UNIT 13 INTRODUCTION TO SIMULTANEOUS EQUATION MODELS

Structure

13.0 OBJECTIVES

After going this unit, you will be able to:

- explain the nature of simultaneous relationship between variables;
- explain the consequences of simultaneous relationship; and
- conceptualize the basic problem of simultaneous equation system.

13.1 INTRODUCTION

The econometric models we studied so far talk about single equation regression models where a dependent variable is related to a set of independent or explanatory variables. In all these regression models, y is the 'dependent' or 'explained' variable and x_1, x_2, ... are the 'independent' or 'explanatory' variables. Thus y depends upon x, but the reverse is not true. In real life, however, we normally find variables that are dependent on each other. In this sort of relationship the equilibrium values of the variables can be determined by considering the simultaneity in relationship.

In simultaneous relationships a single equation under investigation is part of a wider phenomenon. In such cases the variables can be explained with the help of a system of equations. Let us take an example. At the macro level, as you know from macroeconomics course, aggregate consumption expenditure depends upon aggregate disposable income. Aggregate disposable income, on the other hand, depends upon the national income and taxes imposed by the government. Moreover, national income is dependent on aggregate consumption expenditure of the economy. Thus estimation of a single equation between aggregate consumption and disposable income will not provide us with consistent and unbiased estimators.

13.2 SOME EXAMPLES OF SIMULTANEOUS EQUATION MODELS

In regression models one of the crucial assumptions we make is that the exogenous variables x's are independent of the error term u (see Unit 2). This assumption is violated in simultaneous relationships. Let us illustrate this through the demand and supply model.

We can write the demand function as

$$q = \alpha + \beta p + u \qquad \text{...(13.1)}$$

where q represents the quantity demanded, p the price, and u the disturbance term, representing random shifts in the demand function. From microeconomic theory we know that the equilibrium price and quantity are determined in the market by intersection of supply and demand curves (see Fig. 13.1). Therefore, we cannot determine equilibrium price by solving the demand equation independently. Again, if there is random shift in the demand curve we see that it will affect both equilibrium price and quantity (considering the normal slope of supply curve as in panel-a of Fig. 13.1). In case the supply curve is vertical, then a shift in the demand curve affects equilibrium price only (see panel-b of Fig. 13.1). Again, if the supply curve is horizontal, a shift the demand curve affects equilibrium quantity only keeping the level of equilibrium price unchanged (see panel-c of Fig. 13.1). Thus we see that demand curve alone cannot determine the equilibrium price and quantity. Moreover, the slope of the supply curve influences the relationship between price and quantity. If we consider the demand equation only, the slope of the supply curve gets included in the term 'u'. The inclusion of the slope of supply curve in the error term leads to correlation between price and error term, which violates one of the basic assumptions of OLS, that is, $\text{cov}(Xu) = 0$. So if we ignore the supply curve and estimate the demand equation (13.1) by OLS, then it will produce inconsistent estimates of the parameters.

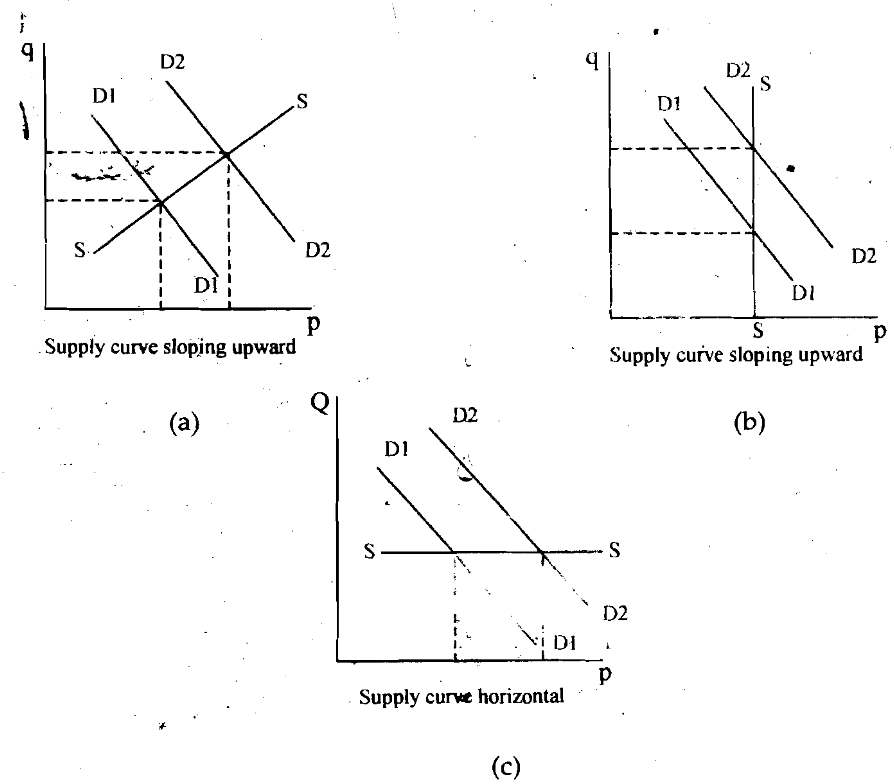

Fig. 13.1: Interaction between Supply and Demand Curves

The problem is that in demand and supply model we cannot consider the demand function in isolation when we are studying the relationship between quantity and price as in equation (13.1). The solution is to bring the supply function into the picture, and estimate the demand and supply functions together. Such models are known as simultaneous equations models.

An econometric model is based on theoretical concepts. Thus, the simultaneous equation model (SEM) can be best understood by some practical examples. Here we present three examples of SEM.

Example 13.1

Wage-price Model

The model of money wage and price determination is a good example of SEM. According to the model the wage rate is determined by unemployment rate in the economy and the rate of price change (that is, inflation) in an unrestricted labour market (reflected in (13.2)). The rate of inflation, on the other hand, is a function of the change in wage rate and money supply (reflected in (13.3)). The model consists of the two equations as follows:

$$W_t = \alpha_0 + \alpha_1 U_t + \alpha_2 P_1 + u_{1t} \qquad \text{...(13.2)}$$

$$P_t = \beta_0 + \beta_1 W_t + \beta_2 R_1 + \beta_3 M_t + u_{2t} \qquad \text{...(13.3)}$$

Where W = rate of change in money wage
U = unemployment rate in percentage
P = rate of change in prices
R = rate of change in cost of capital
M = supply of money
t = time
u_1, u_2 = stochastic disturbances

As the price variable P enters into the wage equation (13.2) and the wage variable W enters into the price equation (13.3), the two variables are jointly dependent on each other. Moreover, the explanatory variable P in (13.2) and W in (13.3) are stochastic in nature. Recall that in OLS it is assumed that the explanatory variables are not stochastic. Therefore, these stochastic explanatory variables are expectedly correlated with the stochastic disturbance terms. This correlation violates the assumption of OLS that error term is independent of independent variable. Thus the classical OLS method is not applicable for estimation of the parameters of the two equations individually.

Example 13.2

Keynesian Model of Income Determination

Consider the simple Keynesian model of income determination:

Consumption function $C_t = \beta_0 + \beta_1 Y_t + u_t \qquad 0 < \beta_1 < 1$ ... (13.4)

Income identity: $Y_t = C_t + I_t (= S_t)$ ... (13.5)

where C = consumption expenditure
Y = income
I = Investment (assumed to be exogenous)
S = Savings
t = time
u = stochastic disturbance term
β_0 and β_1 are parameters

The parameter β_1 represents the marginal propensity to consume (MPC). From macroeconomic theory we know that β_1 is expected to remain between 0 and 1. Equation (13.4) is the consumption function which shows that consumption depends on autonomous consumption expenditure β_0 and income induced consumption $\beta_1 Y$. Equation (13.5) is the national income identity, signifying that total income is equal to total consumption expenditure plus total investment expenditure.

From the consumption function (13.4) it is clear that C and Y are interdependent and that Y_1 is not expected to be independent of the disturbance term, u_t. When u_t changes (because of a variety of factors subsumed in the error term) the consumption function also shifts, which, in turn, affects Y_t. Therefore, once again the classical least-squares method is not applicable to (13.4). If applied, the estimators thus obtained will be inconsistent.

Example 13.3

Macroeconomic Model

The celebrated IS model, or goods market equilibrium model of macroeconomics (see Unit 2 of MEC-002) in its non-stochastic form can be expressed as

Consumption function:	$C_t = \beta_0 + \beta_1 Y_{dt}$	$0 < \beta_1 < 1$	...(13.6)
Tax function:	$T_t = \gamma_0 + \gamma_1 Y_t$	$0 < \gamma_1 < 1$	...(13.7)
Investment function:	$I_t = \theta_0 + \theta_1 r_t$		...(13.8)
Disposable income:	$Y_{dt} = Y_t - T_t$		...(13.9)
Government expenditure:	$G_t = \overline{G}$		...(13.10)
National income identity:	$Y_t = C_t + I_t + G_t$		...(13.11)

where Y = national income

C = consumption spending

I = planned or desired net investment

G = Given level of government expenditure

T = taxes

Y_d = disposable income

r = interest rate

In order to solve the above equation system we first substitute (13.9) and (13.6) into (13.6). Subsequently we put the resulting equation for C and equation (13.8) for I and (13.10) for G into (13.11). Consequently we obtain

$$Y_t = \left[\beta_0 + \beta_1 \left(Y_t - T_t\right)\right] + \left[\theta_0 + \theta_1 r_t\right] + \overline{G}$$

By re-arranging terms in the above we obtain the IS equation as

$$Y_t = \pi_0 + \pi_1 r_t \qquad \text{...(13.12)}$$

$$\pi_0 = \frac{\beta_0 - \alpha_0 \beta_1 + y_0 + \overline{G}}{1 - \beta_1 (1 - \alpha_1)}$$

where

$$\pi_1 = \frac{1}{1 - \beta_1 (1 - \alpha_1)}$$

Equation (13.12) is the equation of the IS curve, or goods market equilibrium, which provides the combinations of the interest rate and level of income such that the goods market is in equilibrium.

What would happen if we were to estimate, say, the consumption function (13.6) in isolation? Could we obtain unbiased and/or consistent estimates of β_0 and β_1? Such a result is unlikely because consumption depends on disposable income, which depends on national income Y; but the latter depends on r and G as well as the other parameters entering in π_1. Therefore, unless we take into account all these influences, a simple regression of C on Y_d is bound to give biased and/or inconsistent estimates of β_0 and β_1.

13.3 ENDOGENOUS AND EXOGENOUS VARIABLES

There are two types of variables in simultaneous equation models: endogenous and exogenous.

Endogenous Variables get determined jointly and interdependently within the model. The endogenous variables are determined by the exogenous variables.

Exogenous Variables are determined outside the model and independently of the endogenous variables. These exogenous variables determine the endogenous variables but the endogenous variables cannot affect the exogenous variables.

Lagged Dependent Variables:

In a model we can include the lagged value of an endogenous variable also. For example, C_t is current year consumption and is assumed to be an endogenous variable. If we consider C_{t-1} in another equation (which implies past year consumption), then it is lagged endogenous variable.

Predetermined Variables:

In addition to the above, often we come across the term 'predetermined variable'. It includes both exogenous variables and the lagged endogenous variables.

Notations

To facilitate our discussion, we introduce the following example of the different notations:

Let us consider a SEM with M endogenous and K exogenous variables.

The general M equations model in M endogenous, or jointly dependent, variables may be written as equation (13.13) given below, which is a time dependent SEM:

$$Y_{1t} = \beta_{12}Y_{2t} + \beta_{13}Y_{3t} + \ldots + \beta_{1M}Y_{Mt} + \gamma_{11}X_{1t} + \gamma_{12}X_{2t} + \ldots\gamma_{1K}X_{Kt} + u_{1t}$$

$$Y_{2t} = \beta_{21}Y_{1t} + \beta_{23}Y_{3t} + \ldots + \beta_{2M}Y_{Mt} + \gamma_{21}X_{1t} + \gamma_{22}X_{2t} + \ldots\gamma_{2K}X_{Kt} + u_{2t}$$

$$Y_{3t} = \beta_{31}Y_{1t} + \beta_{23}Y_{3t} + \ldots\ldots + \beta_{3M}Y_{Mt} + \gamma_{31}X_{1t} + \gamma_{32}X_{2t} + \ldots\gamma_{3K}X_{Kt} + u_{3t}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$Y_{Mt} = \beta_{M1}Y_{1t} + \beta_{M2}Y_{2t} + \ldots + \beta_{M,M-1}Y_{M-1,t} + \gamma_{M1}X_{1t} + \gamma_{M2}X_{2t} + \ldots\gamma_{MK}X_{Kt} + u_{Mt} \qquad \ldots(13.13)$$

where $Y_1, Y_2,\ldots,Y_M$ are M endogenous, or jointly dependent, variables

$X_1, X_2,\ldots,X_K$ are K predetermined variables (one of these X variables may take a value of unity to allow for the intercept term in each equation)

$u_1, u_2,\ldots,u_M$ are M stochastic disturbances

$t =1, \ldots, T$ are Total number of observations

β 's are coefficients of the endogenous variables

γ 's are coefficients of the predetermined variables

As equation (13.13) shows, both endogenous and exogenous variables enter a simultaneous equation model. First, we have endogenous variables, whose values are determined within the model. Next, we have predetermined variables, whose values are determined outside the model. Finally, we have the stochastic error terms. The endogenous variables are regarded as stochastic whereas the predetermined variables are non-stochastic in nature.

The predetermined variables are divided into two categories: i) exogenous variables, exogenous variables may be current as well as lagged; ii) lagged endogenous variables. Thus, X_{1t} is a current (present-time) exogenous variable, whereas $X_{1(t-1)}$ is a lagged exogenous variable. With a lag of one time period, on the other hand, $Y_{(t-1)}$ is a 'lagged endogenous variable' with a lag of one time period. Since the value of $Y_{1(t-1)}$ is known at the current time t, it is regarded as non-stochastic. Hence, it is considered as a predetermined variable[1]. In short, current exogenous, lagged exogenous, and lagged endogenous variables are considered as predetermined variables, as their values are not determined by the model in the current time period.

At this point a question may be taking shape in your mind, how do we construct econometric models? What are the variables that should be included in an equation? In fact, these questions are answered by economic theory. In some cases, which are not established theory as yet, the equations depend upon the logic that the researcher puts forth. Thus behind the inclusion of a variable in an equation, there is some logic. Secondly, it is up to the concepts of economic theory and the model builder to specify which variables are endogenous and which are predetermined. Although non-economic variables (variables not determined by any of the economic theory), such as temperature and rainfall, are clearly exogenous or predetermined, the model building must be exercised with great care while classifying economic variables as endogenous or predetermined. One must be in a position to defend i) the classification of variables into endogenous and exogenous, and ii) inclusion of a particular variable in an equation.

13.4 DIFFERENCE BETWEEN SINGLE EQUATION AND SIMULTANEOUS EQUATION MODELS

From the discussion so far, we understand that the simultaneous equation system is different from the single equation system in many ways. The most prominent difference is that in SEM the independent variables can be both lagged-endogenous

[1] It is assumed implicitly here that the stochastic disturbances, the u's, are serially uncorrelated. If this is not the case, Y_{t-1} will be correlated with the current period disturbance term u_t. Hence, we cannot treat it as pre-determined.

and exogenous whereas in the single equation model the independent variable is always exogenous. Secondly, an equation of a SEM should not be solved independently without considering other equations. We present the main differences between single equation and the SEM in Table 13.1 below.

Table 13.1: Difference between Single Equation Model and Simultaneous Equation Model

Single Equation Model	Simultaneous Equation Model
All the independent variables are exogenous	Independent variables can be both exogenous and lagged-endogenous
Equation of the model can be solved independently	To solve an equation of the model we have to consider the other equations of the system
Dataset represents a specific equation	The exact form of the equation cannot be specified from data set.
No need to identify an equation as the functional form of the equation is specified	Identification is required before solving the equations
OLS estimation is consistent	OLS estimators are inconsistent
Solution technique: OLS, GLS	Solution techniques: ILS, 2SLS, Instrumental variable method, 3SLS, Limited information maximum likelihood estimation

13.5 SIMULTANEOUS EQUATION BIAS: INCONSISTENCY OF OLS ESTIMATORS

We have mentioned earlier that in simultaneous models the explanatory variables and the error terms are not independent. So one of the important assumptions of OLS that there should be no covariance between the explanatory variable and the error term is violated. As a consequence the estimators are not consistent.

In order to show that the parameters are not consistent, let us take a concrete example. Let us go back to the simple Keynesian model of income determination given in Example 13.2. Suppose we want to estimate the parameters of the consumption function given at eq. (13.4) without taking into account the income identity. As you know, the standard assumptions under OLS are

$$E(u_t) = 0,$$

$$E(u_t)^2 = \sigma^2,$$

$$E(u_t u_{t+1}) = 0 \quad (\text{for } t \neq 0),$$

and cov $(X_t u_t) = 0,$

In the case of the consumption function (13.4) we will first show that Y_t and u_t are correlated and then prove that OLS estimate is an inconsistent estimator of β_1.

To prove that Y_t and u_t are correlated, we substitute (13.4) into (13.5) to obtain the following equation.

$$Y_t = \beta_0 + \beta_1 Y_t + u_t + I_t$$

$$\text{or, } Y_t - \beta_1 Y_t = \beta_0 + u_t + I_t$$

$$\text{or, } (1-\beta_1) Y_t = \beta_0 + u_t + I_t \qquad \ldots(13.14)$$

Therefore,

$$Y_t = \frac{\beta_0}{1-\beta_1} + \frac{1}{1-\beta_1} I_t + \frac{1}{1-\beta_1} u_t \qquad \ldots(13.15)$$

and

$$E(Y_t) = \frac{\beta_0}{1-\beta_1} + \frac{1}{1-\beta_1} I_t \qquad \ldots(13.16)$$

Subtracting (13.16) from (13.15) results in

$$Y_t - E(Y_t) = \frac{1}{1-\beta_1} u_t \qquad \ldots(13.17)$$

We have assumed that $E(u_t) = 0$. Therefore, the covariance between Y_t and u_t is given by

$$\operatorname{cov}(Y_t u_t) = E\left[Y_t - E(Y_t)\right]\left[u_t - E(u_t)\right]$$

$$= E\left[\frac{1}{1-\beta_1} u_t \quad u_t\right]$$

$$= \frac{E(u_t)^2}{1-\beta_1} = \frac{\sigma_u^2}{1-\beta_1} \qquad \ldots(13.18)$$

Since σ^2 is positive by definition, the covariance between Y and u given in (13.18) is bound to be different from zero[1]. Let us look back to eq.(13.4), i.e., $C_t = \beta_0 + \beta_1 Y_t + u_t$. Here, the explanatory variable Y_t and the error term u_t are correlated, which violates the assumption of the classical linear regression model that the disturbances are independent or at least uncorrelated with the explanatory variables. As noted previously, the OLS estimators in this situation are inconsistent.

To show that the OLS estimator is inconsistent because of correlation between Y_t and u_t, we proceed as follows:

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[1]It will be greater than zero as long as the MPC lies between 0 and 1, and it will be negative if it is greater than unity. Of course, a value of MPC greater than unity would not make much economic sense. In reality, therefore, the covariance between Y_t and u_t is expected to be

$$\hat{\beta}_1 = \frac{\Sigma c_t y_t}{\Sigma y_t^2}$$

$$= \frac{\Sigma (C_t - \bar{C}) y_t}{\Sigma y_t^2}$$

$$= \frac{\Sigma C_t y_t}{\Sigma y_t^2} \quad \text{Since } \bar{C}\Sigma(y_t) = 0 \qquad \text{...(13.19)}$$

We know that the sum of the deviations from arithmetic mean is zero. Therefore, $\Sigma y_t = 0$.

Let us substitute (13.19) in equation (13.4) for C_t. Thus we obtain

$$\hat{\beta}_1 = \frac{\Sigma(\beta_0 + \beta_1 Y_t + u_t) y_t}{\Sigma y_t^2}$$

$$= \frac{1}{\Sigma y_t^2}[\beta_0 \Sigma y_t + \beta_1 \Sigma y_t y_t + \Sigma y_t u_t]$$

$$= \beta_1 + \frac{\Sigma y_t u_t}{\Sigma y_t^2} \qquad \text{....(13.20)}$$

We obtain (13.20) by using the fact that $\Sigma y_t = 0$ and $\Sigma Y_t y_t \big/ \Sigma y_t^2 = 1$

If we take the expectation of (13.20) we obtain

$$E(\hat{\beta}_1) = \beta_1 + E\left[\frac{\Sigma y_t u_t}{\Sigma y_t^2}\right] \qquad \text{...(13.21)}$$

As expectation operator is a linear operator, we cannot evaluate $E\left[\frac{\Sigma y_t u_t}{\Sigma y_t^2}\right]$, since $E(\frac{A}{B}) \neq \frac{E(A)}{E(B)}$. But intuitively it is clear that $\hat{\beta}_1$ will be unbiased estimator of β_1 only if $\left[\frac{\Sigma y_t u_t}{\Sigma y_t^2}\right] = 0$. Though we have shown in (13.18) that the covariance between Y and u is nonzero that does not mean $\operatorname{cov}(y_t u_t) \neq 0$, since $\operatorname{cov}(y_t u_t)$ is a population concept. It is not exactly equal to $\sum y_t u_t$ which is a sample measure, although as the sample size increases indefinitely the latter will tend toward the former. But if the sample size increases indefinitely, then we can resort to the concept of consistent estimator and thus find out what happens to β_1 as n, the sample size, increases indefinitely. In short, when we cannot explicitly evaluate the expected value of an estimator, as in (13.21), we can turn our attention to its behavior in the large sample.

Now an estimator is said to be consistent if its **probability limit**, or **plim** for short, (see Unit 1 for definition) is equal to its **true (population) value**. Therefore, to show that β_1 of (13.20) is inconsistent; we must show that its plim is not equal to the true β_1.

Applying the rules of probability limit to (13.20), we obtain[2]

$$\text{plim}(\hat{\beta}_1) = \text{plim}(\beta_1) + \text{plim}\left(\frac{\sum y_t u_t}{\sum y_t^2}\right)$$

$$= \text{plim}(\beta_1) + \text{plim}\left(\frac{\sum y_t u_t / n}{\sum y_t^2 / n}\right) \qquad \text{...(13.22)}$$

$$= \beta_1 + \frac{\text{plim}(\sum y_t u_t / n)}{\text{plim}(\sum y_t^2 / n)}$$

Where in the second step we have divided $\sum y_t u_t$ and $\sum y^2_t$ by the total number of observations in the sample n so that the quantities in the parentheses are now the sample covariance between Y and u and the sample variance of Y, respective.

In other words, (13.22) states that the probability limit (plim) of β_1 is equal to true β_1 plus the ratio of the plim of the sample covariance between Y and u to the plim of the sample variance of Y. Now as the sample size n increases indefinitely, one would expect the sample covariance between Y and u to approximate the true population covariance $E[Y_t - E(Y_t)][u_t - E(u_t)]$ which from (13.18) is equal to

$$\left[\sigma^2 / (1-\beta_1)\right]$$

Similarly, as n tends to infinity, the sample variance of Y will approximate its population variance, say σ_y^2. Therefore, equation (13.22) may be written as

$$p\lim\left(\hat{\beta}_1\right) = \beta_1 + \frac{\sigma_u^2 / (1-\beta_1)}{\sigma_y^2}$$

$$= \beta_1 + \frac{1}{(1-\beta_1)}\left(\frac{\sigma_u^2}{\sigma_y^2}\right) \qquad \text{...(13.23)}$$

Given that $0<\beta_1<1$ and that σ^2 and σ_y^2 are both positive, it is obvious from equation (13.23) that plim (β_1) will always be greater than β_1; that is, β_1 will overestimate the true parameter[3], β_1. In other words, β_1 is a biased estimator, and the bias will not disappear no matter how large the sample size.

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[2] As mentioned in Unit 1, the plim of a constant (for example, β_1) is the same constant and the plim of (A/B) = plim(A)/plim(B). Note, however, that $E(A/B) \neq E(A)/E(B)$.

[3] In general, however, the direction of the bias will depend on the structure of the particular model and the true values of the regression coefficients.

13.6 SOLUTION TO THE SIMULTANEOUS EQUATION BIAS

In the preceding section we learnt that if we apply OLS to an equation belonging to a system of simultaneous equations we would not get unbiased and consistent estimates. In order to obtain unbiased and consistent estimates we apply other methods of estimation. There are several methods for this purpose. The most common are:

1) Reduced form method, or indirect least squares (ILS)
2) The method of instrumental variables (IV)
3) Two-stage least squares (2SLS)
4) Limited information maximum likelihood (LIML)
5) The mixed estimation method
6) Three stage least squares (3SLS)
7) Full information maximum likelihood (FIML)

Some of these methods will be discussed in more details in Unit 15.

13.7 STRUCTURAL FORM AND REDUCED FORM

We presented the simultaneous equation system in (13.13). These equations are known as the 'structural equations' or 'behavioral equations' as they represent the structure (of an economic model) or the behavior of an economic agent (e.g., consumer or producer). The β's and γ's in (13.13) are known as the structural parameters or coefficients. Recall that in the structural equations both endogenous and exogenous variables are on the right-hand side of each equation.

By solving for the M endogenous variables from the structural equations we can derive the reduced-form equations and the associated reduced-form coefficients. A reduced-form equation is one where each equation contains one endogenous variable and that endogenous variable is expressed solely in terms of the predetermined variables and the stochastic disturbances. To illustrate, let us consider the Keynesian model of income determination given at Example 13.2 again:

Consumption function: $C_t = \beta_0 + \beta_1 Y_1 + u_t \quad 0<\beta_1<1$ ...(13.24)

Income identity: $Y_t = C_t + I_t$ ...(13.25)

In this model C (consumption) and Y (income) are the endogenous variables and I (investment expenditure) is treated as an exogenous variable. While the first equation is stochastic in nature the second one is an identity. As usual, the MPC (β_1) is assumed to remain between 0 and 1.

If (13.24) is substituted into (13.25), we obtain after simple algebraic manipulation

$$Y_t = C_t + I_t = \beta_0 + \beta_1 Y_1 + u_t + I_t$$

By re-arranging terms in the above, we obtain

$$Y_t - \beta_1 Y_t = \beta_0 + I_t + u_t$$

or,

$$(1-\beta_1)Y_t = \beta_0 + I_t + u_t$$

or

$$Y_t = \frac{\beta_0}{(1-\beta_1)} + \frac{1}{(1-\beta_1)} I_t + \frac{1}{(1-\beta_1)} u_t$$

The above can be presented as

$$Y_t = \Pi_0 + \Pi_1 I_t + w_t \qquad \text{...(13.26)}$$

where $\Pi_0 = \dfrac{\beta_0}{1-\beta_1}$

$$\Pi_0 = \frac{1}{1-\beta_1}$$

$$w_t = \frac{u_1}{1-\beta_1}$$

Equation (13.26) is a reduced-form equation. It expresses the endogenous variable Y solely as a function of the exogenous (or predetermined) variable and the stochastic disturbance term. Π_0 and Π_1 are the associated reduced-form coefficients. Notice that these reduced-form coefficients are nonlinear combinations of the structural coefficients(s).

Substituting the value of Y_t from (13.26) into C_t of (13.24), we obtain another reduced-form equation:

$$C_t = \Pi_2 + \Pi_3 I_t + w_t \qquad \text{... (13.27)}$$

where $\Pi_2 = \dfrac{\beta_0}{1-\beta_1}$ $\Pi_3 = \dfrac{\beta_1}{1-\beta_1}$ $w_t = \dfrac{u_t}{1-\beta_1}$

The reduced-form coefficients, such as Π_1 and Π_2 and Π_3, are also known as impact, or short-run multipliers, because they measure the immediate impact on the endogenous variable of a unit change in the value of the exogenous variable. If in the preceding Keynesian model the investment expenditure is increased by, say, Re.1 and if the MPC (i.e., β_1) is assumed to be 0.8, then from (13.26) we obtain $\Pi_1 = 5$. This result means that increasing the investment by Re.1 will immediately (i.e., in the current time period) lead to an increase in income of Rs. 5, that is, a fivefold increase. Similarly, under the assumed conditions, (13.27) shows that $\Pi_3 = 4$, meaning that Re. 1 increase in investment expenditure will lead immediately to Rs. 4 increase in consumption expenditure.

In the context of econometric models, equations such as (13.25) or $Q_{dt} = Q_{st}$ (Quantity demanded equal to quantity supplied) are known as the equilibrium condition. Identity states that aggregate income Y must be equal to aggregate expenditure (i.e., consumption expenditure plus investment expenditure). When equilibrium is achieved, the endogenous variables assume their equilibrium values[4].

13.8 RECURSIVE SYSTEM

Not all simultaneous equation models give biased estimates when estimated by OLS. An important class of models for which OLS estimation is valid is that of recursive models. These are models in which the errors from the different equations are independent and the coefficients of the endogenous variables show a triangular pattern. For example, let us consider a model with three endogenous variables Y_1, Y_2 and Y_3, and three exogenous variables, Z_1, Z_2 and Z_3, which has the following structure:

$$Y_1 + \beta_{12}Y_2 + \beta_{13}Y_3 + \alpha_1 Z_1 = u_1$$

$$Y_2 + \beta_{23}Y_3 + \alpha_2 Z_2 = u_2$$

$$Y_3 + \alpha_3 Z_3 = u_3$$

where u_1, u_2 and u_3, are independent. The coefficients of the endogenous variables are

$$\begin{matrix} 1 & \beta_{12} & \beta_{13} \\ & 1 & \beta_{23} \\ & & 1 \end{matrix}$$

which form a triangular structure. In such systems, each equation can be estimated by OLS. Suppose in the example above u_2 and u_3 are correlated but they are independent of u_1; then the model is set to the block-recursive. The second and third equations have to be estimated jointly, but the first equation can be estimated by OLS. Note that Y_2 and Y_3 depend on u_2 and u_3 only and hence are independent of the error term u_1.

13.9 LET US SUM UP

Unlike the single equation model, in simultaneous equation models we get more than one dependent, or endogenous variables. This endogenous variable may appear in one equation as endogenous and as exogenous in another equation of the system. Due to presence of the endogenous variables as explanatory variable in some of the equations it becomes correlated with the disturbance term of the equation in which it appears as the explanatory variable.

In this situation the classical OLS regression method is not applied as it may result in inconsistent estimators, that is, they may not converge to the population value no matter how large the sample size is. The simultaneous equation models are used frequently which needs development of alternative estimation techniques. These techniques are discussed in Unit 15.

13.10 KEY WORDS

Endogenous Variables	:	These are variables which get determined jointly and interdependently within the model.
Exogenous Variables	:	These are variables which get determined outside the model and independently of the endogenous variable.
Inconsistent Estimates	:	The value of the estimates may not converge to the population value.
Predetermined Variables	:	The set of exogenous variables and the lagged endogenous variables.
Reduced Form Models	:	Models where each equation contains only one dependent variable.
Reduced Form Parameters	:	Parameters of the reduced form models.
Simultaneity Bias	:	Bias resulting from applying OLS to the simultaneous equation model.
Simultaneous Equation Models	:	Models where the equations are related and cannot be solved independently.
Structural Equation	:	Equations which represents the structure of an economic model or the behavior of an economic agent.
Structural Errors	:	Errors appearing in the structural equations.
Structural Parameters	:	Parameters appearing in the structural equations.

13.11 SOME USEFUL BOOKS/REFERENCES

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Woodridge, Jeffrey M., 2003, *Introductory Econometrics: A Modern Approach*, South Western.

13.12 EXERCISES

1) Explain the meaning of each of the following terms:

   a) Endogenous Variables.

   b) Exogenous Variables.

   c) Structural equations.

   d) Reduced From Equations.

   e) Simultaneity Bias.

   f) Recursive Systems

2) Develop a Simultaneous Equation model of the demand for and supply of economics students in the job market. Specify the exogenous and endogenous variables in the model.

3) For the IS-LM model discussed in the text, find the level of interest rate and income that is simultaneously compatible with the goods and money market equilibrium.

4) Construct a simultaneous equation model describing the market mechanism of an agricultural commodity. Derive the reduced form parameters in terms of structural parameters in your model. What is the economic meaning of the structural and of the reduced form parameters?

5) To study the relationship between inflection and yield of a common stock, the following model is used:

$$R_{bt} = \alpha_1 + \alpha_2 R_{st} + \alpha_3 R_{st-1} + \alpha_4 L_t + \alpha_5 Y_t + u_t$$
$$R_{st} = \beta_1 + \beta_2 R_{bt} + \beta_3 R_{bt-1} + \beta_4 L_t + \beta_5 Y_t + u_t$$

where L : Real per capita monetary base

Y : Real per capita income

R_{bt} : Bound yield

R_{st} : Common stock returns

Which are the endogenous and which are the exogenous variables in the model?

6) G Menges developed the following model of the West German economy :

$$Y_t = \beta_0 + \beta_1 Y_{t-1} + \beta_2 I_t + u_{1t}$$
$$I_t = \beta_3 + \beta_4 Y_t + \beta_5 Q_t + u_{2t}$$
$$C_t = \beta_6 + \beta_7 Y_t + \beta_8 C_{t-1} + \beta_9 P_t + u_{3t}$$
$$Q_t = \beta_{10} + \beta_{11} Q_{t-1} + \beta_{12} R_t + u_{4t}$$

where C = Personal consumption

Y = National income

I = Net Capital Formation

P = Cost of Living Index

R = Industrial Productivity

t = Time

u = Stochastic Disturbance

a) Which of the variables would you regard as exogenous and which as endogenous?

b) Is there any equation in the system that can be estimated by the single equation OLS method?

c) What is the reason behind including the variable P in the consumption function?

7) To study the relation between advertising expenditure and sales of cigarettes, the following model can be studied:

$$y_{1t} = \alpha_1 + \beta_1 Y_{3t} + \gamma_1 X_{1t} + \gamma_2 X_{2t} + u_{1t}$$

$$y_{2t} = \alpha_2 + \beta_2 Y_{3t} + \gamma_3 X_{1t} + \gamma_4 X_{2t} + u_{2t}$$

$$y_{3t} = \alpha_3 + \beta_3 Y_{1t} + \beta_4 Y_{2t} + u_{3t}$$

where Y_1 = Logarithm of sales of filter cigarettes

Y_2 = Logarithm of sales of non-filter cigarettes

Y_3 = Logarithm of advertising dollars for filter cigarettes divided by population over age 20 divided by advertising price index

X_1 = Log of disposable personal income divided by population over age 20 Divided by consumer price index

X_1 = Log of price per package of non filter cigarettes divided by consumer price index

a) In this model Y's are endogenous and X's are exogenous. Why the author assumed X_2 to be exogenous?

b) If X_2 is treated as endogenous variable how would you modify the preceding model?

UNIT 14 IDENTIFICATION PROBLEM

Structure

14.0 OBJECTIVES

After going through this Unit, you should be in a position to:

- explain the nature and significance of identification in simultaneous equations;
- identify the relevance of identification in the simultaneous equation system; and
- explain different techniques of identification.

14.1 INTRODUCTION

The term identification was originally used to denote the possibility (or impossibility) of deducing the values of the parameters[1]. In the preceding Unit while explaining the problem of SEM, we mentioned that in simultaneous equation models the major problem is, we could not understand from the data set which specific relationship it is representing. It is impossible to estimate an equation unless we get a particular form of it. Thus our main problem is to specify the form of the equation and the method of doing this is known as identification. Thus identification comes prior to the estimation. Once the equation is identified the problem of inconsistent estimation can be taken care of as identification ensures unique structure of the equation. Due to this unique specificity, the part that was not being explained earlier in the equation (and getting included as unexplained component causing correlation between dependent variable and error term), gets explained. Thus it also solves the problem of inconsistency in the estimator.

14.2 THE CONCEPT OF IDENTIFICATION

The identification problem is logically prior to estimation. If more than one theory is consistent with the same data then they are said to be observationally equivalent, and

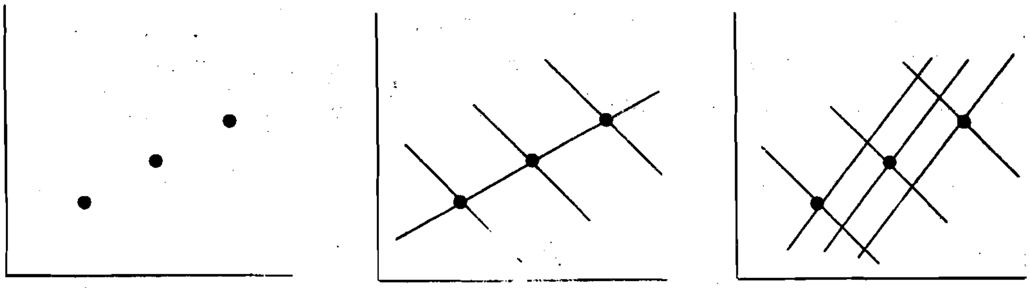
[1] See Johnston and Dinardo (1997).

we cannot distinguish them. In such cases the structure is said to be unidentified or needs to be identified.

Identification is a problem of model formulation, rather than of model estimation or appraisal. We say a model is identified if it is in a unique statistical form, enabling unique estimates of its parameters to be subsequently made from sample data. If a model is not identified then we cannot say exactly what relationship we are estimating.

To measure the coefficients of the demand equation, normally the published time series reporting the quantity bought of the commodity is used. However, the quantity bought is identical with the quantity sold at any particular price. Market data register points of interaction of equilibrium supply and demand at the price prevailing in the market at a certain point of time. A sample of time-series observations shows simultaneously the quantity demanded, and the quantity supplied, at the prevailing market price. That is, it only shows the points of interactions of demand and supply. If we use these data for estimation, we actually measure the coefficients of a function of the form $Q = f(p)$. This equation may be either the demand function or the supply function. But how can we be sure whether this equation represents demand function or supply function? If anyone is interested to measure the demand function then he can use the data. Similarly, the person who is interested to measure the supply equation will also be using the same data. It is clear that we need some criteria, which will enable us to verify that the estimated coefficients belong to the one or the other relationship. Such criteria are known as 'identification conditions' of a function. We will explain these criteria in Section 14.5.

We begin with the concept of identification using the following diagrams.



(a) (b) (c)

Fig. 14.1: Supply and Demand Functions

The observed data consist of the market outcomes shown in Fig. 14.1. We have no knowledge of the conditions of supply and demand beyond our consideration that the data represents equilibrium points that is the interaction points of supply and demand. Fig. 14.1(a) shows three data points. From the Fig. 14.1(b) and 14.1(c) we can see that we can get the same data for the two different situations. In Fig. 14.1(b) the supply curve is stable and the demand curve is shifting. The data represents points on the supply equation under the assumption that supply function is fairly stable and the demand curve shifts. But if both the demand and supply curves are shifting randomly as shown in Fig. 14.1 (c), then if we wish to estimate $Q = f(p)$ from the same data set then we estimate neither demand nor the supply curve; rather we end up estimating the mongrel equation[2].

Thus for estimation of the equation we need to first identify the equation; otherwise we cannot say exactly what function we are estimating. Identification ensures unique structural form of an equation so that we can easily estimate the equation.

[2] A combination of two and more equations is called *mongrel* equation.

It should be noted that identification problems arise only for those equations which contain coefficients which must be estimated statistically (from sample data). Identification difficulties do not arise for definitional equations, identities, or statements of equilibrium conditions, because such relationships do not require measurement.

14.3 PARADOX OF IDENTIFICATION

Discussion in the preceding section suggests that if we want to measure a given function belonging to a simultaneous-equations model, the function must be fairly stable over the sample period, that is, it must shift within a smaller range as compared with other relationships of the same model. In the last section we have shown that we can measure the supply function when it is fairly stable and the demand is shifting similarly we can measure the demand function if it is fairly stable while the supply function shows adequate variability. It can happen when the factor causing shift in one particular function is absent in another function. In other words, in order to identify the demand function, some factors absent from it but included in the supply function (or in other relations of the system) must be changing over the period of the sample.

Similarly, we can trace the supply function if it is fairly stable while demand shows enough variability. This implies that if the supply function is to be identified, some variables absent from it but affecting the demand function must be changing.

So we see that to identify one equation we have to look at the variables absent from it (while at the same time being operative in the other function(s) of the model). This is called the paradox of identification. In identification the relevant variables are those, which are absent from the equation!

14.4 IDENTIFICATION IN A TWO-EQUATION SYSTEM

When we estimate a model by OLS, the key identification condition is that each explanatory variable is uncorrelated with the error term. As we demonstrated in Unit 13, this major condition no longer holds, in general, for SEMs.

Before we consider a general two-equation SEM, it is useful to gain intuition by considering a simple supply and demand example. Let us write the system in equilibrium form (that is, with $q_s = q_d = q$ imposed) as

$$q = \alpha_1 p + \beta_1 z_1 + u_1 \qquad \text{... (14.1)}$$

or

$$q = \alpha_2 p + u_2 \qquad \text{... (14.2)}$$

For concreteness, let q be per capita rice consumption at the country level, let p be the average price per kg of rice in the county, and let z_1 be the price of fertilizer, which we assumed is exogenous to the supply and demand equations for rice. This means that (14.1) must be the supply function, as the price of fertilizer would shift supply (β_1 <0) but not demand. The demand function contains no observed demand shifters.

Intuitively, the fact that the demand equation is identified follows because we have an observed variable, z_1 that shifts the supply equation while not affecting the demand equation. Hence the shift factor for demand is only u_2, which is the unobserved error term, but the estimators will be consistent, provided z_1 is uncorrelated with u_2.

The supply equation cannot be traced out because there are no exogenous observed factors shifting the demand curve. May be in presence of some factor causing shift in the preference of people towards wheat or increase in per capita income we can identify the supply equation. It does not help that there are unobserved factors shifting the demand function; we need something observed.

To summarize: in the system of (14.1) and (14.2), it is the presence of an exogenous variable in the supply equation that allows us to estimate the demand equation.

We can easily extend the identification discussion to a general two-equation model. Let us write the two equations as

$$y_1 = \alpha_1 + \beta_1 y_2 + z_1\beta_2 + u_1 \qquad \text{... (14.3)}$$

and

$$y_2 = \alpha_2 + \beta_3 y_1 + z_2\beta_4 + u_2 \qquad \text{... (14.4)}$$

where y_1 and y_2 are the endogenous variables, and u_1 and u_2 are the error terms. The intercept in the first equation is α_1, and the intercept in the second equation is α_2. The variable z_1 denotes a set of k_1 exogenous variables appearing in the first equation: $z_1 = (z_{11}, z_{12}, \ldots, z_{1k1})$. Similarly, z_2 is the set of k_2 exogenous variables in the second equation : $z_2 = (z_{21}, z_{22}, \ldots, z_{2k2})$.

The fact that z_1 and z_2 generally contain different exogenous variables means that we have imposed exclusion restrictions on the model. In other words, we assume that certain exogenous variables do not appear in the first equation and others are absent from the second equation. As we saw with the previous supply and demand examples, this allows us to distinguish between the two structural equations.

When can we solve equations (14.3) and (14.4) for y_1 and y_2 (as linear functions of all exogenous variables and the structural errors, u_1 and u_2)? The condition is, namely, $\alpha_2\ \alpha_1 \neq 1$. Under this assumption, reduced forms exist for y_1 and y_2.

The key question is: Under what assumptions can we estimate the parameters in, say, (14.4)? This is the identification issue.

14.5 STATUS OF IDENTIFICATION

In econometric theory two possible situations of identifiability can arise: Equation under consideration is identified or not identified:

1) Equation is under-identified

   If the structure of the equation is not unique or there is not sufficient information to uniquely distinguish the equation then the equation is under-identified.

2) Equation is identified

   (a) Exactly identified: If there is sufficient information available to distinguish the equation then it is called just identified or exactly identified equation

   (b) Over identified: If there is more than sufficient information regarding the equation then the equation is called over-identified.

As far as the identification of a system is concerned, we can say an equation is under-identified if its statistical form is not unique. A system is under-identified when one or more of its equations are under-identified.

If an equation has a unique statistical form we say that it is identified. It may be exactly identified or over-identified. But in both cases it is identified. A system is identified if all its equations are identified.

14.5.1 Implications of the Identification State of a Model

Identification is closely related to the estimation of the model.

1) If an equation is identified, its coefficient can, in general, be statistically estimated. In particular:

   (a) If the equation is exactly identified, the appropriate method to be used for its estimation is the method of indirect least squares (ILS, see Unit 15).

   (b) If the equation is over-identified, indirect least squares cannot be applied, because it will not yield unique estimates of the structural parameters. There are various other methods which can be used in this case, for example two-stage least squares (2SLS), or maximum likelihood methods. These methods will be developed in Unit 15.

14.5.2 Formal Rules for Identification

Identification may be established either by the examination of the specification of the structural model, or by the examination of the reduced form of the model.

Traditionally identification has been approached via the reduced form. In the subsequent section we will examine both approaches. However, the reduced form approach is conceptually confusing and computationally more difficult than the structural model approach, because it requires the derivation of the reduced form first and then examination of the values of the determinant formed from some of the reduced form coefficients. The structural form approach is simpler and more useful. In this unit we will only discuss the structural form approach of identification.

In applying the identification rules we should ignore the constant term if it is present in each equations, or, if it is present in some of the equations then we have to retain it and we need to treat it like separate variable. In this case we must include in the set of variables a dummy variable (say X_0), which would always take on the value 1. In this unit we will ignore the constant intercept.

14.6 THE IDENTIFICATION CONDITIONS

We mentioned above that there are two conditions that must be fulfilled for an equation to be identified.

14.6.1 The Order Condition for Identification

This condition is based on a counting rule of the variables included and excluded from the particular equation. It is a necessary but no sufficient condition for the identification of an equation. The order condition may be stated as follows.

For an equation to be identified the total number of variables (endogenous and exogenous) excluded from it must be equal to or greater than the number of endogenous variables in the model less one. Given that in a complete model the number of endogenous variables is equal to the number of equations of the model, the order condition for identification is sometimes stated in the following equivalent form.

For an equation to be identified the total number of variables excluded from it but included in other equations must be at least as greater as the number of equations of the system less one.

Let G = total number of equations (= total number of endogenous variables)

K = number of total variables in the model (endogenous and predetermined)

M = number of variables, endogenous and exogenous, included in a particular equation.

Then the order condition for identification may be symbolically expressed as

$(K-M)$	$\geq \quad (G-1)$
[Excluded variables] ≥	[Total number of equations −1]

For example, if a system contains 12 equations with 20 variables, 12 endogenous and 8 exogenous, an equation containing 10 variables is not identified, while another containing 5 variables is identified.

(a) Let us assume for the first equation we have

$$G = \mathbf{10} \qquad K = \mathbf{15} \qquad M = \mathbf{11}$$

Recall that the **order condition** is given by

$$(K-M) \geq (G-1)$$

In this case, however, we have

$$(15-11) < (10-1)$$

Thus, the order condition is not satisfied and the equation is under-identified.

(b) Let us assume for the second equation we have

$$G = \mathbf{10} \qquad K = \mathbf{15} \qquad M = \mathbf{5}$$

Order condition:

$$(K-M) \geq (G-1)$$

$$(15-5) > (10-1)$$

that is, the order condition is satisfied.

The order condition for identification is necessary for a relation to be identified, but it is not sufficient, that is, it may be fulfilled in any particular equation and yet the relation may not be identified.

14.6.2 The Rank Condition for Identification

The rank condition sates that: in a system of G equations any particular equation is identified if and only if it is possible to construct at least one non-zero determinant of order $(G-1)$ from the coefficients of the variables excluded from that particular equation but contained in the other equations of the model[3].

[3] This condition is called rank condition because it refers to the rank of the matrix of parameters of excluded variables. The rank of a matrix is the order of the largest non-zero

The practical steps for tracing the identifiably of an equation of a structural model may be outlined as follows:

Step1: Write the parameters of all the equations of the model in a separate table, noting that the parameter of a variable excluded from an equation is equal to zero.

For example let a structural model be

$$y_1 = 3y_2 + 2x_1 + x_2 + u_1$$

$$y_2 = y_3 + x_3 + u_2$$

$$y_3 = y_1 - y_2 - 2x_3 + u_3$$

where the y's are the endogenous variables and the x's are the predetermined variables.

This model may be rewritten in the form

$$-y_1 + 3y_2 + 0y_3 - 2x_1 + x_2 + 0x_3 + u_1 = 0$$

$$0y_1 + y_2 + y_3 - 0x_1 + 0x_2 + x_3 + u_2 = 0$$

$$y_1 + y_2 + y_3 - 0x_1 + 0x_2 + 2x_3 + u_3 = 0$$

Ignoring the random disturbances the table of the parameters of the model is as follows.

Table 14.1: Parameters of the Equation System

Equations	Variables					
	Y_1	Y_2	Y_3	X_1	X_2	x_3
1st equation	–1	3	0	–2	1	0
2nd equation	0	–1	1	0	0	1
3rd equation	1	–1	–1	0	0	–2

Step2: Eliminate the row of coefficients of the equation, which is being examined for identification.

For example if we want to examine the identifiably of the second equation of the model we strike out the second row of the table of coefficients.

Step3: Eliminate the columns in which a non-zero coefficient of the equation being examined appears. By deleting the relevant row and columns we are left with the coefficients of variables not included in the particular equation, but contained in the other equations of the model.

For example, if we are examining for identification the second equation of the system, we will eliminate the second, third and the sixth columns of Table 14.1, thus obtaining Table 14.2.

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determinant which can be formed from the matrix. In our case the relevant matrix is the sub-matrix of coefficients of the absent variables. Hence the rank condition may be also stated as follows: a sufficient condition for identification of a relationship is that the rank of the matrix of parameters of all the excluded variables (endogenous and predetermined) from that equation be equal to $(G-1)$.

Table 14.2-(a),(b)

Table of Structural Parameters

	y_1	y_2	y_3	x_1	x_2	x_3
1st	−1	3	0	−2	1	0
2nd	0	−1	1	0	0	1
3rd	1	−1	−1	0	0	−2

Table of parameters left after eliminating the relevant rows and columns

y_1	x_1	x_2
−1	−2	1
1	0	0

Step4: Form the determinant(s) of order $(G-1)$ and examine their value. If at least one of these determinants is non-zero, the equation is identified. If all the determinants of order $(G-1)$ are zero, the equation is under-identified.

In the above example of exploration of the identifiability of the second structural equation we have three determinants of order $(G-1) = 3-1=2$.

They are

$$\Delta_1 = \begin{vmatrix} -1 & -2 \\ 1 & 0 \end{vmatrix} \neq 0$$

$$\Delta_2 = \begin{vmatrix} -2 & 1 \\ 0 & 0 \end{vmatrix} = 0$$

$$\Delta_3 = \begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix} \neq 0$$

(the symbol 'Δ' stands for determinant'). We see that we can form two non-zero determinants of order $G-1 = 3-1 = 2$; hence the second equation of our system is identified.

Step5: To see whether the equation is exactly identified or over-identified we use the order condition $(K-M) > (G-1)$. With this criterion, if the equality sign is satisfied, that is, if $(K-M) = (G-1)$, the equation is exactly identified. If the inequality sign holds, that is, if $(K-M) > (G-1)$, then the equation is over-identified.

In the case of the second equation we have

$$G = 3 \qquad K = 6 \qquad M = 3$$

and the counting rule $(K-M) \geq (G-1)$ gives

$$(6-3) > (3-1)$$

Therefore the second equation of the model is over-identified.

The identification of a function is achieved by assuming that some variables of the model have a zero coefficient in this equation, that is, we assume that some variables do not directly affect the dependent variable in this equation. This, however, is an assumption which can be tested with the sample data. We will examine some tests of identifying restrictions in a subsequent section.

Let us discuss the two formal condition of identification with the help of some examples.

Example 14.1

Assume that we have a model describing the market of an agricultural product. From the theory of partial equilibrium we know that the price in a market is determined by the force of demand and supply. The main determinants of the demand are the price of the commodity, the prices of other commodities, incomes and tastes of consumers. Similarly, the most important determinants of the supply are the price of the commodity, other prices, technology, the prices of factors of production, and weather conditions. The equilibrium condition is that demand be equal to supply. The above theoretical information may be expressed in the form of the following mathematical model

$$D = a_0 + a_1 P_2 + a_2 P_2 + a_3 Y + a_4 t + u$$

$$S = b_0 + b_1 P_1 + b_2 P_2 + b_3 Y + b_4 t + w$$

$$D = S$$

where D = quantity demanded

S = quantity supplied

P_1 = price of the given commodity

P_2 = prices of other commodities

Y = income

C = costs (index of prices of factors of production)

t = time trend. In the demand function it stands for 'taste'; in the supply function it stands for 'technology'

The above model is mathematically complete in the sense that it contains three equations in three endogenous variables, D, S and P_1. The remaining variables, Y, P_2, C, t are exogenous. Suppose we want to identify the supply function. We apply the two criteria for identification:

1) Order condition: $(K-M) \geq (G-1)$

In our example we have

$$K = 7 \quad M = 5 \qquad G = 3$$

Therefore, (K–M) = (G–1)

(7–5) = (3–1) = 2

Consequently the second equation satisfies the first condition for identification.

2) Rank condition

The table of the coefficients of the structural model is as follows.

Table 14.3: Coefficients of the Structural Model

Equations	Variables						
	D	P_1	P_2	Y	t	S	C
1st equation	-1	a_1	a_2	A_2	a_4	0	0
2nd equation	0	b_1	b_2	0	b_4	-1	b_3
3rd equation	-1	0	0	0	0	1	0

Following the procedure explained earlier we eliminate the second row and the second, third, fifth, sixth and seventh columns. Thus we are left with the table of the coefficients of excluded variables (see Table 14.4 (a) and (b)).

Table 14.4-(a), (b):

Complete Table of Structural Parameters

-1	a_1	a_2	a_3	a_4	0	0
0	b_1	b_2	0	b_4	1	b_3
-1	0	0	0	0	1	0

Table of parameters after eliminating the relevant rows and columns

-1	a_3
-1	0

The value of the determinant is non-zero, provided that a_3=0.

$$\Delta = \begin{vmatrix} -1 & a_3 \\ -1 & 0 \end{vmatrix} = (0)(-1) - (-1)(a_3) = a_3$$

We see that both the order and rank conditions are satisfied. Hence the second equation of the model is identified. Furthermore, we see that in the order condition the equality holds: $(7 - 5) = (3 - 1) = 2$. Consequently the second structural equation is exactly identified.

Thus we have seen that the order condition of identification is the necessary condition and the rank condition is both necessary and sufficient condition for identification of an equation. Even if we do not consider the order condition then also we can identify through rank condition.

Even if the rank condition is both necessary and sufficient though order condition is used as it is much more simpler and we can check the identification status of an equation individually while in case of rank condition we need to consider the whole system.

14.7 IDENTIFYING RESTRICTIONS

In many cases we are interested in only one (or a few) of the equations of the model and attempts to measure its parameters statistically without a complete knowledge of the entire model. Under these circumstances as the structure of the model is not known, identification of the relationship cannot be established by applying the order and rank conditions. In such cases identification is established imposing certain restrictions on the values of some parameters in some equations.

The restriction on the values of the parameters can be in any of the following forms:

1) Zero restrictions.

2) Equality restrictions, or other linear relationship between some of the parameters.

3) Extraneous estimates of some parameters.

We discuss each one of these restrictions below.

Zero Restriction

Imposing a zero restriction on the value of a parameter in a particular equation means

A priori knowledge usually enables us to decide that some coefficients must be zero in the particular equation, while they assume non-zero values in other equations of the system. We said that identification of an equation is based on variables not included (not appearing) in it. To be identifiable an equation must be independent of one or more important variables, which are included in other equations of the system. Such excluded variables, if operative during the sample period, will generate shifts in the other equations of the model, which will in turn identify the particular equation from which they are absent (i.e. in which they appear with zero coefficient).

Based on the a priori information a list can be prepared, which should be as complete as possible, of the factors which are relevant to the phenomenon being studied. The list can help us decide which of these factors would normally appear in each relationship. For example, assume that we want to study the demand for an agricultural product. The demand equation belongs to a system of equations describing the market mechanism. The factors relevant to this model and their appearance or exclusion from the demand and supply equations are shown in Table 14.5

From this a priori information it is clear that there are several shift factors in the supply equation which do not affect (do not appear in) the demand equation. This suggests that the demand equation, in which we are interested is identified. Imposition of the restriction of some coefficients being equal to zero in the demand equation assists in the investigation of the identifiability of this equation.

One might argue that one can always 'render' an equation identified by imposing zero restrictions: one can always think of some factors which do not appear in a particular equation while they are, at the same time, included in other equations of the model. However, there are tests for the identifying restrictions, which restrict the freedom of the researcher and prevent the abuse of unjustified restrictions on the values of the parameters.

Table 14.5: *A Priori* Information on the Demand for Commodity *X*

Variables	Notation	Determinants of demand	Determinants of supply
Price of X	P	P	P
Lagged price	P_{t-1}	-	P_{t-1}
Other prices	P_0	P_0	-
Income	Y	Y	-
Weather	W	-	W
Stocks	S	-	S
Tastes	t	t	-
Income distribution	V	V	-
Population	N	N	-
Technology	T	-	T
Factor prices	C	-	C

But we have to keep it in mind that identification should not be cheaply achieved by assuming that some unimportant variables are included in other equations. One must be able to defend identification of an equation by 'adding' to another equation(s) significant factors which were previously mistakenly ignored.

Restrictions on the relative values (or on the relationship) of two or more parameters

The a priori information can also tell us about some relationships among the variables which can help us in identifying an under identified equation. For example, the following restrictions may be known to us *a priori*.

a) **The sum of some parameters**: for example $a_1 + a_2 = c$, where c is a constant. For example, in their original formulation of the production function Cobb and Douglas assumed constant returns to scale which enabled them to set $a_1 + a_2 = 1$ in their model: $X = a_0 L^{a_1} K^{a_2}$; with this restriction the function may be written as $X / L = a_0 \left(K / L\right)^{a_2}$.

b) **The equality of some of the parameters**: For example, in a model containing two consumption functions (one for the agricultural sector and one for the urban areas), we may impose the restriction that the coefficients of income are different while the coefficients of liquid assets are equal in the two functions

c) **The ratio of some parameters:** For example, in a model we may have prior information that $a_1 / a_2 = k$, where k is a constant, and so on.

Extraneous Estimates

If some parameters are identified, while others are not and there exists information on their value from other (extraneous) sources, the researcher may proceed with the estimation of the identified coefficients and use the extraneously known value for the non-identified ones.

Of the above identifying restrictions the most important and most widely used are the restrictions on the values of the structural parameters.

In some cases the investigator may have information regarding the relative chance of fluctuation of a particular equation and this information enables the investigator, on a priori information, to specify which one of the equations of the model is likely to show wider random variation as compared with other equations. For example in the case of many agricultural products the supply is subject to wide random fluctuations due to weather conditions, while their demand is fairly stable over time. The market data on quantity and price would, under such conditions, identity the demand function. The model would be

$$D = a_0 + a_1 P + u$$

$$S = b_0 + b_1 P + v$$

$$D = S$$

$$var\left(u\right) < var\left(v\right)$$

This situation would portray the data as having been generated by the supply curve shifting along the demand curve, which would have remained roughly stable.

In general, knowledge of the conditions of the particular phenomenon may provide a priori information, which would enable the researcher to impose restrictions on the relative size of the variances of the random variables of the equations of the system. However, as the error term is nothing but the conceptualization of different unknown and complex factors in composite form, that assumptions concerning their variances would usually be highly arbitrary.

14.8 LET US SUM UP

The problem of identification is necessary before solving the problem of estimation. The objective of the identification problem is to obtain unique numerical estimates of the structural coefficients from the estimated reduced form coefficients. An equation in a system of simultaneous equations is identified only when we are able to resolve this issue. If not, the equation remains in unidentified or under-identified situation.

An identified equation can be just identified or over-identified. In case of exact identified equation, unique values of structural coefficients can be obtained and in case of over identified equation, more than one value for one or more structural parameters may be obtained. The identification problem arises because the same set of data may be compatible with different sets of structural coefficients, that is, different models. Thus, in the regression of price on quantity only, we can see only bivariate data of price and quantity not their exact relationship implying whether one is estimating the supply function or the demand function, because the same data can be used estimating both the equations.

The technique of reduced-form equations may be used to assess the identifiability of a structural equation. The technique expresses an endogenous variable solely as a function of predetermined variables. The order condition or the rank condition of identification can be used to simplify the problem.

Order condition provides only a necessary condition for identification. On the other hand, the rank condition is both a necessary and sufficient condition for identification. If the rank condition is satisfied, the order condition is satisfied too, although the reverse is not true. In practice, though, the order condition is generally adequate to ensure identifiability.

14.9 KEY WORDS

Identification	:	The possibility (or impossibility) of deducing the values of the parameters
Paradox of Identification	:	The paradox that in identification is done in terms of the variables absent from the equation not in terms of the present variable.
Under-identified Equation	:	If the structure of the equation is not unique or there is not sufficient information to uniquely distinguish the equation then the equation is under-identified.
Exactly-identified Equation	:	If there is sufficient information available to distinguish the equation then it is called just identified or exactly identified equation
Over-identified Equation	:	If there is more than sufficient information regarding the equation then the equation is called over-identified.
Order Condition	:	Necessary Condition for Identification
Rank Condition	:	Necessary and sufficient Condition for identification

14.10 SOME USEFUL BOOKS/REFERENCES

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14.11 EXERCISES

1) Explain the concept of identification.

2) What do you mean by paradox of identification?

3) Why identification is necessary prior to the estimation?

4) Examine the identification state of the following models of income determination.

a)
$$C = a_0 + a_1 Y_d + u_1$$
$$I = b_0 + b_1 r + b_2 Y_{t-1} + u_2$$
$$Y_d = Y - T$$
$$T = \overline{T}$$
$$Y = C + I + \overline{G}$$

(endogenous variables: C, I, Y)

b)
$$C = a_0 + a_1 Y + a_2 Y + u_1$$
$$I = b_0 + b_1 r + b_2 Y_{t-1} + u_2$$
$$M_D = c_0 + c_1 r + c_2 Y + c_3 L + u_3$$
$$Y = C + I + \overline{G}$$

(endogenous variables: C, I, r Y)

where r = interest rate, L = liquid assets,

M_D = demand for money, M_s = supply of money.

5) Suggest ways for rendering any unidentified model exactly identified.

6) Explain what is meant by the "identification problem" in the context of the linear simultaneous equation model.

7) a) What do you understand by an identification problem? Illustrate your answer with examples of your own.

b) State the formal identifiability rules, with reference to your examples.

c) What are the implications of the state of identification of a model for choice of estimation method?

8) Let a model consist of the following equations

$$y_1 = 4y_2 + 3x_1 + u_1$$
$$y_2 = 2y_3 + 2x_3 + u_2$$
$$y_3 = 2y_1 + 3x_2 - x_2 - x_3 + u_3$$

9) Given the simple Keynesian model of income determination

$$C_1 = a_0 + a_1 Y_1 + u_1$$

$$I_t = b_0 + b_1 Y_t + b_2 Y_{t-1} + u_2$$

$$Y_t = C_t + I_t + G_t$$

a) Derive the reduced form coefficients of the behavioural equations.

b) Show that the reduced-form parameters measure the total effect, direct and indirect, of a change in the exogenous variable on the endogenous variables. Use as an example the reduced form of the above investment function.

c) Establish the identification state of each of the behavioural equations, using the order and rank conditions.

d) What is the identification state of the entire model?

e) What estimation method is appropriate for each of the behavioural equations?

f) Show under what conditions could the system become exactly identified.

10) What is a priori information? Explain the role of a priori restrictions in the identification of a generally under identified model.

UNIT 15 ESTIMATION OF SIMULTANEOUS EQUATION MODELS

Structure

15.0 OBJECTIVES

After going through this Unit, you should be in a position to:

- provide the basic understanding of the nature of different solution techniques;
- explaining different techniques of solving a simultaneous equation system; and
- explaining relative applicability of different solution techniques.

15.1 INTRODUCTION

After explaining what is simultaneity, the problem raised by it, and the identification problem in the previous two units we finally come to the estimation of the simultaneous equation system. At the outset it may be noted that the estimation problem is rather complex because there are a variety of estimation techniques with varying statistical properties. To keep our discussion comprehensive we should only focus on few of these techniques.

As simultaneity is relevant only in the system of equations, to preserve the spirit of simultaneous-equation models, ideally we should use the systems methods. In practice, however, such methods are not commonly used for its complication and the computational burden. Another reason for not using systems equation method can be due to the fact that if there is specification error in one of the equation, it gets transmitted into the rest of the system. As a result, the systems methods become very sensitive to specification errors. Therefore, in practice it is always easy to use single equation methods. Single equation methods, in the context of a simultaneous system, may be less sensitive to specification error in the sense that those parts of the system that are correctly specified may not be affected appreciably by errors in specification in another part.

In this unit we begin with the single equation methods then we will briefly explain some of the system methods. In the single equation method most popular are the Indirect Least Squares (ILS) method, Instrumental Variable (IV) method, and the 2-Stage Least Squares (2SLS) method.

15.2 INDIRECT LEAST SQUARES METHOD

In this method we obtain the estimates of the reduced form coefficients by applying OLS and indirectly get the values of structural coefficients in terms of the reduced form coefficients. For this reason the method is called the Indirect Least Square. We normally apply this method to exactly identified equations. The estimates obtained from this technique are called the Indirect Least Squares estimates.

Step 1. We first obtain the reduced-form equations. As noted in Unit 14, these reduced-form equations ensure that in each of the equations the dependent variable will be endogenous variable only and is a function solely of the predetermined (exogenous or lagged endogenous) variables and the stochastic error term(s).

Step 2. We assume that the other usual assumptions about the disturbance term in the OLS are satisfied. We apply OLS to the reduced-form equations individually. This operation is permissible since the explanatory variables in these equations are predetermined and hence uncorrelated with the stochastic disturbances. The estimates thus obtained are consistent.[1]

Step 3. We obtain estimates of the original structural coefficients from the estimated reduced-form coefficients obtained in Step 2. As noted in Unit 14, if an equation is exactly identified, there is a one-to-one correspondence between the structural and reduced-form coefficients. In other words, we can derive unique estimates of the structural parameters from the reduced form parameters. The relationship between reduced form parameters and the structural parameters form a system of equations in which the reduced form coefficients are expressed as functions of the structural parameters.

As this three-step procedure indicates, the name ILS derives from the fact that structural coefficients (the object of primary enquiry in most cases) are obtained indirectly from the OLS estimates of the reduced-form coefficients.

Example 15.1

Consider the demand-and-supply model that is given below:

$$\textit{Demand function:}\quad Q_t = \alpha_0 + \alpha_1 P_t + \alpha_2 X_t + u_{1t} \qquad \ldots (15.1)$$

$$\textit{Supply function:}\quad Q_t = \beta_0 + \beta_1 P_t + u_{2t} \qquad \ldots (15.2)$$

where Q = quantity

P = price

X = income or expenditure

[1] In addition to be[illegible] consistent, the estimates "may be best unbiased and/or asymptotically efficient, de[illegible]spectively upon whether (i) the z's [= X's] are exogenous and not merel[illegible]i.e., do not contain lagged values of endogenous variables] and/or (ii) th[illegible] disturbances is normal." (W.C. Hood and Tjalling C. Koopmans, St[illegible] Methods, John Wiley & Sons, New York, 1953, p. 133)

Assume that X is exogenous. As noted previously, the supply function is exactly identified whereas the demand function is not identified.

By equating (15.1) and (15.2) and rearranging terms we obtain the reduced form equations. These are:

$$P_t = \Pi_0 + \Pi_1 X_t + w_t \qquad \text{... (15.3)}$$

$$Q_t = \Pi_2 + \Pi_3 X_t + v_t \qquad \text{... (15.4)}$$

Where the Π's are the reduced-form parameters. These are in fact (nonlinear) combinations of the structural parameters. Here w and v are linear combinations of the structural disturbances u_1 and u_2.

Notice that each reduced-form equation contains only one endogenous variable, which is the dependent variable and which is a function solely of the exogenous variable X (income) and the stochastic disturbances. Hence, the parameters of the preceding reduced-form equations may be estimated by OLS. These estimates are:

$$\hat{\Pi}_1 = \frac{\sum p_t x_t}{\sum x_t^2} \qquad \text{...(15.5)}$$

$$\hat{\Pi}_0 = \overline{P} - \hat{\Pi}_1 \overline{X} \qquad \text{...(15.6)}$$

$$\hat{\Pi}_3 = \frac{\sum q_t x_t}{\sum x_t^2} \qquad \text{...(15.7)}$$

$$\hat{\Pi}_2 = \overline{Q} - \hat{\Pi}_3 \overline{X} \qquad \text{...(15. 8)}$$

where the lowercase letters, as usual, denote deviations from sample means and where $\overline{Q}$ and $\overline{P}$ are the sample mean values of Q and P. As noted previously, the $\hat{\Pi}_i$'s are consistent estimators and under appropriate assumptions are also minimum variance unbiased or asymptotically efficient.

Since our primary objective is to determine the structural coefficients, let us see if we can estimate them from the reduced-form estimates. The supply function is exactly identified. Therefore, its parameters can be estimated uniquely from the reduced-form parameters as follows:

$$\beta_0 = \Pi_2 - \beta_1 \Pi_0 \quad \text{and} \quad \beta_1 = \frac{\Pi_3}{\Pi_1}$$

Hence, the estimates of the structural parameters can be obtained from the estimates of the reduced-form coefficients as

$$\hat{\beta}_0 = \hat{\Pi}_2 - \hat{\beta}_1 \hat{\Pi}_0 \qquad \text{... (15.9)}$$

$$\hat{\beta}_1 = \frac{\hat{\Pi}_3}{\hat{\Pi}_1} \qquad \text{... (15.10)}$$

These estimates, as mentioned earlier, are the ILS estimators. Note that the parameters of the demand function cannot be thus estimated.

15.2.1 Assumptions of the ILS Method

The method of ILS is based on the following assumptions:

1. The structural equation must be exactly identified. If the structural system is over-identified then we cannot get unique estimates of the reduced form parameters from the structural parameters.

2. The random variable of the reduced form parameter must satisfy all the usual assumptions of the OLS.

3. The exogenous variables in the model must not be perfectly collinear.

Thus we can see that the ILS method is based on all the assumptions of OLS with the additional one that the system of equations should be exactly identified.

15.2.2 Properties of ILS Estimators

If the assumptions of the ILS are fulfilled the estimates of the reduced form parameters will be best, linear and unbiased. However the estimates obtained from the structural parameters, that is those obtained from the system of coefficient relationships, are biased for small samples but they are consistent, that is their bias tends to zero as the size of the sample increases and their distribution converges at the true value of the structural parameters. ILS is generally preferred because of the consistency property and its simplicity compared with other methods.

15.3 INSTRUMENTAL VARIABLE METHOD

The instrumental variable (IV) method is a single equation method, being applied to one equation of the system at a time. It is mainly applicable to over-identified models. The instrumental variable method uses some appropriate exogenous variable as instrument to reduce the dependence between the error term and the explanatory variables. We have already discussed one method of estimation: the indirect least squares method. However, the ILS method is very cumbersome if there are many equations and hence it is not often used. The instrumental variable method is more generally applicable.

Here we are going to discuss single-equation methods only. In these methods we estimate each equation separately using only the information about the restrictions on the coefficients of that particular equation. Before we proceed to explain the concept of this method we should first discuss what this 'instrumental variable' actually means. Broadly speaking, an instrumental variable is a variable that is uncorrelated with the error term but correlated with the explanatory variables in the equation. The crucial point in this method involves choosing the appropriate instrumental variable.

For instance, suppose we have the equation

$$y = \beta x + u \qquad \text{... (15.11)}$$

where x is correlated with u. We cannot estimate this equation by ordinary least squares. The estimate of β is inconsistent because of the correlation between x and u If we can find a variable z that is uncorrelated with u, we can get a consistent estimator for β. We replace the condition cov(z, u) = 0 by its sample counterpart

$$\frac{1}{n}\sum zu = 0$$

or

$$\frac{1}{n}\sum z(y - \beta x) = 0 \qquad \text{... (15.12)}$$

This gives

$$\hat{\beta} = \frac{\sum zy}{\sum zx} = \frac{\sum z(\beta x + u)}{\sum zx} = \beta + \frac{\sum zu}{\sum zx} \qquad \ldots (15.13)$$

But $\sum zu / \sum zx$ can be written as $\dfrac{\frac{1}{n}\sum zu}{\frac{1}{n}\sum zx}$ The probability limit of this expression is

$$\frac{\operatorname{cov}(z,u)}{\operatorname{cov}(z,x)} = 0$$

since $\operatorname{cov}(z,x) \neq 0$. Hence plim $\hat{\beta} = \beta$, thus proving that $\hat{\beta}$ is a consistent estimator for β. Note that we require z to be correlated with x so that $\operatorname{cov}(z,x) \neq 0$.

Example 15.2

Let us consider the following equation system.

$$y_1 = b_1 y_2 + c_1 z_1 + c_2 z_2 + u_1 \qquad \ldots (15.14\text{ a,b})$$

$$y_2 = b_2 y_1 + c_3 z_3 + u_2$$

where y_1, y_2 are endogenous variables, z_1, z_2, z_3 are exogenous variables, and u_1, u_2 are error terms. Consider the estimation of the first equation. Since z_1 and z_2 are independent of u_1, we have $\operatorname{cov}(z_1, u_1) = 0$ and $\operatorname{cov}(z_2, u_1) = 0$. However, y_2 is not independent of u_1. Hence $\operatorname{cov}(y_2, u_1) \neq 0$. Since we have three coefficients to estimate (c_1, c_2, c_3), we have to find a variable that is independent of u_1. Fortunately, in this case we have z_3 and $\operatorname{cov}(z_3, u_1) = 0$. z_3 is the instrumental variable for y_2. Thus, writing the sample counterparts of these three covariances, we have the three equations

$$\frac{1}{n}\sum z_1 (y_1 - b_1 y_2 - c_1 z_1 - c_2 z_2) = 0$$

$$\frac{1}{n}\sum z_2 (y_1 - b_1 y_2 - c_1 z_1 - c_2 z_2) = 0 \qquad \ldots (15.15\text{ a,b, c})$$

$$\frac{1}{n}\sum z_3 (y_1 - b_1 y_2 - c_1 z_1 - c_2 z_2) = 0$$

The difference between the normal equations for the ordinary least squares method and the instrumental variable method is only in the last equation.

Consider the second equation of our model. Now we have to find an instrumental variable for y_1 but we have a choice of z_1 and z_2. Its is because equation (15.15b) is overidentified (by the order condition).

Note that the order condition (counting rule) is related to the question of whether or not we hav. enough exogenous variables elsewhere in the system to use as instruments for the endogenous variables in the equation with unknown coefficients. If the equation is under-identified we do not have enough instrumental variables. If it is exactly identified, we have just enough instrumental variables. If it is overidentified, we have more than enough instrumental variables. In this case we have to use weighted averages of the instrumental variables available. We compute these weighted averages so that we get the most efficient (minimum asymptotic variance) estimators.

It has been shown (proving this is beyond the scope of this book) that the efficient instrumental variables are constructed by regressing the endogenous variables on *all* the exogenous variables in the system (i.e., estimating the reduced-form equations). In the case of the model given by equations (15.15), we first estimate the reduced-form equations by regressing y_1 and y_2 on z_1, z_2, z_3. We obtain the predicted values $\hat{y}_1$ and $\hat{y}_2$ and use these as instrumental variables. For the estimation of the first equation we use $\hat{y}_2$, and for the estimation of the second equation we use $\hat{y}_1$. We can write $\hat{y}_1$ and $\hat{y}_2$ as linear functions of z_1, z_2, z_3. Let us write

$$\hat{y}_1 = a_{11}z_1 + a_{12}z_2 + a_{13}z_3 \qquad \ldots (15.16)$$

$$\hat{y}_2 = a_{21}z_1 + a_{22}z_2 + a_{23}z_3 \qquad \ldots (15.17)$$

where the a's are obtained from the estimation of the reduced-form equations by OLS. In the estimation of the first equation in (15.15) we use $\hat{y}_2$, z_1, and z_2 as instruments. This is the same as using z_1, z_2, and z_3 as instruments because

$$\sum \hat{y}_2 u_1 = 0 \Rightarrow \sum (a_{21}z_1 + a_{22}z_2 + a_{23}z_3)u_1 = 0$$

$$\Rightarrow a_{21}\sum z_1 u_1 + a_{22}\sum z_2 u_1 + a_{23}\sum z_3 u_1 = 0$$

But the first two terms are zero by virtue of the first equations in (15.14). Thus $\sum \hat{y}_2 u_1 = 0 \Rightarrow \sum z_3 u_1 = 0$. Hence using $\hat{y}_2$ as an instrumental variable is the same as using z_3 as an instrumental variable. This is the case with exactly identified equations where there is no choice in the instruments.

The case with the second equation in (15.14) is different. Earlier, we said that we had a choice between z_1 and z_2 as instruments for y_1. The use of $\hat{y}_1$ gives the optimum weighting. The normal equations now are

$$\sum \hat{y}_1 u_2 = 0 \text{ and } \sum z_3 u_2 = 0$$

$$\sum \hat{y}_1 u_2 = 0 \Rightarrow \sum (a_{11}z_1 + a_{12}z_2 + a_{13}z_3)u_2 = 0$$

$$\Rightarrow \sum (a_{11}z_1 + a_{12}z_2)u_2 = 0$$

Since $\sum z_3 u_2 = 0$. Thus the optimal weights for z_1 and z_2 are a_{11} and a_{12}.

15.3.1 Assumptions of Instrumental Variable Method

The instrumental variable method involves the solution of the transformed normal equations of OLS. One of the main assumptions of this method is there should be knowledge about some exogenous variables in other equation of the complete system which can be used as instruments. The instrumental variable method is based on certain assumptions that the instrument should satisfy. These are:

1) The instrument must be strongly correlated with the endogenous variable which it will replace in the structural equation.

2) It must be truly exogenous and hence uncorrelated with the random term of the structural equation.

3) It must be least correlated with the exogenous variable already appearing in the set of explanatory variables of the particular structural equation. Otherwise it could generate the problem of multicollinearity in the variables.

4) If more than one instrumental variable has to be used in the same structural equation then they have to be least correlated with each other for the same reason of multicollinearity.

5) Again the new random term generated in the transformed equation should satisfy all the assumptions of OLS error term.

15.3.2 Properties of Instrumental Variable Estimators

Provided the above assumptions are fulfilled we usually find that for small samples the estimates are biased. Actually despite of the transformation there is some dependence between the error term and the explanatory variable, which renders the estimate statistically biased in small samples. For large samples the estimates are consistent.

The instrumental variable estimates though consistent are not asymptotically efficient; that is they do not possess the minimum variance as compared with the other consistent estimates obtained from alternative econometric techniques.

15.4 TWO-STAGE LEAST SQUARES METHOD

The 2SLS method is also a single equation method and it has provided satisfactory results for the estimates of the structural parameters. It is generally applicable to over-identified models and accepted as the most important of the single equation techniques for the estimation of the over-identified models. Theoretically it can be considered as an extension of ILS and IV method. It differs from the IV method described in Section 15.3 in that the $\hat{y}'s$ are used as regressors rather than as instruments, but the two methods give identical estimates. The 2SLS, like other simultaneous equation techniques, aims at elimination of the simultaneous equation bias as far as possible.

Consider the equation to be estimated:

$$y_t = b_1 y_t + c_1 z_1 + u_1 \qquad \ldots (15.18)$$

The other exogenous variables in the system are z_2, z_3, and z_4.

Let $\hat{y}_2$ be the predicted value of y_2 from a regression on y_2 on z_1, z_2, z_3, z_4 (the reduced-form equation). Then

$$y_2 = \hat{y}_2 + v_2 \qquad \ldots (15.19)$$

where v_2, the residual, is uncorrelated with each of the regressors z_1, z_2, z_3, z_4 and hence with $\hat{y}_2$ as well. (This is the property of least squares regression that we discussed in Block 1.)

The normal equations for the efficient IV method are

$$\sum \hat{y}_2 (y_1 - b_1 y_2 - c_1 z_1) = 0 \qquad \ldots (15.20\ a,b)$$

$$\sum z_1 (y_1 - b_1 y_2 - c_1 z_1) = 0$$

Substituting $y_2 = \hat{y}_2 + v_2$ we get

$$\sum \hat{y}_2(y_1 - b_1\hat{y}_2 - c_1z_1) - b_1\sum \hat{y}_2 v_2 = 0 \qquad \ldots(15.21\text{ a,b})$$

$$\sum z_1(y_1 - b_1\hat{y}_2 - c_1z_1) - b_1\sum z_1 v_2 = 0$$

But $\sum z_1 v_2 = 0$ and $\sum \hat{y}_2 v_2 = 0$ since z_1 and $\hat{y}_2$ are uncorrelated with v_2. Thus equations (15.21) give

$$\sum \hat{y}_2(y_1 - b_1\hat{y}_2 - c_1z_1) = 0 \qquad (15.22\text{a,b})$$

$$\sum z_1(y_1 - b_1\hat{y}_2 - c_1z_1) = 0$$

But these are the normal equations if we replace y_2 by $\hat{y}_2$ in (15.20) and estimate the equations by OLS. This method of replacing the endogenous variables on the right-hand side by their predicted values from the reduced form and estimating the equation by OLS is called the two-stage least squares (2SLS) method. The name arises from the fact that OLS is used in two stages:

Stage 1. Estimate the reduced-form equations by OLS and obtain the predicted $\hat{y}'s$.

Stage 2. Replace the right-hand side endogenous variables by $\hat{y}'s$ and estimate the equation by OLS.

Note that the estimates do not change even if we replace y_1 by $\hat{y}_1$ in equation (15.20). Take the normal equations (15.22). Write

$$y_1 = \hat{y}_1 + v_1$$

where v_1 is again uncorrelated with each of z_1, z_2, z_3, z_4. Thus it is also uncorrelated with $\hat{y}_1$ and $\hat{y}_2$, which are both linear functions of the z's. Now substitute $y_1 = \hat{y}_1 + v_1$ in equations (15.22). We get

$$\sum \hat{y}_2(\hat{y}_1 - b_1\hat{y}_2 - c_1z_1) + \sum \hat{y}_2 v_1 = 0$$

$$\sum z_1(\hat{y}_1 - b_1\hat{y}_2 - c_1z_1) + \sum z_1 v_1 = 0$$

The last terms of these two equations are zero and the equations that remain are the normal equations from the OLS estimation of the equation

$$\hat{y}_1 = b_1\hat{y}_2 + c_1z_1 + w$$

Thus in stage 2 of the 2SLS method we can replace *all* the endogenous variables in the equation by their predicted values from the reduced forms and then estimate the equation by OLS.

What difference does it make? The answer is that the estimated standard errors from the second stage will be different because the dependent variable is $\hat{y}_1$ instead of y_1. However, the estimated standard errors from the second stage are the wrong ones anyway, as we will show presently. Thus it does not matter whether we replace the endogenous variables on the right-hand side or all the endogenous variables by $\hat{y}'s$ in the second stage of the 2SLS method.

Though the preceding discussion has been in terms of a simple model, but the arguments are general because all the $\hat{y}'s$ are uncorrelated with the reduced-form residual $\hat{v}'s$. Since our discussion has been based on replacing y by $\hat{y}+\hat{v}$, the arguments all go through for the general models.

15.4.1 Assumptions of Two Stage Least Squares

All the standard assumptions that we have studied so far which are applicable to the ILS and IV method are also applicable to this method. Here it is also assumed that the sample size is large enough, and in particular that the number of observations is greater than the number of predetermined variables in the structural system. If the sample size is small relative to the total number of exogenous variable then it may not be possible to obtain significant estimates in the first stage.

15.4.2 Properties of the Two Stage Least Squares Method

Provided that all the assumptions are satisfied the 2SLS estimates have the following properties.

1) Unlike ILS, which provides multiple estimates of parameters in the over-identified equations, 2SLS provides only one estimate per parameter.

2) It is easy to apply because all one needs to know is the total number of exogenous or predetermined variables in the system without knowing any other variable in the system.

3) Although specially designed to handle over-identified equations, the method can also be applied to exactly identified equations. In this case ILS and 2SLS will give identical estimates. (Why?)

4) If the R^2 values in the reduced-form regressions (that is, Stage 1 regressions) are very high, say, 0.8-0.9, the classical OLS estimates and 2SLS estimates will be very close. This result should not be surprising because if the R^2 value in the first stage is very high. It means that the estimated values of the endogenous variables are very close to their actual values, and hence the latter are less likely to be correlated with the stochastic disturbances in the original structural equations. If, however, the R^2 values in the first-stage regressions are very low, the 2SLS estimates will be practically meaningless because we shall be replacing the original Y's in the second-stage regression by the estimated $\hat{Y}'s$ from the first-stage regressions, which will essentially represent the disturbances in the first-stage regressions. In other words, in this case, the $\hat{Y}'s$ will be very poor proxies for the original Y's.

15.5 LIMITED-INFORMATION MAXIMUM LIKELIHOOD METHOD

The LIML method, also known as the least variance ratio (LVR) method is the first of the single-equation methods suggested for simultaneous equations models. It was suggested by Anderson and Rubin in 1949 and was popular until the advent of the 2SLS introduced by Theil in the late 1950s. The LIML method is computationally more cumbersome, but for the simple models we are considering, it is easy to use.

We can consider the first equation of the equation system given at (15.14), that is,

$$y_1^{*} = y_1 - b_1 y_2 = c_1 z_1 + c_2 z_2 + u_1 \qquad (15.23)$$

For each b_1 we can construct a y_1^{*}. Consider a regression of y_1^{*} on z_1 and z_2 only and compute the residual sum of squares (which will be a function of b_1). Let us call it ESS_1. Now consider a regression of y_1^{*} on all the exogenous variables z_1, z_2, z_3

and compute the residual sum of squares. Let us call it ESS_2. What equation (15.23) says is that z_3 is not important in determining y_1^*. Thus the extra reduction in ESS by adding z_3 should be minimal. The LIML or LVR method says that we should choose b_1 so that $(ESS_1 - ESS_2)/ESS_1$ or ESS_1/ESS_2 is minimized. After b_1 is determined, the estimates of c_1 and c_2 are obtained by regressing y_1^* on z_1 and z_2. The procedure is similar for the second equation in (15.14).

There are some important relationships between the LIML and 2SLS methods. (We will omit the proofs, which are beyond the scope of this book.)

1) The 2SLS method can be shown to minimize the *difference* $(ESS_1 - ESS_2)$, whereas the LIML minimizes the ratio (ESS_1/ESS_2).

2) If the equation under consideration is exactly identified, then 2SLS and LIML give identical estimates.

3) The LIML estimates are invariant to normalization.

4) The asymptotic variances and covariances of the LIML estimates are the same as those of the 2SLS estimates. However, the standard errors will differ because the error variance σ_u^2 is estimated from different estimates of the structural parameters.

5) In the computation of LIML estimates we use the variances and covariances among the endogenous variables as well. But the 2SLS estimates do not depend on this information. For instance, in the 2SLS estimation of the first equation in (15.14), we regress y_1 on $\hat{y}_2$, z_1, and z_2. Since $\hat{y}_2$ is a linear function of the z's we do not make any use of cov(y_1, y_2). This covariance is used only in the computation of $\hat{\sigma}_u^2$.

15.6 K-CLASS ESTIMATOR

The k-class estimator may be obtained by a generalization of 2-SLS method.

Assume that we have the model

$$y_1 = \beta_2 y_2 + \beta_3 y_3 + \cdots\cdots + \beta_G y_G + \gamma_1 x_1 + \cdots + \gamma_K x_K + u_1 \quad \text{...(15.24)}$$

If we apply OLS we use the original observations on the variables $y_2, y_3, \ldots, y_G$, and we obtain biased and inconsistent estimates.

To avoid this situation we may apply 2SLS, in which we use the estimated values of the endogenous variables applying OLS to the unrestricted reduced form equations.

The k-class estimator is in between these two procedures where from the actual observations of the endogenous variables the estimated non-systematic component is subtracted k times. So in limiting sense k-class estimators converges with the 2SLS when $k = 1$. And it converges with the OLS when $k = 0$.

The scalar k can be set a priori equal to some constant number, or its value can be determined from the observations of the sample according to some rule.

15.7 LET US SUM UP

In this unit we have discussed only single equation methods, that is, estimation of each equation at a time. We have discussed Indirect Least Squares Method, Instrumental Variable Method and 2-Stage Least Squares Method in detail. Along with that we briefly explained the Limited Information Maximum Likelihood Estimation Method and k-Class Estimation Method.

For an exactly identified equation, all the methods are equivalent and give the same results. For an over-identified equation ILS is not applicable and instrumental variable method gives different results depending upon which of the missing exogenous variables are chosen as instruments. The 2-SLS method is a weighted instrumental variable method.

In over-identified equations, the 2-SLS estimates depend on the normalization rule adopted. The LIML estimates do not depend on the normalization. The LIML method is thus truly a simultaneous estimation method, but the 2-SLS is not because strictly speaking since the exogenous variables are jointly determined thus normalization should not matter. The k -class estimator is on the other hand a weighted average of OLS and 2-SLS methods.

15.8 KEY WORDS

Indirect Least Squares : In this method we obtain the estimates of the reduced form coefficients by applying OLS and indirectly get the values of structural coefficients in terms of the reduced form coefficients.

Instrumental Variable Method: The instrumental variable method uses some appropriate exogenous variable as instrument to reduce the dependence between the error term and the explanatory variables.

2-Stage Least Squares : It is generally applicable to over-identified models and accepted as the most important of the single equation techniques for the estimation of the over-identified models.

Limited Information Maximum Likelihood Method: The LIML method, also known as least variance ratio (LVR) method is the first of the single-equation methods suggested for simultaneous equations models.

k-Class Estimator : The k -class estimator is in between these two procedures where from the actual observations of the endogenous variables the estimated non-systematic component is subtracted k times. So in limiting sense k -class estimators converges with the 2SLS when $k = 1$. And it converges with the OLS when $k = 0$.

15.9 SOME USEFUL BOOKS

Gujrati, Damodar N, 1995, *Basic Econometrics*, McGraw Hill, Singapore

Johnston, Jack and John Dinardo, 1997, *Econometric Methods*, The McGraw-Hill Companies Inc., Singapore.

Kmenta, Jan, 1997, *Elements of Econometrics*, University of Michigan Press.

Pindyck, Robert S. and Daniel L. Rubinfield, 1998, *Econometric Models and Economic Forecasts*, Irwin/McGraw Hill, Singapore.

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15.10 EXERCISES

1) Explain each of the following terms:

   a) Indirect least squares

   b) Instrumental variable method

   c) Two stage least squares

   d) Limited information maximum likelihood estimation

2) Examine whether the each of the following sentences are true or false or uncertain, and give a short explanation:

   a) If the R^2 in the 2-SLS method is very low and the R^2 from the OLS is high, it should be concluded that some thing is wrong regarding the specification of the model or the identification of the particular equation.

   b) The R^2 from OLS method will always be higher than R^2 from 2-SLS method, but it does not mean that OLS method is better.

   c) In an exactly identified equation the choice of which variable to normalize does not matter.

   d) In an exactly identified equation we can normalize the equation with respect to an exogenous variable as well.

   e) K-class estimator is a weighted average of OLS and 2-SLS.

   f) The instrument chosen in instrumental variable method should be perfectly correlated with the exogenous variables in the model.

   g) If an equation is exactly identified ILS and 2SLS give identical results.

   h) There is no such thing as R^2 in a simultaneous equation as a whole.

3) Why it is unnecessary to apply 2 Stage Least Squares in an exactly identified equation?

4) Consider the following modified Keyenesian model of income determination:

$$C_t = \beta_{10} + \beta_{11}Y_t + u_{1t}$$
$$I_t = \beta_{20} + \beta_{21}Y_t + \beta_{22}Y_{t-1} + u_{2t}$$
$$Y_t = C_t + I_t + G_t$$

where C, I, G, Y denote their usual meaning and G_t and Y_{t-1} are assumed to be predetermined.

   a) Obtain the reduced-form equations and determine whether the preceding equations are identified and what are the status of their identification?

   b) Which of the methods will you use to estimate the parameters of the over identified equation and of the exactly identified equation?