} x^2 + C.$$

Example: Integrate $\int x \cos 3x dx$

Solution: Put $u = x$ and $dv = \cos 3x dx$.

So that $du = dx$ and $v = \frac{1}{3} \sin 3x$.

$$\begin{aligned} \text{Therefore, } \int x \cos 3x dx &= x \frac{1}{3} \sin 3x - \int \frac{1}{3} \sin 3x dx = \frac{1}{3} x \sin 3x - \\ &\frac{1}{3} \int \sin 3x dx = \frac{1}{3} x \sin 3x - \frac{1}{3} \frac{-\cos 3x}{3} + C \\ &= \frac{1}{3} x \sin 3x + \frac{1}{3} \frac{\cos 3x}{3} + C = \frac{1}{3} x \sin 3x + \frac{1}{9} \cos 3x + C. \end{aligned}$$

Check Your Progress 2

- 1) Since indefinite integration is the anti-derivative, how do you show that $\int \cos ax dx = \frac{1}{a} \sin ax + C$.

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- 2) While solving the integration problem, when to do substitution and when integration by parts?

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- 3) Using the formula for integration by parts to evaluate $\int x^4 \ln x dx$. Why $\ln x$ would be the best choice for u ?

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7.6 SOME USEFUL FORMULAE

We state without proof some useful formulae for integration.

1) $\int x^n dx = \frac{x^{n+1}}{n+1} + c, n \neq -1$

$$2) \quad \int x^{-1} dx = \int \frac{dx}{x} = \log x + c \quad \text{for any } x > 0$$

$$3) \quad \int e^{mx} dx = \frac{e^{mx}}{m} + c \quad \text{for any } m.$$

$$4) \quad \int a^{mx} dx = \frac{a^{mx}}{m \log a} + c$$

$$5) \quad \int \cos ax dx = \frac{\sin ax}{a} + c$$

$$6) \quad \int \sin ax dx = -\frac{\cos ax}{a} + c$$

$$7) \quad \int [K_1 f(x) + K_2 g(x)] dx = K_1 \int f(x) dx + K_2 \int g(x) dx + c \quad \text{for } K_1, K_2 \geq 0$$

7.7 DETERMINATION OF THE CONSTANT OF INTEGRATION

We have noted that the indefinite integral always contains an arbitrary constant. The value of this constant can be precisely determined (and the integral rendered unique thereby) if it is known that the integral function $y = f(x) + c$ obeys some prescribed **initial condition** ($y = y_0$ when $x = 0$) or more generally **boundary condition** ($y = y_0$ when $x = x_0$). To illustrate, we know

$$\int (x^2 + 3) dx = \frac{1}{3} x^3 + 3x + c$$

Here $y = f(x) + c = \frac{1}{3} x^3 + 3x + c$. Suppose, we are given the initial condition, $y = 20$ when $x = 0$. Then

$$y_0 = f(0) + c = c = 20.$$

This fixes the integral as the unique function $y = \frac{1}{3} x^3 + 3x + 20$.

7.8 DEFINITE INTEGRALS

When we use integration to arrive at a number definite integral helps. It has start and end values; in other words we operate in an interval and useful in finding areas, volumes and central points.

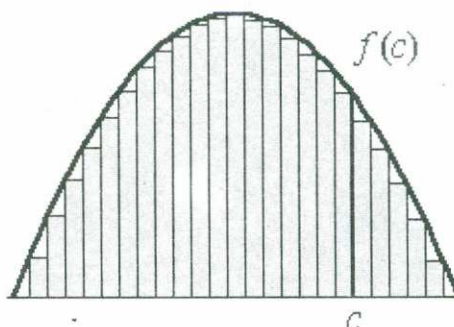
7.8.1 Finding Area

We take a look at the Area Problem before going to deal with definite integrals. It will help us interpret definite integral and lead us to the definition of the definite integral.

Suppose we have a function $f(x)$ that is positive on some interval $[a, b]$. If we want to determine the area of the region between the function and the x -axis,

we do this in two ways: using an approximation (finding areas of rectangles) and using integration.

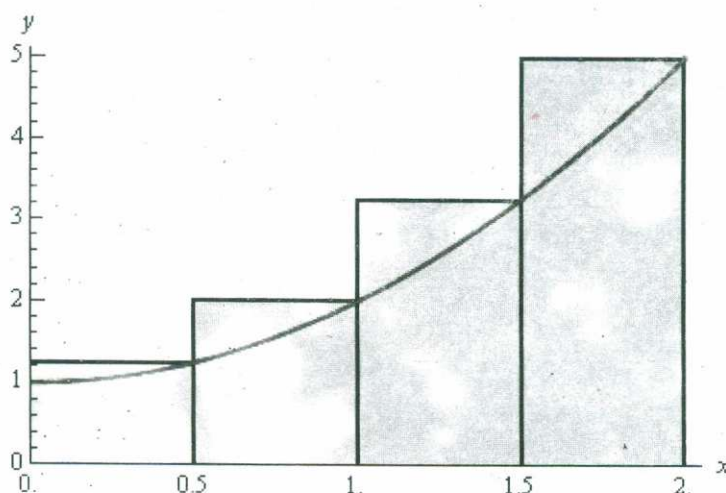
For the method of approximation the space under the function is divided into rectangles and adding areas of those rectangles we get the area under the curve. For example, to determine the area under the curve, that is, the shaded region below we divide the area into rectangles.



The height of each rectangle is found by calculating the function values, as shown for the typical case $x = c$. We get a better result if we take more and more rectangles. In the above diagram, we are approximating the area using **inner rectangles** (each rectangle is inside the curve). We could also find the area using the **outer rectangles** (How?).

To understand the process let us estimate the area by dividing up an interval, $[a, b]$ into n subintervals each of width, $\Delta x = \frac{b-a}{n}$. Then construct a rectangle whose height is given by the function value at a specific point in the interval. The area of each of these rectangles can be derived by $\Delta x \cdot h$ where h is the height of the rectangle in that interval. When we add each of these we get an estimate of the area under the curve using three options of right, left and middle value of the rectangles of the curve.

Suppose you want to determine the area of the function $f(x) = x^2 + 1$ in the interval $[a, b]$ by considering 4 subintervals. The figure below shows the area under examination.

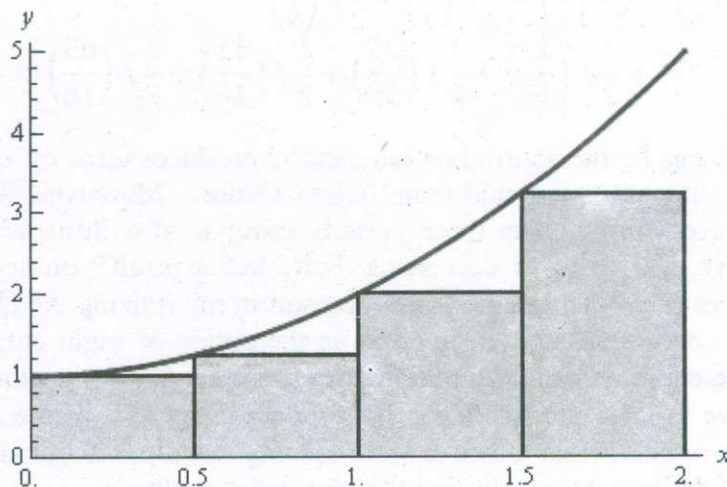


From the figure it is apparent that by choosing the height of each of rectangle as has been shown you over estimate the area. Because each rectangle takes in more area stretching beyond the value of the function as shown by the curve of the graph in each case.

Since the width of each of the rectangles is $\frac{1}{2}$, and the height of each rectangle at the value given by the function at the right endpoint of the concerned interval $f(x_i)$ the estimated total area is

$$\begin{aligned} A_r &= \frac{1}{2}f\left(\frac{1}{2}\right) + \frac{1}{2}f(1) + \frac{1}{2}f\left(\frac{3}{2}\right) + \frac{1}{2}f(2) \\ &= \frac{1}{2}f\left(\frac{5}{4}\right) + \frac{1}{2}f(2) + \frac{1}{2}f\left(\frac{13}{4}\right) + \frac{1}{2}f(5) = 5.75 \end{aligned}$$

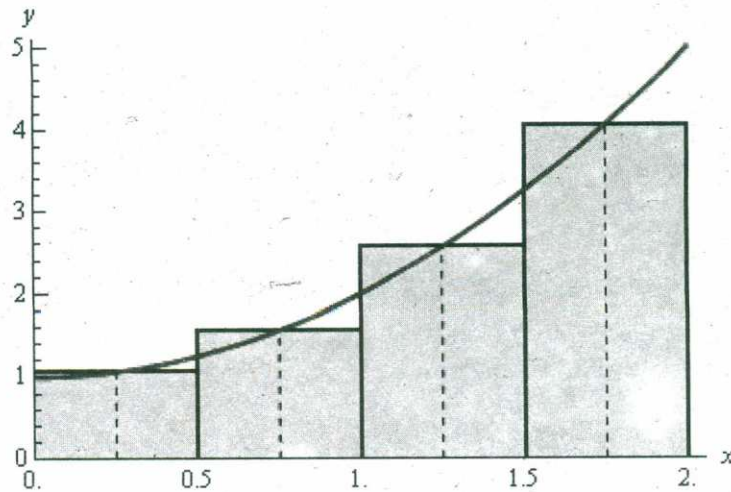
The option of choosing the right end point of the height of a rectangle is not unique as its left end point can also be considered. Exercise of such an option will give the following graph and estimated area.



$$\begin{aligned} A_l &= \frac{1}{2}f(0) + \frac{1}{2}f\left(\frac{1}{2}\right) + \frac{1}{2}f(1) + \frac{1}{2}f\left(\frac{3}{2}\right) \\ &= \frac{1}{2}f(1) + \frac{1}{2}f\left(\frac{5}{4}\right) + \frac{1}{2}f(2) + \frac{1}{2}f\left(\frac{13}{4}\right) = 3.75 \end{aligned}$$

In the left end choice of the height of rectangle the estimation is an underestimation since each rectangle misses some of the area each time.

Instead of using the right or left endpoints of each sub interval, option of taking the midpoint of each subinterval as the height of each rectangle yields the graph as follows:



The estimation of the area in case of mid-point height of the rectangle is

$$\begin{aligned} A_l &= \frac{1}{2}f\left(\frac{1}{4}\right) + \frac{1}{2}f\left(\frac{3}{4}\right) + \frac{1}{2}f\left(\frac{5}{4}\right) + \frac{1}{2}f\left(\frac{7}{4}\right) \\ &= \frac{1}{2}f\left(\frac{17}{16}\right) + \frac{1}{2}f\left(\frac{25}{16}\right) + \frac{1}{2}f\left(\frac{41}{16}\right) + \frac{1}{2}f\left(\frac{65}{16}\right) = 4.625 \end{aligned}$$

Estimating of area by the approximation method produces three estimates viz., right corner, left corner and mid-point height values. Moreover, finding the approximate area through the three options using is also influenced by the nature of curve, increasing or decreasing. For a better result, you need to take more rectangles (i.e., increase n) that will result in minimizing the features of including the curve areas above the curve in the option of right corner height and below the curve areas in the option left corner height. For a more general result, we have to look into functions that produce curves of various types that may lie above or below that x-axis as well. Doing that leads to our adoption of the method of definite integral to find the area under a curve.

Let's start out with a continuous function $f(x) \geq 0$ on $[a, b]$. Note that by specifying the interval we are considering a situation that has start and end points. If we divide the interval into n subintervals each of length we get, $\Delta x = \frac{b-a}{n}$

The endpoints of each subinterval are,

$$x_0 = a$$

$$x_1 = a + \Delta x$$

$$x_2 = a + 2\Delta x$$

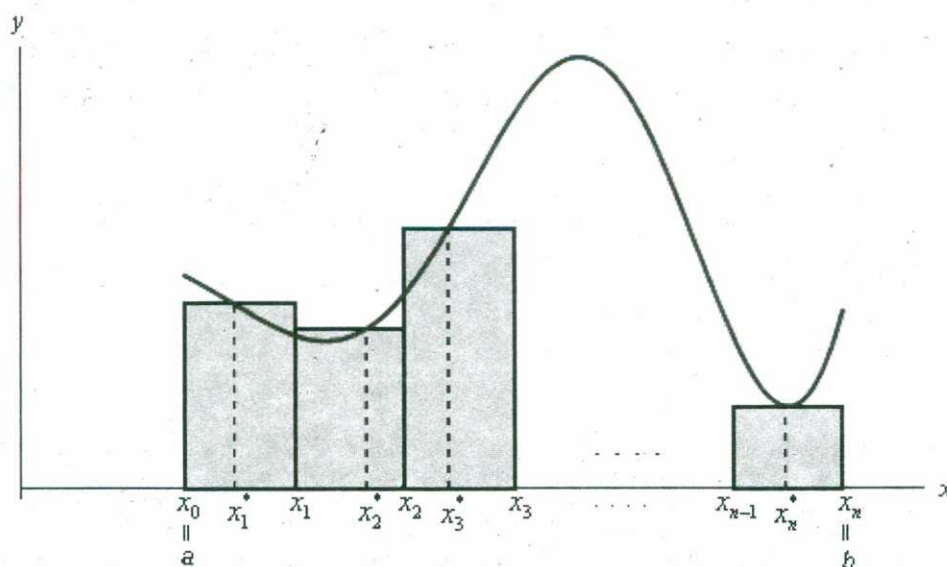
$$x_i = a + i\Delta x$$

$$x_{n-1} = a + (n-1)\Delta x$$

$$x_n = a + n\Delta x = b$$

Thus in each interval we have $[x_0, x_1], [x_1, x_2], \dots [x_{i-1}, x_i], \dots [x_{n-1}, x_n]$. From each we choose a point $x_1^*, x_2^*, \dots x_i^*, \dots x_n^*$ such that these will give the height of the rectangle in each subinterval.

Seen in terms of a graph, our formulation will look like the figure below:



The area under the curve on the given interval is then approximately,

$$A \approx f(x_1^*)\Delta x + f(x_2^*)\Delta x + \dots + f(x_i^*)\Delta x + f(x_n^*)\Delta x$$

$$\text{So } A \approx \sum_{i=1}^n f(x_i^*) \Delta x$$

The summation above is called a **Riemann Sum** and $A = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x$ gives the exact area under the curve.

7.8.2 Definition of Definite Integral

Given a function a continuous function $f(x)$ on the interval $[a, b]$, if we divide the interval into n subintervals of equal width, Δx , and from each interval choose a point, x_i^* , then the **definite integral of $f(x)$ from a to b** is

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x .$$

Example: Using the formula given above, computation of the exact area under

$$f(x) = x^2 + 1 \text{ on } [0, 3].$$

$$\begin{aligned} \int_0^3 (x^2 + 1) dx &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[f(x_i) \cdot \left(\frac{b-a}{n} \right) \right] = \lim_{n \rightarrow \infty} \left(12 + \frac{27}{2n} + \frac{9}{2n^2} \right) \\ &= \left(12 + \frac{27}{2 \cdot \infty} + \frac{9}{2 \cdot \infty^2} \right) = 12 + 0 + 0 = 12 \end{aligned}$$

7.8.3 Properties of Definite Integral

- 1) $\int_a^b 1 dx = b - a$
- 2) $\int_a^b c f(x) dx = c \int_a^b f(x) dx$ if c is a constant.
- 3) $\int_a^b (f(x) + g(x)) dx = \int_a^b f(x) dx + \int_a^b g(x) dx$
- 4) $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$

If the integral of $y = 2x + 3$ from 1 to 3 is 14 and the integral of the same function from 3 to 5 equals 22, then the integral from 1 to 5 of the function $y = 2x + 3$ equals $14 + 22$, or 36 because the two integrals (from 1 to 3 and 3 to 5) make up the whole integral (from 1 to 5). Conversely, since the integral of $y = 4x^3 - 2x$ from 2 to 4 equals 228, then you know that the sum of the integrals from 2 to 3 and 3 to 4 of the same function must also equal 228. If $0 \leq f(x) \leq g(x)$ for all x in $[a, b]$, then $0 \leq \int_a^b f(x) dx \leq \int_a^b g(x) dx$.

$$5) \int_a^a f(x) dx = 0$$

Example: The integral of $y = 3x + 4$ from 2 to 2 is 0 because the integral of any function in which both bounds are the exact same number is zero.

$$6) \text{ If } a < b \text{ then it is convenient to define } \int_b^a f(x) dx = - \int_a^b f(x) dx$$

Example: If the integral from 1 to 4 of the function $y = x^2$ is 21, then the integral from 4 to 1 of x^2 is -21 because when the bounds of an integral are switched, the integral is the same, but has the opposite sign (positive changes to negative, or negative changes to positive).

Check Your Progress 3

- 1) What is a definite integral?

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2) What is Riemann Sum?

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3) Why $\int_a^a f(x)dx = 0$?

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7.9 THE FUNDAMENTAL THEOREM OF CALCULUS

The fundamental theorem of calculus established the relationship between differentiation and integration. We have seen from finding the area that the definite integral of a function can be interpreted as the area under the graph of a function. It justifies our procedure of evaluating an anti-derivative at the upper and lower bounds of integration and taking the difference.

The first part of the theorem shows that indefinite integration can be reversed by differentiation. It also proves that for every continuous function, there is an anti-derivative (integral).

Let F be any anti-derivative, or indefinite integral, for f on $[a, b]$. If f is a continuous function and is defined by

$$F(x) = \int_a^x f(t) dt$$

Then,

$$F'(x) = f(x)$$

And therefore,

$$\frac{d}{dx} \int_a^x f(t) dt = f(x)$$

Every continuous function f has an anti-derivative, and there is one anti-derivative $F(x)$ given by a definite integral of f with a variable upper boundary. Varying the lower boundary of the integrand will produce other anti-derivatives.

One of the first things to notice about the fundamental theorem of calculus is that the variable of differentiation appears as the upper limit of integration in the integral. This makes sense because if we are taking the derivative of the integrand with respect to x , it needs to be in either (or both) the limits of

integration. If we are integrating a function with respect to x , the function we end up with needs to be in terms of x .

The second part of this theorem tells us how to evaluate a definite integral assuming that f has an indefinite integral:

$$\int_a^b f(x) dx = F(b) - f(a)$$

where F is any anti-derivative (or indefinite integral) of continuous function f on a closed interval $[a, b]$. This part of the theorem is one that we use in finding area. It allows us to compute the definite integral of a function by using one of infinitely many anti-derivatives, or indefinite integrals.

Here is a simple example of the fundamental theorem of calculus:

$$\int_0^t \cos(t) dt = \sin(t) \Big|_0^x = \sin(x) - \sin(0) = \sin(x)$$

It is important to note that if $F(x)$ is an anti-derivative of $f(x)$ and the function is defined on some interval, then every other anti-derivative $G(x)$ of $f(x)$ differs from $F(x)$ by a constant:

$$G(x) = f(x) = C$$

$$G(x) = f(x) = C$$

If we recall the general power rule for integration, taking the integral of a function will give us a constant. So there are infinitely many different anti-derivatives for any given function.

7.9.1 Evaluation of Definite Integrals

Let's do a couple of examples using of the fundamental theorem of calculus.

1) Evaluate: $\frac{d}{dx} \int_a^x 4t^3 dt$

Solution: $\frac{d}{dx} \int_a^x 4t^3 dt = \frac{d}{dx} (t^4).$

Use the second part of the theorem and solve for the interval $[a, x]$.

$$\frac{d}{dx} (x^4 - a^4) = 4x^3$$

Taking the derivative with respect to x will leave out the constant giving $f(x) = 4x^3$.

2) Evaluate: $\frac{d}{dx} \int_2^{\sin(x)} e^{t^2} dt$

We need to find out the derivative of the integrand from the interval $[2, \sin(x)]$. In other words, From the first part of the fundamental theorem of calculus, we get the following:

Let $G(x) = \int_2^x e^{t^2} dt$

Then $G'(x) = e^{x^2}$

3) Evaluate: $\int_2^3 x^2 dx$

Solution: $\int_2^3 x^2 dx = x^3|_2^3 = 3^3 - 2^3 = 19$

4) To evaluate $\int_0^2 (t^2 - 3t + 10)dt$.

$$\int (t^2 - 3t + 10)dt = \frac{1}{3}t^3 - \frac{3}{2}t^2 + 10t + c.$$

$$\text{So, } \int_0^2 (t^2 - 3t + 10)dt = \left(\frac{8}{3} - \frac{12}{2} + 20 \right) - 0 = \frac{50}{3}$$

Check Your Progress 4

1) What is fundamental theorem of calculus?

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2) What is difference between integration and differentiation?

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3) What is difference between definite and infinite integrals?

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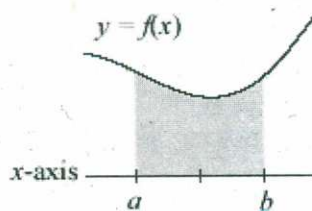
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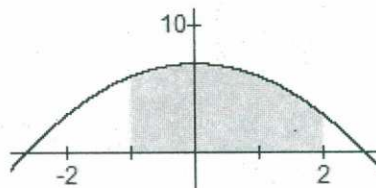
7.10 INTEGRATION WITH CHAIN RULE

We discuss below the area between the graph of $y = f(x)$ and the x - axis by the definite integral. As you will see it gives a positive result for a graph above the x - axis, and a negative result for a graph below the x - axis. If the graph of $y = f(x)$ is partly above and partly below the x - axis, the formula given

below generates the net area. We get the area above the axis minus the area below the axis.

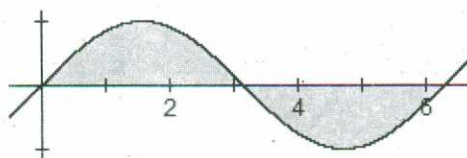


Example 1: Find the area between $y = 7 - x^2$ and the x -axis between the values $x = -1$ and $x = 2$.



$$\begin{aligned} \text{Area} &= \int_{-1}^2 (7 - x^2) dx = \left(7x - \frac{1}{3}x^3 \right) \Big|_{-1}^2 \\ &= \left[7(2) - \frac{1}{3}(8) \right] - \left[7(-1) - \frac{1}{3}(-1) \right] = 18 \end{aligned}$$

Example 2: Find the net area between $y = \sin x$ and the x -axis between the values $x = 0$ and $x = 2\pi$.



$$\begin{aligned} \text{Net Area} &= \int_0^{2\pi} \sin x dx \\ &= (-\cos x) \Big|_0^{2\pi} = (-1) - (-1) = 0 \end{aligned}$$

7.10.1 Integration with Chain Rule and Substitution

When evaluating a definite integral using u -substitution, how does one deal with the limits of integration? There are two possibilities. As before, we have with an integral of the form $\int_a^b f(x)f'(x)dx$. We will again let $u = f(x)$, giving us $du = f'(x)dx$. When we make the substitution in the integral, the limits of integration also change, giving us $\int_{u_1}^{u_2} u du$. Normal integration, as before, gives us $\int_{u_1}^{u_2} u du = \left[\frac{u^2}{2} \right]_{u_1}^{u_2}$. But what are these new limits of integration u_1 and u_2 ? We can regard them simply as placeholders. When we reverse our replacement, substituting $\sin x$ for u , the limits of integration also revert back to the original a and b . This gives us $\int_{u_1}^{u_2} u du = \left[\frac{u^2}{2} \right]_{u_1}^{u_2} =$

$\left[\left(\frac{\sin^2 x}{2}\right)\right]_a^b$, which can be evaluated in the standard fashion.

Alternatively, after finding $\int_{u_1}^{u_2} u du = \left[\left(\frac{u^2}{2}\right)\right]_{u_1}^{u_2}$, we could use the fact that since $u = f(x)$, $u_2 = f(b)$ and $u_1 = f(a)$, to evaluate directly without reversing the u -substitution. Using this method, we have $\int_a^b f(x)f'(x)dx = \int_{u_1}^{u_2} u du = \left[\left(\frac{u^2}{2}\right)\right]_{f(a)}^{f(b)} = \frac{1}{2}((f(b))^2 - (f(a))^2)$.

Example: Evaluate the following using u -substitution:

$$\int_0^2 \sqrt{4-x^2} dx$$

Solution: Let $u = 4 - x^2$. Then we have $du = -2x dx$ and the integral $\int_{u_1}^{u_2} (\sqrt{u}) du$. Integrating, we find

$$\int_0^2 \sqrt{4-x^2} dx = -\int_{u_1}^{u_2} (\sqrt{u}) du = \left(\frac{2}{3}(u)^{\frac{3}{2}}\right)\bigg|_{u_1}^{u_2} = -\left(\frac{2}{3}(4-x^2)^{\frac{3}{2}}\right)\bigg|_0^2 = -\frac{2}{3}(0-8) = \frac{16}{3}.$$

Example: Evaluate $\int_0^{\frac{\pi}{2}} \sin x \cos x dx$.

Solution:

Method 1: Put $u = \sin x$ to get $du = \cos x dx$.

On substitution of u and du the problem reduces to $\int_{u_1}^{u_2} u du$

Integrating, reversing the substitution, and evaluating, we get

$$\left(\frac{u^2}{2}\right)\bigg|_{u_1}^{u_2} = \left(\frac{\sin^2 x}{2}\right)\bigg|_0^{\frac{\pi}{2}} = \frac{1}{2}(\sin^2(\frac{\pi}{2}) - \sin^2(0)) = \frac{1}{2}(1-0) = \frac{1}{2}$$

Method 2:

As in method 1, we substitute and integrate to find $\int_0^{\frac{\pi}{2}} \sin x \cos x dx = \int_{u_1}^{u_2} u du = \left(\frac{u^2}{2}\right)\bigg|_{u_1}^{u_2}$. Since $u = f(x)$, $u_1 = f(0) = \sin(0) = 0$ and $u_2 = f(\frac{\pi}{2}) = \sin(\frac{\pi}{2}) = 1$. Thus, we have $\left(\frac{u^2}{2}\right)\bigg|_{u_1}^{u_2} = \frac{1^2}{2} - \frac{0^2}{2} = \frac{1}{2}$, which is the same as the answer we found using method 1.

Problem: Evaluate $\int_0^{\frac{\pi}{2}} \sin^2 x \cos x dx$.

Solution: We make the same substitution as in examples 1 and 2, $u = \sin x$ $du = \cos x dx$. This gives the integral $\int_{u_1}^{u_2} u^2 du$, which can be evaluated normally exactly as $\int_{u_1}^{u_2} u du$. In this case, we find

$$\int_{u_1}^{u_2} u^2 du = \left\langle \frac{u^3}{3} \right\rangle_{u_1}^{u_2} = \left\langle \frac{\sin^3}{3} \right\rangle_0^{\frac{\pi}{2}} = \frac{1}{3} \langle \sin^3 \rangle_{\frac{\pi}{2}} - \sin^3(0) = \frac{1}{3}(1 - 0) = \frac{1}{3}$$

Check Your Progress 5

- 1) What is the relation between integration and differentiation in the use of chain rule.

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- 2) Suppose you are evaluating integral $\int_{-1}^2 4x(2x^2 + 3)^2 dx$ by u-substitution. If you take $u = 2x^2 + 3$, what are the new limits?

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- 3) Evaluate the above integral, $\int_{-1}^2 4x(2x^2 + 3)^2 dx$ using u-substitution.

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7.11 IMPROPER INTEGRALS

We now discuss briefly one special type of definite integral, the **improper integral**. When one of the limits of integration is $+\infty$ or $-\infty$ a definite integral is called an improper integral. Such integrals are evaluated using the concept of limits according to the following rules:

i) $\int_a^\infty f(x)dx = \lim_{b \rightarrow \infty} \int_a^b f(x)dx.$

ii) $\int_{-\infty}^b f(x)dx = \lim_{a \rightarrow -\infty} \int_a^b f(x)dx$

Example:

Evaluate $\int_1^{\infty} \frac{dx}{x^2}$

Since $\int_1^b \frac{dx}{x^2} = -\frac{1}{b} + 1$, the desired integral is

$$\lim_{b \rightarrow \infty} \int_1^b \frac{dx}{x^2} = \lim_{b \rightarrow \infty} \left(-\frac{1}{b} + 1 \right) = 1.$$

This integral can be interpreted as the area of the unbounded region between $f(x) = \frac{1}{x^2}$, $y = 0$ (the x -axis), and $x = 1$.

Definition. Let f be a function which is continuous on the closed interval $[a, \infty)$. We define

$$\int_a^{\infty} f(x) dx = \lim_{N \rightarrow \infty} \int_a^N f(x) dx$$

If this limit exists and is finite then we say that the integral $\int_a^{\infty} f(x) dx$ is convergent; otherwise, we say that the integral is divergent.

Determine if the following integral is convergent or divergent and if it's convergent find its value.

$$\int_1^{\infty} \frac{1}{x} dx$$

Solution: So, the first thing we do is convert the integral to a limit.

$$\int_1^{\infty} \frac{1}{x} dx = \lim_{a \rightarrow \infty} \int_1^a \frac{1}{x} dx$$

Now, do the integral and the limit.

$$\int_1^{\infty} \frac{1}{x} dx = \lim_{a \rightarrow \infty} \ln(a) - \ln(1) = \infty$$

So, the limit is infinite and the integral is divergent.

7.11.1 Convergent and Divergent Integrals

Definition. Let f be a function which is continuous on the closed interval $[a, \infty)$. We define

$$\int_{-\infty}^b f(x) dx = \lim_{N \rightarrow -\infty} \int_N^b f(x) dx$$

If this limit exists and is finite then we say that the integral $\int_{-\infty}^b f(x) dx$ is convergent; otherwise, we say that the integral is divergent.

Example: Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_{-\infty}^0 \frac{1}{\sqrt{3-x}} dx.$$

Solution: There really isn't much to do with these problems once you know how to do them. We'll convert the integral to a limit/integral pair, evaluate the integral and then the limit.

$$\begin{aligned} \int_{-\infty}^0 \frac{1}{\sqrt{3-x}} dx &= \lim_{b \rightarrow -\infty} \int_{-\infty}^0 \frac{1}{\sqrt{3-x}} dx = \lim_{b \rightarrow -\infty} -2|\sqrt{3-x}|_b^0 = \\ &= \lim_{b \rightarrow -\infty} (-2\sqrt{3} + 2\sqrt{(3-b)}) = -2\sqrt{3} + \infty \end{aligned}$$

So, the limit is infinite and so this integral is divergent.

Definition. Let f be a function which is continuous for all real numbers. If, for some real number c , both of $\int_{-\infty}^c f(x)dx$ and $\int_c^{\infty} f(x)dx$ are convergent, then we define $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^c f(x)dx + \int_c^{\infty} f(x)dx$ and we say that the integral $\int_{-\infty}^{\infty} f(x)dx$ is convergent; otherwise, we say that the integral is divergent.

Example: Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_{-\infty}^{\infty} x e^{-x^2} dx$$

Solution: Since the problem has infinities in both limits we need to split up the integral into two separate integrals. Let us choose $a = 0$ which is a convenient point for the evaluation process. The integral then is,

$$\int_{-\infty}^{\infty} x e^{-x^2} dx = \int_{-\infty}^0 x e^{-x^2} dx + \int_0^{\infty} x e^{-x^2} dx$$

We can now look at each of the individual limits.

$$\begin{aligned} \int_0^{\infty} x e^{-x^2} dx &= \lim_{b \rightarrow \infty} \int_0^b x e^{-x^2} dx = \lim_{b \rightarrow \infty} -\frac{1}{2} e^{-x^2} \Big|_0^b \\ &= \lim_{b \rightarrow \infty} \left(-\frac{1}{2} + \frac{1}{2} e^{-b^2} \right) = \frac{1}{2} \end{aligned}$$

So, the first integral is convergent. Let us examine the second integral.

$$\begin{aligned} \int_{-\infty}^0 x e^{-x^2} dx &= \lim_{b \rightarrow -\infty} \int_b^0 x e^{-x^2} dx = \lim_{b \rightarrow -\infty} \left(-\frac{1}{2} e^{-x^2} \right) \Big|_b^0 \\ &= \lim_{b \rightarrow -\infty} \left(\frac{1}{2} e^{-b^2} + \frac{1}{2} \right) = \frac{1}{2} \end{aligned}$$

Thus, the second integral is convergent and we deal with convergent integral. Its value is,

$$\int_{-\infty}^{\infty} x e^{-x^2} dx = \int_{-\infty}^0 x e^{-x^2} dx + \int_0^{\infty} x e^{-x^2} dx = -\frac{1}{2} + \frac{1}{2} = 0$$

Comparison Theorem. Let f and g be functions which are continuous on the closed interval $[a, \infty)$. Suppose that $f(x) > g(x) > 0$ for all x in $[a, \infty)$.

i) If $\int_a^{\infty} f(x) dx$ is convergent then $\int_a^{\infty} g(x) dx$ is also convergent.

ii) If $\int_a^{\infty} g(x) dx$ is divergent then $\int_a^{\infty} f(x) dx$ is also divergent.

Check Your Progress 6

1) What is improper integral?

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2) Determine if the following integral is convergent or divergent. If it is convergent, find its value.

$$\int_{-2}^{\infty} \sin x dx$$

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3) Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_0^3 \frac{1}{(\sqrt{3-x})} dx$$

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7.12 INTEGRATION WITH PARTIAL FRACTIONS

Suppose we deal with integrals of rational expressions of polynomials. For example, consider an integral like $\int \frac{3x+11}{x^2-x-6} dx$. You may see that substitution method will not work here. However, if you break the integrand as,

$\frac{3x+11}{x^2-x-6} = \frac{4}{x-3} - \frac{1}{x+2}$, then the integration can be performed as follows:

$$\int \frac{3x+11}{x^2-x-6} = \int \left(\frac{4}{x-3} - \frac{1}{x+2} \right) dx = 4\ln|x-3| - \ln|x+2| + c.$$

Such a decomposition of the rational expression is called **partial fraction decomposition**. In what follows is a quick review of such fractions. We will take the help of examples to highlight the solutions.

Start with a rational express of the form: $f(x) = \frac{p(x)}{q(x)}$, where both $p(x)$ and $q(x)$ are polynomials. We can use decomposition to solve the expression if the degree of $p(x)$ is smaller than the degree of $q(x)$. Thus, you need to remember that partial fractions can be only considered only if conditions satisfied are: (i) there are rational expression has polynomials in numerator and denominator and (ii) the degree of the numerator is strictly less than the degree of the denominator.

When the above two conditions satisfied, the crucial point remains is to find the decomposition for a given function. For example, a partial fractions decomposition for $\frac{6}{x^2-1}$ can be performed by factoring the denominator such that $\frac{6}{x^2-1} = \frac{6}{(x-1)(x+1)}$. Let us assume that there are constants A and B , so that

$\frac{6}{(x-1)(x+1)} = \frac{A}{(x-1)} + \frac{B}{(x+1)}$. It can be shown that such constants always exist for the rational function $\frac{p(x)}{q(x)}$ if the above mentioned two conditions are met.

Let's proceed to solve preceding problem. So far we have $\frac{6}{x^2-1} = \frac{6}{(x-1)(x+1)} = \frac{A}{(x-1)} + \frac{B}{(x+1)}$

$$= \frac{A}{x-1} + \frac{B}{x+1} = \frac{A}{x-1} \cdot \frac{(x+1)}{(x+1)} + \frac{B}{x+1} \cdot \frac{(x-1)}{(x-1)} = \frac{A(x+1)}{(x-1)(x+1)} + \frac{B(x-1)}{(x+1)(x-1)} = \frac{A(x+1)+B(x-1)}{(x-1)(x+1)}$$

$$\text{or, } \frac{6}{(x-1)(x+1)} = \frac{A(x+1)+B(x-1)}{(x-1)(x+1)}$$

$$\text{or, } 6 = A(x+1) + B(x-1).$$

Now if we let $x = 1$,

$$\text{then } 6 = A(1+1) + B(1-1) = 2A + (0)$$

so that $A = 3$.

If we choose to let $x = -1$,

$$\text{then } 6 = A(-1+1) + B(-1-1) = (0) - 2B$$

so that $B = -3$.

The decomposition of $\frac{6}{(x-1)(x+1)}$ is equal to $\frac{3}{x-1} + \frac{-3}{x+1}$.

Here is another important point to consider when applying the method of

partial fractions to the rational function $\frac{p(x)}{q(x)}$. If the degree (highest power) of $p(x)$ is equal to or greater than the degree of $q(x)$, then you must rewrite the given rational function as the sum of a polynomial and a new rational function satisfying condition (ii) above. For example, to find a partial fractions decomposition for $\frac{2x-3}{x^3+x}$ begin by factoring the denominator such that $\frac{2x-3}{x^3+x} = \frac{2x-3}{x(x^2+1)}$.

Assuming that there three constants A , B and C , write $\frac{2x-3}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{A(x^2+1)}{x(x^2+1)} + \frac{(Bx+C)x}{x(x^2+1)} = \frac{A(x^2+1)+(Bx+C)x}{x(x^2+1)}$.

Thus, $2x - 3 = A(x^2 + 1) + (Bx + C)x$.

Now if we choose $x = 0$, then $2(0) - 3 = A((0)^2 + 1) + B(0) + C(0) = A$ so that $A = -3$. If we choose to let $x = i$, where i is an imaginary number, then

$$2i - 3 = A(i^2 + 1) + (Bi + C)i = A((-1) + 1) + Bi^2 + Ci = Ci - B$$

so that $B = 3$ and $C = 2$. With these values we get a partial fractions decomposition

We can now say that a partial fractions decomposition for $\frac{2x-3}{x(x^2+1)} = \frac{-3}{x} + \frac{3x+2}{x^2+1}$.

Example: To get partial fraction decomposition of a rational function repeated factors in the denominator see the following example.

$$\text{Let } \frac{2x^3+5x-1}{(x+1)^3(x^2+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} + \frac{Dx+E}{x^2+1} + \frac{Fx+G}{(x^2+1)^2}$$

So, once we've determined that partial fractions can be done we factor the denominator as completely as possible. Then for each factor in the denominator we can use determine the term(s) in the partial fraction decomposition.

Example: To find solution of the integral when degree of the numerator is not less than the degree of the denominator see the example below:

$$\text{Integrate } \int \frac{x^4+x^3+x^2+1}{x^2+x-2} dx$$

$$\text{Solution: } \int \frac{x^4+x^3+x^2+1}{x^2+x-2} dx = \int \left(x^2 + 3 + \frac{-3x+7}{x^2+x-2} \right) dx = \int \left(x^2 + 3 + \frac{-3x+7}{(x+2)(x-1)} \right) dx$$

$$= \int \left(x^2 + 3 + \frac{A}{x+2} + \frac{B}{x-1} \right) dx = A(x-1) + B(x+2) = -3x + 7;$$

$$\text{let } x = -2 \text{ so that } A(-3) + B(0) = 13 \rightarrow A = -\frac{13}{3};$$

$$\text{let } x = 1 \text{ so that } A(0) + B(3) = 4 \rightarrow B = \frac{4}{3}$$

$$\begin{aligned}\text{Now } \int \left(x^2 + 3 + \frac{-13}{x+2} + \frac{4}{x-1} \right) dx &= \int \left(x^2 + 3 - \frac{13}{3} \frac{1}{x+2} + \frac{4}{3} \frac{1}{x-1} \right) dx \\ &= \frac{x^3}{3} + 3x - \frac{13}{3} \ln|x+2| + \frac{4}{3} \ln|x-1| + C.\end{aligned}$$

Example: To find solution of the integral when there is a repeated linear factor see the example below: Integrate: $\int \frac{x+7}{x^2+(x+2)} dx$.

Solution: $\int \frac{x+7}{x^2+(x+2)} dx = \int \left(\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2} \right) dx$

or, $A(x+2) + B(x+2) + Cx^2 = x+7$;

let $x = 0$ so that $A(0) + B(2) + C(0) = 7 \rightarrow B = \frac{7}{2}$;

let $x = -2$ so that $A(0) + B(2) + C(4) = 5 \rightarrow C = \frac{5}{4}$;

let $x = -1$ so that $A(-1) + B(1) + C(1) = -A + \frac{7}{2} + \frac{5}{4} = 6 \rightarrow A = -\frac{5}{4}$

$$\begin{aligned}\text{Now } \int \left(\frac{-5}{4} \frac{1}{x} + \frac{7}{2} \frac{1}{x^2} + \frac{5}{4} \frac{1}{x+2} \right) dx &= \int \left(-\left(\frac{5}{4}\right) \frac{1}{x} + \left(\frac{7}{2}\right) \frac{1}{x^2} + \left(\frac{5}{4}\right) \frac{1}{x+2} \right) dx = \\ &= -\left(\frac{5}{4}\right) \ln|x| + \left(\frac{7}{2}\right) \frac{x^{-1}}{(-1)} + \left(\frac{5}{4}\right) \ln|x+2| + C = \int \left(\frac{-5}{4} \frac{1}{x} + \frac{7}{2} \frac{1}{x^2} + \frac{5}{4} \frac{1}{x+2} \right) dx = \\ &= \int \left(-\left(\frac{5}{4}\right) \frac{1}{x} + \left(\frac{7}{2}\right) \frac{1}{x^2} + \left(\frac{5}{4}\right) \frac{1}{x+2} \right) dx = -\left(\frac{5}{4}\right) \ln|x| + \left(-\frac{7}{2x}\right) + \left(\frac{5}{4}\right) \ln|x+2| + C\end{aligned}$$

Example: To find solution of the integral when there is a non-linear factor see the example below:

Integrate $\int \frac{x^5+1}{x^3(x+2)} dx$.

Solution: $\int \frac{x^5+1}{x^3(x+2)} dx = \int \frac{x^5+1}{x^4+2x^3} dx = \int \left(x-2 + \frac{4x^3+1}{x^4+2x^3} \right) dx =$
 $\int \left(x-2 + \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x+2} \right) dx$

Or, $Ax^2(x+2) + B(x+2) + C(x+2) + Dx^3 = 4x^3 + 1$

Let $x = 0$ so that $A(0) + B(2) + C(2) + D(0) = 1 \rightarrow C = \frac{1}{2}$.

let $x = -2$ so that $A(0) + B(0) + C(0) + D(-8) = -31 \rightarrow D = \frac{31}{8}$;

let $x = 1$ so that $A(3) + B(3) + C(3) + D = 3A + 3B + \frac{3}{2} + \frac{31}{8} = 5 \rightarrow A + B = -\frac{1}{8}$;

let $x = -1$ so that $A(1) + B(1) + C(1) + D(-1) = A - B + \frac{1}{2} - \frac{31}{8} \rightarrow A - B = \frac{3}{8}$;

it follows that $A = \frac{1}{8}$ and $B = -\frac{1}{4}$.

Now
$$\int \left(x - 2 + \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x+2} \right) dx = \int \left(x - 2 + \frac{\frac{1}{8}}{x} + \frac{\frac{-1}{4}}{x^2} + \frac{\frac{1}{2}}{x^3} + \frac{\frac{31}{8}}{x+2} \right) dx$$
$$= \frac{x^2}{2} - 2x + \frac{1}{8} \ln|x| - \left(\frac{1}{4} \right) \frac{x^{-1}}{(-1)} + \left(\frac{1}{2} \right) \frac{x^{-2}}{-2} + \frac{31}{8} \ln|x+2| + C =$$
$$\int \left(x - 2 + \frac{\frac{1}{8}}{x} + \frac{\frac{-1}{4}}{x^2} + \frac{\frac{1}{2}}{x^3} + \frac{\frac{31}{8}}{x+2} \right) dx = \frac{x^2}{2} - 2x + \frac{1}{8} \ln|x| + \frac{1}{4x} - \frac{1}{4x^2} + \frac{31}{8} \ln|x+2| + C.$$

Consider an example of an improper rational function to evaluate the definite integral. We apply the technique of partial fractions first and then apply the limit values as has been done in cases with definite integrals.

$$\int_1^2 \frac{x^3 + 4x^2 - 3x + 3}{x^2 - 3x} dx$$

Solution: This rational function is improper because its numerator has a degree that is higher than its denominator. So take $\frac{x^3 + 4x^2 - 3x + 3}{x^2 - 3x} = (x - 1) + \frac{-6x + 3}{-x^2 - 3x}$.

Now apply partial function decomposition only on the remainder,

$$\frac{-6x + 3}{-x^2 - 3x} = \frac{-6x + 3}{-x(x - 3)} = \frac{A}{x} + \frac{B}{x - 3}$$

For solving the integration the basic equation becomes $-6x + 3 = A(x - 3) + Bx$.

For $x=0$, $3 = -3A + 0 = A$, or, $-1 = A$

For $x=3$, $-18 + 3 = 0 + 3B$

or, $B = -5$

Thus the integral becomes

$$\int_1^2 \frac{x^3 + 4x^2 - 3x + 3}{x^2 - 3x} dx = \int_1^2 \left[(x - 1) + \frac{-6x + 3}{-x^2 - 3x} \right] dx = \int_1^2 \left[x - 1 - \frac{1}{x} - \frac{5}{x - 3} \right] dx$$

Integrating and substituting the limits,

$$= \left[\frac{x^2}{2} - x - \ln|x| - 5\ln|x - 3| \right]_1^2$$
$$= \left(\frac{4}{2} - 2 - \ln|2| - 5\ln|1| \right) - \left(\frac{1}{2} - 1 - \ln|1| - 5\ln|2| \right)$$
$$= 4\ln|2| + \frac{1}{2}.$$

We take up integration of a trigonometric function to solve it with partial fractions.

Example: $\int \sin^5 x dx = \int (\sin x)(\sin^2 x)^2 dx = \int (\sin x)(\sin^2 x)^2 dx = \int (\sin x)(1 - \cos^2 x)^2 dx$

Now by using substitution we can evaluate the integral by letting $u = \cos x$, then $du = -\sin x dx$ and

$$\begin{aligned} \int \sin^5 x dx &= \int (\sin x)(\sin^2 x)^2 dx = \int (\sin x)(\sin^2 x)^2 dx \\ &= \int (\sin x)(1 - \cos^2 x)^2 dx \end{aligned}$$

Using method of substitution

$$= \int -(1 - u^2)^2 du = \int (-u^4 + 2u^2 - 1) du = \frac{-u^5}{5} + 2\frac{u^3}{3} - u + C$$

or, $\frac{-\cos^5 x}{5} + \frac{2\cos^3 x}{3} - \cos x + C$

Check Your Progress 7

1) Evaluate $\int \frac{3x+11}{x^2-x-6} dx$

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2) Find $\int \frac{2x+7}{x^2+4x+3} dx$

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3) Use partial fractions to evaluate: $\int \frac{6x+7}{(x+2)^2} dx$

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7.13 LET US SUM UP

This unit has discussed the technique of integration. The concepts of indefinite and definite integrals are explained. Whereas the indefinite integral is a function and an anti-derivative, the definite integral is a number and represents the area under the curve. The process of finding solution to different functions through substitution, integration by part, trigonometric substitution and partial

7.14 KEY WORDS

Boundary condition: A particular anti-derivatives that satisfies certain initial condition.

Closed Interval: A closed interval is an interval that includes all of its limit points. If the endpoints of the interval are finite numbers a and b , then the interval $x: a \leq x \leq b$ is denoted $[a, b]$. If one of the endpoints is $\pm\infty$, then the interval still contains all of its limit points (although not all of its *endpoints*), so a, ∞ and $(-\infty, b)$ are also closed intervals, as is the interval $(-\infty, \infty)$.

Definite Integral: The definite integral of the function $f(x)$ over the interval (a, b) is expressed symbolically as $\int_a^b f(x)dx$, read as “integral of f with respect

to x from a to b . The smaller number a is termed the **lower limit** and b , the **upper limit** of integration. Geometrically, this definite integral denotes the area under the curve representing $f(x)$ between the points $x = a$ and $x = b$.

Note that the definite integral $\int_a^b f(x)$ is a number.

Improper Integral: It is a special type of definite integral. When one of the limits of integration is $+\infty$ or $-\infty$, a definite integral is called an improper integral. Such integrals are evaluated using the concept of limits.

Indefinite Integral: The indefinite integral is basically reverse differentiation. To differentiate means to find the rate of change (derivative) of a given function. Indefinite integration reverses the process and finds the unknown function whose rate of change (derivative) is given.

Integration by Parts: A special method of **integration** that is often useful when two functions are multiplied together.

Partial Fractions: Each of two or more fractions into which a more complex fraction can be decomposed as a sum.

Rational Substitution: Integrand can be changed to a rational function by using substitutions.

Riemann Sum: A method for approximating the total area underneath a curve on a graph.

Substitution Rule: If $g(x)$ is a differentiable function whose range is the interval I and f is continuous on I , then $\int f(g(x))g'(x)dx = \int f(u)du$ where $u = g(x)$ and $du = g'(x)dx$.

Trigonometric Substitution: Make *substitutions* using *trigonometric* expressions.

7.15 USEFUL BOOKS

IGNOU, 1990, *MTE-01: First Elective Course in Mathematics* (Block 3: Integral Calculus).

K. Sydsaeter and P. Hammond (2002), *Mathematics for Economic Analysis*, Pearson Educational Asia, Delhi.

Kevin Wainwright and Alpha Chiang (2004), *Fundamental Methods of Mathematical Economics*, 4th edition, McGraw-Hill Education.

7.16 ANSWER OR HINTS TO CHECK YOUR PROGRESS

Check Your Progress 1

- 1) We wish to perform the **opposite process** to differentiation and call this as "anti-differentiation". Thus indefinite integral is a function whose derivative is a given function.
- 2) Using the rule of integration of a constant, we write: $\int 4 \, dx = 4x + C$
- 3) $\frac{1}{3}x^3 + 3x + C$
- 4) $-3e^{-3x} + C$
- 5) $\ln(x + 2) + C$

Check Your Progress 2

- 1) $\frac{d}{dx} \sin ax = a \cos ax$. We can say that $\int \cos ax \, dx = \frac{1}{a} \sin ax + C$ where a is an arbitrary constant and C is the constant of integration.
- 2) Always try for substitution to get the integration, if you cannot, go to integration by parts.
- 1) Hint: It simplifies the evaluation (derive the integral by taking $u = \ln x$).

Check Your Progress 3

- 1) An integral expressed as the difference between the values of the integral at specified upper and lower limits of the independent variable.
- 2) Let a closed interval $[a, b]$ be partitioned by points $a < x_1 < x_2 < \dots < x_{n-1} < b$, where the lengths of the resulting intervals between the points are denoted $\Delta x_1, \Delta x_2, \dots, \Delta x_n$. Let x_k^* be an arbitrary point in the k th subinterval. Then the quantity $\sum_{i=1}^n f(x_k^*) \Delta x_k$ is called a Riemann sum for a given function $f(x)$ and partition, and the value Δx_k is called the mesh size of the partition.
- 3) Answer yourself after reading the properties of definite integral.

Check Your Progress 4

- 1) It links the concept of a derivative to that of an integral. As a result, we can use our knowledge of derivatives to find the area under the curve, which is often quicker and simpler than using the definition of the integral.
- 2) They are opposites. If you take the derivative of a function, then integrate the result, you'll end up with the function again. Definition-wise, the derivative of a function is the slope of a function, while the integral is the area between the function and the x-axis.
- 3) The definite integral of $f(x)$ is a number and represents the area under the curve $f(x)$ from $x = a$ to $x = b$. Whereas the indefinite integral is a function $f(x)$ and is an anti-derivative. In other words, it is a function $F(x)$ whose derivative is $f(x)$.

Check Your Progress 5

- 1) Integrating with the chain rule is the opposite of taking the derivative with the chain rule. For the derivative of the chain rule, $\frac{dy}{dx} a(g(x))^n = na(g(x))^{n-1} g'(x)$. So $\int na(g(x))^{n-1} g'(x) = a(g(x))^n$.
- 2) New limits are $a = 2(-1)^2 + 3 = 5$ and $b = 2(2)^2 + 3 = 11$
- 3) Put $u = 2x^2 + 3$ to get $du = 4xdx$
Now

$$\int_{-1}^2 4x(2x^2 + 3)^2 dx = \int_5^{11} 4xu^2 \frac{du}{4x} = \int_5^{11} u^2 du = \frac{1}{3}(u^3)|_5^{11} = \frac{1}{3}(11^3 - 5^3) = \frac{1206}{3} = 402.$$

Check Your Progress 6

- 1) An **improper integral** is a definite **integral** that has either or both limits infinite or an integrand that approaches infinity at one or more points in the range of integration.
- 2) $\int_{-2}^{\infty} \sin x dx = \int_b^0 \sin x dx = \lim_{b \rightarrow \infty} (-\cos x)|_{-2}^b = \lim_{b \rightarrow \infty} (\cos 2 - \cos b)|_{-2}^b$. As the limit doesn't exist, the integral is divergent.
- 3) $\int_0^3 \frac{1}{(\sqrt{3-x})} dx = \lim_{b \rightarrow 3^-} (-2\sqrt{3-x})|_0^b = \lim_{b \rightarrow 3^-} (2\sqrt{3}) - 2\sqrt{3-b} = 2\sqrt{3}$. The limit exists and is finite and so the integral converges and the integral's value is $2\sqrt{3}$.

Check Your Progress 7

- 1) $\int \frac{3x+11}{(x-3)(x+2)} dx = \int \left(\frac{A}{(x-3)} + \frac{B}{(x+2)} \right) dx$. Now work with $\frac{3x+11}{(x-3)(x+2)} = \left(\frac{A}{(x-3)} + \frac{B}{(x+2)} \right) = \left(\frac{A(x+2) + B(x-3)}{(x-3)(x+2)} \right)$ to get $3x + 11 = A(x + 2) + B(x - 3)$.

Let $x = -2$ so that $5 = A(0) + B(-5) \rightarrow B = -1$ and $x = 3$ so that $20 = A(5) + B(0) \rightarrow A = 4$.

Putting the value of A and B in the integration we have

$$\begin{aligned}\int \frac{3x+11}{(x-3)(x+2)} dx &= \int \left(\frac{4}{(x-3)} + \frac{-1}{(x+2)} \right) dx \\ &= \int \left(\frac{4}{(x-3)} \right) dx + \int \left(\frac{-1}{(x+2)} \right) dx \\ &= 4 \ln|x-3| - \ln|x+2| + C\end{aligned}$$

- 2) Set up partial fractions: $\frac{2x+7}{x^2+4x+3} = \frac{A}{x+1} + \frac{B}{x+3}$ so that $2x+7 = A(x+3) + B(x+1)$.

Solve for $x = -1$ and $x = -3$ to get $A = \frac{5}{2}$ and $B = \frac{-1}{2}$.

$$\text{Ans.: } \frac{5}{2} \ln|x+1| - \frac{1}{2} \ln|x+3| + C.$$

- 3) Decompose partial fraction: $6x+7 = A(x+2) + B$.

Now $\frac{6x+7}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}$. If $x = -2$, then $B = -5$; if $x = 0$, then $A = 6$.

So, $\int \frac{6x+7}{(x+2)^2} dx = \int \frac{6}{x+2} dx - \int \frac{5}{(x+2)^2} dx$. Put $u = x+2$ and solve to get the answer as $6 \ln|x+2| + \frac{5}{x+2} + C$.

7.17 EXERCISES AND ANSWER OR HINTS

- 1) Evaluate the following integral: $\int \sin^6 x \cos^3 x dx$

Ans. Hint: $\int \sin^6 x \cos^3 x dx = \int \sin^6 x \cos^2 x \cos x dx = \int \sin^6 x (1 - \sin^2 x) \cos x dx$. Put $\sin x = u$ to get $\int u^6 (1 - u^2) du$. You can integrate the function now.

- 2) Evaluate the following integral:

$$\int \frac{1}{2} \left((1 - \cos(2x))^{\frac{1}{2}} (1 + \cos(2x)) \right) dx$$

Ans. Hint: write $\int \sin^2 x \cos^2 x dx = \int \frac{1}{2} ((1 - \cos(2x)) \left(\frac{1}{2} \right) ((1 + \cos(2x)) dx = \frac{1}{4} \int 1 - \cos^2(2x) dx = \frac{1}{4} \int 1 - \frac{1}{2} (1 + \cos(4x)) dx$ to get the answer $\frac{1}{8} x - \frac{1}{32} \sin(4x) + C$.

- 3) Evaluate the following integral: $\int \sec x dx$

Ans.: $\int \sec x dx = \int \frac{\sec x (\sec x + \tan x)}{\sec x + \tan x} dx = \int \frac{(\sec^2 x + \tan x \sec x)}{\sec x + \tan x} dx$. Put $\sec x + \tan x = u$ for integration which yields $\ln|\sec x + \tan x| + C$.

- 4) Evaluate the integral: $\int x e^{6x} dx$

See that choosing $u = x$ will help as on differentiating x will drop out.

Since $u = x, du = dx$.

Put $dv = e^{6x} dx$ so that $v = \int e^{6x} dx = \frac{1}{6} e^{6x}$

The integral is then $\int e^{6x} dx = \frac{x}{6} e^x dx - \int \frac{1}{6} e^{6x} dx = \frac{x}{6} e^x - \frac{1}{36} e^{6x} + C$

- 5) Evaluate the following integral $\int x\sqrt{x+1} dx$

a) Using Integration by Parts.

b) Using substitution.

Solution: (a) Evaluate using Integration by Parts.

We select the choices for u and dv to handle the problem at hand.

Take $u = x$ and $dv = \sqrt{x+1} dx$ so that $du = dx$ and $v = \frac{2}{3}(x+1)^{\frac{3}{2}}$

Then the integral is $\int x\sqrt{x+1} dx = \frac{2}{3}(x+1)^{\frac{3}{2}} - \frac{2}{3} \int (x+1)^{\frac{3}{2}}(x+1)^{\frac{3}{2}} dx = \frac{2}{3}(x+1)^{\frac{3}{2}} - \frac{4}{15}(x+1)^{\frac{5}{2}} + C$

b) Evaluate using method of substitution

Take $u = x+1$ so that $x = u-1$ and $du = dx$.

Note that you need to use the substitution twice, once for the quantity under the square root and once for the x in front of the square root. The integral is then, $\int x\sqrt{x+1} dx = \int (u-1)\sqrt{u} du = \int u^{\frac{3}{2}} - u^{\frac{1}{2}} du = \frac{2}{5} u^{\frac{5}{2}} - \frac{2}{3} u^{\frac{3}{2}} + C$

$$= \frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}} + C$$

- 6) Integrate $\int \frac{\ln x}{x^5} dx$

Ans. Hints: Put $u = \ln x$ and $dv = \frac{1}{x^5} dx$. Final answer is $\frac{\ln x}{x^4} - \frac{1}{16x^4} + C$.

- 7) Integrate $\int x^2 e^{3x} dx$

Ans. Hints: Put $u = x^2$ and $dv = e^{3x} dx$. Final answer is $\frac{1}{3} x^2 e^{3x} - \frac{2}{9} x^2 e^{3x} + \frac{2}{27} e^{3x} + C$.

Integrate $\int \frac{\ln x}{x^5} dx$

Ans. Hints: Put $u = \ln x$ and $dv = \frac{1}{x^5} dx$. Final answer is $\frac{\ln x}{x^4} - \frac{1}{16x^4} + C$.

8) Integrate $\int x^2 e^{3x} dx$

Ans. Hints: Put $u = x^2$ and $dv = e^{3x} dx$. Final answer is $\frac{1}{3} x^2 e^{3x} - \frac{2}{9} x^2 e^{3x} + \frac{2}{27} e^{3x} + C$.

9) Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_{-2}^3 \frac{1}{x^3} dx$$

Solution: This integrand is not continuous at $x = 0$. So we'll split it at that point.

$$\text{Thus, } \int_{-2}^3 \frac{1}{x^3} dx = \int_{-2}^0 \frac{1}{x^3} dx + \int_0^3 \frac{1}{x^3} dx$$

Now we need to look at each of these integrals and see if they are convergent.

$$\begin{aligned} \int_{-2}^0 \frac{1}{x^3} dx &= \lim_{b \rightarrow 0^-} \int_{-2}^b \frac{1}{x^3} dx \\ &= \lim_{b \rightarrow 0^-} \left(-\frac{1}{2x^2} \right) \Big|_{-2}^b = \lim_{b \rightarrow 0^-} \left(-\frac{1}{2b^2} + \frac{1}{8} \right) = -\infty \end{aligned}$$

One of the integrals is divergent which means the given integral is divergent. We do not need to examine the second integral.

10) Find solution of $\int \frac{2-x}{x^2+5x} dx$

$$\text{Ans.: } \frac{2}{5} \ln|x| - \frac{7}{5} \ln|x+5| + C$$

11) Find solution of $\int \frac{2-x}{x^2+5x} dx$

$$\text{Ans.: } \int \frac{2-x}{x^2+5x} dx = \int \left(1 + \frac{15}{x^2-16} \right) dx \text{ will give } A(x-4) + B(x+4) = 15;$$

Putting $x = -4$ and $x = 4$ you get $A = -15/8$ and $B = 15/8$ leading to the solution of

$$x + \frac{15}{8} (\ln|x-4| - \ln|x+4|) + C.$$

UNIT 8 INTEGRATION AND ECONOMIC DYNAMICS

Structure

- 8.0 Objectives
- 8.1 Introduction
- 8.2 Dynamics and Integration
- 8.3 The Tools of Dynamics
 - 8.3.1 The Indefinite Integral and its Economic Applications
 - 8.3.2 The Definite Integral and its Economic Applications
 - 8.3.3 Differential Equations and its Economic Applications
- 8.4 Let Us Sum Up
- 8.5 Key Words
- 8.6 Some Useful Books
- 8.7 Answers or Hints to Check Your Progress Exercises

8.0 OBJECTIVES

After going through this unit, you should be able to:

- identify the dynamics problems in economics;
- state the rules of adjustment to obtain a dynamic equilibrium; and
- use the mathematical tools of dynamics to solve problems related to economic theory.

8.1 INTRODUCTION

The term **dynamic** refers to events that change over time. Therefore, we would try to introduce time explicitly into certain categories of economic analysis. In order to do that let us see the mathematical techniques of integral calculus and differential equations.

8.2 DYNAMICS AND INTEGRATION

In a dynamic economics model, the basic objective is the identification of the time path of the variable on the basis of its rate of change. For example, national income y of a country changes overtime. To see the rate of change we need to see its change with respect to time and to find the time path followed by y , we look at the rate of change. Thus, if we know the derivative $\frac{dy}{dt}$, it will be possible to get onto the function like $y = y(t)$ through the technique of

integration which happens to be opposite of the process of differentiation. We will return to this process after a while.

8.3 THE TOOLS OF DYNAMICS

The dynamic exercises are usually carried on by differential and difference equations. If we use 'time' as the independent variable, these equations are eminently suited for the formal analysis of behaviour through time. Apart from these, the simple integral, both indefinite and definite, is also useful in many contexts in defining some important concepts of economic dynamics.

In this unit, we begin by introducing the notion of the integral. Then we move on to discuss the technique of differential equations.

From now on, all functions in this unit will be assumed to be continuous and real valued.

8.3.1 The Indefinite Integral and its Economic Applications

As we have pointed out above, basically indefinite integral is reverse differentiation. Recall that differential calculus gives the rate of change (derivative) of a given function. Indefinite integration reverses such a process and finds the unknown function whose rate of change (derivative) is given. If the function $f(x)$ is written symbolically as

$$\int f(x)dx$$

which is read as "the integral of $f(x)$ with respect to x " and $f(x)$ is the 'integrand'. We can write

$$\int f(x)dx = F(x)$$

provided $F(x)$ is the derivative of $f(x)$ i.e. $f(x) = \frac{d}{dx}F(x)$. It is immediately clear that the function $F(x)$ is not unique, because if $F(x)$ has the derivative $f(x)$, then so has $F(x) + c$ where c is any arbitrary constant. That is why the indefinite integral is always written with an arbitrary constant, called 'the constant of integration'. This non-uniqueness is due to the fact that functions, which differ only by an additive constant, have the same derivative.

Examples:

$$1) \quad \int (x^2 + 3)dx = \frac{1}{3}x^3 + 3x + c$$

$$2) \quad \int 9e^{-3x} dx = -3e^{-3x} + c$$

$$3) \quad \int \frac{1}{x+2} dx = \log(x+2) + c$$

It may be verified that in each case the derivative of the right hand side equals the corresponding integrand on the left-hand side. In other words, we check that

$$\frac{d}{dx} \left[\int f(x) dx \right] = f(x)$$

We now state two useful rules of integration.

Rule 1: The integral of a constant time a function equals the constant times the integral of the function.

$$\int kf(x) dx = k \int f(x) dx$$

Example: $\int 5x^2 dx = 5 \int x^2 dx = 5 \left(\frac{x^3}{3} + c \right)$

This rule is obviously related to the rule that the derivative of a constant time a function is the constant times the derivative of the function. For the particular case $k = -1$ we have the result

$$\int [-f(x)] dx = - \int f(x) dx.$$

Rule 2: The integral of a sum of functions equals the sum of the integrals of the functions. For the sum of two functions this can be written as

$$\int [f(x) + g(x)] dx = \int f(x) dx + \int g(x) dx.$$

Example: $\int (4x^3 + 7x + 3) dx = \int (4x^4) dx + \int (7x) dx + \int 3 dx$

$$= \frac{4}{5} x^5 + \frac{7}{2} x^2 + 3x + c$$

The rule is directly related to the rule that the derivative of a sum of function is the sum of the derivatives of the functions.

Some Useful Formulae

We state without proof some useful formulae for integration.

$$1) \quad \int x^n dx = \frac{x^{n+1}}{n+1} + c, n \neq -1$$

$$2) \quad \int x^{-1} dx = \int \frac{dx}{x} = \log x + c \quad \text{for any } x > 0$$

$$3) \quad \int e^{mx} dx = \frac{e^{mx}}{m} + c \quad \text{for any } m.$$

$$4) \quad \int a^{mx} dx = \frac{a^{mx}}{m \log a} + c$$

$$5) \quad \int \cos ax dx = \frac{\sin ax}{a} + c$$

$$6) \quad \int \sin ax dx = -\frac{\cos ax}{a} + c$$

$$7) \quad \int [K_1 f(x) + K_2 g(x)] dx = K_1 \int f(x) dx + K_2 \int g(x) dx + c \quad \text{for } K_1, K_2 \geq 0$$

Determination of the Constant of Integration using Initial or Boundary Conditions

We have noted that the indefinite integral always contains an arbitrary constant. The value of this constant can be precisely determined (and the integral rendered unique thereby) if it is known that the integral function $y = f(x) + c$ obeys some prescribed **initial condition** ($y = y_0$ when $x = 0$) or more generally **boundary condition** ($y = y_0$ when $x = x_0$). To illustrate, we know

$$\int (x^2 + 3) dx = \frac{1}{3} x^3 + 3x + c$$

Here $y = f(x) + c = \frac{1}{3} x^3 + 3x + c$. Suppose, we are given the initial condition, $y = 20$ when $x = 0$. Then

$$y_0 = f(0) + c = c = 20$$

This fixes the integral as the unique function $y = \frac{1}{3} x^3 + 3x + 20$. We will return to the initial condition again in Sub-section 8.3.3.

Some Computational Methods

The standard procedure of integration given above is sometimes inadequate for computational purposes. In such a situation, it becomes necessary to attempt certain kinds of manipulation. Before the function becomes amenable to integration. We have a few standard results to help initiate the process of integration. However, remember that these are not all that can be used and there are no routine manipulations that can be prescribed. Practice opens up the channels of finding a solution to the integral.

1) Method of Substitution

If $F(x) = \int f(x) dx$, the indefinite integral can be obtained by resorting to transformation. If we take $x = g(y)$ then

$$\int f(x) dx = \int [g(y)] g'(y) dy.$$

See that $x = g(y) \Rightarrow dx = g'(y) dy$.

If $\phi(y) = \int f[g(y)]g'(y)dy$, then

$F(x) = \phi[g^{-1}(\cdot)]$ is the inverse of $g(\cdot)$.

Example: (i) Solve $F(x) = \int (1+5x)^{\frac{1}{2}} dx$.

Let $y = 1 + 5x$

$$dy = 5dx, \text{ or } dx = \frac{1}{5} dy$$

$$\therefore F(x) = \frac{1}{5} \int y^{\frac{1}{2}} dy = \frac{1}{5} \cdot \frac{2}{3} y^{\frac{3}{2}} + c$$

$$= \frac{2}{15} \cdot y^{\frac{3}{2}} + c = \frac{2}{15} (1+5x)^{\frac{3}{2}} + c$$

(ii) **Solve** $F(x) = \int \frac{dx}{(3x-1)^2}$

Let $y = 3x - 1$

$$\text{So, } dy = 3 dx, \text{ or, } dx = \frac{1}{3} dy.$$

$$\therefore F(x) = \frac{1}{3} \int \frac{dy}{y^2} = \frac{1}{3} \left(-\frac{1}{y} \right) + c = \frac{1}{3(3x-1)} + c$$

II) Integration by Parts

Let us return to differentiation of product of two functions $u = f(x)$ and $v = f(x)$ which gives,

$$d(uv) = u \frac{dv}{dx} + v \frac{du}{dx}.$$

From this, we can obtain

$$\int (uv)' dx = \int uv' dx + \int vu' dx$$

$$\therefore \int (uv)' dx = uv$$

$$\therefore uv' dx = uv - \int vu' dx$$

$$\text{Solve } F(x) = \int xe^{-x} dx$$

$$\text{Let } u(x) = x, v(x) = -e^{-x}$$

$$\text{So that } u'(x) = 1, v'(x) = e^{-x}$$

$$\therefore f(x) = \int u.v' dx$$

$$= uv - \int vu' dx$$

$$= -xe^{-x} + \int e^{-x} dx$$

$$= -xe^{-x} - e^{-x} + c = -e^{-x}(1+x) + c$$

Economic Applications of Indefinite Integration

In this subsection, we shall consider two examples: the first one from macro-economic theory and the second from micro-economic theory.

a) Investment and the Stock of Capital

Net investment I is the rate of change of the stock of capital K . Treating time as a continuous variable this is expressed as $I(t) = \frac{dK(t)}{dt}$. It follows that if the rate of investment $I(t)$ is known, the capital stock $K(t)$ can be estimated through the formula,

$$K(t) = \int I(t) dt$$

Example: The rate of net investment is given by $I(t) = 14t^{3/4}$ and the initial stock of capital at $t = 0$ is 100 units. Find the equation for the stock of capital.

$$\begin{aligned} K(t) &= \int 14t^{3/4} dt = 14 \left(\frac{4}{7} \right) t^{7/4} + c \\ &= 8t^{7/4} + c \end{aligned}$$

Also $K(0) = c = 100$ given. Therefore,

$$K(t) = 8t^{7/4} + 100$$

b) Obtaining the Total from the Margin

Since the marginal value of a variable in Economics is defined to be the rate of change of the total value, it follows that integration will enable us to recover the total function once the marginal function is known (assuming the relevant variable to vary continuously).

Example: (1) The marginal cost function of a commodity is:

$$MC = 2 + 12q$$

where q is the quantity of output. Find the total cost function, given that the value of fixed cost is 150.

$$\text{Since } MC = \frac{dTC}{dq}, \text{ we have}$$

$$TC = \int MC dq = \int (2 + 12q) dq = 2q + 6q^2 + C$$

Now fixed cost being the level of cost at zero output, $TC = 150$ for $q = 0$. From the above expression, TC , implies $c = 150$. Thus, the required cost function is $TC(q) = 2q + 6q^2 + 150$.

Example 2:

If the marginal revenue function is $MR = 50 - 10q^2$ where q is the level of output and total revenue is 100 at 3 units of output, find the total revenue function.

Since $MC = \frac{dTR}{dq}$, we can write

$$\begin{aligned} TR &= \int MR dq \\ &= \int (50 - 10q^2) dq \\ &= 50q - \frac{10}{3}q^3 + c \end{aligned}$$

At $q = 3$, $TR = 60 + c = 100$ given. So $c = 40$. The required total revenue function is

$$TR(q) = 50q - \frac{10}{3}q^3 + 40$$

8.3.2 The Definite Integral and its Economic Applications

The definite integral of the function $f(x)$ over the interval $[a, b]$ is expressed symbolically as $\int_a^b f(x)dx$, read as “the integral of f with respect to x from a to b ”.

The smaller number a is termed the **lower limit** and b , the **upper limit**, of integration. Geometrically, this definite integral denotes the area under the curve representing $f(x)$ between the points $x = a$ and $x = b$.

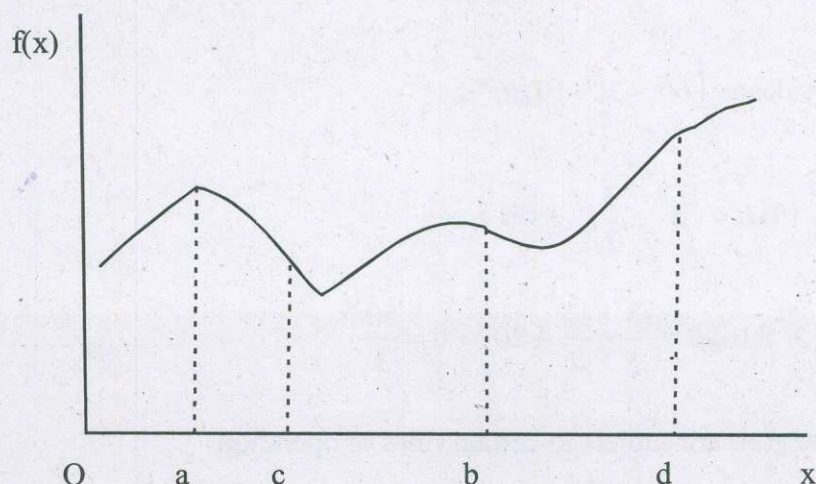


Fig. 8.2

It should be noted that the indefinite integral $\int f(x) dx$ is a **function** of x , whereas the definite integral $\int_a^b f(x) dx$ is a **number**. The numerical value of the definite integral depends on the two limits of integral also changes. This is clear from figure 8.2 where if we change the interval (a, b) to (c, d) the value of the area under the curve will, in general, change.

Another feature of the definite integral is that its value does not depend on the particular symbol chosen to represent the independent variable so long as the form of the function is not changed. That is,

$$\int_a^b f(x) dx = \int_a^b f(t) dt = \int_a^b f(u) du = \dots$$

The following theorem establishes the connection between indefinite and definite integration and supplies the method for evaluating definite integrals.

The Fundamental Theorem of Calculus

If $\int_a^b f(x) dx = f(x) + c$, then

$$\int_a^b f(x) dx = f(b) - f(a).$$

Examples: (1) To evaluate $\int_2^3 3x^2 dx$.

$$\int_2^3 3x^2 dx = x^3 + c$$

$$\text{So, } \int_2^3 3x^2 dx = 3^3 - 2^3 = 19.$$

(2) To evaluate $\int_0^2 (t^2 - 3t + 10) dt$.

$$\int (t^2 - 3t + 10) dt = \frac{1}{3} t^3 - \frac{3}{2} t^2 + 10t + c.$$

$$\text{So, } \int_0^2 (t^2 - 3t + 10) dt = \left(\frac{8}{3} - \frac{12}{2} + 20 \right) - 0 = \frac{50}{3}$$

Definite integrals are subject to certain rules of operation.

Rule 1: If the two limits are equal, the value of the integral is zero.

$$\int_a^b f(x)dx = 0.$$

Rule 2: Reversing the limits of integration changes the sign of the integral.

$$\int_a^b f(x)dx = -\int_b^a f(x)dx.$$

Rule 3: The definite integral can be expressed as the sum of subintegrals.

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

where b is a point within the interval (a, c) .

We now discuss briefly one special type of definite integral, the **improper integral**. When one of the limits of integration is $+\infty$ or $(-\infty)$ a definite integral is called an improper integral. Such integrals are evaluated using the concept of limits according to the following rules:

$$i) \quad \int_a^\infty f(x)dx = \lim_{b \rightarrow \infty} \int_a^b f(x)dx.$$

$$ii) \quad \int_{-\infty}^b f(x)dx = \lim_{a \rightarrow -\infty} \int_a^b f(x)dx$$

Example:

Evaluate $\int_1^\infty \frac{dx}{x^2}$

Since $\int_1^b \frac{dx}{x^2} = -\frac{1}{b} + 1$, the desired integral is

$$\lim_{b \rightarrow \infty} \int_1^b \frac{dx}{x^2} = \lim_{b \rightarrow \infty} \left(-\frac{1}{b} \right) = 1.$$

Economic Applications of the Definite Integral

a) Consumer's Surplus

Alfred Marshall introduced the notion of consumer's surplus (CS) to measure the net benefit that a consumer enjoys from her act of purchasing a particular commodity in the market. The demand function of a consumer $P = f(q)$ may be taken to represent the highest price she is willing to pay (her 'demand price') for any specified quantity. In the figure below, she is willing to pay a price of p_1 per unit for q_1 units, p_2 per unit for q_2 units, and so on. Suppose the market price is $\bar{p}$. At this price she purchases $\bar{q}$ units and her actual expenditure is

$\bar{p} \cdot \bar{q}$, represented by the rectangle $0 \bar{p} E \bar{q}$. Her total willingness to pay for $\bar{q}$ is obtained as the sum of his demand prices for all the units from 0 to $\bar{q}$.

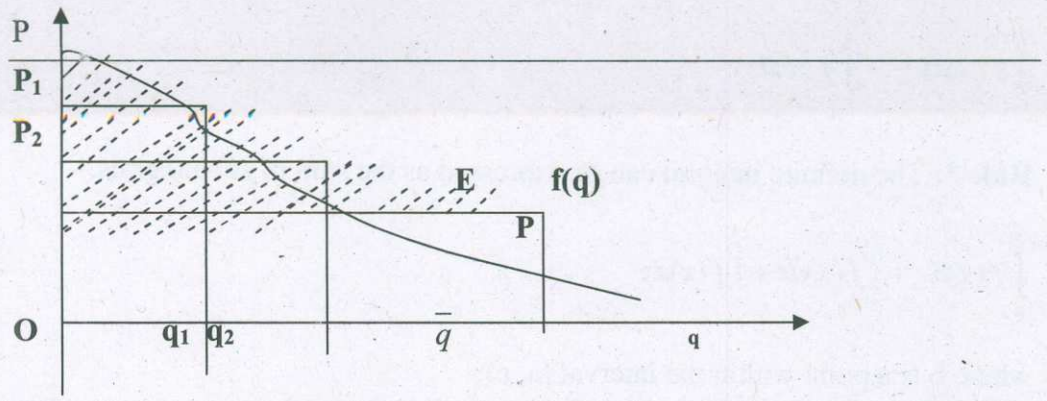


Fig. 8.3

Mathematically, this is the definite integral of the demand function up to $\bar{q}$, or the area under the demand curve up to $\bar{q}$. The excess of this total willingness to pay in units of money over her actual expenditure is her Consumer's surplus.

$$CS = \int_0^{\bar{q}} f(q) dq - \bar{p} \bar{q}.$$

It is represented by the crossed area in the diagram.

Example: To find the CS for the demand curve $p = 20 - 2q$ for $p = 4$. For $p = 4$, we get $q = 8$ from the demand equation. Actual expenditure is $pq = 32$.

$$\begin{aligned} CS &= \int_0^8 (20 - q) dq - 32 \\ &= 96 - 32 = 64. \end{aligned}$$

The consumer's surplus is Rs.64.

b) Capital Accumulation Over a Specified Period

Since $\int I(t) dt = K(t) + c$, we may use the definite integral $\int_a^b I(t) dt = K(b) - K(a)$ to find the total capital accumulation during the time interval $[a, b]$.

Example: Given the rate of net investment $I(t) = 9t^{1/2}$, find the level of capital formation in (i) 16 years and (ii) between the 4th and the 8th years.

$$i) \quad K = \int_0^{16} 9t^{1/2} dt = 6(16)^{3/2} - 0 = 384$$

$$\text{ii) } K = \int_4^8 9t^{1/2} dt = 6(8)^{3/2} - 6(4)^{3/2} = 135.76 - 48 = 87.76.$$

c) Present Value or Discounted Value Under Continuous Compounding of Interest

A basic concept in capital theory is the present or discounted or capital value of a specified sum of money that will be available at a future date. If the annual rate of interest is 100r percent, then the present value Y of Rs. x available next year is $Y = \frac{x}{1+r}$, because Rs. $\left(\frac{x}{1+r}\right)$ now will become Rs.x after one year at the stipulated annual rate of interest of 100r per cent. Similarly, the present value of Rs.x available t years hence is

$$Y = \frac{x}{(1+r)^t}$$

If interest is compounded n times a year at 100r per cent per year then the present value is

$$Y = \frac{x}{\left(1 + \frac{r}{n}\right)^{nt}} = x \left(1 + \frac{r}{n}\right)^{-nt} \quad \dots (1)$$

If interest is compounded continuously, then $n \rightarrow \infty$ and the continuous counterpart of (1) becomes

$$Y = x e^{-rt}$$

using the result: $\lim_{n \rightarrow \infty} \left(1 + \frac{k}{n}\right)^{nx} = e^{kx}$.

Now consider a project that yields an income x(t) at future period t for $t = 1, 2, \dots, T$. That is, the income stream associated with the project for T years is : x(1), x(2), ..., x(T). The present or discounted value of this income stream at annual compounded is:

$$Y = \sum_{t=1}^T \frac{x(t)}{(1+r)^t} \quad \dots (2)$$

When income flows continuously at the rate of x(t) per period up to period T and interest is compounded continuously the expression for present value becomes

$$Y = \int_0^T x(t) e^{-rt} dt \quad \dots (3)$$

Note that the magnitude of present value depends on the size of the income stream, the number of years it flows (the time horizon) and the rate of interest (the discount factor).

You should keep in mind the distinction between the present value of the **sum** $x(T)$ available T periods hence and the present value of the **stream of income** $x(t)$ per period up to period T . In the figure the former is the ordinate at $t = T$, whereas the latter is the shaded area under the curve upto $t = T$.

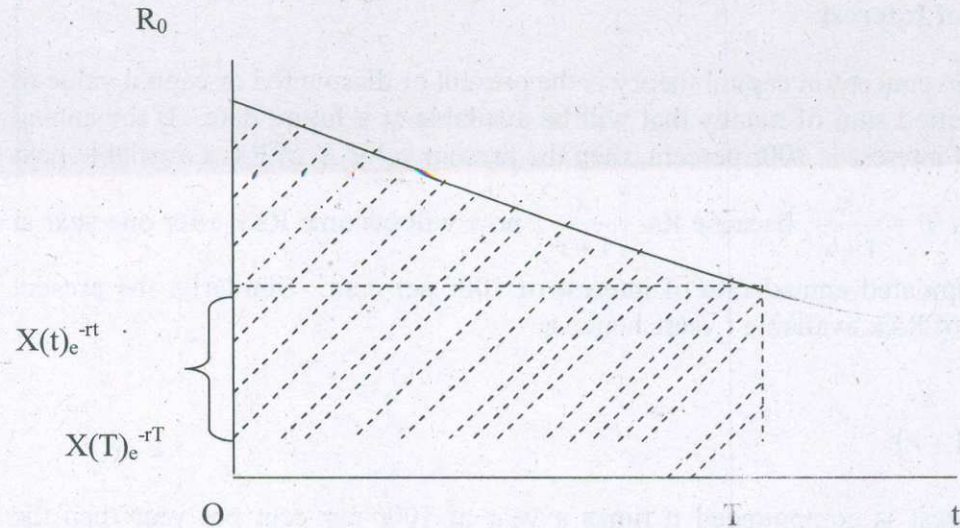


Fig. 8.4

A particular case of interest is the valuation of an asset (a bond or a piece of land) yielding a fixed income Rs. R for ever. The market value Y of such an asset is the present value of the perpetual yield.

$$Y = \int_0^{\infty} R e^{-rt} dt = R \int_0^{\infty} e^{-rt} dt$$

Remembering the rule for evaluating improper integrals.

$$\int_0^{\infty} e^{-rt} dt = \lim_{b \rightarrow \infty} \int_0^b e^{-rt} dt = \lim_{b \rightarrow \infty} \left(-\frac{1}{r} e^{-rb} + \frac{1}{r} \right) = \frac{1}{r}.$$

Hence, the market value is:

$$Y = \frac{R}{r}.$$

To illustrate further, the use of the concept of present value, we consider the following more complex problem of **optimal timing**.

The value of timber planted on a plot of land grows over time according to the function $V(t) = 2^{\sqrt{t}}$. Assuming zero cost of maintenance and a discount factor of r , find the optimal time to cut the timber for sale.

Since cost of production (upkeep) is zero, profit maximisation here is equivalent to the maximisation of sales revenue V . Due to the interest factor, r , different V values, however, are not comparable because they accrue at different points of time. The solution involves discounting each V value to its present value (the value at $t = 0$). The process of discounting puts them on comparable footing.

Assuming continuous compounding, the present value $R(t)$ can be written as

$$R(t) = V(t)e^{-rt} = 2^{\sqrt{t}} e^{-rt}.$$

The optimal time of cutting is the value of t that maximises $R(t)$. Since $f(x)$ and $\log f(x)$ attain their maximum at the same value of x , the problem can equivalently be restated as finding the value of t that maximises $\log R(t)$.

$$\log R(t) = \sqrt{t} \log 2 - rt$$

Differentiating with respect to t and setting the derivative equal to zero, we get

$$\frac{1}{R} \frac{dR}{dt} = \left(\frac{\log 2}{2\sqrt{t}} - r \right) = 0$$

$$\text{or, } \frac{dR}{dt} = R \left(\frac{\log 2}{2\sqrt{t}} - r \right) = 0 (R \neq 0)$$

$$\text{or, } \sqrt{t} = \frac{\log 2}{2r}, \text{ since } R \neq 0.$$

$$\text{or, } t = \left(\frac{\log 2}{2r} \right)^2$$

We leave it to you to check that at this value of t the second order condition for maximisation $\frac{d^2 R}{dt^2} < 0$ is also satisfied. Thus, the expression for the optimum time for cutting the timber is $(\log 2/2r)^2$. It is to be noted that the higher the rate of discount r , the sooner the timber should be cut. This is a general characteristic of all optimal storage or timing problems.

8.3.3 Differential Equations and its Economic Applications

Differential equations are equations involving the derivatives (or differentials) of unknown functions. Solving a differential equation means finding a function that satisfies that equation.

Let us start with some basic ideas behind these equations. If $y = f(x)$ is a function for which derivatives of adequate order exist, then $\frac{dy}{dx} = f'(x)$.

Suppose that we know $f'(x)$ and would like to go back to the function y . Therefore, we try to solve the problem.

$$dy = f'(x)dx$$

$$\Rightarrow y = \int f'(x)dx.$$

Through differential equations, we attempt to solve the problems, which are related to change over time, i.e., dynamic variables. For example, suppose that a hypothetical economy's income (y) is related to time (x). It is given in functional form: $y(x) = 2x^{1/2}$. If the income changes over time, we find the rate

of change as $\frac{dy}{dx} = x^{-\frac{1}{2}}$. Let us work to find the time path of the income change, so that we write $y = y(x)$. The derivative of this function, however, will be same as that of $y = y(x) + c$, where c is any arbitrary constant. In such a situation, we cannot determine a unique time path of the income change. It is necessary, therefore, to work out a definite value of c . Additional information required for that purpose is to have the *initial condition*. If we know the initial income of the economy, say, $y(0)$, i.e., value of y at $x = 0$, then the value of the constant c can be determined.

Thus, from $y(x) = 2x^{1/2} + c$, when $x = 0$,

we get $y(0) = 2(0)^{1/2} + c = c$.

See that constant c is no longer arbitrary as if $y(0) = 10,000$, $c = 10,000$ and

$y(x) = 2x^{1/2} + 10,000$. More generally, for any given initial income, $y(0)$, the time path will be

$y(x) = 2x^{1/2} + y(0)$.

Note that the income example, its dynamic form, consists of the sum of initial condition and another term with time variables.

The **order** of a differential equation is the order of the highest derivative appearing in the equation. Its **degree** is the highest power to which the highest order derivative is raised. A differential equation is **linear** if the dependent variable and derivatives are raised to the first power only and no product term $y \frac{dy}{dx}$ occurs.

Examples: In all the examples that follow the unknown function is $y = f(x)$.

- 1) $\frac{dy}{dx} = 4x - 9$ First order, first degree
- 2) $\left(\frac{dy}{dx}\right)^4 - x^2 = 0$ First order, fourth degree
- 3) $\frac{d^2y}{dx^2} - 2y = 0$ Second order, first degree
- 4) $\frac{d^4y}{dx^4} + x^3 \frac{d^3y}{dx^3} - \log x \frac{d^2y}{dx^2} + e^x \frac{dy}{dx} + x^2y + 10 = 0$ Four order, first degree
- 5) $\left(\frac{d^3y}{dx^3}\right)^4 + \left(\frac{d^2y}{dx^2}\right)^7 = 8 + 2y$ Third order, fourth degree.

Equation (1), (3) and (4) are linear since they are all of the first degree.

Solution to first order differential equation in specific instances can be worked out with (1) separation of variables. Consider the equation,

$$\frac{dy}{dx} = f'(x)$$

$$\text{or, } dy = f'(x)dx$$

If we integrate both the sides,

$\int dy = \int f'(x)dx$, so that variables x and y are separated, it becomes easy for applying appropriate technique of integration.

Example: $\frac{dy}{dx} = x^2$

$$\text{or, } dy = x^2 dx$$

$$\text{So, } \int dy = \int x^2 dx$$

$$\text{or, } y = \frac{x^3}{3} + c$$

Exact equations

$$\text{From } \frac{dy}{dx} = f'(x), \text{ we get}$$

$$y = \int f'(x)dx + c \text{ which can be written as } g(x) + c.$$

Remember that the addition of a constant term, 'c' to the function does not affect its derivative. However, it shifts the function parallelly. Depending upon the different values acquired by the constant term, we get a family of

curves for the function. Take, for example, the above solution, $y = \frac{x^3}{3} + c$

$$\text{with } c = 0. \text{ Then } \frac{dy}{dx} = x^2,$$

$$\text{or, } dy = x^2 dx$$

$$\text{or, } dy - x^2 dx = 0$$

Note that '0' in the right side of this solution is due to a choice like $c = 0$. So we write

$$dy - x^2 dx = c, \text{ (where } c = \text{constant)}$$

Generalising the above result, we can write

$$u(x, y) = c.$$

So, $du = u_x dx + u_y dy = 0$

If $u = \frac{-x^3}{3} + y$, then $du = -x^2 dx + dy = 0$

$$\text{or, } \frac{dy}{dx} = x^2$$

Since $u = \frac{-x^3}{3} + y$, $\frac{\partial u}{\partial x} = -x^2$ and $\frac{\partial u}{\partial y} = 1$.

Therefore, $dx = \frac{\partial u}{\partial x}$ and $dy = \frac{\partial u}{\partial y}$. From these we write the generalised form of differential equation as $P(x, y) dx + Q(x, y) dy = 0$.

If we can find a function $g(x, y)$ such that $P(x, y) = \frac{\partial g}{\partial x}$ and $Q(x, y) = \frac{\partial g}{\partial y}$, we write

$$\begin{aligned} d[g(x, y)] &= \frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial y} dy \\ &= P(x, y) dx + Q(x, y) dy. \end{aligned}$$

Then $g(x, y) = c$ are integral curves of the above differential equation. This class of differential equations is called exact differential equations.

Example:

For $x dx + y dy = 0$, set $g = \frac{1}{2}(x^2 + y^2)$. Then $g_x = x dx$ and $g_y = y dy$. The solutions are $x^2 + y^2 = c$, or circles.

To determine if a differential equation is exact or not, check that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

Example:

From, $(3x^2 + y^2)dx + 2xy dy = 0$, we get

$$P(x, y) = 3x^2 + y^2$$

$$Q(x, y) = 2xy, \text{ so that}$$

$P_y = 2y = Q_x$. The equation, therefore, is exact.

Exercise:

Solve the exact differential equation

$$2yxdy + y^2 dx = 0$$

In this equation

$$P(x, y) = 2yx$$

$$Q(x, y) = y^2$$

$$\text{Now } F(y, x) = \int 2yx dy + \phi(x) = y^2 x + \phi(x)$$

$$\frac{dF}{dx} = y^2 + \phi'(x)$$

$$\text{As } Q = \frac{dF}{dx}, \text{ we equate } Q(x, y) = y^2 \text{ and } \frac{dF}{dx} = y^2 + \phi'(x) \text{ to get } \phi'(x) = 0$$

$$\phi'(x) = \int \phi'(x) dx = \int 0 dx = k, \text{ which give the specific form of } \phi'(x).$$

$$\text{So, } F(y, x) = y^2 x + K$$

The solution of the exact differential equation should then be $F(y, x) = c$. K being a constant can be merged with c , so that $y^2 x = c$, or $yx = cx^{-1/2}$ where c is arbitrary.

A first order linear differential equation is general written as

$$\frac{dF}{dx} = P(x)y + Q(x) \quad \dots (4)$$

where P and Q are two functions of x , and that of y . P and Q may also be expressed in other forms as x^2 and e^x . These may also be constants. In the following, we will discuss homogenous differential equations.

Homogenous Case

If P and Q are constant functions and if Q is identically equal to zero, equation (4) becomes $\frac{dy}{dx} + ay = 0$ where a is some constant $\dots (5)$

Please note that the constant term '0' can be regarded as in the first degree in terms of y because $0y = 0$.

Equation (b) can be written as

$$\frac{1}{y} \frac{dy}{dx} = -a \quad \dots (6)$$

For solution, we write $\frac{dy}{y} = -a dx$ (with $c = -a$) and integrate both the sides, such

that $\int \frac{dy}{y} = \int -a dx$. The left side of the above gives $\log y + c_1$ for $y \neq 0$.

Whereas right side becomes

$$-ax + c_2$$

Bringing together the result of the left and right sides,

$$\log y + c_1 = cx + c_2$$

$$\text{or, } \log y = cdx + c_3 \quad (\text{combining } c_1 \text{ and } c_2 \text{ of both sides})$$

$$\text{or, } e^{\log y} = e^{(cx+c_3)}$$

$$\text{or, } y = e^{cx} \cdot e^{c_3} = A e^{cx} \text{ where } A = e^{c_3}$$

Putting back $c = -a$,

$$\text{we get } y(x) = A e^{-ax} \text{ where } A \text{ is arbitrary} \quad \dots (7)$$

To get rid of the arbitrary constant, set $x = 0$ in the equation $y(x) = A e^{-ax}$, so that

$$y(0) = A e^0 = A.$$

$$\text{Thus, } y(x) = y(0) e^{-ax} \quad \dots (8)$$

In (c), A is an arbitrary constant. The solution, therefore, is a **general solution**. When a particular value is substituted for A , we derived the **particular solution** in (8). There are an infinite number of particular solutions, value of $y(0)$. However, $y(0)$ is important since it can alone satisfy the initial condition. From the feature of giving a definite value to the arbitrary constant, we refer the result in (d) as the definite solution of the differential equation.

Non-homogenous case

When we have a non-zero constant in place of the zero in equation (5) above, it is called a non-homogenous linear differential equation.

$$\text{Thus, } \frac{dy}{dx} + ay = b \quad \dots (9)$$

is a non-homogeneous differential equation. The solution of this class of equations has two parts; (a) complementary function (y_c) and (b) particular integral (y_p). Before we proceed to solve equation (e), it will be useful to point out that homogeneous equation (5) is called a reduced equation of (9) and the non-homogenous equation (9) itself is categorised as the complete equation. Moreover, the complementary function (y_c) is the general solution of the reduced equation, whereas the particular integral (y_p) is any of the particular solution of the complete equation.

Solution to non-homogenous differential equation is seen as a sum of the complementary function and the particular integral.

$$\text{Thus, } y(x) = y_c + y_p.$$

We have noted above that y_c is the general solution of the reduced equation. We take the general solution of the homogeneous differential equation (d) above, which was $A e^{-ax}$. Thus, $y_c = A e^{-ax}$.

Let us come to particular integral. Recall that it is any particular solution of the complete equation. Perhaps the simplest possible type of solution we can think of is to take it being some constant ($y = k$). Taking of as a constant, we get $\frac{dy}{dx} = 0$. Therefore, equation (e) becomes $ay = b$, or $y = \frac{b}{a}$ where $a \neq 0$.

In that case $y_p = \frac{b}{a}$, we get the general solution to the equation as

$$y(x) = A e^{-ax} + \frac{b}{a} \quad \dots (10)$$

See that the solution remains general. The presence of arbitrary constant A is responsible for it. In order to make it definite, we need to take an initial condition. Setting $x = 0$, y can be assigned the value $y(0)$ and we get

$$Y(0) = A + \frac{b}{a}$$

$$\text{or, } A = y(0) - \frac{b}{a}$$

Putting this value in (f), the solution becomes

$$Y(x) = \left[y(0) - \frac{b}{a} \right] e^{-ax} + \frac{b}{a} \quad \dots (11)$$

which is the definite solution as long as $a \neq 0$.

Exercise:

Solve the equation $\frac{dy}{dx} + 2y = 6$ with the initial condition $y(0) = 10$.

We have $a = 2$ and $b = 6$. Hence, according to (g), the solution is

$$y(x) = [10 - 3]e^{-2x} + 3 = 7e^{-2x} + 3$$

Solution when $a = 0$

If $\frac{dy}{dx} + ay = b$ has $a = 0$, then

$$\frac{dy}{dx} = b \quad \dots (12)$$

Its general solution is found by integration, i.e., $y(x) = bx + c$ where $c =$ arbitrary constant.

Complementary function: with $a = 0$

$$y_c = Ae^{-ax} = Ae^0 = A \quad (A = \text{an arbitrary constant})$$

Particular Integral

As $a = 0$, the constant solution $y = k$ does not work and some non-constant solution needs to be tried. Take $y = kx$ so that

$$\frac{dy}{dx} = k.$$

From the complete equation (h) above $k = b$.

$$\therefore y_P = bx$$

$$\text{General solution: } y(x) = y_c + y_P = A + bx \quad \dots (i)$$

Example:

Solve the equation $\frac{dy}{dx} = 2$, with the initial condition $y(0) = 5$.

From (i) above, $y(x) = 5 + 2x$.

Verification of the Solution

You can check the correct answer of your solution to a differential equation by taking its differentiation. Follow the following two steps:

- 1) Test that the derivative of the time path is consistent with the given differential equation.
- 2) Test the definite solution to find that the solution satisfies the initial condition.

Check Your Progress 1

- 1) Solve the equation $\frac{dy}{dx} + 4y = 0$ with initial condition $y(0) = 1$.

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- 2) Find y_c, y_p , the general solution and definite solution of the equation and check its validity:

$$\frac{dy}{dx} + 4y = 12; y(0) = 2$$

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In this section, we discuss one particular type of differential equation with applications to Economics. Consider the equation (first order linear)

$$\frac{dy}{dx} + my = k \quad \dots (4)$$

where m and k are constants. The formula for a **general solution** (assuming $m \neq 0$) is

$$y = Ae^{-mx} + \frac{k}{m} \quad \dots (5)$$

where A is an arbitrary constant. It would be good for you to check that (5) indeed satisfies equation (4). The value of A can be fixed if an initial condition is given. For $m = 0$, $k \neq 0$, the solution is $y = kx + A$ where A is an arbitrary constant.

Example:

Solve $\frac{dy}{dx} - 4y = 20$ with $y(0) = 15$. From the formula (5) a general solution is

$$y = Ae^{4x} - 5.$$

Also, $y(0) = A - 5 = 15$. So $A = 20$. The solution then is

$$y = 20 \cdot e^{4x} - 5.$$

Economic Applications

a) The Harrod -Domar Analysis of Steady Growth

Consider a macroeconomic model consisting of the following equations:

$$S(t) = sY(t), \quad 0 < s < 1$$

$$I(t) = v \frac{dY}{dt}, \quad v > 0$$

$$S(t) = I(t)$$

where Y , S , I stand for the rates of flow of national income, planned saving and planned investment at any point of time t . The first equation says that a constant fraction s of income is saved in each time period. The second equation represents the acceleration theory of investment in which induced investment is proportional to the rate of change of income (v is a constant of proportionality). There is no autonomous investment. For dynamic equilibrium, we need equality between saving and investment as each period. This is the significance of the final equation. The three equations lead to a simple differential equation

$$\frac{dY}{dt} - \frac{s}{v} Y = 0 \quad \dots (6)$$

By formula (5), the general solution is

$$Y(t) = Ae^{(s/v)t}$$

Suppose it is known that $Y = Y_0$ at $t = 0$. Then $A = Y_0$ so that the equation becomes

$$Y(t) = Y_0 e^{(s/v)t} \quad \dots (7)$$

This solution gives the behaviour of national income over time in dynamic equilibrium. For any variable x that changes with time, its **rate of growth** at any point in time is defined to be $\left(\frac{1}{x} \frac{dx}{dt}\right)$, the rate of change of x divided by (as a proportion of) the value of x at that point of time. It is clear from equation (6) that in this model the rate of growth of national income $\left(\frac{1}{Y} \frac{dY}{dt}\right)$

assumes the constant value $\frac{s}{v}$. In other words, national income grows at the constant rate $\frac{s}{v}$ in dynamic equilibrium.

It follows as a corollary from our discussion that if a variable x is changing at a constant rate g over time (growing if $g > 0$, decaying if $g < 0$), then its time path is given by $x = x_0 e^{gt}$ where x_0 is the value of x at initial or base period ($t = 0$).

Our next application is taken from microeconomics. It will help us to understand the notion of stability.

b) The Dynamics of Price in a Single Market

Suppose the demand and supply functions for a particular commodity are given by

$$D_t = a - bP_t$$

$$S_t = c + bP_t; a, b, c, d > 0, a > c$$

The first equation tells us that demand D_t in a particular period t is a decreasing linear function of price prevailing in that period P_t . The second equation, the supply function, has a similar interpretation.

The equilibrium price in this context has the property that it (i) clears the market in each period and (ii) does not change over time. Let us denote it by P^* . (Note that since it is constant through time there is no t -subscript). Writing P^* for P_t in the demand and supply equations and setting $D_t = S_t$ we obtain the expression for the equilibrium price

$$P^* = \frac{a - c}{b + d}$$

The restriction $a > c$ ensures that P^* is positive. Now comes an important point. Knowledge of the equilibrium price tells us nothing about the behaviour of price out of equilibrium. In other words, although we know P^* we do not know what happens when in any period t the price P_t is not equal to P^* . Does price

rise or fall or fluctuate in some unpredictable manner? To answer questions of this type precisely we have to introduce a dynamic **adjustment rule** for price. It seems natural and sensible to assume that price will tend to rise if demand exceeds supply and fall if demand falls short of supply and stay unchanged if demand and supply just balance in any period. This type of adjustment is incorporated in the analysis through a simple linear relationship.

$$\frac{dP}{dt} = \theta(D_t - S_t), \theta > 0 \quad \dots (8)$$

Since θ is a positive constant, this tells us that

- i) $\frac{dP}{dt} = 0$ or P rises if $D_t > S_t$
- ii) $\frac{dP}{dt} = 0$ or P falls if $D_t < S_t$ and
- iii) $\frac{dP}{dt} = 0$ or P stays unchanged if $D_t = S_t$

Substituting the demand and supply functions we obtain the differential equation for price

$$\frac{dP}{dt} = \theta(b + d)P = \theta(a - c).$$

This, being of the form (4), has the solution

$$P = Ae^{-\theta(b+d)t} + \frac{a-c}{b+d}.$$

Suppose it is known that $P = P_0$ at $t = 0$. Then from the above solution

$A = \left(P_0 - \frac{a-c}{b+d} \right)$. But $\frac{a-c}{b+d} = P^*$, the equilibrium price. So the solution can be written as

$$P = (P_0 - P^*) Ae^{-\theta(b+d)t} + P^* \quad \dots (9)$$

This completely describes the time path of price in the market characterised by the given demand and supply curves and the adjustment rule (8). Note that the time path (for any given initial price P_0) is determined by the demand and supply parameters (a, b, c, d) and the coefficient of adjustment θ . The dependence of the solution on θ brings out clearly the importance of adjustment rules in dynamics.

Now let us take up the question of **stability**. Suppose that the initial price P_0 is not the equilibrium price P^* , that is $(P_0 - P^*) \neq 0$. The system is stable if price tends to approach the equilibrium price P^* as time passes, that is if $\lim_{t \rightarrow \infty} P(t) = P^*$.

It is clear that since $\theta > 0$ and $(b + d) > 0$ the term $e^{-\alpha(b+d)t}$ (and hence the first term of (9)) will tend to zero as t tends to infinity, so that P will indeed converge to P^* and we have a system that is dynamically stable.

Check Your Progress 2

- 1) If the rate of change of y with respect to x is $2x$ and $y = 4$ when $x = 1$, find y as a function of x .

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- 2) The rate of change of y with respect to x is $(0.8x - 0.6x^2)$ and $y = 0$ and $x = 0$. Find y as a function of x .

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- 3) Let the consumer's demand function be $P = 20 - 2q$.
Calculate the consumer's surplus for $P = 8$. Is it larger or smaller than the CS for $P = 47$?

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- 4) At the rate of interest of 4 per cent a year, what is the present value of Rs.1000 available 2 years later?

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- 5) A piece of land yields a constant rent of Rs.1000 per year. Find its market value if the rate of interest is 10 per cent per year.

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- 6) Solve the equation $\frac{dy}{dx} - 5y = -25$ with $y(0) = 6$

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- 7) Explain the dynamics of price adjustment process in a single market. What happens if $\theta < 0$?

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- 8) Solve the equation $\frac{dy}{dx} + 4y = 0$, with initial condition $y(0) = 1$.

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- 9) Find y_c , y_p , the general solution and definite solution of the equation and check its validity $\frac{dy}{dx} + 4y = 12$; $y(0) = 1$.

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- 10) Solve $\frac{dy}{dx} - 5y = 0$; $y(0) = 6$ and check its validity.

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8.4 LET US SUM UP

You have seen that dynamics is essentially concerned with change and the effects of change on the behaviour of variables over time. Dynamic analysis

considers the actual process of transition from the initial pre-change position to the final equilibrium. Hence, economic dynamics always has the time period as an independent variable. In this unit, you have been introduced to the following tools of dynamic analysis – the indefinite integral, the definite integral and differential equations. You have been exposed to the application of those tools in solving problems related to economic dynamics.

8.5 KEY WORDS

Consumer's Surplus: This notion was introduced by Alfred Marshall to measure the net benefit that a consumer enjoys from his act of purchasing a particular commodity in the market. It is defined in terms of the excess of the consumer's total willingness to pay in units of money over his actual expenditure.

Definite Integral: The definite integral of the function $f(x)$ over the interval (a, b) is expressed symbolically as $\int_a^b f(x)dx$, read as "integral of f with respect to x from a to b . The smaller number a is termed the **lower limit** and b , the **upper limit** of integration. Geometrically, this definite integral denotes the area under the curve representing $f(x)$ between the points $x = a$ and $x = b$. Note that the definite integral

$\int_a^b f(x)$ is a number.

Differential Equations: Differential equations are equations involving the derivatives (or differentials) of unknown functions. Solving a differential equation means finding a function that satisfies that equation.

Economic Dynamics: Dynamics is essentially concerned with change and the effects of change on the behaviour of variables over time. Economic dynamics deals with economic variables like national income, price, etc. The task of dynamics is to consider the actual process of transition from the initial pre-change position to the final equilibrium.

Improper Integral: It is a special type of definite integral. When one of the limits of integration is $+\infty$ or $-\infty$, a definite integral is called an improper integral. Such integrals are evaluated using the concept of limits.

Indefinite Integral: The indefinite integral is basically reverse differentiation. To differentiate means to find the rate of change (derivative) of a given function. Indefinite integration reverses the process and finds the unknown function whose rate of change (derivative) is given.

8.6 SOME USEFUL BOOKS

Archibald, G.C. and R.G. Lipsey, 1983, *An Introduction to a Mathematical Treatment of Economics* (Third Edition), ELBS London, Chapters 12, 13 and 14.

Chiang, Alpha C., 1983, *Fundamental Methods of Mathematical Economics* (Third Edition) McGraw Hill, International Students Edition Chapters 13, 14.

IGNOU, 1990, *MTE-01: First Elective Course in Mathematics* (Block 3: Integral Calculus).

8.7 ANSWERS OR HINTS TO CHECK YOUR PROGRESS

Check Your Progress 1

1) $y(x) = (1-0)^{-4x} + C = e^{-4x}$

2) $y(x) = e^{-4x} + 3$

Check Your Progress 2

1) $\frac{dy}{dx} = 2x$

$$\text{or } \int dy = \int 2x dx = 2 \int x dx = \frac{2x^2}{2} + c = x^2 + c$$

$$\text{or } y = x^2 + c$$

Now $y = 4$ when $x = 1$ so that $4 = 1 + c$ or $c = 3$.

$$\therefore y = f(x) = x^2 + 3$$

2) $\frac{dy}{dx} = 0.8x - 0.6x^2$

$$\text{or } \int dy = \int (0.8x - 0.6x^2) dx$$

$$= \int 0.8x dx - \int 0.6x^2 dx$$

$$= 0.8 \int x dx - 0.6 \int x^2 dx$$

$$\therefore y = 0.8 \frac{x^2}{2} - 0.6 \frac{x^3}{3} + c$$

$$\text{or } y = 0.4x^2 - 0.2x^3 + c$$

Now $y = 0$ for $x = 0 \Rightarrow c = 0$

$$\therefore y = 0.4x^2 - 0.2x^3$$

3) $P = 20 - 2q = f(q)$

For $P = 8$, $12 = 2q$ or $q = 6$.

$\therefore Pq = 48 = \text{total expenditure when } P = 8.$

$$\int_0^6 f(q) dq = \int_0^6 (20 - 2q) dq$$

$$= 20 \int_0^6 dq - \int_0^6 2q dq$$

$$= 20 \int_0^6 dq - 2 \int_0^6 q dq$$

$$= 20[q]_0^6 - 2 \left[\frac{q^2}{2} \right]_0^6 = 120 - 36 = 84.$$

$$\therefore \text{Consumer's surplus} = \int_0^8 f(q) dq - Pq = 84 - 48 = 36.$$

Now, for $P = 4$, $q = 8$.

$$CS = \int_0^8 (20 - 2q) dq - 32 = 64$$

$\therefore$ Consumer's surplus increases as the price of the commodity falls.

- 4) Let $A = \text{amount}$; $P = \text{principal}$, $r = \text{rate of interest}$, $n = \text{number of years}$.

Then applying the compound interest formula,

$$A = P(1 + r)^n,$$

Here $A = 100$, $r = 0.04$, $n = 2$, $P = ?$

$$\text{Hence } P = \frac{100}{(1 + 0.04)^2} = \frac{100}{(1.04)^2}$$

- 5) Let $Y = \text{market value of land}$

$$Y = \int_0^{\infty} Re^{-rt} dt$$

$= \frac{R}{r}$ (For the steps which you have to do, refer to Example of Economic Applications of the Definite Integral). Here $R = 1000$, $r = 0.1$

$$Y = \frac{1000}{0.1} = 10,000$$

6) $\frac{dy}{dx} - 5y = -25$

Here $m = -5$, $k = -25$

$$\therefore y = Ae^{-mx} + \frac{k}{m} \text{ becomes}$$

$$y = Ae^{5x} + \frac{-25}{-5} = Ae^{5x} + 5.$$

Now, $y(0) = 6 \Rightarrow Ae^0 + 5 = 6$ or $A \cdot 1 = 1$ or $A = 1$.

$\therefore y = e^{5x} + 5$, is the answer.

7) See Section 8.3.3, Example (b).

If $\theta < 0$, the system cannot attain a stable equilibrium, i.e., the system is unstable. You reason why.

8) $y(x) = [1 - 0]e^{-4x} + c = e^{-4x}$

9) $y(x) = -e^{-4x} + 3$

10) $y(x) = 6e^{5x}$

UNIT 9 DIFFERENCE EQUATIONS AND APPLICATIONS IN ECONOMIC DYNAMICS

Structure

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- 9.1 Introduction
- 9.2 Difference Equations in Economics
- 9.3 Solving First Order Difference Equations
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- 9.4 Solving Second Order Difference Equations
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9.0 OBJECTIVES

After going through this unit, you should be able to :

- solve problems of economic dynamics where the time variable takes only discrete values.

9.1 INTRODUCTION

A **difference equation** is used to solve the values of an unknown function $y(x)$ for different discrete values of x . We obtain a function $y(x)$ such that it satisfies the equation for all values of x . In order to understand the process of formulation of the difference equation, you may recall the discussion on differential equation presented in the preceding unit. See that difference and differential equations are exactly analogous with the only difference that the former applies when the independent variable takes only discrete values, whereas the latter when it is continuous.

9.2 DIFFERENCE EQUATIONS IN ECONOMICS

To get an idea about how difference equations may come up in economics consider the case where it is known that the national income y of a particular

country has been growing at a constant rate g over (say) a ten year period starting from some base year. The rate of growth of y at any period t may be represented as $\left(\frac{y_t - y_{t-1}}{y_{t-1}} \right)$. Note that this is the expression that gives the rate

of growth of y at a particular point in time. In contrast, you have seen the expression $\frac{dy(t)/dt}{y}$ when y was treated as a continuous variable. Treating y as

discrete, we find the numerator as the increment of income in current period (period t) over the level attained in the immediately preceding period (period $t - 1$). The ratio of this to the income y_{t-1} of the preceding period gives the current rate of growth. Since we have said that rates of growth of income, g , is constant (i.e., independent of t) over ten year's interval, it can be written as:

$$\frac{y_t - y_{t-1}}{y_{t-1}} = g, t = 1, 2, 3, 4, \dots, 10$$

$$\text{or, } y_t = (1 + g) y_{t-1}, t = 1, 2, \dots, 10 \dots \dots \dots (1)$$

Equation (1) relates the values of the variable y at two distinct periods t and $(t - 1)$. It is an example of a difference equation.

There is a one-period lag in the values of the relevant variable (y_t and y_{t-1}). Therefore, it is an example of a **first order** difference equation. The **order** of a difference equation is determined by the **maximum** number of periods lagged. Some examples of difference equations are given below with the orders noted.

$$y_{t-3} - 3y_{t-4} = 0 \quad \text{order 1.}$$

$$y_t = a(y_{t-1} - y_{t-2}) + 10 \quad \text{order 2.}$$

$$\log y_{t+9} - y_{t+7} (y_{t+6})^3 + 6y_t = 0 \quad \text{order 9.}$$

$$18 y_{t+4} - y_t = 2^t - 5^{5+1} \quad \text{order 4.}$$

$$y_{t+3} + ay_{t+1} = by_{t-1} + c \quad \text{order 4}$$

Consider a difference equation of the following form:

$$y = a_1 y_{t-1} + a_2 y_{t-2} + \dots + a_n y_{t-n} + b$$

where $a_1, a_2, \dots, a_n$ and b are constants.

This is called an **n^{th} order linear constant coefficient** difference equation (assuming $a_n \neq 0$, otherwise the order will be less than n). It is linear because the dependent variable y is not raised to any power and there are no product terms, constant coefficients because $a_1, \dots, a_n$ are constants and do not change with t . This equation will be **homogeneous** if $b = 0$. If $b \neq 0$, then it is **non-homogenous**. In this unit, we shall work only with difference equations of this special type of orders one and two ($n = 1, 2$).

9.3 SOLVING FIRST ORDER DIFFERENCE EQUATIONS

In solving a difference equation, we find a time path $y(t)$ from a given initial condition. As pointed out above, a first order difference equation takes the form

$$y_t = f(t, y_{t-1}) \text{ for all } t.$$

We can solve such an equation by successive calculation, also called recursive method, taking the initial value of y (say y_0) as given. Thus,

$$y_1 = F(1, y_0)$$

$$y_2 = F(2, y_1) = f(2, f(1, y_0))$$

and so on.

Note that given any value y_0 , there exists a unique solution path $y_1, y_2, \dots$

However, resorting to calculation of the solution through such a method doesn't tell us much about the properties of the solution. We should have a general formula, which exists if the form of f is simple.

Let us start with a first-order linear difference equation with constant coefficient. It takes the form

$$y_t = ay_{t-1} + b_t \text{ where } b_t \text{ for } t = 1, \dots \text{ are constants.}$$

When the recursive method is used, you will see a pattern as follows:

$$y_t = ay_0 + \sum_{k=1}^t a^{t-k} b_k \dots\dots\dots(2)$$

and such an equation has a unique solution path. To check that we get the unique solution from the above formulation, verify that it satisfies the original equation.

Since we have

$$ay_{t-1} + b_t = a \left(a^{t-1} y_0 + \sum_{k=1}^{t-1} a^{t-k} b_k \right) + b_t$$

$$= a^t y_0 + \sum_{k=1}^{t-1} a^{t-k} b_k + b_t$$

$$= a^t y_0 + \sum_{k=1}^t a^{t-k} b_k$$

$$= y_t,$$

so that the solution obtained is correct.

Taking the equation (2), we can examine the special case of

$$b_k = b \text{ for all } k = 1, \dots$$

We have

$$y_t = a^t y_0 + b \sum_{j=0}^{t-1} a^{t-1-j}$$

Making use of the result of geometric series summation, the term

$$\sum_{j=0}^{t-1} a^{t-1-j} \text{ may be expanded as } 1 + a + a^2 + \dots + a^{t-1} \text{ to give}$$

$$1 + a + a^2 + \dots + a^{t-1} = (1 - a^t)/(1 - a).$$

if $a \neq 1$. Thus we have

$$y_t = a^t y_0 + b(1 - a^t)/(1 - a)$$

if $a \neq 1$.

For any given value y_0 , the unique solution of the difference equation

$$y_t = a y_{t-1} + b,$$

where $a \neq 1$, is

$$y_t = a^t(y_0 - b/(1 - a)) + b/(1 - a).$$

Equilibrium or Stationary Value

For a given value y_0 , the value of y_t changes with t . But there may be some value of y_0 for which y_t doesn't change. Such a solution exists if

$$y^* = b/(1 - a)$$

and y_t is constant, equal to $b/(1 - a)$.

We call y^* the **equilibrium** value of y and rewrite the solution as

$$y_t = a^t(y_0 - y^*) + y^*.$$

Example: Solve $y_{t+1} = \alpha y_t + \beta, \dots \dots \dots (3)$

where α and β are constants.

Look for a stationary or equilibrium value of y_t over time which can be repeated for any t consistently satisfying the above equation. May be you consider $\bar{y}$ as an equilibrium value of y_t such that

$$\bar{y} = \alpha \bar{y} + \beta$$

$$\text{or, } \bar{y} = \frac{\beta}{1 - \alpha}$$

To understand the above example, we need to remember the dynamic multiplier.

Write

$$C_t = \alpha y_{t-1} + \beta \dots\dots\dots(4)$$

Let the investment be fixed at $\bar{I}$ for every t so that we have

$$\begin{aligned} Y_t &= C_t + I_t = C_t + \bar{I} \\ &= \alpha Y_{t-1} + (\beta + \bar{I}) \\ &= \alpha Y_{t-1} + \beta' \end{aligned}$$

where $\beta' = \beta + \bar{I}$

Use the above relation (4) we have

$$Y_{t+1} = \alpha Y_t + \beta'$$

If an equilibrium income $\bar{Y}$ is found, the solution can be written as

$$\begin{aligned} \bar{Y} &= \alpha \bar{Y} + \beta' \\ &= \frac{\beta'}{1-\alpha} = \frac{\beta + \bar{I}}{1-\alpha} \end{aligned}$$

Note that $\frac{1}{1-\alpha}$ is the Keynesian multiplier.

It is important to remember that we have solved equation (3) for the stationary level of y_t i.e., $\bar{y}$. There is no guarantee that the actual path of y converges to $\bar{y}$. In case y_t approaches $\bar{y}$, then

$$(y_t - \bar{y}) \rightarrow 0.$$

If these values of y_t and y_{t+1} hold, we can write

$$g_t = y_t - \bar{y} \dots\dots\dots(5)$$

Since y_t and $\bar{y}$ satisfy (3), we have

$$\begin{aligned} y_{t+1} &= \alpha y_t + \beta \text{ and} \\ \bar{y} &= \alpha \bar{y} + \beta \end{aligned}$$

Thus,

$$y_{t+1} - \bar{y} = \alpha (y_t - \bar{y}).$$

From (5),

$$g_t = y_t - \bar{y}$$

$$\text{or, } g_{t+1} = y_{t+1} - \bar{y}$$

$$\text{or, } g_{t+1} = y_{t+1} - \bar{y}$$

$$\text{or, } g_{t+1} = \alpha g_t \dots \dots \dots (6)$$

Since

$$g_{t+1} = \alpha g_t$$

$$g_t = \alpha g_{t-1}$$

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$$g_1 = \alpha g_0$$

Substituting backward,

$$g_{t+1} = \alpha^2 g_{t-1} = \alpha^3 g_{t-2} \dots \dots \dots$$

we get

$$g_{t+1} = \alpha^{t+1} g_0$$

$$\text{or } g_t = \alpha^t g_0 \text{ for } t = 0, 1, 2 \dots \dots \dots$$

Thus, any difference equation of the form $y_t = \alpha y_{t-1}$ has a solution $y_t = \alpha^t y_0$, where y_0 is the value of y at some chosen initial point.

General Solution

Suppose we intend to solve the equation

$$y_{t+1} + ay_t = c \dots \dots \dots (7)$$

Its general solution will be consisting of particular solution (y_p) and complementary function (y_c), i.e., $y_g = y_p + y_c$. In this approach, the y_p component represents the inter-temporal equilibrium level of y while that of y_c gives the deviations if the time path from that equilibrium. The solution is called general solution due to the presence of an arbitrary constant. In order to get a definite solution, we need an initial condition.

Let us work with complementary function. From (7), we get its reduced form as

$$y_{t+1} + ay_t = 0 \dots \dots \dots (8)$$

It is seen above that $y_t = \alpha^t y_0$ is a solution to the difference equation. In that case we have $y_{t+1} = \alpha^{t+1} y_0$ as well. We modify this and rewrite

$$y_t = Ab^t \text{ and } y_{t+1} = Ab^{t+1}.$$

Substitution of these into (8) gives

$$Ab^{t+1} + aAb^t = 0$$

$$\text{or, } Ab^t(b+a) = 0$$

$$\text{or, } (b+a) = 0$$

$$\text{or, } b = -a$$

We must have $b = -a$ in the trial solution such that the complementary solution can be written as

$$y_c = Ab^t = A(-a)^t.$$

Particular solution needs to be recasted such that it is in agreement with the general solution. Consider the simplest value of y . If y_t has an equilibrium value k such that it remains constant overtime, we have $y_t = k$ as well as $y_{t+1} = k$. Substitution of these values to the trial solution gives

$$k + a_k = c$$

$$\text{or, } k = \frac{c}{1+a}$$

Since the value, k , satisfies the equation, the particular solution can be written as

$$y_p = k = \frac{c}{1+a} \text{ for } a \neq 0$$

In case $a = -1$, however, the particular solution is not defined and some other solution of (7) must be searched for.

Substituting k into (7), we get

$$k(t+1) + ak_t = c.$$

$$\text{or, } k = \frac{c}{t+1+a_t} = C \text{ and } y_p = C_t.$$

The general solution can now be written in one of the following forms:

$$y_n = A(-a)^t + \frac{C}{1+a} \text{ if } a \neq -1$$

$$\text{or, } y_t = A(-a)^t + C_t = A + C_t \text{ if } a = -1.$$

Notice that the solution above still remains indeterminate. This is due to the presence of arbitrary constant A . We have to take the help of initial condition ($y_t = y_0$) for eliminating it. Thus, taking $t=0$, we have

$$y_0 = A + \frac{C}{1+a}$$

$$\text{or, } A = y_0 - \frac{C}{1+a}$$

The definite solution therefore, becomes

$$y_t = \left(y_0 - \frac{C}{1+a} \right) (-a)^t + \frac{C}{1+a} \quad \text{for } a \neq -1$$

$$\text{or, } y_t = y_0 + C_t \quad \text{for } a = -1$$

9.3.1 Behaviour of Solutions of First Order Equations

The solution of a difference equation gives an expression for the relevant variable as an explicit function of time. In other words, a time path of the variable is obtained. To investigate the nature of this time path of a solution of the first order equation, we write the solution for $a \neq 1$.

The behaviour of the solution path depends on the value of a .

$$|a| < 1$$

y_t converges to y^* and the solution is **stable**. There are two subcases:

$$0 < a < 1,$$

Monotonic convergence.

$$-1 < a < 0$$

Damped oscillations.

$$|a| > 1$$

Divergence:

$$a > 1$$

Explosion.

$$a < -1$$

Explosive oscillations.

To understand these features see Figure 9.1.

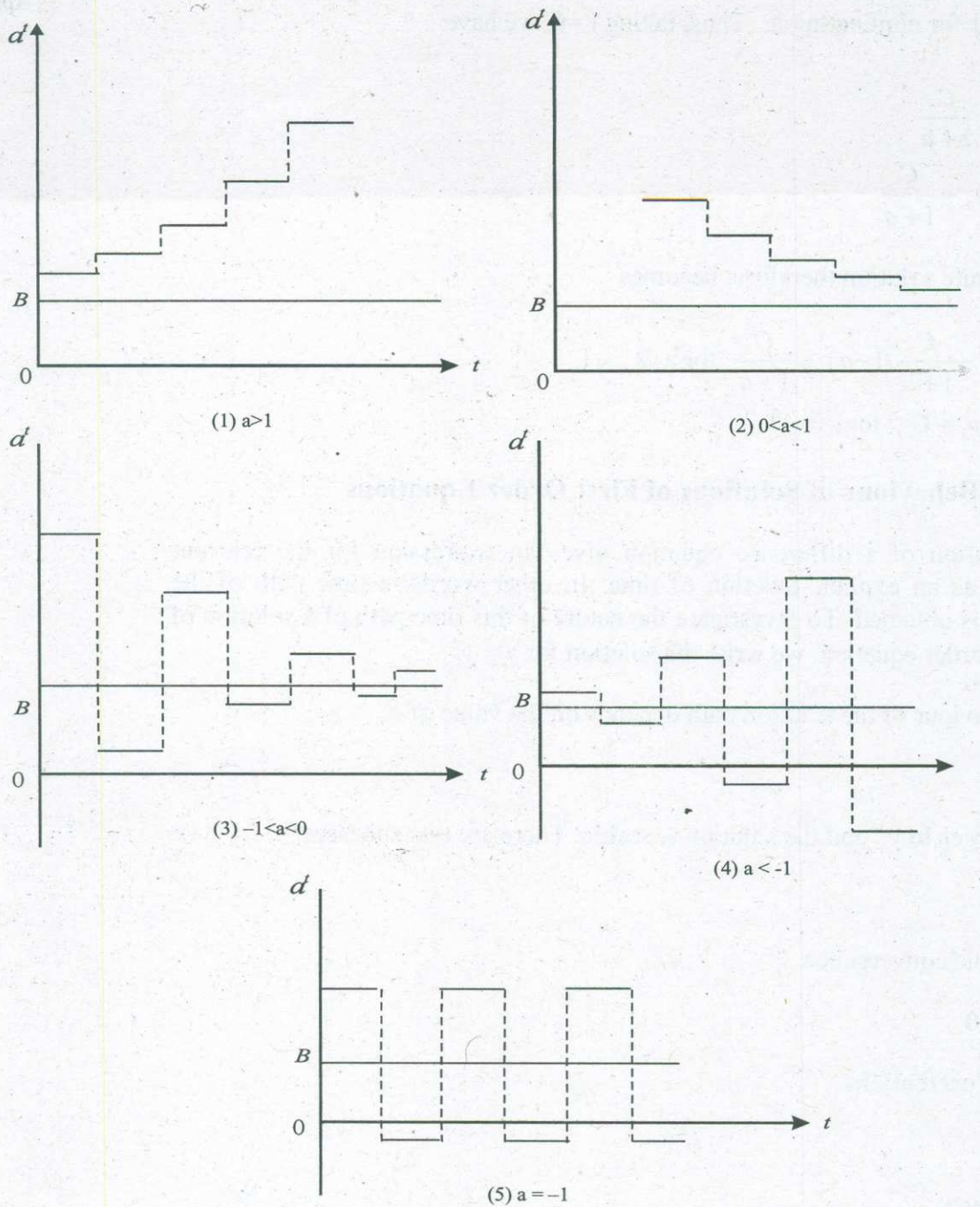


Fig. 9.1

In short,

$|a| > 1$ time path explodes (diverges)

$|a| < 1$ time path converges

$a > 0$ time path non-oscillating

$a < 0$ time path oscillating.

Thus, the condition for stability is $|a| < 1$.

The different cases are shown in Figure 9.1.

9.3.2 Economic Applications of First Order Equations

We consider three applications of the type of equations discussed in the previous section. The first is an analysis one sector Harrod-Domar model while the second is of price dynamics. The last one deals with the amortisation problem of hire purchase of consumer durables.

a) Harrod-Domar One Sector Model

An economy produces one good Q with capital K through a production function $Q_t = bK_t$, where b = constant productivity of capital. Accumulation of capital between t and $t+1$ is given by

$$I_t = K_{t+1} - K_t, \text{ where } I_t = \text{investment in } t.$$

$$\text{Saving } S_t = sQ_t.$$

Equilibrium level of income is determined at the equality of savings and investment. So,

$$S_t = I_t$$

$$\text{or, } sQ_t = K_{t+1} - K_t.$$

$$\text{Since } Q_t = bK_t, \text{ we have } sbK_t = K_{t+1} - K_t$$

or, $K_{t+1} = (1 + sb)K_t$, a homogeneous first order linear difference equation. Therefore, solution to this equation is given by

$$K_t = (1 + sb)^t K_0.$$

Since b is productivity of capital in the model, we write $\frac{1}{b}$ = capital output ratio = v (say).

$$\text{Now } K_t = \left(1 + \frac{s}{v}\right)^t K_0 \text{ and}$$

$$Q_t = \left(1 + \frac{s}{v}\right)^t Q_0$$

Remember that $\frac{s}{v}$ = warranted rate of growth and constituted by two basic parameters s and v . We can find out the output growth rate given s and v .

b) The Cobweb Model

The essential feature of this model is that production or supply responds to price with a one-period lag. This type of lagged supply response is often observed for agricultural products.

We assume: 1) The market demand and supply functions are linear and do not change over time, 2) demand in any period t responds to price prevailing in the same period t , but supply in t depends on price that prevailed in the last period, $(t - 1)$ and 3) the market is competitive in the sense that the price that prevails in each period is the price that equates demand and supply. Thus, the model can be set out as consisting of the following equations.

$$D_t = a - b P_t; a, b > 0$$

$$S_t = \alpha - \beta P_{t-1}; \alpha, \beta > 0, \alpha < a$$

$$D_t = S_t \text{ for all } t.$$

The first equation gives us the simple demand curve in period t . The second displays the lag in supply. Supply in t , S_t , is determined by prices of the immediately preceding period, P_{t-1} . The last equation is the condition of market clearing in each period. The three equations together yield a first order constant coefficient non-homogeneous difference equation in price.

$$P_t = \left(-\frac{\beta}{b} \right) P_{t-1} + \frac{a - \alpha}{b} \quad \dots (9)$$

With a, b, α, β known, a specification of the initial price P_0 allows us to solve the equation as:

$$P_t = \left(P_0 - \frac{a - \alpha}{b + \beta} \right) \left(-\frac{\beta}{b} \right)^t + \frac{a - \alpha}{b + \beta} \quad \dots (10)$$

From our previous discussion, it is clear that the behaviour of P over time depends crucially on the term $\left(-\frac{\beta}{b} \right)$.

Since this term is negative ($b, \beta > 0$) the time path will always be oscillatory. Let us denote the constant $\frac{a - \alpha}{b + \beta}$ by P^* .

Then

$$\left(\frac{\beta}{b} \right) > 1 \quad \text{Price diverges}$$

$$\left(\frac{\beta}{b} \right) = 1 \quad \text{Price oscillates uniformly}$$

$$\left(\frac{\beta}{b} \right) < 1 \quad \text{Price converges to } P^*.$$

Only in the last case (P_t approaches P^* as t increases), the system is **stable**. Thus, the condition for stability is $\left(\frac{\beta}{b} \right) < 1$. Since graphically $\left(\frac{1}{\beta} \right)$ is the

slope of the supply curve and $\left(\frac{1}{b}\right)$ that of the demand curve in absolute value, the stability condition states that the slope of the supply curve must be steeper than the absolute value of the slope of the demand curve.

At this point, we pause to note the significance of the value $\frac{a-\alpha}{b+\beta}$. This is the constant value of price that is a solution of the equation (9). To check, substitute $P_t = P_{t-1} = P^*$ (a constant) in (9).

$$P^* = \left(-\frac{\beta}{b}\right)P^* + \frac{a-\alpha}{b}$$

$$\text{or, } P^* = \frac{a-\alpha}{b+\beta}.$$

Thus, $P_t = \frac{a-\alpha}{b+\beta}$ is a solution of (9). This type of constant solution is called

Stationary solution. The price P^* may be called the **equilibrium price** because it equates demand and supply and stays unchanged over time.

Example: We want to investigate the behaviour of price in a market with the demand and supply functions:

$$D_t = 86 - 0.8 P_t$$

$$S_t = -10 + 0.2 P_{t-1}$$

Assuming market clearing in each period ($D_t = S_t$) we have

$$(-0.8) P_t = 0.2 P_{t-1} - 96$$

$$\text{or, } P_t = (-0.25)P_{t-1} + 120$$

The solution is

$$P_t = \left(P_0 - \frac{120}{1+0.25}\right)(-0.25)^t + \frac{120}{1+0.25}$$

$$= (P_0 - 96)(-0.25)^t + 96$$

Since $|-0.25| = 0.25 < 1$, the time path of P is oscillating but converges. The market is stable and with the passage of time price approaches the equilibrium value 96.

c) The Amortisation Problem

We are all familiar with the practice of hire purchase or purchase by instalments of consumer durables like refrigerators, cars or T.V. sets. The buyer pays a part of the price at the time of purchase (the down payment) and pays the rest in monthly or annual instalments over a specified period. Because the payments are spread over a period of time, an interest cost is included in the

value of instalments. **Amortisation** is the term associated with this method of repaying an initial debt plus interest charges by a series of payments of equal magnitude at equal intervals.

Let the value of the article purchased by V and P the down payment. Then the initial debt of the buyer is $D_0 = V - P$. The contract states that the debt, D_0 is to be paid off over T periods. The rate of interest is r ($100 r\%$). The question we are interested in is: how is the magnitude of periodic instalment to be determined?

Let us denote the value of the instalment (still unknown) by B . This value stays constant over time. The outstanding debt D_t at the end of the period t obeys the equation

$$D_t = (1 + r)D_{t-1} - B \quad \dots\dots\dots(11)$$

This simply says that to find the outstanding debt at the end of the t^{th} period you take the debt outstanding at the end of the previous $((t - 1)^{\text{th}})$ period D_{t-1} , add the interest charge on it, rD_{t-1} , but subtract the payment B made in that period. Given the initial debt of D_0 the solution of (11) is

$$D_t = \left(D_0 + \frac{B}{r} \right) (1 + r)^t - \frac{B}{r} \quad \dots\dots (12)$$

The value of B is to be selected so that the debt disappears at the end of period T , that is, $D_T = 0$. From (12) we get

$$\left(D_0 + \frac{B}{r} \right) (1 + r)^T - \frac{B}{r} = 0$$

$$\text{or, } B = \frac{rD_0}{1 - (1 + r)^{-T}}$$

Thus, we have the exact relationship between the magnitude of the periodic payment and the rate of interest, the magnitude of the initial debt and the time horizon of the contract. The expression:

$$\left(\frac{1 - (1 + r)^{-T}}{r} \right)$$

is referred to as the **amortisation factor** and value of this factor has been extensively tabulated for different values of r and T .

Check Your Progress 1

- 1) What is a difference equation? Distinguish it from a differential equation.

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2) Discuss the nature of the following time paths

i) $y_t = 3^t + 1$ ii) $y_t = 5\left(-\frac{1}{10}\right)^t + 3$

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3) Suppose you find the following the path of y.

$y_t = Aa^t + B; A < 0, B > 0.$

Draw the different cases of the behaviour of y_t for different values of a.

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4) Solve the following equations:

i) $y_{t+1} - \frac{1}{3}y_t = 6$ for $y_0 = 1$

ii) $y_{t+1} - y_t = 3$ for $y_0 = 5$

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9.4 SOLVING SECOND ORDER DIFFERENCE EQUATIONS

A general **second-order difference equation** which we have already mentioned at beginning of this unit takes the form

$y_{t+2} = f(t, y_t, y_{t+1}).$

Just as in the case of first-order equation, a second-order equation will have a unique solution and can be derived by successive (recursive) calculation. We will show that given y_0 and y_1 there exists a uniquely determined value

of y_t for all $t \geq 2$. Note that for a second-order equation we need two starting values, y_0 and y_1 , in place of one taken in the first order counterpart.

9.4.1 Homogeneous Equations

Consider the following second order constant coefficient equation

$$y_{t+2} + ay_{t+1} + by_t = 0 \quad \dots (13)$$

We need to find two solutions of the equation above.

If we make a guess that the solution takes the form $u_t = m^t$

In order for u_t to be a solution, we must have

$$m^t(m^2 + am + b) = 0$$

or, if $m \neq 0$,

$$m^2 + am + b = 0.$$

This is called the **characteristic (or auxiliary) equation** of the difference equation and its solutions are

$$-(1/2)a \pm \sqrt{((1/4)a^2 - b)}.$$

9.4.2 Behaviour of Solutions of Homogeneous Equations

Looking at the component $\sqrt{((1/4)a^2 - b)}$, we distinguish three cases:

i) Distinct real roots

If $a^2 > 4b$, the characteristic equation has distinct real roots, and the general solution of the homogeneous equation is

$$Am_1^t + Bm_2^t,$$

where m_1 and m_2 are the two roots.

ii) Repeated real root

If $a^2 = 4b$, then the characteristic equation has a single root, and the general solution of the homogeneous equation is

$$(A + Bt)m^t,$$

where $m = -(1/2)a$ is the root.

(ii) Complex roots

If $a^2 < 4b$, then the characteristic equation has complex roots, and the general solution of the homogeneous equation is

$$Ar^t \cos(\theta t + \omega),$$

where A and ω are constants, $r = \sqrt{b}$, and $\cos \theta = -a/(2\sqrt{b})$, or, alternatively,

$$C_1 r^t \cos(\theta t) + C_2 r^t \sin(\theta t),$$

where $C_1 = A \cos \omega$ and $C_2 = -A \sin \omega$ (using the formula that $\cos(x+y) = (\cos x)(\cos y) - (\sin x)(\sin y)$).

When the characteristic equation has complex root, the solution **oscillates**. $A r^t$ is the **amplitude** (which depends on the initial conditions) at time t , and r is **growth factor**. $\theta/2\pi$ is the **frequency** of the oscillations and ω is the **phase** (which depends on the initial conditions).

If $|r| < 1$ then the oscillations are **damped**; if $|r| > 1$ then they are **explosive**.

Stability

We say that a system of differential equations is **stable** if its long-run behaviour is not sensitive to the initial conditions.

Consider the second-order equation

$$y_{t+2} + ay_{t+1} + by_t = c_t.$$

Write the general solution as

$$y_t = Au_t + Bv_t + u_t^*,$$

where A and B are determined by the initial conditions.

This solution is **stable** if the first two terms approach 0 as $t \rightarrow \infty$, for all values of A and B . In this case, for *any* initial conditions, the solution of the equation approaches the particular solution u_t^* . If the first two terms approach zero for all A and B , then u_t and v_t must approach zero. You can take $A = 1$ and $B = 0$ to see that u_t approaches zero. On the other hand, take $A = 0$ and $B = 1$ to see that v_t approaches 0. A necessary and sufficient condition for this to be so is that the moduli of the roots of the characteristic equation be both less than 1. Note that the modulus of a complex number $\alpha + \beta i$ is $\sqrt{(\alpha^2 + \beta^2)}$, which is the absolute value of number if the number is real.

There are two cases:

- If the characteristic equation has complex roots then the modulus of each root is $\sqrt{b}$ (the roots are $\alpha \pm \beta i$, where $\alpha = -a/2$ and $\beta = \sqrt{(b - (1/4)a^2)}$). So for stability need $b < 1$.
- If the characteristic equation has real roots then the modulus of each root is its absolute value. So for stability we need the absolute values of each root to be less than 1, or $-a/2 + \sqrt{(a^2/4 - b)} < 1$ and $-a/2 - \sqrt{(a^2/4 - b)} < 1$.

9.4.3 Non-homogeneous Equations

To find the general solution of the original equation

$$y_{t+2} + ay_{t+1} + by_t = c_t$$

we need to find one of its solutions. Suppose that $b \neq 0$.

The form of a solution depends on c_t .

Suppose that $c_t = c$ for all t . Then $y_t = C$ is a solution if $C = c/(1 + a + b)$ and if $1 + a + b \neq 0$;

if $1 + a + b = 0$ then try $y_t = Ct$; if that does not yield a solution, we have to try $y_t = Ct^2$.

9.4.4 An Economic Application of Second Order Non-homogeneous Equation

We discuss now an economic example of a second order non-homogeneous equation. This is Samuelson's model of interaction between the multiplier and the accelerator. Consider the following macro-economic equations:

$$C_t = C_0 + cY_{t-1}, \quad 0 < c < 1.$$

$$I_t = I_0 + v(C_t - C_{t-1}); \quad v > 0.$$

$$Y_t = C_t + I_t$$

The symbols Y , C , I stand for national income, consumption and investment respectively. The first equation is the consumption function with a one-period lag; the second is the investment function of the accelerator type. $C_0 + I_0$ are the levels of autonomous consumption and investment. The marginal propensity to consume c and the accelerator coefficient v are assumed to be constant. The final equation is the condition of macro balance. The three equations together generate the following difference equation in Y

$$Y_t - c(1 + v) Y_{t-1} + cvY_{t-2} = C_0 + I_0 \quad \dots (15)$$

The characteristic equation for the homogeneous part is

$$m^2 - c(1 + v)m + cv = 0$$

The roots are

$$m_1, m_2 = \frac{1}{2} (c(1 + v) \pm \sqrt{c^2(1 + v)^2 - 4cv}) \quad \dots (16)$$

Both m_1 and m_2 are positive because from the theory of quadratic equations we know $m_1 + m_2 = c(1 + v) > 0$ and $m_1 m_2 = cv > 0$. Since $c(1 + v) - cv \neq 1$, the **particular** solution is $\frac{C_0 + I_0}{1 - c}$. Three types of solution are possible depending on the values of c and v .

$$1) \quad c^2(1 + v)^2 > 4cv$$

or, $c(1 + v)^2 > 4v$, the roots are real and distinct:

Here,

$$Y_t = A_1 m_1^t + A_2 m_2^t + \frac{C_0 + I_0}{1 - c}; A_1, A_2 \neq 0 \text{ and constants}$$

- 2) $c(1+v)^2 = 4v$, the roots are real and equal with value $\frac{1}{2}c(1+v)$. In this case

$$Y_t = (A_1 + A_2 t) \left(\frac{c(1+v)^2}{2} \right) + \frac{C_0 + I_0}{1 - c};$$

- 3) $c(1+v)^2 < 4v$, the roots are complex. From (16) we see that the roots are of the form $(a \pm ib)$ with

$$a = \frac{1}{2}c(1+v)$$

$$b = \frac{1}{2}\sqrt{4cv - c^2(1+v)^2}$$

$$(\because \sqrt{c^2(1+v)^2 - 4cv} = \sqrt{i^2(4cv - c^2(1+v)^2)} = i\sqrt{4cv - c^2(1+v)^2} = ib)$$

The modulus of the roots

$$r = \sqrt{a^2 + b^2} = \sqrt{cv}$$

The solution is

$$Y_t = (\sqrt{cv})^t (A_1 \cos(t\theta) + A_2 \sin(t\theta)) + \frac{C_0 + I_0}{1 - c}$$

$$\text{where } \theta = \tan^{-1} \left(\frac{\sqrt{4cv - c^2(1+v)^2}}{c(1+v)} \right)$$

In this case, we have a **cyclical** time path of national income Y . If $\sqrt{cv} < 1$, then $(\sqrt{cv})^t$ will tend to zero as t increases and Y_t will approach the value $\frac{C_0 + I_0}{1 - c}$.

Thus, the condition for stability (damped oscillations in Y) is $\sqrt{cv} < 1$, that is, the product of the marginal propensity to consume and the accelerator coefficient should be less than unity.

Check Your Progress 2

- 1) Solve the following difference equations and determine whether the solution paths are convergent or divergent, oscillating or not.

- a) $y_{t+2} + 3y_{t+1} - (7/4)y_t = 9.$
- b) $y_{t+2} - 2y_{t+1} + 2y_t = 1.$
- c) $y_{t+2} - y_{t+1} + (1/4)y_t = 2.$
- d) $y_{t+2} + 2y_{t+1} + y_t = 9 \cdot 2^t.$
- e) $y_{t+2} - 3y_{t+1} + 2y_t = 3 \cdot 5^t + \sin((1/2) \pi t).$

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2) Find the roots of the equation

$$y_t = ay_{t-1} + by_{t-2}$$

Examine when the roots are

- 1) real, unequal
- 2) real, equal
- 3) complex

What is the auxiliary or the characteristic equation of the equation above?
What are the final forms of general solution of the equation in each case?

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3) Find the solutions of the equations:

- a) $y_t + 4y_{t-2} = 0, y = 12, 11$ at $t = 0, 1$ respectively.
- b) $y_t = 2y_{t-2} - \frac{4}{3}y_{t-2}, y = 0, 1$ at $t = 0, 1$ respectively.

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In continuation with the theme on solving economic problems in a dynamic set up, the present unit took up 'time' as a discrete independent variable and examined tool of simple difference equations. In the process, we considered the solutions of first and second order linear difference equations covering homogeneous and non-homogeneous cases. To see the applications of these equations to economic problems, the time path of adjustment of macro-economic variable – national income – in case of the simple Keynesian multiplier model and Samuelson's multiplier-accelerator interaction model were discussed. We also examined the time path of adjustment of the price variable and looked into the conditions of dynamic stability of the different systems – i.e., whether over time, the economic variables – price or national income – converge to a stable equilibrium. Finally, the conditions when the systems become dynamically explosive – i.e., the variables move further and further away from the equilibrium value were examined.

9.6 KEY WORDS

Amortisation: It is the term associated with the method of repaying an initial debt plus interest charges by a series of payments of equal magnitude at equal intervals.

Cobweb Model: A model where production or supply responds to price with a one-period lag. This model is often used to analyse the demand-supply mechanism for markets of agricultural commodities.

Constant Coefficient Difference Equation: A difference equation has constant coefficient if the coefficients a_i 's associated with the y values are constant and do not change over time.

Difference Equation: A difference equation is an equation involving the values of an unknown function $y(x)$ for different values of x . The independent variable – time in problems of economic dynamics – takes only discrete values. The form of the equation is, $y_t = a_1 y_{t-1} + a_2 y_{t-2} + \dots + a_n y_{t-n} + b$, where $a_1, a_2, \dots, a_n$ and b are constants, is an example of an n -th order linear, constant coefficient, difference equation.

Homogeneous Difference Equation: A difference equation is homogeneous if the constant term b is zero.

Linear Difference Equation: A difference equation is linear if (i) the dependent variable y is not raised to any power and (ii) there are no product terms.

Non-homogeneous Difference Equation: A difference equation is non-homogeneous if the constant term, b , is non-zero.

Order of a Difference Equation: It is determined by the maximum number of periods lagged.

9.7 SOME USEFUL BOOKS

Allén, R.G.D., 1959, *Mathematical Economics* (Second Edition) St. Martin's Press Inc., New York.

Baumol, W.J., 1974, *Economic Dynamics* (Second Edition) Macmillan, New York. Chapters 9, 10 and 11.

Chiang, Alpha C. 1984, *Fundamental Methods of Mathematical Economics* (Third Edition): Mc-Graw Hill International Edition, New Delhi.

9.8 ANSWER HINTS TO CHECK YOUR PROGRESS

Check Your Progress 1

- 1) See Section 9.1
- 2) Non-oscillatory; divergent
- 3) Oscillatory; convergent

Fig. 9.1 shows the time path for $A > 0$. Draw the corresponding figures for $A < 0$.

$$4) \quad i) \quad y_t = -8\left(\frac{1}{3}\right)^t + 9 \quad ; \quad ii) \quad y_t = -2\left(\frac{-1}{4}\right)^t + 4$$

Check Your Progress 2

- 1)
 - a) $A_1\left(\frac{1}{2}\right)^t + A_2\left(\frac{-7}{2}\right)^t + 4$. Nonconvergent oscillations.
 - b) $(\sqrt{2})^t \left(A_1 \cos\left(\frac{\pi}{4}\right)t + A_2 \sin\left(\frac{\pi}{4}\right)t \right) + 1$. Nonconvergent oscillation.
 - c) $A_1\left(\frac{1}{2}\right)^t + A_2t\left(\frac{1}{2}\right)^t + 8$. Convergent, non-oscillating.
 - d) The characteristic equation is $m^2 + 2m + 1 = (m+1)^2 = 0$, which has a double root of -1 . So the general solution of the homogeneous equation is $y_t = (C_1 + C_2t)(-1)^t$. A particular solution is obtained by inserting $u_t^* = A2^t$, which yields $A = 1$. So the general solution of the inhomogeneous equation is $y_t = (C_1 + C_2t)(-1)^t + 2^t$.
 - e) By using the method of undetermined coefficients the constants A , B , and C in the particular solution $u^* = A5^t + B \cos\left(\frac{\pi}{2}\right)t + C \sin\left(\frac{\pi}{2}\right)t$, we

obtain $A = 1/4$, $B = 3/10$, and $C = 1/10$. So the general solution to the equation is $y_t = C_1 + C_2 2^t + \left(\frac{1}{4}\right) 5^t + \left(\frac{3}{10} \cos\left(\frac{\pi}{2}\right)t + \left(\frac{1}{10} \sin\left(\frac{\pi}{2}\right)t\right)$.

- 2) See Section 9.4 and answer.

Note that in the text $y_t = ay_{t-1} + by_{t-1}$

Here you have a slightly changed equation.

3) a) $y_t = 2^t \left[12 \cos\left(\frac{t\pi}{2}\right) + \frac{11}{2} \sin\left(\frac{t\pi}{4}\right) \right]$.

b) $y_t = \left(\frac{2}{\sqrt{3}}\right)^t \left(\frac{\sqrt{3}}{2 \sin \theta}\right) \sin(t\theta)$

where $\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

9.9 EXERCISES

- 1) Investigate the behaviour of price in a market, i.e., the stability of a system with demand and supply functions:

a) $D_t = 86 - 0.8 P_t$

$S_t = -10 + 0.8 P_{t-1}$

b) $D_t = 86 - 0.8 P_t$

- 2) What is amortisation? Derive the exact relationship between the magnitude of the periodic payment B and the rate of interest r , the magnitude of the initial debt D_0 and the time horizon of the contract T .

- 3) Establish the stability condition of Samuelson's multiplier-accelerator interaction model.

- 4) Find the time path represented by the equation $y_t = 2\left(-\frac{4}{5}\right)^t + 9$.

- 5) Find the solution of the equation $y_{t+1} + \frac{1}{4} y_t = 5$ for $y_0 = 2$.

- 6) The demand and supply for cobweb model is given as

$Q_{dt} = 19 - 6P_t$ and $Q_{st} = 6P_{t-1} - 5$. Find the intertemporal equilibrium price and comment on the stability of the equilibrium.

Answer or Hints to Exercises

- 1) Note that for the Cobweb model

$$D_t = a - b P_t \quad a, b > 0$$

$$S_t = \alpha + \beta P_{t-1} \quad \alpha, \beta > 0, \alpha < 0$$

$$D_t = S_t \text{ for all } t.$$

A specification of the initial price P_0 allows us to solve the equation.

$$P_t = \left(-\frac{\beta}{b} \right) P_{t-1} + \frac{a - \alpha}{b}$$

$$\text{as } P_t = \left(P_0 - \frac{a - \alpha}{b + \beta} \right) \left(-\frac{\beta}{b} \right)^t + \frac{a - \alpha}{b + \beta}$$

$$\text{or, } P_t = \left(P_0 - P^* \right) \left(-\frac{\beta}{b} \right)^t + P^*; P^* = \frac{a - \alpha}{b + \beta}$$

If $\left(-\frac{\beta}{b} \right)^t \rightarrow 0$ as $t \rightarrow \infty$, $P_t \rightarrow P^*$, the equilibrium value.

This is possible if and only if $\frac{\beta}{b} < 1$

i.e., $\beta < b$.

a) $\beta = 0.8, b = 0.8$. Hence $\beta = b$.

This results in uniform oscillation, as $\frac{1}{\beta} = \frac{1}{b}$.

or, the slope of the supply curve = the absolute slope of the demand curve.

b) $b = 0.8, \beta = 0.9$

$$\text{Hence } \frac{\beta}{b} = 1$$

or, $\beta > b$.

$$\text{or, } \frac{1}{\beta} < \frac{1}{b}.$$

i.e., the slope of the supply curve is less than the absolute value of the slope of the demand curve.

Hence, price diverges further and further away from the equilibrium and you come across an explosive and oscillatory situation.

c) $B = 0.9, \beta = 0.8$

$$\frac{\beta}{b} < 1 \text{ implies damped oscillation and the system is stable.}$$

You should draw diagrams in each case and satisfy yourself.

- 2) See Example (b) in Sub-section 9.3.4.
- 3) Sub-section 9.4.4 and answer.
- 4) Since $b = -\frac{4}{5} < 0$, the time path is oscillatory. As $|b| = \frac{4}{5} < 1$, the oscillation is damped and it converges to equilibrium level of 9.
- 5) $y_t = -2\left(-\frac{1}{4}\right)^t + 4$.
- 6) $\bar{p} = 2$; discuss on uniform oscillation.

NOTES

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