UNIT 11 CASH-FLOW PROJECTION

Structure

11.0 OBJECTIVES

After going through this unit, you will be able to:

- get some idea about cash-flow techniques and projection;
- know the benefits of life insurance contract;
- estimate the payment of premium; and
- apply stochastic interest rate models in actuarial economics.

11.1 INTRODUCTION

Under cash-flow projections two areas are fundamental. One is theory of *compound interest* and the other is *probability theory*.

In life insurance contract, the benefit of insured consists of a single payment and the sum assured. The time and the amount of payment are functions of the random variable T (or, $T(x)$) where T is the future life time of persons of age x. The present value of the payment is denoted by Z. The expected value of the payment $E(Z)$ is the net single premium of the contract. The premium does not reflect the risk to be carried by the insurer. In order to assess this, one requires the different characteristics of Z.

We are introducing the models of stochastic interest rate and some simple uses in actuarial applications.

11.2 THE MATHEMATICS OF COMPOUND INTEREST

11.2.1 Effective Interest Rates

An interest rate is always related with a *basic time unit* like an annual interest rate of 5%. With the interest rate, the time interval at the end of which interest rate is credited or "Compounded" has to be stated, which is known as *conversion period.*

An interest rate is called *effective* if the basic time unit and the conversion period are identical. In this case interest is credited at the end of basic time unit.

Let i be the effective interest rate and let F_0 be the initial capital. Let r_k be the additional amount invested at the end of year k , k = 1,2,...,n.

We want to find out the balance at the end of n years.

Suppose F_k is the balance at the end of year k including r_k, interest credited to the previous year's balance is $i\ F_{k-1}$.

So,

$$F_k = F_{k-1} + iF_{k-1} + r_k, k = 1,2,...,n \qquad (11.2.1)$$

which implies

$$F_k = (1+i)\ F_{k-1} + r_k \qquad (11.2.2)$$

Multiplying (11.2.2) by $(1+i)^{n-k}$ and summing over all values of k we obtain,

$$F_n = (1+i)^n F_0 + \sum_{i=1}^{n} (1+i)^{n-k} r_k \qquad (11.2.3)$$

The powers of $(1 + i)$ are called *accumulation factors*. The *accumulated value of an initial capital C* after n years is $(1+i)^n\ C$. Equation (11.2.3) gives the capital at the end of interval which is the sum of value of initial capital plus the sum of accumulated values of the intermediate deposits.

The *discount factor* is $v = \dfrac{1}{1+i}$

Equation (11.2.3) can be written as $v^n\ F_n = F_0 + \sum_{k=1}^{n} v^k\ r_k$ (11.2.4)

Hence the present value of a capital C due at time h is $v^h\ C$.

From equation (11.2.1), we have, $F_k - F_{k-1} = i\ F_{k-1} + r_k$ (11.2.5)

Now summing over k we get $F_n - F_0 = \sum_{k=1}^{n} \left(i F_{k-1} + r_k \right)$ (11.2.6)

i.e., *the increment of the fund is the sum of the total interest credited and the total deposits made.*

11.3 NOMINAL INTEREST RATES

If the conversion period does not coincide with the basic time unit, then the interest rate is called *nominal.*

An interest rate of 6% with conversion period 3 months means that interest of $\left(\frac{6\%}{4}\right) = 1\cdot 5\%$ is credited at the end of each quarter. Thus an initial capital of 1 increases to $(1\cdot 015)^4 = 1\cdot 06136$ at the end of one year. Therefore an annual nominal interest rate of 6% is equivalent to an annual effective interest rate of $6\cdot 136\%$.

11.3.1 Relation between Effective Interest Rate and Nominal Interest Rate

Let i denotes a given effective interest rate. We define $i^{(m)}$ as the nominal interest rate convertible m times per year which is equivalent to i.

Equality of accumulation factors for 1 year leads to the following relationship

$$\left(1 + \frac{i^{(m)}}{m}\right)^m = 1 + i \qquad (11.3.1)$$

$$\Rightarrow \qquad i^{(m)} = m\left[(1+i)^{\frac{1}{m}} - 1\right] \qquad (11.3.2)$$

The limiting case correspond to continuous compounding.

Let

$$\delta = \lim_{m \to \infty} i^{(m)} \qquad (11.3.3)$$

It is called the *force of interest* equivalent to i.

So from equation (11.3.2) we have

$$i^{(m)} = \frac{(1 + i)^{\frac{1}{m}} - (1+i)^0}{\frac{1}{m}} \qquad (11.3.4)$$

i.e., δ is a derivative of the function $(1+i)^x$ *at* $x = 0$.

So,

$$\delta = \ln(1+i) \qquad (11.3.5)$$

or

$$e^{\delta} = 1 + i \qquad (11.3.6)$$

From this we conclude that the accumulation factor for a period of h years is $(1+i)^h = e^{\delta h}$ and discount factor for the same period is $v^h = e^{-\delta h}$ where h, the length of the period is any real number.

Example: A deposit of 100,000 is made into a newly established fund. The fund pays nominal interest of 12% convertible quarterly. At the end of each six months, a withdrawal is made from the fund. The first withdrawal is X, the second is 2X, the third is 3X and so on. The last is sixth withdrawal which exactly exhausts the fund. Calculate X.

Solution: Taking the notation of the text to half years we get $F_0 = 100{,}000$ and $F_1 = F_0(1 \cdot 03)^2 - X$ and so on. This can be taken using (11.1.2). The 6th withdrawal exactly exhausts the fund. So $F_6 = 0$. Solving this we get X = 6128.

11.4 DISCOUNTED CASH-FLOW TECHNIQUES

One of the techniques is given by the equation (11.2.4) where the interest is credited at the end of each conversion period in arrears. The other technique is given below, where the interest is credited at the beginning of the conversion period. Interest credited in this way is also referred to as *discount* and the corresponding rate is called *discount rate* or *rate of interest in advance*. Let d be an annual effective discount rate. A person investing an amount of C will be credited interest equal to dC immediately and the invested capital C will be returned at the end of the period. Investing the interest dC at the same conditions, the investor will receive additional interest of $d(dC) = d^2C$, and the additional invested amount will be returned at the end of the year; reinvesting, the interest yields additional interest of $d(d^2C) = d^3\,C$ and so on.

Repeating this process infinitely we find the investor will receive the total sum of

$$C + dC + d^2C + \ldots + \ldots = \frac{1}{1-d}C \qquad (11.4.1)$$

at the end of the year in return for investing the initial capital C. The equivalent effective interest rate i is given by the equation.

$$\frac{1}{1-d} = 1 + i \qquad (11.4.2)$$

which implies

$$d = \frac{i}{1+i} \qquad (11.4.3)$$

This result has the following interpretation. If 1 unit of capital is invested, d (the interst payable at the beginning of the year) is the discounted value of the interst i to be paid at the end of year.

Further, $$i = \frac{d}{1-d}$$

Thus the interest payable at the end of the year is the accumulated value of interest payable at the beginning of the year.

In case of nominal interest rate, we have

$$\frac{1}{1-\frac{d^{(m)}}{m}} = 1+\frac{i^{(m)}}{m} = (1+i)^{\frac{1}{m}} \tag{11.4.5}$$

which gives $$d^{(m)} = \frac{i^{(m)}}{1+\frac{i^{(m)}}{m}} \tag{11.4.6}$$

or $$\frac{1}{d^{(m)}} = \frac{1}{m} + \frac{1}{i^{(m)}} \tag{11.4.7}$$

and $$\lim_{m \to \infty} d^{(m)} = \lim_{m \to \infty} i^{(m)} = \delta \tag{11.4.8}$$

when the interest is compounded continuously the difference between interest in advance and interest in arrears vanishes.

11.5 VALUATION OF CASH-FLOWS OF SIMPLE INSURANCE CONTRACT

In this section we will discuss elementary types of insurance.

11.5.1 Whole Life and Term Insurance

Consider a *whole life insurance.* This provides for payment of 1 unit at the end of year of death. In this case the amount of the payment is fixed, while the time of payment $(K+1)$ is random. Its present value is

$$Z = v^{K+1} \tag{11.5.1}$$

The random variable Z ranges over the values $v, v^2, v^3, \ldots$ and the distribution of Z is determined by (11.5.1) whereas the distribution of K is

$$\Pr\left(Z = v^{(K+1)}\right) = P(K=k) = {}_kp_x\, q_{x+k}, k = 0,1,2,\ldots \tag{11.5.2}$$

where ${}_kp_x$ is the probability that a person aged x survives upto age $x+k$ and q_{x+k} is the probability that a person of exact age $x+k$ will die within one year following the attainment of that age. The net single premium is denoted by A_x and given by

$$A_x = E(Z) = E\left(v^{K+1}\right) = \sum_{k=0}^{\infty} v^{k+1}\, {}_kp_x q_{x+k} \tag{11.5.3}$$

Var(Z) can be calculated by the identity

$$V(Z) = E\left(Z^2\right) - \{E(Z)\}^2$$
$$= E\left(Z^2\right) - A_x^2 \tag{11.5.4}$$

Replacing v by $e^{-\delta}$ we see that

$$E\left(Z^2\right) = E\left[e^{-2\delta(K+1)}\right] \tag{11.5.5}$$

which is the net single premium calculated at twice the original force of interest. Thus calculating the variance is no more difficult than calculating the net single premium.

An insurance which provides for payment only if death occurs within n years is known as *term insurance of duration n.* For example: 1 unit is payable only if death occurs during the first n years, the actual time of payment still being the end of the year of death. So

$$Z = \begin{cases} v^{K+1} \text{ for } & K = 0,1,2, \ \ldots, \ n-1 \\ 0 \quad \text{for} & K = n, \ n+1, \ n+2, \ldots \end{cases} \tag{11.5.6}$$

The net single premium is denoted by

$$A^{1}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^{k+1} \, {}_kp_x q_{x+k} \tag{11.5.7}$$

and

$$E\left(Z^2\right) = E\left[e^{-2\delta(K+1)}\right] \tag{11.5.8}$$

11.5.2 Pure Endowments

A pure endowment of duration n provides for payment of the sum insured only if the insured is alive at the end of n years,

$$Z = \begin{cases} 0 \quad \text{for} & K = 0,1,\ldots,n-1 \\ v^n \quad \text{for} & K = n, \ n+1, n+2,\ldots \end{cases} \tag{11.5.9}$$

The net premium is denoted by $A_{x,\overline{n}|}^{\ \ 1}$ and is given by

$$A_{x,\overline{n}|}^{\ \ 1} = v^n \, {}_np_x \tag{11.5.10}$$

The formula for the variance of a Bernoulli random variable gives

$$Var(Z) = v^{2n} \cdot {}_np_x \cdot {}_nq_x \tag{11.5.11}$$

11.5.3 Endowments

Assume that the sum insured is payable at the end of the year of death, if this occurs within the first n years, otherwise at the end of the nth year

$$Z = \begin{cases} v^{K+1} \text{ for } K = 0,1,2,\ldots,n-1 \\ v^n \text{ for } K = n, \ n+1, \ n+2, \ldots \end{cases} \tag{11.5.12}$$

The net single premium is denoted by $A_{x:\overline{n}|}$. Denoting the present value of (11.5.6) by Z_1 and (11.5.9) by Z_2, we have

$$Z = Z_1 + Z_2 \tag{11.5.13}$$

As a consequence

$$A_{x:\overline{n}|} = A^{1}_{x:\overline{n}|} + A_{x,\overline{n}|}^{\ \ 1} \qquad (11.5.14)$$

and

$$V(Z) = V(Z_1) + V(Z_2) + 2Cov(Z_1, Z_2) \qquad (11.5.15)$$

The product Z_1, Z_2 is always zero, hence

$$Cov(Z_1,\ Z_2) = E(Z_1, Z_2) - E(Z_1),\ E(Z_2)$$

$$= -A^{1}_{x:\overline{n}|}\ A_{x,\overline{n}|}^{\ \ 1} \qquad (11.5.16)$$

Hence

$$V(Z) = V(Z_1)\ + V(Z_2)\ - 2A^{1}_{n:\overline{n}|}\ A_{x,\overline{n}|}^{\ \ 1} \qquad (11.5.17)$$

As a consequence of the last identity the risk in selling an endowment policy, measured by the variance is less than that in selling a term insurance to one person and a pure endowment to another.

Let us finally consider an m year *deferred whole life insurance*. Its present value is

$$Z = \begin{cases} 0 & for \quad K = 0, 1, \ldots, m-1 \\ v^{K+1} & for \quad K = m, m+1, m+2, \ldots \end{cases} \qquad (11.5.18)$$

The net single premium is denoted by $_{m|}A_x$. Alternative formula for its net single premium are

$$_{m|}A_x = {}_m p_x v^m A_{x+m} \qquad (11.5.19)$$

and

$$_{m|}A_x = A_x - A^{1}_{x:\overline{m}|} \qquad (11.5.20)$$

The second moment $E(Z^2)$ again equals the net single premium at twice the original force of interest.

11.5.4 Insurance Payable at the Moment of Death

In previous sections we have discussed that the sum insured was payable at the end of the year of death. This assumption does not reflect insurance practice in a realistic way, but has the advantage that the formulae may be evaluated directly from a life table.

Let us now assume that the sum insured is payable at the instant of death. i.e. at time T. The present value of a payment of payable at the instant of death is at time T. The present value of a payment of 1 payable immediately on death is

$$Z = v^T \qquad (11.5.21)$$

The net single premium is denoted by A_x. Using the formula from force of mortality

$$\overline{A}_x = \int_0^\infty v^t \; {}_tp_x \; \mu_{x+t} \, dt \qquad (11.5.22)$$

The force of mortality at age x is defined as the ratio of instantaneous rate of decrease in l_x to the value of $l_x \cdot is$. $\mu_x = -\frac{1}{l_x}\frac{d}{dx} l_x = -\frac{d}{dx}(\log l_x)$. l_x, is the number of person's living at age x.

One practical approximation of T is given by

$$T = K+S = (K+S) = (K+1)-(1-S) \qquad (11.5.23)$$

where K is the number of complete future years lived by (x) and S is the fraction of a year during which (x) is alive in the year of death.

Making use of assumed independence of K and S, as well as the uniform distribution of S, so that

$$E\left[(1+i)^{1-S}\right] = \int_0^1 (1+i)^u \, du = \overline{S}_{\overline{1}|} = \frac{i}{\delta} \qquad (11.5.24)$$

we find

$$\overline{A}_x = E\left[v^{K+1}\right] E\left[(1+i)^{1-S}\right] = \frac{i}{\delta} A_x \qquad (11.5.25)$$

Thus calculation of $\overline{A}_x$ is a simple extension of A_x.

A similar formula may be derived for term insurance. For endowments the factors $\frac{i}{\delta}$ is only used in the term insurance part.

$$\overline{A}_{x:\overline{n}|} = \overline{A}^{1}_{x:\overline{n}|} + A_{x:\overline{n}|}{}^{\!\!1} = \frac{i}{\delta} A^{1}_{x:\overline{n}|} + A_{x:\overline{n}|}{}^{\!\!1}$$

$$= A_{x:\overline{n}|} + \left(\frac{i}{\delta} - 1\right) A^{1}_{x:\overline{n}|} \qquad (11.5.26)$$

Let us finally assume that the sum insured is payable at the end of the *mth* part of the year in which death occurs i.e. time $K + S^{(m)}$. The present value of a whole life insurance of 1 unit then becomes

$$Z = v^{K+S^{(m)}} \qquad (11.5.27)$$

and $$K + S^{(m)} = (K+1) - \left(1 - S^{(m)}\right) \qquad (11.5.28)$$

So,

$$E\left[(1+i)^{1-S^{(m)}}\right] = S^{(m)}_{\overline{1}|} = \frac{i}{i^{(m)}} \qquad (11.5.29)$$

and we obtain

$$A_x^{(m)} = E\left[v^{K+1}\right] E\left[(1+i)^{1-S^{(m)}}\right] = \frac{i}{i^{(m)}} A_x . \quad (11.5.30)$$

Example: Z_1 is the present value random variable for an n-year continuous endowment insurance of 1 issued to (x). Z_2 is the present value random variable for an n year continuous term insurance of 1 issued to (x). Given $Var(Z_2) = 0.01,\ v^n = 0\cdot 3,\ {}_np_x = 0.8, E(Z_2) = 0\cdot 04$, calculate $Var(Z_1)$.

Solution: Let Z_3 be the present value random variable for the pure endowment

$Z_1 = Z_2 + Z_3$

we want to find out $Var(Z_1)$.

But $\quad Var(Z_1) = Var(Z_2 + Z_3)$

$= Var(Z_2) + 2\operatorname{cov}(Z_2, Z_3) + Var(Z_3).$

$Cov(Z_2, Z_3) = -E(Z_2) \cdot E(Z_3)\ (\because Z_2 Z_3 = 0)$

$$Z_3 = \begin{cases} v^n \cdot 1 \ \textit{with probability}\ 1 \\ \\ 0 \ \textit{otherwise} \end{cases}$$

Now

$E(Z_3) = v^n\ {}_np_x = (0\cdot 3)(0\cdot 8) = (0\cdot 24)$

$Var(Z_3) = v^{2n}\ {}_np_x(1 - {}_np_x)$

$= (0\cdot 3)^2\ (0\cdot 8)\ (0\cdot 2)$

$= (0\cdot 09)(0\cdot 16) = 0\cdot 0144$

$Var(Z_1) = 0\cdot 01 - 2(0\cdot 04)(0\cdot 24) + 0\cdot 01484$

$= 0\cdot 0244 - 0\cdot 192 = 0\cdot 0052.$

Check Your Progress 1

1) What is the difference between effective interest rate and nominal interest rate? Find the relation between them.

....

....

....

....

2) For m = 1,2,3,4,6,12,and ∞ calculate $i^{(m)}$ for i = 6%.

....

....

....

....

3) Z_1 is the present value random variable for an n-year continuous endowment insurance of 1 issued to (x). Z_2 is the present value random variable for an n-year continuous term insurance of 1 issued to (x). Given values $Var(Z_2)$, ${}_np_x$, $E(Z_2)$ obtain a formula to find out $Var(Z_1)$.

....

....

....

11.6 ANALYSIS AND VALUATION OF ANNUITIES, BONDS, LOANS AND OTHER SECURITIES

To have a clear idea on annuity and its analysis we introduce certain types of perpetual streams known as perpetuities. The resulting formulae are very simple and will later be useful for calculating the present value of annuities with a finite term.

11.6.1 Perpetuities

First we consider perpetuities consisting of annual payments of 1 unit. If the first payment occurs at time zero, the perpetuity is called *perpetuity-due* and its present value is denoted by $\ddot{a}_{\overline{\infty}|}$. Thus

$$\ddot{a}_{\overline{\infty}|} = 1 + v + v^2 + \ldots = \frac{1}{1-v} = \frac{1}{d}. \tag{11.6.1}$$

If the first payment is made at the end of year 1, we call the perpetuity an *immediate perpetuity*. Its present value is denoted by $a_{\overline{\infty}|}$ and is given by

$$a_{\overline{\infty}|} = v + v^2 + \ldots = \frac{v}{1-v} = \frac{1}{i} \tag{11.6.2}$$

Let us now consider perpetuity where payments of $\frac{1}{m}$ are made m times each year. If the payments are made in advance (first payment of $\frac{1}{m}$ at time 0), the present value is denoted by $\ddot{a}^{(m)}_{\overline{\infty}|}$ and is

$$\ddot{a}^{(m)}_{\overline{\infty}|} = \frac{1}{m} + \frac{1}{m} v^{\frac{1}{m}} + \frac{1}{m} v^{\frac{2}{m}} + \ldots = \frac{1}{m} \frac{1}{\left(1 - v^{\frac{1}{m}}\right)} = \frac{1}{d^{(m)}} \tag{11.6.3}$$

If the payments are made in arrears (first payment of $\frac{1}{m}$ at time 0), the present value is denoted by $a^{(m)}_{\overline{\infty}|}$ and given by

$$a^{(m)}_{\overline{\infty}|} = \frac{1}{m} v^{\frac{1}{m}} + \frac{1}{m} v^{\frac{2}{m}} + \frac{1}{m} v^{\frac{3}{m}} + \ldots$$

$$= \frac{1}{m} \frac{v^{\frac{1}{m}}}{\left(1 - v^{\frac{1}{m}}\right)} = \frac{1}{m\left[(1+i)^{\frac{1}{m}} - 1\right]} = \frac{1}{i^{(m)}} \tag{11.6.4}$$

The results in (11.6.3) and (11.6.4) lead to an interpretation of the identity (11.4.7). Since a perpetuity-due and an immediate perpetuity differ only by a payment of $\frac{1}{m}$ at time 0, their present value differ by $\frac{1}{m}$.

Let us now consider a continuous perpetuity with constant rate of payment r = 1 and starting at time 0. Its present value is denoted by $\ddot{a}_{\overline{\infty}|}$ and given by

$$\ddot{a}_{\overline{\infty}|} = \int_0^\infty e^{-\delta t} dt = \frac{1}{\delta} \tag{11.6.5}$$

Letting $m \to \infty$ in (11.6.4) or (11.6.3) we will get the same result.

The systematic pattern in formulae (11.6.1) to (11.6.5) is evident.

A certain type of perpetuities with increasing payments is defined by two parameters m and q. i.e. the number of payments per year and the number of increase per year. We assume that q is a factor of m. if $m = 12$, and $q = 4$, it means that payments are made monthly and increase quarterly. In general, the payments of such as increasing perpetuity due are defined as follow.

Time				Payment
0	$\frac{1}{m}$	$\cdots$	$\frac{1}{q} - \frac{1}{m}$	$\frac{1}{mq}$
$\frac{1}{q}$	$\frac{1}{q} + \frac{1}{m}$	$\cdots$	$\frac{2}{q} - \frac{1}{m}$	$\frac{2}{mq}$
$\frac{2}{q}$	$\frac{2}{q} + \frac{1}{m}$	$\cdots$	$\frac{3}{q} - \frac{1}{m}$	$\frac{3}{mq}$

and so on

In particular, the last $\frac{m}{q}$ payments of year k are $\frac{k}{m}$ each.

Let us denote the present value of such a perpetuity by $\left(I^{(q)} \ \ddot{a}\right)^{(m)}_{\overline{\infty}|}$. We can calculate it by representing the sequence of increasing payments as sum of perpetuities with constants payments of $\frac{1}{(mq)}$ payable m times per year and beginning at times $0, \frac{1}{q}, \frac{2}{q}, \ldots$. Thus we obtain the following formula:

$$\left(I^{(q)}\ddot{a}\right)^{(m)}_{\overline{\infty}|} = \frac{1}{q}\ddot{a}^{(m)}_{\overline{\infty}|}\left[1+v^{\frac{1}{q}}+v^{\frac{2}{q}}+\dots\right]$$

$$= \ddot{a}^{(m)}_{\overline{\infty}|}\cdot\ddot{a}^{(q)}_{\overline{\infty}|} = \frac{1}{a^{(m)}}\cdot\frac{1}{d^{(q)}} \qquad (11.6.6)$$

The corresponding immediate annuity differs only in that each payment is made one *mth* year later, thus giving

$$\left(I^{(q)}a\right)^{(m)}_{\overline{\infty}|} = v^{\frac{1}{m}}\left(I^{(q)}\ddot{a}\right)^{(m)}_{\overline{\infty}|} = \frac{v^{\frac{1}{m}}}{d^{(m)}}\cdot\frac{1}{d^{(q)}} = \frac{1}{i^{(m)}}\cdot\frac{1}{d^{(q)}} \qquad (11.6.7)$$

A superscript 1 is always omitted. For instance, the present value of a perpetuity due with annual payments of 1,2,... is

$$\left(I\ddot{a}\right)_{\overline{\infty}|} = \left(I^{(1)}_{\ddot{a}}\right)^{(1)}_{\overline{\infty}|} = \frac{1}{d^2} \qquad (11.6.8)$$

For continuous payment streams

$$\left(I\ddot{a}\right)_{\overline{\infty}|} = \int_0^\infty te^{-\delta t}dt = \frac{1}{\delta^2} \qquad (11.6.9)$$

$$\left(I\ddot{a}\right)_{\overline{\infty}|} = \int_0^\infty [t+1]e^{-\delta t}dt = \frac{1}{\delta d} \qquad (11.6.10)$$

without actually calculating the integrals.

But when the payments are arbitrary with annual payments r_0, r_1, r_2,... (at times 0,1,2,...), its present value is denoted by $\ddot{a}$.

$$\ddot{a} = r_0 + vr_1 + v^2r_2 \qquad (11.6.11)$$

Such a variable perpetuity may be represented as a sum of constant perpetuities in the following way.

Annual Payments	Starts at time
r_0	0
$r_1 - r_0$	1
$r_2 - r_1$	2
$r_3 - r_2$	3
and so an	

The present value of this perpetuity may therefore be expressed as

$$\ddot{a} = \frac{1}{d}\left\{r_0 + v\left(r_1 - r_0\right) + v^2\left(r_2 - r_1\right)\dots\right\}. \qquad (11.6.12)$$

We can use (11.6.11) to calculate the present value of exponentially growing payments with $r_k = e^{\tau k}$ for k = 0,1,2,... and obtain (11.6.13)

$$\ddot{a} = \frac{1}{1 - e^{-(\delta - \tau)}} \tag{11.6.14}$$

provided $\tau < \delta$.

Annuities

In practice annuities are more frequently used than perpetuities. An annuity is defined as a sequence of payments of a limited duration, which we denote by n. Here we consider some standard type of annuities or annuities-certain.

The present value of an annuity due with n-annual payments of 1 starting at time 0, is denoted by $\ddot{a}_{\overline{\infty}|}$. It is given by

$$\ddot{a}_{\overline{n}|} = 1 + v + v^2 + \ldots + v^{n-1} \tag{11.6.15}$$

Representing, the annuity as the difference of two perpetuities (one starting at time 0, the other at time n), we find that

$$\ddot{a}_{\overline{n}|} = \ddot{a}_{\overline{\infty}|} - v^n \ddot{a}_{\overline{\infty}|} = \frac{1}{d} - v^n \frac{1}{d} = \frac{1 - v^n}{d} \tag{11.6.16}$$

This result can be verified by evaluating the geometric sum (11.6.15). Similarly from (11.6.2), (11.6.3) and (11.6.4) the following formulae can be obtained

$$a_{\overline{n}|} = \frac{1 - v^n}{i} \tag{11.6.17}$$

$$\ddot{a}_{\overline{n}|}^{(m)} = \frac{1 - v^n}{d^{(m)}} \tag{11.6.18}$$

$$a_{\overline{n}|}^{(m)} = \frac{1 - v^n}{i^{(m)}} \tag{11.6.19}$$

Note that only the denominator varies, depending on the payment made (immediate / due) and frequency. Note that n must be an integer in (11.6.16) and (11.6.17), and a multiple of $\frac{1}{m}$ in (11.6.18) and (11.6.19).

The final or accumulated value of annuities is also of interest. This is defined as the accumulated value of the payment stream at time n, and the usual symbol used is S. The final value is obtained by multiplying the initial value with the accumulator factor

$$\ddot{S}_{\overline{n}|} = \frac{(1 + i)^n - 1}{d} \tag{11.6.20}$$

$$S_{\overline{n|}} = \frac{(1+i)^n - 1}{i} \tag{11.6.21}$$

$$\ddot{S}_{\overline{n|}}^{(m)} = \frac{(1+i)^n - 1}{d^{(m)}} \tag{11.6.22}$$

$$S_{\overline{n|}}^{(m)} = \frac{(1+i)^n - 1}{i^{(m)}} \tag{11.6.23}$$

Another simple relation between the initial value and final value of a constant annuity may be verified.

$$\frac{1}{a_{\overline{n|}}} = \frac{1}{S_{\overline{n|}}} + i \tag{11.6.24}$$

Let us now consider an increasing annuity due with parameter q and m.

Time				Payment
0	$\frac{1}{m}$	...	$\frac{1}{q}-\frac{1}{m}$	$\frac{1}{mq}$
$\frac{1}{q}$	$\frac{1}{q}+\frac{1}{m}$	...	$\frac{2}{q}-\frac{1}{m}$	$\frac{2}{mq}$
$\frac{2}{q}$	$\frac{2}{q}+\frac{1}{m}$	...	$\frac{3}{q}-\frac{1}{m}$	$\frac{3}{mq}$
...	...	...	...	...
$n-\frac{1}{q}$	$n-\frac{1}{q}+\frac{1}{m}$	...	$n-\frac{1}{m}$	$\frac{n}{m}$

Such an increasing annuity can be represented as an increasing perpetuity starting at time 0, minus an identical increasing perpetuity starting of time *n*, minus a constant annuity starting at *n*.

$$\left(I^{(q)}\ddot{a}\right)_{\overline{n|}}^{(m)} = \left(1^{(q)}\ddot{a}\right)_{\overline{\infty|}}^{(m)} v^n \left(I^{(q)}\ddot{a}\right)_{\overline{\infty|}}^{m} - v^n n\ddot{a}_{\overline{\infty|}}^{(m)} \tag{11.6.25}$$

Substituting (11.6.6) and (11.6.3) and using (11.6.19) we obtain the equation

$$\left(I^{(q)}\ddot{a}\right)_{\overline{n|}}^{(m)} = \frac{\ddot{a}_{\overline{n|}}^{(q)} - nv^n}{d^{(m)}} \tag{11.6.26}$$

Similarly the present value of the corresponding immediate annuity is calculated as

$$\left(I^{(q)}a\right)_{\overline{n|}}^{(m)} = \frac{\ddot{a}_{\overline{n|}}^{(q)} - nv^n}{i^{(m)}} \tag{11.6.27}$$

In these equations n must be a multiple of $1/q$. Special cases are the combinations of $m = 1$ and $q = 1$, $m = 12$ and $q = 12$, $m = \infty$ and $q = 1$, and $m = \infty$ and $q = \infty$.

The annuities just considered are known as *standard increasing annuities* (I). *Standard decreasing annuities* (D) are similar, but the payments are made in the reversed order. The sum of these two annuities is a constant annuity. This relation carries over to the present values and we obtain

$$\left(I^{(q)}\ddot{a}\right)_{\overline{n}|}^{(m)} + \left(D_{\ddot{a}}^{(q)}\right)_{\overline{n}|}^{(m)} = \left(n + \frac{1}{q}\right)\ddot{a}_{\overline{n}|}^{(m)} \tag{11.6.28}$$

Using (11.6.26) and (11.6.27)

$$\ddot{a}_{\overline{n}|}^{(q)} = a_{\overline{n}|}^{(q)} + \left(1 - v^n\right)\frac{1}{q} \tag{11.6.29}$$

we get

$$\left(D^{(q)}\ \ddot{a}\right)_{\overline{n}|}^{(m)} = \frac{n - a_{\overline{n}|}^{(q)}}{d^{(m)}} \tag{11.6.30}$$

The standard decreasing annuity due may be interpreted as a constant perpetuity with m^{th} ly payments of n/m, minus a series of deferred perpetuities-due, each with constant m^{th} ly payments of $\left(\frac{1}{mq}\right)$ and starting at times $\frac{1}{q}, \frac{2}{q}, \ldots, n$.

Loans, Bonds and Other Securities

Let S be the value at 0 of a loan that is to be repaid by payments $r_1, r_2, \ldots, r_n$, made at the end of years $k = 1, 2, \ldots, n$. Then S must be the present value of the payments.

$$S = vr_1 + v^2 r_2 + \ldots + v^n r_n \tag{11.6.31}$$

Let S_k be the principal outstanding i.e., the remaining debt immediately after r_k has been paid. It consists of the previous years debt, accumulated for one year minus r_k.

$$S_k = (1 + i)\ S_{k-1} - r_k,\ k = 1, 2, \ldots, n \tag{11.6.32}$$

or, $$r_k = i S_{k-1} + \left(S_{k-1} - S_k\right) \tag{11.6.33}$$

It consists of two components, one is interest on the *running debt* and *reduction of principal*. Substituting $-S_k$ for F_k (11.6.32) is equivalent to (11.2.2). From (11.2.3)

$$S_k = (1 + i)^k\ S - \sum_{h=1}^{k} (1 + i)^{k-h}\ r_k \tag{11.6.34}$$

From this one can get $S_n = 0$ and $S_k = v\, r_{k+1} + v^2 r_{k+2} + \ldots + v^{n-k} r_n$ (11.6.35)

Formula (11.6.34) is the *retrospective formula* and (11.6.35) is *the prospective formula* for the outstanding principal.

The payments $r_1, r_2, \ldots, r_k$ may be chosen arbitrarily, subject to the constraint (11.6.31). By proper choice of payment steam the following formula of annuities can be obtained. For example, $S = 1$ can be repaid by the payments, The payments $r_1, r_2, \ldots, r_k$ may be chosen arbitrarily, subject to the constraint (11.6.31). By proper choice of payment steam the following formula of annuities can be obtained. For example, $S = 1$ can be repaid by the payments,

$$r_1 = r_2 = \ldots = r_{n-1} = i,\ r_n = 1 + i \tag{11.6.36}$$

In this case only interest is paid for the first (n – 1) years and the entire debt together with the last year's interest, is paid of the end of the n^{th} year. From (11.6.31)

$$1 = i\, a_{\overline{n|}} + v^n \tag{11.6.37}$$

The debt $S = 1$ may also be repaid by constant payments of

$$r_1 = r_2 = \ldots = r_n = \frac{1}{a_{\overline{n|}}} \tag{11.6.38}$$

As an alternative to repaying the creditor at times $1,2,\ldots,n$-1, one could pay only the interest on S as in (11.6.36). In order to cover the final repayment one can make equal deposits to a fund that is to accumulate to 1 of the end of n years, from this it is obvious that the annual deposit must be $\frac{1}{S_n}$. Since the total annual outgo must be the same in both the cases and we arrive at

$$\frac{1}{a_{\overline{n|}}} = \frac{1}{S_{\overline{n|}}} + i.$$

Suppose now that we repay a debt of $S = n$, so that the principal outstanding decreases linearly to D, $S_k = n - k$ for $k = 0, 1,2,\ldots,n$. From (11.6.33) it is evident that $r_k = i(n-k+1) + 1$. Using (11.6.31)

$$n = i(Da)_{\overline{n|}} + a_{\overline{n|}} \tag{11.6.39}$$

$$(Da)_{\overline{n|}} = \frac{n - a_{\overline{n|}}}{i} \tag{11.6.40}$$

This result is a special case ($m = q = 1$) of (11.6.30)

The loan itself may consist of a series of payments. Assume that equal payments of 1 are received by the debtor at times $0,1,2,\ldots,n$-1. At the end of each year interest on the received amounts is paid and the total amount received is repaid at time n.

$$r_k = ik \ \ for \ \ k = 1,2,\ldots,n-1,\ r_n = in + n \tag{11.6.41}$$

From the equality of present values

$$\ddot{a}_{\overline{n|}} = i(Ia)_{\overline{n|}} + nv^n \tag{11.6.42}$$

For security, suppose an investor pays a price p which entitles him to n future payments. The payments are denoted by $r_1, r_2, \ldots, r_n$ and payment r_k is due at time T_k, for $k = 1,2,\ldots,n$. What is the resulting rate of return?

The present value of the payment stream to be received by the investor is a function of the force of interest δ. Define

$$a(\delta) = \sum_{k=1}^{n} \exp(-\delta T_k)\, r_k \tag{11.6.43}$$

Let t be the solution of the equation

$$a(t) = p \tag{11.6.44}$$

The *internal rate of return* or *investment yield* is defined as $i = e^t - 1$.

Equation (11.6.44) can be solved by standard numerical methods such as interval bisection or the Newton-Raphson method. We shall present a more efficient and simpler method than other methods.

Consider

$$f(\delta) = \ln\left(\frac{a(\delta)}{r}\right), \tag{11.6.45}$$

here $r = r_1 + \ldots + r_n$ denotes the undiscounted sum of payments. It is not difficult to verify that

$$f(0) = 0, \; f'(\delta) = \frac{a'(\delta)}{a(\delta)} < 0$$

$$f''(\delta) = \frac{a''(\delta)}{a(\delta)} - \left(\frac{a'(\delta)}{a(\delta)}\right)^2 > 0 \tag{11.6.46}$$

The last inequality may be verified by interpreting $f''(\delta)$ as a variance. Interpreting $\frac{f(\delta)}{\delta}$ as the slope of the secant and noting that f is a convex function by (11.6.46) we see that $\frac{f(\delta)}{\delta}$ is an increasing function of δ. Hence for $0 < s < t < u$,

$$\frac{f(s)}{s} < \frac{f(t)}{t} < \frac{f(u)}{u} \tag{11.6.47}$$

giving
$$\frac{f(t)}{f(s)} \cdot s < t < \frac{f(t)}{f(u)} \cdot u \tag{11.6.48}$$

Thus we have proved that

$$\frac{\ln\left(p/r\right)}{\ln\left(a(s)/r\right)} s < t < \frac{\ln\left(p/r\right)}{\ln\left(a(u)/r\right)} u \tag{11.6.49}$$

The process may be iterated yielding the following algorithm. Starting with δ_0, calculate

$$\delta_{k+1} = \frac{\ln\left(p/r\right)}{\ln\left(a(\delta_k)/r\right)} \delta_k, \; k = 0,1,\ldots \tag{11.6.50}$$

recursively.

For $\delta_0 < t$, the resulting sequence will be monotonically increasing and will converge to t. For $\delta_0 > t$, the sequence will decrease monotonically to t. Thus any arbitrary positive value may be chosen for δ_0.

Example: Security = Rs.5000/- (Face amount) yearly coupons = Rs.300/-. Assume that the security has been bought for Rs.5250 for a remaking running time of 9 years. Thus we have

$$P = 5250$$

$$T_k = k(k = 1,2,\ldots,9)$$

$$r_k = 300(k = 1,2,\ldots,8)$$

$$r_9 = 5300$$

$$r = 7700.$$

Assuming we know that the current yield of similar securities lies between 5% and 5.5% we have $s = \ln 1\cdot 05$, $u = \ln 1\cdot 055$. The bounds for the investment yield is

$$0\cdot 051461 < t < 0\cdot 051572$$

and

$$5\cdot 2808\% < i < 5\cdot 2925\% \quad (i = i' - 1) \tag{11.6.51}$$

The algorithm defined by (11.6.50) may be used to obtain greater precision.

11.7 YIELD CURVES

There are many classes of bonds in financial markets. For example, we can restrict ourselves to various risk classes, such as Treasury bonds, municipal bonds, corporate bonds or junk bonds. Given the class of Treasury bonds, we see Treasury bills that mature within a year, Treasury notes with maturities between one and five years and Treasury bonds with maturities extending to 30years.

If risk-fee instantaneous interest rates were constant at r, the time t price of a pure discount bond with maturity u would be given by

$$B(u,t) = 100e^{-r(u-t)} \tag{11.7.1}$$

where $B(\cdot)$ denotes the present value of 100 discounted at a rate r. A pure discount bond does not make any coupon payment, does not contain any implicit option and has a par value of 100. The function $e^{-r(u-t)}$ at time t plays the role of a discount factor. At time

$$t = u \tag{11.7.2}$$

the bond matures and the exponential function equals 1. At all t < u, it is less than 1.

The bond price formula depends on the whole spectrum of future short rates. Hence the yield curve at time t contains all the available information concerning future short rates and the bond prices depend on the whole yield curve or on the term structure of interest rates, which we define below.

Definition: Assume that at time t there exists zero coupon bonds with a spectrum of maturities $u \in [t,T]$. Let their price be $B(u,t)$ and their yield be given by R_t^u. Then the spectrum of yields $\{R_t^u, u \in [t, T]\}$ is called the term structure of interest rates.

Here the yield R_t^u is defined as the number that satisfies the equality

$$B(u,t) = 100e^{-R_t^u(u-t)}, t < u \tag{11.7.3}$$

or
$$R_t^u = \frac{\log B \cdot (u,t) - \log 100}{t - u} \tag{11.7.4}$$

Examples of yield curves

In practice, there are two ways of proceedings. Often a market participant assumes a functional form for the yield curve and from there obtains the implied forward rates. A second method is to go the opposite direction. One can assume a dynamic behavior for the forward rates and from there obtain a yield curve. In most common case one assumes that the yields $R(\cdot)$ depends on one variable r_t which represents the short rate observed now, and they are given by the functional form.

$$R(r_t, u, t) = A(u, t) - C(u,t)\, r_t \tag{11.7.8}$$

Functions $A(u,t)$ and $C(u,t)$ can be constructed in various ways, so that the yield curve can be upword, downward sloping or hump shaped.

11.8 INTRODUCTION TO STOCHASTIC INTEREST RATE MODELS AND ACTUARIAL APPLICATIONS

The interest rate that will apply in future years is of course not known. Thus it seems reasonable to ask why future interest rates have not been modelled as stochastic process. Two reasons are there: (1) Life insurance is particularly concerned with the long term development of interest rates and no commonly employed stochastic models exist for making long term predictions. (2) A reasonable assumption is that the remaining life times of the insured lives are essentially independent random variables. The probability distribution of aggregate loss can be obtained by convolution. The variance of the aggregate loss is the sum of the individual variances, which facilitates the use of normal approximation. Stochastic independence between policies would be lost with the introduction of stochastic interest rate, since all policies are affected by the same interest development.

The mathematics of derivative assets assume that time passes continuously. As a result, new information is revealed continuously and decision makers may face instantaneous changes in randomness. Hence technical tools for pricing derivative products require ways of handling random variables over infinitesimal time intervals. To have a clear understanding on stochastic interest rate, we must have clear concept on continuous payments.

Continuous Payments

Let us consider that payments are made continuously with an annual instantaneous rate of payment of $r(t)$. Thus the amount deposited to the fund during the infinitesimal time interval from $t + dt$ is $r(t)$ dt. Let $F(t)$ denote the balance of the fund at time t. We assume that interest is credited continuously, according to a time-dependent force of interest $\delta(t)$. Interest credited in the infinitesimal time interval from t to $t + dt$ is $F(t)\ \delta(t)$ dt.

The total increase in the capital during this interval is thus

$$dF(t) = F(t)\,\delta(t)\,dt + r(t)\,dt \tag{11.8.1}$$

To solve the corresponding differential equation

$$F'(t) = F(t)\,\delta(t) + r(t) \tag{11.8.2}$$

we write

$$\frac{d}{dt}\left[e^{-\int_0^t \delta(s)\,ds} F(t)\right] = e^{-\int_0^t \delta(s)\,ds}\, r(t) \tag{11.8.3}$$

Integration with respect to t from 0 to h gives

$$e^{-\int_0^h \delta(s)\,ds} F(h) - F(0) = \int_0^h e^{-\int_0^t \delta(s)\,ds}\, r(t)\,dt \tag{11.8.4}$$

Thus the value at time 0 of a payment to be made at time t (i.e., its present value) is obtained by multiplication with the factor $e^{-\int_0^t \delta(s)\,ds}$

From (11.8.4) we obtain

$$F(h) = e^{\int_0^h \delta(s)\,ds}\left[F(0) + \int_0^h e^{\int_0^h e(s)\,ds}\, r(t)\,dt\right] \tag{11.8.5}$$

Thus the value at time h of a payment made at time $t < h$ (its accumulated value), is obtained by multiplication with the factor

$$e^{\int_0^h \delta(s)\,ds}.$$

In the case of a constant force of interest i.e., $\delta(t) = \delta$, the factors $e^{-\int_0^h \delta(s)\,ds}$ and $e^{\int_0^h \delta(s)\,ds}$ are reduced to the discount factors and accumulation factors.

Now we consider equation (11.7.1) where $B(\cdot)$ is the present value of 100 discounted at a rate 'r'. When interest rates become stochastic, the formula (11.7.1) would change. R_t would represent the risk free rate earned instantaneously, and to obtain the price of a discount bond with par value equal to 100, we would have to use

$$B(u,t) = 100\, E\left[e^{-\int_0^t r_s\,ds} \middle| I_t\right] \tag{11.8.6}$$

First of all, we need to use the conditional expectations operator E[.|.], since r_s represents the instantaneous rate at some future time s > t and consequently is not known with certainty at time t, it can only be predicted.

Secondly, there is the issue of which probability measure is used to calculate these expectations. One can argue that Treasury bonds are risk – free assets and that there is no need to use the equivalent martingale measure. This would not be correct. As interest rates become stochastic, prices of Treasury bonds with larger maturities are subject to more shocks and, everything else being the same, longer maturities are riskier. In order to eliminate the corresponding risk premia, we need to use equivalent martingale measure in evaluating the expressions shown in (11.8.6).

Third, note a heuristic explanation for where the integral in (11.8.6) comes from. Consider the case of a 3 year bond. In discrete time frame-work the price of a 3-period discount bond will be given by

$$B(3,1) = E\left[\frac{100}{(1+r_1)(1+r_2)(1+r_3)}\middle| I_1\right] \tag{11.8.7}$$

where r_1 is the current short rate, r_2 is the unknown short rate during year 2 and r_3 is the (unknown) short rate during year 3. According to equation (11.8.7) the bonds price is equal to the expectation under risk neutral probabilities of the discounted value of the par value. The discount factor is obtained by multiplying the one period discounts during the present and the two future years. Since future discounts are unknown, an expectation operator needs to be used.

This example may be more useful in interpreting the continuous-time bond price given in (11.8.6). As time becomes continuous, discrete-time discount factors need to be replaced by the exponential function. But because interest rates are continuously changing an integral has to be used in the exponent.

Check Your Progress 2

1) Define annuity. Represent annuity as the difference of two perpetuities one at time 0 and the other at time n. Show that

$$\ddot{a}_n = \frac{1-v^n}{d}$$

....

....

....

....

2) Prove that in repayment of debt, each payment consists of two components, one is interest on running debt and the other is reduction in principal.

....

....

....

....

3) Discuss the method of continuous payments. Give an expression for yield.

....

....

....

....

11.9 LET US SUM UP

This unit discusses the Cash-Flows like interest, annuities, bonds, loans and other payments which arise due to insurance contracts. An important theme covered is the projection of interest in deterministic and stochastic framework. While dealing with the deterministic component, we have included the derivation of effective and nominal interest rates. These rates are extended to include the techniques of discounting to evaluate their present values. The general Cash-Flows in insurance is discussed taking various life insurance forms such as whole life and pure endowments.

11.10 KEY WORDS

Annuity: A sequence of payments of a limited duration.

Complete Expectation of Life or Complete Future Life Time: The average number of years a person of given age can be expected to live under the prevailing mortality conditions. It gives the number of years of life entirely completed and includes the fraction of the year survived in the year in which death occurs which on the average is taken as ½ year. So complete future life time = curtate future life time + ½ .

Conversion Period: The time interval at the end of which interest is credited or compounded.

Curtate Future Life Time: The average number of completed future years lived by the group attaining age x.

Discount Rate: When interest is credited of the beginning of conversion period.

Effective Interest Rate: When the conversion period and basic time unit are identical, interest rate is credited at the end of basic time unit.

Life Table: A conventional method of expressing the most essential facts about the age distribution of mortality in a tabular form and it gives the life history of a hypothetical population which is gradually diminished by deaths. It is a powerful tool for measuring the probability of life and death at various age sectors.

Net Single Premium: The expected value of present value of the payment.

Nominal Interest Rate: When the conversion period does not coincide with basic time unit.

Perpetuity: A sequence of payments of an unlimited duration.

11.11 SOME USEFUL BOOKS

Batten,R.W.(1978) Mortality Table Construction, Prentice-Hall Englewood diffs, New Jersey

Butcher, M.V., Nesbitt C.J. (1971) Mathematics of Compound Interest. Ulrich's Books, Ann Arbor, Michigan

Jordan, C.W. (1967) Life Contingencies, Second Edition, Society of Actuaries Chicago, III

Salih, N. Neftci (1996), An Introduction to the Mathematics of Financial Derivatives, Academic Press. San Diego-California, USA

11.12 ANSWER OR HINTS TO CHECK YOUR PROGRESS

Check Your Progress 1

1) Read 11.2 and 11.3 and answer.

2) Use equation 11.3.4 and tabulate the result.

3) Read 11.5 with the example and answer.

Check Your Progress 2

1) Read 11.6 and use equation and 11.6.15 and 11.6.16 and answer.

2) Read 11.6 and use equation 11.6.32 and 11.6.33 and answer.

3) Read 11.8 and answer.

11.13 EXERCISES

1) Show that $i^{(m)} - d^{(m)} = \dfrac{i^{(m)} \cdot d^{(m)}}{m}$.

2) A deposit of 100,000 is made into a newly established fund. The fund pays nominal interst of 12% convertible quarterly. At the end of each six months a withdrawal is made from the fund. The first withdrawal is X, the second is 2X, the 3rd is 3X and so on. The last is sixth withdrawal, which exactly exhausts the fund, Calculate X.

3) Given: (i) $A_{x:\overline{n}|} = 4$, (ii) $A^{\,1}_{x:\overline{n}|} = y$; (iii) $A_{x+n} = z$. Determine the value of A_x in terms of u, y and x.

4) Z_1 is the present value random variable for an n year continuous endowment insurance of 1 issued to x. Z_2 is the present value random variable for an n year continuous term insurance of 1 issued to x. Given

(i) $Var(Z_2) = 0 \cdot 01$, (ii) $v^n = 0 \cdot 30$.

(iii) ${}_np_x = 0 \cdot 8$, (iv) $E(Z_2) = 0 \cdot 04$.

Calculate $Var(Z_1)$.

5) A loan of 1000 at a nominal rate of 12% convertible monthly is to be repaid by six monthly payments with the first payment due at the end of one month. The first three payments are x each, and the final three payments are $3x$ each, calculate x.

UNIT 12 LIFE CONTINGENCIES

Structure

12.0 OBJECTIVES

After going through this unit, you will be able to:

- calculate probabilities of death and the future life time of a person at a particular age;
- determine the benefit reserves for life insurance; and
- know the theory of life contingencies and multiple decrement theory.

12.1 INTRODUCTION

Probability and expected values of interest can be expressed in terms of the functions of future lifetime or probabilities of death and survival. Life insurance contracts primarily depend on a random variable i.e., the future lifetime. Life annuity consists of a series of payments, which are made while the beneficiary lives i.e., on probability of survival, which occurs as one of the columns of life table. An insurance policy specifies the benefits payable to the insurer and the premium payable by the insured. At the time of policy issue, the expected present value of future premiums equals the expected present value of future benefit payments making the expected loss of the insurer zero. When the person under consideration leaves one specific status at a particular age due to one of some mutually exclusive causes of decrement like death and accident, we study pairs of random variables, the remaining life time of the specified status and the causes of decrement. All these are studied under one important theory known as multiple decrement theory (see Unit 8).

12.2 SURVIVAL TO COSTS AND RISKS OF LIFE ASSURANCES

The Model

Let us consider a person aged x years, also called a life aged x and denoted by (x). We denote his or her future lifetime by T or, more explicitly by $T(x)$. Thus $x+T$ will be the age at death of the person. The future lifetime T is a random variable with a probability distribution function

$$G(t) = Pr(T \leq t), \quad t \geq 0, \tag{12.2.1}$$

The function G (t) represents the probabilities that the person will die within t years, for any fixed t. We assume that G, the probability distribution of T, is known. We also assume that G is continuous and has a probability density $g(t) = G'(t)$. Thus

$$g(t)\, dt = Pr(t < T \leq t + dt), \tag{12.2.2}$$

being the probability that death will occur in the infinitesimal time interval from t to $t+dt$ or that (x)'s age at death will fall between $x+t$ and $x+t+dt$.

Probabilities and expected values of interest may be expressed in terms of the functions of g and G. Let ${}_tq_x$ be the probability that a life aged x will die within t years. Thus

$${}_tq_x = G(t) \tag{12.2.3}$$

and ${}_tp_x$ is the complementary of ${}_tq_x$. So

$${}_tp_x = 1 - G(t) \tag{12.2.4}$$

and

$${}_{s/t}q_x = \Pr(s < T \leq s + t) = G(s+t) - G(s) = {}_{s+t}q_x - {}_tq_x \tag{12.2.5}$$

This gives the probability that the life aged x will survive s years and subsequently die within t years. ${}_tp_{x+s}$ denotes the conditional probability that the person will survive another t years after having attained the age $x + s$. Thus

$${}_tp_{x+s} = \Pr(T \geq s + t | T > s) = \frac{1 - G(s+t)}{1 - G(s)} \tag{12.2.6}$$

Similarly,

$${}_tq_{x+s} = p_r(T \leq s + t | T > s) = \frac{G(s+t) - G(s)}{1 - G(s)} \tag{12.2.7}$$

Identities in frequent use are

$${}_{s+t}p_x = 1 - G(s+t) = [1 - G(s)]\frac{1 - G(s+t)}{1 - G(s)} = {}_sp_x\, {}_tp_{x+s} \tag{12.2.8}$$

and

$$_{s/t}q_x = G(s+t) - G(s) = \left[1 - G(s)\right] \frac{G(s+t) - G(s)}{1 - G(s)} = {}_sp_x \;\; {}_tq_{x+s} \qquad (12.2.9)$$

The expected remaining lifetime of a life aged x is E(T) and denoted by e_x^0

$$e_x^0 = \int_0^\infty t\, g(t)\, dt \qquad (12.2.10)$$

or, in terms of the distribution function

$$e_x^0 = \int_0^\infty [1 - G(t)]\, dt = \int_0^\infty {}_tp_x \; dt \qquad (12.2.11)$$

If $t = 1$, the index t is usually omitted in the symbols ${}_tq_x$, ${}_tp_x$, ${}_{s|t}q_x$. Thus q_x is the probability of dying within 1 year and ${}_{s|}q_x$ is the probability of surviving s years and subsequently dying within 1 year.

The Force of Mortality

The force of mortality of (x) at the age $x+t$ is defined by

$$\mu_{x+t} = \frac{g(t)}{1 - G(t)} = -\frac{d}{dt} \ln [1 - G(t)] \qquad (12.2.12)$$

An alternative expression may be derived using (12.2.2) and (12.2.4) for the probability of dying in the interval t and $t + dt$

$$\Pr(t < T < t + dt) = {}_tp_x \;\; \mu_{x+t} \;\; dt \qquad (12.2.13)$$

and

$$e_x^0 = \int_0^\infty t \;\; {}_tp_x \; \mu_{x+t} \;\; dt \qquad (12.2.14)$$

Curtate Future Life Time of (*x*)

The random variables $K = K(x)$, $S = S(x)$ *and* $S^{(m)} = S^{(m)}(x)$ are closely related to the original random variable T. We shall discuss them in the following.

We define $K = [T]$, the number of completed future years to be lived by (x) or the curtate future lifetime of (x). The probability distribution of K is

$$\Pr(K = k) = \Pr\,(k \le T \le k + 1) = {}_kp_x \;\; q_{x+k} \text{ for } k = 0,1,2,\ldots. \qquad (12.2.15)$$

The expected value of K is called the expected curtate future lifetime of (x) and is denoted by e_x. So

$$e_x = \sum_{k=1}^{\infty} k \Pr(K = k) = \sum_{k=1}^{\infty} k \;\, {}_kp_x \;\; q_{x+k} \qquad (12.2.16)$$

or $$e_x = \sum_{x=1}^{\infty} \Pr(K \ge k) = \sum_{k=1}^{\infty} {}_kp_x \qquad (12.2.17)$$

Let S be the fraction of a year during which (x) is alive in the year of death i.e.,

$$T = K + S \qquad (12.2.18)$$

The random variable S has a continuous distribution between 0 and 1. Approximating its expected value by ½ , we find from (12.2.18) the approximation

$$e_x^0 \approx e_x + \frac{1}{2} \qquad (12.2.19)$$

which is used in practice for the expected future lifetime of (x).

Let us assume that K and S are independent random variables, so that the conditional distribution of S given K is independent of k; thus,

$$\Pr\left(S \le u \middle| K = k\right) = \frac{{}_u q_{x+k}}{q_{x+k}} \qquad (12.2.20)$$

will not depend on the argument k, so that one can write

$$ {}_u q_{x+k} = 1 + (u)\ q_{x+k} \text{ for } k = 1,2,\ldots, \text{ and } 0 \le u \le 1 \qquad (12.2.21)$$

and some function $H(u)$. If we assume $H(u) = u$. (uniform distribution between 0 and 1). Then the approximation (12.2.21) is exact. Moreover, from (12.2.18)

$$Var(T) = Var(K) + \frac{1}{12} \qquad (12.2.22)$$

For positive integers m we define the random variable

$$S^{(m)} = \frac{1}{m}[mS + 1] \qquad (12.2.23)$$

Thus $S^{(m)}$ is derived from S by rounding to the next higher multiple of $1/m$. The distribution of $S^{(m)}$ has its mass in the points $1/m$, $2/m$, ..., 1. Independence between K and S implies independence between K and $S^{(m)}$. If S has uniform distribution between 0 and 1, then $S^{(m)}$ has a uniform discrete distribution.

Probabilities of Death for Fraction of Year

The distribution of K and its related quantities may be calculated from a life table. For example

$$ {}_k p_x = p_x\ p_{x+1} \cdots p_{x+k-1},\quad k = 1,2,3,\ldots \qquad (12.2.24)$$

To obtain the distribution of T by interpolation, assumptions are made regarding the pattern of the probabilities of death, ${}_u q_n$ or the force of mortality μ_{x+u} at intermediate ages $x + u$ (x an integer, $0 < u < 1$)

(i) when ${}_u q_x$ is linear i.e., ${}_u q_x = u\, q_x$

then $$\mu_{x+u} = \frac{q_x}{1 - uq_x} \qquad (12.2.25)$$

(ii) when μ_{x+u} is constant $\mu_{x+\frac{1}{2}} = -\ln p_x$ or, ${}_u P_x = e^{-u\mu_{x+\frac{1}{2}}} = (p_x)^u$; then

$$P\left(S \le u \middle| K = k\right) = \frac{1 - p_{x+k}^u}{1 - p_{x+k}} \qquad (12.2.26)$$

(iii) when ${}_{1-u} q_{x+u}$ is linear, then

$$P\left(S \le u \middle| K = k\right) = \frac{u}{1-(1-u)\, q_{x+k}} \tag{12.2.27}$$

Example: Suppose we want to find out μ_{45} when ${}_tp_x = \frac{100-x-t}{100-x}$ for $0 \le x < 100 \quad 0 \le t \le 100 - x$.

We know $\mu_{45} = -\frac{d}{dt}\ln({}_tp_x)$ evaluated at 45 – x. Thus.

$$\mu_{45} = -\frac{d}{dt}\ln\left(\frac{100-x-t}{100-x}\right)$$

$$= \left.\frac{1}{100-x-t}\right|_{t=45-x} = \frac{1}{55}.$$

12.3 LIFE ANNUITIES

A life annuity consists of a series of payments, which are made while the beneficiary (of initial age) lives. Thus a life annuity may be represented as an annuity certain, with a term dependent on the remaining lifetime T. Its present value thus becomes a random variable, which we shall denote by Y.

The net single premium of a life annuity is its expected present value $E(Y)$. So the distribution Y and its moments are of great use.

Elementary Life Annuities

We consider whole life annuity due which provides for annual payments of 1 unit as long as the beneficiary lives. Payments are made at the time points $0,1,2,\ldots,k$. The present value of this payment stream is

$$Y = 1 + v + v^2 + \ldots + v^K = \ddot{a}_{\overline{K+1|}} \tag{12.3.1}$$

The probability distribution of this random variable is

$$\Pr\left(Y = \ddot{a}_{\overline{K+1|}}\right) = \Pr(K = k) = {}_kp_x \; q_{x+k},\; k = 0, 1, 2, \ldots . \tag{12.3.2}$$

The net single premium denoted by $\ddot{a}_x$, is the expected value of (12.3.1)

$$\ddot{a}_x = \sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1|}} \; {}_kp_x \; q_{x+k}. \tag{12.3.3}$$

The present value (12.3.1) may also be expressed as

$$Y = \sum_{k=0}^{\infty} v^k I_{(K \ge k)} \tag{12.3.4}$$

where I_A is the indicator function of an event. The expectation of (12.3.4) is

$$\ddot{a}_k = \sum_{k=0}^{\infty} v^k \; {}_kp_x. \tag{12.3.5}$$

Thus we have found two expressions for the net single premium for a whole life annuity due. In (12.3.3) we consider the whole annuity as a unit, while in (12.3.5) we think the annuity as a pure endowment.

The net single premium may also be expressed in terms of the net single premium for a whole life insurance, the later being given by $z = v^{k+1}$ and

$$\Pr\left(Z = v^{k+1}\right) = p_r\left(K = k\right) = {}_k p_x \; q_{x+k}$$

using

$$\ddot{a}_{\overline{n}|} = \frac{1 - v^n}{d}$$

$$Y = \frac{1 - v^{k+1}}{d} = \frac{1 - Z}{d} \tag{12.3.6}$$

(This formula can also be obtained by viewing the life annuity as the difference of two perpetuities due, one starting at time 0 and the other at time k+1). Taking expectations yield

$$\ddot{a}_{\overline{x}|} = \frac{1 - A_x}{d} \tag{12.3.7}$$

where $$A_x = E\left[v^{k+1}\right]$$

or $$1 = d\ddot{a}_x + A_x \tag{12.3.8}$$

We may interpret it in terms of a debt of 1 unit with annual interest in advance, and a final payment of 1 unit at the end of the year of death. Of course the higher order moments of Y may also be derived from (12.3.6), so that, for instance,

$$\operatorname{var}(Y) = \frac{\operatorname{var}(Z)}{d^2} \tag{12.3.9}$$

The present value of an n-year temporary life annuity-due is

$$Y = \begin{cases} \ddot{a}_{\overline{k+1}|} \quad for \;\; k = 0, 1, 2, \ldots, (n-1) \\ \\ \ddot{a}_{\overline{n}|} \quad for \;\; k = n\,,\, n+1,\, n+2, \ldots . \end{cases} \tag{12.3.10}$$

Similarly, using (12.2.3) and (12.2.5) the net single premium can be expressed by either

$$\ddot{a}_{\overline{x:n}|} = \sum_{k=0}^{n-1} \ddot{a}_{\overline{k+1}|} \; {}_k p_x \; q_{x+k} + \ddot{a}_{\overline{n}|} \; {}_n p_x \tag{12.3.11}$$

or $$\ddot{a}_{\overline{x:n}|} = \sum_{k=0}^{n-1} v^k \; {}_k p_x \tag{12.3.12}$$

Now we have

$$Y = \frac{1 - Z}{d} \tag{12.3.13}$$

where $$Z = \begin{cases} v^{k+1} \quad for \;\; k = 0,\, 1,\, 2,\, \ldots,\, n-1 \\ \\ v^n \quad for \;\; k = n,\, n+1,\, n+2, \ldots \end{cases}$$

As a consequence,

$$\ddot{a}_{\overline{x:n}|} = \frac{1 - A_{\overline{x:n}|}}{d} \tag{12.3.14}$$

or $$1 = d \cdot \ddot{a}_{\overline{x:n}|} + A_{\overline{x:n}|} \tag{12.3.15}$$

The corresponding immediate life annuities provide for payments at times 1,2...k.

$$Y = v + v^2 + ... + v^k = a_{\overline{k}|} \tag{12.3.16}$$

The random variable (12.3.1) and (12.3.16) differ only by the constant term 1. Thus the net single premium a_x is given by

$$a_x = \ddot{a}_x - 1 \tag{12.3.17}$$

From equation $1 = ia_{\overline{n}|} + v^n$ with $n = k + 1$, we obtain

$$1 = ia_{\overline{k}|} + (1+i)\, v^{k+1} \tag{12.3.18}$$

Taking expectations we get

$$1 = i\, a_k + (1+i) A_x \tag{12.3.19}$$

in analogy to (12.3.8).

The present value of an m year deferred life annuity due with annual payments of 1 unit is

$$Y = \begin{cases} 0 \;\; for \;\; k = 0, 1, 2, ..., m-1 \\ \\ v^m + v^{m+1} ... + v^k \;\; for \;\; k = m, m+1, ... \end{cases} \tag{12.3.20}$$

The net single premium may be obtained from the above known relations as

$$_{m|}\ddot{a}_x = {}_m p_x \;\; v^m \; \ddot{a}_{x+m}$$

$$_{m|}\ddot{a}_n = \ddot{a}_k - \ddot{a}_{\overline{x \cdot m}|}\,.$$

Example: Suppose Y is the present value random variable of a whole life annuity due of 1 per year issued to (x). Given $\ddot{a}_x = 10$, evaluated with $i = \dfrac{1}{24} = e^{\delta} - 1$ and $\ddot{a}_x = 6$ evaluated with $i = e^{2\delta} - 1$, calculate the var (Y).

Using formula $\operatorname{var}(Y) = \dfrac{\operatorname{var}(Z)}{d^2}$

$$= \frac{1}{d^2} E\left[e^{-2\delta(k+1)} - A_x^2\right],$$

$$\operatorname{var}(Z) = E(Z^2) - A_x^2 \text{ and } E(Z^2) = E\left[e^{-2\delta(k+1)}\right].$$

we have, A_x evaluated at δ is $1 - (0 \cdot 04)10 = 0 \cdot 6 \quad (A_x = 1 - d\,\ddot{a}_x)$.

The discount corresponding to 2δ is $1 - v^2 = 1 - (0 \cdot 96)^2 = 0 \cdot 0784$.

So $E\left[e^{-2\delta(k+1)}\right] = A_x$ evaluated at 2δ is $1-(0\cdot784)(6)=0\cdot5296$.

$$\text{Therefore, } \operatorname{var}(Y) = \frac{\left\{0\cdot5296 \ \ (0\cdot6)^2\right\}}{(0\cdot04)^2} = 106.$$

12.4 ANALYSIS OF SURVIVAL DISTRIBUTIONS

Analysis of survival distribution means the analytical distribution of *T*. We call the function *G*, an analytical or mathematical probability distribution if it can be expressed by a simple formula. There are different reasons for postulating an analytical distributor for *T*.

An analytical formula has the advantage that *G(t)* can readily be calculated from a small number of numeric parameters.

Analytical formulae also have some attractive theoretical properties. Their popularity is akin to the popularity of normal distribution in statistics. A normal model is often used. Some examples of analytical distribution are given below.

De Moivre (1724) postulated the existence of a maximum age w for human being and assumed that *T* was uniformly distributed between the ages of 0 and $w-x$ leading to $g(t) = \dfrac{1}{w-x}$ *for* $0 < t < w-x$.

The force of mortality then becomes

$$\mu_{x+t} = \frac{1}{w-x-t}, \ 0<t<w-x \qquad (12.4.1)$$

which is an increasing function of t. Gompertz (1825) postulated that the force of mortality would grow exponentially

$$\mu_{x+t} = BC^{x+t}, \ t>0 \qquad (12.4.2)$$

which reflects the aging process better than De Moivre's law and in addition removes the assumption of a maximum age w.

Makeham (1860) generalized (12.4.2). He postulated the law

$$\mu_{x+t} = A + BC^{x+t}, \ t>0 \qquad (12.4.3)$$

Makeham's mortality law adds a constant age independent component $A > 0$ to the exponentially growing force of mortality.

A special case of the mortality laws of Gompertz (by putting $c = 1$) and Makeham (by making $B = 0$) is that of a constant force of mortality. The probability distribution of *T*, then becomes the exponential distribution. While mathematically very simple, this distribution does not reflect human mortality in a realistic way. From (12.4.3) and ${}_tp_x = e^{-\int_0^t \mu_{x+s}\,ds}$ and putting $m = \dfrac{B}{\ln C}$, the survival probability in Makeham's model may be derived

$${}_tp_x = \exp\left\{-At - mc^x\left(C^t - 1\right)\right\} \qquad (12.4.4)$$

Weibull (1939) suggested that the force of mortality grows as the power of t, instead of exponentially:

$$\mu_{x+t} = k(x+t)^n \qquad (12.4.5)$$

with the fixed parameters k > 0 and n > 0. The survival probability then becomes

$$_t p_x = \exp\left(-\frac{k}{n+1}\left[(x+t)^{n+1} - x^{n+1}\right]\right) \qquad (12.4.6)$$

12.5 INTERNATIONAL ACTUARIAL NOTATIONS

International actuarial community uses time-honoured notations. For example the probability that a life aged x will die within n years is denoted by the symbol $_n q_x$. Life table uses international actuarial notations in different columns which are given below. l_x is the number of persons living at any specified age x, dx is the number of persons among the l_x persons who die before reaching the age $(x+1)$. So,

$$d_x = l_x - l_{x+1} = -\Delta l_x \qquad (12.5.1)$$

where Δ is the difference operator, $_n p_x$ is the probability that a person aged x survives upto age $(x+n)$, so that

$$_n p_x = \frac{l_{x+n}}{lx} \qquad (12.5.2)$$

In particular for $n = 1$,

$$p_x = \frac{l_{x+1}}{l_x}, \qquad (12.5.3)$$

Note the following notations:

$q_x = 1 - p_x$ is complementary probability of survival i.e., q_x is the probability that a person of exact age x will die within one year following the attainment of that age. Thus

$$q_x = \frac{dx}{lx}. \qquad (12.5.4)$$

L_x is the number of years lived in the aggregate by the cohort of l_0 persons between age x and $x+1$. So,

$$L_x = \int_0^1 l_{x+t}\, dt = \int_0^1 (l_x + td_x)\, dt = l_x - \frac{1}{2} d_x = \frac{1}{2}(l_x + l_{x+1}) \qquad (12.5.5)$$

T_x is the total number of years lived by the cohort after attaining the age x i.e., T_x is the total future life time of the l_x persons who reach age x

and $$T_x = L_x + L_{x+1} + L_{x+2} + \ldots = L_x + T_{x+1} \qquad (12.5.6)$$

e_x^0 is the complete expectation of life

$$e_x^0 = \frac{T_x}{l_x} \qquad (12.5.7)$$

e_x is the curtate expectation of life and

$$e_x = e_x^0 - \frac{1}{2}. \tag{12.5.8}$$

m_x is central mortality rate i.e., the probability that a person whose exact age is not known but lies in between x and $(x+1)$ and will die within one year following the attainment of that age and

$$m_x = \frac{2q_x}{2-q_x} \tag{12.5.9}$$

μ_x is the force of mortality i.e., the ratio of instantaneous rate of decrease in l_x to the value of l_x and

$$\mu_x = -\frac{d}{dx}(\log l_x) \tag{12.5.10}$$

$(12.5.10)$ gives nominal annual rate of mortality i.e., the probability of a person of age x exactly dying within the year if the risk of dying is same at every moment of the year as it is during the moment following the attainment of age x. So,

$$\mu_{x+\frac{1}{2}} = m_x \tag{12.5.11}$$

These notations are used in commutation functions, which are used in life insurance and to determine life annuities. Table of commutation functions simplifies the calculation of numerical values for many actuarial functions. Expected values such as net single premiums may be derived within a deterministic model closely related to commutation functions, some such models are contained in (12.5.3), (12.5.4), (12.5.6) etc.

Life Annuities

We first consider a life annuity-due with annual payment of 1 unit. We have

$$\ddot{a}_x = \sum_{k=0}^{\infty} v^k \, {}_kp_x = \sum_{k=0}^{\infty} v^k \frac{l_{x+k}}{l_x}$$

$$\ddot{a}_x = \frac{l_x + vl_{x+1} + v^2 l_{x+2} + \ldots}{l_x} \tag{12.5.12}$$

Multiplying both the numerator and the denominator by v^x

$$\ddot{a}_x = \frac{v^x l_x + v^{x+1} l_{x+1} + v^{x+2} l_{x+2} + \ldots}{v^x l_x} \tag{12.5.13}$$

But we know that

$$D_x = v^x l_x \, , N_x = D_x + D_{x+1} + D_{x+2} + \ldots \tag{12.5.14}$$

So,

$$\ddot{a}_x = \frac{N_x}{D_x} \tag{12.5.15}$$

The function D_x is called the discounted number of survivors. Similarly, the net single premium of a temporary life annuity is given as

$$\ddot{a}_{\overline{x \cdot n}|} = \frac{N_x - N_{x+n}}{D_x} \tag{12.5.16}$$

Immediate life annuities is written as

$$a_x = \frac{N_{x+1}}{D_x} \tag{12.5.17}$$

While general annuities with annual payments will be

$$E(Y) = \frac{r_0\ D_v + r_1\ D_{x+1} + r_2\ D_{x+2} + \ldots}{D_x} \tag{12.5.18}$$

If $r_k = k + 1$, then $\quad (I\ddot{a})_x = \dfrac{S_x}{D_x}$,

where $S_x = D_x + 2D_{x+1} + 3D_{x+2} + \ldots\ldots = N_x + N_{x+1} + N_{x+2} + \ldots$

In life Insurance, other formulas are

$$\left.\begin{aligned} C_x &= v^{x+1}\, d_x \\ M_x &= C_x + C_{x+1} + C_{x+2} + \ldots \\ R_x &= C_x + 2C_{x+1} + 3c_{x+2} + \ldots = M_x + M_{x+1} + M_{x+2} + \ldots \end{aligned}\right\} \tag{12.5.19}$$

Replacing ${}_k p_x\ q_{x+k}$ in the formula of A_x

$$A_x = \frac{kd_x + v^2\ d_{x+1} + v^3\ d_{x+1} + \ldots.}{1_x} = \frac{C_x + C_{x+1}\, C_{x+2} + \ldots}{D_x}$$

$$= \frac{M_x}{D_x} \tag{12.5.20}$$

Similarly,

$$(IA_x) = \frac{vd_x + 2v^2\, d_{x+1} + 3v^2\ d_{x+2} + \ldots.}{1_x}$$

$$= \frac{C_x + 2C_{x+1} + 3C_{x+2} + \ldots}{D_x} = \frac{R_x}{D_x} \tag{12.5.21}$$

Endowment insurances are presented as

$$A^1_{\overline{x.n}|} = \frac{M_x - M_{x+n}}{D_x}$$

$$A_{\overline{x.n}|} = \frac{M_x - M_{x+n} + D_{x+n}}{D_x}$$

$$(IA)^1_{\overline{x.n}|} = \frac{C_{x+2}\ C_{x+1} + 3C_{x+2} + \ldots + nC_{x+n-1}}{D_x}$$

$$= \frac{M_x + M_{x+1} + M_{x+2} + \ldots + M_{x+n-1} - n\, M_{x+n}}{D_x} = \frac{R_x - R_{x+n} - n\, M_{x+n}}{D_x} \qquad (12.5.22)$$

Other commutation functions in the area are

$$C_x = v\, D_x - D_{x+1} \qquad (12.5.23)$$

$$M_x = vN_x - (N_x - D_x) = D_x - dN_x \qquad (12.5.24)$$

$$R_x = N_n - d\, S_n \qquad (12.5.25)$$

$$A_x = \frac{R_x}{D_x} = 1 - d\ddot{a}_x \qquad (12.5.26)$$

$$(IA)_x = \ddot{a}_x - d(I\ddot{a})_x \qquad (12.5.27)$$

Check Your Progress 1

1) What do you mean by life annuity? For whole life annuity due, find two expressions for net single premium.

....

....

....

2) Discuss the meaning of survival distribution.

....

....

....

3) Derive formula of $\ddot{a}_x$ in terms of commutation functions.

....

....

....

12.6 ANNUAL BENEFIT PREMIUM

Consider the example of a term insurance for a life of age 40 (duration 10 years; sum insured C; payable at the end of year of death; premium π payable annually in advance while the insured is alive, but not longer than 10 years). The loss L of the insurer is given by

$$L = \begin{cases} Cv^{k+1} - \pi\ddot{a}_{\overline{k+1}|} & for\ \ k = 0,1,\ldots,9 \\ \\ -\pi\ddot{a}_{\overline{10}|} & for\ \ k \geq 10 \end{cases} \qquad (12.6.1)$$

Here k denotes the curtate future lifetime of (40). The random variable L has a discrete distribution concentrated in 11 points, So that

$$\Pr\left(L = Cv^{k+1} - \pi\ddot{a}_{k+1}\right) = {}_kp_{40}\; q_{40+k}, \quad k = 0,1,\ldots 9$$

$$\Pr(L = -\pi\ddot{a}_{10}) = {}_{10}p_{40} \qquad (12.6.2)$$

Now we shall determine the net annual premium. From (12.6.1) one obtains the condition

$$CA^{1}_{40:\overline{10|}} - \pi \ddot{a}_{40:\overline{10|}} = 0 \qquad (12.6.3)$$

which gives

$$\pi = c \cdot \frac{A^{1}_{40:\overline{10|}}}{\ddot{a}_{40:\overline{10|}}} \qquad (12.6.4)$$

As an illustration take $i = 4\%$ and assume that the mortality follows De Moivre's law with terminal age $w = 100$.

$$\text{Then, } A^{1}_{40:\overline{10|}} = \frac{1}{60} v + \frac{1}{60} v^{2} + \ldots + \frac{1}{60} v^{10} = \frac{1}{60} a_{\overline{10|}} = 0.1352$$

$$A_{40:\overline{10|}}^{\;\;\,1} = \frac{5}{6} v^{10} = 0.5630 \qquad (12.6.5)$$

so that $A_{40:\overline{10|}} = 0.6982$ (12.6.6)

$$\ddot{a}_{40:\overline{10|}} = \left(1 - A_{40:\overline{10|}}\right) \frac{1}{d} = 7.8476\,.$$

Hence $\pi = 0.0172C$. (12.6.7)

The insurer cannot be expected to pay benefits in return for net premiums and there should be a safety loading which reflects the assumed risk. So there is a method for determining premiums which take account of the incurred risk.

Premiums are determined by a utility function $u(.)$. This is a function satisfying $u'(x) > 0$ and $u''(x) < 0$ and measuring the utility that the insurer has of a monetary amount x. More specifically, we assume that the utility function is exponential, i.e.,

$$u(x) = \frac{1}{a}\left(1 - e^{-ax}\right) \qquad (12.6.8)$$

The parameter $a > 0$ measures the risk aversion of the insurer. See that the condition (12.6.1) is now replaced by the condition

$$E\left[u\,(-L)\right] = u(0) \qquad (12.6.9)$$

i.e., the premium should now be determined in such a way that the expected utility loss is zero. With the utility function given by (12.6.8) the annual premium must satisfy

$$E\left[e^{aL}\right] = 1 \qquad (12.6.10)$$

12.7 DETERMINATION OF BENEFIT RESERVES FOR LIFE INSURANCE AND ANNUITIES

Consider an insurance policy which is financed by net premiums. At the time of policy issue, the expected present value of the future premiums equals the expected present value of future benefit premiums, making the expected loss L of the insurer zero.

This equivalence between future payments and future benefits does not in general, exist at a later time. Thus we define a random variables ${}_tL$ as the difference at time t between the present value of future benefit payments and the present value of future premium payments. We assume that ${}_tL$ is not identically equal to zero and $T > t$. The net premium reserve at time t is defined by ${}_tV$, and it is defined as the conditional expectation of ${}_tL$ given that $T > t$.

Life insurance policies are usually designed in such a way that the net premium reserve is positive or at least non-negative. For, the insured should at all times have an interest in continuing the insurance. Thus the expected value of future benefits will always exceed the expected value of future premium payment. To compensate for this liability the insurer should always reserve sufficient funds to cover the difference of those expected values, i.e., the net premium reserve ${}_tV$

Consider the following example. The net premium reserve at the end of the kth policy year for an endowment insurance (duration : n, sum insured:1, payable after n years or at the end of the year of death, annual premium) is denoted by ${}_kV_{\overline{n:n|}}$ and given by the expression,

$${}_kV_{x:\overline{n|}} = A_{x+k:\overline{n-k|}} - P_{x:\overline{n|}}\, \ddot{a}_{n+k,\, n-k}, k = 0,1,2,\ldots,n-1. \qquad (12.7.1)$$

Obviously ${}_0V_{x:\overline{n|}} = 0$ because of the definition of net premium.

The net premium reserve at the end of year k of the corresponding term insurance is denoted by ${}_kV^1_{x:\overline{n|}}$. It is given by

$${}_kV^1_{x:\overline{n|}} = A^1_{x+k:\overline{n-k|}} - P^1_{x:\overline{n|}}\, \ddot{a}_{x+k,\overline{n-k|}} \qquad (12.7.2)$$

Net Premium Reserve of a Whole Life Insurance

Consider the whole life insurance discussed in Unit 8. Its net premium reserve of the end of year k is denoted by ${}_kV_x$ and is by definition.

$${}_kV_x = A_{k+x} - P_x\, \ddot{a}_{n+k} \qquad (12.7.3)$$

Replacing A_{x+k} by $1 - d\ddot{a}_{x+k}$, we have

$${}_kV_x = 1 - (P_x + d)\, \ddot{a}_{x+k} \qquad (12.7.4)$$

Now, replacing $P_x + d$ by $\frac{1}{\ddot{a}_x}$, we obtain

$${}_kV_x = 1 - \frac{\ddot{a}_{x+v}}{\ddot{a}_x} \qquad (12.7.5)$$

Again putting $\ddot{a}_x = \frac{1 - A_x}{d}$ *and* $\ddot{a}_{k+k} = \frac{1 - A_{x+k}}{d}$, we have

$${}_kV_x = \frac{A_{x+k} - A_x}{1 - A_x} \qquad (12.7.6)$$

The identity $P_{x+k}\, \ddot{a}_{x+k} = A_{x+k}$ with (12.7.3) gives

$$_kV_x = \left(1 - \frac{P_x}{P_{x+k}}\right) A_{x+k} \tag{12.7.7}$$

$$_kV_x = \left(P_{x+k} - P_x\right) \ddot{a}_x \tag{12.7.8}$$

Finally, we replace $\ddot{a}_{x+k}$ by $\frac{1}{P_{x+k} + d}$ to find

$$_kV_x = \frac{P_{x+k} - P_x}{P_{x+k} + d} \tag{12.7.9}$$

Formula (12.7.2), (12.7.7) and (12.7.8) are important because they are easily interpreted and may be generalized to other types of insurance. Equation (12.7.3) may be interpreted by recognising that the future premiums of P_x may serve to finance a whole life insurance with face amount P_x / P_{x+k}; the net premium reserve is then used to finance the remaining face amount of $\left(1 - P_x / P_{x+k}\right)$.

If the whole life insurance were to be bought at age $(x+k)$ the premium would be P_{x+k}. The premium difference formula (12.7.4) shows that the net premium reserve is the expected present value of the shortfall of the premiums.

Net Premium Reserves at Fractional Duration

Let us assume that the insured is alive at time $k+u$, (k an integer, $0<u<1$), and denote the net premium reserve by $_{k+u}V$. The net premium reserve can be expressed by

$$_{k+u}V = {}_{k+1}Vv^{1-u} + \left(C_{k+1} - {}_{k+1}V\right) u^{1-u}\ {}_{t-u}q_{x+k+u} \tag{12.7.10}$$

where C_{k+1} is the sum insured in the $(k+1)th$ year after policy issued.

Assuming

$$_{1-u}q_{n+k+u} = \frac{(1-u)q_{x+k}}{1-u\,q_{x+k}} \tag{12.7.11}$$

$_{k+u}V$, can be evaluated. $_{k+u}V$ can also be expressed in terms of $_kV$. Thus,

$$_{k+u}V = \left({}_kV + \pi_k^s\right)(1+i)^u + \frac{1-u}{1-uq_{x+k}}\ \pi_k^r (1+i)^u \tag{12.7.12}$$

A third possible formula is

$$_{k+u}V = \frac{1-u}{1-uq_{x+k}}\left({}_kV + \pi_k\right)(1+i)^u + \left\{1 - \frac{1-u}{1-u\,q_{x+k}}\right\} {}_{k+1}Vv^{1-4} \tag{12.7.13}$$

In practical applications an approximation based on linear interpolation is often used. Thus,

$$_{k+u}V \approx (1-u)\left({}_kV + \pi_k\right) + u_{k+1}\,V \tag{12.7.14}$$

12.8 ANALYSIS OF INSURANCE LOSS RANDOM VARIABLES

Define $\Lambda_k\left(k=0,1,2,\ldots\right)$ to be the loss incurred by the insurer during the year $k+1$; thus the beginning of the year is used as reference point on the time scale. Three cases can be distinguished:

(1) The insured has died before time k:

(2) The insured dies during year $k+1$; and

(3) The insured survives to $k+1$.

The random variable Λ_k is thus defined by

$$\Lambda_k = \begin{cases} 0 \ if \ K \leq k-1 \\ c_{k+1}\, v - \left({}_kV + \pi_k \right) if \ K = k \\ {}_{k+1}V \cdot v - \left({}_kV + \pi_k \right) if \ K \geq k+1 \end{cases} \tag{12.8.1}$$

Replacing π_k by $\pi_k^s + \pi_k^r$ and using

$\pi_k^s = {}_{k+1}V_v - {}_kV$, we find

$$\Lambda_k = \begin{cases} 0 \ if \ K \leq k+1 \\ -\pi_k^r + \left(c_{k+1} - {}_{k+1}V\right) u \ if \ K = k \\ -\pi_k^r \ if \ K \geq k+1 \end{cases} \tag{12.8.2}$$

The over all loss of the insurer is given by

$$L = \sum_{k=0}^{\infty} \Lambda_k \, v^k \tag{12.8.3}$$

The sum is finite and running from 0 to k.

Using (12.8.2) and $\Pi_k^r = \left(C_{k+1} - {}_{k+1}V\right) vq_{x+k}$

we find

$$E\left[\Lambda_k \middle| K \geq k\right] = 0 \tag{12.8.4}$$

which implies

$$E\left(\Lambda_k\right) = E\left(\Lambda_k \middle| K \geq k\right) \Pr\left[K \geq k\right) = 0 \tag{12.8.5}$$

while (12.8.3) is generally valid, the validity of (12.8.5) requires that each years payments are offset against the net premium reserve of that year. The classical Hattendorff's theorem states that

$$Cov\left(\Lambda_k, \Lambda_j\right) = 0 \ for \ k \neq j \tag{12.8.6}$$

$$Var\left(L\right) = \sum_{k=0}^{\infty} v^{2k} \, Var\left(\Lambda_k\right) \tag{12.8.7}$$

The second formula states that the variance of the insurer's over all loss can be allocated to individual policy years and it is a direct consequence of the first formula and (12.8.3). The first formula is not directly evident since the random variables Λ_0, Λ, ... are not independent.

In proof of (12.8.6) we may assume $k \neq j$ without loss of generality. Then one has

$$Cov(\Lambda_k, \Lambda_j) = E[\Lambda_x . \Lambda_j) = E[\Lambda_k \ \Lambda_j | k \geq j] \Pr(k \geq j)$$

$$= -\pi_k^r \ E[\Lambda_j | k \geq j] \Pr(k \geq j) = 0 \qquad (12.8.8)$$

Here (12.8.4) has been used in last step.

The variance of Λ_k may be calculated as follows:

$$Var(\Lambda_k) = E[\Lambda_k^2] = E[\Lambda_k^2 | K \geq k] \Pr(K \geq k)$$

$$= Var(\Lambda_k | K \geq k) \Pr(K \geq k)$$

$$= (C_{k+1} - {}_{k+1}V)^2 \ v^2 \ p_{x+k} \ q_{x+k} \Pr(K \geq k)$$

$$= (C_{k+1} - {}_{k+1}V)^2 \ v^2 \ {}_{k+1}p_x \ q_{x+k} \qquad (12.8.9)$$

Substituting this into (12.8.7) we finally find

$$Var(L) = \sum_{k=0}^{\infty} v^{2k+2} (C_{k+1} - {}_{k+1}V)^2 \ {}_{k+1}p_x \ q_{x+k} \qquad (12.8.10)$$

Equation (12.8.10) is useful in evaluating the influence of financing method on the variance of L, when the benefit plan is fixed. Consider for instance a pure endowment, with $C_1 = C_2 = \ldots = 0$. The variance of L increases with the net premium reserve. Thus financing by a net single premium leads to a greater variance than financing by net annual premiums.

12.9 THEORY OF LIFE CONTINGENCIES FOR MULTIPLE LIVES

Consider m lives with initial ages $x_1, x_2, ..., x_m$. For simplicity we denote the future lifetime of kth life $T(x_k)$ by $T_k (k = 1,2,...,m)$. On the basis of these m elements we shall define a status u with a future life time $T(u)$. We shall accordingly denote by ${}_tp_x$ the conditional probability that the status u is still intact at time t given that the status existed at time 0. The symbols q_x, μ_{x+t}, ... etc are defined in a similar way. We shall also consider annuities which are defined in terms of u. The symbol $\ddot{a}_u$ denotes the net single premium of an annuity due with 1 unit payable annually, as long as u remains intact. We shall also analyse insurances with a benefit payable at the failure of the status u. The symbol $\ddot{A}_u$ would, for instance, denote the net single premium of an insured benefit of 1 unit payable immediately upon the failure of u.

Joint -Life Status

The status

$$u = x_1 : x_2 : ....: x_m \qquad (12.9.1)$$

is defined to exist as long as all m participating lives survive. The failure time of this joint status is

$$T(u) = Minimum(T_1, T_2, ..., T_m) \tag{12.9.2}$$

We shall assume that the random variables $T_1, T_2, ..., T_m$ are independent. The probability distributions of the failure time of status (12.9.1) is given by

$${}_tp_{x_1 : x_2 : x_m} = \Pr(t + (u) > t) = \Pr(T_1 > t, T_2 > t, ..., T_m > t)$$

$$= \prod_{k=1}^{\infty} \Pr(T_k > t) = \prod_{k=1}^{\infty} + p_{x_k} \tag{12.9.3}$$

The instantaneous failure rate of joint life status is

$$\mu_{u+t} = -\frac{d}{dt} \ln {}_tp_u = -\frac{d}{dt} \sum_{k=1}^{\infty} \ln \ {}_tp_{x_k} = \sum_{k=1}^{\infty} \mu_{x_x + t} \tag{12.9.4}$$

This identity presupposes that $T_1, T_2, ..., T_m$ are independent.

The net single premium for an insurance payable on the first death is

$$A_{x_1 : x_2, ..., : x_m} = \sum_{k=0}^{\infty} v^{k+1} \ {}_kp_{x_1 : x_2 ....: x_x} \ q_{x_1 + k : x_2 + k ... : x_m + k} \tag{12.9.5}$$

The net single premium for a joint life annuity due is

$$\ddot{a}_{x_1 : x_2 : ... x_n} = \sum_{k=0}^{\infty} v^k \ {}_kp_{x_1 : x_2 : ... x_m} \tag{12.9.6}$$

or, we may write

$$1 = d\ddot{a}_{x_1 : x_2 : ... x_m} + A_{x_1 : x_2 : ... : x_m} \tag{12.9.7}$$

If we denote $\overline{n|}$ the status which fails at time n i.e.,

$$T(\overline{n|}) = n \tag{12.9.8}$$

then $T(x : \overline{n|}) = minimum\ (T(x), n)$. It is then evident that the net single premium symbols $\ddot{A}_{x:\overline{n|}}$ (endowment) and $\ddot{a}_{x:\overline{n|}}$ (temporary annuity) are in accordance with the joint life notation.

Simplifications

A significant simplification results if all lives are subject to the same Gompertz mortality law, viz.,

$$\mu_{x_k + t} = BC^{x_k + t}, \ t \geq 0, \ k = 1,2,...,m \tag{12.9.9}$$

After solving the equation

$$C^{x_1} + C^{x_2} + ... + C^{x_m} = C^{w} \tag{12.9.10}$$

where w, the instantaneous joint life failure rate may be expressed by

$$\mu_{u+t} = \mu_{w+t}, \ t \geq 0 \tag{12.9.11}$$

This implies that the failure rate of joint life status follows the same Gompertz mortality law as an individual life with initial age w. All calculations in respect of the joint-life status may then be performed in terms of the single life w. As an example, we have

$$A_{x_1 : x_2 : \dots : x_m} = A_w \tag{12.9.12}$$

and

$$a_{x_1 : x_2 : \dots : x_n} = a_w \tag{12.9.13}$$

Some simplification also results if all lives follow the Makeham mortality law, viz.,

$$\mu_{x_k + t} = A + BC^{x_k + t} \tag{12.9.14}$$

Let w be the solution of the equation

$$C^{x_1} + C^{x_2} + \dots + C^{x_m} = me^{w} \tag{12.9.15}$$

Then

$$\mu_{u+t} = m\mu_{w+t} = \mu_{w+t:\, w+t, \dots,\, w+t,}\; t \geq 0 \tag{12.9.16}$$

This means that m lives aged $x_1, x_2, \dots, x_m$ may be replaced by m lives of the same initial age w. As an example,

$$a_{x_1 : x_2 : \dots : x_m} = q_{w:w:\dots:w} \tag{12.9.17}$$

Note that the age w defined by (12.9.15) is a sort of mean of the component ages $x_1, x_2, \dots, x_m$, while the age w defined by (9.9.10) exceeds all component ages $x_1, x_2, \dots, x_m$.

Last Survivor Status

The last survivor status is defined to be intact while at least one of the m lives survives, so that it fails with the last death. That is,

$$u = \overline{x_1 : x_2 : \dots : x_n} \tag{12.9.18}$$

and

$$T(u) = Maximum\,(T_1, T_2, \dots, T_m). \tag{12.9.19}$$

Probabilities and net single premium in respect of a last-survivor status may be calculated using certain joint life-status. Here the inclusion-exclusion formula of probability theory is applied.

$$\Pr(B_1 \cup B_2 \cup \dots \cup B_m) = S_1 - S_2 + S_3 \dots + (-1)^{m-1} S_m \tag{12.9.20}$$

where $B_1, B_2, \dots, B_m$ are any events and S_k denotes the symmetric sum

$$S_k = \sum \Pr(B_{j1} \cap B_{j2} \dots \cap B_{jk}) \tag{12.9.21}$$

where the summation ranges over all $\binom{m}{k}$ subsets of k events.

Again, B_k denotes the event that the kth life still alives at time t. From (12.9.20) we obtain

$$ {}_{t}p_{\overline{x_1 : x_2 : \ldots : x_m}} = S_1^t - S_2^t + S_3^t - \ldots + (-1)^{m-1} S_m^t \tag{12.9.22} $$

with the notation

$$ S_k^t : \sum {}_{t}p_{x_{j1}\, x_{j2} \ldots x_{jk}} \tag{12.9.23} $$

Multiplying (12.9.22) by u^t and summing over t we obtain the analogous formula for the net single premium of a last survivor annuity

$$ \ddot{a}_{\overline{x_1 : x_2 \ldots : x_n}} = S_1^{\ddot{a}} - S_2^{\ddot{a}} + S_3^{\ddot{a}} \ldots + (-1)^{m-1} S_m^{\ddot{a}} \tag{12.9.24} $$

where

$$ S_k^{\ddot{a}} = \sum \ddot{a}_{x_{j2} : x_{j2} \ldots x_{jk}} \tag{12.9.25} $$

Consider now an insured benefit of 1 payable upon the last death. Its net single premium may be calculated as follows:

$$ A_{\overline{x_1 : x_2 \ldots : x_m}} = 1 - d\ddot{a}_{\overline{x_1 : x_2 : \ldots : x_m}} = 1 - d\left(S_1^{\ddot{a}} - S_2^{\ddot{a}} + \ldots\right) \tag{12.9.26} $$

Let us define the symmetric sum

$$ S_k^A = \sum A_{x_{j1} : x_{j2} \ldots x_{jk}} . \tag{12.9.27} $$

Substituting

$$ S_k^{\ddot{a}} = \frac{\binom{m}{k} - S_k^A}{d} \tag{12.9.28} $$

in (12.9.26) one obtains the formula

$$ A_{\overline{x_1 : x_2 : \ldots : x_m}} = S_1^A - S_2^A + S_3^A \ldots + (-1)^{m-1} S_m^A \tag{12.9.29} $$

A similar formula may be derived for net single premium of fractional annuities payable immediately on the last death. As an illustration for 3 lives

$$ \ddot{a}_{\overline{x:y:z}} = S_1^{\ddot{a}} - S_2^{\ddot{a}} + S_2^{\ddot{a}} \tag{12.9.30} $$

with

$$ \begin{aligned} S_1^{\ddot{a}} &= \ddot{a}_x - \ddot{a}_y + \ddot{a}_z \\ S_2^{\ddot{a}} &= \ddot{a}_{x:y} - \ddot{a}_{x:y} + \ddot{a}_{y:z} \\ S_3^{\ddot{a}} &= \ddot{a}_{x:y:z} \end{aligned} \tag{12.9.31} $$

$\ddot{a}_{x:y}$, $\ddot{a}_{x:z}$, $\ddot{a}_{y:z}$ as well as $\ddot{a}_{x:y:z}$ may be calculated using equation (12.9.3) and (12.9.6).

12.10 MULTIPLE DECREMENT THEORY

Consider a person is in a specific status at age x. The person leaves that status at time T due to one of m mutually exclusive causes of decrement (numbered conveniently 1 to m). We shall study a pair of random variables, the remaining life time in the specified status T and the cause of decrement J.

In classical example of disability insurance, the initial status is "Active" and possible causes of decrement are "Disablement" and "Death".

In another setting, T is the remaining life time of (x), distinguishing between two causes of decrement, death by "Accident" and by "Other causes". This model is appropriate in connection with insurances, which provide double indemnity on accidental death.

The joint probability distribution of T and J can be written in terms of the density function $g_1(t),\ g_2(t),\ \ldots,\ g_m(t)$, so that

$$g_j(t)\,dt = \Pr(t < T < t + dt,\ J = j) \tag{12.10.1}$$

is the probability of decrement with j in infinitesional time interval $(t,\ t + dt)$. Obviously $g(t) = g_1(t) + g_2(t) + \ldots + g_m(t)$ (12.10.2)

If the decrement occurs at time t, the conditional probability of j being the cause of decrement is

$$\Pr(J = j \mid T = t) = \frac{g_j(t)}{g(t)} \tag{12.10.3}$$

We introduce the symbols

$${}_tq_{jx} = \Pr(T < t,\ J = j) \tag{12.10.4}$$

or, more generally

$${}_tq_{j,x+s} = \Pr(T < s + t,\ J = j \mid T > s) \tag{12.10.5}$$

can be calculated as follows

$${}_tq_{j,x+s} = \left(\int_0^{s+t} g_j(z)\,dz\right) \Big/ 1 - G(s) \tag{12.10.6}$$

Forces of Decrement

For a life (x) the force of decrement at age $x + t$ in respect of the cause j is defined by

$$\mu_{j,\,x+t} = \frac{g_j(t)}{1 - G(t)} = \frac{g_j(t)}{{}_tp_x}. \tag{12.10.7}$$

The aggregate force of decrement is

$$\mu_{x+t} = \mu_{1,x+t} + \ldots + \mu_{m,x+t} \tag{12.10.8}$$

See (12.10.2) and definition of μ_{x+t}. Equation (12.10.1) can be expressed as

$$P_r\,(t < T < t + dt,\ J = j) = {}_tp_x\ \mu_{j,\,x+t}\,\mathrm{dt} \tag{12.10.9}$$

Further,

$$\Pr(J = j \mid T = t) = \frac{\mu_{j,\,x+t}}{\mu_{x+t}} \tag{12.10.10}$$

If all forces of decrement are known, the joint distribution of T and J may be determined by first using (12.10.2) and ${}_t p_x = e^{-\int_0^t \mu_{x+s} ds}$ and then determining $g_j(t)$ from (12.10.7).

Curtate life time of (x)

If the one-year probabilities of decrement

$$q_{j\ x+k} = \Pr(T < k+1, J = j | T > k) \qquad (12.10.11)$$

are known for $k = 0,1,\ldots$, and $j = 1,2,\ldots m$, the joint probability distribution of the curtate time $K = [T]$ and the cause of decrement J may be evaluated. Start by observing that

$$q_{x+k} = q_{1,\ x+1} + \ldots + q_{m,\ x+k} \qquad (12.10.12)$$

from which ${}_k p_x$ can be calculated; then

$$\Pr(K = k,\ J = j) = {}_k p_x\ q_{j,\ x+k} \qquad (12.10.13)$$

for $k = 0,1,2,\ldots$ and $j = 1,2,\ldots,m$.

The joint distribution of T and J can be computed under suitable assumptions concerning probabilities of decrement at fractional ages. A popular assumption is that

$${}_u q_{j,x+k} = u.q_{j,u+k},\quad 0 < u < 1,\ k \text{ an integer} \qquad (12.10.14)$$

which implies

$$g_j(k+u) = {}_k p_x\ q_{j,x+k} \qquad (12.10.15)$$

This with the identity ${}_{k+u} p_x = {}_k p_x (1 - u\ q_{u+k})$ yields.

$$\mu_{j,x+k+u} = \frac{q_{j,x+k}}{1 - u\ q_{x+1x}} \qquad (12.10.16)$$

Assumption (12.10.14) has the obvious advantage known from the previous section, viz. That K and S become independent random variables and that S will have a uniform distribution between 0 and 1. In addition one has

$$\Pr(J = j | K = k,\ S = u) = \frac{q_{j,x+k}}{q_{x+k}}, \qquad (12.10.17)$$

a consequence of (12.10.10) and (12.10.16). The relation (12.10.17) states that the conditional probability of decrement by cause j is constant during the year.

A General Type of Insurance

Consider an insurance which provides for payment of the amount $C_{j,\ k+1}$ at the end of year $k+1$, if the decrement by cause j occurs during that year. The present value of the insured benefit is thus

$$Z = C_{j,\ k+1}\ v^{k+1} \qquad (12.10.18)$$

and the net single premium

and $$E(Z) = \sum_{j=1}^{m} \sum_{k=1}^{\infty} c_{j,k+1}, v^{k+1} \; {}_k p_x \; q_{j,x+k} \cdot \tag{12.10.19}$$

If the insurance provides for payment immediately on death, the present value of the insured benefit is

$$Z = c_j(T)\, v^T \tag{12.10.20}$$

and the net single premium is

$$E(Z) = \sum_{j=1}^{m} \int_0^{\infty} g_i(t)\, v^t g_i(t)\, dt \tag{12.10.21}$$

This can be evaluated by splitting each of the m integrals.

$$E(Z) = \sum_{j=1}^{m} \sum_{k=0}^{\infty} \int_0^1 c_j\,(k+u)\, v^{k+u}\; g_j(k+u)\, du. \tag{12.10.22}$$

Use of (12.10.14) allows to substitute (12.10.5) in the above expression. Thus (12.10.22) assumes the form (12.10.9) if

$$C_{j,k+1} = \int_0^1 c_j(k+u),\, (1+i)^{1-x}\, du \tag{12.10.23}$$

In practical calculation

$$c_{j,k+1} \approx c_{j,\left(k+\frac{1}{2}\right)(1+i)^{\frac{1}{2}}} \tag{12.10.24}$$

will be accurate.

The above results show that the evaluation of the net single premium in the continuous model (12.10.20) can be reduced to a calculation within the discrete model (12.10.18).

The insured's exit from the initial status will not always result in a single payment; another possibility is the initiation of a life annuity. If for instance, the cause $j = 1$ denotes disablement, then $c_1(t)$ could be the net single premium of a temporary life annuity starting at age $x + t$. Thus in the general model, the payment $c_{j,k+1}$ (respectively $c_j(t)$ may themselves be expected values, however the formulae (12.10.19) and (12.10.21) remain valid.

Check Your Progress 2

1) Explain the net premium reserve of a whole life insurance

....

....

....

2) Distinguish and discuss the joint life status and last-survivor status in multiple life insurance

....

....

....

3) What do you mean by multiple decrement theory?

....

....

....

12.11 LET US SUM UP

In this unit we have discussed contingency measures incorporated in insurance to deal with risks. In the process we covered the important functions on probabilistic of death and survival and defined the concepts used for planning and execution of insurance. The standard notations used by actuaries as communicated functions are discussed and their use in determination of reserves, annuities, loss analysis, multiple lives and multiple decrements highlighted.

12.12 KEY WORDS

Central Mortality Rate: Probability that a person whose exact age is not known but lies in between x and $x + 1$ will die within one year following the attainment of that age,

Cohort: A hypothetical group of population gradually diminished by deaths.

Curtate Future Life Time: Curtate future lifetime is the number of completed years lived by a person aged x years.

Force of Mortality: Ratio of instantaneous rate of decrease in the number of persons living at age x to the value of the number of persons living at age x.

Joint-Life Status: To exist as long as all participating lives survive.

Last-Survivor Status: It is defined to be intact when at least are of all the participating lives survive so that it fails with the last death.

Life Annuity: A series of payments, which are made while the beneficiary (of initial age) lives.

Life Table: A conventional method of expressing the most fundamental facts about the age distribution of mortality in a tabular form.

Net Premium Reserve: The net premium reserve at time t is defined as the conditional expectation of the difference between the present value of the future benefit payments and the present value of the future premium payment given that the future life time is greater than the given time.

Net Premium: The expected value of the loss is zero.

12.13 SOME USEFUL BOOKS

Batten R.W. (1978), Mortality Table Construction, Prentice-Hall, Englewood diffs, New Jersay

Elandt-Johnson, R.C., Johnson N.L. (1967), Survival Models and Data Analysis, John Wiley and Sons, New York, London, Sydney

Jordan, C.W. (1967), Life Contingencies, Second Edition, Society of Actuaries, Chicago, III

Nill, A. (1977), Life Contingencies, William Heinemann, London

Woltheris, H. Van Hock. I. (1986), Stochastic Models for Life Contingencies, Insurance: Mathematics and Economics, 5, 1986 (3), 217-254.

12.14 ANSWER OR HINTS TO CHECK YOUR PROGRESS

Check Your Progress 1

1) Read Section 12.3 and answer.
2) Read 12.4 and answer.
3) Read 12.5 and answer.

Check Your Progress 2

1) Read Section 12.7 and answer
2) Read Section 12.9 and answer
3) Read Section 12.10 and answer

12.15 EXERCISES

1) Given $l_x = (121 - x)^{\frac{1}{2}}$ *for* $0 \le x \le 121$. Calculate the probability that a life age 21 will die after attaining age 40 but before attaining age 57.

2) Given (i) i = 0.10. (ii) $a_{30:\overline{q|}} = 5.6$ (iii) $v^{10}\ {}_{10}P_{30} = 0.35$, Calculate $1000\ P^{1}_{30:\overline{10|}}$.

3) Given ${}_{10}V_{25} = 0.1$ *and* ${}_{10}V_{35} = 0.2$ calculate ${}_{20}V_{25}$.

4) In a certain population, smokers have a force of mortality twice of non-smokers. For non-smokers, $s(x) = 1 - x/75,\ 0 \le x \le 75$, calculate $e^{0}_{55:65}$ for a smoker (55) and a non-smoker (65).

5) Given $\mu_{j,x+t} = \dfrac{j}{150}$, *for* $j = 1,2,3$ & $t > 0$ calculate $E[T|J = 3]$.

UNIT 13 CREDIBILITY THEORY

Structure

13.0 OBJECTIVES

After going through this unit, you will be able to:

- appreciate the role credibility in insurance;
- predict future events or costs;
- limit the random fluctuations in the actuarial business;
- estimate the cost to provide future insurance coverage; and
- determine the criterion for credibility.

13.1 INTRODUCTION

Credibility theory provides tools to deal with the randomness of data that are used for predicting future events or costs. For this purpose we need other information together with the recent observations. For example, suppose that the recent experience indicates that skilled workers should be charged a rate of Rs.5 (per Rs.100 of pay roll) for workers compensation insurance. Assume that current rate is Rs.10. What should the new rate be? Should it be Rs.5, Rs.10 or in between? Credibility is used to weigh together these two estimates.

The basic formula for calculating credibility weighted estimate is:

Estimate: $Z \times$ [observation] + $(1 - Z)$ [other information], $0 \leq z \leq 1$. Z is called the credibility assigned to the observation whereas $(1 - Z)$ is called the complement of credibility. In the above example Rs.10 (current rate) is the other information Thus, the skilled workers rate of workers compensation insurance is

$Z\times$ Rs.5 + (1 – Z) Rs.10. Under this we calculate Z for solution we which is given in next sections.

The credibility Z is a function of the expected variance of the observations versus the selected variance to be allowed in the first term of the credibility formula, $Z\times$[observation]. Bühlmann credibility is referred to as least square credibility. Another approach that combines current observations with prior information to produce a better estimate is Bayesian analysis. Bayes theorem is the basis of this analysis.

13.2 CLASSICAL CREDIBILITY

In classical credibility, one determines how much data will be needed before assigning to it 100% credibility. Such amount of data is referred to as full credibility criterion or the standard for full credibility criterion or the standard for full credibility. If one has this much of data or more, then z = 1.00; if one has observed less than this amount of data then $0 \le z < 1$ and we call this as partial credibility.

There are four basic concepts from classical probability that we will cover.

1) How to determine the criterion for full credibility when estimating frequencies.
2) How to determine the criterion for full credibility for estimating severities.
3) How to determine the criterion for full credibility when estimating pure premiums (loss costs).
4) How to determine the amount of partial credibility to be assigned when one has less data than is needed for full credibility.

Full Credibility for Frequency

Here we mainly use normal approximation to Poison process. The probability P that observation X is within $\pm k$ of the mean μ is

$$P = \Pr ob\left[\mu - k\mu \le X \le \mu + k\mu\right]$$

$$= \Pr ob\left[-k\left(\frac{\mu}{\sigma}\right) \le \frac{X-\mu}{\sigma} \le k\left(\frac{\mu}{\sigma}\right)\right] \qquad (13.2.1)$$

If we assure $u = \dfrac{X-\mu}{\sigma}$ is normally distributed for a Poison distribution with expected number of claims n, then $\mu = n$ and $\sigma = \sqrt{n}$. The probability that the observed number of claims N is within $\pm k$ of the expected number $\mu = n$ is:

$$P = \Pr ob\left[-k\sqrt{n} \le u \le k\sqrt{n}\right] = \Phi\left(k\sqrt{n}\right) - \Phi\left(-k\sqrt{n}\right)$$

$$= \Phi\left(k\sqrt{n}\right) - \left[1 - \Phi\left(k\sqrt{n}\right)\right]$$

$$= 2\Phi\left(k\sqrt{n}\right) - 1 \qquad (13.2.2)$$

$$\Rightarrow \Phi\left(k\sqrt{n}\right) = \frac{(1+P)}{2} \qquad (13.2.3)$$

Example 1: If the number of claims has a Poisson distribution, compute the probability of being within $\pm 5\%$ of a mean of 100 claims using the normal approximation to the Poison distribution.

Solution: $2\Phi\left(0 \cdot 05\sqrt{100}\right) - 1 = 38 \cdot 3\%$.

To compute the number of expected claims n_0 such that the chance of being within $\pm k$ of the mean is P. Let $y = \sqrt{k}$ be such that $\Phi(y) = \dfrac{1+P}{2}$.

y is determined from normal table, which yields

$$n_0 = \frac{y^2}{k^2} \tag{13.2.4}$$

Example 2: For $P = 95\%$ and for $k = 5\%$, what is the number of claims required for full credibility for estimating the frequency?

Solution: $y = 1 \cdot 960$. Since $\Phi(1 \cdot 960) = \dfrac{1+P}{2} = 97.5\%$

$$n_0 = \frac{y^2}{k^2} = \left(\frac{1 \cdot 96}{0 \cdot 05}\right)^2 = 1537$$

Full Credibility for Severity

The classical credibility ideas can also be applied to estimate the claim severity, the average size of a claim.

Suppose a sample of N claims $X_1, X_2, X_3, \ldots X_N$ are each independently drawn from a loss distribution, with mean μ_s and variance σ_s^2. The severity, i.e., mean of the distribution can be estimated by $(X_1 + X_2 + \ldots + X_N)/N$. The variance of the observed severity is

$$Var\left(\sum_{i=1}^{N} \frac{X_i}{N}\right) = \left(\frac{1}{N^2}\right) \sum_{i=1}^{N} Var(X_i) = \frac{\sigma_s^2}{N}.$$

Therefore, standard deviation is $\dfrac{\sigma_s}{\sqrt{N}}$.

The probability that the observed severity S is within $\pm$k of the mean μ_s is

$$P = \Pr ob\left[\mu_s - k\,\mu_s \le S \le \mu_s + k\mu_s\right] \tag{13.2.5}$$

Subtracting throughout μ_s, and dividing by $\dfrac{\sigma_s}{\sqrt{N}}$ and substituting $u = \dfrac{S - \mu_s}{\dfrac{\sigma_s}{\sqrt{N}}}$

we get

$$P = \Pr ob\left[-k\sqrt{N}\left(\frac{\mu_s}{\sigma_s}\right) \le u \le k\sqrt{N}\left(\frac{\mu_s}{\sigma_s}\right)\right] \tag{13.2.6}$$

According to central limit theorem the distribution of $\left(\sum_{i=1}^{M}\frac{X_i}{N}\right)$ can be approximated by a normal distribution for large N. Now define $\Phi(y)=\frac{1+P}{2}$.

We want $y=k\sqrt{N}\left(\frac{\mu_s}{\sigma_s}\right)$. Solving for N

$$N=\left(\frac{y}{k}\right)^2\left(\frac{\sigma_s}{\mu_s}\right)^2 \tag{13.2.7}$$

But $\frac{\sigma_s}{\mu_s}=CVs$, the coefficient of variation of the claim size distribution. Letting n_0 be the full credibility standard for frequency, given p and k produces

$$N = n_0 \ CVs^2$$

This is the standard for full credibility for severity.

Example 3: The coefficient of variation of the severity is 3. For P = 95% and k = 5%, what is the number of claims required for full credibility for estimating the severity?

Solution: From example 2,

$n_0 = 1537$. So $N = 1537\cdot(3)^2 = 13{,}833$ claims.

Full Credibility for Pure Premiums

Suppose that N claims of sizes $X_1, X_2..., X_N$ occur during the observation period. The following quantities are useful in analysing the cost of insuring a risk or group of risks.

Aggregate losses $L=(X_1+X_2+...+X_N)$

Pure Premium: $PP = \frac{(X_1 + X_2 \ ..... + X_N)}{Exposures}$

Exposures can be obtained by dividing the standard of credibility by the no of claims.

Loss Ratio: $LR=\frac{(X_1+X_2+...+X_N)}{Earned\ Premium}$

$$\text{Pure Premium} = \frac{Losses}{Exposures}$$

$$=\left(\frac{Number\ of\ claims}{Exposures}\right)\left(\frac{Losses}{Number\ of Claims}\right)$$

$$= (Frequency)\ (Severity) \tag{13.2.8}$$

When frequency and severity are not independent,

process variance of pure premium

= (Mean frequency) (Variance of Severity)

+ (Mean severity)^2 (Variance of frequency)

or $\sigma_{pp}^2 = \mu_f \, \sigma_s^2 + \mu_s^2 \, \sigma_f^2$ (13.2.9)

When frequency and severity are not independent, process variance can be obtained using the variance formula, i.e.,

$$V(X) = E(X^2) - \{E(X)\}^2$$

where X is pure premium.

In (13.2.9) if we use Poisson frequency then

$$\sigma_{PP}^2 = \mu_f(\sigma_s^2 + \mu_s^2) = \mu_f(2nd \quad moment \quad of \quad severity) \quad (13.2.10)$$

The subscripts indicate the means and variance of the frequency (f) and severity (s). Assuming normal approximation, full credibility standards can be calculated in the same manner as given earlier.

The probability that the observed pure premium PP is within $\pm k$ of the mean μ_{pp} : is given by

$$P = \Pr ob[\mu_{PP} - k\mu_{PP} \le PP \le \mu_{PP} + k\mu_{PP}]$$

$$= \Pr ob\left[-k\left(\frac{\mu_{PP}}{\sigma_{PP}}\right) \le u \le k\left(\frac{\mu_{PP}}{\sigma_{PP}}\right)\right]$$

where $u = \dfrac{PP - \mu_{PP}}{\sigma_{PP}}$ is a unit normal variable, assuming normal approximation.

Define y such that $\Phi(y) = \dfrac{1+P}{2}$ and

$$y = k\left(\frac{\mu_{PP}}{\sigma_{PP}}\right)$$

Assuming that frequency is a Poisson process and that n_F is the expected number of claims required for full credibility, we get

$$\mu_f = \sigma_f^2 = n_F .$$

Further, suppose that frequency and severity are independent Then

$$\mu_{PP} = \mu_f \quad \mu_s = n_F \quad \mu_s \text{ and } \sigma_{PP}^2 = \mu_f \quad (\sigma_s^2 + \mu_s^2) = n_F(\sigma_s^2 + \mu_s^2)$$

substituting for μ_{pp} & σ_{pp}

$$y = k\left[\frac{n_F \, \mu_s}{\{n_F \, (\sigma_s^2 + \mu_s^2)\}^{\frac{1}{2}}}\right]$$

Solving for n_F

$$n_F = \left(\frac{y}{k}\right)^2 \left[1+\left(\frac{\sigma_s^2}{\mu_s^2}\right)\right] = n_0\left(1+CV_s^2\right) \tag{13.2.12}$$

This is the standard for full credibility of the pure premium. $CV_s = \left(\frac{\sigma_s}{\mu_s}\right)$ is the coefficient of variation of the severity.

Example 4: The number of claims has a Poisson distribution. The mean of the severity distribution is 2000 and the standard deviation is 4000. For $P = 90\%$ and $k = 5\%$ what is the standard for full credibility of the pure premium?

Solution: $n_0 = 1082$ claims CVs = 2.

So, $n_F = 1082\left(1+2^2\right) = 5410$ claims.

It is interesting to note that

$$n_F = n_0 + n_0 CV_s^2$$

= standard for full credibility of frequency + standard for full credibility of severity.

Partial Credibility

When one has at least the number of claims needed for full credibility, then one assigns 100% credibility to the observations. However, when there is less data than is needed for full credibility, less than 100% credibility is assigned.

Let n be the (expected) number of claims for the volume of data and n_F be the standard for Full credibility. Then the partial credibility assigned is $Z = \sqrt{\frac{n}{n_F}}$, if $n \geq n_F$, and $Z = 1.00$. Use the square root rule for partial credibility for either frequency, severity or pure premium.

Example 5: The standard for full credibility is 683 claims and one has observed 300 claims. How much credibility is assigned to this data?

Solution: $\sqrt{\frac{300}{683}} = 66.3\%$.

13.3 BAYESIAN CREDIBILITY THEORY

Bayesian analysis is another technique to update a prior hypothesis based on the observations. This theory is based on Baye's theorem. The statement of the theorem is given below.

If one has a set of mutually disjoint events A_i, then one can write the marginal distribution function $P[B]$ in terms of the conditional distribution $P\left[B|A_i\right]$ and the probabilities $P[A_i]$

$$P[B] = \sum_i P\left(B|A_i\right)\ P[A_i] \text{ and}$$

$$P[A_i|B] = \frac{P(A_i).\ P[B|A_i]}{\sum_i P[B|A_i]\ .\ P(A_i)} \qquad (13.3.1)$$

$P[A_i|B]$ is the conditional probability of A_i given that B has already occurred.

Conditional Expectation

In order to compute conditional expectation, we take the weighted average over all the possibilities x:

$$E[X|B] = \sum_x x\ .\ P[X = x|B] \qquad (13.3.2)$$

Example 13.3.1

Let G be the result of rolling a green 6 sided die. Let R be the result of rolling a red 6 sided die. G and R are independent of each other. Let M be the maximum of G and R. What is the expectation of the conditional distribution of M if $G = 3$?

Solution: $M = Max(G,R)$

The conditional distribution of M if $G = 3$

$$f(3) = \frac{3}{6}\ ,\ f(4) = \frac{1}{6}\ ,\ f(5) = \frac{1}{6}\ \&\ f(6) = \frac{1}{6}.$$

Note that if $G = 3$, then M = 3 if $R = 1, 2$ or 3,

So $f(3) = \frac{3}{6}$. Thus the conditional expectation of M if $G = 3$ is

$$(3)\left(\frac{3}{6}\right) + 4\left(\frac{1}{6}\right) + 5\left(\frac{1}{6}\right)\ 6\left(\frac{1}{6}\right) = 4.$$

Bayesian Analysis

Take the following simple example. Assume that there are two types of risks each with Bernoulli claim frequencies. One type of risk has 30% chance of a claim and a 70% chance for no claims. The second type has a 50% chance of having a claim. Of the universe of risks, ¾ are of the first type with a 30% chance of claim, while ¼ are of second type with a 50% chance of having a claim.

Type of Risk	A Priori probability that a risk is of this type	Chance of a claim occurring for a risk of this type
1	¾	30%
2	¼	50%

If a risk is chosen at random, then the chance of having a claim is

$$\left(\frac{3}{4}\right)(30\%) + \frac{1}{4}(50\%) = 35\%.$$

Thus chance of no claims is 65%. Assume that we pick a risk at random and observe no claim. Then what is the chance that we have risk type 1?

$$P(Type1|n = 0) = P(Type\ 1\ and\ n = 0)/P(n = 0).$$

However P (Type = 1 and n = 0) = P (n = 0 | Type = 1). P (Type = 1)

= (0.7) (0.75).

Therefore, $P(Type = 1|n = 0) = \dfrac{P(n = 0|Type1) \cdot P(Type = 1)}{P(n = 0)}$

$= \dfrac{(0.7)(0.75)}{0.65} = 0.8077$.

This is a special case of Baye's Theorem

$$P(A|B) = P(B|A)\ \frac{P(A)}{P(B)}.$$

Or, $P(RiskType|Observ) \ = \ P(Observ|RiskType) \times P(Risk\ Type) \big/ P(Observ)$

Example 13.3.2

Assume we pick a risk at random and observe no claim. Then what is the chance that we have risk Type 2?

$$P(Type = 2|n = 0) = P(n = 0|Type2).P(Type = 2) \big/ P(n = 0)$$

$$= \frac{(0.5)(0.25)}{0.65} = 0.1923.$$

Posterior estimates

When we have probabilities posterior to an observation, this can be used to estimate a claim if the same risk is observed again. For example, if there is no claim, the estimated claim frequency for the same risk is:

(Posterior prob. Type 1) (Claim freq. Type 1)

+ (Posterior prob. Type 2) (Claim freq. Type 2)

= (0.8077) (30%) + (0.1923) (50%) = 33.85%.

Thus the posterior estimate is a weighted average of the hypothetical means for the different types of risks. The posterior estimate of 33.85% lies between 30% and 50%. But it is not necessarily true while applying to credibility.

Example 13.3.3

If a risk is chosen at random and one claim is observed, what is the posterior estimate of the chance of a claim from this same risk?

Solution: (0.6429) (0.3) + (0.3571) (0.5) = 37.14 %.

A	B	C	D	E	F
Type of risk	Apriori chance of this type of risk	Chance of the observation	Prob. Weight = product of the columns B & C	Posterior chance of this type of risk	Mean annual freq.
1	0.75	0.3	0.225	64.29%	0.30
2	0.25	0.5	0.125	35.71%	0.50
Over all			0.35	1.000	37.14%

$$P(\text{Type} = 1 \text{ and } n = 1) \Big/ P(n = 1)$$

$$= \frac{(0.75)(0.3)}{0.35} = 0.643\,.$$

Similarly, P (Type = 2 | n = 1) = 0.357.

Here we note that the estimated posterior to the observation of one claim is 37.14%, which is greater than the apriori estimate of 35%. Thus we infer that the future chance of a claim from this risk is higher than it was prior to the observation.

We had 65% chance of observing no claim and 35% chance of one claim, weighting together the two posterior estimate: (65%) (33.85%) + (35%) (37.14%) = 35%. The weighted average of the posterior estimates is equal to the overall apriori mean. This is referred to as "the estimates in balance". If D_i are the possible outcomes, then the Bayesian estimates are $E[X \mid D_i]$ and

$$\sum_i P(D_i)\, E[X|D_i] = E[X] = \text{a priori mean}$$

i.e., $$\sum_i P(D_i)\, E[X|D_i] = E[X] \qquad (13.3.3)$$

The sum of the products of the a priori chance of each outcome times its posterior Bayesian estimate is equal to the apriori mean.

Multi-Sided Dice Example

Assume that there are a total of 100 multisided dice for which 60 are 4-sided, 30 are 6-sided and 10 are 8-sided. For a given die, each side has an equal chance of being rolled i.e., the die is fair.

One person has picked at random a multi-sided die (you do not know what sided die he has picked). He then rolled the die and told you the result. You are to estimate the result when he rolls that same die again.

If the result is a 3, then the estimate of the next roll of the same die is 2.853.

A	B	C	D	E	F
Type of Die	Apriori chance of this type of die	Chance of the observation	Prob. Weights = product of columns *B* & *C*	Posterior chance of this type of die	Mean die roll
4-sided	0.600	0.250	0.1500	70.6%	2.5
6-sided	0.300	0.167	0.0500	23.5%	3.5
8-sided	0.100	0.125	0.0125	5.9%	4.5
Overall			0.2125	1.000	2.853

Example 13.3.4

If instead a 6 is rolled, what is the estimate of the next roll of the same die?

Solution:

A	B	C	D	E	F
Type of die	Apriori chance of this type of die	Chance of the observation	Prob. Weights = product of the columns *B* & *C*	Posterior chance of this type of die	Mean die roll
4-sided	0.600	0.000	0.0000	0.0%	2.5
6-sided	0.300	0.167	0.0500	80.0%	3.5
8-sided	0.100	0.125	0.0125	20.0%	4.5
Overall			0.625	1.000	3.700

Thus the estimate of the next roll of the same die is 3.7.

For this example we get the following set of estimates corresponding to each possible observation:

Observation	1	2	3	4	5	6	7	8
Bayesian estimate	2.853	2.853	2.853	2.853	3.7	3.7	4.5	4.5

13.4 BUHLMANN CREDIBILITY

Bühlmann credibility is also known as least squares credibility or the greatest accuracy credibility. Credibility is given by the formula: $Z = N / (N+K)$ where N is number of observations and K is called Bühlmann credibility parameter which can be determined from the expected value of the process variance and the variance of hypothetical means using analysis of variance.

Analysis of Variance

Consider the example of multisided dice: There are a total of 100 multi-sided dice of which 60 are 4 sided, 30 are 6-sided and 10 are 8-sided. For a given die, each side has an equal chance of being rolled; i.e., the die is fair. One person picked at random a multisided die. He then rolled the die and told you the result. You are to estimate the result when he rolls the same die again.

In order to apply Bühlmann credibility to this problem one will first have to calculate the items that would be used in "analysis of variance". One needs to compute the expected value of the process variance (*EPV*) and the variance of the hypothetic means, which together sum to the total variance.

Expected Value of the Process Variance (EPV)

We can compute the mean and variance for each type of die. Mean and variance for a 6-sided die is given below.

A	B	C	D
Roll of die	APriori Probability	Column A × Column B	Square of Column A × Column B
1	0.16667	0.16667	0.166667
2	0.16667	0.33333	0.66667
3	0.16667	0.50000	1.50000
4	0.16667	0.66667	2.66667
5	0.16667	0.83333	4.16667
6	0.16667	1.00000	6.00000
Sum	1	3.5	15.16667

Thus the mean is 3.5 and variance is $15.16667 - (3.5)^2 = 2.91667 = 35/12$. Thus the conditional variance if a 6-sided die is picked is: Var $[X \mid \text{6-sided}] = 35/12$. Similarly the mean and variance of a 4-sided die is 2.5 and 15/12 respectively and the mean and variance of a 8-sided die is 4.5 and 63/12 respectively.

Expected value of the process variance can be computed by weighting together the process variance for each type of risk using as weights the chance of having each type of risk. Expected value of the process variance is:

$$(60\%)\left(\frac{15}{12}\right)+(30\%)\left(\frac{35}{12}\right)+(10\%)\left(\frac{63}{12}\right)=\frac{25.8}{12} = 2.15$$

Variance of Hypothetical Means: (VHM)

A	B	C	D
Type of die	APrior Chance of this type of die	Mean for this type of die	Square of the mean of this type of Die
4-sided	0.6	2.5	6.25
6-sided	0.3	3.5	12.25
8-sided	0.1	4.5	20.25
Average		3	9.45

Variance of the hypothetical mean is:

$$9.45 - 3^2 = 0.45.$$

This is the variance for a single observation i.e., one roll of a die.

Total Variance

One can compute the total variance if one were to do this repeatedly. In this case, there is a $60\% \times \left(\frac{1}{4}\right) = 15\%$ chance that a 4-sided die will be picked and then a 1 is rolled. Similarly, this chance for 6-sided die is $30\% \times \left(\frac{1}{6}\right) = 5\%$ and 8-sided die is $(10\%) \times \frac{1}{8} = 1.25\%$. The total chance of a 1 is 15% + 5% + 1.25% = 21.25%.

A	B	C	D	E	F	G
Roll of a die	Probability due to 4 sided die	Probability due to 6-sided die	Probability due to 8-sided die	Aprior probability = $B+C+D$	Column A × Column E	(Square of A) × E
1	0.15	0.05	0.0125	0.2125	0.2125	0.2125
2	0.15	0.05	0.0125	0.2125	0.4250	0.8500
3	0.15	0.05	0.00125	0.2125	0.6375	1.9125
4	0.15	0.05	0.0125	0.2125	0.8500	3.4000
5		0.05	0.0125	0.0625	0.3125	1.5625
6		0.05	0.0125	0.0625	0.3750	2.2500
7			0.0125	0.0125	0.0875	0.6125
8			0.0125	0.0125	0.1000	0.8000
Sum	0.6	0.3	0.1	1	3	11.6

Total variance $= 11.6 - 3^2 = 2.6$.

Note that $EPV + VHM = 2.15 + 0.45 = 2.6$.

So $EPV + VHM =$ Total variance

$$EPV = E_\theta\left[Var\left[X \mid \theta\right]\right]$$

$$VHM = Var_\theta\left[E\left[X \mid \theta\right]\right]$$

For *EPV*, first we separately compute the process variance for each of the type of risks and then take the expected value over all types of risks. For *VHM* first we compute the expected value for each type of risk and then take the variance overall types of risks.

Example10.4.1

The following information will be used in a series of examples involving the frequency, severity and pure premium.

Type	Portion of risks in this type	Bernoulli (Annual) frequency distribution	Severity Gamma distribution
1	50%	$P = 40\%$	$\alpha = 4$, $\lambda = 0.01$
2	30%	$P = 70\%$	$\alpha = 3$, $\lambda = 0.01$
3	20%	$P = 80\%$	$\alpha = 2$, $\lambda = 0.01$

We assume that the types are homogeneous i.e., every insured of a given type has the same frequency and severity process. Further, assume that for an individual insured, frequency and severity are independent. We will show how to compute the expected value of the process variance and the variance of the hypothetical mean in each case i.e., frequency, severity and pure premium.

Expected Value of the Process Variance, Frequency example

For type 1 process variance of Bernoulli frequency is pq = (0.4) (1-0.4) = 0.24. For type 2, process variance for the frequency = (0.7) (0.3) =0.21 and for type 3 process variance for the frequency =(0.8) (1 – 0.8) = 0.16.

$$EPV = (50\%)\,(0.24) + (30\%)\,(0.21) + (20\%)\,(0.16) = 0.215.$$

Variance of the Hypothetical Mean Frequencies

For type 1 mean = p = 0.4, for type 2 it is 0.7 and for type 3 it is 0.8.

First moment = (50%) (0.4) + (30%) (0.7) + (20%) (0.8) = 0.57

Second moment = (50%) $(0.4)^2$ + (30%) $(0.7)^2$ + (20%) $(0.8)^2$ = 0.355

$VHM = .355 - (0.57)^2 = 0.0301.$

Excepted Value of the Process Variance Severity Example

In this case one has to weigh together the process variances of the severities for the individual types using the chance that a claim comes from each type. The chance that a claim comes from an individual of a given type is proportional to the product of the apriori chance of an insured being of that type and the mean frequency for that type.

For type 1, the process variance of Gamma severity is $\dfrac{\alpha}{\lambda^2} = \dfrac{4}{0.01^2} = 40{,}000$; for type 2 it is $\dfrac{3}{0.01^2} = 30{,}000$ and, for type 3, it is $\dfrac{2}{0.01^2} = 20{,}000$. The mean frequencies are 0.4, 0.7 and 0.8. The apriori chance of each type are 50%, 30% and 20% respectively. Thus, the weights for EPV are (0.4) (50%) = 0.2, (0.7)(30%) = 0.21 and (0.8) (20%) = 16. The sum of these weights 0.2 +0.21 + 0.16 = 0.57. The probability that a claim came from each class is

$$\frac{0\cdot 2}{0\cdot 57} = 0\cdot 351, \quad \frac{0\cdot 21}{0\cdot 57} = 0.368, \quad \frac{0\cdot 16}{0\cdot 57} = 0\cdot 281$$

EPV of the severity = {(0.2) (40,000) + (0.21) (30000) +(0.16) (20,000)} / (0.2 + 0.21 + 0.16) = 30,702.

Variance of Hypothetical Mean Severities

A	B	C	D	E	F	G	H
Class	Apriori probability	Mean frequency	Weights = B x C	Gamma parameters α λ		Mean severity	Square of mean severity
1	50%	0.4	0.20	4	0.01	400	160,000
2	30%	0.7	0.21	3	0.01	300	90,000
3	20%	0.8	0.16	2	0.01	200	40,000
Average			0.57			307.02	100,526

Mean of Gamma severity = $\frac{\alpha}{\lambda} = \frac{4}{0.01} = 400$

First moment = $\frac{(0\cdot2)\ (400)\ +\ (0\cdot21)\ (300)\ +\ (0\cdot16)\ (200)}{0\cdot20\ +0\cdot21+\ 0\cdot16} = 307\cdot2$

Second moment =

$$\frac{(0\cdot2)\ (400)^2\ +\ (0\cdot21)\ 300^2\ +\ (0\cdot16)\ 200^2}{0\cdot57} = 100.526$$

VHM of the severity = 100. 526 – 307.02^2 = 6,265.

Expected Value of the Process Variance, Pure Premium Example

A	B	C	D	E	F	G
Class	A priori probability	Mean frequency	Variance of frequency	Mean severity	Variance of severity	Process variance
1	50%	0.4	0.24	400	40,000	54,400
2	30%	0.7	0.21	300	30,000	39,900
3	20%	0.8	0.16	200	20,000	22,400
Average						43650

Since frequency and severity are independent, the process variance of the pure premium= (Mean Frequency) (Variance of severity) + (Mean severity)^2 (Variance of frequency) = (0.4) (40,000) + $(400)^2$ (0.24) = 54, 400.

Variance of Hypothetical Mean Pure, Premium case

Class	Probability	Frequency	Severity	Mean pure premium	Square of the pure premium
1	50%	0.4	400	160	25,600
2	30%	0.7	300	210	44,100
3	20%	0.8	200	160	25,600
Average				175	31,150

Mean of pure premium = (Mean Frequency) (Mean Severity)

Variance of hypnotically mean of pure premium = 31,150 – 175^2 = 525.

Bühlmann Credibility

Using Bühlmann credibility, the new estimate = *Z*(observation)+ (1-*Z*) (prior mean).

In 100 multisided dice example, the prior mean is 3. The apriori expected value of selecting a die at random and rolling it is: 0.6(2.5) +0.3(3.5) + 0.1(4.5) = 3.00. Bühlmann credibility parameter is calculated as $K = EPV/VHM$. For the example of 100 multisided die

$$K = \frac{EPV}{VHM} = \frac{2 \cdot 15}{0 \cdot 45} = 4 \cdot 778 = \frac{43}{9}.$$

For N observations, the Bühlmann credibility is:

$$Z = \frac{N}{N+k}.$$

In this case for one observation, $Z = \frac{1}{1+4.778} = 0 \cdot 1731$. Thus if we observe a roll of a 5, then the new estimate is $(0 \cdot 1731)\ (5) + (1-0.1731)\ (3) = 3 \cdot 3462$. The Bühlmann credibility estimate is a linear function of the observation:

$0 \cdot 1731$ (observation)+2.4807.

Note that if $N = 1$, then $Z = 1/1+K$

$$= \frac{VHM}{VHM + EPV} = \frac{VHM}{Total\ \text{var}iance}$$

Observation	1	2	3	4	5	6	7	8
New estimate	2.6538	2.8269	3	3.1713	3.3462	3.5193	3.6924	3.8655

When $N = 3$, $Z = 3 / 3 + 4.778 = 0.386$. As $N \to \infty$, $Z \to 1$. But unlike classical credibility, Bühlmann credibility never reaches 100%.

In general one computes *EPV* and *VHM* for a single observation and then plugs into the formula for Bühlmann credibility with the number of observation *N*. If one is estimating claim frequencies or pure premiums then *N* is in exposures. If one is estimating claim severities then *N* is in the number of claims (*N* is in the units of whatever is in the denominator of the quantity one is estimating)

Example 13.4.2

In example 13.4.1. we assumed that the types were homogeneous i.e., every insured of a given type has the same frequency and severity process. Assume that for an individual insured, frequency and severity are independent. Let us compute Bühımann credibility parameter in each case.

Suppose that an insured is picked at random and we do not know what type he is. For this randomly selected insured during 4 years one observes 3 claims for a total of $450. Then one can use Bühlmann credibility to predict the future frequency, severity or pure premium of this insured.

Frequency Example

$$K = \frac{EPV}{VHM} = \frac{0.215}{0.0301} = 7.14.$$

Thus, 4 years of experience are given a credibility of $\frac{4}{4+K} = \frac{4}{11.14} = 35.9\%$.

The observed frequency is 0.75 and the apriori mean frequency is 0.57. Thus, the estimate of the future frequency for this insured is (0.359) (0.75) + (1-0.359) (0.57) = 0.635.

Severity Example

$$K = \frac{EPV}{VHM} = \frac{30{,}702}{6{,}265} = 4.9.$$

3 observed claims are given a credibility of 3 / 3+K = 38.0%. The observed severity is \$450 / 3 = \$150. Thus the estimate of the future severity for this insured is:

$$(0.380)(150) + (1 - 0.380)(307) = \$247.3.$$

Pure Premium Example:

$$K = \frac{EPV}{VHM} = \frac{43{,}650}{525} = 83.1$$

4 years of experience is given a credibility of 4 / 4+K = 4/87.1 = 4.6%. The observed pure premium is \$450 / 4 = \$112.5. The apriori mean pure premium is \$175. Thus the estimate of the future pure premium for this insured is: (0.046) (112.5) + (1-0.046) (175) = \$172.

Assumptions Underlying Z

1) The complement of credibility is given to the overall mean.
2) The credibility is determined as the slope of the weighted least squares line to the Bayesian estimates.
3) The risk parameters and risk process do not shift over time.
4) The expected value of the process variance of the sum of N observations increases as N. Therefore, the expected value of the process variance of the average of N observations decreases as 1/N.
5) The variance of the hypothetical means of the sum of N observations increases as N^2. Therefore the variance of the hypothetical means of the average of N observations is independent of N.

13.5 BÜHLMANN-STRAUB CREDIBILITY MODEL

The Bühlmann-Straub model assumes that the means of the random variables are equal for the selected risk. But the process variances are inversely proportional to the size (i.e., exposure) of the risk during each observation period. For example, when tne risk is twice as large, the process variance is halved. These assumptions are summarised in the following table.

Assumptions of Bühlmann-Starub Credibility

Exposure	Period $1/m_1$	...	Period N/m_N...
Hypothetical mean for risk θ per unit of exposure	$\mu(\theta) = E_{X\|\Theta}[X_1\|\theta] =$	...	$= E_{X\|\Theta}[X_N\|\theta]$...
Process variance for risk θ	$Var_{X\|\Theta}[X_1\|\theta] = \frac{\sigma^2(\theta)}{m_1}$	...	$Var_{X\|\Theta}[X_1\|\theta] = \frac{\sigma^2(\theta)}{m_N}$

$$\mu = 2{,}400,\ m = 240,\ \bar{X} = 3000$$

$$Z = \frac{240}{240+500} = 0 \cdot 3243$$

Estimated pure premium

$$= (0 \cdot 3243)\ (3000) + (1-0 \cdot 3243)\ (2{,}400)$$

$$= 2{,}594 \cdot 58.$$

Check Your Progress 1

1) Discuss full credibility for frequency and severity

....

....

....

2)

Type of Risk	Apriori chance of type of risk	Chance of a claim
A	60%	20%
B	25%	30%
C	15%	40%

A risk is selected at random from the above information. Calculate the expected annual claim frequency from the same risk if you observe no claim in a year using Bayesian analysis.

....

....

....

3) Define and explain Bühlmann Credibility parameter

....

....

....

13.6 ESTIMATION OF CREDIBILITY FORMULA PARAMETERS

The selection of credibility parameters requires a balancing of responsiveness versus stability. Larger credibility weights put more weight on the observations, which means that the current data have a larger impact on the estimate. Then the estimates are more responsive to current data. But this comes at the expense of less stability in the estimates. Credibility parameters often are selected to reflect the actuary's desired balance between responsiveness and stability.

In classical credibility, P and k values must be chosen where P is the probability that observation X is within $\pm k$ percent of the mean μ. Usually P = 90% and k = 5%.

Example 13.6.1

A sample of 100 claims was distributed as follows:

Size of claim	Number of claim
\$1,000	85
\$5,000	10
\$10,000	3
\$25,000	2

Estimate the coefficient of variation of the claim severity based on this distribution.

Solution: The sample mean is

$\hat{\mu} = (0.85)(1000)+(0\cdot 10)(5000)+0\cdot 03(10,000)+0\cdot 02(25,000) = 2,150.$

$\hat{\sigma} = \left[\frac{1}{(100-1)}\cdot\{85\,(1000-2150)^2 + 10(5000-2,150)^2 + 3(10,000-2,150)^2 +\right.$

$\left.2(25,0000-2,150)^2\}\right]^{\frac{1}{2}} = 3.791.$ We are dividing by $n-1$ to calculate an unbiased estimate. Thus

$$CVs = \frac{\hat{\sigma}}{\hat{\mu}} = 1\cdot 76$$

$$P = 90\% \;\&\; k \;\; 5\%.$$

From the table of standards for full credibility for frequency (claims) $n_0 = 1,082$. So, the full credibility standard for the pure premium is $n_F = n_0(1+CVs)^2 = 1,082(1+1\cdot 76^2) = 4,434.$

Suppose that there are M risks in a population and they are similar in size. Assume that we tracked the annual frequency year by year for Y years for each of the risks. The frequencies are

$$\begin{bmatrix} X_{11} & X_{12} & \cdots & X_{1Y} \\ X_{21} & X_{22} & \cdots & X_{2Y} \\ \cdots & \cdots & \cdots & \cdots \\ X_{M1} & X_{M2} & \cdots & X_{MY} \end{bmatrix}$$

X_{ij} represents the frequency of i^{th} risk and j^{th} year. Lists of estimators are given below

	Symbol	Estimator
Mean frequency for risk i	$\overline{X}_i$	$\frac{1}{Y}\sum_{j=1}^{Y} X_{ij}$
Mean frequency for population	$\overline{X}$	$\frac{1}{M}\sum_{i=1}^{M}\overline{X}_i$
Process variance for risk i	$\hat{\sigma}_i^2$	$\left[\frac{1}{(n-1)}\right]\sum_{j=1}^{Y}\left(X_{ij}-\overline{X}_i\right)^2$
Expected value of process variance	EPV	$\frac{1}{M}\sum_{i=1}^{M}\hat{\sigma}_i^2$
Variance of the hypothetical means	VHM	$\left[\frac{1}{(M-1)}\right]\sum_{i=1}^{M}\left(\overline{X}_i-\overline{X}\right)^2-\frac{EPV}{Y}$

$K=\frac{EPV}{VHM}$ If sample VHM is zero or negative then the credibility is assigned a value of 0.

Example 13.6.2

There are two auto drivers in a particular rating class. The first driver had the following sequence of claims in year 1 through 5, 2,0,0,1,0 and the second had 1,1,2,0,2. Estimate each of the values in the prior table.

Solution: For first driver, $\overline{X}_1 = 0.6$ and $\hat{\sigma}_1^2 = 0.80$ For the second driver $\overline{X}_2 = 1.2$ and $\hat{\sigma}_2^2 = 0.7$

$$\overline{X}=\frac{\overline{X}_1+\overline{X}_2}{2}=0.900$$

$$EPV=\frac{\hat{\sigma}_1^2+\hat{\sigma}_2^2}{2}=0.75.$$

$$VHM=(0.6-0.9)^2+\frac{(1.2-0.9)^2}{1}-\frac{0.75}{5}=0.03$$

$$K=\frac{EPV}{VHM}=25$$

$$Z=\frac{5}{5+25}=\frac{1}{6}$$

Thus the estimated future claim frequency for first driver is

$$\left(\frac{1}{6}\right)(\cdot 6)+\left(\frac{5}{6}\right)(\cdot 9)=0\cdot 85$$ and for the second driver is

$$\left(\frac{1}{6}\right)(1\cdot 2)+\left(\frac{5}{6}\right)(\cdot 9)=0\cdot 95.$$

Bühlmann Credibility: *(EstimateK from Best Fit to Data)*

A credibility formula of the form $Z = \dfrac{N}{N+K}$ is usually used to weigh the policyholder's experience.

One can estimate K by observing which values of K would have worked well in the past. The goal is for each policyholder to have the same expected loss ratio after the application of experience rating. Let LR_i be the loss ratio for policyholder i where, in the denominator we use the premiums after the application of experience rating. Let LR_{AVE} be the average loss ratio for all policyholders. Then define $D(K)$ to be

$$D(K) = \sum_{all\ i}\left(LR_i - LR_{AVE}\right)^2 \tag{13.6.1}$$

The sum of squares of the differences is a function of K, the credibility parameter that was used in the experience rating. The goal is to find a K that minimizes $D(K)$. This requires recomputing the premium that each policyholder would have been charged under a different K' value. This generates new LR_i's that are then put into the formula above and $D(K')$ is computed. Using techniques from numerical analysis, a K that minimizes $D(K)$ can be found out.

Another approach to calculate credibility parameters is linear regression analysis of a policyholder's current frequency, pure premium etc. Using historical data for many policyholders we set up the regression equation:

Observation in the year Y

$= m$[observation in year $(Y - 1)$] + constant (13.6.2)

The slope m from a least squares fit to the data turns out to be the Bühlmann credibility Z. The constant term is $(1 - Z)$ (coverall average). After we have calculated the parameters in our model using historical data, we can estimate future results using the model and recent data. Regression models can also be built using multiple years.

13.7 COMPARISION OF CLASSICAL AND BÜHLMANN CREDIBILITY

Although the formulae look very different, classical credibility and Bühlmann credibility can produce similar results. The most significant difference between the two models is that Bühlmann credibility never reaches $Z = 1.00$, which is an asymptote at the curve. Both models can be effective at improving the stability and accuracy of estimates.

Classical credibility and Bühlmann credibility formulae will produce approximately the same credibility weights, if the full credibility standard for classical credibility n_0, is about 7 to 8 times larger than the Bühlmann credibility parameter K.

For a particular application, the actuary can choose the model out of the three models (classical, Bühlmann and Baysian) appropriate to the goals and data. If the goal is to generate the most accurate insurance rates, with least squares as the measure of fit, then Bühlmann credibility may be the best choice. Biihlmann credibility forms the basis of most experience-rating plan. It is often used to calculate class rates in a classification plans. The use of Bühlmann credibility requires an estimate of the *EPV* and *VHM*.

Classical credibility might be used if estimates for the *EPV* and *VHM* are unknown or difficult to calculate. Classical credibility is often used in the calculation of overall rate increases. Often it is simpler to work with Bayesian analysis. Bayesian analysis may be an option if the actuary has a reasonable estimate of the prior distribution. However, Bayesian analysis may be complicated to apply and the most difficult of the methods to explain to non actuaries.

13.8 THE MAXIMUM AGGREGATE LOSS AND THE GENERAL SOLUTION

Aggregate losses, pure premiums and loss ratios depend on both the number of claims and the size of claim; they have more reasons to vary than either frequency and severity. Because they are more difficult to estimate than frequencies, all other things being equal, the standard for full credibility is larger than that for frequencies. Formulae for the variance of the pure premium as seen in section 13.2 were calculated as follows:

General case: $\sigma_{pp}^2 = \mu_f\, \sigma_S^2 + \mu_S^2\, \sigma_f^2$ (13.8.1)

Poisson frequency:

$$\sigma_{pp}^2 = \mu_f\left(\sigma_s^2 + \mu_s^2\right) = \mu_f\,(2^{nd}\text{ moment of the severity}) \qquad (13.8.2)$$

The subscripts indicate the means and variance of the frequency (*f*) and severity (*S*). Assuming the normal approximation, full credibility standards can be calculated following the same steps as in the previous sections.

The probability that the observed pure premium *PP* is within $\pm k$ of the mean μ_{PP} is

$$P = \Pr ob\left[\mu_{PP} - k\mu_{PP} \le PP \le \mu_{PP} + k\mu_{PP}\right]$$

$$= \Pr ob\left[-k\left(\frac{\mu_{PP}}{\sigma_{PP}}\right) \le u \le k\left(\frac{\mu_{PP}}{\sigma_{PP}}\right)\right]$$

where $u = \dfrac{PP - \mu_{PP}}{\sigma_{PP}}$ is a unit normal variable, assuming the normal approximation. Consequently, we have the following results:

I) Define y such that $\phi(y) = \dfrac{1+P}{2}$. Then, in order to have probability P that the observed pure premium will differ from the true pure premium by less than $\pm$k μ_{PP}: