UNIT 5 VALUING RISK MANAGEMENT

Structure

5.0 OBJECTIVES

After going through this unit, you will be able to:

- appreciate the methods used for valuation of insurance claims:
- see the interplay between finance and insurance in risk valuation; and
- understand the methods of valuing unit-linked life insurance contracts through risk-minimisation

5.1 INTRODUCTION

The actuarial policy combines elements of insurance risk and financial risk to formulate its risk management. To appreciate the underlying idea, you may think of schemes such as unit-linked life insurance contracts, catastrophe insurance futures and bonds, and integrated risk-management solutions. Therefore, there is a need to understand the developments involving methods for valuation of risk. Particularly, we have to appreciate the mechanism through which the risk in a portfolio of unit-linked life insurance contracts can be analysed.

Remember that the theory of insurance was formulated mainly to compute the premiums for life insurance contracts. It was suggested that the methods of evaluation of life insurance contracts could be carried on combining interest and mortality. Daniel Bernoulli pointed out that risks should not be measured by their expectations. He made use of gambling to argue that the preferences of an individual might depend on her economic situation. Specifically, it could be quite reasonable for a poorer individual to prefer an uncertain future payment to another more uncertain payment with a large expected value. On the other hand, a wealthier person would prefer the payment with the largest expected value. Recall that we have already seen this argument while studying the expected utility theory in MEC-01. We have also studied there that individuals could to buy insurance contracts at a price, which might exceed the expected value of the payments from the contract.

In order to draw insight from the similar decision making scenario, we may look at the stock markets. Prices there undergo frequent fluctuations. Such a feature can be described as a process akin to Brownian motion.

Samuelson (1965) developed a framework where the stock price was modelled by a geometric Brownian motion i.e., the exponential function of a Brownian motion. It had the advantage of not generating negative prices. Black and Scholes (1973) and Merton (1973) extended this framework with the additional assumption that money could be deposited on a savings account. Their results show that options on stocks should be priced such that no sure profits could arise from composing portfolios of long and short positions in the underlying stock and in the option itself. Cox, Ross and Rubinstein (1979) showed that the change in the value of the stock between two trading times could attain two different values. In that setting, they derived option prices and obtained the pricing formulae of Black, Scholes and Merton as limiting cases by letting the length of the time intervals between trading times tend to 0. Building on concepts and ideas in Harrison and Kreps (1979) for discrete time models, Harrison and Pliska (1981) gave a mathematical theory for pricing of options under continuous trading and clarified the role of martingale theory in the pricing of options and its connection to key concepts such as absence of arbitrage and completeness.

The net result of insights thrown by pricing in insurances and finance market in the presence of risk is the schemes like catastrophe futures and options and financial stop-loss reinsurance contracts. Thus, the recent methodology of pricing principles in many insurance products in based on the theories behind risk and combined approach of insurance and finance.

As you will see we have followed Møller (2001) in developing the present unit. For an easy comprehension same notation as well as sequence of the above work is retained. In the process we have borrowed freely from it. Keeping in view the explorative nature of the theme required by the present course we have discussed the basic ideas contained in Møller (2001). However, for reference it will be useful to go through the entire work.

To appreciate the tools used for analysing the insurance valuation you should have understood the stochastic calculus and financial mathematics we have given in units 3 and 4.

5.2 CLASSICAL VALUATION OF INSURANCE CONTRACTS

Following the tradition of actuarial theory let us consider life insurance and non-life insurance separately. Note that there are fundamental differences between the two areas. For example, in respect of time horizon, the individual contract for life insurance can extend up to 50 years, whereas for non-life insurance, it is typically limited to one year. Accordingly, the calculation of premiums is undertaken differently. We review some notions and key concepts of life and non-life insurance, placing focus on the valuation techniques used.

5.2.1 Life Insurance

We start with some basic concepts of life insurance. Consider a portfolio of n lives aged say, y, to be insured at time 0 with i.i.d. remaining life times $T_1,...,T_n$. Let us assume that there exists a continuous function (called the hazard rate function) μ_{y+t} such that the survival probability is of the form

$$_t p_y = p(T_i > t) = \exp\left(-\int_0^t \mu_y + u^{du}\right).$$

A pure endowment contract with sum insured K and term T stipulates that the amount K, the insurance benefit, is payable at time T contingent on survival of the policy-holder. Assume that the contract is paid by a single premium, say, at time 0. Assume further that the seller of the contract (the insurance company) invests the premium in some asset, which pays a rate of return $r = (r_t)_{0 \le_t \ge T}$ during $[0,T]$. For the i^{th} policyholder, the obligation of the insurance company is given by the present value

$$\mathrm{H}_i = 1_{\{T_i > T\}} K e^{\int_0^T r_t dt} \tag{2.1}$$

which is obtained by discounting the amount payable at T, $1_{\{T_i > T\}}K$, using a rate of return r. Equation (2.1) is a random variable. The fundamental principle of equivalence states that the premiums should be chosen such that the present values of premiums and benefits balance on average. If we assume in addition that r is stochastically independent of the remaining life times, the principle of equivalence reduces to

$$k = E[H_i] = {}_T p_y KE\left[e^{-\int_0^T r_t dt}\right], \tag{2.2}$$

in case of the single premium. This principle is justified due to the law of large numbers. As the size of the portfolio n is increased, the relative number of survivors $\frac{1}{n}\sum_{i-1}^{n} = 1_{\{T_n > T\}}$ converges almost surely (a.s.) towards the probability ${}_T p_y$ of survival to T by the strong law of large numbers since the lifetimes $T_1, \ldots, T_n$ are stochastically independent. Thus, for n sufficiently large, the actual number of survivors $\frac{1}{n}\sum_{i=1}^{n} {}_{\{T_i > T\}}$ will be approximately equal to the expected number, $n_T p_y$.

When the amount to be paid to the policyholders is based on a deterministic rate of return, the principle of equivalence is justified directly by use of the law of large numbers, which guarantees that the actual number of survivors is close to the expected number.

However, it will be better to consider a more realistic situation by taking the rate of return (r) as a stochastic process. Then the simple accumulating premium(nk) will not generate the amount to be paid to policyholders since $\bar{e}^{\int_0^T rt^{dt}}$ will differ considerably from its expected value. To deal with this problem it is necessary to replace the "true" rate of return process r in (2.2) with some deterministic rate of return process r which is such that the single premium n accumulated by the true rate of return r is larger than K times the expected number of survivors with a large probability. The excess (if any) should then be added to the amount paid to the policy-holder However, this approach raises the problem of existence of any deterministic and strictly positive r in a very long time horizon. It has the property that it will be larger than the actual return on investments with a very large probability. An alternative to this approach is therefore to replace r by short rate of interest and then replace the last term in (2.2) by the price on the financial market of a financial asset which pays one unit at time T, a zero coupon bond.

5.2.2 Non-life Insurance

In the non-life insurance, for premium calculation takes a relatively short time horizon. For this reason discounting plays a less significant role there. Thus, if H denotes the claim payable at a time (T), then a premium $\tilde{u}$(H) charged is equal to the expected value E [H] of the claim and we write

$$\tilde{u}(H) = E[H] + A(H).$$

The most important examples of such premium calculation principles are:

- net premium principle or the principle of equivalence $A(H) = 0$;
- expected value principle $A(H) = aE[H]$,
- standard deviation principle $A(H) = a(Var[H])^{1/2}$;
- variance principle $A(H) = aVar[H]$; and
- semi-variance principle $A(H) = aE\left[\left((H - E[H])^{+}\right)^{2}\right]$.

In addition to these, there are some other principles, which we list below:

Fair Premium Principle

Under fair premium principle the calculation aims at a zero increase in expected utility. If u is the utility function with $u'(x) \geq 0$ and $u''(x) \leq 0$ for any $x \in \mathbb{R}$, and if V_0 denotes the insurer's initial capital at time 0, then

$$E\left[u\left(V_0 + u(H) - H\right)\right] = E\left[_{u}(V_0)\right] \tag{2.5}$$

That is, the expected utility of the final wealth $V_0 + \tilde{u}(H) - H$ from selling the claim H at the premium $\tilde{u}(H)$ should equal the expected utility of V_0. Here V_0 can be interpreted as the wealth associated with not selling the claim H and therefore, selling the claim leaves the expected utility unaffected, i.e., it leads neither to an increase nor a decrease in expected utility.

Exponential Principle

In practice use is made of the exponential principle, which is obtained for the exponential utility function

$$u(x) = \frac{1}{a}\left(1 - e^{-ax}\right).$$

In particular, when V_0 is constant P-a.s., the solution to (2.5) does not depend on V_0 and is given by

$$\tilde{u}(H) = \frac{1}{a}\log\left(E\left[e^{aH}\right]\right).$$

Quadratic Utility Function Based Principle

A frequently used utility function is the quadratic utility function which is defined by

$$u(x) = x - \frac{x^2}{2s}, x \leq s, \text{and } u(x) = \tfrac{s}{2} \text{ for x > s.}$$

Esscher Principle

The Esscher principle is based on an exponential scaling of the claim (H) such that

$$\bar{u}(H)=\frac{E\left[He^{aH}\right]}{E\left[e^{aH}\right]} \tag{2.6}$$

Percentile (Generalised) Principle

The premium calculation principles are sometimes based on the generalisations of the **maximal loss principle.** For $\varepsilon\in(0,1)$ and $p\in[0,1]$, the (generalised) percentile principle states that the premium should be computed as

$$\tilde{u}(H)=pE[H]+(1-p)F^{-1}(1-\varepsilon),$$

where F is the distribution function of H and F^{-1} is its generalised inverse, i.e., $F^{-1}(y)=\inf\{xF(x)\geq y\}$. Thus the premium is a weighted average of the expected value of H and the $(1-\varepsilon)$-percentile of the distribution of H. The maximal loss principle is obtained when $\varepsilon=0$ and $p=0$.

Check Your Progress 1

1) What is fundamental principle of equivalence in insurance?

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2) How would you formulate the fair premium principle?

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3) Write the equations used for exponential principles and Esscher principle.

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5.3 FINANCIAL VALUATION PRINCIPLES

While developing this section we follow the formulation of Møller (2001). Let us discuss some basic concepts of finance theory with a view to see their implementation in actuarial science. Take a market with two traded assets: a riskless asset B, some bond (or money market account) and a risky asset (a stock). Also, take T a fixed finite time horizon and the price processes are $S=(S_t)_{0\le t\le T}$ and $B=(B_t)_{0\le t\le T}$ with the underlying probability space defined on some $(\Omega,\mathcal{F},P)$. Again introduce the discounted price processes $X=S/B$ and $X^0=B/B=1$. To complete the description of the above formulation, let us specify the information availability at a particular point of time. The agents have access to information contained in $\mathcal{F}_t$ at that point of time t. We also assume that the stock price is adapted and follows a specific sample path. In such a setting, we try to see the payoff of a contingent claim from a dynamic trading strategy. Consider a 2-dimensional process $\varphi_t=(\vartheta_t\eta)_{0<t<T}$ satisfying certain integrability conditions, where ϑ is predictable and η is adapted with respect to some filtration $\mathbb{F}=(\mathcal{F}_t)_{0\le t\le T}$ which describes the evolution of available information up to time t. The variables in $\varphi_t=(\vartheta_t,\eta_t)$ are the portfolio held at time t. That is, ϑ_t is the number of shares of the stock held at t and η_t is the discounted amount invested in the savings account. The discounted value at time t of φ_t is given by $V_t(\varphi)=\vartheta_t+\eta_t$. The strategy φ is self-financing if

$$V_t(\varphi=V_0(\varphi)+\int_0^t \vartheta_s dX_s \tag{3.1}$$

Let us consider the terms in (3.1). $V_0(\varphi)$ is the amount invested at time 0 and $\int_0^t \vartheta_s dX_s$ is the accumulated trading gains generated up to time t. Thus, for a self-financing strategy φ, the current value of the portfolio φ_t at time t is exactly the initially invested amount plus trading gains, so that no in- or out-flow of capital which has taken place during $[0,t]$. Let H be interpreted as payoff for some financial contract and investor such as an investment bank who considers selling a contingent claim, i.e., F_T measurable random variable H_0. Then a contract (or claim) specifying the discounted payoff H at time T is said to be attainable if there exists a self-financing strategy φ such that $V_T(\varphi) = H$ a.s., that is, if H coincides with the terminal value of a self-financing strategy. Thus, a claim is attainable if and only if it can be represented as a constant H_0 plus a stochastic integral for the discounted stock price process

$$H = H_0 + \int_0^T \vartheta_s^H dX_s . \tag{3.2}$$

We need the initial investment $V_0(\varphi) = H_0$ for this perfect replication of H, which is called the unique no-arbitrage price of H. To see that any other price will lead to an arbitrage possibility, i.e., to a risk-free gain, let us suppose that the price of H at time 0 is given by $H_0 + \varepsilon$, where $\varepsilon > 0$. Then risk-free gain of can be generated as noted by Møller (2001) in the following way:

- Sell the claim H at time 0 and receive $H_0 + \varepsilon$. Thus, at time T we have to pay H to the buyer of the claim.
- Construct a dynamic hedging strategy by investing H_0 through the self-financing strategy $\varphi = (\vartheta, \eta)$ by taking $\vartheta_t = \vartheta_t^H$ and by choosing η_t such that (3.1) is satisfied, i.e.,

  $$\eta_t = V_0(\varphi) + \int_0^t \vartheta_s dX_s - \vartheta_t X_t .$$

- The value of the portfolio φ_T at time T is now exactly equal to H. From (3.1) and (3.2), we get this condition where the payment is made to the buyer of the contract.

The net result of all these steps is to find the estimation of gain ε from the trade. If the price of H is $H_0 + \varepsilon$, with $\varepsilon < 0$, the gain ε can be obtained by buying H and using the hedging strategy φ. We have shown how the amount H_0 received at time 0 can be transformed into the amount H at time T by following a self-financing strategy. There is no further payment and hence no more risk. This implies, H_0 is the fair price.

5.3.1 Complete Financial Market

A financial market is said to be complete if all claims are attainable. There are two models dealing with continuous and discrete time, which can be seen in this context.

Continuous Time Trading: The Black-Scholes model with continuous trading is an example of the complete market. It consists of two assets, a stock whose price process is described by a geometric Brownian motion and a savings account, which pays a deterministic and constant rate of return.

Discrete Time Trading: We have the Cox-Ross-Rubinstein (1979) model, which is called the binomial model.

In complete market we have no arbitrage possibilities and there is a unique *risk-neutral measure.* A risk-neutral measure is a probability measure Q which is equivalent to P. The existence of an equivalent martingale measure is equivalent to the absence of arbitrage opportunities. We have, therefore, the economic justification for the 'First Fundamental Theorem of Asset Pricing', given by Harrision and Kreps (1979).

5.3.2 Incomplete Financial Market

There are some claims, which are not attainable. In other words, these do not allow a representation of the form (3.2) and hence cannot be replicated by means of any self-financing trading strategy. Then the market is said to be incomplete. In this case there are infinitely many risk-neutral measures. In the discrete time case, incompleteness comes in if we replace the binomial model with a trinomial model, i.e., a model where the change in the value of the stock between two trading times can attain three different values. Under continuous trading, the incompleteness is present if we have a Poisson-driven jump component to the geometric Brownian motion. In the following we list some different approaches to pricing and hedging in incomplete markets.

5.3.3 Pricing in Incomplete Market

5.3.3.1 Super-replication

In super-replication we try to find the cheapest self-financing strategy where the terminal value is no smaller than the payoff of the contingent claim. That is, for a given contingent claim H, we try to find the smallest number, say, V_0^* such that there exists a self-financing strategy

$$\tilde{\varphi} \qquad \text{with } V_0\left(\tilde{\varphi}\right) = V^*_0, \text{ and}$$

$$V_T\left(\bar{\varphi}\right) \geq H, P-a.s.$$

When the price V_0^* is charged and the strategy $\tilde{\varphi}$ is adopted, the hedger can generate an amount, which exceeds the needed amount H, P-a.s. Consequently a major advantage of this approach is to have no risk to the hedger. That is, there is no need of additional capital to pay the amount H to the buyer of the contract after making the initial investment.

5.3.4 Marginal Utility Approach

Fair prices can also be derived invoking utility functions that describe the preferences of the buyers and sellers. A marginal utility approach defines the fair price of a claim H such that one makes investors indifferent between investing an extra unit of their funds in the contract and not investing in it. In particular, let u be a utility function, c the investor's initial capital at time 0, p the price charged at time 0 per unit of some claim H and z the amount invested in H. Then,

$$W(z,p,c) = W(z,p,c) = \sup_{\vartheta} E\left[u\left(c - z + \int_0^T \vartheta_u dX_u + \tfrac{z}{p} H\right)\right],$$

where the supremum is taken over all strategies ϑ from some suitable space of processes. The number $W(z,p,c)$ is the maximum obtainable expected utility for an investor with initial capital c when she makes an investment in z/p units of the risk H. The fair price $\tilde{u}(H;c)$ of H is then defined as the solution $\tilde{p}$ to the equation

$$\frac{\partial}{\partial z} W(0, p, c = 0),$$

with the condition that the relevant quantities exist.

5.3.5 Quadratic Approaches

Quadratic methods are applied to pricing and hedging in incomplete markets with an emphasis on risk minimization. In the process of application we divide these methods into (i) (local) risk-minimization for X (which is a discount process and is a martingale) and (ii) mean-variance hedging. In case of the first approach, X martingale can be generalised to semi martingales. In case of the second, an attempt is made to approximate the claim H as closely as possible by the terminal value of a self-financing strategy using a quadratic criterion. Thus, we try to find a self-financing strategy $\varphi^* = (\vartheta^*, \eta^*)$ which minimises

$$E\left[\left(H - V_T(\varphi)\right)^2\right] = \| H - V_T(\varphi) \|^2_{L^2(p)} \tag{3.3}$$

This strategy is completely determined by the pair $\left(V_0(\varphi^*), \vartheta^*\right)$, so that the solution to the problem of minimizing (3.3) is obtained by projecting the random variable H in $L^2(p)$. Note that the optimal initial capital $V_0(\varphi^*)$ is called the approximation price for H, and the optimal strategy is the mean-variance hedging strategy.

Quantile hedging

An important character of the quadratic approach is that it punishes losses and gains equally. As an alternative to it we use quantile hedging and the objective is to hedge the claim with a certain probability. Another alternative is the criterion of minimising the expected shortfall risk, i.e., expected losses from hedging. In this method a loss function $l : [0,\infty 0 \to [0,\infty)$ is introduced, which is taken to be an increasing convex function with $l(0=0)$, and we consider the problem of minimising

$$E\left[l\left(\left(H - V_T(\varphi)\right)^+\right)\right], \tag{3.5}$$

over the self-financing hedging strategies. Usually, loss functions are power functions, $l(x) = x^p, p \geq 1$, and (3.5) is related to minimising the lower partial moments in such a situation.

Check Your Progress 2

1) Explain the meaning of self-financing principle.

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2) What do you mean by complete financial market?

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3) List the pricing principles in incomplete market.

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5.4 PRICING: INSURANCE AND FINANCE

We consider some specific areas of the interplay between finance and insurance in the following discussion. Before doing that it may be useful to remember that traditionally finance and insurance have a common tool of analysis in stochastic processes only. However, increasing collaboration between insurance companies and banks representing all types of finance in recent years has made it possible for an interplay between these two fields.

5.4.1 Unit-linked Insurance Contracts

A unit-linked life insurance contract differs from the traditional ones. In this product, benefits depend on the development of some stock index or the value of some specified portfolio. Such a feature allows for greater flexibility as compared to the traditional life insurance products as the policyholder is offered the opportunity of deciding how much of her premiums are to be invested. We discuss a little more on its features.

Let S_t be the value of the stock index at time t. Following the notations used in Møller (2001), we refer to the entire development of the stock as S. Consider a portfolio consisting of n policyholders with remaining life times $T_1 ... T_1$. To simplify the analysis assume that they all buy the same form of unit-linked pure endowment contract at time 0. Let the life times be stochastically independent of the development on the financial market. The contracts specify the payment of some (non-negative) amount $f(S)$ to the policyholder at time T if she is still alive at this time; the function f describes nature of the dependence on the development of the stock price. Under such a set up, the present value at time 0 of the insurance company's liability towards the n policyholders is given as

$$H = \sum_{i=1}^{n} 1_{\{T_i > T\}} f(S) e^{-\int_0^T r_u du}, \tag{4.1}$$

where the payment is discounted by the short rate of interest r. It is possible that the amount paid could be a function of the terminal value of the stock only. In that case we have

$$f(S) = S_T. \tag{4.2}$$

Another formulation could be to set the terminal value guaranteed against falling short of some prefixed amount K. Then we have

$$f(S) = \max(S_T, K) \tag{4.3}$$

From (4.2) and (4.3) above we get the contracts which are called pure unit-linked contracts and (4.3) is called unit-linked with guarantee, the guaranteed benefit being K. We may extend f to be a more complex function of the process S. For example, we may have a guaranteed annual return

$$f(S) = K.\prod_{j=1}^{T} \max\left(1 + \frac{S_j - S_{j-1}}{S_{j-1}}, 1 + \delta_j\right).$$

In such a case $\frac{S_j - S_{j-1}}{S_{j-1}}$ gives the return in year j on the asset S and δ_j is the guaranteed return in year j. The amount payable at time T is guaranteed at the time of taking the contract (i.e., at time 0) against falling short of $K.\prod_1^T (1 + \delta_j)$. However, the guarantee goes beyond this in worst-case scenario.

Thus we see that the modern theories of finance offer new valuation principles and investment strategies for unit-linked insurance contracts. It combines the traditional (law of large numbers) arguments from life insurance with the methods of Black and Scholes (1973) and Merton (1973). In the process we could (i) replace the uncertainty of the insured lives by their expected values, (ii) include the actual insurance claims by taking into account mortality risk as well as financial risk. Such modified claims have contained financial uncertainty only. Thus, instead of considering the claim (4.1), this method examines

$$H = \eta_T p_y f(S) e^{-\int_0^T r_u du} \tag{4.4}$$

where the notation ${}_T p_y = P(T_1 > T)$. Such modified claims can be treated as options, with a long maturity period. As a result, these could be priced and hedged using the basic principles of Black and Scholes (1973) and Merton (1973).

Let us look at the insurance related results of unit-linked contracts:

In case of pure unit-linked contract (4.2), the claim (4.4) is proportional to the terminal value of the stock S_T and hence can be hedged by a buy-and-hold strategy which consists in buying $n_T p_y$ units of the stock at time 0 and holding these until time T.

Case of no guarantee

When there is no guarantee, the unique no-arbitrage price of H' is simply $n_T p_y s_0$. Thus, you can prescribe one possible fair premium for each policyholder is ${}_T p_y s_0$ with the probability of survival to T times the value at time 0 of the stock index.

Case of contract with benefit

Write $f(S) = \max(S_T, K) = (S_T - K)^+ + K$ and consider the pricing of a European call option. Then from (4.4) it follows that the premium is $n_T p_y V_0^f$, where V_0^f is the price at time 0 of the purely financial contract which pays $f(S)$ at time T

5.4.2 Other Insurance Derivatives

There are products known as insurance derivatives which combine financial derivatives and insurance products. These have the features of traditional insurance risk and financial derivatives. For example, consider reinsurance valuation such as catastrophe futures, catastrophe-linked bonds, financial stop-loss contracts and stop-loss contracts with a barrier.

5.4.2.1 Catastrophe Insurance (CAT) Futures

Occurrence of several catastrophes of large magnitude in 1980s and 1990s impaired the capacity of reinsurers offering traditional catastrophe covers. As a result, there was a move to increase the reinsurance premiums. In 1992 the catastrophe insurance (CAT) futures and options on CAT futures were introduced. These instruments standardised catastrophe insurance risk and transformed these into tradeable securities, providing a new tool for insurers seeking cover against catastrophe risk.

5.4.2.2 Catastrophe-Linked Bonds

Individual insurance companies can choose to securitise part of their insurance risk directly. For example, they may issue bonds that are linked to insurance losses from certain insurance portfolios. One example of such an arrangement is the Winterthur Insurance Convertible Bond, also called WinCAT bond (see Møller (2001) for greater discussion). This bond was introduced by Winterthur in 1997. Under this scheme, investors receive annual coupons as long as certain catastrophic events related to one of Winterthur's own insurance portfolios have not occurred. Thus, the investors receive a return from the bond which exceeds the market interest rate as long as no catastrophe has occurred and a lower return in the case of a catastrophe. The difference between the return under no catastrophe and the interest rate on the market was essentially a premium that Winterthur paid investors for putting their money at risk. Similarly, the low return in connection with a catastrophic event implied that the investors had covered part of Winterthur's losses.

5.4.2.3 Financial Stop-loss Contracts

We have seen that CAT futures and Catastrophe-linked bonds are aimed at a larger group of investors. The reinsurance contracts in these combine elements of insurance and financial derivatives. There are some new contracts described by 'Integrated Risk Management Solutions'. One example init is financial stop-loss contract. It pays at some fixed time T the amount

$$H = \left(U_T + Y_T - K\right)^+ \tag{4.5}$$

where U_T is the aggregate claim amount during [0, T] on some insurance portfolio, Y_T is some financial loss and K is some retention limit. The financial stop-loss contracts cover large losses due to fluctuations within the insurance portfolio (insurance risk) and adverse development of the financial markets (financial risk). Reinsurance companies would be able to sell spreads on the form

$$\left(U_T + Y_T - K_1\right)^+ - \left(U_T + Y_T - K_2\right)^+,$$

where $0 \le K_1 \le K_2$, which covers the $(K_1, K_2]$ layer of the losses $U_T + Y_T$.

The insurance contract (4.5) helps provide cover for the insurer's total risk, i.e., the combined insurance risk from the insurance portfolio and the financial risk from the financial portfolio. With a traditional stop-loss contract, the reinsurer would cover insurance losses exceeding the level K. However, the financial stop-loss contract is designed so that the cover is only paid provided that the insurance loss augmented by the financial loss exceeds this level. Thus, a large financial gain $-\ Y_T$ may compensate for large insurance losses, and in this situation, the buyer does not really need additional compensation from the reinsurer.

5.4.3 Combining Theories for Financial and Actuarial Valuation

Consider the basic difference between financial valuation techniques, (more precisely, pricing by no-arbitrage) and the classical actuarial valuation principles reviewed above. We have seen that the financial valuation principles are formulated within a framework which includes the possibility of trading certain assets. In contrast to these the classical actuarial valuation principles are based on considerations involving the law of large numbers. In addition, the financial valuation principles are based on dynamic trading, whereas many decision problems in insurance are traditionally analysed taking a static view.

Dynamic Reinsurance Markets

The dynamic reinsurance markets in a continuous time framework using no-arbitrage conditions can be analysed. This is possible as some of the processes related to an insurance risk process are tradeable.

From Actuarial to Financial Valuation Principles

Gerber and Shiu (1996) among others have examined the situation where the logarithm of the stock price process is a Levy process, i.e., a process with independent and stationary increments. Within this setting, they demonstrate how the Esscher transform (2.6) can be used in the pricing of options. They also give a very simple option pricing formula which involves Esscher transforms and which, for a European call option, specialises to the Black-Scholes formula in the case of a geometric Brownian motion. Moreover, they also demonstrate how this pricing formula can be derived through a simple utility indifference argument in the case of power utility function $u(x) = \frac{x^{1-a}}{1-a}$ with parameter $a > 0$.

In Schweizer (2001), the starting points are the traditional standard deviation and variance principles, which are of the form (2.4). These principles are taken as measures of riskiness, which assign to each claim a premium. It is then argued that the measures can equivalently be viewed as measures of preferences which operate on the insurer's terminal wealth by simply changing the sign on the loading factor.

Check Your Progress 3

1) Which technique of finance will you use for insurance pricing?

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2) How the modern theories of finance offer new valuation principles for unit-linked insurance?

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3) Which insurance derivations combine financial derivatives and insurance products?

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4) Which actuarial concepts are relevant to be used for solving financial markets problems?

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5.5 LET US SUM UP

In this unit we have discussed the methods used for hedging and valuing insurance claim with financial risk. An attempt is made to focus on pricing rules that make an investor indifferent between investment in insurance and finance. In the process we have addressed the important themes that bring out the interdependency between insurance and finance. We have dealt with the application of financial pricing techniques to insurance problems. Following the similar logic, the actuarial concepts that are reference to finance problems are pointed out.

5.6 KEY WORDS

Actuarial Equivalent: An alternative form of a benefit equal in value. For example, a Lump Sum payout of a benefit rather than smaller monthly payments.

Actuarial Valuation: In an actuary of a pension or a benefit plan trust, this is the total amount needed to meet promised benefits. This set of mathematical procedures typically calculates the value of benefits to be paid, the funds available, the annual contribution required, and the amount of expense that an organization can take on for accounting and tax purposes.

Complete market : A market is complete with respect to a trading strategy if there exists a self-financing trading strategy such that at any time t, the returns of the two strategies are equal. That is, all cash flows for a trading strategy can be replicated by a similar synthetic trading strategy. Because a trading strategy can be simplified into a set of simple contingent claims (a trading strategy that pays 1 in one state and zero in every other state), a complete market can be generalized as the ability to replicate cash flows of all simple contingent claims.

Contingent Claims: Assets or securities whose prices depend on the values of other assets or numerical indices.

Financial Market : A mechanism which allows people to trade money for securities.

Financial Risk : Any risk associated with money.

Hazard Function : By calculating the failure rate for smaller and smaller intervals of time Δt, the interval becomes infinitesimally small. This results in the **hazard function**, which is the *instantaneous* failure rate at any point in time:

$$h(t) = \lim_{\Delta t \to 0} \frac{R(t) - R(t + \Delta t)}{\Delta t \cdot R(t)}$$

Failure Distribution: Continuous failure rate depends on a **failure distribution**, $F(t)$, which is a cumulative distribution function that describes the probability of failure prior to time t,

$$P(t \le t) = F(t) = 1 - R(t), t \ge 0.$$

Failure Distribution Function: The failure distribution function is the integral of the failure density function ,$f(x)$,

$$F(t) = \int_0^t f(x)dx.$$

The hazard function can be defined as

$$h(t) = \frac{f(t)}{R(t)}$$

Degenerate Function: A variable taking a certain value so as to be considered nonrandom. But it satisfies the definition of random vaiable. This is useful because it puts deterministic variables and random variables in the same format.

Hedge : An investment made to reduce the risk in another investment.

Hedging: A strategy designed to minimise exposure to an unwanted business risk, while still allowing the business to profit from an investment activity.

Law of Large Numbers: An outcome with repeated, independent trials with the same probability of success in each trial where the percentage of successes is increasingly likely to be close to the chance of success as the number of trials increases.

Self-Financing: A strategy when a company is able to release liquidity in order to finance operating charges and development. It is equal to the net profit to which you reintegrate all the entries that are not associated with the movements of cash such as depreciation and provisions, appreciation and depreciation of asset disposals.

Unit-linked Insurance: The premium is invested in units of a unitised insurance fund. Units are encashed to cover the cost of the life assurance.

Market Risk - the potential to experience financial losses due to fluctuations in the prices at which equities, foreign currencies, interest rate linked securities, and commodities can be bought or sold.

5.7 SOME USEFUL BOOKS AND JOURNALS

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5.8 ANSWER OR HINTS TO CHECK YOUR PROGRESS

Check Your Progress 1

1) See Sub-section 5.2.1 and answer.

2) See Sub-section 5.2.2 and answer.

3) See Sub-section 5.2.2 and answer.

Check Your Progress 2

1) See Section 5.3 and answer.

2) See Sub-section 5.3.1 and answer.

3) See Sub-section 5.3.3 and answer.

Check Your Progress 3

1) See Section 5.4 and answer.

2) See Sub-section 5.4.1 and answer.

3) See Sub-section 5.4.2 and answer.

4) See Sub-section 5.4.3 and answer.

UNIT 6 STOCHASTIC MODELS IN INSURANCE

Structure

6.0 OBJECTIVES

After going through this unit you will be able to:

- understand the use of basic probability based models in insurance that take into account the time factor; and
- evaluate insurance claims and premiums using stochastic techniques.

6.1 INTRODUCTION

Insurance operates from the platform of probability theory. Therefore, it is necessary to understand the formulation of claim and premium processes in the framework of probability analysis. Particularly, probability distributions of these variables have to be studied.

6.2 INSURANCE PROCESS

In order to model an insurer's portfolio of risk over time we consider variables such as claim number, claim size and premium rate. We will see below that the nature of these variables differs. For example, in non-life insurance not only the claims arrival times are random but also the claim severities are random. Needless to point out that these are different in case of life insurance. Moreover, the task of modelling risk involves various stochastic processes. Consequently, we have at hand a difficult job.

To consider a portfolio for risk evaluation, we often take into account the sequence of claim arrival times and the sequence of claim severities, the claim arrival process and the claim size process. To develop the analytical framework it is often required to assume that variables considered under a portfolio have independent identical distribution (iid).

6.3 SURVIVAL FUNCTION

Let us see the preliminary idea of survival function given in the Wakipedia.

Survival analysis in insurance is used to deal with death of a life. It attempts to answer questions such as: what is the fraction of a population which will survive past a certain time? If it survives, at what rate will it die? Can multiple causes of death or failure be taken into account?

Take survival function as a property of any random variable that maps a set of events associated with mortality onto time. Through it we get the probability that the life will survive beyond a specified time. To see the functional form consider the following (as given in the above source):

Let X be a continuous random variable describing life time. It has a cumulative distribution $F(t)$ on the interval $[0,\infty)$. Then the survival function is given as

$$R(t) = P\{T \geq t\} = \int_t^{\infty} f(u)du = 1 - F(t)$$

The survival function $R(t)$ is monotone decreasing, i.e.,

$$R(u) < R(t) \text{ for } u > t$$

The time, $t = 0$ represents starting of the insurace. We will assume that $\lim_{t \to \infty} R(t)$ is zero.

Take a failure rate $\lambda(t)$ as the probability that a failure occurs in a specified interval, given no failure before time t. It can be defined with the aid of the survival function $R(t)$, the probability of no failure before time t, as:

$$\lambda = \frac{R(t_1) - R(t_2)}{(t_2 - t_1).R(t_1)} = \frac{R(t_1) - R(t_1 + \Delta t)}{\Delta t.R(t_1)}$$

where t_1 (or t) and t_2 are respectively the beginning and ending of a specified interval of time spanning Δt.

In the above equation as $\lim_{\Delta t \to 0} \lambda(t) \to 0$. Thus the failure rate becomes infinitesimally small. Then we have the hazard function, which gives the *instantaneous* failure rate at any point in time:

$$h(t) = \lim_{\Delta t \to 0} \frac{R(t) - R(t + \Delta t)}{\Delta t . R(t)}$$

Continuous failure rate depends on a failure distribution, $F(t)$, which is a cumulative distribution function that describes the probability of failure prior to time t,

$$P(t \le t) = F(t) = 1 - R(t), \quad t \ge 0$$

The failure distribution function is the integral of the failure density, $f(x)$,

$$F(t) = \int_0^t f(x) dx$$

Then the hazard function can be defined as

$$h(t) = \frac{f(t)}{R(t)}$$

A common failure distribution is the exponential failure distribution,

$$F(t) = \int_0^t \lambda e^{-\lambda x} dx = 1 - e^{-\lambda t},$$

which is based on the exponential density function. Consequently, we get a constant hazard rate.

6.4 LIFE INSURANCE CONTRACTS

In life insurance the company has to pay predetermined amount of insurance claims. Such claims are the sums insured by policyholders. The amount of claim is defined in the contract to cover the events such as death, survival, disability etc. Claims could occur either in policy period or after the expiry date of the insurance. Since claims are payment by the company it depends premiums charged from them to meet the costs. The premiums are determined through *equivalent principle* at the time of issuing the policy. Essentially, the principle strikes the balance between the expected value of discounted future benefits payable by the insurer plus the administrative costs with that of the expected value of discounted future premiums payable by the policyholder. The factors taken into account for premium calculation are interest rate, probability distributions and cost shares. In the following discussion, we will ignore the costs for simplicity of analysis.

We will use the following notations:

r	interest rate,
$1 + r$	interest factor,
$v = \frac{1}{1 + r}$	discount factor,

n integer duration of years of a contract,

$\tilde{L}_x$ the residual life time of a person aged x,

L_x the number of completed years of a person aged x, i.e., $L_x = \left[\tilde{L}_x\right]$ the integer part of $\tilde{L}_x$.

An annuity is a sequence of payments of limited duration n (we will discuss details in Unit 7). The present value of an annuity with n annual payments of 1 starting at time 0 is

$$\ddot{a}_{\overline{n}|} = \sum_{j=0}^{n-1} v^j = \frac{1-v^n}{1-v}.$$

Let us denote the lifetime of a newborn person as $\tilde{L}_0$. It is a random variable with distribution function F with range $[0,\omega]$, where ω is the end of the predetermined value assigned to a life. We assume F to be a smooth function, that is, without jumps. In this set up the future lifetime of a person aged x is a random variable $\tilde{L}_x$ with probability distribution function F_x, where

$$F_x(t) = \frac{F(x+t)-F(x)}{1-F(x)}$$

We use the $L_x = \lfloor \tilde{L}_x \rfloor$ for integer value of x so that L_x is the number of completed years lived by the life after the x^{th} birthday. Then the one-year survival probability of the life is given as

$$p_x = {}_1p_x = \mathbb{P}\{L_x > 0\} = \mathbb{P}\{\tilde{L}_x \geq 1\} = 1 - F_x(1),$$

and a similar duration death probability is

$$q_x = \mathbb{P}\{L_x = 0\} = \mathbb{P}\{0 \leq \tilde{L}_x < 1\} = 1 - p_x = F_x(1)$$
$$= \frac{F(x+1)-F(x)}{1-F(x)}$$

Setting $p_\omega = 0$ and $q_\omega = 1$ we have

$${}_kp_x = \mathbb{P}\{L_x \geq k\} = p_x \cdot p_{x+1} \ldots p_{x+k-1}.$$

The values of q_x can be found in mortality tables issued by statistical bureaus. If one assumes that these values do not change over time, then the distribution of L_x can be calculated. Thus,

$${}_kr_x = \mathbb{P}\{L_x = k\}$$
$$= \mathbb{P}\{L_x \geq k\} - \mathbb{P}\{L_x \geq k+1\}$$
$$= {}_kp_x - {}_kp_x \cdot p_{x+k} = {}_kp_x \cdot q_{x+k}$$

which denotes the probability that the person aged x will survive k years but not $k+1$ years, for $0 \leq k \leq \omega - x$.

We will introduce a refinement to model and have estimations of cohort dependent mortalities in Unit 8.

6.4.1 Simple Life Insurance Contracts

To understand the life insurance contracts, we include the basic tools in the following discussion. The notations that will be used are:

Z^b = discounted benefits
Z^c = discounted contributions.

Taking Z^b and Z^c as random variables, we get the equivalence principle:

$$E(Z^c) = E(Z^b) + \text{costs}.$$

We follow the simplification norm as mentioned above and ignore the costs. Moreover, we assume that all contracts start at the x^{th} birthday of the insured.

Usually, benefits in life insurance contracts consist of at most one payment of the sum insured. To see the underlying formulation we consider two basic types of life insurance viz., (a) term insurance and (b) pure endowment.

Term Insurance

A term insurance of duration n consists of payment of 1 unit at the end of the year of death, if it falls within the contract period. The sum insured is fixed at the time of contract. However, the time of payment L_x is random. The net present value (NPV) of the contract is

$$Z^b = v^{L_x+1} 1_{\{L_x \le n-1\}}. \qquad (4.1)$$

The probability distribution of Z^b is given by

$$\mathbb{P}\{Z^b = v^{k+1}\} = \mathbb{P}\{L_x = k\}\ {}_k r_x = {}_k p_x \cdot q_{x+k}$$

for $k \le n-1$. Otherwise, the value of Z^b is zero.

Therefore, the *expected net present value* (ENPV) is

$$_{|n}A_x = E(Z^b) = \left(v^{L_x+1} 1_{\{L_x \le n-1\}}\right) = \sum_{k=0}^{n-1} v^{k+1} \cdot {}_k r_x.$$

The ENPV is called *net single premium.*

In the above equation when $n \to \infty$ we get a "whole life insurance". In that case the net single premium A_x is given by

$$A_x = \mathbb{E}\left(v^{L_x+1}\right) = \sum_{k=0}^{\infty} v^{k+1} \cdot {}_k r_x.$$

When benefits c_j vary from year to year in a life insurance, we have

$$Z^b = c_{L_x+1} \cdot v^{L_x+1} 1_{\{L_x \le n-1\}},$$

and the ENPV equals to

$$E\left(Z^b\right)=\sum_{k=0}^{\infty} c_{k+1}\cdot v^{k+1}\cdot {}_k r_x.$$

Pure Endowment

Let us consider a life insurance of duration n that consists of payment of 1 unit only if the insured is alive at the end the n^{th} year. Then

$$Z^b = v^n 1_{\{L_x > n-1\}}.$$

We call it pure endowment. Its ENPV denoted by ${}_nE_x$ is given as

$${}_nE_x = v^n \cdot {}_np_x.$$

Endowments

Consider a payment of 1 unit at the end of the year of death, if this occurs within the first n years, otherwise at the end of the n^{th} year. We call it endowment life insurance. Under this form the discounted benefit is given as

$$Z^b = v^{\min(L_x+1,n)}.$$

This has the features of a term insurance and a pure endowment. As a result we can derive it by adding ENPV of these two forms.

Premiums can be paid periodically leading to as a series of payments which have to be paid while the insured person is alive. For such payments we have to include the duration and the frequency of premiums and the additional amount to be paid. The regular premiums can by interpreted as life annuities payable in advance. If there is yearly payment of 1, then the discounted ENPV is given by

$$Z^c = \sum_{k=0}^{n-1} v^k . 1_{\{L_x > k-1\}}.$$

Check Your Progress 1

1) Define the term survival function.

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2) Write the meaning of term life insurance.

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3) How endowment life insurance is different from pure endowment?

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6.5 NON-LIFE INSURANCE AND RUIN PROBABILITY

Lundberg (1903) provided the mathematical foundation of non-life insurance. He developed a model considering premiums and claims. Subsequently, he gave another model by considering a homogenous Poisson process, which we present below.

The following discussion is developed taking some select concepts given in Embrechts et al (1999). We have taken the definitions as given there.

Definition: The stochastic counting process $N = (N(t))_t$ is a homogeneous Poisson process with rate the intensity rate of $\lambda > 0$ if

(a) $N(0) = 0$ *a.s.*,

(b) N has stationary, independent increments, and

(c) for all $0 \le s < t < \infty : N(t) - N(s) \sim \text{POIS}(\lambda(t-s))$, i.e.,

$$P(N(t) - N(s) = k) = e^{-\lambda(t-s)} \frac{(\lambda(t-s))^k}{k!}, \quad k \in \mathbb{N}. \tag{5.1}$$

In insurance applications, $N(t)$ is used for the number of claims in the interval $(0,t]$ in a well defined portfolio. If we denote the claim arrival of the n^{th} claim by S_n, then

$$N(t) = \sup\{n \ge 1 : S_n \le t\}, \quad t \ge 0.$$

The inter-arrival times T_1, $T_k = S_k - S_{k-1}$, $k = 2,3...$ are independent, identically exponentially distributed $(\exp(\lambda))$ with finite mean $EY_1 = 1/\lambda$. The latter property also characterises the homogeneous Poisson process. The main size process $(X_k)_{k\in\mathrm{N}}$ is at first assumed to be iid with distribution function (df) F ($F(0) = 0$) and finite mean $\mu = EX_1$. The rv X_k denotes the claim size occurring at time S_k. As a result, the total claim amount up to time t is given by $S(t) = \sum_{k=1}^{N(t)} X_k$. The latter rv is referred to as compound Poisson rv. Its df can be written as

$$G_t(x) = P(S(t) \le x) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} F^{n^*}, \quad x,t \ge 0, \tag{5.2}$$

where df F^{n^*} in (5.2) denotes the n-fold convolution of F and F^{0^*} is the Dirac measure in 0 (see Unit 9 for meaning).

In addition to the liability process $(S(t))_{t\ge 0}$, an insurance company depends on premiums in order to meet the losses. In the above standard the premium process $(P(t))_t$ is assumed to be linear (deterministic), i.e., $P(t) = u + ct$, where $u \ge 0$ is the initial capital and $c > 0$. That is, we have a constant premium rate chosen in such a way that the company (or portfolio) has a fair chance of "survival". With this set up we present the Cramér-Lundberg model in the following where random variable (rv) plays a crucial role. Let τ be the ruin time of the risk process and

$$U(t) = u + ct - S(t), \quad t \ge 0. \tag{5.3}$$

i.e., $$\tau = \inf\{t \ge 0 : U(t) < 0\} \tag{5.4}$$

We assume that $\inf \emptyset = \infty$. The ruin probabilities in this formulation are

are defined as $\Psi(u,T) = P(\tau \le T), \quad T \le \infty.$

When $\Psi(u,\infty)$, the infinite horizon ruin probability, we write $\Psi(u)$. With the net-profit condition

$$c - \lambda\mu > 0, \tag{5.5}$$

$\lim_{u\to\infty} \Psi(u) = 0$. It is implied by (5.5) that on the average we obtain a higher premium income than a claim loss. The basic risk process (5.3) can now be rewritten as

$$U(t) = u + (1+\vartheta)\lambda\mu t - S(t),$$

where $\lambda\mu t = ES(t)$ and $\vartheta = c/(\lambda\mu) - 1 > 0$ is the safety loading which guarantees "survival".

Definition. The stochastic process $(U(t))_t$ defined in (5.3) with the net-profit condition (5.5) is called the Cramér-Lundberg risk process.

Let us denote $\bar{H} = 1 - H$ for any df H concentrated on $[0, \infty)$. Then

$$1 - \Psi(u) = (1-\rho)\sum_{n=0}^{\infty} \rho^n F_I^{n*}(u), \quad u \geq 0, \tag{5.6}$$

where $\rho = \lambda\mu / c < 1$ and the integrated tail df F_I is defined as

$$F_I(x) = \frac{1}{\mu}\int_0^x \bar{F}(y)\,dy, \quad x \geq 0. \tag{5.7}$$

The function $\Psi(u)$ in (5.6) also allows a compound df expression.

6.6 STOCHASTIC PROCESSES IN INSURANCE

Some Basic Results

The Cramér-Lundberg model $U(t) = u + ct - S(t), \ t \geq 0$; (see (5.3)) prompts us consider two related results, viz., Lundberg coefficient and Cramér-Lundberg approximation which we give below:

Lundberg Coefficient

Definition. Let the claim df F allows for a constant $R > 0$ to exist for which $\theta(R) = 0$, then R is called the Lundberg coefficient (or Lundberg adjustment coefficient) of the risk process $(U(t))_t$.

Examples where R exists are the exponential and gamma distributions. However, R does not exist for Pareto or lognormal distributions.

The Cramér-Lundberg Approximation

Theorem. Assume that the Lundberg coefficient R exists and that

$$\int_0^\infty x e^{Rx} \bar{F}(x)\,dx < \infty. \tag{6.1}$$

Then

$$\lim_{u\to\infty} \Psi(u) e^{Ru} = \frac{c - \lambda\mu}{\lambda m_X'(R) - c}. \tag{6.2}$$

The moment condition (6.1) is satisfied in all examples where R exists. The asymptotic estimate (6.2) is called the Cramér-Lundberg approximation.

The important assumption in the Cramér-Lundberg approximation is that the exponential moments of the claim size distribution exist for some $r > 0$. This means that the right tail of F decreases at least exponentially fast. However, analysis of insurance and financial data typically indicates the presence of heavy tails.

6.6.1 Generalisation of the Claim Number Process

The classical risk process has been generalized to obtain more realistic description of the insurance closer to reality. The homogeneous Poisson process, which is a stationary implying that the size of the portfolio cannot change (increase or decrease) has been extended to consider inhomogeneous process. Moreover, since some of the insurance products such as fire and automobile insurances, which require consideration for inclusion of risk fluctuations in the modelling have been complied with. As already mentioned in Section 6.5 the simplest way to take size fluctuations into account is to consider in homogenous Poisson processes with intensity measure $\Lambda(t)$. The purpose of this section is mainly to discuss the choice of point processes describing such risk fluctuations.

6.6.1.1 Mixed Poisson Processes

Definition

Let $\tilde{N}$ be a homogeneous Poisson process with intensity 1 and Λ a random variable with $P(\Lambda > 0) = 1$, independent of $\tilde{N}$. Then the process

$$N = \tilde{N} \circ \Lambda = \left(\tilde{N}(\Lambda t)\right)_t$$

is called a mixed Poisson Process. The random variable Λ is called structure variable.

A mixed Poisson process has stationary increments. However, the independent increments condition can no longer be retained. The stochastic variation of the claim number intensity can be interpreted as random changes of the Poisson parameter from its expected value λ. To model this type of distribution of the structure gamma distribution is used. The density function of the gamma distribution is given by

$$f_\Lambda(x) = \frac{\delta^\gamma}{\Gamma(\gamma)} x^{\gamma-1} e^{-\delta x}, \quad x \geq 0. \tag{6.3}$$

The notation $\Lambda \sim \Gamma(\gamma, \delta)$ indicates that the random variable Λ has a gamma distribution with density function given in (61).

Definition

A mixed Poisson process N is called a negative binomial process or Polya process if $\Lambda \sim \Gamma(\gamma, \delta)$

We then have for a Polya process N

$$
\begin{aligned}
P\left(N(t)=n\right) &= P\left(\tilde{N}(\Lambda t)=n\right) \\
&= \int_0^\infty P\left(\tilde{N}(\Lambda t)=n \mid \Lambda=\lambda\right) f_\Lambda(\lambda) \\
&= \int_0^\infty e^{-\lambda t}\frac{(\lambda t)^n}{n!}\frac{\delta^\gamma}{\Gamma(\gamma)}\lambda^{\gamma-1}e^{-\delta\lambda} \\
&= \binom{\gamma+n-1}{n}\left(\frac{\delta}{\delta+t}\right)^\gamma\left(\frac{t}{\delta+t}\right)^n,
\end{aligned}
$$

i.e., $N(t)$ has a negative binomial distribution. If we compare the total claim amount up to time t for the Poisson model keeping means equal, then the variance of the Polya model is found to be bigger than in the Poisson model. This is called an over-dispersion and is which often found in real insurance data (see Embrechts et al (1999)).

6.6.1.2 Ordinary Renewal Processes

Definition

Let the occurrence of claims be described by N. Take T_k to be the interarrival times between two successive claims. A point process on $\mathbb{R}_+$ is called a renewal process if the variables $(T)_{k\geq 1}$ are independent and if $T_1, T_2, T_3 \ldots$ have the same df G. N is called ordinary renewal process if also has df G.

We call N a stationary renewal process if G has definite mean $\frac{1}{\lambda}$ and if G_0 of N_1 satisfies $G_0(x)=\lambda\int_0^x \bar{G}(s)\,ds.$

Ordinary Renewal Process

Take N to be an ordinary renewal process and assume that T_k has finite mean $1/\lambda$. In this formulation N is not stationary, and $EN(t)\neq\lambda t$ unless T_k has an exponential distribution. We consider the associated random walk

$$S_n=\sum_{k=1}^n Y_k, (S_0=0),$$

where $Y_k=-cT_k+X_k$. We assume that $EY_k=-c/\lambda+u<0$, implying that the random walk S_n drifts to $-\infty$. The safety loading ϑ is defined as $\vartheta=c/(\lambda\mu)-1$. Since ruin can occur only at renewal epochs, we have that

$$\Psi(u)=P\left(\max_n S_n>u\right).$$

Check Your Progress 2

1) Why Lundberg model considered Poisson process?

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2) How do you define ruin probabilities?

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3) Explain Lundberg coefficient.

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4) Why do you use renewal process in insurance modelling?

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6.7 LET US SUM UP

In this unit we have discussed the modelling techniques of insurance in the stochastic framework. While claim arrival time and claim size are pointed out to be random, the premium determination is seen as in the context of equivalent principle which states that the expected value of discounted future benefits payable by the insurer is equal to the expected value of discounted future premiums payable by the policyholder. Survival function has been defined to get the probability that the life will survive beyond a specified time and expected net present value of (or net single premium) in life insurace contracts of term insurance and pure endowment have been studied. The classical risk model formulated by Lundberg for non-life insurance considering the homogenous Poisson process have been examined and ruin probabilities have been derived. The classical risk model has been seen by relaxing the assumption of homogenous statonary process.

6.8 KEY WORDS

Compound Poisson Process: A stochastic process with rate $\lambda > 0$ and jump size distribution G in a continuous-time. If $Y(t): t \geq 0$ then $Y(t) = \sum_{i}^{N(t)} D_i$ where, $N(t): t \geq 0$ is a Poisson process with rate λ, and $\{D_i : i \geq 1\}$ are independent and identically distributed random variables, with distribution function G, which are also independent of $N(t): t \geq 0$.

Convolution: A mathematical operator which takes two functions (f and g) and produces a third function that in a sense represents the amount of overlap between f and a reversed and translated version of g. A convolution is a kind of very general moving average.

Dirac Measure: A measure δ_x on a set X (with any σ - algebra of subsets of X) that gives the singleton set $\{x\}$ the measure 1, for a chosen element $x \in X : \delta_x(\{x\}) = 1$.

Endowments: An insurance contract that requires payment of 1 unit at the end of the year of death, if this occurs within the first n years, otherwise at the end of the n^{th} year.

Equivalence Principle: A premium fixation principle in insurance where the expected value of discounted future benefits payable by the insurer is equal to the expected value of discounted future premiums payable by the policyholder.

Erlang Distribution: A probability distribution describing waiting times in queuing systems.

Erlang Unit: A unit to measure telecommunications (or other) traffic.

Erlang Unit: a unit to measure telecommunications (or other) traffic

Law of Large Numbers: In repeated, independent trials with the same probability of success in each trial, the percentage of successes is increasingly likely to be close to the chance of success as the number of trials increases.

Markov Process: A stochastic process where the future expected value of a value (such as an asset price) is dependent only on the current value.

Net Present Value: Present value of cash flow minus initial investment.

Pure Endowment: An insurance contract of duration n that consists of payment of 1 unit only if the insured is alive at the end the n^{th} year.

Renewal Process : A generalisation of the Poisson prcess. The Poisson process is a continuos–time Markov process on the positive integers (usually starting at zero) which has iid *holding times* at each integer i exponentially distributed before advancing (with probability 1) to the next integer:$i + 1$. We may define a renewal process to be similar except that the holding times take on a more general distribution.

Renewal Theory: A theory used for the purpose of comparing the long term benefits of different insurance policies.

Survival Function: A relation between mortality and time. Through it we get the probability that the life will survive beyond a specified time.

Term Insurance: An insurance contract of duration n that consists of payment of 1 unit at the end of the year of death, if it falls within the contract period.

6.9 SOME USEFUL BOOKS

Butcher. M.V., Nesbitt. C.J. (1971), Mathematics of Compound Interest. Ulrich's Books. Ann. Arbor. Michigan

Embrechts, Paul, RüdigerFrey, Hansjörg Furrer(1999), Stochastic Process in Insurance and Finance, ETH Zürich, Switzerland (see internet)

Gerbar H. U. (1997), Life Insurance, Third Edition. Springer

Salih N. Neftci.(1996), An Introduction to the Mathematics of Financial Derivatives, Academic Press. San Diego-California, USA

6.10 ANSWER OR HINTS TO CHECK YOUR PROGRESS

Check Your Progress 1

1) See Section 6.3 and answer.

2) See Sub-section 6.4.1 and answer.

3) See Sub-section 6.4.1 and answer.

Check Your Progress 2

1) Read Section 6.5 and answer.
2) Read Section 6.5 and answer.
3) Read Sub-section 6.6 and answer.
4) Read Sub-section 6.6.1.2 and answer.

UNIT 7 THEORY OF INTEREST

Structure

7.0 OBJECTIVES

After going through this unit, you will be able to:

- calculate interest with discrete and continuous time;
- model fixed and variable interest rate;
- understand interest payments in financial transactions; and
- differentiate between deterministic and stochastic interest rate modelling.

7.1 INTRODUCTION

The behaviour of interest rate is key to the solvency of insurance companies and other financial institutions. It is therefore, important to understand the concepts, notation and computation of interest rate in deterministic and stochastic environments. We will discuss all the themes given here in Units 11-13 in the context of life contingencies. Parts of present discussion follow Veeh (2001) and Slud (2001) and makes use of select examples given in therein. It will be fruitful to understand the mathematics of deterministic interest rate portion discussed by the first study mentioned and the theory of interest by the second .

7.2 ELEMENTS OF INTEREST CALCULATION

Interest rate is the time value of money. As an insurance contract's due insured pays the insurer a fixed premium periodically, annually to semi-annually, the present and future value of such payments needs to be balanced. Thus, \$1 in hand today is to be viewed as more valuable than \$1 you would receive one year later. Insurance companies take this into account.

We are familiar with the computation of compound interest. It runs as follows: an amount A is invested at interest rate i per year. At the end of the period, the amount will be $A + iA = A(1+i)$ subject to the condition that the interest is compounded annually. For the computation of second year amount, the interest earned during the first year enters along with the principal.

In such a process, after n years, the amount will be $A(1+i)^n$. Note that the accumulation factor is $(1+i)^n$. We derive the nominal annual rate of interest, if the interest is compounded daily so that after n years the amount becomes $A\left(1+\frac{i}{365}\right)^{365n}$ and effective annual rate of interest is $A\left(1+\frac{i}{365}\right)^{365} - 1$.

Example 1.

Suppose that there prevails an interest rate of 5% compounded quarterly. Find the effective rate of interest.

Since $\left(1+\frac{0.05}{4}\right)^4 = 1+i$

effective interest rate $= i = \left(1+\frac{0.05}{4}\right)^4 - 1$

Example 2.

You are interested to invest \$1 for which two banks A and B are willing offer two types of compounded process. While A pays 5% interest compounded monthly, B follows a process of compounding daily and offers an interest that makes you indifferent between selecting the banks. What is the interest rate in B that makes you equally attractive for you investment plan?

Suppose that your investment is for t years. Then the amount to be given by A and B will be

$$\left(1+\frac{0.05}{4}\right)^{12t} \text{ and } \left(1+\frac{i}{365}\right)^{365t}$$

Thus $\left(1+\frac{0.05}{12}\right)^{12t} = \left(1+\frac{i}{365}\right)^{365t}$

or, $$\left(1+\frac{0.05}{4}\right)^{\frac{12}{365}}=\left(1+\frac{i}{365}\right)$$

or, $$\left(1+\frac{0.05}{4}\right)^{\frac{12}{365}}-1=\frac{i}{365}$$

$$i=365\left[\left(1+\frac{0.05}{12}\right)\frac{12}{365}-1\right]=0.04989$$

Example 3.

Suppose you plan to have an amount A at a future date n if the interest rate is i. How much should be deposited today so as to reach the target amount?

See that the amount required is equal to $A(1+i)^{-n}$. This quantity is called the present value of A and the factor $(1+i)^{-1}$ is the discount factor denoted by v.

Example 4.

Annual interest rate is given to be 5%. You are entitled to receive Rs.2000 after 10 years from today. Find the present value of that amount.

Present value = Rs.2000 $(1+0.05)^{-10}$ = Rs.1227.83

Thus, you may receive either Rs.1227.83 today or Rs.2000 after 10 years. You are indifferent between these two choices if the interest rate of 5% is compounded annually.

Note that an interest rate at the end of the period is equivalent to the payment of d at the beginning of the period. Such a payment is called discount if paid at the beginning of period.

Example 5.

The discount rate for 3 months Treasury bill is given to be 5%. Find the equivalent rate of interest.

$$d^{(4)}=0.05$$

Since $$i=\left(1-\frac{0.05}{4}\right)^{-4}-1=0.0516$$

Cash-Flow Valuation

Often cash payments of varying amounts are received at different point of time. Such payments may be positive or negative. Whereas a positive amount denotes an inflow of cash, a negative amount denotes a cash outflow.

Suppose a payment c_j is made at time j. This is equivalent to payments of $c_j v^{j-t}$ at time t. Thus, the value of the cash-flow at tiKe is $\sum_{j=0}^{n} c_j v^{j-t}$

Example 6.

A loan with interest rate of 6% compounded annually requires payments of 300, 400 and 700 at the end of year 1, 2, and 3. The borrower prefers to make a single payment of 1400 and is interested to find out the appropriate time at which the option could be exercised. The she has to proceed by computing all the present values at time 0. The equation satisfying the time is

$$300(1.006)^{-1} + 400(1.06)^{-2} + 700(1.06)^{-3} = 1400(1.06)^{-t}$$

Thus, $t = 2.267$

Example 7.

The effective rate carried by a loan is 4%. It is required that the borrower repays 1000 after 1 year, 2000 after 2 years, 3000 after 3 years and 4000 after 4 years. Suppose the borrower plans to make a single payment of 10,000 to repay the loan. We proceed to find the appropriate time at which the option can be exercised as follows:

$$1000v + 2000v^2 + 3000v^3 + 4000v^4 = 10000v^t$$

where $v = \dfrac{1}{1.04}$

Thus, $t = 2.98$ years.

7.3 THEORY OF INTEREST

Preparation of this portion (i.e., Section 7.3) has been almost an adaptation of Slud (2001). We have depended heavily on this work borrowing the notations, examples as well as sequencing. Credit for the formulation of the theme the way it has been done goes to the author.

We have seen above that a theory of interest helps in valuing streams of payments made over time. Since payments are based on unpredictable occurrences or contingencies for insured lives, these can occur at different times. So we have to study the basic development. Let us start with the case constant interest. Here you deposit money in a bank account, which grows according to a schedule determined by both the interest rate and the occasions when interest amounts are compounded. Thus, it is deemed to be added to the account. See the importance of compounding rules: they determine when new one on previously earned interest amounts will be included.

Consider compounding at time-intervals $h = \frac{1}{m}$, with nominal interest rate $i^{(m)}$. It means that a unit amount accumulates to $(1 + i^m / m)$ after a time $h = \frac{1}{m}$. The principal or account value $(1 + i^m / m)$ at time $h = \frac{1}{m}$ accumulates over the time-interval from $\frac{1}{m}$ until $\frac{2}{m}$, to $(1 + i^m / m)$ $(1 + i^m / m)(1 + i^m / m) = (1 + i^m / m)^2$. Thus, a unit amount accumulates to $(1 + i^m / m)^n = (1 + i^m / m)^{Tm}$ after the time $T = nh$, which is a multiple of n whole units of h. As $m \to \infty$, the unit amount compounds to $e^{\delta T}$ after time T, where the instantaneous annualised nominal interest rate $\delta = \lim_m i^{(m)}$. This is called the **force of interest**. In either case of compounding, the actual **Annual Percentage Rate (APR)** or effective annual interest rate is defined as the amount (minus 1, and multiplied by 100 if it is to be expressed as a percentage) to which a unit compounds after a single year, i.e., respectively as

$$i_{APR} = \left(1 + \frac{i^{(m)}}{m}\right)^m - 1 \quad \text{or,} \quad e^{\delta} - 1$$

A unit invested at time 0 accumulates at the effective interest rate i_{APR} over a time-duration T (still assumed to be a multiple of $1/m$) is, therefore,

$$\left(1 + i_{APR}\right)^T = \left(1 + \frac{i^{(m)}}{m}\right)^{mT} = e^{\delta T}$$

which is called the **accumulation factor** operating over the interval of duration T at the fixed interest rate. With different periods of compounding all types of nominal interest rates $i^{(m)}$ are related

$$(1 + i^{(m)} / m)^m = 1 + i = 1 + i_{APR}, \quad i^{(m)} = m\left\{(1 + i)^{1/m} - 1\right\} \tag{3.1}$$

Similarly, interest can be said to be governed by the *discount rates* for various compounding periods, defined by

$$1 - d^{(m)/m} = \left(1 + i^{(m),m}\right)^{-1}$$

Solving the last equation for $d^{(m)}$ gives

$$d^{(m)} = i^{(m)} / \left(1 + i^{(m)} / m\right) \tag{3.2}$$

We are familiar with the idea of discount rates. It shows if \$1 is loaned out at interest, then the amount $d^{(m)} / m$ is the correct amount to be repaid at the beginning rather than the end of each fraction $1/m$ of the year. The principal of \$1 is repaid at the end of the year. The reason is that, according to the definition, the amount $1 d^{(m)}/m$ accumulates at nominal interest $i^{(m)}$

(compounded m times yearly) to $\left(1-d^{m}/m\right).\left(1+i^{(m)}/m\right)=1$ after a time-period of $1/m$.

Basically the interest payments are made at the ends and the beginnings of successive time-periods $1/m$ in order that the principal owed at each time j/m on an amount $1 borrowed at time 0 will always be $1. To define these terms consider first the simplest case of $m=1$.

If $1 is to be borrowed at time 0, then the single payment at time 1, which fully compensates the lender, if that lender could alternatively have earned interest rate i, is $\$(1+i)$. Take this as a payment of $1 principal i.e., the face amount of the loan and $\$_{i}$ interest. In exactly the same way, if $1 is borrowed at time 0 for a time-period $1/m$, then the repayment at time $1/m$ takes the form of $1 principal and $\$_{i^{(m)}}/m$ interest.

Thus, if $1 was borrowed at time 0, an interest payment of $\$\frac{i^{(m)}}{m}$ at time $1/m$ leaves an amount $1 still owed, which can be viewed as an amount borrowed on the time-interval $(1/m, 2/m)$. Then a payment of $\$\frac{i^{(m)}}{m}$ at time $2/m$ still leaves an amount $1 owed at $2/m$, which is deemed borrowed until time $3/m$, and so forth, until the loan of $1 on the final time-interval $\left(\frac{1}{m},\frac{2}{m}\right)$ is paid off at time 1 with a final interest payment of $\$\frac{i^{(m)}}{m}$ together with the principal repayment of $1. Thus, we have $1 at time 0 is equivalent to the stream of m payments of $\$\frac{i^{(m)}}{m}$ at times $1/m, 2/m, \ldots, 1$ plus the payment of $1 at time 1.

Take the case when interest is to be paid at the beginning of the period of the loan instead of the end. Thus interest paid at time 0 for a loan of $1 would be $d=i/(1+i)$, and principal would be repaid at time 1. Is it correct? Since interest d is paid at the same instant as receiving the loan of $1, the net amount actually received is $1-d=(1+i)^{1}$, which accumulates in value to $(1-d)(1+i)=\$1$ at time 1.

Similarly, if interest payments are to be made at the beginnings of each of the intervals $\left(j/m,(j+1)/m\right]$ for $j=0,1,\ldots,m-1$, with a final principal repayment of $1 at time 1, then the interest payments should be $d^{(m)}/m$. This follows because the amount effectively borrowed (after the immediate interest payment) over each interval $(j/m, (j+1)/m$ is $\$(1-d(m)/m)$, which accumulates in value over the interval of length 1/m to an amount $\left(1-d^{(m)}/m\right)\left(1+i^{(m)}\right)$. So all along the year-long life of the loan, the principal owed at (or just before) each time $(j+1)/m$ is exactly \$1. The net result is \$1 at time 0 is equivalent to the stream of m payments of $\$d^{(m)}/m$ at times $0, 1/m, 2/m, \ldots, (m-1)/m$ plus the payment of \$1 at time 1.

Check Your Progress 1

1) Show that $Lt_{m=\infty}\, d^{(m)} = Lt_{m\to}\, i^{(m)} = s$

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2) Show that the nominal rate of interest convertible once in every four years is equivalent to a nominal rate of discount convertible quarterly.

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3) Show that $d = 1 - v$.

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4) Show that $S = 10g(1+i)$

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7.3.1 Variable Interest Rates

Let us consider the case of non-constant instantaneously varying nominal interest rates $\delta(t)$. It is called the time-varying force of interest. Here, set the compounding-interval $\left[kh,(k+1)h\right]$ of length $h=1/m$, use the instantaneous continuously compounding interest-rate $\delta(kh)$ available at the beginning of the interval and calculate an equivalent annualised nominal interest-rate over the interval. Effectively you are trying to find a number $r_m(kh)$ such that

$$\left(1+\frac{r_{m(kh)}}{m}\right)=\left(e^{\delta(kh)}\right)^{1/m}=\exp\left(\frac{\delta(kh)}{m}\right)$$

So if $m\to\infty$ there is an essentially constant principal amount over each interval of length $1/m$, such that, over the interval $[b,b+t]$, with instantaneous compounding, the unit principal amount accumulates to

$$\exp(\lim_{m}\frac{1}{m}\sum_{k=0}^{[mt]-1}\delta(b+k/m)=\exp)\left(\int_0^t\delta(b+s)\,ds\right)$$

To specify continuous-time varying interest rates in terms of effective or *APR* rates, instead of the instantaneous nominal rates $\delta(t)$, we require the simple conversion,

$$r_{APR}(t)=e^{\delta(t)}-1,\delta(t)=\ln\left(1+r_{APR^{(t)}}\right)$$

Next, consider the case of deposits $s_0,s_1,\ldots,s_k,\ldots,s_n$ these are made at times $0,h,\ldots,kh,\ldots,nh$, where $h=1/m$ is the given compounding-period, and where nominal annualised instantaneous interest-rates $\delta(kh)$ (with compounding-period h) apply to the accrual of interest on the interval $\left[kh,(k+1)h\right]$ after time kh_h. If the accumulated account is denoted by B_k, then these can be expressed in terms of s_j and $\delta(jh)$. That is,

$$B_{k+1}=B_k\left(1+\frac{i^{(m)}(kh)}{m}\right)+S_{k+1},\quad B_0=S_0$$

See that we have a difference equation here which can be solved in terms of successive summation and product operations acting on the sequences s_j and $\delta(jh)$ as follows: First define a function A_k to denote the accumulated bank balance at time kh for a unit invested at time 0 and earning interest with instantaneous nominal interest rates $\delta(jh)$ (or equivalently, nominal rates $r_m(jh)$. For compounding at multiples of $h=1/m$) apply over the whole compounding-intervals $\left[jh,(j+1)h\right]$, $j=0,\ldots,k-1$. Then by definition, A_k

satisfies a homogeneous equation analogous to the previous one, which together with its solution is given by

$$A_{K+1} = A_K \cdot \left(1 + \frac{r_m(kh)}{m}\right), \quad A_0 = 1, \quad A_k = \prod_{j=0}^{k-1}\left(1 + \frac{r_m(jh)}{m}\right)$$

Equivalent investment and present value

Suppose that a stream of deposits s_j accruing interest with annualised nominal rates $r_m(jh)$ with respect to compounding periods $(jh,(j+1)h)$ for $j = 0,...,n$ is such that a single deposit D at time 0 would accumulate by compound interest to give exactly the same final balance F_n at time $T = nh$.. Then the present cash amount D in hand is said to be equivalent to the value of a contract to receive s_j at time $jh, j = 0,1,...,n$. In other words, the present value of the contract is precisely D. We have just calculated that an amount 1 at time 0 compounds to an accumulated amount A_n at time $T = nh$. Therefore, an amount a at time 0 accumulates to aA_n at time T, and in particular, $1/A_n$ at time 0 accumulates to 1 at time T. Thus, the present value of 1 at time $T = nh$ is $1/A_n$.

Define G_k to be the present value of the stream of payments s_j at time jh for $j = 0,1,...,k$. Since B_k was the accumulated value just after time kh of the same stream of payments, and since the present value at 0 of an amount B_k at time kh is just B_k / A_k, we conclude

$$G_{k+1} = \frac{B_{k+1}}{A_{k+1}} = \frac{B_k\left(1 + r_m(kh)/m\right)}{A_{k\left(1+r_{m(kh)}/m\right)}} + \frac{S_{k+1}}{A_{k+1}}, \quad k \geq 1, G_0 = s_0$$

Thus, $G_{k+1} - G_k = S_{k+1}/A_{k+1}$, and

$$G_{k+1} = S_0 + \sum_{i=0}^{k} \frac{S_{i+1}}{A_{i+1}} = \sum_{j=0}^{k+1} \frac{S_j}{A_j}$$

Thus, the solution for the accumulated balance B_k just after time kh and for the present value G_k at time 0 can be given as

$$G_k = G_k = \sum_{i=0}^{k} \frac{S_i}{A_i}, \quad B_k = A_k . G_k \quad k = 0,...,n$$

To appreciate the underlying idea of the above formulae, note following observation:

(a) The present value at fixed interest rate i of a payment of \$1 exactly t years in the future, must be equal to the amount which must be put in the bank at time 0 to accumulate at interest to an amount 1 exactly t

years later. Since $(1+t)^t$ is the factor by which today's deposit increases in exactly t years, the present value of a payment of \$1 delayed t years is $(1+i)^{-t}$. Here t may be an integer or positive real number.

(b) Present values superpose adaptively: that is, if I am to receive a payment stream C which is the sum of payment streams A and B, then the present value of C is simply the sum of the present value of payment stream A and the present value of payment stream B.

(c) As a consequence of (a) and (b), the present value for constant interest rate i at time 0 of a payment stream consisting of payments s_j at future times t_j, $j = 0, \ldots, n$ must be the summation

$$\sum_{j=0}^{n} s_j (1+i)^{-t_j}$$

(d) Finally, to combine present values on distinct time intervals, at possibly different interest rates, remark that if fixed interest-rate i applies to the time-interval $[0,s]$ and the fixed interest rate i applies to the time-interval $[s,t+s]$, then the present value at time s of a future payment of a at time $t+s$ is $b = a(1+i)^{-t}$, and the present value at time 0 of a payment b at time s is $b(1+i)^{-s}$. The idea of present value is that these three payments, a at time $s+t$, $b = a(1+I)^{-t}$, at time s, and $b(1+i)^{-s} = a(1+i)^{-t}(1+i)^{-s}$ at time 0, are all equivalent.

(e) Applying the idea of paragraph (d) repeatedly over successive intervals of length $h = 1/m$ each, we find that the present value of a payment of \$1 at time t (assumed to be an integer multiple of h), where $r(kh)$ is the applicable effective interest rate on time-interval $[kh,(k+1)h]$, is

$$1/A(t) = \prod_{j=1}^{mt} (1+r(jh))^{-h}$$

where $A(t) = A_k$ is the amount previously derived as the accumulation-factor for the time-interval $[0,t]$.

The formulae just developed can be used to give the internal rate of return r over the time-interval $[0,T]$ of a unit investment, which pays amount s_k at times $t_k; k = 0, \ldots, n,\ 0 \le t_k \le T$. This constant (effective) interest rate r is the one such that

$$\sum_{k=0}^{n} s_k \left(1+r\right) = 1$$

with respect to the $APR\ r$, the present value of a payment s_k at a time t_k time-units in the future is $s_k(1+r)^{-tk}$. Therefore, the stream of payments s_k at times t_k, $(k = 0,1,...,n)$ becomes equivalent, for the uniquely defined interest rate r, to an immediate (time-0) payment of 1.

Example 8.

Suppose that you are offered an investment option under which a \$10,000 investment made now is expected to pay \$300 yearly for 5 years (beginning 1 year from the date of the investment), and then \$800 yearly for the following five years, with the principal of \$10,000 returned to you (if all goes well) exactly 10 years from the date of the investment (at the same time as the last of the \$800 payments). If the investment goes as planned, what is the effective interest rate you will be earning on your investment.

In the effective interest rate, the present value of the payment-stream, at the unknown interest rate $r = i_{APR}$, must be balanced with the value (here \$10, 000), which is invested. That is because the indicated payment stream is being regarded as equivalent to bank interest at rate r.) The balance equation in the example is obviously

$$10,000 = 300\sum_{j=m}^{5}(1+r)^{-j} + 800\sum_{j=6}^{10}(1+r)^{-j} + 10,000(1+r^{-10}).$$

The right-hand side can be simplified somewhat, in terms of the notation $x = (1+r)^{-5}$, to

$$\frac{300}{1+r}\left(\frac{1-x}{1-(1+r)^{-1}}\right) + \frac{800x}{(1+r)}\left(\frac{1-x}{1-(1+r)^{-1}}\right) + 10,000x^2$$

$$= \frac{1-x}{r}(300+800x) + 10,000x^2 \tag{3.3}$$

7.3.2 Continuous-time Payment Streams

We consider now continuous-time deposit streams with continuous compounding. Let $D(t)$ be the rate per unit time at which savings deposits are made such that D (t) = $D(t) = \lim_{m\to\infty} ms_{[mt]}$, where [.] denotes greatest-integer. If we take δ (t) to be the time-varying nominal interest rate with continuous compounding, and $B(t)$ to be the accumulated balance as of time t (analogous to the quantity $B_{[TM]} = B_k$ from before, when $t = \frac{k}{m}$, then the previous difference-equation can be replaced by

$$B(t+h) = B(t)(1+h)\delta(t) + hD(t) + o(h)$$

where $o(h)$ denotes a remainder with $\frac{o(h)}{o(h)} \to 0$ as $h \to 0$.

Subtracting $B(t)$ from both sides of the last equation, dividing by h, and letting h decrease to 0, yields a differential equation at times $t > 0$:

$$B'(t) = B(t)\delta(t) + D(t), \quad A(0) = s_0 \tag{3.4}$$

Solution to (3.4) can be obtained by multiplying an integrating factor. The integrating factor $\frac{1}{A(t)} = \exp\left(-\int_0^t \delta(s)ds\right)$ is the present value at time 0 of a payment of 1 at time t, and the quantity $\frac{B(t)}{A(t)} = G(t)$ is then the present value of the deposit stream of s_0 at time 0 followed by continuous deposits at rate D(t). The ratio-rule of differentiation yields

$$G'(t) = \frac{B'(t)}{A(t)} - \frac{B(t)A'(t)}{A^{2(t)}} = \frac{B'^{(t)} - B(t)\delta(t)}{A(t)} = \frac{D(t)}{A(t)}$$

where the substitution $\frac{A'(t)}{A(t)} \equiv \delta(t)$ has been made in the third expression. Since $G(0) = B(0) = s_0$, the solution to the differential equation (3.4) is

$$G(t) = s_0 + \int_0^t \frac{D(s)}{A(s)}ds, \quad \mathrm{B}(t) = A(t)G(t)$$

For the case of a constant unit-rate payment stream

$$\left(D(x) = 1, \delta(x) = \delta = In(1+i), \ 0 \le x \le T\right)$$

with no initial deposit (i.e., $s_0 = 0$), $A(t) = \exp\left(t\ln(1+i)\right) = (1+i)^t$, and the present value of such a payment stream is

$$\int_0^T 1.\exp\left(-tIn(1+i)\right)dt = \frac{1}{\delta}\left(1-(1+i)^{-T}\right)$$

Recall that the force of interest $\delta = In(1+i)$ is the limiting value of the nominal interest rate $i^{(m)}$ the difference-quotient representation yields

$$\lim_{m\to\infty} i^{(m)} = \lim_{m\to\infty} \frac{\exp((1/m)\ln(1+i))-1}{1/m} = \ln(1+i)$$

The present value of a payment at time T in the future is, as expected,.

$$\left(1+\frac{i^{(m)}}{m}\right)^{-mT} = (1+i)^{-T} = \exp(-\delta T)$$

Example 9. (*due to Slud, 2001*)

How many years does it take for money to triple in value at interest rate i?

The equation to solve is $3=(1+i)^t$, so the answer is $\left(\ln 3/\ln(1+i)\right)$, with numerical answer given by

$$t = \begin{cases} 22.52 & for \quad i=0.05 \\ 16.24 & for \quad i=0.07 \\ 11.53 & for \quad i=0.10 \end{cases}$$

Example 10.

Suppose that a sum of $1000 is borrowed for 5 years at 5%, with interest deducted immediately in a lump sum from the amount borrowed, and principal due in a lump sum at the end of the 5 years. Suppose further that the amount received is invested and earns 7%. What is the value of the net profit at the end of the 5 years? what is its present value (at 5%) as of time 0?

First, the amount received is $1000(1-d)^5 = 1000/(1.05)^5 = 783.53$ where $d=$ .05/1.05, since the amount received should compound to precisely the principal of $1000 at 5% interest in 5 years. Next, the compounded value of 783.53 for 5 years at 7% is 783.53 (1.07) 5 = 1098.94, so the net profit at the end of 5 years, after paying o the principal of 1000, is $98.94. The present value of the profit ought to be calculated with respect to the 'going rate of interest', which in this problem is presumably the rate of 5% at which the money is borrowed, so is $98.94/(1.05)^5 = 77.52$.

Check Your Progress 2

1) What do you mean by variable rate of interest?

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2) What is the meaning of equivalent rate of interest?

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3) Write the meaning of continuous-time payment-stream.

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7.4 STOCHASTIC INTEREST RATE MODELS

The following discussion is based on a lecture note entitled, *Hilary Term Lecture 5, Stochastic interest rate models*, available in internet.

When we assume that all interest rates are known, it is possible for us to compare investments under different interest rates. Any uncertainty of investment proceeds will be expressed modelling cash-flows by random variables. However interest rates themselves are uncertain. Therefore, we need to model interest rates by random variables.

7.4.1 Basic Model for Stochastic Interest Rate

One Interest Rate Case

Let us take a simple example. Suppose, you invest £100 for 1 year at an interest rate i which is not known in advance. Assume further that the present interest rates are at 3% . However, you expect one of three possibilities to happen in that rate:

a fall by 1%, no change or a rise by 1%, (each equally likely).

Thus, P (I = 2%) = P (I = 3%) = P (I = 4%) = 1/3

Consequently, the returns from the investment $R = 100(1+I)$ at the end of the year are random, and we have

$$P(R = 102) = P(R = 103) = P(R = 104) = 1/3$$

You can calculate your expected proceeds as

$$E(R) = (1/3)\{(102+103+104)\} = 103.$$

If you take an average interest rate $E(I) = 3\%$, then you get the same result.

However, calculation works with a term of 1 year. If the term is, say, $t \in (0,\infty)$ years, $t \neq 1$, we get $R = 100(1+I)^t$. This is because

$$P\left(R = (102)^t\right) = P\left(R = (103)^t\right) = \left(R = (104)^t\right) = 1/3$$

and $\quad E(R)=(100/3)\left((1.02)^t+(1.03)^t+(1.04)^t\right)=(1.03)^t$

Instead of assuming that I can only take 3 values, we may as well consider any random variable, discrete or continuous, I that ranges $(-1,\infty)$.

7.4.2 Independent Annual Interest Rates

Let us consider a situation where the interest rate changes. Then, we get the model of independent annual interest rates.

To develop it, assume that the interest rate is a collection of independent identically distributed (iid) random variable with annual interest rates I_j, $j \geq 1$, taking values in $(-1,\infty)$; rate I_j being applied the j^{th} year.

Given an iid interest rate model, any simple cash flow (s,c); $s \in \mathbb{N}$ has an expected value at time $t \geq s$, $t \in \mathbb{N}$ given by

$$E\left(Vatl_t\left((s,c)\right)\right)=c\prod_{j=s+1}^{t}\left(1+\mathrm{E}\left(\mathrm{I}_j\right)\right)=c\left(1+E\left(I_1\right)\right)^{t-s}$$

And at time $t \geq s, t \in \mathbb{N}$ given by

$$E\left(Val_t\left((s,c)\right)\right)=c\prod_{j=t+1}^{s}E\left(\left(1+I_j\right)^{-1}\right)=c\left(E\left(\left(1+I_1\right)^{-1}\right)\right)^{s-t}$$

In this class of interest rate modelling, log-normal distribution is used frequently which we discuss in the following:

Log-normal Distribution

Definition: A random variable X is said to have a log-normal distribution if $Z=\log(X)$ has a normal variate. This distribution, $\log N\left(\mu,\sigma^2\right)$, has two parameters $\mu=E\left(\log(X)\right)$ and $\sigma^2=\operatorname{var}\left(\log(X)\right)$.

Proposition: If $(1+I)$ has a log-normal distribution with parameters μ and σ^2 then

$$j=E(1)=\exp\left\{\mu+\frac{1}{2}\sigma^2\right\}-1 \qquad \text{(a)}$$

and $\quad$ s^2 = Vary(I) = exp $s^2 \doteq \operatorname{var}(I)=\exp\left\{2\mu+\sigma^2\right\}\left(\exp\left\{\sigma^2\right\}\right)-1.$ $\qquad$ (b)

To see (a) above consider the moment generating function of the normal distribution

$$E(I) = E\left(\exp\{\log(1+I)\}\right) - 1 = E\left(\exp\{Z\} - 1\right)$$
$$= \exp\left\{\mu + \frac{1}{2}\sigma^2\right\} - 1.$$

For (b), consider

$$Var(I) = Var(1+I) = E\left\{(1+I)^2 - \left(E(1+I)\right)^2\right\} = E\left(\exp\{2Z\}\right) - \left(E\left(\exp\{Z\}\right)\right)^2$$
$$= \exp\{2\mu + 2\sigma^2\} - \exp\{2\mu + \sigma^2\}$$

Thus, we see that if $(1+I)$ has a log-normal distribution, then $\Delta = \log(1+I)$ also follows normal distribution.

Example 11.

Let $1+I,\ldots,1+I_n$ be independent log-normal random variables with common parameters μ and σ^2. Then the distribution of the accumulated value at time n of a unit investment at time 0 is found to be

$$S_n = \prod_{j=1}^{n}\left(1+I_j\right) = \exp\left\{\sum_{j=1}^{n}\Delta_j\right\} \sim \log N\left(n\mu, \sigma^2\right)$$

Let us take $\mu = 0.07$, $\sigma^2 = 0.006$ and $n = 10$ and plan to accumulate at least Rs. 600,000 with probability 99%, Then we have to invest A so that

$$.99 = P\left(AS_n > 600{,}000\right) = P\left(Z > \frac{\log\{600{,}000/A\} - n\mu}{\sqrt{n\sigma^2}}\right)$$

$$\Rightarrow -2.33 = \frac{\log\{600{,}000/\mathrm{A}\} - n\mu}{\sqrt{n\sigma}} \Rightarrow A = 600{,}000\exp\left\{2.33\sqrt{n\sigma - n\mu}\right\}$$
$$= 527242.59$$

Here we used that $P(Z > -2.33) = 0.99$ for a standard normal random variable Z.

7.4.3 Dependent Annual Interest Rates

Interest rates do not fluctuate as strongly as modelled in the iid formulation above. It is commonly observed that an interest rate prevailing currently may lead to a still higher rate the following year. The case could be the same in low interest scenario. To model such situations we need to centre the new interest rate around the current interest rate, or between the current and a general long term mean interest rate. Such a formulation uses the theories of random walks, time series models and Markov chains etc. We have not discussed these in the unit.

Example 12. (2-step dependent log-normal model).

Let $1+I_1$ be a log-normal variable with parameters μ and σ^2 as above, and $1+I_2$, a conditionally log-normal variable with parameters

$$\mu_2 = k\ \log(1+\mathrm{I}_1)+(1-k)\mu \text{ and } \sigma^2$$

for some $k \in [0,1]$. Then given $\log(1+I_1)$, $\log(1+I_2)$. is normal with these parameters, i.e.,

$$\log(1+I_2) = k\log(1+I_1)+(1-k)\mu + N_2,$$

where N_2 is independent $N(0,\sigma^2)$. Consequently,

$$\log(1+I_2) \sim N\left(\mu,\left(1+k^2\right)\sigma^2\right), \text{ and}$$

$$\log\left((1+I_1)(1+I_2)\right)$$
$$=(1+k)\log(1+I_1)+(1-k)\mu+N \sim N\left(2\mu+2k+k^2\sigma^2\right)$$

That is, the accumulated value at time 2 of a unit investment at time 0 has a log-normal distribution with parameters 2μ and $\left(2\mu+2k+k^2\sigma^2\right)$.

In this model if you take $k=0$ it reduces to the iid model given above with mean μ. When $k=1$ we get the result that the centering of the second rate is around the first rate. On the other hand, k is assigned an intermediate value between 0 and 1 it represents different strengths of reversion to mean μ.

Check Your Progress 3

1) What is stochastic interest rate?

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2) In stochastic interest rate model, how do you formulate dependent annual interest rates?

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3) Write the meaning of 2-step dependent log-normal model

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7.5 LET US SUM UP

This unit deals with the cash-flow in insurance. Interest rate in deterministic and stochastic modelling has been discussed in that connection. Introducing the compounding methods, the effective interest rate has been derived for fixed as well as time-varying situations. Use of discounting to estimate the present (time) value of money is analysed and returns from investment has been compared. The expected value of interest with a single rate, year-wise independent and dependent rates is seen in stochastic interest rate modelling. The unit closes with a discussion on one of popular methods, i.e., log-normal distribution, used in stochastic interest rate modelling.

7.6 KEY WORDS

Cash Flow: Amounts of cash being received and spent by a business during a defined period of time, sometimes tied to a specific project.

Compound Interest: Interest added to the original principal. New interest is then calculated, not only on the principal, but also on the interest that has been added. The more frequently interest is compounded, the faster the principal grows

Discr The process of finding the present value of an amount of cash at

Effective Annual Rate of Discount: The amount of discount paid at the beginning of year when the anount invested at the end of the year is a unit amount.

Effective Annual Rate of Interest: Amount of money that one unit invested at the bigining of the year will earn during the year when amount earned is paid at the end of the year.

Force interest : Rate of instantaneous compounding.

Nominal Interest Rate: Interest rate not adjusted for compounding. An interest rate is called *nominal* if the period of time after that the interest is credited (e.g., a month) is not identical to the *basic time unit* (normally a year).

Present Value : Amount of money to change hands at some future date, discounted to account for the time value of money.

7.7 SOME USEFUL BOOKS

Butcher. M.V., Nesbitt. C.J. (1971), Mathematics of Compound Interest. Ulrich's Books. Ann. Arbor. Michigan

Salih N. Neftci.(1996), An Introduction to the Mathematics of Financial Derivatives, Academic Press. San Diego-California, USA

Slud Eric V. (2001), Actuarial Mathematics and Life-Table Statistics, Mathematics Department, University of Maryland, College Park (see internet)

Veeh, Jerry Alan (2001), Lecture Notes on Actuarial Mathematics (see Internet)

7.8 ANSWER OR HINTS TO CHECK YOUR PROGRESS

Check Your Progress 1

1) Since $i^{(m)} = m\left[(1+i)^{\frac{1}{m}} - 1\right]$ the limit can be evaluated directly using L 'Hospital' rule or definition of derivative.

2) $\left[1+4i^{\frac{1}{4}}\right]^{\frac{1}{4}} = \left(1-d^{\frac{(4)}{4}}\right)^{-4}$

3) Since d = iv which interest paying is moved back in time to the equivalent payment of iv at time 0.

4) Take $e^{\delta} = 1+i$

Check Your Progress 2

1) Read 7.3.1 and answer

2) Read 7.3.1 and answer
3) Read 7.3.2 and answer

Check Your Progress 3

1) Read Section 7.4 and answer
2) Read Sub-section 7.4.3 and answer
3) Read Sub-section 7.4.3 and answer

7.9 EXERCISES

1) (*From Assignments on the theme*) An insurance company calculates the single premium for a contract paying 10,000 in ten years' time as the present value of the benefits payable, at the expected rate of interest it will earn on its fund. The annual effective rate of interest over the whole of the next ten years will be 7%, 8% or 10% with probabilities 0.3, 0.5 and 0.2 respectively.

   a) Calculate the single premium.
   b) Calculate the expected profit at the end of the term of the contract.

2) (*From Assignments on the theme*) Let I_j denote the effective rate of interest in the year $j-1$ to j. It is assumed that, for $j \geq 1$

$$I_{j+1} = \begin{cases} I_j + 0.02 & & 0.25 \\ I_j & \textit{with probability} & 0.5 \\ I_j - 0.02 & & 0.25 \end{cases}$$

Given that $I_1 \equiv 0.06$, calculate the probability that an investment of 1 at time 0 accumulates to more than 1.2 at time 3.

3) (*From Assignments on the theme*) Let $1+I$ be a lognormal random variable with parameters μ and σ^2, mean J and variance s^2. Show that

$$\sigma^2 = \log\left(1+\left(\frac{s}{1+j}\right)^2\right) \textit{and } \mu = \log\left(\frac{1+j}{\sqrt{1+\left(\frac{s}{1+j}\right)^2}}\right)$$

For the exercises 4 and 5 you will need some vales of the $N(0,1)$ distribution. Let $Z \sim N(0,1)$, then

Z	-2.4778	-1.8349	-0.039	0.233
$P(Z \leq z)$	0.00661	0.03326	0.48445	0.59211

4) The rate of interest earned in the year form time $j-1$ to J is denoted by I_j. Assume $1+I_j$ is lognormal distribution. The expected value of the rate of interest is 5% and the standard deviation is 11%.

   a) Calculate the parameters of the lognormal distribution of $1+I_j$.

b)Calculate the probability that the rate of interest in the year from time $j-1$ to j lies between 4% and 7%.

5) A company is adopting a particular investment strategy such that the expected annual effective return from investments if 7% and the standard deviation of annual returns is 9%. Annual returns are independent and $(1+I_j)$ is log-normally distributed where I_j is the return in the j^{th} year. The company has received a premium of 10,000 and will pay the policyholder 1400 after 10 years.

6) Calculate the expected value and standard deviation of an investment of 1000 over 19 years, deriving all formulae that you use.

7) Calculate the probability that the accumulation of the investment will be less than 50% of its expected value in ten years' time.

8) The company has invested 1200 to meet its liability in 19 years time. Calculate the probability that it will have insufficient funds to meet its liability.

9) *(Due to Slud, 2001)* For how long a time should \$100 be left to accumulate at 5% interest so that it will amount to twice the accumulated value (over the same time period) of another \$100 deposited at 3%?

10) *(Due to Slud, 2001)* Suppose you sell for \$6,000 the right to receive for 10 years the amount of \$1,000 per year payable quarterly (beginning at the end of the first quarter). What effective rate of interest makes this a fair sale price? (You will have to solve numerically or graphically, or interpolate a tabulation, to find it.)

11) *(Due to Slud, 2001)* \$100 deposited 20 years ago has grown at interest to \$235. The interest was compounded twice a year. What were the nominal and effective interest rates?

NOTES