UNIT 1 FIRST AND HIGHER ORDER EQUATIONS

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1.1 INTRODUCTION

From your knowledge of differential equations which you must have studied at the undergraduate level, you are familiar with various methods of solving first/second or even higher order linear, ordinary differential equations (ODEs) with constant coefficients. Also, for linear ODEs upto second order with variable coefficients you know certain transformations of the dependent and independent variables, which reduce the equation to a form, which is solvable by the known methods. However, in all the cases, solutions to ODEs were obtained in terms of known elementary functions. To this day, no method of getting a solution of general first order initial value problem (IVP) of the form $\frac{dy}{dx} = f(x, y)$, in closed form or, in terms of elementary functions exists. In this unit we shall give a method of finding an approximate solution to this general first order IVP.

In Sec.1.2 we shall start with defining the initial and boundary value problems and illustrate them through various examples giving geometrical interpretation of their solutions. In Sec.1.3, we have discussed the Picard's method of successive approximation for solving a first order initial value problem (IVP), $y' = f(x, y)$ subject to $y(x_0) = y_0$. We have also stated and proved Picard's theorem on existence and uniqueness of solution of IVP here. We have shown, through examples, how the nature of solutions of first order IVPs can be examined even without solving them. In Sec 1.4, we shall do a quick recap of methods of solving linear differential equations of any order with constant coefficients and illustrate them through a number of examples.

Objectives

After studying this unit you should be able to

- identify initial value problems and boundary value problems;
- obtain an approximate solution of an IVP of the first order;
- describe the nature of solution of first order differential equation with reference to the existence and uniqueness of solutions;
- solve linear differential equations of first/second/higher orders with constant coefficients.

1.2 INITIAL AND BOUNDARY VALUE PROBLEMS

Consider the first order linear differential equation

$$y' = 2x\ (1 + y^2)$$

By the usual process of integration, it can be seen that the solution of the given differential equations is

$$y(x) = \tan(x^2 + c),$$

where c is an arbitrary constant. Since c is arbitrary, for each value of c, we get a distinct solution of the given equation. Hence, a one-parameter family consisting of an infinite number of solutions is obtained. Geometrically, this equation represents a one-parameter family of curves, called integral curves of the given equation. Each integral curve is the geometric representation of the corresponding solution of the differential equation. In physical applications, we often require a specific solution out of this family representing a particular physical phenomenon. Specifying a particular solution is equivalent to picking out a particular integral curve from the one-parameter family. It is usually convenient to do this by prescribing a point (x_0, y_0) through which the integral curve must pass; that is, we seek a solution such that

$$y(x_0) = y_0.$$

Such a condition is called an **initial condition.** A first order differential equation together with an initial condition form an **initial value problem**. This terminology is suggested by the fact that the independent variable often denotes time, the initial condition defines the situation at some fixed instant, and the solution of the initial value problem describes what happens later. In the case of the equation considered above we may be interested in a solution which satisfies the requirement

$$y(0) = 0.$$

This condition means that we obtain that solution $y(x)$ which passes through the point $(0,0)$. Thus,

$$y(0) = \tan(0 + c) = \tan c.$$

and $\quad y(0) = 0$ gives $c = 0$.

The desired solution is given by

$$y(x) = \tan x^2, \ (0 < x < \sqrt{\pi/2}).$$

Next consider the differential equation of order two, namely,

$$y'' + 5y' + 6y = 0. \qquad (y' = \frac{dy}{dx}, y'' = \frac{d^2y}{dx^2}).$$

It can be verified that for the above equation

$$y(x) = c_1e^{-2x} + c_2e^{-3x}$$

is a solution, existing on $-\infty < x < \infty$. Here c_1 and c_2 are arbitrary constants. For each choice of c_1 and c_2, we get a distinct solution. Hence, $c_1e^{-2x} + c_2e^{-3x}$ is a two parameter family of solutions. To find a specific solution of this family, we require two conditions, the value of the solution $y(x_0)$ and its derivative $y'(x_0)$ at a point x_0. Thus to determine uniquely an integral curve of a second order equation it is necessary to specify not only a point through which it passes, but also the slope of the curve at that point. Suppose for the above second order equation the conditions are

$$y(0) = 0 \text{ and } y'(0) = 1$$

Then, it follows that

$$y(0) = c_1 + c_2 = 0$$

$$y'(0) = -2c_1 - 3c_2 = 1$$

These are simultaneous equations in c_1 and c_2 and solving them we get $c_1 = 1, \ c_2 = -1$. The specific solution is, therefore, given by

$$y(x) = e^{-2x} - e^{-3x}, -\infty < x < \infty.$$

We thus observe that to determine a particular solution of a first order equation, we need one condition while in the case of a second order equation, we need two conditions at the same value of x. Similarly, we conclude that for the n^{th} order equation of the form

$$P_0(x)\,y^{(n)}(x) + P_1(x)\,y^{(n-1)}(x) + \cdots + P_{n-1}(x)\,y' + P_n(x)\,y = G(x)$$

where the functions $P_0, P_1, \ldots P_n$ and G are functions of x only on some interval $x_1 < x < x_2$, we need n conditions

$$y(a) = \alpha_0, y'(a) = \alpha_1, \ldots, y^{(n-1)}(a) = \alpha_{n-1},$$

where, $x_1 < a < x_2$ and $\alpha_0, \alpha_1 \ldots, \alpha_{n-1}$ is any set of prescribed real constants. These conditions are referred to as **initial conditions**.

A differential equation together with initial conditions is called an Initial Value Problem (IVP).

You may **observe** that an IVP is determined by the differential equation and the initial conditions. A change in either of them would mean a different IVP.

Next, consider a second order differential equation

$$y'' + y = 0$$

It can be verified that solutions of this equations is given by

$$y(x) = c_1 \cos x + c_2 \sin x, \qquad -\infty < x < \infty.$$

where c_1 and c_2 are arbitrary constants. Let us assume that the following pairs of conditions are required to be satisfied by the solutions

(i) $y(0) = 0, \quad y(\pi/2) = 0$

(ii) $y(0) = 0, \quad y'(\pi/2) = 0$

(iii) $y(0) = 0, \quad y'(\pi/2) = 1$

Since the general solution involves two constants c_1 and c_2, two conditions are required to determine them. It may be noted that in this case the two conditions in each of the pair above are given at two different points, namely, $x = 0$ and $x = \frac{\pi}{2}$. The conditions given in (i), (ii) and (iii) are referred to as boundary conditions stated at $x = 0$ and $x = \frac{\pi}{2}$.

A differential equation together with boundary conditions is called Boundary-Value Problem (BVP).

If you apply these pair of conditions to the general solution $y(x) = c_1 \cos x + c_2 \sin x$, then you would see that corresponding solutions for the three cases are given respectively, by

(i) $y(x) = 0$ (ii) $y(x) = c_2 \sin x$ (iii) No solution, existing on $-\infty < x < \infty$.

You may **note** that in (i) above there is only one solution viz., trivial solution $y = 0$. In (ii), since c_2 in an arbitrary constant, we get an infinite number of solutions. In (iii), $y(0) = 0$ gives $c_1 = 0$ and $y'(\pi/2) = 1$ gives $1 = c_2 0 = 0$, which is a contradiction and hence no solution. A boundary-value problem may thus have infinite solution, single solution or no solution at all.

Similarly, for an initial value problem (IVP) the following questions may come to your mind

i) Does an initial value problem always have a solution?
ii) Can it have more than one solution?
iii) Is the solution valid for all x, or only for some restricted interval about the initial point?

Such questions are answered by the existence and uniqueness theorem for the solution of an IVP which we shall be discussing in the next section, but before that you may try this exercise:

E1) Discuss the solution of the boundary value problem

$$y'' + \lambda y = 0$$

with boundary conditions $y(0) = 0$ and $y(\pi) = 0$.

Let us now discuss the existence and uniqueness of solutions of IVPs.

1.3 EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR INITIAL VALUE PROBLEMS

We know that only a few simple type of differential equations can be solved explicitly in terms of known elementary functions. Some of these types are discussed in Blocks 1 and 2 of MTE-08. However, many differential equations fall outside this category and nothing has been done so far to suggest a procedure that might work in such cases. Let us examine the initial value problem (IVP).

$$\frac{dy}{dx} = f(x, y), \qquad y(x_0) = y_{0,} \tag{1}$$

where $f(x, y)$ is an arbitrary function defined and continuous in some neighborhood of the point (x_0, y_0). Geometrically, we want to obtain a function $y = y(x)$ satisfying the differential equation $y' = f(x, y)$ in some neighborhood of x_0 and whose graph passes through the point (x_0, y_0) (see. Fig.1).

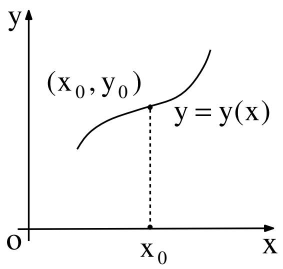

Fig.1

We know that elementary procedures will not work here. We will have to evolve a process to construct a sequence of functions that converges to a limit function satisfying the IVP. This process is provided by **Picard's method of successive approximations**.

Picard's methods for solving differential equations is quite different from any method that you might have studied so far. This method consists in replacing the initial value problem (1) by the equivalent integral equation

$$y(x) = y_0 + \int_{x_0}^{x} f[t, y(t)]\, dt \tag{2}$$

Eqn.(2) is called an i**ntegral equation** because the unknown function occurs under the integral sign.

It can be easily seen that Eqns.(1) and (2) are equivalent. For if $y(x)$ is a solution of Eqn.(1), then $y(x)$ is continuous and so is the right hand side of

$$y'(x) = f[x, y(x)] \tag{3}$$

Now we integrate Eqn.(3) w.r.t. x, from x_0 to x and obtain

$$y(x) = C + \int_{x_0}^{x} f[t, y(t)]\, dt.$$

Using the initial condition $y(x_0) = y_0$, we obtain

$$y(x_0) = C + \int_{x_0}^{x_0} f[t, y(t)]\, dt = y_0$$

implies $C = y_0$. Thus,

$$y(x) = y_0 + \int_{x_0}^{x} f[t, y(t)]\, dt,$$

which is our Eqn.(2). **Note** that t is a dummy variable here.
Thus any solution of Eqn.(1) is a continuous solution of Eqn.(2).

Conversely, if $y(x)$ is a continuous solution of Eqn.(2), then $y(x_0) = y_0$ because the integral vanishes when $x = x_0$. Also by differentiation of Eqn.(2) w.r.t. x we recover the differential equation

$$y'(x) = f[x, y(x)].$$

We can thus say that the integral Eqn.(2) and the IVP(1) are equivalent.

We now try to solve Eqn.(2) by a method of successive approximation i.e., we begin with a guess or approximation to the solution of Eqn.(2) and improve it step-by-step by applying a repeatable operation, which we hope will bring us as close as we please to an exact solution. The primary advantage that Eqn.(2) has over Eqn.(1) is that the integral equation provides a convenient mechanism for carrying out this process as we see below.

The first or a rough approximation to a solution of Eqn.(2) is given by the constant function $y(x_0) = y_0$, which is simply a horizontal straight line through the point (x_0, y_0). We put this approximation in the right hand side of Eqn.(2) to obtain a new approximation $y_1(x)$ as follows:

$$y_1(x) = y_0 + \int_{x_0}^{x} f[t,\ y_0(t)]\ dt.$$

We hope that this new function is a better approximation to the solution. We then use $y_1(x)$ to generate another and perhaps even better approximation $y_2(x)$ in the same way and obtain

$$y_2(x) = y_0 + \int_{x_0}^{x} f\,[t,\ y_1(t)]\ dt.$$

In this manner we obtain the iterants i.e., a sequence of functions $y_1(x), y_2(x), y_3(x), \ldots$ whose n^{th} term is defined by the relation

$$y_n(x) = y_0 + \int_{x_0}^{x} f\,[t, y_{n-1}(t)]\,dt, \qquad n = 1, 2, 3\ldots \tag{4}$$

The repetitive use of formula (4) is known as **Picard's method of successive approximations/iterations.**

We shall now show how this method works by means of an example.

Example 1: Apply Picard's iteration method to the initial value problem

$$y' = x^2 y, \quad y(0) = 1$$

and compare with the exact solution, if possible.

Solution : For an initial value problem

$$y' = x^2 y, \quad y(0) = 1,$$

the corresponding integral equation is

$$y(x) = 1 + \int_{x_0}^{x} t^2\ y(t)\ dt$$

Picard's iterations are

$$y_0(x) = 1$$

$$y_1(x) = 1 + \int_0^x t^2\ 1dt = 1 + \frac{x^3}{3}$$

$$y_2(x) = 1 + \int_0^x t^2\left(1 + \frac{t^3}{3}\right)dt = 1 + \frac{x^3}{3} + \frac{x^6}{18} = 1 + \frac{x^3}{3} + \frac{1}{2!}\left(\frac{x^3}{3}\right)^2$$

$$y_3(x) = 1 + \int_0^x t^2\left(1 + \frac{t^3}{3} + \frac{t^6}{18}\right)dt = 1 + \frac{x^3}{3} + \frac{x^6}{18} + \frac{1}{18}.\frac{x^9}{9}$$

$$= 1 + \frac{x^3}{3} + \frac{1}{2!}\left(\frac{x^3}{3}\right)^2 + \frac{1}{3!}\left(\frac{x^3}{3}\right)^3$$

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$$y_n(x) = 1 + \frac{x^3}{3} + \frac{1}{2!}\left(\frac{x^3}{3}\right)^2 + \frac{1}{3!}\left(\frac{x^3}{3}\right)^3 + \cdots + \frac{1}{n!}\left(\frac{x^3}{3}\right)^n$$

The exact solution of the given problem can easily be obtained as

$$y = e^{\left(\frac{x^3}{3}\right)},$$

to which the above approximate solution converges when $n \to \infty$.

* * *

We would like to **remark** here that you should not be deceived by the relative ease with which the iterants $y_n(x)$ were obtained in Example-1. In general, it is seen that the integration involved in generating each $y_n(x)$ can become complicated very quickly. Even if it is possible to generate $y_n(x)$, it may not always converge to an explicit function or the exact solution. Let us consider the following example.

Example 2: Apply Picard's method of successive approximations to the initial value problem.

$$y' = x\,(y - x^2 + 2), \qquad y(0) = 1.$$

Solution : The integral equation, equivalent to the given initial value problem is

$$y(x) = 1 + \int_0^x t(y(t) - t^2 + 2)\,dt.$$

The approximate solutions are

$$y_0(x) = 1$$

$$y_1(x) = 1 + \int_0^x t(3 - t^2)\,dt = 1 + \frac{3}{2}x^2 - \frac{1}{4}x^4$$

$$y_2(x) = 1 + \int_0^x t(3 + \frac{t^2}{2} - \frac{t^4}{4})\,dt = 1 + \frac{3x^2}{2} + \frac{x^4}{8} - \frac{x^6}{24}$$

$$y_3(x) = 1 + \int_0^x t(3 + \frac{t^2}{2} + \frac{t^4}{8} - \frac{t^6}{24})\,dt$$

$$= 1 + \frac{3x^2}{2} + \frac{x^4}{8} + \frac{x^6}{48} - \frac{x^8}{192}$$

and so on.

The exact solution of the given problem can be easily obtained as

$$y = x^2 + e^{\frac{x^2}{2}}$$

$$= x^2 + 1 + \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^6}{48} + \frac{x^8}{192} + \cdots$$

The approximate solution in this case differs from the exact solution.

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After going through Examples 1 and 2, you might ask the question:
Is Picard's method a practical means of solving a first order equation $y' = f(x, y)$ subject to $y(x_0) = y_0$? In most cases the answer is no. The question then arises, what is it good for? Picard's methods of iteration is a theoretical tool used in the consideration of existences and uniqueness of solutions of differential equations. Under certain conditions on $f(x, y)$ it can be shown that as $n \to \infty$, the sequence $\{y_n(x)\}$ defined by Eqn.(4) converges to a function $y(x)$ that satisfies the integral Eqn.(2) and hence the IVP (1) and the solution obtained is unique. The theorem that makes precise assertions of this kind is called Existence and Uniqueness Theorem which we shall discuss now but before that you may try the following exercise.

E2) Apply Picard's iteration method to the following initial value problems and compare the results obtained with the exact solutions wherever possible. Perform at least three iterations.

(a) $y' = x + y,\ y(0) = 1$ (b) $y' = x - y^2,\ y(0) = \frac{1}{2}$

(c) $y' = 2x(1+y),\ y(0) = 0$ (d) $y' = y,\ y(0) = 1$

(e) $y' = y^2,\ y(0) = 1.$

We shall now state and prove **Picard's theorem on existence and uniqueness of solution of IVP.**

Theorem 1: Let $f(x, y)$ and $\frac{\partial f}{\partial y}$ be continuous functions of x and y on a closed rectangle R with sides parallel to the axes (see. Fig.2). If (x_0, y_0) is an interior point of R, then there exists a number $h > 0$ with the property that the initial value problem

$$y' = f(x, y),\ y(x_0) = y_0 \tag{5}$$

has one and only one solution $y = y(x)$ on the interval $|x - x_0| \le h.$

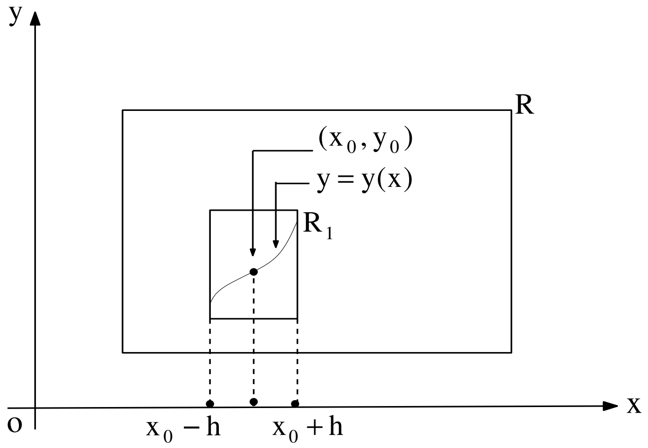

Fig.2

Proof: We know that every solution of IVP (1) is also a continuous solution of the integral equation

$$y(x) = y_0 + \int_{x_0}^{x} f[t,\ y(t)]\,dt \tag{6}$$

and conversely. We can thus say that Eqn.(5) has a unique solution on an interval $|x - x_0| \le h$**,** if and only if, Eqn.(6) has a unique continuous solution on the same interval.

By Picard's method of successive approximations, the sequence of functions $y_n(x)$ defined by

$$y_0(x) = y_0$$

$$y_1(x) = y_0 + \int_{x_0}^{x} f[t,\ y_0(t)]\,dt$$

$$y_2(x) = y_0 + \int_{x_0}^{x} f[t,\ y_1(t)]\,dt$$

....

$$y_n(x) = y_0 + \int_{x_0}^{x} f[t,\ y_{n-1}(t)]\,dt. \tag{7}$$

may converge to a solution of Eqn.(6) . Here you may observe that $y_n(x)$ is the n^{th} partial sum of the series of functions

$$y_0(x) + \sum_{n=1}^{\infty}[y_n(x) - y_{n-1}(x)] = y_0(x) + [y_1(x) - y_0(x)] + [y_2(x) - y_1(x)] + \cdots + [y_n(x) - y_{n-1}(x)] + \cdots \quad (8)$$

and thus the convergence of the sequence (7) is equivalent to the convergence of series (8). In order to prove Theorem 1, we are now required to produce a number $h > 0$ defined on the interval $|x - x_0| \le h$, and show that on this interval, the following statements are true

(i) the series (8) converges to a function y(x)
(ii) y(x) is a continuous solution of Eqn.(6)
(iii) y(x) is the only continuous solution of Eqn.(6).

We shall now prove the above three statements one by one but first of all let us try to produce a positive number h.

From the hypothesis of Theorem 1 we know that $f(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous functions on the rectangle R. Now since R is closed (as it includes its boundary) and bounded, each of these functions is necessarily bounded on R. Thus there exist constants M and N such that

$$|f(x, y)| \le M \quad (9)$$

and

$$\left|\frac{\partial}{\partial y} f(x, y)\right| \le N \quad (10)$$

for all points (x, y) is R. Now if (x, y_1) and (x, y_2) are two distinct points in R with the same x-coordinate, then the mean value theorem guarantees that

$$|f(x, y_1) - f(x, y_2)| = \left|\frac{\partial}{\partial y} f(x, y^*)\right| |y_1 - y_2| \quad (11)$$

for some number y^* between y_1 and y_2. It is clear from Eqns.(10) and (11) that

$$|f(x, y_1) - f(x, y_2)| \le N |y_1 - y_2| \quad (12)$$

for any points (x, y_1) and (x, y_2) in R lying on the same vertical line. Let us choose h to be any positive number such that

$$N h < 1 \quad (13)$$

and the rectangle R_1 defined by the inequalities $|x - x_0| \le h$ and $|y - y_0| \le Mh$ is contained in R. As (x_0, y_0) is an interior point of R, we can say that such an h exists.

Now after choosing h we shall now prove statements (i) – (iii) above. Here onwards, we shall confine our attention to the interval $|x - x_0| \le h$.

In order to prove (i), it is sufficient to show that the series

$$|y_0(x)| + |y_1(x) - y_0(x)| + |y_2(x) - y_1(x)| + \cdots + |y_n(x) - y_{n-1}(x)| + \cdots \quad (14)$$

converges and to achieve this, we estimate the terms $|y_n(x) - y_{n-1}(x)|$. First we show that the graph of each of the functions $y_n(x)$ lies in R_1 and hence in R. Obviously, $y_0(x) = y_0$, and so the point $[t, y_0(t)]$ lies in R_1. Eqn.(9) gives $|f(t, y_0(t))| \le M$, and

$$|y_1(x) - y_0| = \left|\int_{x_0}^{x} f[t, y_0(t)]\, dt\right| \le Mh,$$

which proves that the point $y_1(x)$ lies in R_1. It follows consequently from this inequality, that the point $[t, y_1(t)]$ is in R_1, so that $\left|f[t, y_1(t)]\right| \le M$ and

$$|y_2(x) - y_0| = \left|\int_{x_0}^{x} f[t, y_1(t)]\, dt\right| \le Mh.$$

Similarly, we can say that

$$|y_3(x) - y_0| = \left|\int_{x_0}^{x} f[t, y_2(t)]\, dt\right| \le Mh.$$

and so on. Now we have estimated the terms $|y_n(x) - y_{n-1}(x)|$.

We know that a continuous function has a maximum on a closed interval. Since $y_1(x)$ is continuous, we can define a constant $'\alpha'$ by $\alpha = \max|y_1(x) - y_0|$ and write

$$|y_1(x) - y_0| \le \alpha.$$

Next, the point $[t, y_1(t)]$ and $[t, y_0(t)]$ lie in R_1 and so relation (12) yields

$$\left|f[t, y_1(t)] - f[t, y_0(t)]\right| \le N\left|y_1(t) - y_0(t)\right| \le N\alpha,$$

and we have

$$|y_2(x) - y_1(x)| = \left|\int_{x_0}^{x} \{f[t, y_1(t)] - f[t, y_0(t)]\}\, dt\right| \le N\alpha h = \alpha(Nh)$$

Similarly, we obtain

$$|f[t, y_2(t)] - f[t, y_1(t)]| \le N|y_2(t) - y_1(t)| \le N^2\alpha h$$

and

$$|y_3(x) - y_2(x)| = \left|\int_{x_0}^{x} \{f[t, y_2(t)] - f[t, y_1(t)]\}dt\right|$$

$$\le (N^2\alpha h)\, h = \alpha(Nh)^2$$

Continuing this way, we find that

$$\left|y_n(x) - y_{n-1}(x)\right| \le \alpha(Nh)^{n-1},$$

for every $n = 1, 2, \ldots$. You can thus observe that each term of the series (14) is less than or equal to the corresponding term of the series of constants

$$\left|y_0\right| + \alpha + \alpha\,(Nh) + \alpha\,(Nh)^2 + \cdots + \alpha\,(Nh)^{n-1} + \cdots$$

But relation (13) guarantees that this series converges. So series (14) converges by the comparison test and hence Eqn.(8) converges to a sum, which we denote by $y(x)$, and thus $y_n(x) \to y(x)$. Also, since the graph of each $y_n(x)$ lies in R_1, the graph of $y(x)$ also has this property. This completes the proof of statement (i).

Next, we prove statement (ii). From the proof of (i) it is clear that not only $y_n(x)$ converges to $y(x)$ in the interval, but also that this convergence is uniform. This means that by choosing n to be sufficiently large, we can make $y_n(x)$ as close as we please to $y(x)$ for all x in the interval. This means that if we are given $\in > 0$, then there exists a positive integer n_0 such that, for $n \ge n_0$, we have $\left|y(x) - y_n(x)\right| < \in$, for all x in the interval. Since each $y_n(x)$ is continuous, this uniformity of the convergence implies that the limit function $y(x)$ is also continuous. To prove that $y(x)$ is actually a solution of Eqn.(6), we must show that

$$y(x) - y_0 - \int_{x_0}^{x} f[t, y(t)]\, dt = 0, \qquad (15)$$

From relations (7) we know that

$$y_n(x) - y_0 - \int_{x_0}^{x} f[t, y_{n-1}(t)]\,dt = 0, \tag{16}$$

Subtracting the left side of Eqn.(16) from the left side of Eqn.(15) we get

$$y(x) - y_n(x) + \int_{x_0}^{x} \{ f[t, y_{n-1}(t)] - f[t, y(t)] \}dt = 0,$$

We can thus write

$$y(x) - y_0 - \int_{x_0}^{x} f[t, y(t)]\,dt = y(x) - y_n(x) + \int_{x_0}^{x} \{f[t, y_{n-1}(t)] - f[t, y(t)]\}\,dt.$$

Taking modulus of both sides, we obtain

$$\left| y(x) - y_0 - \int_{x_0}^{x} f(t, (t))\,dt \right| \le \left| y(x) - y_n(x) \right| + \left| \int_{x_0}^{x} \{f(t, y_{n-1}(t)) - f(t, y(t))\}dt \right|$$

Since the graph of $y(x)$ lies in R_1 and hence in R, using relations (12) we can write

$$| y(x) - y_0 - \int_{x_0}^{x} f(t, y(t))dt | \le | y(x) - y_n(x) | + Nh. \max | y_{n-1}(x) - y(x) | \tag{17}$$

The uniform convergence of $y_n(x)$ to $y(x)$ implies that the right side of inequality (17) can be made as small as we please by taking n large enough. The left side of inequality (17) must, therefore, be equal to zero and this completes the proof of statement (ii).

To prove statement (iii), let us assume that $\bar{y}(x)$ is also a continuous solution of Eqn.(6) on the interval $| x - x_0 | \le h$. We shall then be through, if, we show that $\bar{y}(x) = y(x)$ for every x in the interval. As a first step we shall try to show that $\bar{y}(x)$ lies in R_1 and hence in R. If possible, let us suppose that the graph of $\bar{y}(x)$ leaves R_1 see Fig. 3.

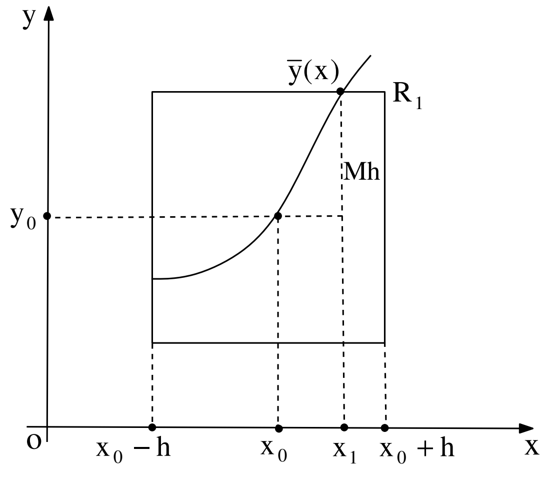

Fig.3

Now since $\bar{y}(x)$, is continuous and the fact that $\bar{y}(x_0) = y_0$ (being a solution) implies that there exists an x_1 such that $| x_1 - x_0 | < h$, $| \bar{y}(x_1) - y_0 | = Mh$ and $| \bar{y}(x) - y_0 | < Mh$ if $| x - x_0 | < | x_1 - x_0 |$. For such an x_1, it follows that

$$\frac{| \bar{y}(x_1) - y_0 |}{| x_1 - x_0 |} = \frac{Mh}{| x_1 - x_0 |} > \frac{Mh}{h} = M.$$

Thus by the mean value theorem, there exists a number x^* between x_0 and x_1, such that

$$\frac{| \bar{y}(x_1) - y_0 |}{| x_1 - x_0 |} = | \bar{y}'(x^*) |$$

But, $|\bar{y}(x^*)| = \left| f[x^*, \bar{y}(x^*)] \right| \le M$, since $[x^*, \bar{y}(x^*)]$ lies in R_1.

$$\therefore \quad \frac{|\bar{y}(x_1) - y_0|}{|x_1 - x_0|} \le M$$

which is a contradiction. This shows that no such point x_1 exists and the graph of $\bar{y}(x)$ lies in R_1. Finally, to show that the two solutions $y(x)$ and $\bar{y}(x)$ of Eqn.(6) are equal we write

$$|\bar{y}(x) - y(x)| = \left| \int_{x_0}^{x} \{ f[t, \bar{y}(t)] - f[t, y(t)] \} dt \right|$$

Since the graphs of $\bar{y}(x)$ and $y(x)$ both lie in R_1, relation (12) yields

$$|\bar{y}(x) - y(x)| \le Nh \max \left| \bar{y}(x) - y(x) \right|.$$

$$\therefore \qquad \max |\bar{y}(x) - y(x)| \le Nh \max |\bar{y}(x) - y(x)|$$

This implies that max. $|\bar{y}(x) - y(x)| = 0$, for otherwise we would have $1 \le Nh$ in contradiction to relation (13). This proves that $\bar{y}(x) = y(x)$ for every x in the interval $|x - x_0| \le h$, and Picard's Theorem is completely proved.

We will now make the following remarks.

Remark 1: Picard's theorem can be strengthened in various ways by weakening its hypotheses. For example, our assumption that $\frac{\partial f}{\partial y}$ is continuous on R is stronger than the proof requires, and is used only to obtain the inequality (12). We can, therefore, introduce this inequality (12) into the theorem as an assumption which replaces the assumption of continuity of $\frac{\partial f}{\partial y}$ on R. This would lead us to a stronger form of the theorem as in practice, there are number of functions that lack a continuous partial derivative but satisfy relation (12) for some constant N. This inequality viz; $|f(x, y_1) - f(x, y_2)| \le N |y_1 - y_2|$, which says that the difference quotient

$$\frac{f(x, y_1) - f(x, y_2)}{y_1 - y_2}$$

is bounded on R is called a **Lipschitz condition in the variable y.** The constant N is called the **Lipschitz constant.**

Remark 2: If we drop the Lipschitz condition, and assume only that $f(x, y)$ is continuous on R, then it is still possible to prove that the IVP (5) has a solution. This result is known as **Peano's theorem** named after Italian logician and mathematician Guiseppe Peano (1858 – 1932). But with only this assumption the solution whose existence this theorem guarantees may not necessarily be unique. Consider for example, the problem $y' = 3y^{\frac{2}{3}}$, $y(0) = 0$ and let R be the rectangle $|x| \le 1, |y| \le 1$. Here $f(x, y) = 3y^{\frac{2}{3}}$ is continuous on R. Also $y_1(x) = x^3$ and $y_2(x) = 0$ are two different solution of this problem valid for all x. The given problem therefore has a solution but it is not unique. The reason being that $f(x, y)$ does not satisfy the Lipschitz condition on the rectangle R, because the difference quotient

$$\frac{f(0, y) - f(0, 0)}{y - 0} = \frac{3y^{\frac{2}{3}}}{y} = \frac{3}{y^{\frac{1}{3}}}$$

is unbounded in every neighborhood of the origin.

Remark 3: Theorem 1, is called a **local** existence and uniqueness theorem because it guarantees the existence of a unique solution only on some interval $\left| x - x_0 \right| \le h$, where h may be very small. However, there are several important cases in which this restriction can be removed. For example, consider the first order linear equation

$$y' + P(x)\, y = Q(x),$$

where $P(x)$ and $Q(x)$ are defined and continuous on an interval $a \le x \le b$.
If we compare this equation with the IVP(5), then we have

$$f(x, y) = -P(x)\, y + Q(x),$$

and if $N = \max\left|P(x)\right|$ for $a \le x \le b$, it is clear that

$$\left|f(x, y_1) - f(x, y_2)\right| = \left|-P(x)(y_1 - y_2)\right| \le N\left|y_1 - y_2\right|$$

The function $f(x, y)$ is therefore continuous and satisfies a Lipshitz condition with **Lipshitz constant** $N = \max\left|P(x)\right|$, on the infinite vertical strip defined by $a \le x \le b$ and $-\infty < y < \infty$. Under these circumstances, the IVP

$$y' + P(x)\, y = Q(x), \quad (x_0) = y_0$$

has a unique solution on the entire interval $a \le x \le b$.

We shall now take up a few examples to illustrate the existence and uniqueness theorem.

Example 3: Show that $f(x, y) = x^2\left|y\right|$ satisfies a Lipschitz condition on the rectangle $\left|x\right| \le 1$ and $\left|y\right| \le 1$, but $\dfrac{\partial f}{\partial y}$ fails to exist at many points of this rectangle.

Solution: Here rectangle R is defined by $R : \left|x\right| \le 1$ and $\left|y\right| \le 1$.
Now,

$$\begin{aligned}
|f(x, y_1) - f(x, y_2)| &= |x^2\, |y_1| - x^2\, |y_2||\\
&= |x^2|\ \ ||y_1| - |y_2||\\
&\le |x^2|\ |y_1 - y_2|\\
&\le |y_1 - y_2| \text{ in } R,
\end{aligned}$$

Hence $f(x, y)$ satisfies a Lipschitz condition, with Lipschitz constant as 1.
Let $(a, 0)$ be any point in R for $a \ne 0$.

Here, $\lim\limits_{h \to 0} \dfrac{f(a, 0+h) - f(a, 0)}{h} = \lim\limits_{h \to 0} \dfrac{a^2\left|h\right| - a^2 0}{h} = \lim\limits_{h \to 0} \dfrac{a^2\left|h\right|}{h}$, which does not exist.

Hence, $\dfrac{\partial f}{\partial y}$ does not exist for many points $(a, 0)$ in R for $a \ne 0$.

***

Example 4: Examine the IVP $y' = e^{-x^2} y^2 \sin x$, for Lipschitz condition on the region R where, R is the strip $0 \le y \le 2$ in the xy-plane.

Solution: The function $f(x, y) = e^{-x^2} y^2 \sin x$ is continuous for all x and y.
Let (x, y_1) and (x, y_2) be the two points of R, then

$$\begin{aligned}
\left|f(x, y_2) - f(x, y_1)\right| &= \left|e^{-x^2} \sin x\right|\left|y_2 + y_1\right|\left|y_2 - y_1\right|\\
&\le 4\left|y_2 - y_1\right|
\end{aligned}$$

because $\left|e^{-x^2} \sin x\right| \le 1$ for all x and $\left|y_2 + y_1\right| \le 4$, if y_1 and y_2 are both in the interval $[0, 2]$. Thus $f(x, y)$ satisfies the Lipschitz condition with $N = 4$ in the strip R.

***

Example 5: Discuss the nature of solutions of the initial value problem $y' = \sqrt{|y|}$, $y(0) = 0$, on the rectangle $|x| \le 1$ and $|y| \le 1$.

Solution: The function $f(x,y)=\sqrt{|y|}$ is continuous for all values of y. Taking $y_1=0$ and $y_2>0$, as two points, we get

$$\frac{|f(x,y_2)-f(x,y_1)|}{|y_2-y_1|}=\frac{\sqrt{|y_2|}}{|y_2|}=\frac{1}{\sqrt{y_2}}$$

Now $\frac{1}{\sqrt{y_2}}$ can be made as large as we please by choosing the sufficiently small values of y_2. Therefore, there exists no N, independent if x, y such that

$$|f(x,y_2)-f(x,y_1)|<N\,|y_2-y_1|.$$

$\Rightarrow$ Lipschitz condition is not satisfied in any region containing $y=0$.

It may be observed that given IVP has solutions $y(x)=0$ and $y(x)=\frac{x^2}{4}$. Thus continuity of $f(x,y)$ is not sufficient for the unique solution of the IVP.

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Example 6: Discuss the nature of solution of the IVP $y'=y^2$, $y(0)=b>0$ on the infinite strip R consisting of the points (x,y) for which $0\le x\le 1$ and y is arbitrary.

Solution: Function $f(x,y)=y^2$ is continuous everywhere. Taking (x,y_1) and (x,y_2) as two points of the region R, we get

$$\begin{aligned}\left|f(x,y_2)-f(x,y_1)\right| &= \left|y_2^2-y_1^2\right|\\ &= \left|y_2+y_1\right|\left|y_2-y_1\right|\end{aligned}$$

Because $\left|y_2+y_1\right|$ can be made arbitrarily large, it follows that $f(x,y)$ does not satisfy Lipschitz condition on the infinite strip R. However, when we solve the given IVP by separation of variable, we get

$$y(x)=\frac{b}{1-bx}$$

Now because the denominator vanishes for $x=\frac{1}{b}$, the above solution of IVP is only valid for $x<\frac{1}{b}$. In particular, if b is large, then we have a solution only on a very small interval to the right of $x=0$.

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You may now try some exercises.

E3) For IVP

$$y'=\begin{cases}\frac{2y}{x}, & x>0\\ 0, & x=0.\end{cases}\qquad y(0)=0$$

show that the Lipschitz condition is not satisfied in any closed rectangle containing (0, 0).

E4) Examine the existence and uniqueness of solution of IVP
$y'=f(x,y), y(0)=1,$
where, (a) $f(x,y)=x$
(b) $f(x,y)=-\left|y\right|$.

E5) Show that $f(x,y)=xy^2$
(a) satisfies a Lipschitz condition on any rectangle $a\le x\le b$ and $c\le y\le d$.
(b) does not satisfy a Lipschitz condition on any strip $a\le x\le b$ and $-\infty<y<\infty$.

E6) Show that $y' = \frac{(y-1)}{x}$, $y(0) = 1$ has an infinity of solutions.

E7) Examine IVP $y' = \begin{cases} \frac{4x^3 y}{x^4 + y^2} & \text{when x \& y are not both zero} \\ 0, & \text{when } x = y = 0 \end{cases}$, $y(0) = 0$

for uniqueness of solutions.

In the next section we shall quickly recall various properties of solutions of linear differential equations. We shall also give here a quick recap of various methods of finding solutions of these equations when the coefficients are constants. For details, you can refer Block-2 of MTE-08. The case when the coefficients are variable will be discussed in Unit 2.

1.4 LINEAR DIFFERENTIAL EQUATIONS

The most general **linear, non-homogeneous differential equation** of order n is of the form

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = f(x) \qquad (18)$$

where $a_0(x) \neq 0$. Further, if the functions $a_0(x)$, $a_1(x), \ldots, a_n(x), f(x)$ are continuous on an interval $[a, b]$, where the interval $[a, b]$ can be of finite or infinite length or, it can be open or closed at either end, then a solution $y(x)$ of Eqn.(18) over $[a, b]$ together with its derivatives $y'(x), y''(x), \ldots, y^{(n-1)}(x)$ will be continuous on this interval. From Eqn.(18) it follows that $y^{(n)}(x)$ will also be continuous on the $[a, b]$ (ref. Unit 5, MTE-08).

We shall now state a theorem (without proof) which gives the conditions under which we can expect a solution of Eqn.(18).

Theorem 2: Let $x = x_0$ be a point of the interval $[a, b]$ and let $b_0, k_1, \ldots, k_{n-1}$ be arbitrary set of n real constants. If the functions $a_0(x), a_1(x), \ldots, a_n(x), f(x)$ are continuous on $[a, b]$ and $a_0(x)$ does not vanish at any point of the interval then there exists one and only one solution $y(x)$ of Eqn.(18) with the property

$$y(x_0) = k_0, y'(x_0) = k_1, \ldots, y^{(n-1)}(x_0) = k_{n-1}.$$

Further, this solution is defined over the entire interval $[a, b]$.

For example, let the interval $[a, b]$ be $0 \le x \le 1$ and that $n = 3$. Then at the point $x = \frac{1}{3}$, we may prescribe $y = \sqrt{2}, y' = \pi^2, y'' = 10^{10}$. Then the theorem asserts the existence of one and only one solution $y(x)$ taking on these values at $x = \frac{1}{3}$. This solution will, further, be defined at every point of $0 \le x \le 1$.

If $f(x) \equiv 0$ on $[a, b]$ Eqn.(18) becomes

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = 0 \qquad (19)$$

Eqn.(19) is called a **homogeneous** equation. Before talking about the properties of the solutions of Eqn.(18) and (19) let us quickly recap linear dependence and independence of set of functions on a given interval.

Consider a set of n functions, namely $y_1(x), y_2(x), \ldots, y_n(x)$ of x defined over the interval $[a, b]$. These n functions are said to be **linearly dependent** in an interval $[a, b]$ if for every x in the interval, there exists a relation

$$c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) = \sum_{i=1}^{n} c_i y_i(x) = 0, \qquad (20)$$

where the constants c_i's are not all zero.

If such a relation does not exist, then the functions are said to be **linearly independent** in $[a, b]$, i.e., none of the functions can be expressed as a linear combination of the others.

For example, the functions $\cosh x, e^x$ and e^{-x} are linearly dependent, since

$\cosh x = \frac{1}{2}(e^x + e^{-x})$ or, $\cosh x - \frac{1}{2}e^x - \frac{1}{2}e^{-x} = 0$. Similarly, function

$y_1(x) = \sin 2x$ and $y_2(x) = \sin x \cos x$ are linearly dependent on the interval $]\infty, \infty[$ since $c_1 \sin 2x + c_2 \sin x \cos x = 0$ is satisfied for every real x with $c_1 = \frac{1}{2}$ and $c_2 = -1$.

Note that in the consideration of linear dependence or linear independence, the interval on which the functions are defined is important. You can check this for yourself that the functions $y_1(x) = x$ and $y_2(x) = |x|$ are linearly independent on the interval $]-\infty, \infty[$ whereas, they are linearly dependent on $]0, \infty[$.

The above procedure of examining the linear dependence or independence of a set of functions appears to be quite involved. We, therefore, give below a **sufficient condition** for examining the linear independence of a set of n functions.

Suppose that $y_1(x), y_2(x), \ldots, y_n(x)$ are n functions defined on an interval $[a, b]$ possessing derivatives upto $(n-1)^{th}$ order. If the determinate

$$W(y_1(x), y_2(x), \ldots, y_n(x) = \begin{vmatrix} y_1 & y_2 \cdots & y_n \\ y_1' & y_2' \cdots & y_n' \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ y_1^{(n-1)} & y_2^{(n-1)} \cdots & y_n^{(n-1)} \end{vmatrix}$$

is not zero for at least one point in the interval $[a, b]$, then the functions $y_1(x), \ldots, y_n(x)$ are linearly independent on the interval. The determinant $W(y_1(x), y_2(x), \ldots, y_n(x))$ is called the **Wronskian** of the functions. It is named after a Polish mathematician Josef Maria Hoene Wronski (1778-1853).

The functions $y_1(x) = e^{kx}, y_2(x) = e^{-kx}$ and $y_3(x) = \sinh kx$, for instance are linearly dependent on $]-\infty, \infty[$ because

$$W(y_1, y_2, y_3) = \begin{vmatrix} e^{kx} & e^{-kx} & \sinh kx \\ ke^{kx} & -ke^{-kx} & k\cosh kx \\ k^2e^{kx} & k^2e^{-kx} & k^2 \sinh x \end{vmatrix} = 0.$$

Remember that in the above condition the non-vanishing of the Wronskian at a point in the interval provides only a sufficient condition. In other words, if $W(y_1, y_2, \ldots, y_n) = 0$ for some x in the interval $[a, b]$ then it does not necessarily mean that the functions are linearly dependent on the interval. For example, if $y_1(x) = x^2$ and $y_2(x) = x|x|$ then $W(y_1(x), y_2(x)) = 0$ for every real number whereas, $y_1(x)$ and $y_2(x)$ are linearly independent on $]-\infty, \infty[$.

With the above background let us now come back to the properties of solutions of Eqns.(18) and (19).

We can think of the forms of the solutions of Eqn.(19) by making use of the following elementary theorems.

Theorem 3: If $y = y_1$ is a solution of Eqn.(19) on an interval $[a, b]$, then $y = c\,y_1$ is also its solution on $[a, b]$, where c is any arbitrary constant.

Theorem 4: If $y = y_1, y_2, \ldots, y_n$ are n solutions of homogeneous differential Eqn.(19) on an interval $[a, b]$, then $y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$ is also a solution of Eqn.(19) on $[a, b]$, where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

Theorem 4 is known as the **superpostion principle**. Theorems 3 and 4 represent properties that non-linear differential equations, in general, do not possess. This will become more clear to you after doing E8).

Let us now consider the following definition which involves a linear combination of solutions.

Definition: Let $y_1, y_2, \ldots, y_n$ be n linearly independent solutions of homogenenous linear differential Eqn.(19) of order n on an interval $[a, b]$. Then

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x),$$

where $c_i, i = 1, 2, \ldots, n$ are arbitrary constants, is defined to be the general solution of Eqn.(19) on $[a, b]$.

The above definition generates our interest in knowing when n solutions $y_1, y_2, \ldots, y_n$ of the homogeneous differential Eqn.(19) are linearly independent. Surprisingly, the non-vanishing of the Wronskian of a set of n such solutions on an interval $[a, b]$ is both **necessary** and **sufficient** for linear independence. That is, if $y_1, y_2, \ldots, y_n$ be n solutions of homogeneous linear n^{rh} order differential Eqn.(19) on $[a, b]$, then the set of solutions is linearly independent on $[a, b]$, if and only if

$$W(y_1, y_2, \ldots, y_n) \neq 0$$

For every x in the interval. Such a set $y_1, y_2, \ldots, y_n$ of n linearly independent solutions of Eqn.(19) on $[a, b]$ is said to be a **fundamental set of solutions** on the interval. For instance, the second order equation $\frac{d^2y}{dx^2} - k^2 y = 0$, has two solutions $y_1 = e^{kx}$ and $y_2 = e^{-kx}$ since

$$W(e^{kx}, e^{-kx}) = \begin{vmatrix} e^{kx} & e^{-kx} \\ ke^{kx} & -ke^{-kx} \end{vmatrix} = -2k \neq 0$$

for every value of x. Functions y_1 and y_2 form a fundamental set of solution on $]-\infty, \infty[$. The general solution of the differential equation on the interval is $y = c_1 e^{kx} + c_2 e^{-kx}$.

We now state one more theorem wich pertains to the solution of non-homogeneous linear differential Eqn.(18).

Theorem 5: If $y = Y_0(x)$ is any solution of differential Eqn.(18) on an interval $[a, b]$ and if $y = Y(x)$ is the general solution of the corresponding homogeneous Eqn.(19) on the interval, then

$$y = Y_0(x) + Y(x)$$

is the general solution of Eqn.(18) on the given interval.

We shall not prove Theorems 3, 4 and 5 here but illustrate them through various examples. If you are interested in the proofs of these theorems then you can refer to Unit 5, Block 2 of MTE-08.

Let us consider the following examples.

Example 7: Show that if $y_1 = x^2$ and $y_2 = x^2 \ln x$ are both solutions of the equation $x^3 y''' - 2xy' + 4y = 0$, on the interval $]0, \infty[$. Then $y = c_1 x^2 + c_2 x^2 \ln x$ is also a solution of the equation on the given interval.

Solution: We have $y = c_1x^2 + c_2x^2 \ln x$

$$\therefore y' = 2c_1x + 2c_2x\ln x + c_2x,$$

$$y'' = 2c_1 + 2c_2 \ln x + 3c_2\ ,\ y''' = \frac{2c_2}{x}.$$

$$\therefore\ x^3y''' - 2xy' + 4y = x^3\left(\frac{2c_2}{x}\right) - 2x(2c_1x + 2c_2x\ln x + c_2x) + 4c_1x^2 + 4c_2x^2\ln x$$

$$= 0$$

Thus $y = c_1x^2 + c_2x^2 \ln x$ is also a solution of the equation.

***

Example 8: Show that linearly independent solutions of

$$y'' - 2y' + 2y = 0$$

on any interval are $e^x \sin x$ and $e^x \cos x$. What is the general solution? Find the solution $y(x)$ with the property $y(0) = 2, y'(0) = -3.$

Solution : It can be verified that $y_1 = e^x \sin x$ and $y_2 = e^x \cos x$ satisfy the given equation and hence, are solutions of given equation.
Here

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x \sin x & e^x \cos x \\ e^x \cos x + e^x \sin x & e^x \cos x - e^x \sin x \end{vmatrix} = -e^x \neq 0.$$

Hence $e^x \sin x$ and $e^x \cos x$ are linearly independent solutions of the given equation. The general solutions can then be written as

$$y(x) = c_1\, e^x \sin x + c_2\, e^x \cos x.$$

where, c_1 and c_2 are arbitrary constants.

Here $\quad y'(x) = c_1(e^x \sin x + e^x \cos x) + c_2(e^x \cos x - e^x \sin x)$

Now $\quad y(0) = 2$ and $y'(0) = -3$ imply

$$2 = c_2 \text{ and } -3 = c_1 + c_2$$

$$\text{i.e, } c_1 = -5, c_2 = 2$$

Hence, in this case the solution is $y(x) = -5\,e^x \sin x + 2\,e^x \cos x.$

* * *

You may now try the following exercises.

E8) Functions $y_1 = 1$ and $y_2 = \ln x$ are solutions of the non-linear differential equation $y'' + (y')^2 = 0$, on the interval $]0, \infty[$. Then

a) is $y_1 + y_2$ a solution of the equation?

b) is $c_1y_1 + c_2y_2$, a solution of the equation, where c_1 and c_2 are arbitrary constants?

E9) Show that $\sin ax$ and $\cos ax$ are linearly independent solutions of $y'' + a^2y = 0$, a being a positive constant on the interval $]-\infty, \infty[$. Obtain the general solution of the equation on the interval.

E10) Show that on the interval $0 < x < \infty, \sin\frac{1}{x}$ and $\cos\frac{1}{x}$ are linearly independent solutions of the equation $x^4\, y'' + 2x^3y' + y = 0.$ Find the solution $y(x)$ of the differential equation with the property $y\left(\frac{1}{\pi}\right) = 1$ and $y'\left(\frac{1}{\pi}\right) = -1.$

In practice, equations of the form (18) and (19), where coefficients are functions of x, do not usually have solutions expressible in terms of elementary functions, and even when they do, it is very difficult to find them. However, if coefficients in Eqns.(18) and (19) are constants then they are called **linear equations with constant coefficients**, and their solutions in terms of elementary functions can be obtained. In the next section we shall quickly recall some of the methods of finding these solutions.

1.4.1 Solutions of Homogeneous, Linear Differential Equations with Constant Coefficients

Consider the differential equation

$$a_0\frac{d^n y}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_{n-1}\frac{dy}{dx}+a_n y=0 \qquad (21)$$

where $a_0, a_1, \ldots, a_n$ are all constants and $a_0 \neq 0$.
Suppose that a possible solution of Eqn.(21) is

$$y = e^{mx} \qquad (22)$$

Differentiating y w.r.t. x, n times we get

$$\frac{dy}{dx}=me^{mx}, \frac{d^2y}{dx^2}=m^2e^{mx},\ldots,\frac{d^ny}{dx^n}=m^ne^{mx}.$$

With these values, Eqn.(21) takes the form

$$a_0m^ne^{mx}+a_1m^{n-1}e^{mx}+\cdots+a_{n-1}me^{mx}+a_ne^{mx}=0 \qquad (23)$$

Since $e^{mx} \neq 0$ for all m and x, we can divide Eqn.(23) by e^{mx} to obtain

$$a_0m^n+a_1m^{n-1}+\cdots+a_{n-1}m+a_n=0 \qquad (24)$$

Now each value of m, for which Eqn.(24) holds, will make $y=e^{mx}$ a solution of Eqn.(21). Eqn(24) is called the **auxiliary equation** (A.E.) or, the **characteristic equation** (C.E.) of Eqn.(21). **Observe** that Eqn.(24) is an algebraic equation in m of degree n and, therefore, by the fundamental theorem of algebra, it has at least one and not more than n distinct roots. We denote these roots by $m_1, m_2, \ldots, m_n$, where m's need not all be distinct or real.
The following three cases arise while solving the A.E. (24).

1. All the roots of A.E. (24) are **real** and **distinct.**
2. All the roots of A.E. (24) are **real** but **some** are **repeated.**
3. **Some** of the roots of A.E. (24) are **complex.**

We, now, discuss these three cases one by one.
Case I: If the n roots $m_1, m_2, \ldots, m_n$ of A.E.(24) are distinct, then n solutions of Eqn.(21) are

$$y_1=e^{m_1x}, y_2=e^{m_2x},\ldots,y_n=e^{m_nx}$$

But these n solutions are different and linearly independent and, thus, the general solution of Eqn.(21) is

$$y_c=y=c_1e^{m_1x}+c_2e^{m_2x}+\cdots+c_ne^{m_nx} \qquad (25)$$

Here y_c is known as the **complementary function.**

Let us look at an example now.

Example 9: Solve $\frac{d^3y}{dx^3}-\frac{d^2y}{dx^2}-6\frac{dy}{dx}=0.$

Solution: A.E. is

$$m^3-m^2-6m=0$$

i.e., $m(m^2-m-6)=0$, or, $m=0,-2,3$

Thus the general solution is

$$y=c_1e^{0x}+c_2e^{-2x}+c_3e^{3x}=c_1+c_2e^{-2x}+c_3e^{3x}.$$

* * *

You may now try the following exercises:

E11) Solve $y''' + 6y'' + 11y' + 6y = 0$

E12) Solve $y'' - 3y' + 2y = 0$ with $y = 0$ and $y' = 0$ when $x = 0$.

Case II: If A.E. (24) has a root m_1, which repeats r times, then the part of the solution corresponding to $m = m_1$ is

$$\left(c_1 + x c_2 + x^2 c_3 + \cdots + c_r x^{r-1}\right)e^{m_1 x}.$$

Now if A.E.(24) has r roots each equal to m_1 and the remaining (n – r) roots are all distinct, then solution of Eqn.(21) is

$$y_c = y = \left(c_1 + c_2 x + c_3 x^2 + \cdots + c_r x^{r-1}\right)e^{m_1 x} + c_{r+1} e^{m_{r+1} x} + \cdots + c_n e^{m_n x}$$

We now illustrate this case with the help of the following examples.

Example 10: Solve $\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + a^2 y = 0$, a being a positive constant.

Solution: Here A.E. is

$$m^2 - 2am + a^2 = 0$$

and has a double root $m = a$. The general solution is

$$y(x) = (c_1 + c_2 x)e^{ax}.$$

* * *

Example 11: Solve $y''' - 3y' + 2y = 0.$

Solution: The A.E. is

$$m^3 - 3m + 2 = 0 \Rightarrow (m+2)(m^2 - 2m + 1) = 0$$

which gives $m = 1, 1, -2$ and the general solution is given by

$$y(x) = (c_1 + c_2 x)e^{x} + c_3 e^{-2x}$$

* * *

We now discuss the case when some of the roots of A.E.(24) are complex.

Case III: We know from the theory of equations that if all the coefficients of Eqn.(21) are real, then any complex root it may have must occur in conjugate pairs. Thus, if $\alpha + i\beta$ is one root, then $\alpha - i\beta$ must be another root. If $\alpha \pm i\beta$ are the two complex roots of an A.E.(24), then the corresponding part of the general solution for constants A and B is given by

$$A e^{(\alpha+i\beta)x} + B e^{(\alpha-i\beta)x}$$
$$= e^{\alpha x}\left[A(\cos\beta x + i\sin\beta x) + B(\cos\beta x - i\sin\beta x)\right]$$
$$= e^{\alpha x}\left[(A+B)\cos\beta x + i(A-B)\sin\beta x\right] = e^{\alpha x}\left[c_1\cos\beta x + c_2 \sin\beta x\right],$$

where $c_1 = A + B$ and $c_2 = i(A - B)$ are the new constants.

If $\alpha + i\beta$ and $\alpha - i\beta$ each occurs twice as a root of A.E. Eqn.(24), then the corresponding part of the general solution is

$$e^{\alpha x}\left[(c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x\right].$$

Let us consider the following examples to illustrate this case.

Example 12: Solve $\left(\frac{dy}{dx} - y\right)^2\left(\frac{d^2y}{dx^2} + y\right)^2 = 0$

Solution: The A.E. is

$$(m-1)^2(m^2+1)^2 = 0,$$

which gives $m = 1, 1, \pm i, \pm i$. Then the general solution is

$$y(x) = (c_1 + c_2 x)e^{x} + (c_3 + c_4 x)\cos x + (c_5 + c_6 x)\sin x.$$

* * *

Example 13: Solve $\frac{d^3y}{dx^3} + y = 0.$

Solution : The A.E. is $m^3 + 1 = 0$, or, $(m+1)(m^2 - m + 1) = 0$

which gives $m = -1, \frac{1 \pm i\sqrt{3}}{2}$ and the solution is

$$y(x) = c_1 e^{-x} + e^{\frac{1}{2}x}\left[c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right].$$

***

How about trying the following exercises now?

E13) Solve the following differential equations:

a) $16y'' + 24y' + 9y = 0$

b) $y^{(4)} - 2a^2 y'' + a^4 y = 0$, a being a constant.

c) $y''' + y'' = 0$ with $y(0) = 1, y'(0) = 0, y''(0) = 1.$

E14) Solve the following differential equations.

a) $\frac{d^4y}{dx^4} + 8\frac{d^2y}{dx^2} + 16y = 0$

b) $y'' + 4y = 0$

c) $\frac{d^3y}{dx^3} - 2\frac{dy}{dx} + 4y = 0$

d) $y^{(6)} + 9y^{(4)} + 24y^{(2)} + 16y = 0.$

We next take up the solution of the non-homogeneous linear DEs with constant coefficients.

1.4.2 Solution of Non-Homogeneous Linear Differential Equations with Constant Coefficients

The general solution of a non-homogeneous linear DE

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_n y = f(x), \tag{26}$$

where $a_0 \neq 0, f(x) \neq 0$ and $a_0, a_1, \ldots, a_n$ are constants is $y(x) = y_c + y_p$.

Here y_c and y_p are respectively, known as the **complementary function** (C.F.) and **particular integral** (P.I.) or, particular solution. In Sec.1.4.1, we discussed, for $f(x) = 0$, the methods of finding the general solution $y = y_c$. It now remains to find the P.I. of a given differential equation.

Before going into the details of the procedures of finding the P.I. of a given DE, let us recall the definition of the differential operator.

Definition: A mathematical device by means of which we can convert one function into another is known as an **operator.**
The operation of differentiation is an operator as it converts a differentiable function $f(x)$ into a new function $f'(x)$. The letter D, which we shall use to denote the differentiation, is called the **differential operator**.

If y is an n^{th} order differentiable function, then

$$D^0 y = y, Dy = y', D^2 y = y'', \ldots, D^n y = y^{(n)} \tag{27}$$

If $F(D)$ is a **polynomial operator** of order n defined by

$$F(D) = a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n, a_0 \neq 0, \tag{28}$$

and y is an n^{th} order differentiable function, then

$$F(D)\,y = (a_0D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n)\,y$$
$$= a_0D^n y + c_1D^{n-1}y + \cdots + a_{n-1}Dy + a_n y$$
$$= a_0y^n + a_1y^{n-1} + \cdots + a_{n-1}y' + a_n y, \qquad (29)$$

and Eqn.(26) can be written as

$$F(D)y = f(x), a_0 \neq 0.$$

If $F(D)$ is defined by the relation (28), then

$$F(D) = a_0\,(D - m_1)(D - m_2)\ldots(D - m_n), \qquad (30)$$

where $m_1, m_2, \ldots, m_n$ are the real or complex roots of A.E. corresponding to $F(D)y = 0$. We can thus say that **a polynomial operator with constant coefficients can be factored just the way we factor an ordinary polynomial.**

Remark 1: $F(D)$ is a polynomial operator given by Eqn.(28) , then

$$F(D) = F_1(D).F_2(D),$$

where $F_1(D)$ and $F_2(D)$ may be composite factors of $F(D)$, i.e., F_1 and F_2 may be products of factors of Eqn.(30).

Remark 2: The polynomial operator satisfies the commutative law for multiplication, viz; $(D - m_1)(D - m_2) = (D - m_2)(D - m_1)$.

Remark 3: If $F(D)$ is a polynomial operator defined by Eqn.(28) and $g(x)$ is an n^{th} order differentiable function of x , then

$$F(D)[g(x)e^{ax}] = e^{ax}\,F(D + a)\,g(x),$$

where 'a' is a constant.

We are now in a position to give a method for solving Eqn(26) by making use of the polynomial operator. The procedure is illustrated by means of the following example

Example 14: Solve the differential equation $y''' + 2y'' - y' - 2y = e^{2x}$

Solution : In the operator notation, the given DE can be written as

$$(D^3 + 2D^2 - D - 2)y = e^{2x}$$

or, $$(D - 1)(D + 1)(D + 2)\,y = e^{2x} \qquad (31)$$

Let $$u = (D + 1)(D + 2)\,y \qquad (32)$$

Then Eqn.(31) becomes

$$(D - 1)\,u = e^{2x}, \qquad (33)$$

which is a linear differential equation and its solution is

$$u = e^{2x} + c_1\,e^x.$$

Putting this value of u in Eqn.(32), we get

$$(D + 1)(D + 2)\,y = e^{2x} + c_1e^x \qquad (34)$$

Let $$v = (D + 2)\,y \qquad (35)$$

Then Eqn.(34) becomes

$$(D + 1)\,v = e^{2x} + c_1e^x$$

which is a linear equation and its solution is

$$v = \frac{1}{3}e^{2x} + \frac{c_1}{2}e^x + c_2\,e^{-x}.$$

putting the value of v in Eqn.(35), we obtain

$$Dy + 2y = \frac{1}{3}e^{2x} + \frac{c_1}{2}e^x + c_2\,e^{-x},$$

which is again, a linear equation with the solution as

$$y = \frac{1}{12}e^{2x} + \frac{c_1}{6}e^x + c_2\,e^{-x} + c_3e^{-2x}$$

Here $\frac{1}{12}e^{2x}$ is a P.I. (free from constant) and $\frac{c_1}{6}e^x + c_2e^{-x} + c_3e^{-2x}$ is C.F.

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In general, if the non-homogeneous linear differential equation

$$a_0 y^n + a_1 y^{n-1} + \cdots + a_{n-1} y + a_n = f(x), a_0 \neq 0,$$

of order n is written as

$$(D - m_1)(D - m_2)\ldots(D - m_n)y = f_1(x), \tag{36}$$

where $m_1, m_2, \ldots, m_n$ are the roots of A.E., then a general solution can be obtained as follows:

$$\text{Let} \quad u = (D - m_2)(D - m_3)\ldots(D - m_n)y \tag{37}$$

Then Eqn.(36) takes the form

$$(D - m_1)u = f_1(x), \tag{38}$$

which is a linear equation in u. Obtain its solution and put it in Eqn.(37) to get

$$(D - m_2)(D - m_3)\ldots(D - m_n)y = u(x) \tag{39}$$

Now, let $v = (D - m_3)(D - m_4)\ldots(D - m_n)y$ (40)

Then Eqn.(39) becomes

$$(D - m_2)v = u(x), \tag{41}$$

which is linear in v. Obtain its solution and put it in Eqn.(40) to get

$$(D - m_3)\ (D - m_4)\ldots(D - m_n)y = v(x)$$

Repeat this process $(n-2)$ times to get the solution for y.

We shall now define the inverse operator and use it to find the P.I. We shall also give some shortcut methods to find P.I. when the right hand side, i.e., the non-homogeneous term of the given equation is a polynomial in x, an exponential function, sine or cosine function or, combination of these functions.

Inverse Operation

We start with the definition of inverse operator

Definition : Let $F(D)y = f(x)$, where $F(D)$ is the polynomial operator defined as

$$F(D) = a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n$$

and $f(x)$ is a function of x. The **inverse operator of F(D)**, written as $F^{-1}(D)$ or $\frac{1}{F(D)}$, is then defined as an operator, which when operated on $f(x)$, gives the P.I. y_p of $F(D) = f(x)$, i.e.,

$$F^{-1}(D) f(x) = y_p \quad \text{or} \quad y_p = \frac{1}{F(D)} f(x)$$

Remark 1: From the above definition, we conclude that $D^{-n} f(x)$ will mean integration of $f(x)$ n-times by ignoring constants of integration.

Remark 2: If $F(D)y = 0$, then

$$y_p = \frac{1}{F(D)}(0) = 0.$$

Remark 3: From the above definition and general method stated above,

$$\frac{1}{(D - m_1)(D - m_2)\ldots(D - m_n)} f(x) = e^{m_{n-1}} \int e^{(m_{n-1} - m_n)x} \left(\int e^{(m_{n-2} - m_{n-1})x} \int \ldots \right.$$

$$\left. \left(\int e^{(m_1 - m_2)x} \left(\int e^{-m_1 x} f(x) dx \right) \cdots dx \right) dx \right.$$

Remark 4: If we write P.I. of $F(D)y = f(x)$ as

$y_p = \frac{1}{F(D)} f(x) = \frac{1}{(D - m_1)(D - m_2)\ldots(D - m_n)} f(x)$, then following the ordinary rules of algebra, for distinct $m_1, m_2, \ldots m_n$, we can write

$$y_p = \left(\frac{\alpha_1}{D - m_1} + \frac{\alpha_2}{D - m_2} + \cdots + \frac{\alpha_n}{D - m_n} \right) f(x)$$

$= \alpha_1 e^{m_1 x}\int e^{-m_1 x} f(x)dx + \alpha_2 e^{m_2 x}\int e^{-m_2 x} f(x)dx + \cdots \alpha_n e^{m_n x}\int e^{-m_n x} f(x)dx$, where coefficients $\alpha_1, \alpha_2, \ldots, \alpha_n$ can be obtained by a simple algebraic manipulations (as in partial fractions).

In case a root m_1 of A.E. is repeated r times, the corresponding partial fraction will be

$$\frac{\alpha_1}{D-m_1} + \frac{\alpha_2}{(D-m_1)^2} + \cdots + \frac{\alpha_r}{(D-m_1)^r}$$

and the corresponding terms in the integral will be

$$\alpha_1 e^{m_1 x}\int e^{-m_1 x} f(x)dx + \alpha_2 e^{m_1 x}\iint e^{-m_1 x} f(x)(dx)^2 + \cdots$$

$$\cdots + \alpha_r e^{m_1 x}\iint \ldots \int e^{-m_1 x} f(x)(dx)^r.$$

We illustrate the above theory and remarks through the following examples:

Example 15: Find the particular integral of $\frac{d^2y}{dx^2} + y = \sec^2 x$.

Solution : Here,

$$\text{P.I.} = y_p = \frac{1}{D^2+1}\sec^2 x = \frac{1}{(D+i)(D-i)}\sec^2 x$$

$$= \frac{1}{2i}\left[\frac{1}{D-i} - \frac{1}{D+i}\right]\sec^2 x$$

$$= \frac{1}{2i}\left[e^{ix}\int e^{-ix}\sec^2 x\,dx - e^{-ix}\int e^{ix}\sec^2 x\,dx\right]$$

$$= \frac{1}{2i}\left[e^{ix}\int \frac{\cos x - i\sin x}{\cos^2 x}dx - e^{ix}\int \frac{\cos x + i\sin x}{\cos^2 x}dx\right]$$

$$= \frac{1}{2i}\left[e^{ix}\int (\sec x - i\sec x\tan x)dx - e^{ix}\int (\sec x + i\sec x\tan x)\,dx\right]$$

$$= \frac{1}{2i}\left[\left(e^{ix} - e^{-ix}\right)\int \sec x\,dx - i\left(e^{ix} + e^{-ix}\right)\int \sec x\tan x\,dx\right]$$

$$= \frac{1}{2i}\left[(2i\sin x)\ln|\sec x + \tan x| - (2i\cos x)\sec x\right]$$

$$= \sin x\ln|\sec x + \tan x| - 1$$

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Example 16: Find the particular integral of $(D-1)^2(D+1)^2 y = e^x$.

Solution: The particular integral is

$$y_p = \frac{1}{(D-1)^2(D+1)^2}e^x = \frac{1}{4}\left[\frac{-1}{D-1} + \frac{1}{(D-1)^2} + \frac{1}{D+1} + \frac{1}{(D+1)^2}\right]e^x$$

$$= \frac{1}{4}\left[-e^x\int e^{-x}e^x dx + e^x\int\left(\int e^{-x}e^x dx\right)dx + e^{-x}\int e^x e^x dx + e^{-x}\int\left(\int e^x e^x dx\right)dx\right]$$

$$= \frac{1}{4}\left[-xe^x + \frac{x^2}{2}e^x + \frac{1}{2}e^x + \frac{1}{4}e^x\right].$$

***

You may now try the following exercises.

E15) Find the complete solution of the following equations:

a) $y''' + y' = x^3 + \cos x$

b) $(D^3 + 3D^2 - D - 3)y = \cosh x$

c) $(D^3 + D^2 + 4D + 4)y = \sin 2x$

d) $y'' + n^2 y = \sec nx$

e) $(D^3 - D^2 - 8D + 12)y = X(x)$.

The general method of computing a particular integral as given above is quite laborious and requires a lot of calculations as can be seen from Examples 15 and 16 above. However, in certain cases, the P.I. can be obtained by shorter methods. We shall now consider these shorter methods.

Short Methods of Finding Particular Integrals

Consider the general n^{th} order linear DE, namely,

$F(D)y = (a_0D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n)y = f(x),\ a_0 \neq 0,$

where $a_0, a_1, \ldots, a_n$ are constants.

For certain particular forms of the non-homogeneous term $f(x)$, shorter methods of finding P.I. are available which we are stating below.

Case I: When $f(x) = e^{\alpha x}$, α constant.

$$\frac{1}{F(D)}e^{\alpha x} = \frac{1}{F(\alpha)}e^{\alpha x}, F(\alpha) \neq 0$$

When, $F(\alpha) = 0$ and $F(D) = (D-\alpha)^p\,\phi(D)$, $\phi(\alpha) \neq 0$ for some p, then

$$\frac{1}{F(D)}e^{\alpha x} = \frac{1}{(D-\alpha)^p\,\phi(D)}e^{\alpha x},\ \phi(\alpha) \neq 0$$

$$= \frac{x^p}{p!}.\frac{1}{\phi(\alpha)}e^{\alpha x}.$$

Case II: when $f(x) = \cos(ax+b)$, or, $\sin(ax+b)$.

If $f(D) = \phi(D^2)$,

then $\dfrac{1}{\phi(D^2)}\cos(ax+b) = \dfrac{1}{\phi(-a^2)}\cos(ax+b)$, if $\phi(-a^2) \neq 0$

and, $\dfrac{1}{\phi(D^2)}\sin(ax+b) = \dfrac{1}{\phi(-a^2)}\sin(ax+b)$, if $\phi(-a^2) \neq 0$

If, $\phi(-a^2) = 0$ and $\phi(D^2) = (D^2+a^2)^p\,\psi(D^2)$ for some p and $\psi(-a^2) \neq 0$

$$\text{Then, } \frac{1}{\phi(D^2)}\cos(ax+b) = \frac{1}{(D^2+a^2)^p\,\psi(D^2)}\cos(ax+b)$$

$$= \frac{1}{\psi(-a^2)}\left[\frac{1}{(D^2+a^2)^p}\cos(ax+b)\right] \text{if, } \psi(-a^2) \neq 0.$$

$$\text{and, } \frac{1}{\phi(D^2)}\sin(ax+b) = \frac{1}{(D^2+a^2)^p\,\psi(D^2)}\sin(ax+b)$$

$$= \frac{1}{\psi(-a^2)}\left[\frac{1}{(D^2+a^2)^p}\sin(ax+b)\right] \text{if, } \psi(-a^2) \neq 0.$$

Note that the terms in the brackets above are evaluated by the general method. Cosine and sine functions in Case II above can also be dealt as exponential functions by writing them in the form:

Symbols Re and Im are read as 'real part of' and 'imaginary part of' respectively.

$$\cos(ax+b) = \text{Re }\ e^{i(ax+b)},$$

and $\quad \sin(ax+b) = \text{lm }\ e^{i(ax+b)}$

Case III: When $f(x)$ is a polynomial in x.

Let $F(D)$ be a polynomial in D of degree n and let $f(x)$ be a polynomial in x of degree say, K. Then

$$\frac{1}{F(D)} f(x) = \frac{1}{a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n} f(x)$$

$$= \frac{1}{a_n}\left(1 + \frac{a_{n-1}}{a_n} D + \cdots \frac{a_1}{a_n} D^{n-1} + \frac{a_0}{a_n} D^n\right)^{-1} f(x)$$

$$= \left(c_0 + c_1 D + c_2 D^2 + \cdots + c_k D^k\right) f(x) + 0(D^{k+1}) f(x)$$

$$= \left(c_0 + c_1 D + c_2 D^2 + \cdots + c_k D^k\right) f(x)$$, the second term on r.h.s. is zero.

Case IV: When $\mathbf{f(x) = e^{\alpha x} V(x)}$, $\boldsymbol{\alpha}$ constant.

$$\frac{1}{F(D)}\left[e^{\alpha x} V(x)\right] = e^{\alpha x} \frac{1}{F(D+\alpha)} V(x)$$

This result is known as **shifting theorem**.

Remark 1: We may remark that Euler's equation

$$(a_0 x^n D^n + x^{n-1} a_1 D^{n-1} + \cdots + a_{n-1} x D + a_n) y = f(x)$$

can be reduced to an equation with constant coefficients with the help of substitution $x = e^z$.

Remark 2: Differential Equation

$$[a_0 (ax+b)^n D^n + a_1 (ax+b)^{n-1} D^{n-1} + \cdots + a_{n-1}(ax+b) D + a_n] y = f(x)$$

can be either reduced to Euler's equation by the substitution $ax + b = z$, or, it can be reduced to an equation with constant coefficients by the transformation $ax + b = e^z$.

We, now, take up some examples to illustrate the shorter methods of finding P.I. stated above.

Example 17: Find P.I. of $y''' + y'' + y' + y = x^4 + 2x + 1$

Solution: Here P.I. $= \frac{1}{1 + D + D^2 + D^3}(x^4 + 2x + 1)$

$$= \frac{1-D}{1-D^4}(x^4 + 2x + 1)$$

$$= (1 - D^4)^{-1}\left[x^4 + 2x + 1 - 4x^3 - 2\right]$$

$$= (1 + D^4 + D^8 + \cdots)(x^4 - 4x^3 + 2x - 1)$$

$$= (x^4 - 4x^3 + 2x - 1) + 24$$

$$= x^4 - 4x^3 + 2x + 23$$

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Example 18: Find P.I. of $(D^4 - 2D^3 + D^2) y = x$

Solution : Here P.I. $= \frac{1}{D^4 - 2D^3 + D^2} x$

$$= \frac{1}{D^2}[1 - (2D - D^2)]^{-1} \; x$$

$$= \frac{1}{D^2}\left[1 + 2D - D^2 + \cdots\right] x$$

$$= \frac{1}{D^2}\left[x + 2\right]$$

$$= \frac{1}{D}\left(\frac{x^2}{2} + 2x\right)$$

$$= \frac{x^3}{6} + x^2.$$

***

Example 19: Solve $(D^2-2D+1)y = x\ \sin x$

Solution : A.E. is $m^2-2m+1=0 \Rightarrow m=1,1$

$$\therefore\ \text{C.F.} = y_c = (c_1+c_2\,x)e^x$$

$$\text{P.I.} = \frac{1}{D^2-2D+1}x\,e^x\sin x = e^x\frac{1}{(D+1)^2-2(D+1)+1}x\sin x$$

$$= e^x D^{-2}(x\sin x)$$

$$= e^x\left[-x\sin x - 2\cos x\right]$$

$\therefore$General solution is, $y(x) = (c_1+c_2x)e^x + e^x(-x\sin x - 2\cos x)$.

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Example 20: Find the particular integral of $(D^2+1)y = x^2\sin 2x$.

Solution : Here P.I. $= \dfrac{1}{D^2+1}x^2\ \sin 2x = \text{Im}\dfrac{1}{D^2+1}(x^2e^{2ix})$

$$= \text{Im of } e^{2ix}\frac{1}{(D+2i)^2+1}x^2 = \text{Im of } e^{2ix}\frac{1}{D^2+4iD-4+1}x^2$$

$$= \text{Im of } e^{2ix}\frac{1}{-3}\left[1-\left(\frac{4iD+D^2}{3}\right)\right]^{-1}x^2$$

$$= \text{Im of } \frac{e^{2ix}}{-3}\left[1+\frac{4iD+D^2}{3}-\frac{16}{9}D^2+\cdots\right]x^2$$

$$= \text{Im of } \frac{e^{2ix}}{-3}\left[1+\frac{4i}{3}D-\frac{13}{9}D^2+0(D^3)\right]x^2 = \text{Im of } \frac{e^{2ix}}{-3}\left[x^2+\frac{8i}{3}x-\frac{13}{9}.2\right]$$

$$= \text{Im of } \left[\left(\frac{\cos 2x+i\sin 2x}{-3}\right)\left(x^2-\frac{26}{9}+i\frac{8}{3}x\right)\right] = -\frac{8}{9}x\cos 2x - \frac{1}{3}\left(x^2-\frac{26}{9}\right)\sin 2x$$

$$= -\frac{1}{27}\left[24x\ \ \cos 2x + (9x^2-26)\sin 2x\right].$$

***

Example 21: Find P.I. of $(D^4-1)y = \sin x$.

Solution: Here P.I. $= \dfrac{1}{D^4-1}\ \sin x = \dfrac{1}{(D^2-1)(D^2+1)}\ \sin x$

$$= \frac{1}{(-1-1)}\frac{1}{D^2+1}\ \sin x$$

$$= -\frac{1}{2}\left[-\frac{x}{2}\cos x\right] = \frac{x}{4}\cos x$$

***

Example 22: Solve $(1+x)^2\dfrac{d^2y}{dx^2}+(1+x)\dfrac{dy}{dx}+y = 2\sin[\ \ln\ (1+x)\,]$

Solution: Let $1+x=e^t, t=\ln\ (1+x)$ and $D\equiv\dfrac{d}{dt}$. Then the given equation becomes

$$[D(D-1)+D+1]y = 2\sin t\text{, i.e.,}$$

$$(D^2+1)y = 2\sin t,$$

which is a linear equation with constant coefficients with

$$\text{C.F.} = y_c = c_1\cos t + c_2\sin t$$

and $\text{P.I.} = y_p = \dfrac{1}{D^2+1}\ 2\sin t = -t\ \cos t$

Hence, the complete solution is

$$y = y_c + y_p = c_1 \cos t + c_2 \sin t - t \cos t$$
$$= c_1 \cos[\ln(1+x)] + c_2 \sin[\ln(1+x)] - \ln(1+x)\cos[\ln(1+x)].$$

***

Example 23: Solve $x^3D^3y + 3x^2D^2y + x\,Dy + y = x + \ln x$

Solution: Let $x = e^t, \frac{d}{dt} \equiv D_1$, so that the given equation becomes

$$[D_1(D_1-1)(D_1-2) + 3D_1(D_1-1) + D_1 + 1]\; y = e^t + t$$

i.e., $(D_1^3 + 1)y = e^t + t$

A.E. is, $m^3 + 1 = 0$, which has the roots $m = -1, \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. Thus,

$$\text{C.F.} = y_c = c_1e^{-t} + e^{\frac{t}{2}}\left[c_2 \cos\frac{\sqrt{3}}{2}t + c_3 \sin\frac{\sqrt{3}}{2}t\right]$$
$$= c_1x^{-1} + \sqrt{x}\left[c_2 \cos\left(\frac{\sqrt{3}}{2}\ln x\right) + c_3 \sin\left(\frac{\sqrt{3}}{2}\ln x\right)\right]$$

$$\text{P.I.} = y_p = \frac{1}{D_1^3+1}e^t + \frac{1}{D_1^3+1}t = \frac{1}{1+1}e^t + (1 - D_1^3 + \cdots)t$$
$$= \frac{1}{2}e^t + t = \frac{1}{2}x + \ln x$$

Therefore, the required solution is

$$y = y_c + y_p = c_1x^{-1} + \sqrt{x}\left[c_2 \cos\left(\frac{\sqrt{3}}{2}\ln x\right) + c_3 \sin\left(\frac{\sqrt{3}}{2}\ln x\right)\right] + \frac{1}{2}x + \ln x.$$

***

You may now try the following exercises.

E16) Solve the following differential equations

a) $(D^4 - 2D^3 + 2D^2 - 2D + 1)y = 0$

b) $(D^4 + 2D^3 - 3D^2 + 4D + 4)y = 0$

c) $(D^2 - 2D + 5)y = 0$, given that $y = 0$ and $\frac{dy}{dx} = 4$, when $x = 0$.

E17) Obtain the general solution of the following differential equations

a) $y''' + 3y'' + 3y' + y = e^{-x}(2 - x^2)$.

b) $(D^2 + 1)(D^2 + 4)y = \cos\frac{x}{2}\,\cos\frac{3x}{2}$.

c) $(D^5 - D)y = 12e^x + 8\sin x - 2x$.

d) $(D^2 - 1)y = e^{-x} + \cos x + x^3 + e^x \cos x$.

e) $(x+3)^2 y'' - 4(x+3)y' + 6y = \ln(x+3)$.

f) $(D^2 + a^2)y = \tan\ ax$

We now end this unit by giving a summary of what we have covered in it.

1.5 SUMMARY

In this unit, we have shown you the following:

1. The differential equation together with the initial conditions is called an initial value problem (IVP).

2. The differential equation together with the boundary conditions is called boundary value problem (BVP).

3. IVP, $\frac{dy}{dx} = f(x,y), y(x_0) = y_0$ is equivalent to the integral equation

$$y(x) = y_0 + \int_{x_0}^{x} f[t, y(t)]dt.$$

4. Picard's Method of successive Approximations for IVP,
$\frac{dy}{dx} = f(x,y),\ y(x_0) = y_0$, is given by

$$y_n(x) = y_0 + \int_{x_0}^{x} f[t, y_{n-1}(t)]dt,\ n = 1, 2, 3, \ldots.$$

5. If $f(x,y)$ and $\frac{\partial f}{\partial y}$ be continuous functions of x and y on a closed rectangle R and $f(x_0, y_0)$ is an interior point of R, then there exists a number $h > 0$ with the property that the initial value problem $\frac{dy}{dx} = f(x,y), y(x_0) = y_0$ has one and only one solution $y = y(x)$ on the interval $|x - x_0| \le h$. It is called the **Picard's theorem on existence and uniqueness** of solution of IVP.

6. The inequality $\left|\frac{f(x,y_1) - f(x,y_2)}{y_1 - y_2}\right| < N$ (a constant), is called a **Lipschitz condition** in the variable y and the constant N is called a Lipschitz constants.

7. Continuity of $f(x,y)$ guarantees a solution of IVP, $y' = f(x,y), y(x_0) = y_0$, whereas, Lipschitz condition or, continuity of $\frac{\partial f}{\partial y}$ ensures a unique solution.

8. For a linear differential equation of order n of the form
$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = f(x), a_0(x) \ne 0$,
let $x = x_0$ be a point of interval $[a,b]$ and let $k_0, k_1, \ldots, k_{n-1}$ be an arbitrary set of n constants. Then there exists one and only one solution $y(x)$ of the equation in $[a,b]$, with the property

$$y(x_0) = k_0, y'(x_0) = k_1, \ldots, y^{(n-1)}(x_0) = k_{n-1,}$$

where continuity of $a_0(x), a_1(x), \ldots, a_n(x)$ in $[a,b]$ has been assumed.

9. The 'n' functions $y_1(x), \ldots, y_n(x)$ defined on $[a,b]$ are said to be **linearly dependent** in [a, b] if, for all x in that interval, there exists a relation

$$c_1y_1 + c_2y_2 + \cdots + c_ny_n = 0$$

in which the c_i are constants and not all are zero. If such a relation does not exist, then the functions are said to be **linearly independent** in [a, b].

10. If $a_0(x), a_1(x), \ldots, a_n(x), f(x)$ are continuous functions of x on [a, b] and $a_0(x) \ne 0$, then

a) for a homogeneous equation
$a_0(x)y^{(n)} + a_1(x)y^{(n-`1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = 0$,
if, $y_1(x), y_2(x), \ldots, y_n(x)$ are n linearly independent solutions of the equation in [a, b] then $y(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)$ is the general solution of the equation for constants $c_1, c_2, \ldots, c_n$.

b) for a non-homogeneous equation
$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = f(x)$
If $Y(x)$ is a particular integral and $y_1, y_2, \ldots, y_n$ are n linearly independent

solutions of the corresponding homogeneous equation then,
$y(x) = Y(x) + c_1 y_1(x) + \cdots + c_n y_n(x)$ is the general solution for constants $c_1, c_2, \ldots, c_n$

11. Solution y, of an n^{th} order linear DE
$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0, a_0 \neq 0,$
with constant coefficients $a_0, a_1, \ldots, a_n$ having n roots $m_1, m_2, \ldots, m_n$ when
   a) roots are all real and distinct, is
   $y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$.
   b) roots are real and repeated, say $m_1 = m_2 = \cdots = m_r$, is
   $y = (c_1 + c_2 x + \cdots + c_r x^{r-1}) e^{m_1 x} + c_{r+1} e^{m_{r+1} x} + \cdots + c_n e^{m_n x}$.
   c) roots are complex and one such pair is $\alpha \pm i\beta$, then corresponding part of solution is $y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$.

12. For a non-homogeneous equation
$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = f(x), a_0 \neq 0$, with constant coefficients $a_0, a_1, \ldots, a_n$
   a) the complete solution of the corresponding homogeneous part is called its complementary function (C.F.).
   b) particular solution of the non-homogeneous part involving no arbitrary constant is called its particular integral (P.I.).
   c) complementary function and particular integral together constitute its general solution.

13. A mathematical device by means of which we can convert one function into another is known as an **operator**, e.g., $D = \frac{d}{dx}$ is differential operator.

14. If $F(D)$ is a **polynomial operator** of order n, then it is defined as
$F(D) = a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n, a_0 \neq 0.$
A polynomial operator with constant coefficients can be factored like an ordinary polynomial.

15. Euler's equation $(a_0 x^n D^n + a_1 x^{n-1} D^{n-1} + \cdots + a_{n-1} x D + a_n) y = f(x)$
can be reduced to an equation with constant coefficients by substituting $x = e^z$.

16. D.E. $[a_0 (ax+b)^n D^n + a_1 (ax+b)^{n-1} D^{n-1} + \cdots + a_{n-1}(ax+b)d + a_n]\ y = f(x)$
can be either reduced to Euler's equation by the substitution $ax + b = z$ or it can be reduced to an equation with constant coefficients by using $ax + b = e^z$.

1.6 SOLUTIONS/ANSWERS

E1) If λ is negative say $\lambda = -\alpha$ then the solution is
$$y = c_1 e^{-\sqrt{-\alpha}x} + c_2 e^{\sqrt{-\alpha}x}$$
when $y(0) = 0$ and $y(\pi) = 0$, we get $c_2 = 0 = c_1$
Hence only solution of given problem is the trivial solution $y = 0$.
If $\lambda = 0$, then general solution is $y(x) = c_1 x + c_2$
Again $y(0) = 0$ and $y(\pi) = 0$ give the trivial solution $y = 0$.
If λ is positive, the general solution is
$$y(x) = c_1 \sin\sqrt{\lambda}x + c_2 \cos\sqrt{\lambda}x.$$
For $y(0) = 0$, we get $0 = c_2$ and the solution reduces to $y(x) = c_1 \sin\sqrt{\lambda}x$.
For the second b.c. $y(\pi) = 0$, we get $0 = c_1 \sin\sqrt{\lambda}\pi$
For the non-trivial solution, we must have $\sin\sqrt{\lambda}\pi = \sin n\pi$ for some positive integer n, so that $\lambda = n^2$, i.e, λ must be one of the numbers $1, 4, 9, \ldots$.
Hence the non-trivial solution of given problem are

$y(x) = a_1 \sin x$, or, $a_2 \sin 2x$, or, $a_3 \sin 3x$, or $\ldots$.

And the solution vanishes at the end points 0 and π of interval $[0, \pi]$.

E2) a) Here $\frac{dy}{dx} = x + y, \quad y(0) = 1$. Thus $x_0 = 0, \quad y_0 = 1$, $f(x, y) = x + y$

$$\therefore y_1 = y_0 + \int_0^x f(x, y_0)\,dx = 1 + \int_0^x (x+1)\,dx = 1 + x + \frac{x^2}{2}$$

$$y_2 = y_0 + \int_0^x f(x, y_1)\,dx = 1 + \int_0^x \left(x + 1 + x + \frac{x^2}{2} \right) dx = 1 + x + x^2 + \frac{x^3}{6}$$

$$y_3 = y_0 + \int_0^x f(x, y_2) = 1 + \int_0^x \left(x + 1 + x + x^2 + \frac{x^3}{6} \right) dx$$

$$= 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}$$

b) $\frac{dy}{dx} = x - y^2, \quad y(0) = \frac{1}{2}$. Here $x_0 = 0, \quad y_0 = \frac{1}{2}, \quad f(x, y) = x - y^2$

$$\therefore y_1 = y_0 + \int_0^x (x - y_0^2)\,dx = \frac{1}{2} + \int_0^x \left(x - \frac{1}{4} \right) dx = \frac{1}{2} - \frac{1}{4}x + \frac{x^2}{2}$$

$$y_2 = y_0 + \int_0^x (x - y_1^2)\,dx = \frac{1}{2} + \int_0^x \left\{ x - \left(\frac{1}{4} + \frac{1}{16}x + \frac{x^4}{4} + \frac{x^2}{2} - \frac{1}{4}x - \frac{1}{4}x^3 \right) \right\} dx$$

$$= \frac{1}{2} - \frac{1}{4}x + \frac{19}{32}x^2 - \frac{x^3}{6} + \frac{1}{16}x^4 - \frac{x^5}{20}$$

$$y_3 = y_0 + \int_0^x (x - y_2^2)\,dx$$

$$= \frac{1}{2} - \frac{1}{4}x + \frac{5}{8}x^2 - \frac{21}{96}x^3 + \frac{89}{768}x^4 - \frac{1531}{3072}.\frac{x^5}{5} + \frac{67}{1440}x^6 - \frac{1303}{80640}x^7$$

$$+ \frac{77}{7480}x^8 - \frac{77}{8640}x^9 + \frac{1}{1600}x^{10} - \frac{x^{11}}{4400}.$$

c) $\frac{dy}{dx} = 2x(1+y), \quad y(0) = 0$. Here $x_0 = 0, \ y_0 = 0, \ f(x, y) = 2x(1+y)$

$$\therefore y_1 = y_0 + \int_0^x f(x, y)\,dx = 0 + \int_0^x 2x \ dx = x^2$$

$$y_2 = y_0 + \int_0^x 2x(1 + x^2)\,dx = 0 + x^2 + \frac{x^4}{2}$$

$$y_3 = y_0 + \int_0^x 2x\left(1 + x^2 + \frac{x^4}{2} \right) dx = x^2 + \frac{x^4}{2} + \frac{x^6}{6}$$

d) $\frac{dy}{dx} = y, \quad y(0) = 1$. Here $x_0 = 0, \ y_0 = 1, \ f(x, y) = y$

$$y_1 = y_0 + \int_0^x f(x, y_0)\,dx = 1 + \int_0^x dx = 1 + x$$

$$y_2 = y_0 + \int_0^x f(x, y_1)\,dx = 1 + \int_0^x (1+x)\ dx = 1 + x + \frac{x^2}{2}$$

$$y_3 = y_0 + \int_0^x f(x, y_2)\,dx = 1 + \int_0^x \left(1 + x + \frac{x^2}{2} \right) dx = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}.$$

e) $\frac{dy}{dx} = y^2, \quad y(0) = 1$. Here $x_0 = 0, \quad y_0 = 1, \quad f(x, y) = y^2$

$$y_1 = y_0 + \int_0^x f(x, y_0)\,dx = 1 + x$$

$$y_2 = y_0 + \int_0^x f(x, y_1)\,dx = 1 + x + x^2 + \frac{x^3}{3}$$

$$y_3 = y_0 + \int_0^x f(x, y_2)\,dx = 1 + x + x^2 + x^3 + \frac{2x^4}{3} + \frac{x^5}{3} + \frac{x^6}{9} + \frac{x^7}{63}.$$

E3) Here $f(x, y) = \begin{cases} \frac{2y}{x}, \text{for } x > 0, \quad y(0) = 0 \\ 0, \text{for } x = 0 \end{cases}$

Thus, $\frac{|f(x, y_1) - f(x, y_2)|}{|y_1 - y_2|} = \frac{\left|\frac{2}{x}\right| |y_1 - y_2|}{|y_1 - y_2|} = \left|\frac{2}{x}\right|$.

Hence $f(x, y)$ doesn't satisfy Lipschitz conditions in any closed rectangle containing $(0, 0)$.

E4) a) $f(x, y) = x$ and $\frac{\partial f}{\partial y} = 0$ are continuous on a whole plane and in every rectangle. Thus $f(x, y)$ satisfies the Lipschitz condition in every such rectangle and there exists a unique solution $y = \frac{x^2}{2} + 1$.

b) $f(x, y) = -|y|$ is not continuous and solution does not exist at any point in the neighbourhood of origin.

E5) Here $f(x,) = xy^2, \quad \frac{\partial f}{\partial y} = 2xy$ and,

$$\frac{|f(x, y_1) - f(x, y_2)|}{|y_1 - y_2|} = \frac{|xy_1^2 - xy_2^2|}{|y_1 - y_2|} = \frac{|x(y_1 - y_2)(y_1 + y_2)|}{|y_1 - y_2|} = |x||y_1 + y_2|$$

and the right hand side is $\leq N$ for, $a \leq x \leq b, \ c \leq y \leq d$ but is not less than equal to some finite constant for $a \leq x \leq b, \ -\infty < y < \infty$. Hence Lipschitz's condition is satisfied for any rectangle $a \leq x \leq b, \ c \leq y \leq d$ but is not satisfied for any strip $a \leq x \leq b, \ -\infty < y < \infty$.

E6) Here $f(x, y) = \frac{y - 1}{x}$. Thus $\frac{\partial f}{\partial y} = \frac{1}{x}$, which does not exist at the origin. Also

$$\frac{|f(x, y_1) - f(x, y_2)|}{|y_1 - y_2|} = \frac{\left|\frac{y_1 - 1}{x} - \frac{y_2 - 1}{x}\right|}{|y_1 - y_2|} = \frac{\left|\frac{1}{x}\right| |y_1 - y_2|}{|y_1 - y_2|} = \left|\frac{1}{x}\right|$$, which is

unbounded in every neighbourhood of origin.
Hence the Lipschitz condition is not satisified. The given problem may have an infinity of solutions.

E7) Here $f(x, y) = \frac{4x^2 y}{x^4 + y^2}$, when x and y are not both zero

$= 0$, when $x = y = 0$

Hence function $f(x, y)$ is continuous function of x and y (prove it.)
On the other hand,

$$|f(x,y_1)-f(x,y)| = \left|\frac{4x^3(x^4-y_1y)}{(x^4+y^2)(x^4+y_1^2)}\,(y_1-y)\right|$$

$$= \left|\frac{4(1-pq)}{(1+p^2)(1+q^2)}\frac{(y_1-y)}{x}\right|, \text{ where, } y=px^2, y_1=qx^2$$

$$=4\frac{|1-pq|}{|(1+p^2)(1+q^2)|}\cdot\frac{|y_1-y|}{|x|}$$

and therefore the Lipschitz condition is not satisfied in any region containing the origin. The equation admits the solution $y=c^2-\sqrt{x^4+c^4}$, where c is an arbitrary real constant. Thus there is an infinity of solutions satisfying the intial conditions $x=0,\ y=0$.

E8) a) y_1+y_2 is a solution of the equation

b) $c_1y_1+c_2y_2$ (for arbitrary c_1, c_2) is not a solution of the equation when $c_2 \ne 0,1$.

E9) It can be easily verified that $=\sin ax$ and $y_2=\cos ax$ satisfy the equation

$$y''+a^2y=0$$

and, hence, are the solutions of the given equation on the interval $]-\infty,\infty[$.

Here $W(y_1,y_2)=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \sin ax & \cos ax \\ a\cos ax & -a\sin ax \end{vmatrix}$

$$=-a\sin^2 ax - a\cos^2 ax = -a \ne 0.$$

Hence $\sin ax$ and $\cos ax$ are linearly independent solutions of the given equation. The general solution can then be written as $y=c_1\sin ax+c_2\cos ax$.

E10) Let $y_1=\sin\frac{1}{x}$ and $y_2=\cos\frac{1}{x}, y_1'=-\frac{1}{x^2}\cos\frac{1}{x}$, $y_2'=\frac{1}{x^2}\sin\frac{1}{x}$

$$y_1''=-\frac{1}{x^4}\sin\frac{1}{x}+\frac{2}{x^3}\cos\frac{1}{x}, \quad y_2''=\frac{-1}{x^4}\cos\frac{1}{x}-\frac{2}{x^3}\sin\frac{1}{x}$$

Here $x^4y_1''+2x^3y_1'+y_1$

$$=x^4\left(-\frac{1}{x^4}\sin\frac{1}{x}+\frac{2}{x^3}\cos\frac{1}{x}\right)+2x^3\left(-\frac{1}{x^2}\cos\frac{1}{x}\right)+\sin\frac{1}{x}=0$$

Hence y_1 satisfies given equation for $0<x<\infty$.

Similarly $x^4y_2''+2x^3y_2'+y_2=0$, thus y_2 is also a solution of given equation.

Here $W(y_1,y_2)=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \sin\frac{1}{x} & \cos\frac{1}{x} \\ -\frac{1}{x^2}\cos\frac{1}{x} & \frac{1}{x^2}\sin\frac{1}{x} \end{vmatrix} = \frac{1}{x^2} \ne 0$, for $0<x<\infty$.

Hence y_1 and y_2 are linearly independent solutions of given equation and the general solution can be written as

$$y=c_1\sin\frac{1}{x}+c_2\cos\frac{1}{x} \text{ and } y'=-c_1\frac{1}{x^2}\cos\frac{1}{x}+c_2\frac{1}{x^2}\sin\frac{1}{x}$$

Now $y\left(\frac{1}{\pi}\right)=1 \Rightarrow 1=c_1\sin\pi+c_2\cos\pi=-c_2 \Rightarrow c_2=-1$

$$y'\left(\frac{1}{\pi}\right)=-1 \Rightarrow -1=-c_1\pi^2\cos\pi+c_2\pi^2\sin\pi \Rightarrow -1=c_1\pi^2 \Rightarrow c_1=-\frac{1}{\pi^2}$$

Hence the required solution is $y=-\frac{1}{\pi^2}\sin\frac{1}{x}-\cos\frac{1}{x}$.

E11) A.E. is $m^3+6m^2+11m+6=0$, $m=-1,-2,-3$

Thus the general solution is, $y = c_1e^{-x} + c_2e^{-2x} + c_3e^{-3x}$.

E12) A.E. is $m^2 - 3m + 2 = 0$, $m = 1,2$

Thus the general solution is, $y = c_1e^x + c_2e^{2x}$

Satisfying the conditions $y = 0$ and $y' = 1$ when $x = 0$, we get $c_1 = -1,\ c_2 = 1$

Hence the solution of given problem is $y = -e^x + e^{2x}$

E13) a) A.E. is $16m^2 + 24m + 9 = 0$, $m = -3/4,\ -3/4$

$y = (c_1 + c_2x)e^{\frac{-3}{4}x}$

b) A.E. is $m^4 - 2a^2m^2 + a^4 = 0$, $m = a, a, -a, -a$.

Hence the solution is

$y = (c_1 + c_2x)e^{ax} + (c_3 + c_1x)e^{-ax}$.

c) A.E. is $m^3 + m^2 = 0$, $m = 0,0,-1$

Thus, solution is $y = (c_1 + c_2x) + c_3e^{-x}$

Initial conditions $y(0) = 1, y'(0) = 0,\ y''(0) = 1$ give $c_1 = 0,\ c_2 = 1,\ c_3 = 1$

Hence required solution is $y = x + e^{-x}$.

E14) a) A.E. is $m^4 + 4m^2 + 16 = 0$, $m = 2i,\ 2i,\ -2i,\ -2i$

Hence the solution is

$y = (c_1 + xc_2)\cos 2x + (c_3 + xc_4)\sin 2x$.

b) $y = c_1 \cos 2x + c_2 \sin 2x$.

c) $y = c_1e^{-2x} + e^x(c_2 \cos x + c_3 \sin x)$

d) A.E. is $m^6 + 9m^4 + 24m^2 + 16 = 0$, $m = \pm\ i,\ \pm 2i,\ \pm 2i$

Hence the solution is

$y = (c_1 \cos x + c_2 \sin x) + (c_3 + xc_4)\cos 2x + (c_5 + xc_6)\sin 2x$.

E15) a) A.E. is $m^3 + m = 0$, $m = 0,\ m = \pm\ i$

$\text{C.F.} = c_1 + c_2 \cos x + c_3 \sin x$

$$\text{P.I.} = \frac{1}{D^3 + D}(x^3 + \cos x) = \frac{1}{D(D+i)(D-i)}(x^3 + \cos x)$$

$$= \left(\frac{1}{D} + \frac{1}{2}.\frac{1}{D-i} - \frac{1}{2}.\frac{1}{D+i}\right)(x^3 + \cos x)$$

$$= \int (x^3 + \cos x)\ dx + \frac{1}{2}e^{ix}\int e^{-ix}(x^3 + \cos x)dx - \frac{1}{2}e^{-ix}\int e^{ix}(x^3 + \cos x)dx$$

$$= \frac{x^4}{4} - 3x^2 - \frac{1}{2}x \cos x$$

Hence $y = c_1 + c_2 \cos x + c_3 \sin x + \dfrac{x^4}{4} - 3x^2 + x \cos x$

b) C.F. $= c_1e^x + c2\ e^{-x} + c_3\ e^{-3x}$

$$\text{P.I.} = \frac{1}{(D-1)\ (D+1)\ (D+3)}\cosh x = \left(\frac{1}{8}.\frac{1}{D-1} - \frac{1}{4}.\frac{1}{D+1} + \frac{1}{8}.\frac{1}{D+3}\right).\frac{1}{2}(e^x + e^{-x})$$

$$= \frac{1}{16}e^x\int e^{-x}(e^x + e^{-x})dx - \frac{1}{8}e^{-x}\int e^x(e^x + e^{-x})dx + \frac{1}{16}e^{-3x}\int e^{3x}(e^x + e^{-x})dx$$

$$= \frac{1}{16}x\,e^x - \frac{1}{8}x\,e^{-x} - \frac{3}{64}e^x$$

Hence $y = c_1e^x + c_2e^{-x} + c_3e^{-3x} + \dfrac{1}{16}x\,e^x - \dfrac{1}{8}x\,e^{-x}$

c) $\text{C.F.} = c_1 e^{-x} + c_2 \cos 2x + c_3 \sin 2x$

$$\text{P.I.} = \frac{1}{(D+1)(D^2+4)} \sin 2x = \left[\frac{1}{5}.\frac{1}{D+1} - \frac{1}{4(i+2)}.\frac{1}{D+2i} + \frac{1}{4(i-2)}\frac{1}{D-2i}\right] \sin 2x$$

$$= \frac{1}{5} e^{-x} \int e^x \sin 2x\, dx - \frac{1}{4(i+2)} e^{-2ix} \int e^{2ix} \sin 2x\, dx + \frac{1}{4(i-2)}.e^{2ix} \int e^{-2ix} \sin 2x\, dx$$

$$= \frac{1}{5} e^{-x}\left[\frac{1}{5} e^x \sin 2x - \frac{2}{5} e^x \cos 2x\right] - \frac{1}{8i(i+2)} e^{2ix} \int e^{2ix}(e^{2ix} - e^{-2ix})\, dx$$

$$+ \frac{1}{8i(i-2)} e^{2ix} \int e^{-2ix} - e^{-2ix})\, dx$$

$$= \frac{-x}{20}(\cos 2x + 2 \sin 2x)$$

d) $y = c_1 \cos nx + c_2 \sin nx + \frac{x}{n} \sin(nx) + \frac{\ln \cos(nx)}{n^2}.(\cos nx)$

e) $\text{C.F.} = (c_1 + c_2 x) e^{2x} + c_3 e^{-3x}$

$$\text{P.I.} = \frac{1}{(D-2)^2 (D+3)} X(x)$$

$$= \left(-\frac{1}{25}.\frac{1}{D-2} + \frac{1}{5}.\frac{1}{(D-2)^2} + \frac{1}{25}.\frac{1}{D+3}\right) X$$

$$= -\frac{1}{25} e^{2x} \int e^{-2x} X\, dx + \frac{1}{5} e^{2x} \iint e^{-2x} X (dx)^2 + \frac{1}{25} e^{-3x} \int e^{3x} X\, dx$$

Hence, $y = (c_1 + c_2 x) e^{2x} + c_3 e^{-3x} - \frac{1}{25} e^{2x} \int e^{-2x} X dx + \frac{1}{5} e^{2x} \iint e^{-2x} X (dx)^2$

$$+ \frac{1}{25} e^{-3x} \int e^{3x} X\ dx.$$

E16) a) $y = (c_1 + c_2 x) e^x + c_3 \cos x + c_4 \sin x$

b) $y = (c_1 + c_2 x) e^x + (c_3 + c_4 x) e^{-2x}$

c) $y = 2e^x \sin 2x$

E17) a) $\text{C.F.} = (c_1 + c_2 x + c_3 x^2) e^{-x}$

$$\text{P.I.} = \frac{1}{(D+1)^3} e^{-x} (2 - x^2)$$

$$= e^{-x} \frac{1}{[(D-1)+1]^3} (2 - x^3) = e^{-x} \frac{1}{D^3} (2 - x^2)$$

$$= e^{-x} \frac{1}{D^2}\left(2x - \frac{x^3}{3}\right) = e^{-x} \frac{1}{D}\left(x^2 - \frac{x^4}{12}\right) = e^{-x}\left(\frac{x^3}{3} - \frac{x^5}{60}\right)$$

Hence, $y = (c_1 + c_2 x + c_3 x^2) e^{-x} + e^{-x}\left(\frac{x^3}{3} - \frac{x^5}{60}\right).$

b) $\text{C.F.} = c_1 \cos x + c_2 \sin x + c_3 \cos 2x + c_4 \sin 2x$

$$\text{P.I.} = \frac{x}{12}\left(\sin x - \frac{1}{2} \sin 2x\right)$$

c) $y = c_1 + c_2 e^{-x} + c_3 e^x + c_4 \cos x + c_5 \sin x + 2x^2 + 2x \sin x + 3x e^x$.

d) $y = c_1 e^{-x} + c_2 e^x - \frac{x}{2} e^{-x} - \frac{1}{2} \cos x - x^3 - 6x - \frac{1}{5} e^x (\cos x - 2 \sin x)$

e) Let $x+3=e^t$, $\frac{d}{dt}=D$, then the given equation becomes

$$(D^2-5D+6)\,y=t$$

A.E. is $m^2-5m+6=0 \Rightarrow m=2,\ 3$

$$\text{C.F.}=c_1\,e^{2t}+c_3\,e^{3t}=c_1(x+3)^2+c_2(x+3)^3$$

$$\text{P.I.}=\frac{1}{6-5D+D^2}\,t=\frac{1}{6}\left(1-\frac{5}{6}D+\frac{1}{6}D^2\right)^{-1}t$$

$$=\frac{1}{6}\left(1+\frac{5}{6}D+\cdots\right)t$$

$$=\frac{1}{6}\left(t+\frac{5}{6}\right)=\frac{1}{6}t+\frac{5}{36}=\frac{1}{6}\ln(x+3)+\frac{5}{36}$$

Hence $y=c_1(x+3)^2+c_2(x+3)^3+\frac{5}{6}+\frac{1}{6}\ln(x+3)$

f) $\text{C.F.}=c_1\cos ax+c_2\sin a\,x$

$$\text{P.I.}=\frac{1}{D^2+a^2}\tan ax=\frac{1}{2ia}\left(\frac{1}{D-i\,a}-\frac{1}{D+i\,a}\right)\tan ax$$

$$=\frac{1}{2ia}.e^{iax}\int e^{-iax}\,\tan ax\,dx-\frac{1}{2ia}e^{-iax}\int e^{iax}\,\tan ax\,dx$$

$$=\frac{1}{2ia}.e^{iax}\left[\int(\sin ax-i\sec ax+i\cos ax)\,dx\right]$$

$$-\frac{1}{2ia}e^{-iax}\left[\int(\sin ax+i\sec ax-i\cos ax)\,dx\right]$$

$$=-\frac{1}{a^2}\sin ax\cos ax+\frac{1}{a^2}\sin ax\cos ax-\frac{1}{a^2}\cos ax\ln\left|\sec ax+\tan ax\right|$$

$$=-\frac{1}{a^2}\cos\,(ax).\ln\left|\sec(ax)+\tan(ax)\right|$$

Hence $y=c_1\cos ax+c_2\sin ax-\frac{1}{a^2}\cos(ax).\ln\left|\sec(ax)+\tan(ax)\right|$.

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UNIT 2 POWER SERIES SOLUTIONS

Structure **Page No.**

2.1 INTRODUCTION

In Unit 1, we recalled the methods of finding explicit solutions of linear differential equations with constant coefficients and Euler-Cauchy equations which are reducible to equations with constant coefficients by a change of independent variable and illustrated these methods through various examples. The class of functions, appearing as solution of these equations involved elementary functions, i.e., exponential, trigonometric, logarithmic and polynomial functions, all of which can be written in terms of exponential functions. In many engineering, mathematical and physical problems, the equations governing the physical situations turn out to be homogeneous, linear differential equations with variable coefficients. Solutions of such equations cannot be expressed in terms of elementary functions. But since elementary functions can be expressed in terms of power series; this gives us an idea that for a general homogeneous, linear differential equation with variable coefficients, we might try a function defined by a general power series

$$y = \sum_{n=0}^{\infty} a_n \ (x - x_0)^n$$

as a solution, where the series on the right being meaningful only when it is convergent. We shall see that this gives a powerful method of solving linear homogeneous differential equations with variable coefficients and leads to the introduction of some new ‘standard functions’, commonly known as ‘higher transcendental functions’ or ‘special functions’ such as Legendre, Bessel, Hypergeometric, Hermite functions etc. The appearance of these special functions as solutions of differential equations describing various phenomena in physics and engineering indicates that these functions are as ‘standard’ as our earlier ‘elementary functions’. In this unit we shall discuss methods of finding power series solutions of linear homogeneous differential equations with variable coefficients.

In Sec.2.2, we have started the discussion with the introduction of power series and its radius of convergence. We have discussed ordinary and singular points of a differential equation in Sec.2.3. In Sec.2.4 we have discussed power series solutions of a differential equation about an ordinary point and series solutions about a regular singular point – in particular, the Frobenius method. We shall be discussing the applications of these power series methods to various physical situations, in Units 3 and 4.

Objectives

After studying this unit you should be able to

- describe the need for a series solution of homogeneous linear differential equation with variable coefficients;
- define the radius of convergence and interval of convergence, of a power series;
- distinguish between an ordinary point, a regular singular point and an irregular singular point of a differential equation;

- obtain a power series solution of a given differential equation about an ordinary point;
- obtain a series solution of a given differential equation about a regular singular point using Frobenius method.

2.2 POWER SERIES AND RADIUS OF CONVERGENCE

You are already familiar with the series of **constant** terms i.e., series of the form,

$$\sum_{n=1}^{\infty} n = 1 + 2 + 3 + \cdots$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

If the terms of a series are **variable**, say, function of a variable x, then they assume definite values when x is given a fixed value. The basic idea of the power series method for solving differential equations, which is our concern in this unit, is very simple and natural. Given a differential equation, we represent all the functions in the equation by a power series. This process involves various operations like differentiation, addition and multiplication of power series. Therefore, to justify the method let us first consider the theoretical basis of these operations.

A power series in powers of $(x - a)$ is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots \tag{1}$$

where x is a variable, $a_0, a_1 \ldots$ are constants, called the **coefficients**, and a is a constant, called the **centre** of the series. For discussion in this unit we shall assume that all variables and constants are real.

In particular, if $a = 0$ then we get a power series in powers of x as

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \tag{2}$$

Remark: The term 'power series' usually refers to series of the form (1), including the case (2), but does not include series of negative powers of x such as $c_1 x^{-1} + c_2 x^{-2} + \cdots$ or, series involving fractional powers of x.

The convergence behaviour of a power series can be characterized in a simple way. We know from d'Alembert's ratio test for absolute convergence of a series that series (2) is absolutely convergent or divergent according as

$$\lim_{n\to\infty} \frac{|a_n x^n|}{|a_{n+1} x^{n+1}|} = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| \left| \frac{1}{x} \right|$$

is greater or less than one, provided the above limit exists.

Let $$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|, \tag{3}$$

then subject to the existence of this limit, series (2) converges absolutely if

$$\frac{R}{|x|} > 1,\ x \neq 0, \text{ i.e., } |x| < R$$

and diverges if , $\frac{R}{|x|} < 1$, i.e., $|x| > R$

Thus the region of convergence is the interval $-R < x < R$ within which the series (2) is absolutely convergent and outside this interval it is divergent. Accordingly, series (1) converges in the interval.

$$a - R < x < a + R \tag{4}$$

with centre $x = a$. The number R is known as the **radius of convergence** of series (1) and interval (4) is known as its **interval of convergence.** Here R is always non-negative and may assume the value 0 or ∞, or, any value in between.

Usually we put R equal to 0 when the series converges only for $x = 0$ and equal to ∞ when it converges for all x. This convention allows us to cover all possibilities in a single statement: each power series in x has a radius of convergence R, where $0 \le R \le \infty$ with the property that the series converges if $|x| < R$ and diverges if $|x| > R$. You may note that if $R = 0$ then no x satisfies $|x| < R$, and if $R = \infty$ then no x satisfies $|x| > R$. We can thus say that all power series in x fall into one, or, another, or three major categories which are typified by the following examples

$$\sum_{n=0}^{\infty} n!x^n = 1 + x + 2!x^2 + 3!x^3 + \cdots \tag{5}$$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \tag{6}$$

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \tag{7}$$

Series(5) diverges (i.e. fails to converge) for all $x \neq 0$. Series (6) converges for all x and series (7) converges for $|x| < 1$ and diverges for $|x| > 1$. How R is calculated for the above three series will be clear to you after going through the following examples.

Example 1: Find the radius of convergence of the series

$$\sum_{n=0}^{\infty} n^n x^n = 1 + x + 2^2 x^2 + 3^3 x^3 + \cdots$$

Solution: The given series has radius of convergence R given by

$$R = \lim_{n\to\infty} \left| \frac{n^n}{(n+1)^{n+1}} \right| = \lim_{n\to\infty} \left| \frac{1}{(n+1)\left(1 + \frac{1}{n}\right)^n} \right| = 0$$

Hence the series does not converge for any non-zero x. However, it obviously converges for $x = 0$.

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Example 2: Find the interval of convergence of the series $\sum_{n=0}^{\infty} n\, x^n$.

Solution: The radius of convergence of the given series is

$$R = \lim_{n\to\infty} \left| \frac{n}{n+1} \right| = \lim_{n\to\infty} \left| \frac{1}{1 + \frac{1}{n}} \right| = 1$$

Thus the given series converges for $|x| < 1$, i.e., $-1 < x < 1$ and diverges for $|x| > 1$.

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Let us now suppose that series (2) converges for $|x| < R$ with $R > 0$, and denote its sum by $f(x)$ then ,

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \tag{8}$$

Now $f(x)$ is obviously continuous and has derivatives of all orders for $|x| < R$. Also the series can be differentiated termwise in the sense that

$$f'(x) = \sum_{n=1}^{\infty} n\, a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

$$f''(x) = \sum_{n=2}^{\infty} n\,(n-1) a_n x^{n-2} = 2a_2 + 3\ 2a_2 x + \cdots$$

and so on, and each of the resulting series converges for $|x| < R$. The formula linking the a_n to $f(x)$ and its derivatives can be obtained from these successively differentiated series as follows:

$$a_n = \frac{f^{(n)}(0)}{n!} \tag{9}$$

Just as we have differentiated series (8) it can also be integrated termwise provided the limits of integration lie inside the interval of convergence. Further, if we have a second power series in x that converges to a function $g(x)$ for $|x| < R$, so that

$$g(x) = \sum_{n=0}^{\infty} b_n x^n = b_0 + b_1 x + b_2 x^2 + \cdots \tag{10}$$

then series (8) and (10) an be added or subtracted termwise as

$$f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n) x^n = (a_0 \pm b_0) + (a_1 \pm b_1) x + \cdots$$

They can also be multiplied like polynomials, such that

$$f(x) g(x) = \sum_{n=0}^{\infty} c_n x^n$$

where $c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0$. If series (8) and (10) converge to the same functions, so that $f(x) = g(x)$ for $|x| < R$, then according to formula (8) they have the same coefficients: $a_0 = b_0,\ a_1 = b_1, \ldots$

In particular, if $f(x) = 0$ for $|x| < R$ then $a_0 = 0, a_1 = 0, \ldots$.

Now let us suppose that $f(x)$ be a continuous function that has derivatives of all orders for $|x| < R$ with $R > 0$. Then you may wonder whether $f(x)$ can be represented by a power series or not. If we use formula (9) to define the a_n then we can hope that the expansion

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0) x + \frac{f''(0)}{2!} x^2 + \cdots \tag{11}$$

will hold throughout the interval. This is often true but sometimes it is false. To check the validity of this expansion for a specific point x in the interval we can use Taylor's formula:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} + R_n(x)$$

where the remainder $R_n(x)$ is given by

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\ x^{n+1}$$

for some point ξ between 0 and x. To verify (11), it suffices to show that $R_n(x) \to 0$ as $n \to \infty$. The above procedure can be used quite easily to obtain the following familiar expansions, which are valid for all x.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \tag{12}$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n\ \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{31} + \frac{x^5}{5!} \cdots \tag{13}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n\ \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots \tag{14}$$

You may now try the following exercises.

E1) Verify that $R = 0$, $R = \infty$ and $R = 1$, for the series (5), (6) and (7), respectively.

E2) Discuss the convergence of the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

E3) Obtain the radius of convergence of the following series

(a) $1 - 2x + 3x^2 - 4x^3 + \cdots$

(b) $\sum_{n=0}^{\infty} \frac{(kn)!}{(n!)^k} x^n$, k is a positive integer.

As we have mentioned earlier our aim here is to find the general solution of a linear homogeneous differential equation with variable coefficients. In particular, we shall try to find the solution of the equation

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)\,y = 0, \quad a_0(x) \neq 0.$$

or,

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = 0 \qquad (15)$$

where, $P(x) = \frac{a_1(x)}{a_0(x)}$, $Q(x) = \frac{a_2(x)}{a_0(x)}$ and $a_0(x), a_1(x)$ and $a_2(x)$ are continuous functions of x on some interval I. You have studied some properties of Eqn.(15) in Unit 1 when a_0, a_1, a_2 are constants and obtained solutions of such equations in terms of elementary functions. But for equations with variable coefficients when solutions cannot be obtained in terms of these elementary functions, power series solutions come to our rescue. For studying the power series method we need to study some definitions, which we shall take up next.

2.3 ORDINARY AND SINGULAR POINTS OF AN EQUATION

Here we start with recalling the definition of an analytic function which you have studied in your complex analysis course (MMT-005).

Analytic Function: A function $f(x)$ with the property that a power series expansion of the form

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n \qquad (16)$$

is valid in some neighbourhood of the point a is said to be **analytic** at a. In this case the a_n's are given by

$$a_n = \frac{f^{(n)}(a)}{n!}, \quad n = 0, 1, 2, \ldots$$

and Eqn.(16) is called the Taylor series of $f(x)$ at $x = a$. Thus, Eqns. (12), (13) and (14) show that e^x, $\sin x$ and $\cos x$ are analytic at $x = 0$, and the given series are the Taylor series of these functions at this point. You must also be familiar with the following facts about analyticity:

i) Polynomials and the functions e^x, $\sin x$ and $\cos x$ are analytic at all points.

ii) If $f(x)$ and $g(x)$ are analytic at a, then $f(x) + g(x), f(x)g(x)$, and $f(x)/g(x)$ [if $g(a) \neq 0$] are also analytic at a.

iii) The sum of a power series is analytic at all points inside the interval of convergence.

We shall now use this definition to introduce some more definitions which will be of use for our further discussions.

Ordinary Point: A point $a \in I$ is called an **ordinary point** of Eqn.(15) if , $P(x)$ and $Q(x)$ are analytic at $x = a$. Ordinary point is also called a **regular point** of the equation.

Equation $y'' - 2xy' + 2py = 0$, where p is a constant is called Hermite's equation and has an ordinary point at $x = 0$, since $-2x$ and $2p$ are analytic at $x = 0$.

The point $x = 2$ is not an ordinary point for the equation $(x-2)\,y'' + y = 0$, because the function $\dfrac{1}{x-2}$ does not admit a power series around 2.

Singular Point: Any point of I which is not an ordinary point of Eqn.(15) is said to be a **singular point** of it. Thus for $x = a$ to be a singular point of Eqn.(15), at least one of the coefficients $P(x)$, $Q(x)$ must fail to be analytic at $x = a$.

Regular Singular Point: A singular point a of Eqn.(15) viz.,

$$y'' + \frac{a_1(x)}{a_0(x)}\,y' + \frac{a_2(x)}{a_0(x)}\,y = 0,\quad a_0(x) \neq 0,$$

is said to be regular singular point, if $(x-a)\dfrac{a_1(x)}{a_0(x)}$ and $(x-a)^2\dfrac{a_2(x)}{a_0(x)}$ are analytic at $x = a$.

In other words, we say that in such cases, the singular points are 'weak', in the sense that the coefficient functions are only mildly non-analytic and, with simple modifications, can be made analytic and can have a Taylor series expansion about the singular point.

A singular point, which is not regular singular point is called an **irregular singular point.**

As a simple example, the origin is clearly a regular singular point of the equation

$$y'' + \frac{2}{x}\,y' - \frac{2}{x^2}\,y = 0.$$

Consider the equation

$$(1 - x^2)\,y'' - 2xy' + p(p+1)\,y = 0,$$

where p is a constant. This equation is Legendre's equation about which we shall be studying in detail in the next unit. The equation can be put in the form

$$y'' - \frac{2x}{1-x^2}\,y' + \frac{p(p+1)}{1-x^2}\,y = 0.$$

It is clear that $x = 1$ and $x = -1$ are its singular points. The point $x = 1$ is regular because

$$(x-1)\,P(x) = (x-1)\left[-\frac{2x}{1-x^2}\right] = \frac{2x}{x+1},$$

and $(x-1)^2\,Q(x) = (x-1)^2\left[\dfrac{p(p+1)}{1-x^2}\right] = -\dfrac{(x-1)\;p(p+1)}{x+1}$,

are analytic at $x = 1$. For similar reasons $x = -1$ is also a regular singular point.

As another example, consider Bessel's equation of order p, where p is a non-negative constant

$$x^2\,y'' + x\,y' + (x^2 - p^2)\,y = 0.$$

Writing the equations in the form

$$y'' + \frac{1}{x}\,y' + \frac{x^2 - p^2}{x^2}\,y = 0,$$

it is apparent that origin is a regular singular point of the equation because

$$x\,P(x) = x\,\frac{1}{x} = 1 \quad \text{and} \quad x^2 Q(x) = x^2\,\frac{x^2 - p^2}{x^2} = x^2 - p^2,$$

are analytic at $x = 0$.

You may now try the following exercises.

E4) Find the regular and singular points of the following DEs. Also, specify the nature of singular points.

(a) $y'' + 4x\,y' + y = 0$.

(b) $x^3(x-2)\,y'' + x^3 y' + 6y = 0$.

(c) $(x+1)\,y''' + x\,y'' + x^2 y' + x^3\,y = 0$.

(d) $x^4 y'' + x^2 y' + 2y = 0$.

(e) $x\,y'' + (\sin x)\,y' + e^x y = 0$.

In the next section we shall discuss the series solution methods for solving differential equations with variable coefficients about an ordinary point and about a regular singular point.

2.4 SERIES SOLUTION METHODS

Let us consider Eqn.(15) viz.,

$$y'' + P(x)\,y' + Q(x)\,y = 0$$

where, $P(x) = \dfrac{a_1(x)}{a_0(x)}$, $Q(x) = \dfrac{a_2(x)}{a_0(x)}$ and $a_0(x), a_1(x)$ and $a_2(x)$ are continuous functions of x over some interval I and $a_0(x) \neq 0$.

The series solution methods of solving such equations are classified into two categories: Power series method and general series solution method-in particular, the Frobenius method. We first discuss the power series solution of Eqn.(15) about an ordinary point.

2.4.1 Power Series Solution About an Ordinary Point

We start with a **sufficient condition** for the existence of a power series solution of Eqn.(15).

Theorem 1: Let $x = a$ be an ordinary point (regular point) of Eqn.(15). Then, every solution of the equation is analytic at $x = a$ and hence has a power series expansion about the point $x = a$ of the form

$$y(x) = \sum_{m=0}^{\infty} c_m (x-a)^m \tag{17}$$

where c_0, c_1 are constants and c_k for $k \geq 2$ are obtained in terms of c_0 and c_1.

The **proof** is obvious since $a_0(x) \neq 0$, $P(x)$ and $Q(x)$ are analytic at $x = a$. Hence $y'(a),\ y''(a), y'''(a), \ldots$ exist and the Taylor expansion of $y(x)$, that is, power series solution about $x = a$ exists. Further, note that every function which is analytic in the region $|x-a| < R$ admits a converging power series representation $\sum_{m=0}^{\infty} c_m (x-a)^m$ in the region.

Let us for the sake of convenience, restrict our argument to the case in which $a = 0$. This allows us to work with power series in x rather than $x - a$, and involves no real loss of generality. With this slight simplification, Eqn.(15) will now have solution of the form

$$y(x)=\sum_{m=0}^{\infty}c_m x^m \tag{18}$$

where c_m's are constants to be determined. Also now $P(x)$ and $Q(x)$ are analytic at the origin and therefore have power series expansions of the form

$$P(x)=\sum_{m=0}^{\infty}p_m x^m \tag{19}$$

and,
$$Q(x)=\sum_{m=0}^{\infty}q_m x^m \tag{20}$$

where p_m's and q_m's are known constants. The above expansion will converge on an interval $|x|<R$ for some $R>0$.

In order to find a solution of Eqn.(15), term by term differentiation of Eqn.(18) yields

$$y'=\sum_{m=1}^{\infty}mc_m x^{m-1}=\sum_{m=0}^{\infty}(m+1)c_{m+1}x^m=c_1+2c_2x+3c_3x^2+\cdots$$

and
$$y''=\sum_{m=2}^{\infty}m(m-1)c_m x^{m-2}=\sum_{m=0}^{\infty}(m+1)(m+2)c_{m+2}x^m$$

$$=2c_2+2.\ 3c_3x+3.4c_4x^2+\cdots \tag{21}$$

From the rule for multiplying power series, it follows that

$$P(x)y'=\left(\sum_{m=0}^{\infty}p_m x^m\right)\left[\sum_{m=0}^{\infty}(m+1)c_{m+1}x^m\right]$$

$$=\sum_{m=0}^{\infty}\left[\sum_{k=0}^{m}p_{m-k}(k+1)c_{k+1}\right]x^m \tag{22}$$

and
$$Q(x)\,y=\left(\sum_{m=0}^{\infty}q_m x^m\right)\left(\sum_{m=0}^{\infty}c_m x^m\right)$$

$$=\sum_{m=0}^{\infty}\left(\sum_{k=0}^{m}q_{m-k}c_k\right)x^k \tag{23}$$

On substituting Eqns(21), (22) and (23) into Eqn. (15) and adding the series term by term, we obtain

$$\sum_{m=0}^{\infty}\left[(m+1)(m+2)c_{m+2}+\sum_{k=0}^{m}p_{m-k}(k+1)c_{k+1}+\sum_{k=0}^{m}q_{m-k}c_k\right]x^m=0.$$

Thus we have the following recursion formula for c_m:

$$(m+1)\ (m+2)c_{m+2}=-\sum_{k=0}^{m}[(k+1)\,p_{m-k}c_{k+1}+q_{m-k}c_k\ ] \tag{24}$$

For $m=0,1,2,\ldots$ this formula becomes

$$2c_2=-(p_0c_1+q_0c_0)$$
$$2.3c_3=-(p_1c_1+2p_0c_2+q_1c_0+q_0c_1)$$
$$3.4c_4=-(p_2c_1+2p_1c_2+3p_0c_2+q_2c_0+q_1c_1+q_0c_2)$$

These formulas determine $c_2,c_3,\ldots$ in terms of c_0 and c_1, and the series (18), which satisfies Eqn.(15), is uniquely determined.

We illustrate the above method by means of the following examples.

Example 3: Solve $y''-x\,y'+3y=0$ (25)

Solution: We observe that Eqn.(25) is a linear equation and its coefficients are analytic everywhere. In particular, $x=0$ is an ordinary point of Eqn.(25) and hence we try power series solution

$$y(x)=\sum_{m=0}^{\infty} c_m x^m \tag{26}$$

where the unknown constants c_m's are to be determined. Here

$$y'(x)=\sum_{m=1}^{\infty} m\, c_m x^{m-1}$$
$$y''(x)=\sum_{m=2}^{\infty} m\,(m-1)\; c_m x^{m-2} \tag{27}$$

Substituting from Eqns.(26) and (27) the values of y, y', y'' into Eqn.(25), we get

$$\sum_{m=2}^{\infty} m(m-1)\, c_m x^{m-2} - x\sum_{m=1}^{\infty} mc_m x^{m-1} + 3\sum_{m=0}^{\infty} c_m x^m = 0\,.$$

or, $$\sum_{m=0}^{\infty}(m+2)\,(m+1)c_{m+2}x^m - \sum_{m=1}^{\infty} m\, c_m x^m + 3\sum_{m=0}^{\infty} c_m x^m = 0 \tag{28}$$

Equating coefficients of various powers of x on both sides of Eqn.(28), we get,

$$2.1\, c_2 + 3c_0 = 0 \tag{29}$$

and, $$(m+2)\,(m+1)\, c_{m+2} - (m-3)\, c_m = 0,\ m \ge 1 \tag{30}$$

From Eqns.(29) and (30), we get

$$c_2 = -(3/2)\, c_0 \tag{31}$$

and, $$c_{m+2} = [\,(m-3)\,/(m+1)\ (m+2)\,]\, c_m \ \text{ for }\ m \ge 1 \tag{32}$$

Setting $m = 1, 2, 3, \ldots,$ we get

$$c_3 = -(1/3)\ c_1 \tag{33}$$

$$c_4 = -\frac{1}{4.3} c_2 = -\frac{1}{4.3}\left(-\frac{3}{2}\right) c_0 = \frac{1}{8} c_0 \tag{34}$$

$$c_5 = 0 \tag{35}$$

We observe that once $c_5 = 0$, all the coefficients

$$c_7 = c_9 = \cdots = 0\,. \tag{36}$$

Remaining even indexed coefficients $c_6, c_8, \ldots$ can be determined in terms of c_0 using Eqn.(32). Recurrence relation (32) yields

$$\begin{aligned}
c_{2r+4} &= \frac{2r-1}{(2r+4)\,(2r+3)} c_{2r+2} \\
&= \frac{(2r-1)}{(2r+4)\,(2r+3)} \cdot \frac{(2r-3)}{(2r+2)\,(2r+1)} c_{2r} \\
&= \frac{(2r-1)\,(2r-3)}{(2r+4)\,(2r+3)\,(2r+2)\,(2r+1)}\ \frac{(2r-5)}{2r\,(2r-1)} c_{2r-2} = \cdots\cdots \\
&= \frac{(2r-1)\,(2r-3)\cdots .3.1}{(2r+4)\,(2r+3)\cdots 7.6.5.} c_4 \\
&= \frac{(2r-1)\,(2r-3)\cdots 3.1}{(2r+4)\,(2r+3)\cdots 7.6.5} \cdot \frac{1}{4.3} \cdot \frac{3}{2} c_0 \\
&= \frac{3(2r-1)\,(2r-3)\cdots .3.1}{(2r+4)!} c_0, \text{for } r = 1, 2, \ldots .
\end{aligned} \tag{37}$$

Thus from Eqn.(26), the solution becomes

$$\begin{aligned}
y(x) &= c_0 + c_2 x^2 + c_4 x^4 + \sum_{r=1}^{\infty} \frac{3(2r-1)\,(2r-3)\ldots .3.1}{(2r+4)!} c_0 x^{2r+4} + c_1 x + c_3 x^3 \\
&= c_0\left[1 - \frac{3}{2}x^2 + \frac{1}{8}x^4 + 3\sum_{r=1}^{\infty} \frac{(2r-1)\,(2r-3)\ldots 3.1}{(2r+4)!} x^{2r+4}\right] + c_1\left(x - \frac{1}{3}x^3\right)
\end{aligned} \tag{38}$$

The radius of convergence R of the series, obtained by using ratio test, is

$$R = \lim_{r\to\infty}\left|\frac{(2r-1)(2r-3)\ldots 3.1}{(2r+4)!}\cdot\frac{(2r+6)!}{(2r+1)(2r-1)\ldots .3.1}\right|$$

$$= \lim_{r\to\infty}\left|\frac{(2r+5)\ (2r+6)}{(2r+1)}\right| \to \infty$$

Therefore the series in Eqn.(38) converges for all x and the solution is valid for all x.

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Example 4: Find the power series solution about the origin of the equation

$$(1-x^2)\,y'' - 4x\,y' + 2y = 0 \tag{39}$$

Solution: We know that $x = 0$ is an ordinary point of Eqn.(39) and therefore the power series solution exists. Substituting the expressions for y, y', y'' from Eqns.(26) and (27) in Eqn.(39), we get

$$(1-x^2)\sum_{m=2}^{\infty} m(m-1)\,c_m x^{m-2} - 4x\sum_{m=1}^{\infty} m\,c_m x^{m-1} + 2\sum_{m=0}^{\infty} c_m x^m = 0\,.$$

or, $\displaystyle\sum_{m=2}^{\infty} m(m-1)c_m x^{m-2} - \sum_{m=2}^{\infty} m(m-1)c_m x^m - 4\sum_{m=1}^{\infty} mc_m x^m + 2\sum_{m=0}^{\infty} c_m x^m = 0$

or, $\displaystyle 2c_2 + 6c_3 x + \sum_{m=4}^{\infty} m(m-1)c_m x^{m-2} - \sum_{m=2}^{\infty} m(m-1)c_m x^m - 4c_1 x$

$$-4\sum_{m=2}^{\infty} mc_m x^m + 2(c_0 + c_1 x) + 2\sum_{m=2}^{\infty} c_m x^m = 0 \tag{40}$$

In the third term of Eqn.(40), setting $m-2 = t$ or, $m = t+2$, we get

$$2(c_2 + c_0) + 2(3c_3 - c_1)\,x + \sum_{t=2}^{\infty}(t+2)\,(t+1)\,c_{t+2}\ x^t$$

$$-\sum_{m=2}^{\infty} m(m-1)\,c_m x^m - 4\sum_{m=2}^{\infty} m\,c_m x^m + 2\sum_{m=2}^{\infty} c_m x^m = 0$$

Since t is dummy variable, we can write

$$2(c_2 + c_0) + 2(3c_3 - c_1)x + \sum_{m=2}^{\infty}[\,(m+2)\,(m+1)\,c_{m+2} - (m^2 + 3m - 2)\,c_m\,]x^m = 0$$

Setting the coefficients of successive powers of x to zero, we obtain

$$c_2 + c_0 = 0,\, 3c_3 - c_1 = 0,\, (m+2)\,(m+1)\,c_{m+2} = (m^2 + 3m - 2)c_m\,,\, m \ge 2$$

Solving, we obtain

$$c_2 = -c_0\,,\, c_3 = \frac{1}{3}c_1\,,\, c_{m+2} = \frac{(m^2 + 3m - 2)}{(m+2)\,(m+1)}c_m\,,\, m \ge 2$$

where c_0 and c_1 are arbitrary constants. We thus have

$$c_4 = \frac{2}{3}c_2 = \frac{-2}{3}c_0\,,\, c_5 = \frac{4}{5}c_3 = \frac{4}{15}c_1\,,$$

$$c_6 = \frac{13}{15}c_4 = \frac{-26}{45}c_0\,,\, c_7 = \frac{19}{21}c_5 = \frac{76}{315}c_1\,,\ldots$$

The power series solution is

$$y(x) = c_0\left[1 - x^2 - \frac{2}{3}x^4 - \frac{26}{45}x^6 - \cdots\right] + c_1\left[x + \frac{x^3}{5} + \frac{4}{15}x^5 + \frac{76}{315}x^7 + \cdots\right] \tag{41}$$

***

Remarks: In Example 4, $x = \pm 1$ are the singular points of Eqn.(39). Thus, the power series solution cannot exist in any interval I which contains $+1$ or -1. Therefore the interval of largest length, in which the power series solution (41) holds is $]-1,1[$

which is its interval of convergence. Further, since the radius of convergence R is the distance between the point $x = a$, about which the power series solution is sought and the nearest singularity of the equation therefore, in this case, $R = 1$.

You may now try some exercises.

E5) Solve $(x^2 - 2x + 2)\, y'' - 4(x-1)\, y' + 6y = 0$, about the point $x = 1$.

E6) Solve $y'' + x\, y' + 3y = 0$, by power series method about the point $x = 0$.

E7) Find power series solution in powers of $(x-1)$ of the initial value problem

$$\begin{aligned} &x\, y'' + y' + 2y = 0 \\ &y(1) = 1,\ y'(1) = 2 \end{aligned}.$$

E8) Find the power series solution of the equation $y'' + xy' + y = 0$, about $x = 2$.

E9) The equation $y'' - 2xy' + 2ny = 0$, where n is a positive integer, is called Hermite's equation of order n. Using the power series method show that if

$$z_n(x) = \sum_{k=0}^{\infty} a_k x^k$$

is a solution of Hermite's equation, then

$$z_n(x) = a_0\left[1 + \sum_{k=1}^{\infty} \frac{2^k(-n)(-n+2)\cdots(-n+2k-2)}{(2k)!} x^{2k}\right]$$
$$+ a_1\left[x + \sum_{k=1}^{\infty} \frac{2^k(1-n)(1-n+2)\cdots(1-n+2k-2)}{(2k+1)!} x^{2k+1}\right].$$

So far, we have discussed a method of finding series solution of a differential equation about an ordinary point, that is, power series solution of the equation. Power series is inadequate to represent a solution about a regular singular point $x = a$, as the solution may also contain negative powers and/or fractional powers of $(x - a)$. We shall now discuss the Frobenius method for obtaining a series solution about a regular singular point of the equation.

2.4.2 Series Solution about a Regular Singular Point: Frobenius Method

The method we are going to discuss here is due to Ferdinand George Frobenius (1849-1917), a German mathematician who made several valuable contribution to the theory of elliptic functions and differential equations. We shall be restricting ourselves to second order equations only because most of the important equations, in physical science and engineering, such as Bessel's equation, Legendre's equation, Hermite's equation or, the hypergeometric equations for which the method works are of second order only. Also, to simplify the matter we shall assume that the singular point a is located at the origin, for if it is not, then we can always move it to the origin by changing the independent variable from x to $x - a$.

Method of Frobenius

Consider linear differential equation of order two of the form

$$f(x)\frac{d^2y}{dx^2} + g(x)\frac{dy}{dx} + r(x)\, y = 0 \tag{42}$$

Let $x = 0$, be a regular singular point of Eqn.(42).

The method of Frobenius to solve Eqn.(42) consist of the following steps:-

Step I: Take a trial solution of Eqn.(42) of the form

$$y(x) = x^k(c_0 + c_1 x + c_2 x^2 + \cdots + c_m x^m + \cdots)$$

i.e., $$y(x) = x^k \sum_{m=0}^{\infty} c_m x^m .$$

$$= \sum_{m=0}^{\infty} c_m x^{m+k} \text{, where k is any real number and } c_0 \neq 0 \qquad (43)$$

Step II: Differentiate relation (43) and obtain

$$\left.\begin{aligned} y'(x) &= \sum_{m=0}^{\infty} (m+k)c_m x^{m+k-1} \\ \text{and} \quad y''(x) &= \sum_{m=0}^{\infty} (m+k)\ (m+k-1)c_m x^{m+k-2} \end{aligned}\right\} \qquad (44)$$

Using Eqns.(43) and (44), Eqn.(42) reduces to an identity.

Step III: Equate to zero the coefficient of the smallest power of x in the identity in Step II above. This would yield a quadratic equation in k. The quadratic equation so obtained is called the **indicial equation.** The two roots of the indicial equation are called **exponents** of Eqn.(42) and are the only possible values of k in Eqn.(43).

Step IV: Solve the indicial equation. The following cases are possible:

1) Indicial roots are distinct and do not differ by an integer.
2) Indicial roots are distinct and differ by an integer.
3) Indicial roots are equal.

Step V: Equate to zero the coefficient of general power of x (e.g. x^{k+m} or x^{k+m-1}, whichever may be the lowest) in the identity obtained in Step II. The equation so obtained is called the **recurrence relation** because it connects the coefficients c_m, c_{m-2} **or,** c_m, c_{m-1}, etc.

Step VI: If the recurrence relation connects c_m and c_{m-2}, then, in general, determine c_1 by equating to zero the coefficient of the term having next higher power in x i.e., the term in x next to the term already used for getting the indicial equation. On the other hand, if the recurrence relation connects c_m and c_{m-1}, this step may be omitted.

Step VII: Obtain all the coefficients with the help of Steps V and VI above. Substitute them in relation (43) and obtain the required solutions corresponding to two roots of the indicial equation.

We shall illustrate various steps involved in the method through examples. Depending upon the nature of the indicial roots listed in Step IV, different types of solution procedures are followed. We shall now illustrate these procedures one-by-one. You may **notice** here that the two series solutions obtained in each case are linearly independent.

Case 1: Indicial roots are distinct and do not differ by an integer.

Example 5: Obtain the series solution about the origin of the equation

$$9x(1-x)\,y'' - 12y' + 4y = 0 \qquad (45)$$

Solution: Evidently $x = 0$ is a regular singular point of Eqn.(45).

Let $$y(x) = \sum_{m=0}^{\infty} c_m x^{k+m},\ c_0 \neq 0 .$$

Differentiating $y(x)$ twice w.r.t. x, we get

$$y'(x) = \sum_{m=0}^{\infty} (k+m)\,c_m x^{k+m-1}$$

and $$y''(x) = \sum_{m=0}^{\infty} (k+m)\,(k+m-1)\,c_m x^{k+m-2}$$

Substituting the above values of y, y', y'' in Eqn.(45), we get

$$9x(1-x)\sum_{m=0}^{\infty}(k+m)(k+m-1)c_m x^{k+m-2} - 12\sum_{m=0}^{\infty}(k+m)c_m x^{k+m-1}$$

$$+4\sum_{m=0}^{\infty}c_m x^{k+m} = 0$$

or, $$9\sum_{m=0}^{\infty}(k+m)(k+m-1)c_m x^{k+m-1} - 9\sum_{m=0}^{\infty}(k+m)(k+m-1)c_m x^{k+m}$$

$$-12\sum_{m=0}^{\infty}(k+m)c_m x^{k+m-1} + 4\sum_{m=0}^{\infty}c_m x^{k+m} = 0 \qquad (46)$$

Combining the first and third term and the second and fourth term of Eqn.(46), we get

$$3\sum_{m=0}^{\infty}(k+m)(3k+3m-7)c_m x^{k+m-1} - \sum_{m=0}^{\infty}[9(k+m)^2 - 9(k+m) - 4]c_m x^{k+m} = 0$$

$$\text{or, } 3\sum_{m=0}^{\infty}(k+m)(3k+3m-7)c_m x^{k+m-1} - \sum_{m=0}^{\infty}(3k+3m-4)(3k+3m+1)c_m x^{k+m} = 0$$

$$\text{or, } 3\sum_{m=-1}^{\infty}(k+m+1)(3k+3m-4)c_{m+1} x^{k+m} - \sum_{m=0}^{\infty}(3k+3m-4)(3k+3m+1)c_m x^{k+m} = 0$$

$$\text{or, } 3k(3k-7)c_0 x^{k-1} + \sum_{m=0}^{\infty}x^{k+m}\big[3(k+m+1)(3k+3m-4)c_{m+1}$$

$$-(3k+3m-4)(3k+3m+1)c_m\big] = 0 \qquad (47)$$

Equating to zero the coefficient of smallest power of x (i.e., x^{k-1}) in identity (47), we get **indicial equation** as

$$3k(3k-7) = 0 \Rightarrow k = 0,\ 7/3 .$$

Clearly roots of indicial equation are distinct, not differing by an integer. Equating to zero the coefficient of general power of x, i.e., x^{k+m}, in identity (47), we get the recurrence relation as

$$3(k+m+1)(3k+3m-4)c_{m+1} = (3k+3m-4)(3k+3m+1)c_m, \text{for } m \ge 0$$

$$\text{i.e., } c_{m+1} = \frac{(3k+3m-4)(3k+3m+1)}{3(k+m+1)(3k+3m-4)}c_m = \frac{3k+3m+1}{3(k+m+1)}c_m \qquad (48)$$

For $k = 0$, $c_{m+1} = \dfrac{3m+1}{3(m+1)}c_m,\ m \ge 0$

Putting $m = 0, 1, 2, 3, \ldots$

$$c_1 = \frac{1}{3}c_0, c_2 = \frac{4}{6}.c_1 = \frac{1.4}{3.6}c_0, c_3 = \frac{7}{9}c_2 = \frac{1.4.7}{3.6.9}c_0, \cdots$$

Therefore, series solution corresponding to $k = 0$ is

$$y_1(x) = c_0\left(1 + \frac{1}{3}x + \frac{1.4}{3.6}x^2 + \frac{1.4.7}{3.6.9}x^3 + \cdots\right), \qquad (49)$$

For $k = 7/3$, we get from Eqn.(48),

$$c_{m+1} = \frac{3m+8}{3m+10}C_m, \text{ for } m \ge 0$$

Putting $m = 0, 1, 2, \ldots,$, we get

$$c_1 = \frac{8}{10}c_0, c_2 = \frac{11}{13}c_1 = \frac{11.8}{13.10}c_0, c_3 = \frac{14}{16}c_2 = \frac{14.11.8}{16.13.10}c_0, \cdots$$

Therefore, series solution corresponding to $k = 7/3$ is

$$y_2(x) = c_0 x^{7/3}\left(1 + \frac{8}{10}x + \frac{8.11}{10.13}x^2 + \frac{8.11.14}{10.13.16}x^3 + \cdots\right) \tag{50}$$

Since the solutions $y_1(x)$ and $y_2(x)$ given by Eqns.(49) and (50) are linearly independent, the general solution of Eqn.(45) may be written as

$$y(x) = \alpha_1 y_1(x) + \alpha_2 y_2(x),$$

where α_1 and α_2 are arbitrary constants.

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Note: In general, if indicial equation has two distinct roots, differing not by an integer, two linearly independent solutions are obtained by substituting the values of k in the series for $y(x)$.

Let us look at another example.

Example 6: Find a series solution about $x = 0$, of the equation $2x^2y'' - xy' + (1 - x^2)y = 0$.

Solution: Observe that the given equation

$$2x^2y'' - x\,y' + (1 - x^2)\,y = 0 \tag{51}$$

has $x = 0$ as a regular singular point.

Let $$y(x) = x^k(c_0 + c_1x + c_2x^2 + \cdots) = \sum_{m=0}^{\infty} c_m x^{m+k} \tag{52}$$

be a solution of Eqn.(51).

Differentiating Eqn.(52) w.r.t. x, we get

$$y'(x) = \sum_{m=0}^{\infty}(k+m)c_m x^{m+k-1} \tag{53}$$

and $$y''(x) = \sum_{m=0}^{\infty}(k+m)(k+m-1)\,c_m x^{k+m-2} \tag{54}$$

Putting the values of y, y', y'' from Eqns.(52), (53) and (54) in Eqn.(51), we get

$$2x^2\sum_{m=0}^{\infty}(k+m)(k+m-1)c_m x^{k+m-2} - x\sum_{m=0}^{\infty}(k+m)c_m x^{k+m-1} + (1-x^2)\sum_{m=0}^{\infty}c_m x^{k+m} = 0$$

$$\text{or, } \sum_{m=0}^{\infty}2(k+m)(k+m-1)c_m x^{k+m} - \sum_{m=0}^{\infty}(k+m)c_m x^{k+m} + \sum_{m=0}^{\infty}c_m x^{k+m} - \sum_{m=0}^{\infty}c_m x^{k+m+2} = 0 \tag{55}$$

Equating the coefficient of lowest power of x, i.e., of x^k to zero in identity (55), we get the **indicial equation** as

$$2c_0k(k-1) - k\,c_0 + c_0 = 0, \text{ i.e., } c_0[2k(k-1) - k + 1] = 0$$

i.e., $c_0(k-1)\;(2k-1) = 0 \Rightarrow k = 1,\ 1/2$, i.e., roots of indicial equation are distinct, but difference is not an integer.

To get a recurrence relation, we equate the coefficients of general power of x, i.e., x^{k+m} in identity (55) equal to zero and obtain

$$[2(k+m)(k+m-1) - (k+m) + 1]\,c_m = c_{m-2}$$

i.e., $$c_m = \frac{c_{m-2}}{(k+m-1)(2k+2m-1)} \text{ for all } m \ge 2. \tag{56}$$

Again, equating the coefficient of x^{k+1} in Eqn.(55) to zero, we get

$$2(k+1)kc_1 - (k+1)c_1 + c_1 = 0 \Rightarrow c_1[2k(k+1) - (k+1) + 1] = 0$$
$$\Rightarrow c_1\, k(2k+1) = 0.$$

Now for $k = 1, 1/2,\ k(2k+1) \neq 0$, hence $c_1 = 0$ (57)

From Eqns.(56) and (57), we get

$$c_1 = 0 = c_3 = c_5 = c_7 = \ldots, = c_{2m+1} = \ldots$$

Putting $m = 2, 4, 6, \ldots$ in relation (56), we get

$$c_2 = \frac{1}{(2k+3)(k+1)} c_0.$$

$$c_4 = \frac{1}{(k+3)(2k+7)} c_2 = \frac{1}{(2k+3)(2k+7)(k+1)(k+3)} c_0.$$

$$c_6 = \frac{1}{(k+5)(2k+11)} = \frac{1}{(2k+3)(2k+7)(2k+11)(k+1)(k+3)(k+5)} c_0.$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Putting these values of $c_0, c_1, \ldots$ in Eqn.(52), we get

$$y(x) = x^k \left[c_0 + \frac{c_0}{(2k+3)(k+1)} x^2 + \frac{c_0}{(2k+3)(2k+7)(k+1)(k+3)} x^4 + \right.$$
$$\left. + \frac{c_0}{[(2k+3)(2k+7)(2k+11)(k+1)(k+3)(k+5)]} x^6 + \cdots \right].$$

Putting $k = 1$, we get

$$y_1 = x\, c_0 \left[1 + \frac{x^2}{2.5} + \frac{x^4}{2.4.5.9} + \frac{x^6}{2.4.6.5.9.13} + \cdots \right] = c_0 u \qquad (58)$$

Putting $k = 1/2$ in $y(x)$, we get

$$y_2 = c_0 x^{1/2} \left[1 + \frac{x^2}{2.3} + \frac{x^4}{2.3.4.7} + \frac{x^6}{3.7.11.2.4.6} + \cdots \right] = c_0 v, \qquad (59)$$

Hence the general solution of Eqn.(51) is

$$\begin{aligned} y(x) &= A_1 y_1(x) + B_1 y_2(x) \\ &= Au + Bv \end{aligned} \qquad (60)$$

where $A = A_1 c_0$ and $B = B_1 c_0$ are arbitrary constants and

$$u = x \left[1 + \frac{1}{2.5} x^2 + \frac{1}{2.4.5.9} x^4 + \frac{1}{2.4.6.5.9.13} x^6 + \cdots \right] \qquad (61)$$

$$v = x^{1/2} \left[1 + \frac{1}{2.3} x^2 + \frac{1}{2.4.3.7} x^4 + \frac{1}{2.4.6.3.7.11} x^6 + \cdots \right] \qquad (62)$$

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You may now try the following exercises.

E10) Find a series solution about $x = 0$, of the following equations

a) $2x(1-x)y'' + (1-x)y' + 3y = 0$.

b) $(2x + x^2)y'' - y' - 6xy = 0$.

Case 2: Indicial roots are distinct and differ by an integer

Let the roots of the indicial equation be k_1, k_2 such that $k_1 - k_2 = n$, an integer. In this case two types of problems arise. In **one type** two linearly independent solutions are obtained by substituting $k = k_1$ and $k = k_2$ in $y(x)$ as in Case 1. This case is illustrated through the following example.

Example 7: Find a series solution about $x = 0$, of the equation

$$x^2 y'' + x^3 y' + (x^2 - 2)\, y = 0 \tag{63}$$

Solution: Evidently $x = 0$, is a regular singular point of Eqn.(63).

Let $$y = \sum_{m=0}^{\infty} c_m x^{k+m}, \quad c_0 \neq 0. \tag{64}$$

$$\therefore y' = \sum_{m=0}^{\infty} (k+m) c_m x^{k+m-1} \quad \text{and} \quad y'' = \sum_{m=0}^{\infty} c_m (k+m)(k+m-1)\, x^{k+m-2}$$

Putting the values of y, y', y'' in Eqn.(63), we get

$$x^2 \sum_{m=0}^{\infty} c_m (k+m)(k+m-1) x^{k+m-2} + x^3 \sum_{m=0}^{\infty} (k+m) c_m x^{k+m-1} + (x^2 - 2) \sum_{m=0}^{\infty} c_m x^{k+m} = 0$$

i.e., $$\sum_{m=0}^{\infty} c_m \left[(k+m)(k+m-1) - 2\right] x^{k+m} + \sum_{m=0}^{\infty} c_m (k+m+1)\; x^{k+m+2} = 0. \tag{65}$$

Equating to zero the coefficients of smallest power of x, namely x^k, Eqn.(65) gives

$$c_0[k(k-1) - 2] = 0 \Rightarrow k^2 - k - 2 = (k-2)\;(k+1) = 0, \;\text{ or }\; k = 2, -1,\; c_0 \neq 0.$$

These are distinct roots of **indicial equation**, which differ by an integer. Equating to zero the coefficient of next power of x, namely x^{k+1}, we get

$$[(k+1)k - 2]\; c_1 = 0, \;\text{ or }\; c_1 = 0 \;\text{ for }\; k = 2, -1.$$

The remaing terms are given by

$$\sum_{m=2}^{\infty} [\,(k+m)\;(k+m-1) - 2]\; c_m\; x^{k+m} + \sum_{m=0}^{\infty} (k+m+1) c_m x^{k+m+2} = 0$$

From here we get the recurrence relation

$$[(m+k+1)\;(m+k+2) - 2] c_{m+2} + (m+k+1) c_m = 0, \; m \geq 0$$

or, $$c_{m+2} = -\frac{(m+k+1)\; c_m}{(m+k+1)\;(m+k+2)}, \;\; m \geq 0 \tag{66}$$

Putting $m = 1, 2, 3, \ldots,$ we get

$$c_2 = \frac{-(k+1) c_0}{(k+1)\;(k+2) - 2}, \; c_3 = \frac{-(k+2) c_1}{(k+2)\;(k+2) - 2} = 0, \; c_4 = \frac{-(k+3) c_2}{(k+3)\;(k+4) - 2}, \cdots.$$

Therefore from Eqn.(64), we get

$$y(x) = c_0 x^k \left[1 - \frac{(k+1)}{(k+1)\;(k+2) - 2} x^2 + \frac{(k+3)\;(k+1)}{[(k+1)\;(k+2) - 2]\;[(k+3)(k+4) - 2\,]} x^4 - \cdots\right] \tag{67}$$

For $k = 2$, we get

$$y(x) = c_0 x^2 \left[1 - \frac{3}{10} x^2 + \frac{3}{56} x^4 - \cdots\right] = y_1$$

For $k = -1$, coeffieicents $c_2 = 0,\; c_3 = 0,\; c_4 = 0, \ldots$ and

$$y(x) = \frac{c_0}{x} = y_2.$$

Hence the derived solution is $y = a\, y_1 + b\, y_2$, where a and b are arbitrary constants.

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In the **second type** of problems again two cases arise. We now discuss them as I and II and give the rules for solving such problems.

I: Indicial roots are distinct, differ by an integer and make a coefficient of y indeterminate:

In this situation the following rule is used to obtain the two linearly independent solutions of the given problem.

Rule: Let the indicial equation has two roots k_1 and k_2 such that $k_2 - k_1 = n$, an integer. Now if one of the coefficients of y becomes indeterminate when $k = k_1$ (say), the complete solution is given by putting $k = k_1$ in y, which then contains two arbitrary constants. The result of putting $k = k_2$ in y merely gives a numerical multiple of one of the series contained in the first solution hence producing a linearly dependent solution, which is rejected.

We illustrate the method for this case through the following example:

Example 8: Obtain a series solution about $x = 0$, of the equation

$$x^2 y'' + 6xy' + (6 + x^2)\, y = 0 \tag{68}$$

Solution: We find $x = 0$ is a regular singular point of Eqn.(68).

$$\text{Let } y(x) = \sum_{m=0}^{\infty} c_m x^{m+k}, \quad c_0 \neq 0. \tag{69}$$

Substituting for y, y', y'' in Eqn.(68), we get

$$x^2 \sum_{m=0}^{\infty} (m+k)\,(m+k-1) c_m x^{m+k-2} + 6x \sum_{m=0}^{\infty} (m+k) c_m x^{m+k-1} + (6 + x^2) \sum_{m=0}^{\infty} c_m x^{m+k} = 0$$

or, $$\sum_{m=0}^{\infty} (m+k)\,(m+k-1) c_m x^{m+k} + 6\sum_{m=0}^{\infty} (m+k) c_m x^{m+k} + 6\sum_{m=0}^{\infty} c_m x^{m+k} + \sum_{m=0}^{\infty} c_m x^{m+k+2} = 0.$$

or, $$\sum_{m=0}^{\infty} \left[(m+k)\,(m+k+5) + 6\right] c_m x^{m+k} + \sum_{m=0}^{\infty} c_m x^{m+k+2} = 0. \tag{70}$$

Setting the coefficient of lowest degree term, namely x^k to zero, we obtain

$$[k(k+5) + 6]\, c_0 = 0, \text{ or } (k+2)\,(k+3) = 0,\ c_0 \neq 0 \tag{71}$$

The **indicial** roots are $k = -2, -3$. Setting the coefficient of x^{k+1} to zero, we get

$$[\,(k+1)\,(k+6) + 6\,]\, c_1 = 0.$$

For $k = -2$, $c_1 = 0$ and for $k = -3$, c_1 is **indeterminate**. Therefore, $k = -3$ gives the complete solution, which contains two arbitrary constants. Equating the coefficients of general term to zero, we get

$$c_{m+2} = \frac{-c_m}{(m+k+2)\,(m+k+7) + 6}, \quad m \geq 0 \tag{72}$$

Putting $m = 1, 2, 3, \ldots$ in Eqn.(72), we get

$$c_2 = \frac{-c_0}{(k+2)\,(k+7) + 6} \qquad c_3 = \frac{-c_1}{(k+3)\,(k+8) + 6}$$

$$c_4 = \frac{-c_2}{(k+4)\ (k+9)+6} = \frac{c_0}{[(k+2)\ (k+7)+6]\ [(k+4)\ (k+9)+6]}$$

and so on.

Therefore from Eqn.(69), we get

$$y(x) = c_0 x^k \left[1 - \frac{1}{(k+2)\ (k+7)+6} x^2 + \frac{1}{[(k+2)\ (k+7)+6]\ [(k+4)\ (k+9)+6]} x^4 - \cdots\right]$$

$$+ c_1 x^k \left[x - \frac{1}{(k+3)\ (k+8)+6} x^3 + \frac{1}{[(k+3)\ (k+8)+6]\ [(k+5)\ (k+10)+6} x^5 - \cdots\right]$$

For $k = -3$, we get

$$y(x) = c_0 x^{-3}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + c_1 x^{-3}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)$$

$$= c_0\left(\frac{\cos x}{x^3}\right) + c_1\left(\frac{\sin x}{x^3}\right) = c_0 y_1(x) + c_1 y_2(x) \tag{73}$$

For $k = -2$, we know $c_1 = 0 = c_3 = c_5$ and so on.

$$\therefore \quad y(x) = c_0 x^{-2}\left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right] = c_0 x^{-3}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right] = c_0 y_2(x) \tag{74}$$

which is a linearly dependent solution.

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You may now attempt a few exercises.

E11) Obtain the series solution about $x = 0$, of the equation
$(2 + x^2)y'' + xy' + (1 + x)\,y = 0$.

E12) Find a series solution about $x = 0$, of the differential equation
$(1 - x^2)\,y'' + 2xy' + y = 0$.

We, now take up II for the second type of problem.

II: Indicial roots are distinct, differ by an integer and make a coefficient of y infinite:

In this case, the following rule is used:

Rule: Let the indicial equation has two roots k_1 and k_2 such that $k_1 - k_2 = n$, an integer. Now if one of the coefficients of y becomes infinite for $k = k_2$ (say), then the form of y in Eqn.(43) is modified by replacing c_0 by $d_0(k - k_2)$ where $d_0 \neq 0$. Two linearly independent solutions are then obtain by substituting $k = k_2$ in y and $\frac{\partial y}{\partial k}$. The solution corresponding to $\left.\frac{\partial y}{\partial k}\right)_{k=k_2}$ will always contain a logarithmic term. The result of putting $k = k_1$ in y merely gives a solution that is a numerical multiple of the one obtained for $k = k_2$, leading to a linearly dependent solution of the given equation.

Let us take up an example to understand this case.

Example 9: Find the series solution about $x = 0$, of the differential equation

$$(x + x^2 + x^3)\, y'' + 3x^2 y' - 2y = 0 \qquad (75)$$

Solution: Evidently, $x = 0$, is a regular singular point of Eqn.(75).

$$\text{Let} \quad y = \sum_{m=0}^{\infty} c_m x^{k+m}, \; c_0 \neq 0. \qquad (76)$$

$$\text{Then } y' = \sum_{m=0}^{\infty} (k+m) c_m x^{k+m-1} \text{ and } y'' = \sum_{m=0}^{\infty} (k+m)(k+m-1) c_m x^{k+m-2} \qquad (77)$$

Substituting for y, y', y'' in Eqn.(75), we get

$$(x + x^2 + x^3) \sum_{m=0}^{\infty} c_m (k+m)(k+m-1)\, x^{k+m-2} + 3x^2 \sum_{m=0}^{\infty} c_m (k+m) x^{k+m-1}$$
$$- 2\sum_{m=0}^{\infty} c_m x^{k+m} = 0$$

$$\text{i.e., } \sum_{m=0}^{\infty} c_m (k+m)(k+m-1)\, x^{k+m+1} + \sum_{m=0}^{\infty} c_m 3(k+m) x^{k+m+1}$$
$$+ \sum_{m=0}^{\infty} c_m [(k+m)(k+m-1) - 2]\, x^{k+m} + \sum_{m=0}^{\infty} c_m (k+m)(k+m-1)\, x^{k+m-1} = 0.$$

$$\text{i.e., } \sum_{m=0}^{\infty} c_m (k+m)(k+m+2) x^{k+m+1} + \sum_{m=0}^{\infty} c_m \{(k+m)(k+m-1) - 2\} x^{k+m}$$
$$+ \sum_{m=-1}^{\infty} c_{m+1} (k+m+1)(k+m) x^{k+m} = 0$$

$$\text{i.e., } \sum_{m=1}^{\infty} c_{m-1} (k+m-1)(k+m+1) x^{k+m} + \sum_{m=0}^{\infty} \{(k+m)(k+m-1) - 2\} c_m x^{k+m}$$
$$+ \sum_{m=-1}^{\infty} (k+m+1)(k+m)\, c_{m+1} x^{m+k} = 0$$

$$\text{i.e., } k(k-1)\, c_0 x^{k-1} + [\,(k+1)k\, c_1 + \{\, k(k-1) - 2\} c_0\,]\; x^k$$
$$+ \sum_{m=1}^{\infty} [\,(k+m+1)(k+m)\, c_{m+1} + (m+k-1)(m+k+1)\, c_{m-1}$$
$$+ \{(k+m)(k+m-1) - 2\} c_m\,] x^{m+k} = 0 \qquad (78)$$

This is an identity. Here the **indicial equation** obtained by equating to zero the coefficients of x^{k-1} is

$$c_0 k(k-1) = 0 \Rightarrow k = 0, 1 \quad (\because c_0 \neq 0). \qquad (79)$$

Equating to zero the coefficients of x^k and x^{m+k} in Eqn.(78), we get

$$(k+1)k\, c_1 + \{k(k-1) - 2\}\, c_0 = 0 \qquad (80)$$

and,
$$(k+m+1)(k+m)\, c_{m+1} + (m+k+1)(m+k-1)\, c_{m-1}$$
$$+ \{\,(k+m)(k+m-1) - 2\}\, c_m = 0, \text{ for } m \geq 1 \qquad (81)$$

From Eqn.(80), we get $c_1 = -\dfrac{k(k-1) - 2}{k(k+1)} c_0$

Putting $m = 1, 2, 3, 4, \ldots$ in Eqn.(81), we get the following:

For $m = 1$

$$(k+2)(k+1)\, c_2 + k(k+2)\, c_0 + \{k(k+1) - 2\}\, c_1 = 0$$

i.e., $$c_2 = -\frac{k}{k+1}\, c_0 + \frac{[k\,(k-1)-2][k\,(k+1)-2]}{k\,(k+1)^2\,(k+2)} c_0 = -\frac{3k-2}{k(k+1)} c_0$$

For $m = 2$

$$(k+3)(k+2)c_3 + (k+3)(k+1)c_1 + \{(k+2)(k+1)-2\}\; c_2 = 0$$

i.e., $$c_3 = -\frac{k+1}{k+2} c_1 - \frac{k^2+3k}{(k+3)(k+2)} c_2 = -\frac{k+1}{k+2} c_1 + \frac{(3k-2)}{(k+1)(k+2)} c_0$$

$$= \left[\frac{k(k-1)-2}{k(k+2)} + \frac{3k-2}{k+1}\right] c_0 = \frac{k^3+3k^2-5k-2}{k(k+1)(k+2)} c_0$$

and so on.

Substituting the values of $c_1, c_2, c_3, \ldots$ in Eqn.(76), we get

$$y(x) = c_0 x^k \left[1 - \frac{k^2-k-2}{k(k+1)} x - \frac{3k-2}{k(k+1)} x^2 + \frac{k^3+3k^2-5k-2}{k(k+1)(k+2)} x^3 - \cdots -\right] \tag{82}$$

It is seen that **for $k = 0$**, the coefficients in solution (82) become infinite. So we, replace c_0 by $k\,r_0$ ($r_0 \neq 0$). Thus Eqn(82) reduces to

$$y(x) = r_0 x^k \left[k - \frac{k^2-k-2}{k+1} x - \frac{3k-2}{k+1} x^2 + \frac{k^3+3k^2-5k-2}{(k+1)(k+2)} x^3 - \cdots -\right] \tag{83}$$

Substituting from Eqn.(83) in L.H.S. of Eqn.(75), we get

$$(x + x^2 + x^3)y'' + 3x^2 y' - 2y = r_0 k^2 (k-1) x^{k-1}$$

(where the right hand side is simply the indicial equation). Here the presence of k^2 shows that y and $\frac{\partial y}{\partial k}$ satisfy Eqn.(75) for $k = 0$.

From Eqn.(83), we get

$$\frac{\partial y}{\partial k} = r_0 x^k \ln x \left[k - \frac{k^2-k-2}{k+1} x - \frac{3k-2}{k+1} x^2 + \frac{k^3+3k^2-5k-2}{(k+1)(k+2)} x^3 - \cdots\right]$$
$$+ r_0 x^k \left[1 - \left\{\frac{2k-1}{k+1} - \frac{k^2-k-2}{(k+1)^2}\right\} x - \left\{\frac{3}{k+1} - \frac{3k-2}{(k+1)^2}\right\} x^2\right.$$
$$\left. + \left\{\frac{3k^2+6k-5}{(k+1)(k+2)} - \frac{k^3+3k^2-5k-2}{(k+1)^2(k+2)} - \frac{k^3+3k^2-5k-2}{(k+1)(k+2)^2}\right\} x^3 - \cdots\right] \tag{84}$$

Putting $k = 0$ in Eqns.(83) and (84), we get

$$y = r_0[2x + 2x^2 - x^3 - \cdots] = y_1, \text{say} \tag{85}$$

and $$\frac{\partial y}{\partial k} = (\ln x) y_1 + r_0[1 - x - 5x^2 - x^3 - \cdots] = y_2, \text{say} \tag{86}$$

Putting $k = 1$ in Eqn.(83), we get

$$y = r_0 x\left[1 + x - \frac{1}{2}x^2 - \frac{1}{2}x^3 - \cdots\right] = y_3, \text{say} \tag{87}$$

Evidently, $2y_3 = y_1$.

Hence, out of three solutions y_1, y_2 and y_3 obtained above, only two are linearly independent. Hence general solution is

$$y = \alpha y_1 + \beta y_2,$$

where α and β are arbitrary constants.

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You may now try the following exercise.

E13) Find a series solution of the following differential equations about $x = 0$.

a) $xy'' + (3 - x)y' + y = 0$.

b) $x(1 - x)y'' - 3xy' - y = 0$.

We shall now take up the case when the roots of indicial equation are equal.

Case 3: Indicial roots are equal

In this case, the following rule works:

Rule: If indicial equation has two equal roots $k_1 = k_2$, obtain two linearly independent solution by substituting this value of k in y and $\frac{\partial y}{\partial k}$.

We illustrate this rule through an example.

Example 10: Solve, in series, the differential equation

$$xy'' + y' + xy = 0 \tag{88}$$

Solution: Here $x = 0$ is a regular singular point.
Let the series solution of Eqn.(88) be of the form

$$y = \sum_{m=0}^{\infty} c_m x^{k+m}, \quad c_0 \neq 0 \tag{89}$$

Thus, $y' = \sum_{m=0}^{\infty} c_m (k+m) x^{k+m-1}$ and $y'' = \sum_{m=0}^{\infty} c_m (k+m)(k+m-1) x^{k+m-2}$ (90)

Putting these values of y, y', y'' in Eqn.(88), we get

$$\sum_{m=0}^{\infty} c_m (k+m)(k+m-1) x^{k+m-1} + \sum_{m=0}^{\infty} c_m (k+m) x^{k+m-1} + \sum_{m=0}^{\infty} c_m x^{k+m+1} = 0$$

or, $$\sum_{m=0}^{\infty} c_m (k+m)^2 x^{k+m-1} + \sum_{m=0}^{\infty} c_m x^{k+m+1} = 0, \tag{91}$$

which is an identity. Equating to zero the coefficient of least power x, in identity (91) **indicial equation** is obtained as

$$c_0 k^2 = 0 \Rightarrow k = 0, 0 \quad (\because c_0 \neq 0) \tag{92}$$

Thus indicial equation has equal roots.
Again from Eqn.(91), the recurrence relation is obtained as

$$c_m (k+m)^2 + c_{m-2} = 0$$

i.e., $$c_m = -\frac{1}{(k+m)^2} c_{m-2} \tag{93}$$

Again equating the coefficient of x^k in Eqn.(91) to zero, we get

$$c_1 (k+1)^2 = 0, \text{but since } k = 0, k+1 \neq 0 \text{ and hence } c_1 = 0. \tag{94}$$

For $c_1 = 0$, we get from Eqn.(93), $c_1 = c_3 = 0 = c_5 = c_7 = \cdots$.
For $m = 2, 4, 6, \ldots$ Eqn.(93) yields

$$c_2 = -\frac{1}{(k+2)^2} c_0,$$

$$c_4 = -\frac{1}{(k+4)^2} c_2 = \frac{1}{(k+2)^2 (k+4)^2} c_0$$

$$c_6 = -\frac{1}{(k+2)^2 (k+4)^2 (k+6)^2} c_0$$

and so on.

Putting these values of c's in Eqn.(89), we get

$$y = c_0 x^k \left[1 - \frac{x^2}{(k+2)^2} + \frac{x^4}{(k+2)^2 (k+4)^2} - \frac{x^6}{(k+2)^2 (k+4)^2 (k+6)^2} + \cdots\right] \quad (95)$$

Putting $k = 0$ in Eqn.(95), we get

$$y = c_0 \left(1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \frac{x^6}{2^2.4^2.6^2} + \cdots\right) = c_0 u, \text{say} \quad (96)$$

To get another independent solution, substituting Eqn.(95) in L.H.S. of Eqn.(88) and simplifying, we get

$$x^2 y'' + y' + xy = c_0\, k^2 x^{k-1} \quad (97)$$

Differentiating both sides of Eqn.(97) partially w.r.t. to k gives

$$\frac{\partial}{\partial k}\left[x^2 y'' + y' + xy\right] = \frac{c_0}{x} \frac{\partial}{\partial k}(k^2 x^k) = \frac{c_0}{x}\left[2kx^k + k^2 x^k \ln x\right] \quad (98)$$

The right hand side of Eqn.(98) vanishes for $k = 0$, this suggests that a second solution of the given equation is $\left.\frac{\partial y}{\partial k}\right)_{k=0}$, where y is given by Eqn.(95).

Differentiating Eqn.(95) partially w.r.t. to k, we get

$$\frac{\partial y}{\partial k} = c_0 x^k \ln x \left[1 - \frac{x^2}{(k+2)^2} + \frac{x^4}{(k+2)^2 (k+4)^2} - \frac{x^6}{(k+2)^2 (k+4)^2 (k+6)^2} \cdots\right]$$
$$+ c_0 x^k \left[-\frac{(-2)x^2}{(k+2)3} + \frac{x^4}{(k+2)^2 (k+3)^2}\left\{\frac{-2}{k+2} - \frac{2}{k+4}\right\} - \frac{x^6}{(k+2)^2 (k+4)^2 (k+6)^2}\left\{-\frac{2}{k+2} - \frac{2}{k+4} - \frac{2}{k+6}\right\} + \cdots\right] \quad (99)$$

Putting $k = 0$ in Eqn.(99), we get

$$\left.\frac{\partial y}{\partial k}\right)_{k=0} = c_0 \ln x \left(1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \frac{x^6}{2^2.4^2.6^2} + \cdots\right)$$
$$+ c_0 \left[\frac{x^2}{2^2} - \frac{x^4}{2^2.4^2}\left(1 + \frac{1}{2}\right) + \frac{x^6}{2^2.4^2.6^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right) + \cdots\right]$$
$$= c_0 \left[u \ln x + \left\{\frac{x^2}{2^2} - \frac{x^4}{2^2.4^2}\left(1 + \frac{1}{2}\right) + \frac{x^6}{2^2.4^2.6^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right) + \cdots\right.\right]$$
$$= c_0\, v, \text{ say} \quad (100)$$

Thus the required general series solution of the given equation is,

$$y = A_1 c_0 u + B_1 c_0 v$$
$$= A\, u + B\, v$$

Where A_1, B_1, A and B are arbitrary constants, such that $A_1 c_0 = A$ and $B_1 c_0 = B$.

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How, about trying a few exercises now?

E14) Find a series solution about $x = 0$, of the differential equation
$x(1-x)y'' + (1-x)\,y' - y = 0$.

E15) Solve in series, $xy'' + (p - x)y' - y = 0$ when,

(a) $p = 1$ (b) p is not an integer.

E16) Find a series solution about $x = 0$, of the following equations:

a) $xy'' + (1 + x)y' + 2y = 0$.

b) $(x - x^2)y'' + (1 - 5x)y' - 4y = 0$.

c) $4(x^4 - x^2)y'' + 8x^3y' - y = 0$.

In the next subsection we discuss the case when for an equation of the type (15), we are required to find a series solution for large values of the independent variable.

2.4.3 Series Solution about a Point at Infinity

Often we come across physical situations where we need to find the solution of equations of the type

$$y'' + P(x)y' + Q(x)y = 0 \tag{101}$$

for large values of the independent variable x. For instance, if the variable is time, we may want to know how the physical system described by Eqn.(101) behaves in the distant future, when transient disturbances have faded out. In such cases, we can use our previous ideas and broaden them up by studying solutions near a **point at infinity**. If we change the independent variable in Eqn.(101) from x to t by the relation

$$t = \frac{1}{x},$$

then large x's correspond to small t's. The transformed equation can then be solved near $t = 0$ and then replacing t by $\frac{1}{x}$ in the solution obtained, we get solution of Eqn.(101) that are valid for large values of x. To carry out this, we need the formulas

$$y' = \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\left(\frac{-1}{x^2}\right) = -t^2\frac{dy}{dt} \tag{102}$$

and $$y'' = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = \left(-t^2\frac{d^2y}{dt^2} - 2t\frac{dy}{dt}\right)(-t^2) \tag{103}$$

When these expressions for y' and y'' are substituted in Eqn.(101) then it becomes

$$y'' + \left[\frac{2}{t} - \frac{P(1/t)}{t^2}\right]y' + \frac{Q(1/t)}{t^4}y = 0 \tag{104}$$

where primes in Eqn.(104) denote derivatives w.r. to t.

We then say that Eqn.(101) has $x = \infty$ as an ordinary point, a regular singular point with exponents k_1 and k_2, or an irregular singular point, if the point $t = 0$ has the corresponding character for the transformed Eqn.(104).

We now illustrate the above procedure through an example.

Example 11: Obtain a series solution of the following equation for large values of x.

$$x^2y'' + (3x - 1)y' + y = 0. \tag{105}$$

Solution: Here $x = 0$, is an irregular singular point and Frobenius method can not be applied.

We change the independent variable x to t by putting $x = \frac{1}{t}$

Then $$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} = -t^2\frac{dy}{dt}$$

and $$\frac{d^2y}{dx^2} = -t^2\frac{d}{dt}\left(-t^2\frac{dy}{dt}\right) = t^4\frac{d^2y}{dt^2} + 2t^3\frac{dy}{dt}$$

Substituting the above values in the given Eqn.(105), we get

$$\frac{1}{t^2}\left(t^4\frac{d^2y}{dt^2}+2t^3\frac{dy}{dt}\right)+\left(\frac{3}{t}-1\right)\left(-t^2\frac{dy}{dt}\right)+y=0$$

i.e., $$t^2\frac{d^2y}{dt^2}-t(1-t)\frac{dy}{dt}+y=0 \tag{106}$$

Now $t=0$ is a regular singular point of the above equation and we look for its series solution.

Let $$y(t)=\sum_{m=0}^{\infty}c_m t^{m+k},\ \ (c_o\neq 0) \tag{107}$$

Then $$y'(t)=\sum_{m=0}^{\infty}(m+k)\,c_m t^{m+k-1} \tag{108}$$

and $$y''(t)=\sum_{m=0}^{\infty}(m+k)\,c_m t^{m+k-2} \tag{109}$$

On substituting the above expression for y, y' and y'' in Eqn.(106), we get

$$\sum_{m=0}^{\infty}(m+k-1)^2 c_m t^{m+k}+\sum_{m=1}^{\infty}(m+k-1)\,c_{m-1}t^{m+k}=0 \tag{110}$$

The **indicial equation** is $c_0(k-1)^2=0\Rightarrow k=1,\,1$.

Equating coefficients of t^{m+k} in Eqn.(110) to zero, we get the recurrence relation as

$$c_m=\frac{(-1)\,c_{m-1}}{m+k-1},\ \text{ for } m\geq 1. \tag{111}$$

$$\Rightarrow c_m=\frac{(-1)^m c_0}{k(k+1)\cdots(k+m-1)},\ \text{ for } m\geq 1.$$

Substituting the values of $c_0, c_1, \ldots$ etc. in Eqn.(107), we obtain

$$y(t)=c_0t^k+\sum_{m=1}^{\infty}\frac{(-1)^m c_0 t^{m+k}}{k(k+1)\cdots((k+m-1)}. \tag{112}$$

Putting $k=1$ in the above expression for $y(t)$ we get one solution of Eqn.(106).

Since roots of indicial equation are equal, to find second linearly independent solution, we need to **put $k=1$** in

$$\frac{\partial y}{\partial k}=y\ln t-\sum_{m=1}^{\infty}\frac{(-1)^m\left\{\frac{1}{k}+\frac{1}{k+1}+\cdots+\frac{1}{k+m-1}\right\}}{k(k+1)\cdots(k+m-1)}c_0t^{m+k} \tag{113}$$

Thus, we get two linearly independent series solution of Eqn.(106) as

$$y(t)\big|_{k=1}=c_0t+\sum_{m=1}^{\infty}c_0\frac{(-1)^m t^{m+1}}{(m)!}=u,\ \text{ say}$$

and $$\left.\frac{\partial y}{\partial k}\right)_{k=1}=u\ \ln t-\sum_{m=1}^{\infty}c_0\frac{(-1)^m H_m\, t^{m+1}}{(m)!}=v,\ \text{ say}$$

where, $H_m=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m}=\sum_{p=1}^{m}\left(\frac{1}{p}\right)$.

Therefore, two linearly independent solutions of Eqn.(105) are

$$u=\sum_{m=0}^{\infty}c_0\frac{(-1)^m}{(m)!}\frac{1}{x^{m+1}}=\frac{c_0}{x}\sum_{m=0}^{\infty}\frac{(-1)^m}{(m)!}x^{-m}=\frac{c_0}{x}e^{\frac{1}{x}}$$

and $$v=u\ln\left(\frac{1}{x}\right)-\sum_{m=1}^{\infty}c_0\frac{(-1)^m H_m}{(m)!}.\frac{1}{x^{m+1}},\ \text{ where } H_m=\sum_{p=1}^{m}\left(\frac{1}{p}\right).$$

The general solution of Eqn.(105) is then given by $y(x) = au + bv$, where a and b are arbitrary constants.

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You may now try this exercise.

E17) Find a series solution of the following equations for large values of x.

a) $4x^3y'' + 6x^2y' + y = 0$.

b) $2x^2(1-x)y'' - 5x(1+x)y' + (5-x)y = 0$.

In this unit we discussed methods of finding series solutions of second order linear, homogeneous differential equations with variable coefficients. In the next two units, we shall apply these methods to solve some differential equations, which occur frequently in physical and engineering problems. In particular, we shall solve Legendre's, Bessel's, Laguerre's and Hermite's differential equations.

We now end this unit by giving a summary of what we have covered in it.

2.5 SUMMARY

In this unit, we have learnt the following points:

1. Second order linear homogeneous differential equation with variable coefficients cannot be, in general, solved by methods giving closed form solutions and there is a need for series solutions for such equations.
2. For a power series, $\sum_{m=0}^{\infty} c_m x^m$, if $R = \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right| \cdot \left|\frac{1}{x}\right|$ exists, then **the power series converges if** $|x| < R$ **and diverges if** $|x| > R$ and the **region of convergence is** $-R < x < R$. Here **R is** known as **radius of convergence** of the power series.
3. A power series can be differentiated term by term any number of times without affecting its radius of convergence.
4. For the power series $f(x) = \sum_{m=0}^{\infty} c_m (x-a)^m$, **the r-times differentiated series has the sum** $\frac{d^r[f(x)]}{dx^r}$, for each positive integer r and the coefficients c_m are completely determined by $c_m = \frac{f^{(m)}(a)}{(m)!}$.
5. A function $f(x)$ is said to be **analytic** at $x = a$ if, $f(x)$ can be expanded in a power series $\sum_{m=0}^{\infty} c_m (x-a)^m$ with a positive radius of convergence.
6. A point $x = a$ is called an **ordinary point** of the differential equations $y'' + d_1(x)y' + d_2(x) = b_1(x)$ if $d_1(x)$ and $d_2(x)$ are analytic at $x = a$.
7. If $x = a$ is not an ordinary point of differential equation $y'' + d_1(x)y' + d_2(x)y = 0$, then it is called a **singular point** of the differential equation. Further if $(x-a)d_1(x)$ and $(x-a)^2 d_2(x)$ are analytic at $x = a$, then $x = a$ is called a **regular singular point** of the differential equation. A singular point, which is not regular, is called an **irregular singular point.**
8. If $x = 0$ is an **ordinary point** of the differential equation

   $$y'' + d_1(x)y' + d_2(x)y = 0,$$

   then power series solution is assumed as $y(x) = \sum_{m=0}^{\infty} c_m x^m$. Expressions for y, y' and y'' are substituted in the given equation and $c_1, c_2 \ldots$ are determined by comparing the coefficient of different power of x.

9. If $x = 0$ is a **regular singular** point of the differential Eqn(42), viz.,

$$f(x)\,y''(x) + g(x)\,y'(x) + r(x)\,y = 0$$

then series solution of the equation is obtained by using the Frobenius method where the series solution assumed is of the form

$$y(x) = \sum_{m=0}^{\infty} c_m\, x^{m+k}, c_0 \neq 0.$$

10. In the Frobenius method, equation obtained by equating to zero the coefficient of smallest power of x is a quadratic equation in k, called the **indicial equation**.
11. When the roots of indicial equation k_1, k_2 (say) are **unequal** and **do not differ by an integer** then two linearly independent series solutions of Eqn.(42) are obtained corresponding to two values of k viz., k_1, k_2 in the expression for $y(x)$.
12. When the roots of indicial equation k_1, k_2 (say) are **unequal**, **differ by an integer** and make a coefficient of y **indeterminate** for $k = k_2$ (say) then two linearly independent solutions of Eqn(42) are obtained by putting $k = k_2$ in $y(x)$, which then contains two arbitrary constants.
13. When the roots of indicial equation k_1, k_2 (say) are **unequal**, **differ by an integer** and make a coefficient of y **infinite** for $k = k_2$ (say), then the form of y is modified by replacing c_0 by $d_0(k - k_2)$. Two linearly independent solutions are then obtained by substituting $k = k_2$ in $y(x)$ and $\frac{\partial y}{\partial k}$.
14. When the roots of indicial equation are **equal**, then two linearly independent solution of Eqn.(42) are obtained by substituting this value of k in $y(x)$ and $\frac{\partial y}{\partial k}$.

2.6 SOLUTIONS/ANSWERS

E1) For series (5), $R = \lim\limits_{n\to\infty}\left|\frac{n!}{(n+1)!}\right| = \lim\limits_{n\to\infty}\left|\frac{n}{n+1}\right| = 0$.

For series (6), $R = \lim\limits_{n\to\infty}\left|\frac{\frac{1}{n!}}{\frac{1}{(n+1)!}}\right| = \lim\limits_{n\to\infty}\left|\frac{(n+1)!}{n!}\right| = \lim\limits_{n\to\infty}|n+1| \to \infty$

For series (7), $R = \lim\limits_{n\to\infty}\left|\frac{1}{1}\right| = 1$.

E2) For the series $\sum \frac{x^n}{n!}$, $\quad R = \lim\limits_{n\to\infty}\left|\frac{\frac{1}{n!}}{\frac{1}{(n+1)!}}\right| = \lim\limits_{n\to\infty}(n+1) \to \infty$.

Hence the given series converges for all x.

E3) a) We have, $1 - 2x + 3x^2 - 4x^3 + \cdots = \sum (-1)^n (n+1)\, x^n$

$$R = \lim_{n\to\infty}\left|\frac{(-1)^n\,(n+1)}{(-1)^{n+1}(n+2)}\right| = \lim_{n\to\infty}\left|\frac{-\left(1+\frac{1}{n}\right)}{\left(1+\frac{2}{n}\right)}\right| = 1.$$

b) We have, $R = \lim_{n\to\infty}\left|\frac{(kn)!}{(n!)^k}\cdot\frac{[(n+1)!]^k}{[k(n+1)]!}\right|$

$$= \lim_{n\to\infty}\left|\frac{(n+1)^k}{k(n+1)}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^{k-1}}{k}\right| \to \infty, \text{ for } k > 1$$

$$\text{and} \quad \lim_{n\to\infty}\left|\frac{(n+1)^{k-1}}{k}\right| = 1 \text{ for}, k = 1.$$

E4) Comparing the given equations with $a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$,

a) Here $a_0(x) = 1$, $a_1(x) = 4x$, $a_2(x) = 1$

Now $(x-a)\frac{a_1(x)}{a_0(x)} = (x-a)4x$ and $(x-a)^2\frac{a_2(x)}{a_0(x)} = (x-a)^2$, are analytic at $x = a$. Hence $x = a$ is an ordinary point of given differential equation for all real finite a, including $a = 0$.

b) Here $a_0(x) = x^3(x-2)$, $a_1(x) = x^3$, $a_2(x) = 6$

Now $(x-a).\frac{a_1(x)}{a_0(x)} = \frac{(x-a)x^3}{x^3(x-2)}$, is analytic for $a = 2$

Also, $(x-a)^2.\frac{a_2(x)}{a_0(x)} = \frac{(x-a)^2.6}{x^3(x-2)}$ is analytic for $a = 2$ and singular for $x = 0$.

Hence $x = 0$ is an irregular singular point, and $x = 2$ is a regular singular point for the given differential equation.

c) $x = -1$, is singular point.

d) $x = 0$, is an irregular singular point, since $x^2\frac{a_2(x)}{a_0(x)} = x^2.\frac{2}{x^4} \to \infty$ at $x = 0$.

e) $x = 0$, is a regular singular point, since $x\frac{\sin x}{x}$ and $x^2.\frac{e^x}{x}$ are analytic at $x = 0$.

E5) The given D.E. is

$$(x^2 - 2x + 2)y'' - 4(x-1)y' + 6y = 0$$

In this case, we change the independent variable x to u by the relation $x - 1 = u$, the given D.E. becomes

$$(u^2 + 1)y'' - 4uy' + 6y = 0, \tag{114}$$

where y' and y'' now denote the derivatives of y w.r.t. u. We are now required to solve Eqn.(114) about the point $u = 0$.

We see that the coefficients of y' and y, i.e., $-\frac{4u}{u^2+1}$ and $\frac{6}{u^2+1}$ are analytic everywhere. Hence we assume solution of Eqn.(114) as

$$y = \sum_{m=0}^{\infty} c_m u^m \tag{115}$$

Substituting for expressions for y, y', y'' from (115) into Eqn.(114), we get

$$(u^2+1)\sum_{m=2}^{\infty} c_m m(m-1)u^{m-2} - 4u\sum_{m=1}^{\infty} c_m m u^{m-1} + 6\sum_{m=0}^{\infty} c_m u^m = 0$$

or,

$$(2c_2 + 6c_0) + (2c_1 + 6c_3)u + \sum_{m=2}^{\infty}\left[(m+2)(m+1)c_{m+2} + (m^2 - 5m + 6)c_m\right]u^m = 0$$

Setting the coefficients of successive powers of u to zero, we get $2c_2 + 6c_0 = 0$, $2c_1 + 6c_3 = 0$ and $(m+2)(m+1)c_{m+2} + (m-2)(m-3)c_m = 0$ for $m \ge 2$.

where c_0, c_1 are arbitrary constants. Solving, we get

$$c_2 = -3c_0,\ c_3 = \frac{-c_1}{3} \text{ and } c_{m+2} = -\frac{(m-2)(m-3)}{(m+2)(m+1)} c_m, \text{ for } m \geq 2.$$

$$\Rightarrow c_4 = 0,\ c_5 = 0,\ c_6 = -\frac{2.1}{6.5} c_4 = 0,\ c_7 = -\frac{3.2}{7.6} c_5 = 0$$

$\therefore$ The solution of given D.E. is

$$y = c_0 + c_1 u + c_2 u^2 + c_3 u^3 = c_0 (1 - 3u^2) + c_1 \left(u - \frac{1}{3} u^3 \right)$$

$$= c_0 \left[1 - 3(x-1)^2 \right] + c_1 \left[(x-1) - \frac{1}{3}(x-1)^3 \right].$$

E6) The given D.E. is

$$y'' + xy' + 3y = 0 \tag{116}$$

For the given equations, $x = 0$ is an ordinary point and we shall obtain the solution about $x = 0$.

Let $\quad y(x) = \sum_{m=0}^{\infty} c_m x^m$.

Then Eqn.(116) becomes

$$\sum_{m=2}^{\infty} m(m-1) c_m x^{m-2} + x \sum_{m=1}^{\infty} m c_m x^{m-1} + 3 \sum_{m=0}^{\infty} c_m x^m = 0$$

or, $\quad (2c_2 + 3c_0) + \sum_{m=1}^{\infty} \{ (m+2)(m+1) c_{m+2} + (3+m) c_{m=0} \} x^m$

Equating the coefficients of similar powers of x, we get

$$2c_2 + 3c_0 = 0 \Rightarrow c_2 = \frac{-2}{3} c_0$$

$$6c_3 + 4c_1 = 0 \Rightarrow c_3 = \frac{-2}{3} c_1$$

Indeed, $(m+2)(m+1) c_{m+2} + (m+3) c_m = 0$, for $m \geq 1$.

$$\therefore \quad c_4 = -\frac{5}{4.3} c_2 = (-1)^2 . \frac{5.2}{4.3.3} c_0$$

$$c_5 = -\frac{6}{5.4} c_3 = (-1)^2 \frac{2.6}{3.4.5} c_1$$

$$c_6 = -\frac{7}{6.5} c_4 = (-1)^3 \frac{7.5.2}{6.5.4.3.2} c_0$$

$$c_7 = -\frac{8}{7.6} c_5 = (-1)^3 \frac{8.6.2}{7.6.5.4.3} c_1$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

where c_0 and c_1 are arbitrary constants.

Hence the general solution is

$$y(x) = c_o \left[1 - \frac{2}{3} x + (-1)^2 \frac{5}{4.3} \left(\frac{2}{3} \right) x^2 + (-1)^3 \frac{7.5}{6.5.4.3} \left(\frac{2}{3} \right) x^3 + \cdots \right]$$
$$+ c_1 \left[x - \left(\frac{2}{3} \right) x^3 + (-1)^2 \frac{6}{4.5} \left(\frac{2}{3} \right) x^4 + (-1)^3 \frac{8.6}{7.6.5.4} \left(\frac{2}{3} \right) x^5 + \cdots \right]$$

E7) The given initial value problem is

$$\left. \begin{aligned} xy'' + y' + 2y &= 0 \\ y(1) &= 1 \\ y'(1) &= 2 \end{aligned} \right\} \tag{117}$$

Evidently, $x = 1$ is an ordinary point.

Let $\quad y = \sum_{m=0}^{\infty} c_m (x-1)^m \qquad (118)$

Differentiating twice in succession, we get

$$y' = \sum_{m=1}^{\infty} mc_m (x-1)^{m-1} \qquad (119)$$

and $\quad y'' = \sum_{m=2}^{\infty} m(m-1) c_m (x-1)^{m-2} \qquad (120)$

Substituting for y, y', y'' in Eqn.(117), we get

$$x\sum_{m=2}^{\infty} m(m-1)c_m (x-1)^{m-2} + \sum_{m=1}^{\infty} m c_m (x-1)^{m-1} + 2\sum_{m=0}^{\infty} c_m (x-1)^m = 0$$

or, $\quad (2c_2 + c_1 + 2c_0) + \sum_{m=1}^{\infty} \{m(m+1)c_{m+1} + (m+2)(m+1) c_{m+2}$

$$+ (m+1)c_{m+1} + 2c_m\}(x-1)^m = 0$$

$\therefore \quad 2c_2 + c_1 + 2c_0 = 0$ i.e., $c_2 = -\left[c_0 + \frac{c_1}{2}\right] \qquad (121)$

and $(m+1)mc_{m+1} + (m+2)(m+1)c_{m+2} + (m+1)c_{m+1} + 2c_m = 0$, for $m \ge 1$

i.e., $(m+1)(m+2)c_{m+2} + (m+1)^2 c_{m+1} + 2c_m = 0$, for $m \ge 1 \qquad (122)$

using initial conditions given by (117) in relations (118) and (119), we get

$$c_0 = 1 \text{ and } c_1 = 2$$

From (121) and above, $c_2 = -2$

For $m = 1, 2, 3, \ldots$ in Eqn.(122) and using above values, we get

$$6c_3 + 4c_2 + 2c_1 = 0 \Rightarrow c_3 = 2/3$$

$$12c_4 + 9c_3 + 2c_2 = 0 \Rightarrow c_4 = -1/6$$

$$20c_5 + 16c_4 + 2c_3 = 0 \Rightarrow c_5 = 1/15.$$

and so on.

Hence $y = 1 + 2(x-1) - 2(x-1)^2 + \frac{2}{3}(x-1)^3 - \frac{1}{6}(x-1)^4 + \frac{1}{15}(x-1)^5 + \cdots$.

E8) The given D.E. is $y'' + xy' + y = 0 \qquad (123)$

Here $x = 2$, is an ordinary point.

Let $y = \sum_{m=0}^{\infty} c_m (x-2)^m \qquad (124)$

Substituting for y, y', y'' in Eqn.(123), we get

$$\sum_{m=2}^{\infty} c_m (x-2)^{m-2} m(m-1) + x\sum_{m=1}^{\infty} m c_m (x-2)^{m-1} + \sum_{m=0}^{\infty} c_m (x-2)^m = 0$$

or, $\quad (2.1.c_2 + 2.1.c_1 + c_0) + \sum_{m=1}^{\infty} \{(m+2)(m+1)c_{m+2} + mc_m$

$$+ 2(m+1)c_{m+1} + c_m\}(x-2)^m = 0.$$

$$\therefore \left.\begin{aligned} 2c_2 + 2c_1 + c_0 = 0 &\Rightarrow c_2 = -c_1 - \frac{1}{2}c_0 \\ \text{and } (m+2)(m+1)c_{m+2} + 2(m+1)c_{m+1} + (m+1)c_m &= 0, \text{ for } m \ge 1 \end{aligned}\right\} \qquad (125)$$

For $m = 1, 2, 3, \ldots$ and using Eqn.(125), we get

$$3.2.c_3 + 2.2c_2 + 2c_1 = 0 \Rightarrow c_3 = \frac{1}{3}(c_1 + c_0)$$

$$4.3.c_4 + 2.3c_3 + 3c_2 = 0 \Rightarrow c_4 = \frac{1}{3.4}c_1 - \frac{1}{2.4.3}c_0$$

$$5.4.c_5 + 2.4\,c_4 + 4\,c_3 = 0 \Rightarrow c_5 = \frac{1}{5.6}c_1 + \frac{1}{12}c_0$$

and so on. Hence Eqn.(124) yields

$$y(x) = c_0\left[1 - \frac{1}{2}(x-2)^2 + \frac{1}{3}(x-2)^3 - \frac{1}{24}(x-2)^4 + \frac{1}{12}(x-2)^5 + \cdots\right]$$
$$+ c_1\left[(x-2) - (x-2)^2 + \frac{1}{3}(x-2)^3 + \frac{1}{12}(x-2)^4 + \frac{1}{30}(x-2)^5 + \cdots\right].$$

E9) The given Hermite equation of order n is

$$y'' - 2xy' + 2ny = 0 \tag{126}$$

If $z_n(x) = \sum_{k=0}^{\infty} a_k x^k$ is a solution of Eqn.(126), then substituting for y, y', y'' in Eqn.(126), we get

$$\sum_{k=2}^{\infty} a_k k(k-1)x^{k-2} - 2x\sum_{k=1}^{\infty} k\,a_k x^{k-1} + 2n\sum_{k=0}^{\infty} a_k x^k = 0$$

or, $$2.1.a_2 + 2n\,a_0 + \sum_{k=1}^{\infty}\left[(k+2)(k+1)a_{k+2} - 2k\,a_k + 2n\,a_k\right]x^k = 0$$

Equating coefficients of successing powers of x, we get

$$2a_2 + 2n\,a_0 = 0 \Rightarrow a_2 = -n\,a_0$$

and $(k+2)(k+1)a_{k+2} - 2(k-n)a_k = 0$, for $k \ge 1$

putting $k = 1, 2, 3, \ldots$ in the above relation, we get

$$3.2.a_3 - 2(1-n)a_1 = 0 \Rightarrow a_3 = \frac{2(1-n)}{2.3}a_1$$

$$4.3\,a_4 - 2(2-n)a_2 = 0 \Rightarrow a_4 = \frac{2(2-n)}{4.3}a_2 = -\frac{2n(2-n)}{4.3}a_0$$

$$5.4\,a_5 - 2(3-n)a_3 = 0 \Rightarrow a_5 = \frac{2(3-n)}{5.4}a_3 = \frac{2^2(1-n)(3-n)}{2.3.4.5}a_1$$

$$6.5\,a_6 - 2(4-n)a_4 = 0 \Rightarrow a_6 = \frac{2(4-n)}{6.5}a_4 = -\frac{2^2 n(2-n)(4-n)}{6.5.4.3}a_0$$

and so on.

Hence the solution is

$$z_n(x) = y(x) = a_0\left[1 + (-n)x^2 + \frac{2(-n)(2-n)}{4.3}x^4 + \frac{2^2(-n)(2-n)(4-n)}{6.5.4.3}x^6 + \cdots\right]$$
$$+ a_1\left[x + \frac{2(1-n)}{2.3}x^3 + \frac{2^2(1-n)(3-n)}{2.3.4.5}x^5 + \cdots\right]$$
$$= a_0\left[1 + \frac{2(-n)}{2}x^2 + \frac{2^2(-n)(2-n)}{4.3.2}x^4 + \frac{2^3(-n)(2-n)(4-n)}{6.5.4.3.2}x^6 + \cdots\right]$$
$$+ a_1\left[x + \frac{2(1-n)}{2.3}x^3 + \frac{2^2(1-n)(3-n)}{2.3.4.5}x^5 + \cdots\right]$$
$$= a_0\left[1 + \sum_{k=1}^{\infty}\frac{2^k(-n)(-n+2)\ldots(-n+2k-2)}{(2k)!}x^{2k}\right]$$
$$+ a_1\left[x + \sum_{k=1}^{\infty}\frac{2^k(1-n)(1-n+2)\ldots(1-n+2k-2)}{(2k+1)!}x^{2k+1}\right]$$

E10) a) The given equation

$$2x(1-x)y'' + (1-x)y' + 3y = 0 \tag{127}$$

has $x = 0$, as a regular singular point

Let $y(x)=\sum_{m=0}^{\infty} c_m x^{k+m}$

Putting the values of y, y', y'' in Eqn.(127), we get

$$\sum_{m=0}^{\infty}\{3-(k+m)-2(k+m)(k+m-1)\}c_m x^{k+m}$$
$$+\sum_{m=0}^{\infty}\{2(k+m-1)+1\}(k+m)c_m x^{k+m-1}=0 \qquad (128)$$

Equating the coefficients of lowest power of x to zero in Eqn.(128), we get the **indicial equation** as $c_0(2k-2+1)k=0$ i.e., $c_0k(2k-1)=0 \Rightarrow k=0$ and $k=1/2$, i.e., the roots of indicial equation are **distinct**, and do not differ by an integer. To get the recurrence relation, we equate coefficients x^{k+m} in Eqn.(128) to zero and obtain

$$[3-(k+m)-2(k+m)(k+m-1)]c_m+\{2(k+m+1-1)+1\}(k+m+1)c_{m+1}=0$$

i.e., $$c_{m+1}=\frac{2(k+m)(k+m-1)+(k+m)-3}{(k+m+1)(2k+2m+1)}c_m$$

Putting $m=0,1,2,3,\ldots$

$$c_1=\frac{2k^2-k-3}{(k+1)(2k+1)}c_0$$
$$c_2=\frac{\{2(k+1)k+(k+1)-3\}\{2k(k-1)+k-3\}}{(k+1)(k+2)(2k+1)(2k+3)}c_0$$
$$c_3=\frac{\{2(k+2)(k+1)+(k+2)-3\}\{2(k+1)k+(k+1)-3\}\{2k(k-1)+k-3\}}{(k+1)(k+2)(k+3)(2k+1)(2k+3)(2k+5)}c_0$$

Putting values of $c_1, c_2, c_3, \ldots$ in $y(x)=\sum_{m=0}^{\infty} c_m x^{k+m}$, we get

$$y(x)=x^k\left[c_0+\frac{2k^2-k-3}{(k+1)(2k+1)}c_0x+\frac{(2k^2+3k-2)(2k^2-k-3)}{(k+1)(k+2)(2k+1)(2k+3)}c_0x^2\right.$$
$$\left.+\frac{(2k^2+7k+3)(2k^2+3k-2)(2k^2-k-3)}{(k+1)(k+2)(k+3)(2k+1)(2k+3)(2k+5)}c_0x^3+\cdots\right] \qquad (129)$$

Putting $k=0$ in Eqn.(129), we get

$$y_1=c_0\left[1-3x+\frac{(-2)(-3)}{1.2.1.3}x^2+\frac{3(-2)(-3)}{1.2.3.1.3.5}x^3+\cdots\right]$$
$$=c_0\left[1-3x+\frac{3}{1.3}x^2+\frac{3}{1.3.5}x^3+\cdots\right]=c_0u\text{, say} \qquad (130)$$

Putting $k=1/2$ in Eqn.(129), we get

$$y_2=c_0x^{1/2}[1+(-x)+0+0+\cdots]=c_0x^{1/2}(1-x)=c_0v\text{, say} \qquad (131)$$

Hence the general solution is $y(x)=A_1y+B_1y_2=Au+Bv$

where $A=A_1c_0,\ B=B_1c_0$ are arbitrary constants and u and v are given by Eqn.(130) and Eqn.(131) respectively.

b) The given equation

$$(2x+x^2)y''-y'-6xy=0 \qquad (132)$$

has $x=0$ as a regular singular point.

Let $y(x)=\sum_{m=0}^{\infty} c_m x^{k+m}$ be a solution of Eqn.(132). Putting for y, y', y'' in Eqn.(132), we get

$$-6\sum_{m=0}^{\infty} c_m x^{k+m+1} + \sum_{m=0}^{\infty} (k+m)\ (k+m-1)\, x^{k+m}$$

$$+\sum_{m=0}^{\infty} (k+m)(2k+2m-3)\, c_m x^{k+m-1} = 0 \qquad (133)$$

Equating the coefficient of lowest power of x to zero in Eqn.(133), we get the **indicial equations** as $k(2k-3)c_0 = 0 \Rightarrow k = 0$ and $k = 3/2$, i.e., the roots of indicial equation are **distinct**, and do not differ by an integer. To get the recurrence relation, equate the coefficient of x^{k+m+1} to zero is relation (133) and obtain

$$-6c_m + (k+m+1)(k+m)c_{m+1} + (k+m+2)(2k+2m+1)c_{m+2} = 0 \quad (134)$$

Equating coefficient of x^k in (133) to zero, we get

$$k(k-1)c_0 + (k+1)(2k-1)c_1 = 0 \Rightarrow c_1 = -\frac{k(k-1)}{(k+1)(2k-1)} c_0$$

Putting $m = 0,1,2,3,\ldots$ in Eqn.(134), we get

$$c_2 = \frac{6(2k-1) + k^2(k-1)}{(k+2)(2k-1)(2k+1)} c_0$$

$$c_3 = -\frac{6k(k-1)}{(k+1)(2k-1)(k+3)(2k+3)} c_0 - \frac{(k+1)[k^3 - k^2 + 12k - 6]}{(k+3)(2k-1)(2k+1)(2k+3)} c_0$$

and so on.

For $k = 0$, we get

$$c_1 = 0, c_2 = \frac{-6}{2(-1)(1)} c_0 = 3c_0, c_3 = -\frac{1(-6)}{3(-1).1.3} c_0 = -\frac{2}{3} c_0, \cdots$$

For $k = 3/2$, we get

$$c_1 = \frac{3}{4.5} c_0\ ,\ c_2 = \frac{15}{32} c_0, \cdots$$

Hence $y_1 = c_0\left[1 + 3x - \frac{2}{3}x^2 + \cdots\right] = c_0\, u$, say $\qquad (135)$

and $y_2 = c_0 x^{3/2}\left[1 + \frac{3}{4.5}x + \frac{15}{32}x^2 + \cdots\right] = c_0\ v$, say $\qquad (136)$

Hence the general solution is $y(x) = A_1 y_1 + B_1 y_2 = Au + Bv$, where $A = c_0 A_1$ and $B = c_0 B_1$ are arbitrary constant and u and v are given by relations (135) and (136) respectively.

E11) The given D.E. is

$$(2 + x^2)y'' + xy' + (1+x)y = 0$$

Let $y = \sum_{m=0}^{\infty} c_m x^{k+m}, c_0 \neq 0$, be a series solution of the given equation. Then putting the values of y, y', y'' in the given equation, we get

$$(2+x^2)\sum_{m=2}^{\infty} (m+k)(m+k-1)c_m x^{k+m-2} + x\sum_{m=1}^{\infty} (m+k)c_m x^{m+k-1}$$

$$+ (1+x)\sum_{m=0}^{\infty} c_m x^{m+k} = 0$$

$$\Rightarrow 2c_0 k(k-1)x^{k-2} + 2c_1 k(k+1)x^{k-1}\left[2c_2(k+2)(k+1) + c_0 k(k-1) + c_0 k + c_0\right]x^k$$

$$+\sum_{m=3}^{\infty}\left[2c_m(k+m)(k+m-1) + c_{m-2}(k+m-2)(k+m-3)\right.$$

$$\left. + c_{m-2}(k+m-2) + c_{m-2} + c_{m-3}\right]x^{k+m-3} = 0$$

Equating coefficients of lowest powers of x, we get **indicial equation** as

$$2c_0k(k-1)=0 \Rightarrow k=0,1 \quad \text{as} \quad c_0 \neq 0$$

Equating to zero coefficients of other powers of x, we get

$$2c_1(k+1)k=0$$

This shows that c_1 becomes **indeterminate for $k=0$**.

$$\therefore \quad 2c_2(k+2)(k+1)+c_0k(k-1)+c_0k+c_0=0$$

$$c_2=-\frac{k^2+1}{2(k+1)(k+2)}c_0$$

The general recurrences relation is

$$2c_m(k+m)(k+m-1)+c_{m-2}\left[(k+m-2)^2+1\right]+c_{m-3}=0 \quad \text{for} \quad m \geq 3$$

Putting $m=3,4,5,\ldots$, we get

$$c_3=-\frac{[(k+1)^2+1]c_1}{2(k+3)(k+2)}-\frac{c_0}{2(k+3)(k+2)}$$

$$c_4=-\frac{c_1}{2(k+3)(k+4)}+(-1)^2\frac{(k^2+1)(k^2+4k+5)c_0}{2^2(k+1)(k+2)(k+3)(k+4)}$$

and so on.

Putting these values in the assumed solution, we get

$$y=x^k\left[c_0+c_1x-\frac{(k^2+1)c_0}{2(k+1)(k+2)}x^2-\left\{\frac{c_0}{2(k+2)(k+3)}+\frac{\{(k+1)^2+1\}c_1}{2(k+3)(k+2)}\right\}x^3\right.$$
$$\left.+\left\{-\frac{c_1}{2(k+4)(k+3)}+\frac{(k^2+1)\{(k+2)^2+1\}}{2^2(k+1)(k+2)(k+3)(k+4)}c_0\right\}x^4+\cdots\right]$$

Putting $k=0$, we get

$$y=c_0\left[1-\frac{1}{2}x^2-\frac{1}{2.2.3}x^3+\frac{5}{2^2.1.2.3.4}x^4+\cdots\right]$$
$$+c_1\left[x-\frac{1}{6}x^3-\frac{1}{2.3.4}x^4+\cdots\right]$$

as the required solution, where c_0 and c_1 are arbitrary constants.

E12) The given equation is $(1-x^2)y''+2xy'+y=0$.
Let the series solution of given equation be

$$y(x)=\sum_{m=0}^{\infty}c_mx^{m+k}, \quad c_0 \neq 0 \tag{137}$$

substituting for y, y', y'' in the given equation, we get

$$c_0k(k-1)x^{k-2}+c_1k(k-1)x^{k-1}+\sum_{m=0}^{\infty}\left[-(m+k)(m+k-1)c_m+2(m+k)c_m+\right.$$
$$\left.c_m+(m+2+k)(m+k+1)c_{m+2}\right]x^{m+k}=0$$

The **indicial equation** is $c_0k(k-1)=0 \Rightarrow k=0,1$ as $c_0 \neq 0$. (138)

Thus indicial roots are **distinct** and **differ by an integer**.
Equating the coefficients of x^{k-1} to zero, we get

$$c_1k(k+1)=0 \tag{139}$$

When $k=0, c_1$ becomes **indeterminate** $(0/0)$. But in this case, we get the identity $c_1.0=0$, which is satisfied by every values of c_1. Thus, we can also take c_1 as arbitrary constant.
The recurrence relation is

$$(m+k+2)(m+k+1)c_{m+2}+\left[1-(m+k)(m+k-1)+2(m+k)\right]c_m=0$$

$$\Rightarrow c_{m+2} = \frac{(m+k)^2 - 3(m+k) - 1}{(m+k+2)(m+k+1)} c_m = \frac{(m+k)(m+k-3) - 1}{(m+k+2)(m+k+1)} c_m$$

Putting $m = 0, 1, 2, \ldots$, we get

$$c_2 = \frac{k^2 - 3k - 1}{(k+2)(k+1)} c_0 = \frac{k(k-3) - 1}{(k+2)(k+1)} c_0$$

$$c_3 = \frac{(k+1)(k-2) - 1}{(k+3)(k+2)} c_1$$

$$c_4 = \frac{\{(k+2)(k-1) - 1\}\{k(k-3) - 1\}}{(k+4)(k+2)^2(k+1)} c_0$$

$$c_5 = \frac{\{(k+3)(k) - 1\}\{(k+1)(k-2) - 1\}}{(k+5)(k+3)^2(k+2)} c_1$$

....

Substituting these values of coefficients in series for $y(x)$, we get

$$y = x^k \Bigg[c_0 + c_1 x + \frac{k(k-3) - 1}{(k+2)(k+1)} c_0 x^2 + \frac{(k+1)(k-2) - 1}{(k+3)(k+2)} c_1 x^3 + \frac{\{(k+2)(k-1) - 1\}\{k(k-3) - 1\}}{(k+4)(k+2)^2(k+1)} c_o x^4 + \frac{\{(k+3)k - 1\}\{(k+1)(k-2) - 1\}}{(k+5)(k+3)^2(k+2)} c_1 x^5 + \cdots \Bigg] \quad (140)$$

Substituting $\mathbf{k = 0}$ in this, we get

$$y = c_0 \left[1 - \frac{1}{2} x^2 - \frac{3}{16} x^4 + \cdots \right] + c_1 \left[x - \frac{1}{2} x^3 + \frac{(-3)}{90} x^5 - \cdots \right] \quad (141)$$

where c_0 and c_1 are arbitrary constants.

This is the required general series solution.

Note that if we substitute $k = 1$ in the series for $y(x)$, we get from relation (139) that $c_1 = 0$ and hence $c_3, c_5, \ldots$ all will vanish and then relation (140) gives

$$y = x\left[c_0 + \left(-\frac{1}{2} \right) x^2 c_0 + \left(-\frac{3}{90} \right) x^4 c_0 + \cdots \right] \quad (142)$$

which is the first series in Eqn.(141) and hence it is not a new series.

E13) a) The given equation is

$$xy'' + (3 - x)y' + y = 0$$

Here $x = 0$, is a regular singular point.

$$\text{Let } y(x) = \sum_{m=0}^{\infty} c_m x^{m+k}, \; c_0 \neq 0. \quad (143)$$

Substituting for y, y', y'' in the given equation, we get

$$[k(k-1) + 3k] c_0 x^{k-1} + \sum_{m=0}^{\infty} [\{(m+k+1)(m+k) + 3(m+k+1)\} c_{m+1} - (m+k-1) c_m] x^{m+k} = 0$$

Indicial equation is

$$k(k-1) + 3k = 0, \text{ as } c_0 \neq 0, \; k = 0, -2$$

The recurrence relation is

$$[(m+k+1)(m+k) + 3(m+k+1)] c_{m+1} - (m+k-1) c_m = 0, \; m \geq 0$$

or, $$c_{m+1} = \frac{m+k-1}{(m+k+1)(m+k+3)} c_m, \quad m \geq 0$$

Putting $m = 0,1,2,3,\ldots$

$$c_1 = \frac{k-1}{(k+1)(k+3)} c_0$$

$$c_2 = \frac{k(k-1)}{(k+1)(k+2)(k+3)(k+4)} c_0$$

$$c_3 = \frac{(k-1)k}{(k+2)(k+3)^2(k+4)(k+5)} c_0$$

$$c_4 = \frac{(k-1)k}{(k+3)^2(k+4)^2(k+5)(k+6)} c_0$$

and so on.

$$\therefore y(x) = c_0 x^k \left[1 + \frac{k-1}{(k+1))k+3)} x + \frac{(k-1)k}{(k+1)(k+2)(k+3)(k+4)} x^2 \right.$$
$$+ \frac{(k-1)k}{(k+2)(k+3)^2(k+4)(k+5)} x^3$$
$$\left. + \frac{(k-1)k}{(k+3)^2(k+4)^2(k+5)(k+6)} x^4 + \cdots \right]$$

For $k = -2$, coefficients of x^2 and x^3 become **infinite**. Replacing c_0 by $a(k+2)$, the above series solution takes the form

$$y(x) = ax^k \left[(k+2) + \frac{(k-1)(k+2)}{(k+1)(k+3)} x + \frac{(k-1)k}{(k+1)(k+3)(k+4)} x^2 \right.$$
$$\left. + \frac{(k-1)k}{(k+3)^2(k+4)(k+5)} x^3 + \frac{(k-1)k(k+2)}{(k+3)^2(k+4)^2(k+5)(k+6)} x^4 + \cdots \right] \qquad (144)$$

$$\therefore \frac{\partial y}{\partial k} = ax^k \ln x \left[(k+2) + \frac{(k-1)(k+2)}{(k+1)(k+3)} x + \cdots \right]$$
$$+ ax^k \left[1 + \left\{ \frac{2k+1}{(k+1)(k+3)} - \frac{(k-1)(k+2)}{(k+1)(k+3)} \left(\frac{1}{k+1} + \frac{1}{k+3} \right) \right\} x \right.$$
$$+ \left\{ \frac{2k-1}{(k+1)(k+3)(k+4)} - \frac{k(k-1)}{(k+3)(k+4)(k+1)} \left(\frac{1}{k+1} + \frac{1}{k+3} + \frac{1}{k+4} \right) \right\} x^2$$
$$\left. + \left\{ \frac{2k-1}{(k+3)^2(k+4)(k+5)} - \frac{k(k-1)}{(k+3)^2(k+4)(k+5)} \left(\frac{2}{k+3} + \frac{1}{k+4} + \frac{1}{k+5} \right) \right\} x^3 + \cdots \right] \qquad (145)$$

Putting $k = -2$ in Eqns.(144) and (145), we get

$y = ax^{-2}[-3x^2 + x^3] = a\, y_1$, say

$$\frac{\partial y}{\partial k} = a(\ln x) y_1 + ax^{-2} \left[1 + 3x + 4x^2 - \frac{11}{3} x^3 + \frac{1}{8} x^4 - \cdots \right] = ay_2 \text{, say}$$

$\therefore$ The general solution is

$$y(x) = c_1 y_1 + c_2 y_2 .$$

b) Given equation is

$$x(1-x)y'' - 3xy' - y = 0$$

Here $x = 0$, is a regular singular point.

Let $y(x) = \sum_{m=0}^{\infty} c_m x^{m+k}$, $c_0 \neq 0$

Substituting for y, y', y'' in the given equation, we get

$$\sum_{m=0}^{\infty} c_m (m+k)(m+k-1)\, x^{m+k-1} - \sum_{m=0}^{\infty} (m+k+1)^2 c_m x^{m+k} = 0$$

Equating to zero the coefficients of lowest power of x, i.e., x^{k-1}, the **indicial equation** is

$$c_0\, k(k-1) = 0 \text{ or } k = 0,1 \;\; (\because c_0 \neq 0)$$

There are **unequal roots**, differing by an integer.

The recurrence relation is

$$c_m (m+k+1)^2 - c_{m+1}(m+k+1)\,(m+k) = 0 \quad \text{for } m \geq 0$$

$$\therefore c_{m+1} = \frac{m+k+1}{m+k} c_m \quad \text{for } m \geq 0$$

Putting $m = 0,1,2,3,\ldots$, we get

$$c_1 = \frac{k+1}{k} c_0 \;,\;\; c_2 = \frac{k+2}{k} c_0 \,,\; c_3 = \frac{k+3}{k} c_0$$

and so on.

Putting these values in the series solution for $y(x)$, we get

$$y(x) = x^k \left[c_0 + \frac{k+1}{k} c_0 x + \frac{k+2}{k} c_0 x^2 + \frac{k+3}{k} c_0 x^3 + \cdots \right]$$

If we put $\mathbf{k = 0}$, we find that coefficients of $x, x^2, x^3, \ldots$ become **infinite**.

To remove this difficulty, we put $c_0 = k\, d_0$ in the above series, which then becomes $y(x) = d_0 x^k \left[1 + (k+1)\,x + (k+2)\,x^2 + (k+3)\,x^3 + \cdots\right]$.

Putting $\mathbf{k = 0}$, and replacing d_0 by a, we get

$$y(x) = a\,(x + 2x^2 + 3x^3 + \cdots) = a\,u_1 \text{, say.}$$

If we put $k = 1$ to obtain second solution, we obtain

$y(x) = c_0 (x + 2x^2 + 3x^3 + \cdots)$, which is not distinct from above solution.

In such a case, second independent solution is given by $\left(\frac{\partial y}{\partial k}\right)_{k=0}$.

Now $\frac{\partial y}{\partial k} = d_0 x^k \ln x\{k + (k+1)x + (k+2)x^2 + \cdots\} + d_0 x^k [1 + x + x^2 + \cdots]$

Putting $\mathbf{k = 0}$ and replacing d_0 by b in the above, we get

$$\left(\frac{\partial y}{\partial k}\right)_{k=0} = b[\ln x\,(x + 2x^2 + 3x^3 + \cdots) + (1 + x + x^2 + \cdots)]$$

$$= b\,[u \ln x + (1 + x + x^2 + \cdots)] = bv \text{, say}$$

and required solution is $y(x) = au + bv$, where a and b are arbitrary constants.

E14) The given differential equation is

$$x(1-x)\, y'' + ((1-x)y' - y = 0 \qquad (146)$$

Here $x = 0$, is a regular singular point of the above equation.

Let $y(x) = \sum_{m=0}^{\infty} c_m x^{m+k}$, $c_0 \neq 0$.

Substituting for y, y', y'' in the given equation, we get

$$k^2 c_0 x^{k-1} + \sum_{m=0}^{\infty} \left[(m+k+1)^2 c_{m+1} - \{(m+k)^2 + 1\} c_m\right] x^{m+k} = 0$$

Equating coefficients of least power of x, the **indicial equation** is

$$c_0 k^2 = 0 \Rightarrow k = 0, 0 \quad (\because c_0 \neq 0).$$

The recurrence relation is

$$c_{m+1} = \frac{(m+k)^2+1}{(m+k+1)^2} c_m \ , \ m \geq 0 .$$

Hence $\quad c_1 = \dfrac{k^2+1}{(k+1)^2} c_0$

$$c_2 = \frac{(k^2+1)[(k+1)^2+1]}{(k+1)^2(k+2)^2} c_0$$

$$c_3 = \frac{(k^2+1)[(k+1)^2+1]\,[(k+2)^2+1]}{(k+1)^2(k+2)^2(k+3)^2} c_0$$

and so on.

$$\therefore y(x) = c_0 x^k \left[1 + \frac{k^2+1}{(k+1)^2} x + \frac{(k^2+1)\{(k+1)^2+1\}}{(k+1)^2\,(k+2)^2} x^2 \right.$$
$$\left. + \frac{(k^2+1)\{(k+1)^2+1\}\{(k+2)^2+1\}}{(k+1)^2\,(k+2)^2\,(k+3)^2} x^3 + \cdots \right] \qquad (147)$$

Hence solution **for $k = 0$** is

$$[y(x)]_{k=0} = c_0 x \left[1 + x + \frac{1.2}{1^2.2^2} x^2 + \frac{1.2.5}{1^2.2^2.3^2} x^3 + \cdots \right] = y_1 \text{, say} \qquad (148)$$

with y given by Eqn.(147), left hand side of Eqn.(146) becomes $k^2 c_0 x^{k-1}$

$$\therefore \frac{\partial}{\partial k}\left[x(1-x)y'' + (1-x)y' - y \right] = 2k\,c_0 x^{k-1} + c_0 k^2 x^{k-1} \ln x$$

or, $\left[x(1-x)\dfrac{d^2}{dx^2} + (1-x)\dfrac{d}{dx} - 1 \right] \dfrac{\partial y}{\partial k} = 2kc_0 x^{k-1} + c_0 k^2 x^{k-1} \ln x$

Therefore, for $k = 0, \dfrac{\partial y}{\partial k}$ is a second solution of Eqn.(146).

From Eqn.(147), we get

$$\left(\frac{\partial y}{\partial k} \right)_{k=0} = c_0 \ln x \left[1 + \frac{1}{1^2} x + \frac{1.2}{1^2.2^2} x^2 + \frac{1.2.5}{1^2.2^2.3^2} x^3 + \cdots \right] + c_0 \left[-2x - x^2 - \cdots \right]$$
$$= y_1 \ln x + c_0 \left[-2x - x^2 - \cdots \right] = y_2 \text{, say}$$

since y_1 and y_2 are linearly independent, hence the general solution is

$$y = \alpha_1 y_1 + \alpha_2 y_2,$$

where α_1, α_2 are arbitrary constants.

E15) a) Proceeding as in E14) when $p = 1$, $y = au + bv$

where, $u = \left(1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots \right)$

and $v = u \ln x - \left[\dfrac{x}{1} + \dfrac{x^2}{2!}\left(1 + \dfrac{1}{2}\right) + \dfrac{x^3}{3!}\left(1 + \dfrac{1}{2} + \dfrac{2}{3}\right) + \cdots \right].$

b) When p is not a integer, $y = AU + BV$

where, $U = 1 + \dfrac{x}{p} + \dfrac{x^2}{p(p+1)} + \dfrac{x^3}{p(p+1)(p+2)} + \cdots$

$$V = x^{1-p}\left[1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \right].$$

E16) a) Proceeding as in E14), the solution is $y = au + bv$, where, $u = 1 - 2x - \frac{3}{2}x^2 - \frac{4}{3}x^3 - \cdots$

and $v = u \ln x + \left[2\left(2 - \frac{1}{2}\right)x - \frac{3}{2!}\left(2 + \frac{1}{2} - \frac{1}{3}\right)x^2 - \frac{4}{3!}\left(2 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right)x^3 + \cdots\right]$

b) The given equation is

$$(x - x^2)y'' + (1 - 5x)y' - 4y = 0$$

Evidently $x = 0$ is a regular singular point of the equation.

Let $y(x) = \sum_{m=0}^{\infty} c_m x^{m+k}, c_0 \neq 0$. (149)

Substituting for y, y', y'' in the given equation, we get

$$\sum_{m=0}^{\infty} c_m (m+k)^2 x^{m+k-1} - \sum_{m=0}^{\infty} c_m (m+k+2)^2 x^{m+k} = 0,$$

Equating to zero, the coefficients of x^{k-1}, the indicial equation is

$$c_0 k^2 = 0 \text{ or } k = 0, 0 \quad (\because c_0 \neq 0).$$

The recurrence relation is $c_m = \frac{(m+k+1)^2}{(m+k)^2} c_{m-1}$, for $m \geq 1$.

Putting $m = 1, 2, 3, \ldots$, we get $c_1 = \frac{(k+2)^2}{(k+1)^2} c_0$, $c_2 = \frac{(k+3)^2}{(k+1)^2} c_0$

$$c_3 = \frac{(k+4)^2}{(k+1)^2} c_0, \quad c_4 = \frac{(k+5)^2}{(k+1)^2} c_0, \ldots$$

Putting the values of coefficients $c_0, c_1, c_2, \ldots$ in solution (149), we get

$$y(x) = x^k c_0 \left[1 + \frac{(k+2)^2}{(k+1)^2} x + \frac{(k+3)^2}{(k+1)^2} x^2 + \frac{(k+4)^2}{(k+1)^2} x^3 + \cdots\right] \quad (150)$$

Putting $k = 0$ in Eqn.(150), we get

$$y(x)\big|_{k=0} = c_0\left[1 + 2^2 x + 3^2 x^2 + 4^2 x^3 + \cdots\right] = c_0 u, \text{ (say)} \quad (151)$$

Differentiating Eqn.(150) partially, w.r.t. k and putting $k = 0$, the second solution is obtained as

$$\left(\frac{\partial y}{\partial k}\right)_{k=0} = c_0 \ln x\left[1 + 2^2 x + 3^2 x^2 + 4^2 x^3 + \cdots\right]$$

$$+ c_0\left[2^2 x(1-2) + 3^2 x^2\left(\frac{2}{3} - 2\right) + 4^2 x^3\left(\frac{2}{3} - 2\right) + \cdots\right]$$

$$= c_0 u \ln x - c_0\left[1.2x + 2.3x^2 + 3.4x^3 + \cdots\right] = c_0 v, \text{ (say)} \quad (152)$$

Hence the required general solution is

$$y = au + bv.$$

c) Proceeding as in E14), the general solution is $y = au + bv$,

where, $u = x^{1/2}\left\{1 + \frac{1.3}{4^2}x^2 + \frac{1.3.5.7}{4^2.8^2}x^4 + \cdots\right\}$

and $v = u \ln x + 2x^{1/2}\left\{\frac{1.3}{4^2}\left(1 + \frac{1}{3} - \frac{1}{2}\right)x^2 + \frac{1.3.5.7}{4^2.8^2}\right.$

$$\left.\left(1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4}\right)x^4 + \cdots\right\}$$

E17) a) The given equation is

$$4x^3y'' + 6x^2y' + y = 0 \qquad (153)$$

Here $x = 0$, is an irregular singular point. We change variable x to t by $x = \frac{1}{t}$. Then

$$4t\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0 \qquad (154)$$

Here $t = 0$ is a regular singular point.

Assuming $y = \sum_{m=0}^{\infty} c_m t^{m+k}$ and using Frobenius method, the solution of D.E. (154) is obtained as

$$y = a\left(1 - \frac{t}{2!} + \frac{t^2}{4!} - \cdots\right) + bt^{1/2}\left(1 - \frac{1}{3!}t + \frac{1}{5!}t^2 - \frac{1}{7!}t^3 + \cdots\right)$$

Putting $t = \frac{1}{x}$ in the above solution, we obtain

$$y(x) = a\left(1 - \frac{x^{-1}}{2!} + \frac{x^{-2}}{4!} - \cdots\right) + bx^{-1/2}\left(1 - \frac{x^{-1}}{1!} + \frac{x^{-2}}{5!} - \frac{x^{-3}}{71} + \cdots\right),$$

which is the required series solution of the given equation.

b) The given equation is

$$2x^2(1 - x)y'' - 5x(1 + x)y' + (5 - x)y = 0 \qquad (155)$$

Since we have to find solution for large values of x, putting $x = \frac{1}{t}$, Eqn.(155) becomes

$$2t^2(t - 1)\frac{d^2y}{dt^2} + t(9t + 1)\frac{dy}{dt} + (5t - 1)y = 0. \qquad (156)$$

Clearly $t = 0$ is a regular singular point. Using Frobenius method, the solution of Eqn.(156) will be

$$y = a\left[\sum_{n=0}^{\infty} (n + 1)(2n + 3)(2n + 5)t^{n+1}\right] + b\left[\sum_{n=0}^{\infty} (n + 1)(n + 2)(2n + 1)t^{n+1/2}\right]$$

Hence solution of given equation (155) is

$$y = a\sum_{n=0}^{\infty} (n + 1)(2n + 3)(2n + 5)x^{-n-1} + b\sum_{n=0}^{\infty} (n + 1)(n + 2)(2n + 1)x^{-n-1/2}.$$

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UNIT 3 LEGENDRE, HERMITE AND LAGUERRE POLYNOMIALS

Structure **Page No.**

3.1 INTRODUCTION

In Unit 2, we discussed methods of finding series solutions of second order, linear, homogeneous differential equations with variable coefficients. In this unit and the unit to follow, we shall apply these methods to a few special second order equations occurring frequently in applied mathematics, engineering and physics. These equations are Legendre's, Bessel's, Hermite's and Laguerre's equations. The solution to these equations, that occur in applications, are referred to as **special functions**. They are called 'special' as they are different from the standard functions like sine, cosine, exponential, logarithmic etc. In this unit, we shall concentrate on Legendre, Hermite and Laguerre polynomials which are polynomial solutions to Legendre's, Hermite's and Laguerre's differential equations.

Legendre polynomials first arose in the problem of expressing the Newtonian potential of a conservative force field in an infinite series involving the distance variable of two points and their included central angle. Other similar problems dealing with either gravitational potential or electrostatic potentials and steady-state heat conduction problems in spherical solids, also lead to Legendre polynomials. Other polynomials which commonly occur in applications are the Hermite and Laguerre. They play an important role in quantum mechanics, and in probability theory.

We have started the unit by obtaining the power series solutions of Legendre's differential equation in Sec.3.2 and introduced Legendre polynomials and Legendre functions of both first and second kind. Various properties of Legendre polynomials are also discussed in this section. Polynomials solutions of Hermite's and Laguerre's equations and their properties are discussed in Sec.3.3. Applications of the Legendre and Hermite polynomials to physical situtations are discussed in Sec.3.4.

Objectives

After studying this unit you should be able to

- obtain the power series solutions of Legendre's differential equation;
- derive Rodrigue's formula, for Legendre polynomials;
- obtain Legendre polynomials through generating function;
- use recurrence relations for Legendre polynomials and its orthogonality property in various applications;
- derive Rodrigue's formula, generating function, recurrence relations and orthogonal property of Hermite and Laguerre polynomials and use them in various applications.

3.2 LEGENDRE POLYNOMIALS

The differential equation of the form

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0 \qquad (1)$$

is called **Legendre's differential equation** or simply **Legendre's equation**, where 'n' is a real constant. We can also write Legendre's equation in the form

$$\frac{d}{dx}\left\{(1-x)^2\frac{dy}{dx}\right\}+n(n+1)y=0 \qquad (2)$$

In Eqn.(1), if we divide by $(1-x^2)$ throughout then we can write the equation as

$$\frac{d^2y}{dx^2}-\frac{2x}{1-x^2}\frac{dy}{dx}+\frac{1}{1-x^2}n(n+1)y=0,\ x\neq\pm1. \qquad (3)$$

Here $-\frac{2x}{1-x^2}$ and $\frac{n(n+1)}{1-x^2}$ are analytic functions for $x=0$. Also if we use Binomial expressions, then both these converge for $|x|<1$. Hence $x=0$ is an ordinary point of Legendre's Eqn.(1) and this suggests that Eqn.(1) has a power series solution about $x=0$.

Assume the series solution

$$y(x)=\sum_{k=0}^{\infty}c_k x^k \qquad (4)$$

Differentiting Eqn.(4) w.r. to x, we get

$$y'(x)=\sum_{k=1}^{\infty}c_k k\, x^{k-1}$$

and, $$y''(x)=\sum_{k=2}^{\infty}c_k k(k-1)\, x^{k-2}$$

Substituting for $y(x), y'(x)$ and $y''(x)$ in Eqn.(1), we get

$$(1-x^2)\sum_{k=2}^{\infty}c_k k(k-1)x^{k-2}-2x\sum_{k=1}^{\infty}c_k k\, x^{k-1}+n(n+1)\sum_{k=0}^{\infty}c_k x^k=0$$

i.e., $$\sum_{k=2}^{\infty}c_k\, k(k-1)x^{k-2}-\sum_{k=2}^{\infty}c_k k(k-1)x^k-2\sum_{k=1}^{\infty}c_k kx^k+n(n+1)\sum_{k=0}^{\infty}c_k x^k=0$$

i.e., $$\sum_{k=0}^{\infty}c_{k+2}(k+2)(k+1)x^k-\sum_{k=0}^{\infty}c_{k+2}(k+2)(k+1)x^{k+2}-2\sum_{k=0}^{\infty}c_{k+1}(k+1)x^{k+1}$$

$$+n(n+1)\sum_{k=0}^{\infty}c_k x^k=0 \qquad (5)$$

Equating the coefficients of various powers of x on both sides of Eqn.(5), we get

For x^0 : $c_2(2.1)+n(n+1)c_0=0$ i.e., $c_2=-\frac{n(n+1)}{2}c_0$ (6)

For x^1 : $c_3(3.2)-2c_1+n(n+1)c_1=0$ i.e., $c_3=\frac{2-n(n+1)}{3.2}c_1$ (7)

For x^k : $c_{k+2}(k+2)(k+1)-c_k\, k(k-1)-2c_k k+n(n+1)c_k=0$

$$\text{i.e., } c_{k+2}=\frac{k(k-1)+2k-n(n+1)}{(k+2)\,(k+1)}c_k,\ \text{ for } k\geq0$$

$$\text{i.e., } c_k=\frac{(k-2)\,(k-1)-n(n+1)}{k\,(k-1)}c_{k-2},\ \text{for } k\geq2$$

$$=\frac{-[n-(k-2)\,][n+(k-1)\,]}{k\,(k-1)}c_{k-2},\ \text{ for } k\geq2 \qquad (8)$$

From recurrence relation (8), alongwith Eqns.(6) and (7), we observe that coefficients c_k for k even are multiples of c_0 while those for k odd are multiples of c_1. Thus, in general, we have

$$c_{2k} = \frac{-(n-2k+2)(n+2k-1)}{2k(2k-1)} c_{2k-2}, \text{ for } k \geq 1$$

$$= (-1)^k \frac{(n-2k+2)(n-2k+4)\dots(n-2).n(n+1)(n+3)\cdots(n+2k-1)}{(2k)!} c_0,$$

$$\text{for } k \geq 1 \qquad (9)$$

and, $$c_{2k+1} = \frac{-(n-2k+1)(n+2k)}{(2k+1)(2k)} c_{2k-1}, \text{ for } k \geq 1$$

$$= (-1)^k \frac{(n-2k+1)(n-2k+3)\dots(n-1)(n+2)(n+4)\dots(n+2k)}{(2k+1)!} c_1,$$

$$\text{for } k \geq 1 \qquad (10)$$

Substituting from relations (9) and (10) in relation (4), the power series solution of Eqn.(1) can be written as

$$y(x) = c_0\left[1 - \frac{n(n+1)}{(n)!}x^2 + \frac{(n+3)(n+1)n(n-2)}{(4)!}x^4 - \cdots\right]$$

$$+ c_1\left[x - \frac{(n+2)(n-1)}{(3)!}x^3 + \frac{(n+4)(n+2)(n-1)(n-3)}{(5)!}x^5 - \cdots\right]$$

$$= c_0\left[1 + \sum_{k=1}^{\infty}(-1)^k \frac{(n-2k+2)(n-2k+4)\cdots(n-2).n(n+1)(n+3)\dots(n+2k-1)}{(2k)!}x^{2k}\right]$$

$$= c_1\left[x + \sum_{k=1}^{\infty}(-1)^k \frac{(n-2k+1)(n-2k+3)\cdots(n-1).(n+2)(n+4)\dots(n+2k)}{(2k+1)!}x^{2k+1}\right]$$

$$= c_0\, y_1(x) + c_1\, y_2(x) \qquad (11)$$

where,

$$y_1(x) = 1 + \sum_{k=1}^{\infty}(-1)^k \frac{(n-2k+2)(n-2k+4)\dots(n-2).n(n+1)(n+3)\dots(n+2k-1)}{(2k)!}x^{2k} \qquad 12)$$

and

$$y_2(x) = x + \sum_{k=1}^{\infty}(-1)^k \frac{(n-2k+1)(n-2k+3)\dots(n-1).n(n+2)(n+4)\dots(n+2k)}{(2k+1)!}x^{2k+1} \qquad (13)$$

when n is not an integer both $y_1(x)$ and $y_2(x)$ converge for $|x|<1$.

Since c_0 and c_1 are arbitrary constants, and y_1 and y_2 are linearly independent solutions of Eqn.(1), solution (11) can be considered as the general solution of Legendre's Eqn.(1). The functions defined by Eqns.(12) and (13) are called **Legendre Functions.**

In case, n is a non-negative integer, then $y_1(x)$ reduce to a polynomial of degree n, if n even; and the same is true for $y_2(x)$ if n is odd. For example, **for n even**, we have

For $n = 0$: $y_1(x) = 1$

For $n = 2$: $y_1(x) = 1 - 3x^2$

For $n = 4$: $y_1(x) = 1 - 10x^2 + \frac{35}{3}x^4$

and so on. Thus $y_1(x)$ reduces to a polynomial of even powers. For these values of $n, y_2(x)$ remains an infinite series.

In general, for $n = 2m$, where m is an integer,

$$y_1(x) = 1 + \sum_{k=1}^{m} (-1)^k \frac{(2m-2k+2)(2m-2k+4)\ldots 2m(2m+1)(2m+3)\ldots(2m+2k-1)}{(2k)!} x^{2k} \quad (14)$$

and for $k = m+1, k = m+2, \ldots$ etc. the terms in the summation on the R.H.S. of Eqn.(14) are zero.

When n **is odd**, $y_1(x)$ remains an infinite power series and $y_2(x)$ reduces to polynomial of odd powers. For example,

For $n = 1$: $y_2(x) = x$,

For $n = 3$: $y_2(x) = x - \frac{5}{3}x^3$.

For $n = 5$: $y_2(x) = x - \frac{14}{3}x^3 + \frac{21}{5}x^5$.

In general, for $n = 2m+1$, we may obtain

$$y_2(x) = x + \sum_{k=1}^{\infty} (-1)^k \frac{(2m-2k+2)(2m-2k+4)\ldots 2m.(2m+3)(2m+5)\ldots(2m+2k+1)}{(2k+1)!} x^{2k+1} . \quad (15)$$

These polynomials multiplied by suitable constants are called **Legendre polynomials.** The Legendre polynomials are denoted by $P_n(x)$ where n denotes the order of the polynomial. Therefore, when n takes positive integral values, one of the linearly independent solutions of Eqn.(1) is a Legendre polynomial and the second solution is an infinite series. In order to explicitly write the expressions for the Legendre polynomials, we need to evaluate the multiplicative constants. The values of the multiplicative constants are obtained by setting $P_n(1) = 1$. It is too cumbersome to use the recurrence relation given by Eqn.(8) or the polynomials given above to determine the multiplicative constants. It is easy to use the Rodrigue's formula which we shall discuss now to find the expressions for the Legendre Polynomials.

3.2.1 Rodrigue's Formula

We shall now show that a polynomial of degree n, obtained by applying the binomial theorem to $(x^2-1)^n$ and differentiating it n times, is a solution of the Legendre's differential equation. The expression for $P_n(x)$ can then be obtained by using $P_n(1) = 1$.

Consider the function

$$f(x) = (x^2-1)^n . \quad (16)$$

Differentiating it w.r.t. to x, we get

$$f'(x) = n(x^2-1)^{n-1}.2x$$

$$\Rightarrow (x^2-1)f'(x) - 2n f(x)x = 0 . \quad (17)$$

Differentiating Eqn.(17), $(n+1)$ times by using the Leibnitz rule, we get

$$\left[(x^2-1)f^{(n+2)}(x) + (n+1)(2x)f^{(n+1)}(x) + \frac{n(n+1)}{2}.2\, f^{(n)}(x)\right]$$
$$- 2n[x f^{n+1}(x) + (n+1).1.f^{(n)}(x)] = 0,$$

or, $\quad (x^2-1)f^{(n+2)}(x) + 2x f^{(n+1)}(x) - n(n+1)f^{(n)}(x) = 0$,

or, $\quad (1-x^2)\frac{d^2}{dx^2}[f^{(n)}(x)] - 2x\frac{d}{dx}[f^{(n)}(x)] + n(n+1)[f^{(n)}(x)] = 0$.

This shows that $f^{(n)}(x) = \frac{d^n}{dx^n}[f(x)] = \frac{d^n}{dx^n}[(x^2-1)^n]$ satisfies Lagendre's Eqn.(1).

Thus Legendre polynomial $P_n(x)$ must be a constant multiple of $\frac{d^n}{dx^n}[(x^2-1)^n]$, i.e.,

$P_n(x) = A\frac{d^n}{dx^n}[(x^2-1)^n]$, where A is a constant. The constant A is determined by setting $P_n(1)=1$.

$$\text{Now } P_n(x) = A\,D^n[(x^2-1)^n] = A\,D^n[(x-1)^n(x+1)^n], \text{ where } D=\frac{d}{dx}$$

$$= A\sum_{r=0}^{n} {}^nC_r\,[D^{n-r}\{(x-1)^n\}].[D^r\{(x+1)^n]$$

$$= (n)!A(x+1)^n + \text{terms with } (x-1) \text{ as factor.}$$

Thus, $\quad P_n(1)=1 \Rightarrow A.(n)!.2^n = 1$

i.e., $\quad A = \frac{1}{(n)!2^n}$.

Hence, $\quad \mathbf{P_n(x) = \frac{1}{(n)!2^n}\frac{d^n}{dx^n}\{(x^2-1)^n\}}$. (18)

It is called **Rodrigue's Formula** for Legendre Polynomials.
Using the Binomial theorem we can write

$$(x^2-1)^n = \sum_{r=0}^{n} {}^nC_r(x^2)^{n-r}(-1)^r = \sum_{r=0}^{n}\frac{(-1)^r n!x^{2n-2r}}{r!(n-r)!}$$

Differentiating the above expression n times and substituting in Eqn.(18), we get

$$P_n(x) = \frac{1}{n!2^n}\sum_{r=0}^{n}\frac{(-1)^r n!}{r!(n-r)!}\frac{d^n(x^{2n-2r})}{dx^n}$$

$$= \mathbf{\frac{1}{2^n}\sum_{r=0}^{N}\frac{(-1)^r(2n-2r)!x^{n-2r}}{r!(n-r)!(n-2r)!}} \tag{19}$$

where, $N = n/2$ or $(n-1)/2$ whichever is an integer. In notation we usually write

$$N=[n/2], \text{ where } \left[\frac{n}{2}\right] = \begin{cases} n/2, & \text{if } n \text{ is even} \\ \frac{n-1}{2}, & \text{if } n \text{ is odd} \end{cases}.$$

This function $P_n(x)$ as given by Eqn.(19) is called **Legendre function of the first kind** or **Legendre polynomial of degree n.**

Example 1: Show that

$$P_n(0) = \begin{cases} (-1)^{n/2}\ \frac{1.3.5.\dots(n-1)}{2.4.\dots n}, & \text{if } n \text{ is even} \\ 0 & , \text{ if } n \text{ is odd} \end{cases}.$$

Solution: From Rodrigue's formula for Legendre polynomials ,

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left[\left(x^2-1\right)^n\right]$$

$$= \frac{1}{2^n n!}\frac{d^n}{dx^n}\left[(x+1)^n(x-1)^n\right]$$

$$= \frac{1}{2^n n!}\Big[n!(x+1)^n + {}^nC_1 n(x+1)^{n-1}(x-1)n! + {}^nC_2 n(n-1)(x+1)^{n-2}\frac{n!}{2}(x-1)^2$$

$$+\cdots+n!(x-1)^n]$$

$$= \frac{1}{2^n n!}\Big[n!(x+1)^n + ({}^nC_1)^2(x+1)^{n-1}(x-1)n! + ({}^nC_2)^2(x+1)^{n-2}(x-1)^2 n!$$

$$+\cdots+n!(x-1)^n]$$

Putting $x = 0$, we get

$$P_n(0) = \frac{1}{2^n n!} n!\left[({}^nC_0)^2 - ({}^nC_1)^2 + ({}^nC_2)^2 + \cdots + (-1)^n({}^nC_n)^2\right]$$

$$= \frac{1}{2^n n!} n!\begin{cases} 0 & , \text{ if } n \text{ is odd} \\ (-1)^{n/2}.{}^nC_{n/2}, & \text{if } n \text{ is even} \end{cases} \qquad [\because {}^nC_k = {}^nC_{n-k}]$$

$\therefore \quad P_n(0) = 0$ if n is odd

and $\quad P_n(0) = \frac{1}{2^n}(-1)^{n/2}\ {}^nC_{n/2}$, if n is even

$$= \frac{1}{2^n}(-1)^{n/2}\frac{n!}{\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}$$

$$= (-1)^{n/2}\frac{1.3.5.\ldots(n-1)}{2.4.6.\ldots n}$$

***

You may now try the following exercises.

E1) Using Rodrigue's Formula, show that

a) $\int_{-1}^{1} f(x)P_n(x)dx = \frac{(-1)^n}{2^n n!}\int_{-1}^{1}(x^2-1)^n f^{(n)}(x)dx.$

b) $\int_{-1}^{1} x^n P_n(x)dx = \frac{2^{n+1}(n!)^2}{(2n+1)!}.$

E2) Prove that all the roots of $P_n(x) = 0$ are real, distinct and lie between -1 and $+1$.

E3) Show that $\int_{-1}^{1} P_n(x)\,dx = 0$, except when $n = 0$, in which case, the value of the integral is 2.

Thus you have seen that Legendre polynomials are obtained either as polynomial solutions of Legendre's equation or by Rodrigue's formula. We shall now discuss the historical discovery of the Legendre polynomials through expansion of the **generating function** into a particular type of series. We shall show that these polynomials can also be obtained as coefficients of t^n in the expansion of $(1-2xt+t^2)^{-1/2}$.

3.2.2 Generating Function

The subject of potential theory is concerned with the forces of attraction due to the presence of a gravitational field. Central to the discussion of problems of gravitational attraction is **Newton's law of universal gravitation.**

Every particle of matter in the universe attracts every other particle with a force whose direction is that of the line joining the two, and whose magnitude is directly as the product of their masses and inversely as the square of their distance from each other.

The force field generated by a single particle is usually considered to be conservative. That is, there exists a potential function V such that the gravitational force **F** at a point of free space (i.e., free of point masses) is related to the potential function according to

$$\mathbf{F} = -\nabla V \tag{20}$$

where the minus sign is conventional. If r denotes the distance between a point mass and a point of free space, the potential function can be shown to have the form

$$V(r) = \frac{k}{r} \tag{21}$$

where k is a constant whose numerical value does not concern us. Because of spherical symmetry of the gravitational field, the potential function V depends on only the radial distance r.

To obtain Legendre's results, let us suppose that a particle of mass m is located at point P, which is 'a' units from the origin of our coordinate system (see Fig.1).

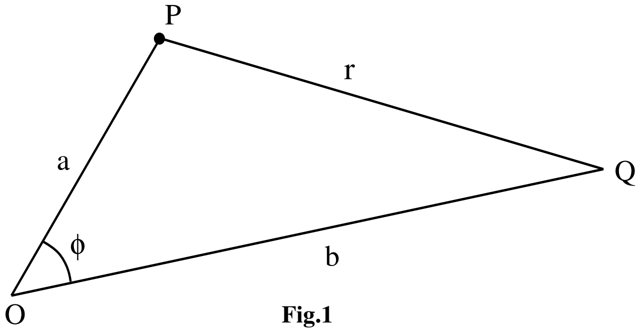

Fig.1

Let the point Q represent a point of free space r units from P and b units from the origin O. Let us assume that $b > a$, then, from the law of cosines, we find the relation

$$r^2 = a^2 + b^2 - 2ab\cos\phi \tag{22}$$

where ϕ is the central angle between segments OP and OQ. By rearranging the terms and factoring out b^2, it follows that

$$r^2 = b^2\left[1 - 2\frac{a}{b}\cos\phi + \left(\frac{a}{b}\right)^2\right], \quad a < b \tag{23}$$

We introduce the parameters

$$t = \frac{a}{b}, \quad x = \cos\phi \tag{24}$$

and taking the square root of Eqn.(23), we obtain

$$r = b\left(1 - 2xt + t^2\right)^{1/2} \tag{25}$$

Finally, substituting Eqn.(25) into Eqn.(21), we obtain the potential function

$$V = \frac{k}{b}\left(1 - 2xt + t^2\right)^{-1/2}, \quad 0 < t < 1 \tag{26}$$

We refer to the function $w(x,t) = (1 - 2xt + t^2)^{-1/2}$ as the **generating function** of the Legendre polynomials. We shall now develop $w(x,t)$ in a power series in the variable t and show that

$$\frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{n=0}^{\infty} t^n P_n(x), \qquad t \neq 1 \tag{27}$$

We write

$$(1 - 2xt + t^2)^{-1/2} = [1 - (2x - t)t]^{-1/2} = (1 - u)^{-1/2}$$

where, $u = (2x - t)t$

Expanding in a binomial series, we get

$$(1-u)^{-1/2} = 1 + \frac{1}{2}u + \frac{(1/2)(3/2)}{2!}u^2 + \frac{(1/2)(3/2)(5/2)}{3!}u^3 + \cdots$$

$$= \frac{1}{2} + \frac{2!}{(1!)2^2}u + \frac{4!}{(2!)^2 2^4}u^2 + \frac{6!}{(3!)^2 2^6}u^3 + \cdots + \frac{(2n)!}{(n!)^2 2^{2n}}u^n + \cdots$$

Substituting $u = (2x - t)t$, we get

$$(1-2xt+t^2)^{-1/2} = 1 + \frac{2!}{(1)^2 2^2}(2x-t)\,t + \frac{4!}{(2!)^2 2^4}(2x-t)^2 t^2 + \cdots$$

$$+ \frac{(2n-2r)!}{[(n-r)!]^2 2^{2n-2r}}(2x-t)^{n-r} t^{n-r} + \cdots + \frac{(2n)!}{(n!)^2 2^{2n}}(2x-t)^n t^n + \cdots \qquad (28)$$

Since we are interested in the coefficients of t^n in the above series, we consider the $(n-r)^{th}$ term of the series and expand the product $(2x-t)^{n-r}t^{n-r}$ in powers of t.

We have

$$t^{n-r}[\,{}^{n-r}C_r(-t)^r(2x)^{n-2r}\,] = \frac{(n-r)!2^{n-2r}}{r!(n-2r)!}(-1)^r t^n x^{n-2r}$$

The $(n-r)^{th}$ term of series (28) then becomes

$$\frac{(2n-2r)!}{[(n-r)!]^2 2^{2n-2r}} \cdot \frac{(n-r)!2^{n-2r}}{r!(n-2r)!}(-1)^r t^n x^{n-2r} = \frac{(-1)^r(2n-2r)!t^n x^{n-2r}}{2^n (n-r)!r!(n-2r)!}\cdots \qquad (29)$$

Summing the r.h.s of Eqn(29) for $r = 0, 1, \ldots, N$, where $N = \frac{n}{2}$ or $\frac{(n-1)}{2}$, whichever is an integer, we get on using Eqn.(19)

$$\sum_{r=0}^{N} \frac{(-1)^r(2n-2r)!x^{n-2r}}{2^n r!(n-r)!(n-2r)!}\,t^n = P_n(x)t^n \qquad (30)$$

Therefore, $(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n, t \neq 1$

Thus $P_n(x)$ can be obtained by the generating function. You may **observe** here that when n is an even number the polynomial $P_n(x)$ is an even function; and when n is odd, the polynomial is an odd function and the relation

$$P_n(-x) = (-1)^n P_n(x), \qquad (31)$$

holds for $n = 0, 1, 2, \ldots$.

In Fig.2 below, we have given the graphs of $P_n(x),\ n = 0, 1, 2, 3, 4,$ over the interval $-1 \le x \le 1$.

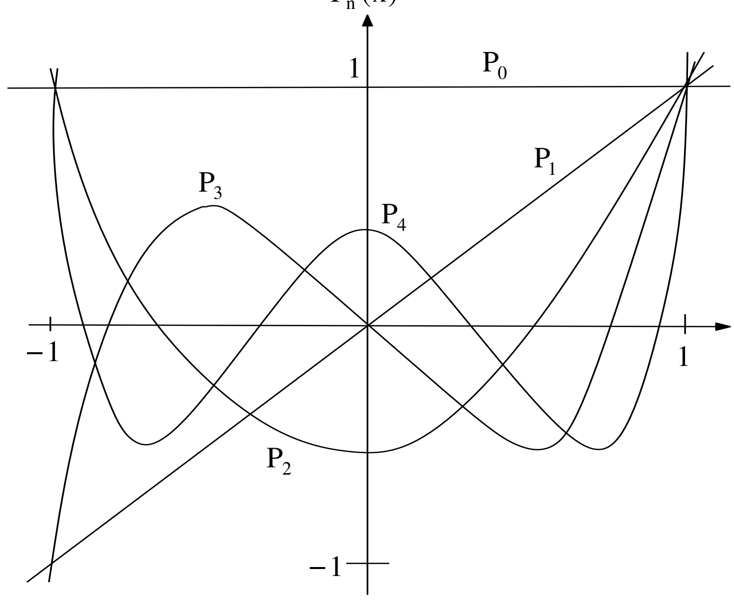

Fig.2: Graph of $P_n(x), n = 0, 1, 2, 3, 4$

Further, considering Eqn.(26) with $x = \cos\phi$ and $t = a/b$, we find that the potential function has the series expansion

$$V = \frac{k}{b}\sum_{n=0}^{\infty} P_n(\cos\phi)\left(\frac{a}{b}\right)^n, \quad a < b \tag{32}$$

In terms of the argument $\cos\phi$, the Legendre polynomials can be expressed as trigonometric polynomials of the form shown below

$$P_0(\cos\phi) = 1$$

$$P_1(\cos\phi) = \cos\phi$$

$$P_2(\cos\phi) = \frac{1}{2}(3\cos^2\phi - 1) = \frac{1}{4}(3\cos 2\phi + 1)$$

$$P_3(\cos\phi) = \frac{1}{2}(5\cos^3\phi - 3\cos\phi) = \frac{1}{8}(5\cos 3\phi + 3\cos\phi)$$

Let us now take up the following examples.

Example 2: Show that

(a) $P_n(-x) = (-1)^n P_n(x)$

(b) $P_n(1) = 1$

(c) $P_n(-1) = (-1)^n$

Solution: From Generating Function for Legendre Polynomials, we get

(a) $$\sum_{n=0}^{\infty} P_n(-x)\,h^n = (1 + 2xh + h^2)^{\frac{-1}{2}}$$

$$= \left[1 - 2x(-h) + (-h)^2\right]^{\frac{-1}{2}}$$

$$= \sum_{n=0}^{\infty} P_n(x)(-h)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n P_n(x)\,h^n$$

$$\therefore P_n(-x) = (-1)^n P_n(x)$$

(b) $$\sum_{n=0}^{\infty} P_n(1)\,h^n = (1 - 2h + h^2)^{-\frac{1}{2}}$$

$$= (1-h)^{-1}$$

$$= 1 + h + h^2 + h^3 + \cdots + h^n + \cdots$$

$$= \sum_{n=0}^{\infty} h^n$$

$$\therefore\ P_n(1) = 1.$$

(c) $$\sum_{n=0}^{\infty} P_n(-1)\,h^n = (1 + 2h + h^2)^{-\frac{1}{2}}$$

$$= (1+h)^{-1}$$

$$= \sum_{n=0}^{\infty} (-1)^n h^n$$

$$\therefore\ P_n(-1) = (-1)^n$$

***

Example 3: Using the generating function for the Legendre polynomial show that

$$P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = \frac{3x^2 - 1}{2},\ P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3).$$

Solution: From generating function for Legendre polynomials we know that

$$\sum_{n=0}^{\infty} h^n P_n(x) = (1-2hx+h^2)^{-\frac{1}{2}} = [1-h(2x-h)]^{-1/2}$$

$$= 1 + \frac{1}{2}h(2x-h) + \frac{1.3}{2.4}h^2(2x-h)^2 + \frac{1.3.5}{2.4.6}h^3(2x-h)^3 + \frac{1.3.5.7}{2.4.6.8}h^4(2x-h)^4 + \cdots$$

$$\Rightarrow P_0(x) + hP_1(x) + h^2P_2(x) + h^3P_3(x) + h^4P_4(x) + \cdots$$

$$= 1 + x.h + \frac{1}{2}(3x^2-1)h^2 + \frac{1}{2}(5x^3-3x)h^3 + \frac{1}{8}(35x^4-30x+3)h^4 + \cdots$$

Equating the coefficients of like powers of h, we get

$$P_0(x) = 1, P_1(x) = x, P_2(x) = \frac{1}{2}(3x^2-1), P_3(x) = \frac{1}{2}(5x^3-3x)$$

$$P_4(x) = \frac{1}{8}(35x^4-30x^2+3).$$

***

Example 4: Express $f(x) = x^4 + 3x^3 + 4x^2 - x + 2$ in terms of Legendre polynomials.

Solution: We have shown in Example 3, that

$$P_0(x) = 1, P_1(x) = x, P_2(x) = \frac{3x^2-1}{3},\ P_3(x) = \frac{1}{2}(5x^3-3x)$$

$$P_4(x) = \frac{1}{8}(35x^4-30x^2+3).$$

The above results can also be obtained by substituting $n = 0,1,2,3,$ and 4 in relation (19).

From these relations we can write

$$x^2 = \frac{P_0}{3} + \frac{2}{3}.P_2(x),\ \ x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}x = \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x) \text{ and}$$

$$x^4 = \frac{8}{35}P_4(x) + \frac{6}{7}x^2 - \frac{3}{35}$$

$$= \frac{8}{35}P_4(x) + \frac{4}{7}P_2(x) + \frac{1}{5}P_0$$

Thus, $x^4 + 3x^3 + 4x^2 - x + 2$

$$= \left(\frac{8}{35}P_4(x) + \frac{4}{7}P_2(x) + \frac{P_0}{5}\right) + 3\left(\frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)\right) + 4\left(\frac{2}{3}P_2(x) + \frac{P_0}{3}\right) - P_1(x) + 2P_0(x)$$

$$= \frac{8}{35}P_4(x) + \frac{6}{5}P_3(x) + \frac{68}{21}P_2(x) + \frac{4}{5}P_1(x) + \frac{53}{15}P_0(x).$$

***

You may now attempt the following exercises.

E4) Prove that

$$P_n\left(-\frac{1}{2}\right) = P_0\left(-\frac{1}{2}\right)P_{2n}\left(\frac{1}{2}\right) + P_1\left(-\frac{1}{2}\right)P_{2n-1}\left(\frac{1}{2}\right) + \cdots + P_{2n}\left(-\frac{1}{2}\right)P_0\left(\frac{1}{2}\right).$$

E5) Show that

a) $P_n'(1) = \frac{1}{2}n(n+1)$

b) $P_n'(-1) = (-1)^{n+1}\frac{1}{2}n(n+1)$

E6) Show that

a) $P_{2n+1}(0) = 0$, $n = 0,1,2,\ldots$

b) $P_{2n}(0) = \frac{(-1)^n 2n!}{2^{2n}(n!)^2}, \ n = 0, 1, 2, \ldots$

E7) Verify that

a) $P_0(\cos\phi) = 1$

b) $P_1(\cos\phi) = \cos\phi$

c) $P_2(\cos\phi) = \frac{1}{4}(3\cos 2\phi + 1)$

d) $P_3(\cos\phi) = \frac{1}{8}(5\cos 3\phi + 3\cos\phi)$

We now give some relations between Legendre polynomials of different order.

3.2.3 Recurrence Relations

We shall now establish a few useful relations involving Legendre polynomials of different order or between Legendre polynomials and their derivatives, which are called recurrence relations.
From Rodrigue's formula, we have

$$P_{n+1}(x) = \frac{1}{2^{n+1}(n+1)!} D^{(n+1)}[(x^2-1)^{n+1}], \text{ where } D = \frac{d}{dx} \tag{33}$$

Differentiating Eqn.(33) w.r. to x, we get

$$P'_{n+1}(x) = \frac{1}{2^{n+1}(n+1)!} D^{n+1}[(n+1)(x^2-1)^n . 2x] \tag{34}$$

$$= \frac{1}{2^n (n)!} D^n [(x^2-1)^n + 2nx^2(x^2-1)^{n-1}]$$

$$= \frac{1}{2^n (n)!} D^n [(x^2-1)^n + 2n(x^2-1+1)(x^2-1)^{n-1}]$$

$$= \frac{1}{2^n} \frac{1}{n!} D^n [(2n+1)(x^2-1)^n + 2n(x^2-1)^{n-1}]$$

$$= \frac{2n+1}{2^n \, n!} D^n (x^2-1)^n + \frac{1}{2^{n-1}(n-1)!} D\{D^{n-1}(x^2-1)^{n-1}\}$$

$$= (2n+1) P_n(x) + P'_{n-1}(x)$$

Thus $\mathbf{P'_{n+1}(x) = (2n+1) P_n(x) + P'_{n-1}(x)}$ (35)

Now we use Leibnitz rule on the right hand side of Eqn.(34) to obtain

$$P'_{n+1}(x) = \frac{1}{2^n (n)!}[\{D^{n+1}(x^2-1)^n\} x + (n+1)\{D^n (x^2-1)^n\}.1]$$

$$= x.D\left[\frac{1}{2^n (n)!} D^n \{(x^2-1)^n\}\right] + (n+1). \frac{1}{2^n (n)!}\{D^n (x^2-1)^n\}$$

$$= x\, P'_n(x) + (n+1) P_n(x)$$

$$\Rightarrow \qquad \mathbf{P'_{n+1}(x) = x\, P'_n(x) + (n+1) P_n(x)} \tag{36}$$

Subtracting Eqn.(36) from Eqn.(35), we get

$$\mathbf{n\, P_n(x) = x P'_n(x) - P'_{n-1}(x)} \tag{37}$$

Changing $(n+1)$ to n in relation (35) and using it in relation (37), we get

$$n\, P_n(x) = x[(2n-1) P_{n-1}(x) + P'_{n-2}(x)] - P'_{n-1}(x)$$

$$= (2n-1)\, x\, P_{n-1}(x) + [x\, P'_{n-2}(x) - P'_{n-1}(x)]$$

$$= (2n-1)\, x\, P_{n-1}(x) - (n-1) P_{n-2}(x) \quad (\text{using relation } (37))$$

We can thus write

$$\mathbf{(n+1)\,P_{n+1}(x) = (2n+1)x\,P_n(x) - n\;P_{n-1}(x)} \qquad (38)$$

which is a three-term recurrence relation.

From Eqn.(36) and (37) following relations can also be obtained

$$\mathbf{(1-x^2)\,P_n'(x) = n\{\,P_{n-1}(x) - x\;P_n(x)\;\}} \qquad (39)$$

and $$\mathbf{(1-x^2)\;P_n(x) = (n+1)\{x\;P_n(x) - P_{n+1}(x)\;\}} \qquad (40)$$

We are leaving them here for you to do it yourself.

E8) Establish relations (39) and (40).

E9) Derive the identity
$(1-x^2)\,P_n'(x) = (n+1)[x\,P_n(x) - P_{n+1}(x)]$, $n = 0, 1, 2, \ldots$.

Legendre polynomials belong to an important class of orthogonal polynomials which we shall discuss now.

3.2.4 Orthogonality Property

We shall now show that the Legendre polynomials $P_n(x)$ satisfy the following orthogonal property

$$\int_{-1}^{+1} P_m(x)\,P_n(x)\,dx = \begin{cases} 0 & ,\text{if } \; m \neq n \\ \dfrac{2}{2n+1} & ,\text{if } \; m = n \end{cases}, \qquad (41)$$

Consider the Legendre equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)\,y = 0.$$

This can be written as

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + n(n+1)y = 0 \qquad (42)$$

Legendre Polynomials $P_m(x)$ and $P_n(x)$ satisfy Eqn.(42), thus

$$\frac{d}{dx}\left[(1-x^2)\;P_n'\right] = -n(n+1)\,P_n \qquad (43)$$

and $$\frac{d}{dx}\left[(1-x^2)\;P_m'\right] = -m(m+1)\,P_m \qquad (44)$$

Multiplying Eqn.(43) by P_m and Eqn.(44) by P_n and substracting the resulting expressions, we get

$$\frac{d}{dx}\left[(1-x^2)(P_m\,P_n' - P_n P_m')\right] = \left[m(m+1) - n(n+1)\right]P_n P_m$$

Integrating both sides w.r. to x from $x = -1$ to $x = 1$, we get

$$\left|(1-x^2)(P_m P_n' - P_n P_m')\right|_{-1}^{+1} = [m(m+1) - n(n+1)]\int_{-1}^{1} P_n(x)\,P_m(x)\,dx$$

The left hand term vanishes at $x = \pm 1$

$$\therefore \qquad (n+m+1)(m-n)\int_{-1}^{+1} P_m(x)\,P_n(x)dx = 0$$

Since m and n are non-negative integers, $\therefore n+m+1 \neq 0$. Thus if $m \neq n$, i.e., $m - n \neq 0$, then from above

$$\int_{-1}^{+1} P_m(x)P_n(x)dx = 0, \quad \text{for } m \neq n.$$

To prove the property for $m = n$, we use Rodrigue's formula

$$P_n(x)=\frac{1}{(n)!2^n}\frac{d^n}{dx^n}\{(x^2-1)^n\}$$

Let $f(x)$ be any function with at least n continuous derivatives on the interval $-1\le x\le 1$ and consider the integral

$$I=\int_{-1}^{1} f(x)\,P_n(x)\,dx.$$

Using Rodrigue's formula

$$I=\frac{1}{2^n n!}\int_{-1}^{1} f(x)\frac{d^n}{dx^n}(x^2-1)^n\,dx$$

Integrating by parts, we get

$$I=\frac{1}{2^n n!}\left[f(x)\frac{d^{n-1}(x^2-1)^n}{dx^{n-1}}\right]_{-1}^{1}-\frac{1}{2^n n!}\int_{-1}^{1} f'(x)\frac{d^{n-1}(x^2-1)^n}{dx^{n-1}}\,dx$$

The first expression in the above equation on the right hand side, vanishes at both limits, so

$$I=-\frac{1}{2^n n!}\int_{-1}^{1} f'(x)\frac{d^{n-1}(x^2-1)^n}{dx^{n-1}}\,dx$$

and continuing to integrate by parts, we obtain

$$I=\frac{(-1)^n}{2^n n!}\int_{-1}^{1} f^{(n)}(x)(x^2-1)^n\,dx.$$

Now if we put $f(x)=P_n(x)$ in the above integral then since $P_n^{(n)}(x)=\dfrac{2n!}{2^n n!}$, it follows that

$$I=\int_{-1}^{1} P_n^2(x)\,dx=\frac{2n!}{2^{2n}(n!)^2}\int_{-1}^{1}(1-x^2)^n\,dx$$

$$=\frac{2(2n)!}{2^{2n}(n!)^2}\int_{0}^{1}(1-x^2)^n\,dx$$

Using the substitution $x=\sin\,\theta$, and recalling the reduction formula,

$$\int\cos^{2n+1}\theta\,d\theta=\frac{1}{2n+1}\cos^{2n}\theta\sin\theta+\frac{2n}{2n+1}\int\cos^{2n-1}\theta\,d\theta\,.$$

Then the definite integral in I becomes

$$\int_0^1(1-x^2)^n\,dx=\int_0^{\pi/2}\cos^{2n+1}\theta\,d\theta=\frac{2n}{2n+1}\int_0^{\pi/2}\cos^{2n-1}\theta d\theta$$

$$=\frac{2n}{2n+1}\,\frac{2n-2}{2n-1}\cdots\frac{2}{3}\int_0^{\pi/2}\cos\theta\,d\theta$$

$$=\frac{2^n n!}{1.3\ldots(2n-1)(2n+1)}=\frac{2^{2n}(n!)^2}{(2n)!(2n+1)}$$

Thus, we conclude that

$$I=\frac{2(2n)!}{2^{2n}(n!)^2}\int_0^1(1-x^2)^n\,dx$$

$$=\frac{2(2n)!}{2^{2n}(n!)^2}\,\frac{2^{2n}(n!)^2}{(2n)!(2n+1)}=\frac{2}{2n+1}$$

or, $$\int_{-1}^{1}P_n^2(x)\,dx=\frac{2}{2n+1} \tag{45}$$

Example 5: Assuming the orthogonality property of Legendre polynomials, show that

$$\int_{-1}^{1} (1-x^2) P_n^{(1)}(x) P_m^{(1)}(x)\, dx = \begin{cases} 0 & , \text{ if } m \neq n \\ \dfrac{2n(n+1)}{2n+1}, & \text{ if } m = n \end{cases}$$

where m and n are integers.

Solution: $\text{L.H.S.} = \int_{-1}^{1} (1-x^2) P_n^{(1)}(x)\, P_m^{(1)}(x)\, dx$

$$= \left| (1-x^2) P_m^{(1)}(x).P_n(x) \right|_{-1}^{1} - \int_{-1}^{1} P_n(x) \left[\frac{d}{dx} \left\{ (1-x^2) P_m^{(1)}(x) \right\} \right] dx$$

Since $P_m(x)$ is a solution of Legendre's equation, thus

$$\frac{d}{dx} \left[(1-x^2) P_m^{(1)}(x) \right] = -m(m+1) P_m(x)$$

$$\therefore\ \text{L.H.S.} = m(m+1) \int_{-1}^{1} P_n(x) P_m(x)\, dx = \begin{cases} 0 & , \text{ if } n \neq m \\ n(n+1) \displaystyle\int_{-1}^{1} P_n^2(x)\, dx, & \text{ if } n = m \end{cases}$$

$$= \begin{cases} 0 & , \text{ if } n \neq m \\ \dfrac{2n(n+1)}{2n+1}, & \text{ if } n = m \end{cases} \quad \text{[using result (41)]}$$

***

Many problems of potential theory depend on the possibility of expanding a given function in a series of Legendre polynomials. It is easy to see that this can always be done when the given function is itself a polynomial. For example, any third-degree polynomials $p(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3$ can be written as

$$p(x) = b_0 P_0(x) + b_1 P_1(x) + b_2 \left[\frac{1}{3} P_0(x) + \frac{2}{3} P_2(x) \right] + b_3 \left[\frac{3}{5} P_1(x) + \frac{2}{3} P_3(x) \right] \text{(see Example 4)}$$

$$= \left(b_0 + \frac{b_2}{3} \right) P_0(x) + \left(b_1 + \frac{3b_3}{5} \right) P_1(x) + \frac{2b_2}{3} P_2(x) + \frac{2b_3}{5} P_3(x)$$

$$= \sum_{n=0}^{3} a_n P_n(x)$$

More generally, since $P_n(x)$ is a polynomial of degree n for every positive integer n, it can easily be seen that x^n can always be expressed as a linear combination of $P_0(x), P_1(x), \ldots, P_n(x)$, so any polynomial $p(x)$ of degree k has an expansion of the form

$$p(x) = \sum_{n=0}^{k} a_n P_n(x) \tag{46}$$

But when the given function is not a polynomial then it may not be easy to expand an 'arbitrary' function $f(x)$ in series of the form:

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x) \tag{47}$$

We then need a new procedure for calculating the coefficients a_n in Eqn.(47), and the key lies in formula (41). We multiply Eqn.(47) by $P_m(x)$ and integrate term by term from -1 to 1 and obtain

$$\int_{-1}^{1} f(x) P_m(x)\, dx = \sum_{n=0}^{\infty} a_n \int_{-1}^{1} P_m(x) P_n(x)\, dx$$

Using formula (41), it reduces to

$$\int_{-1}^{1} f(x)P_m(x)\,dx = \frac{2a_m}{2m+1}$$

Coefficients a_n in Eqn.(47) are then obtained as

$$a_n = \left(n+\frac{1}{2}\right)\int_{-1}^{1} f(x)P_n(x)\,dx \qquad (48)$$

The above manipulations are easy to justify if $f(x)$ is known in advance to have a series expansion of the form (47) and this series is integrable term by term on the interval $-1 \le x \le 1$. Both the conditions are obviously satisfied when $f(x)$ is a polynomial. However, we shall not discuss here the conditions under which the representation of form (47) and (48) is valid for any other type of functions. For now it suffices to say that for certain functions the series (47) will converge throughout the interval $-1 \le x \le 1$, even at points of finite discontinuities of the given function. Series of this type are called **Legendre series.**

In practice, the evaluation of integrals like (48) is performed numerically. However, if the function f is not too complicated, we can sometimes use various properties of Legendre polynomials to evaluate such integrals in closed form. We illustrate this through the following example.

Example 6: Find the Legendre series for

$$f(x) = \begin{cases} -1, & -1 \le x < 0 \\ 1\ , & 0 < x \le 1 \end{cases}.$$

Solution: The given function f is an odd function. Hence $f(x)P_n(x)$ is an odd function when n is even, and in this case

$$a_n = \left(n+\frac{1}{2}\right)\int_{-1}^{1} f(x)P_n(x)\,dx = 0,\ \ n = 0, 2, 4, \ldots$$

For n odd, the product $f(x)P_n(x)$ is an even function, and therefore

$$a_n = \left(n+\frac{1}{2}\right)\int_{-1}^{1} f(x)P_n(x)\,dx$$

$$= (2n+1)\int_{0}^{1} f(x)P_n(x)\,dx = (2n+1)\int_{0}^{1} P_n(x)\,dx,\ \ n = 1, 3, 5, \ldots$$

Now let us use the identity (35)

$$P_n(x) = \frac{1}{2n+1}\left[P'_{n+1}(x) - P'_{n-1}(x)\right]$$

and set $n = 2k+1$ for $k = 0, 1, 2, \ldots$

$$a_{2k+1} = (4k+3)\int_{0}^{1} P_{2k+1}(x)\,dx$$

$$= \int_{0}^{1} [P'_{2k+2}(x) - P'_{2k}(x)]\,dx$$

$$= \left[P_{2k+2}(x) - P_{2k}(x)\right]_0^1$$

$$= P_{2k}(0) - P_{2k+2}(0) \qquad [\because P_n(1) = 1 \text{ for all } n]$$

Using the result of E6), we have

$$a_{2k+1} = \frac{(-1)^k (2k)!}{2^{2k}(k!)^2} - \frac{(-1)^{k+1}(2k+2)!}{2^{2k+2}\left[(k+1)!\right]^2}$$

$$= \frac{(-1)^k (2k)!}{2^{2k} (k!)^2} \left[1 + \frac{(2k+2)(2k+1)}{2^2 (k+1)^2} \right]$$

$$= \frac{(-1)^k (2k)!}{2^{2k} (k!)^2} \left(1 + \frac{2k+1}{2k+2} \right) = \frac{(-1)^k (2k)!(4k+3)}{2^{2k+1} k!(k+1)!}$$

and thus

$$f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (2k)!(4k+3)}{2^{2k+1} k!(k+1)!} P_{2k+1}(x) \ , \quad -1 \le x \le 1 \, .$$

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You may try a few exercises now.

E10) If $f(x)$ is a polynomial of degree less then n, prove that

$$\int_{-1}^{1} f(x) P_n(x) \, dx = 0 \, .$$

E11) Express $f(x) = x^3 + a x^2 + b x + c$ as a linear combination of Legendre polynomials.

E12) Show that $(2n+1)(x^2 - 1) P_n' = n(n+1)\ (P_{n+1} - P_{n-1})$.

E13) Prove that $\displaystyle\int_{-1}^{1} \frac{P_n(x)}{\sqrt{1 - 2x\,t + t^2}} \, dx = \frac{2t^n}{2n+1}$.

E14) Prove that $1 + \frac{1}{2} P_1(\cos\theta) + \frac{1}{3} P_2(\cos\theta) + \cdots = \ln \left\{ \left(1 + \sin\frac{\theta}{2} \right) \Big/ \sin\frac{\theta}{2} \right\}$.

E15) Make the change of variable $x = \cos\phi$ in the DE

$$\frac{1}{\sin\phi} \frac{d}{d\phi} \left(\sin\phi \frac{dy}{d\phi} \right) + n(n+1) y = 0$$

and show that it reduces to Legendre's Eqn.(1).

E16) Determine the values of n for which $y = P_n(x)$ is a solution of

a) $(1 - x^2) y'' - 2xy' + n(n+1) y = 0, \ y(0) = 0, \ y(1) = 1$

b) $(1 - x^2) y'' - 2xy' + n(n+1) y = 0, \ y'(0) = 0, \ y(1) = 1$.

3.2.5 Legendre Function of the Second Kind

We have shown that for positive integral values of n the Legendre polynomials $P_n(x)$ represents one of the linearly independent solutions of Legendre's Eqn.(1), viz.,

$$(1 - x^2) y'' - 2xy' + n(n+1) y = 0 \quad .$$

Because the equation is of second order, we know from the theory of differential equations that there exists a second linearly independent solution $Q_n(x)$ (say), such that

$$y = c_1 P_n(x) + c_2 Q_n(x) \tag{49}$$

where c_1 and c_2 are arbitrary constants, is a general solution of Eqn.(1).

Also from the theory of second-order linear differential equations it is well known that if $y_1(x)$ is a non-trivial solution of

$$y'' + a(x) y' + b(x) y = 0 \tag{50}$$

then a second linearly independent solution can be defined by

$$y_2(x) = y_1(x) \int \frac{\exp\left[-\int a(x) \, dx \right]}{y_1^2(x)} \, dx \quad \text{[ref. Unit 7, MTE-08].} \tag{51}$$

Hence, if we express Eqn.(1) in the form

$$y'' - \frac{2x}{1-x^2}y' + \frac{n(n+1)}{(1-x^2)}y = 0$$

and let $y_1(x) = P_n(x)$, it follows that

$$y_2(x) = P_n(x)\int \frac{dx}{(1-x^2)\,[P_n(x)]^2} \tag{52}$$

is a second solution, linearly independent of $P_n(x)$. Because any linear combination of solutions is also a solution of a homogeneous differential equation, we defined the second solution of Eqn.(1) as follows:

$$Q_n(x) = P_n(x)\left\{A_n + B_n\int \frac{dx}{(1-x^2)\,[P_n(x)]^2}\right\}, \tag{53}$$

where A_n and B_n are constants to be chosen for each n. We refer to $Q_n(x)$ as the **Legendre function of the second kind** of integral order.

Accordingly, when $n = 0$, we choose $A_0 = 0$ and $B_0 = 1$, and hence

$$Q_0(x) = \int \frac{dx}{1-x^2}$$

which leads to

$$Q_0(x) = \frac{1}{2}\ln\frac{1+x}{1-x}, \quad |x| < 1 \tag{54}$$

For $n = 1$, we set $A_1 = 0$ and $B_1 = 1$, from which we obtain

$$Q_1(x) = x\int \frac{dx}{(1-x^2)x^2}$$

$$= x\int \left(\frac{1}{1-x^2} + \frac{1}{x^2}\right) dx$$

$$= \frac{1}{2}x\ln\frac{1+x}{1-x} - 1 \tag{55}$$

or, $\quad Q_1(x) = x\,Q_0(x) - 1,\ |x| < 1 \qquad (56)$

If we continue in this fashion then we will have to evaluate more difficult integrals. Since it is assumed that all solutions of Legendre's equation satisfy the recurrence formulas for $P_n(x)$, hence we select the Legendre functions $Q_n(x)$ so that,

$$Q_{n+1}(x) = \frac{2n+1}{n+1}x\,Q_n(x) - \frac{n}{n+1}Q_{n-1}(x) \tag{57}$$

for $n = 1, 2, 3, \ldots$ with $Q_0(x)$ and $Q_1(x)$ already defined. The substitution $n = 1$ into Eqn.(57) yields

$$Q_2(x) = \frac{3}{2}x\,Q_1(x) - \frac{1}{2}Q_0(x)$$

$$= \frac{1}{2}(3x^2 - 1)\,Q_0(x) - \frac{3}{2}x$$

or, $\quad Q_2(x) = P_2(x)\,Q_0(x) - \frac{3}{2}x,\ |x| < 1 \qquad \left[\because P_2(x) = \frac{1}{2}(3x^2 - 1)\right] \qquad (58)$

For $n = 2$, we find

$$Q_3(x) = P_3(x)\,Q_0(x) - \frac{5}{2}x^2 + \frac{2}{3}, \ |x| < 1 \tag{59}$$

whereas in general, we state the result

$$Q_n(x) = P_n(x)\,Q_0(x) - \sum_{k=0}^{[(n-1)/2]} \frac{(2n-4k-1)}{(2k+1)(n-k)}P_{n-2k-1}(x),\ |x| < 1 \tag{60}$$

for $n = 1, 2, 3, \ldots$.

Because of the logarithmic term in $Q_0(x)$, $Q_n(x)$ has infinite discontinuities at $x = \pm 1$. However, within the interval $-1 < x < 1$, these functions are well defined.

Without giving the details, we state that the Legendre functions $Q_n(x)$ satisfy all recurrence relations given in Sec.3.2.3 for $P_n(x)$. In addition, there are several relations that directly involve both $P_n(x)$ and $Q_n(x)$. For example, if $|t| < |x|$, then

$$\frac{1}{x-t} = \sum_{n=0}^{\infty} (2n+1)\, P_n(t)\, Q_n(x) \tag{61}$$

From this result, it can be easily shown that

$$Q_n(x) = \frac{1}{2}\int_{-1}^{1} \frac{P_n(t)}{x-t}\, dt, \; n = 0, 1, 2, \ldots \tag{62}$$

which is called the **Neumann formula.**

You may now try the following exercises.

E17) Verify result (62).

E18) Find the general solution of the following differential equations in terms of $P_n(x)$ and $Q_n(x)$

a) $(1-x^2)\,y'' - 2xy' = 0$

b) $(1-x^2)\,y'' - 2xy' + 12y = 0$

c) $(1-x^2)\,y'' - 2xy' + 2y = 0$

d) $(1-x^2)\,y'' - 2xy' + 30y = 0$.

E19) Given $P_0(x) = 1$ and $Q_0(x) = \frac{1}{2}\ln\frac{1+x}{1-x^2}$, find $W[P_0(x), Q_0(x)]$ where , W is the wronskian.

E20) Using Eqn.(53), show that the wronskian of $P_n(x)$ and $Q_n(x)$ is given by

$$W\left(P_n(x), Q_n(x)\right) = \frac{B_n}{1-x^2}\;, \; n = 0, 1, 2, \ldots.$$

Other polynomials which commonly occur in applications are the Hermite and Laguerre polynomials which we shall be discussing now.

3.3 HERMITE AND LAGUERRE POLYNOMIALS

Let us first consider the Hermite polynomials.

Hermite Polynomials

The Hermite polynomials play an important role in problems involving Laplace's equations in cylinderical coordinates, in various problems in quantum mechanics and in probability theory.

The Hermite polynomials are polynomial solutions to **Hermite's equation**

$$y'' - 2xy' + 2n\, y = 0 \tag{63}$$

where n is a constant.

Using the power series method it can be easily shown that $y(x) = a_0 y_1(x) + a_1 y_2(x)$ is the general solution of Eqn.(63), where

$$y_1(x) = 1 - \frac{2n}{2!}x^2 + 2^2 n\frac{(n-2)}{4!}x^4 - \frac{2^3 n(n-2)\,(n-4)}{6!}x^6 + \cdots \tag{64}$$

and

$$y_2(x) = x - \frac{2(n-1)}{3!}x^3 + \frac{2^2(n-1)\,(n-3)}{5!}x^5 - \frac{2^3(n-1)(n-3)(n-5)}{7!}x^7 + \cdots \tag{65}$$

and both the series converge for all x. We shall not be going into the details of the series solution here and leave it for you to verify it yourself.

If n is a non-negative integer, then one of these series terminates and is thus a polynomial- $y_1(x)$ if n is even, and $y_2(x)$ if n is odd, while the other remains an infinite series. It can be easily verified that for $n = 0, 1, 2, 3, 4, 5,$ these polynomials are

$$1, x, 1-2x^2, x-\frac{2}{3}x^3, 1-4x^2+\frac{4}{3}x^4, x-\frac{4}{3}x^3+\frac{4}{15}x^5, \text{ respectively.}$$

The polynomial solutions of Hermite's Eqn.(63) are constant multiples of these polynomials. The constant multiples with the property that the terms containing the highest powers of x are of the form $2^n x^n$ are denoted by $H_n(x)$ and called the **Hermite polynomials.**

Let us now define the Hermite polynomials

Generating Function

We define the Hermite polynomials $H_n(x)$ by means of the relation

$$\mathbf{e^{(2xt-t^2)} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}}, \quad \text{for all finite } x \text{ and } t. \tag{66}$$

The function on the left hand side is called the generating function of the Hermite polynomials. This definition of Hermite polynomials is used often in statistical applications.

We have

$$e^{(2xt-t^2)} = e^{2xt}.e^{-t^2} = \left[\sum_{m=0}^{\infty}\frac{(2xt)^m}{m!}\right]\left[\sum_{k=0}^{\infty}\frac{(-t^2)^k}{k!}\right]$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]}\frac{(-1)^k (2x)^{n-2k}}{k!\,(n-2k)!}\, t^n \tag{67}$$

where the last step follows from the index change $m = n-2k$ and $[n/2]$ is the standard notation to the greatest integer $\le n/2$. From Eqns.(66) and (67) it follows that

$$\mathbf{H_n(x) = \sum_{k=0}^{[n/2]}\frac{(-1)^k\, n!}{k!(n-2k)!}(2x)^{n-2k}} \tag{68}$$

where $\left[\frac{n}{2}\right] = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ \frac{n-1}{2}, & \text{if } n \text{ is odd} \end{cases}$

Eqn.(68) shows that $H_n(x)$ is a polynomials of degree n in x and that

$$H_n(x) = 2^n x^n + \pi_{n-2}(x),$$

where $\pi_{n-2}(x)$ is a polynomial of degree $(n-2)$ in x. Further, it follows that it is an even function of x for even n and an odd function of x for odd n. Thus, it follows that

$$\mathbf{H_n(-x) = (-1)^n H_n(x)} \tag{69}$$

The first few Hermite polynomials are listed in Table-1.

Table 1

$H_0(x) = 1$
$H_1(x) = 2x$
$H_2(x) = 4x^2 - 2$
$H_3(x) = 8x^3 - 12$
$H_4(x) = 16x^4 - 48x^2 + 12$
$H_5(x) = 32x^5 - 160x^3 + 120x$

In addition to series (68), the Hermite polynomials can be defined by the Rodrigues formula also.

The Rodrigue's Formula

We prove the formula $H_n(x) = (-1)^n e^{x^2} \dfrac{d^n}{dx^n}(e^{-x^2})$ (70)

We have the relation

$$e^{2tx-t^2} = e^{x^2-(t-x)^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n$$

If we examine this relation in the light of Maclaurin's theorem, we see that

$$\frac{\partial^n}{\partial t^n}(e^{2tx-t^2})\Big|_{t=0} = e^{x^2} \frac{\partial^n}{\partial t^n}(e^{-(t-x)^2})\Big|_{t=0} = H_n(x)$$

or, $$\frac{\partial^n}{\partial t^n}[e^{-(t-x)^2}]\Big|_{t=0} = e^{-x^2} H_n(x)$$

If we introduce a new variable $z = x - t$ and use the fact that $\dfrac{\partial}{\partial t} = -\dfrac{\partial}{\partial z}$, then since $t = 0$ corresponds to $z = x$, the above expression reduces to

$$(-1)^n \frac{d^n}{dz^n}(e^{-z^2})\Big|_{z=x} = (-1)^n \frac{d^n(e^{-x^2})}{dx^n} = e^{-x^2} H_n(x)$$

$$\therefore \mathbf{H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2})}$$

which proves the Rodrigue's formula (70).

Hermite polynomials satisfy a number of properties, we list some of these here without proving them. You can verify them yourself.

Recurrence Relations

Using the generating function for Hermite polynomials it can be shown that

$$\mathbf{x H_n'(x) = n\ H_{n-1}'(x) + n\ H_n(x)} \tag{71}$$

The relation

$$e^{(2xt-t^2)} = \sum_{n=0}^{\infty} \frac{H_n(x)t^n}{n!} \tag{72}$$

on differentiating with respect to x, yields

$$2t\, e^{(2xt-t^2)} = \sum_{n=0}^{\infty} \frac{H_n'(x)\, t^n}{n!} \tag{73}$$

From Eqns.(72) and (73), we get

$$\sum_{n=0}^{\infty} \frac{2H_n(x)\, t^{n+1}}{n!} = \sum_{n=0}^{\infty} \frac{H_n'(x)\ t^n}{n!}$$

which with a shift of index on the left, yields $\mathbf{H_0'(x) = 0}$, and for $n \geq 1$,

$$\mathbf{H_n'(x) = 2n\, H_{n-1}(x)} \tag{74}$$

Combining relations (71) and (74), we obtain

$$\mathbf{H_n(x) = 2x\ H_{n-1}(x) - H_{n-1}'(x)} \tag{75}$$

Further, using relations (74) and (75), we obtain the relations

$$\mathbf{H_n(x) = 2x\ H_{n-1}(x) - 2(n-1)\ H_{n-2}(x)} \tag{76}$$

and Hermite's differential equation

$$\mathbf{H_n''(x) - 2x\ H_n'(x) + 2n\ H_n(x) = 0} \tag{77}$$

Relation (77) can be used to prove the orthogonality property of Hermite polynomials.

Orthogonality

The Hermite polynomials $H_n(x)$ satisfy the following orthogonality property

$$\int_{-\infty}^{\infty} \mathbf{e^{-x^2}\ H_n(x)\ H_m(x)dx} = \begin{cases} \mathbf{0\ ,} & \mathbf{if\ \ m \neq n} \\ \mathbf{2^n n!\sqrt{\pi}\ ,} & \mathbf{if\ \ m = n} \end{cases} \tag{78}$$

Eqn.(77) can be written in the form

$$[e^{-x^2}\ H'_n(x)]' + 2n\,e^{-x^2}\ H_n(x) = 0 \tag{79}$$

Similarly, for a polynomial $H_m(x)$, we can write

$$[e^{-x^2} H'_m(x)]' + 2m\,e^{-x^2} H_m(x) = 0 \tag{80}$$

Multiplying Eqn.(79) by $H_m(x)$ and Eqn.(80) by $H_n(x)$ and adding the two, we get

$$\begin{aligned} 2(n-m)\,e^{-x^2} H_n(x)H_m(x) &= H_n(x)[e^{-x^2} H'_m(x)]' - H_m(x)[e^{-x^2} H'_n(x)]' \\ &= [e^{-x^2}\{H_n(x)\ H'_m(x) - H'_n(x)\ H_m(x)\}]' \end{aligned}$$

Therefore, it follows that

$$2(n-m)\int_{-\infty}^{\infty} e^{-x^2} H_n(x)\ H_m(x)\,dx = [e^{-x^2}\{H_n(x)H'_m(x) - H'_n(x)H_m(x)\}]_{-\infty}^{\infty} \tag{81}$$

Since the product of any polynomial in x by $e^{-x^2} \to 0$ as $x \to \infty$ or as $x \to -\infty$, we conclude that

$$\int_{-\infty}^{\infty} e^{-x^2}\ H_n(x)\ H_m(x)\,dx = 0,\ \ m \neq n \tag{82}$$

This proves the first part of formula, that is, the Hermite polynomials form an orthogonal set over the interval $]-\infty,\ \infty[$ with weight function e^{-x^2}.

To establish the second part of (78), we use Rodrigue's formula (70) and integrate

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x)\ H_n(x)\ dx = (-1)^n \int_{-\infty}^{\infty} H_n(x)\frac{d^n e^{-x^2}}{dx^n}\ dx$$

by parts with $u = H_n(x),\ \ du = H'_n(x)\,dx$

$$dv = \frac{d^n}{dx^n}(e^{-x^2})\,dx,\quad v = \frac{d^{n-1}}{dx^{n-1}}(e^{-x^2})$$

Since uv is the product of e^{-x^2} and a polynomial, it vanishes at both limits and

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-x^2}[H_n(x)]^2 dx &= (-1)^{n+1}\int_{-\infty}^{\infty} H'_n(x)\frac{d^{n-1}}{dx^{n-1}}(e^{-x^2})\ dx \\ &= (-1)^{n+2}\int_{-\infty}^{\infty} H''_n(x)\frac{d^{n-2}}{dx^{n-2}}(e^{-x^2})\,dx \\ &= \cdots\cdots = (-1)^{2n}\int_{-\infty}^{\infty} H_n^{(n)}(x)\ \ e^{-x^2}\ \ dx \end{aligned}$$

Now the term containing the highest power of x in $H_n(x)$ is $2^n x^n$, so $H_n^{(n)}(x) = 2^n\ n!$ and the last integral is

$$2^n\,n!\int_{-\infty}^{\infty} e^{-x^2}\ dx = 2^n\,n!\,2\int_{0}^{\infty} e^{-x^2}\ dx = 2^n\ n!\sqrt{\pi}$$

which gives the desired result

$$\int_{-\infty}^{\infty} e^{-x^2}\ \ H_n^2(x)\ dx = 2^n\ n!\sqrt{\pi}$$

These orthogonality properties can be used to expand an arbitrary function $f(x)$ in a series

$$\mathbf{f(x)=\sum_{n=0}^{\infty} a_n\ H_n(x)} \tag{83}$$

The coefficients a_n can be found by multiplying Eqn.(83) by $e^{-x^2}H_m(x)$ and integrating term by term from $-\infty$ to ∞. Using formula (78), this gives

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x)\ f(x)\ dx = \sum_{n=0}^{\infty} a_n \int_{-\infty}^{\infty} e^{-x^2} H_m(x)\ H_n(x)\, dx = a_m 2^m m! \sqrt{\pi}$$

Replacing m by n, in the above equation, we get

$$\mathbf{a_n = \frac{1}{2^n n! \sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} H_n(x)\ f(x)\ dx} \tag{84}$$

The series (83), with its coefficients calculated by formula (84), is called the **Hermite series** and is useful in the theory of orthogonal functions.

Let us consider the following example.

Example 7: Express $f(x)=e^{2bx}$ in a Hermite series, and use this result to deduce the value of the integral

$$\int_{-\infty}^{\infty} e^{-x^2+2bx}\ H_n(x)\ dx\,.$$

Solution: Set $t=b$ in the generating function (66) to obtain

$$e^{2bx-b^2} = \sum_{n=0}^{\infty} \frac{b^n}{n!} H_n(x)$$

We can then write the series

$$f(x)=e^{2bx}=e^{b^2}\sum_{n=0}^{\infty} \frac{b^n}{n1} H_n(x) = \sum_{n=0}^{\infty} a_n H_n(x)\text{, where, } a_n = \frac{b^n}{n!} e^{b^2}$$

Also, from formula (84), we obtain

$$a_n = \frac{1}{2^n n! \sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2+2bx} H_n(x)\, dx, \quad n=0,1,2,\ldots$$

Thus, it follows that

$$\int_{-\infty}^{\infty} e^{-x^2+2bx} H_n(x)\, dx = \sqrt{\pi}\, 2^n \frac{n! b^n e^{b^2}}{n!} = \sqrt{\pi}\,(2b)^n e^{b^2}, \quad n=0,1,2,\ldots$$

***

You may now try the following exercises.

E21) Verify that $y_1(x)$ and $y_2(x)$ given by Eqns.(64) and (65) respectively, are two linearly independent solutions of Hermite's Eqn.(63).

E22) Use the generating function for the Hermite polynomials to find $H_0(x)$, $H_1(x)$, $H_2(x)$ and $H_3(x)$.

E23) Prove the following relations for Hermite polynomials $H_n(x)$.

a) $x H_n'(x) = n\ H_{n-1}'(x) + n\ H_n(x)$

b) $H_n(x) = 2x\ H_{n-1}(x) - 2(n-1)\ H_{n-2}\ (x)$

c) $H_n''(x) - 2x\ H_n'(x) + 2n\ H_n(x) = 0$

E24) Show that

a) $H_{2n}(0) = (-1)^n \dfrac{2n!}{n!}$ b) $H_{2n+1}(0) = 0$

c) $H'_{2n}(0) = 0$ d) $H'_{2n+1}(0) = (-1)^n \dfrac{(2n+2)!}{(n+1)!}$

E25) Use the generating function (66) to derive the relation

$$x^n = \sum_{k=0}^{[n/2]} \frac{n!H_{n-2k}(x)}{2^n\, k!(n-2k)!}, \quad n = 0, 1, 2, \ldots$$

E26) Expand $f(x) = x^3 - 3x^2 + 2x$ in a series of the form $\sum_{n=0}^{\infty} a_n \ H_n(x)$.

Let us now consider the Laguerre polynomials.

Laguerre Polynomials

The Laguerre polynomials $L_n(x)$ are polynomial solutions to Laguerre's equation

$$xy'' + (1-x)\,y' + ny = 0 \tag{85}$$

where n is a constant. These polynomial solutions are obtained when n is a non-negative integer. An important application involving Laguerre polnomials in quantum mechanics is to find the wave function associated with the electron in a hydrogen atom.

We now define the Laguerre polynomials and list some of their useful properties.

We shall not be proving these properties here. We shall leave them for you to verify yourself.

Generating Function

We define the Laguerre polynomials $L_n(x)$ by means of the relations

$$\mathbf{(1-t)^{-1} e^{\frac{-xt}{1-t}} = \sum_{n=0}^{\infty} L_n(x) t^n, \ \ |t| < 1, \ \ 0 \le x \le \infty} \tag{86}$$

The function on the left hand side is called the generating function of the Laguerre polynomials.

We have

$$\begin{aligned}(1-t)^{-1} e^{\frac{-xt}{1-t}} &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} (xt)^k (1-t)^{-k-1} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} (xt)^k \sum_{m=0}^{\infty} \binom{-k-1}{m} (-1)^m \ t^m \end{aligned} \tag{87}$$

But since $\dbinom{-k-1}{m} = \dfrac{(-k-1)(-k-2)(-k-3)\ldots(-k-m)}{m!}$

$$= (-1)^m \frac{(k+1)(k+2)\ldots(k+m)}{m!} = (-1)^m \binom{k+m}{m},$$

Eqn.(87) becomes

$$(1-t)^{-1} e^{\frac{-xt}{1-t}} = \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k (k+m)!\, x^k}{(k!)^2\, m!} \ t^{k+m} \tag{88}$$

where we have reversed the order of summation.

Using the change of index $m = n - k$ in Eqn.(88) and comparing with Eqn.(86), the Laguerre polynomials are defined by

$$\mathbf{L_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k\, n!\, x^k}{(k!)^2 (n-k)!}} \tag{89}$$

In Table-2, we have listed the first few polynomials $L_n(x)$

Table 2

$L_0(x) = 1$
$L_1(x) = -x + 1$
$L_2(x) = \frac{1}{2!}(x^2 - 4x + 2)$
$L_3(x) = \frac{1}{3!}(-x^3 + 9x^2 - 18x + 6)$
$L_4(x) = \frac{1}{4!}(x^4 - 16x^3 + 72x^2 - 96x + 24)$

The Rodrigue's Formula

The Rodrigue's formula for Laguerre polynomials $L_n(x)$ is given by

$$\mathbf{L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n}(x^n e^{-x}), \quad n = 0, 1, 2, \ldots} \tag{90}$$

which can be verified by the application of the Leibniz formula

$$\frac{d^n}{dx^n}(fg) = \sum_{k=0}^{n} \binom{n}{k} \frac{d^{n-k} f}{dx^{n-k}} \frac{d^k g}{dx^k}, \quad n = 1, 2, 3, \ldots \tag{91}$$

Recurrence Relations

We now list a few important properties satisfied by Laguerre polynomials

$$\mathbf{(n+1) L_{n+1}(x) + (x - 1 - 2n) L_n(x) + n\, L_{n-1}(x) = 0} \tag{92}$$

$$\mathbf{L_n'(x) - L_{n-1}'(x) + L_{n-1}(x) = 0} \tag{93}$$

$$\mathbf{x\, L_n'(x) = n L_n(x) - n\, L_{n-1}(x)} \tag{94}$$

and

$$\mathbf{x L_n''(x) + (1 - x)\, L_n'(x) + n\, L_n(x) = 0} \tag{95}$$

Relation (95) can be used to prove the following orthogonality property of Laguerre polynomials.

Orthogonality

The Laguerre polynomials $L_n(x)$ satisfy the orthogonality property

$$\mathbf{\int_0^{\infty} e^{-x} L_n(x) L_m(x)\, dx = 0, \text{ if } m \neq n} \tag{96}$$

You may now try the following exercises.

E27) Determine the Laguerre polynomials $L_0(x), L_1(x), L_2(x)$ and $L_3(x)$.

E28) Prove the recurrence relations (92) – (95).

E29) Prove that $\int_0^{\infty} e^{-x} L_n(x)\, L_m(x)\, dx = 0$, if $m \neq n$.

We shall now discuss some applications of these polynomials to physical situations.

3.4 APPLICATIONS TO PHYSICAL SITUATIONS

We shall first illustrate the role played by Legendre polynomials in solving certain boundary value problems of mathematical physics.

Steady-state temperature in a sphere

Consider a solid sphere of radius a. Let origin be its centre. Let the surface of the sphere be held at a prescribed temperature say, $g(\phi)$, which is assumed to be independent of coordinate θ, until within the sphere a steady state for temperature $T(\rho,\phi)$ is produced by the flow of heat. Here (ρ,θ,ϕ) are the spherical polar coordinates and we wish to find the steady-state temperature $T(\rho,\phi)$ which satisfies Laplace equation.

You may recall that Laplace equation for any function $T(x,y,z)$ is given by

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}=0.$$

and in the spherical polar coordinates (ρ,θ,ϕ) where ,
$x=\rho\sin\phi\cos\theta,\ y=\rho\sin\phi\sin\theta,\ z=\rho\cos\phi$, it reduces to

$$\frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial T}{\partial\rho}\right)+\frac{1}{\rho^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial T}{\partial\phi}\right)+\frac{1}{\rho^2\sin^2\phi}\frac{\partial^2 T}{\partial\theta^2}=0 \tag{97}$$

In the given situation since T does not depend on θ, Eqn.(97) reduces to

$$\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial T}{\partial\rho}\right)+\frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial T}{\partial\phi}\right)=0 \tag{98}$$

We have to find the solution of Eqn(98) subject to the boundary condition

$$T(a,\phi)=g(\phi) \tag{99}$$

To solve Eqn.(98) , we use the method of separation of variable and take the product solution of the form

$$T(\rho,\phi)=R(\rho)\,P(\phi) \tag{100}$$

Using Eqn.(100), Eqn.(98) becomes

$$P\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right)+R\frac{1}{\sin\phi}\frac{d}{d\phi}\left(\sin\phi\frac{dP}{d\phi}\right)=0$$

$$\Rightarrow \frac{1}{R}\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right)=-\frac{1}{P}\frac{1}{\sin\phi}\frac{d}{d\phi}\left(\sin\phi\frac{dP}{d\phi}\right) \tag{101}$$

The left side of Eqn.(101) is a function of ρ only whereas right hand side of Eqn.(101) is a function of ϕ only. This is possible only when each side is equal to the same constant, say . Then Eqn.(101) is equivalent to the equations.

$$\rho^2\frac{d^2R}{d\rho^2}+2\rho\frac{dR}{d\rho}-\lambda R=0 \tag{102}$$

and, $$\frac{1}{\sin\phi}\frac{d}{d\phi}\left(\sin\phi\frac{dP}{d\phi}\right)+\lambda P=0 \tag{103}$$

Eqn.(102) is an Euler equation, its auxiliary equation is

$$m(m-1)+2m-\lambda=0,\ \text{or},\ m^2+m-\lambda=0.$$

$$\Rightarrow m=\frac{-1\pm\sqrt{1+4\lambda}}{2}.$$

General solution of Eqn.(102) can be written as

$$R=c_1\,\rho^{\frac{-1}{2}+\sqrt{\frac{1}{4}+\lambda}}+c_2\rho^{\frac{-1}{2}-\sqrt{\frac{1}{4}+\lambda}} \tag{104}$$

In order that R is single-valued and bounded near $\rho = 0$, we put in Eqn.(104)

$c_2 = 0$ and $-\frac{1}{2} + \sqrt{\quad + \frac{1}{4}} = n$, where n is a non-negative integer. It then follows that $= n(n+1)$ and Eqn.(104) reduces to

$$R = c_1 \rho^n \qquad (105)$$

and Eqn.(103) becomes $\frac{d^2P}{d\phi^2} + \frac{\cos\phi}{\sin\phi}\frac{dP}{d\phi} + n(n+1)P = 0$.

In this equation if we let $x = \cos\phi$, then we get

$$(1-x^2)\frac{d^2P}{dx^2} - 2x\frac{dP}{dx} + n(n+1)P = 0 \qquad (106)$$

which is Legendre's equation.

By the physics of the problem, the function P must be bounded for $0 \le \phi \le \pi$ i.e., $-1 \le x \le 1$ and we know that, the solution of Eqn.(106) are constant multiples of Legendre polynomials, $P_n(x)$. If we combine this result with Eqn.(105), then we have particular solution of Eqn.(98) of the form

$$\alpha_n \rho^n P_n(\cos\phi), \text{ for each } n = 0, 1, 2, 3\ldots$$

where α_n are arbitrary constants.

The general solution of Eqn.(98) can be written as

$$T(\rho, \phi) = \sum_{n=0}^{\infty} \alpha_n \rho^n P_n(\cos\phi) \qquad (107)$$

The boundary condition (99) now requires

$$g(\phi) = \sum_{n=0}^{\infty} \alpha_n a^n P_n(\cos\phi)$$

or equivalently,

$$g(\cos^{-1} x) = \sum_{n=0}^{\infty} \alpha_n a^n P_n(x)$$

Multiplying both sides of the above equation by $P_m(x)$ and integrating w.r.t. to x from $x = -1$ to $x = 1$ and using orthogonality property of Legendre polynomials, we get

$$\alpha_n = \frac{1}{a^n}\left(n + \frac{1}{2}\right)\int_{-1}^{1} g(\cos^{-1} x) P_n(x)\, dx \qquad (108)$$

Hence relation (107) gives the required solution where coefficients are given by relation (108).

Let us look at another application of Legendre polynomials.

Electrostatic Dipole Potential

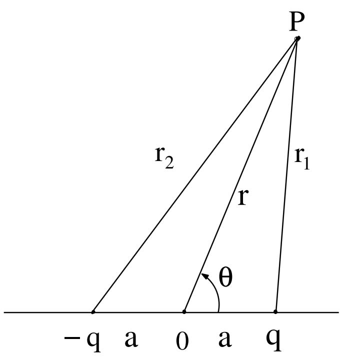

Fig.3

Consider two point charges of equal magnitude, say q, but of opposite sign.

Let these charges be placed in a polar coordinate system (see Fig.3). With suitable units of measurement, the potential U at the point P is

$$U = \frac{q}{r_1} - \frac{q}{r_2} \qquad (109)$$

where, $r_1 = \sqrt{r^2 + a^2 - 2ar\cos\theta}$

and, $r_2 = \sqrt{r^2 + a^2 + 2ar\cos\theta}$

From generating function relation for the Legendre's polynomials, we have

$$(1 - 2xt + t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(x) \qquad (110)$$

when $r > a$, we can use Eqn.(110) to write

$$\frac{1}{r_1} = \frac{1}{r}\frac{1}{\sqrt{1-2(a/r)\cos\theta+(a/r)^2}} = \frac{1}{r}\sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{a}{r}\right)^n \tag{111}$$

and, similarly,

$$\frac{1}{r_2} = \frac{1}{r}\frac{1}{\sqrt{1-2(a/r)\cos(-\theta)+(a/r)^2}} = \frac{1}{r}\sum_{n=0}^{\infty} P_n(-\cos\theta)\left(\frac{a}{r}\right)^n \tag{112}$$

From Eqns.(109), (110) and (111), we get

$$U = \frac{q}{r}\sum_{n=0}^{\infty}[P_n(\cos\theta) - P_n(-\cos\theta)]\left(\frac{a}{r}\right)^n \tag{113}$$

From the properties of Legendre's polynomial $P_n(x)$, we know that $P_n(x)$ is even if n is even and it is odd if n is odd. Thus $[P_n(\cos\theta) - P_n(-\cos\theta)]$ equals 0 or, $2P_n(\cos\theta)$ according as n is even or odd. Thus relation (113) becomes

$$U = \frac{2q}{r}\sum_{n=0}^{\infty} P_{2n+1}(\cos\theta)\left(\frac{a}{r}\right)^{2n+1}$$

$$= \frac{2q}{r}\left[P_1(\cos\theta)\left(\frac{a}{r}\right) + P_3(\cos\theta)\left(\frac{a}{r}\right)^3 + \cdots\right]. \tag{114}$$

When r is large compared with a, we assume that all terms except the first can be neglected and since $P_1(x) = x$, relation (114) becomes

$$U = 2a\,q\left(\frac{\cos\theta}{r^2}\right)$$

Physicists use this approximation for the dipole potential.

We shall now consider the application of Hermite polynomials and understand the role played by the Hermite polynomials $H_n(x)$ and the corresponding Hermite function $e^{-x^2/2}H_n(x)$ in the theory of the linear harmonic oscillator in quantum mechanics.

Simple Harmonic Oscillator

In wave mechanics, the basic equation for describing the location of a particle attracted by an energy potential $V(z)$ is the **Schrödinger's** (time-independent) **equation**

$$\frac{d^2\psi}{dz^2} + \frac{8m\pi^2}{h^2}[V(z) - E]\,\psi = 0 \tag{115}$$

where m is the mass of the particle, E is the total energy, and h is Plank's constant. The unknown quantity ψ is called the **wave function** i.e. the amplitude of the wave whose intensity gives the probability of finding the particle at any given point in space. A fundamental problem in wave mechanics concerns the motion of a particle bounded in a potential well. It has been established that bounded solutions of Schrodinger's equation for such problems are obtained only for certain discrete energy levels of the particle within the well. A particular example of this important class of problems is the linear oscillator (also called the simple harmonic oscillator), the solution of which lead to Hermite polynomials.

If the restoring force on a particle displaced a distance z from its equilibrium position is $-kz$, where k can be associated with the spring constant of the classical oscillator, then its potential energy is $V(z) = \frac{1}{2}kz^2$. Substituting this expression into Eqn.(115) and introducing the dimensionless parameters

$$x = z\left(\frac{4mk\pi^2}{h^2}\right)^{1/4}, \quad \lambda = \frac{4\pi E}{h}\sqrt{\frac{m}{k}} = \frac{4\pi E}{h\omega}$$

where $\omega = \sqrt{k/m}$ is the angular frequency of the classical oscillator, we find that Eqn.(115) becomes

$$\psi'' + (\lambda - x^2)\psi = 0, \quad -\infty < x < \infty \tag{116}$$

where the primes denote differentiation with respect to x. In addition to Eqn.(116), the wave function ψ must satisfy the boundary condition

$$\lim_{|x| \to \infty} \psi(x) = 0 \tag{117}$$

In order to look for bounded solutions of Eqn.(116), we start with the observation that λ becomes negligible compared with x^2 for large values of x. Thus, asymptotically we expect the solution of Eqn.(116) to behave like

$$\psi(x) \sim e^{\pm x^2/2}, \quad |x| \to \infty$$

where only the negative sign in the exponent is appropriate in order that Eqn.(117) be satisfied. Based on this observation, we make the assumption that Eqn.(116) has solutions of the form

$$\psi(x) = y(x)\, e^{-x^2/2} \tag{118}$$

for suitable y. The substitution of Eqn.(118) into Eqn.(116) yields the differential equation

$$y'' - 2xy' + (\lambda - 1)y = 0 \tag{119}$$

The boundary condition (117) suggests that whatever functional form y assumes, it must either be finite for all x or approach infinity at a rate slower than $e^{-x^2/2}$ approaches zero. It is seen that the only solution of Eqn.(119) satisfying this condition are those for which $\lambda - 1 = 2n$, or

$$\lambda = \lambda_n = 2n + 1, \quad n = 0, 1, 2, \ldots \tag{120}$$

called eigenvalues . In terms of E, we find

$$E_n = \frac{\left(n + \frac{1}{2}\right)h\omega}{2\pi}, \quad n = 0, 1, 2, \ldots \tag{121}$$

with $E_0 = h\omega / 4\pi$ being the lowest or minimum energy level. With λ so restricted, we see that Eqn.(119) becomes

$$y'' - 2xy' + 2ny = 0$$

which is Hermite's equation with solution $y = H_n(x)$. Hence, we conclude that to each eigenvalue λ_n given by Eqn.(120) there corresponds the solution of Eqn.(116) (called an eigenfunction) given by

$$\psi_n(x) = H_n(x)\, e^{-x^2/2}, \quad n = 0, 1, 2, \ldots \tag{122}$$

You may now try the following exercises.

E30) Show that the functions $\psi_n(x) = e^{-x^2/2} H_n(x)$ satisfy the relations.

a) $2n\,\psi_{n-1}(x) = x\,\psi_n(x) + \psi_n'(x)$

b) $2x\,\psi_n(x) - 2n\,\psi_{n-1}(x) = \psi_{n+1}(x)$

c) $\psi_n'(x) = x\,\psi_n(x) - \psi_{n+1}(x)$

E31) By writing $\psi_n(x) = c_n e^{-x^2/2} H_n(x)$, show that the normalized eigen function, that is $\int_{-\infty}^{\infty} [\psi_n(x)]^2\, dx = 1$, assumes the form

$$\psi_n(x) = \left[\sqrt{\pi}\, 2^n\, n!\right]^{-1/2} e^{-x^2/2} H_n(x), \quad n = 0, 1, 2, \ldots .$$

We now end this unit by giving a summary of what we have covered in it.

3.5 SUMMARY

In this unit, we have learnt the following points:

1. The equation $(1-x^2)y''-2xy'+n(n+1)y=0$ is called **Legendre's Equation** where n is a real constant. The polynomial solutions of this equation for integral values of n are **Legendre polynomials .**
2. Legendre polynomial of degree n, having value 1 at $x=1$, is denoted by $P_n(x)$ and is called **Legendre function of the first kind** and is given by

$$P_n(x)=\sum_{k=0}^{[n/2]}\frac{(-1)^k(2n-2k)!}{2^n(k)!(n-k)!(n-2k)!}x^{n-2k}$$

where $[n/2]=\begin{cases} n/2 & , \text{ if } n \text{ is even} \\ \dfrac{n-1}{2} & , \text{ if } n \text{ is odd.}\end{cases}$

3. **Legendre function of the second kind** is denoted by $Q_n(x)$, when n is an integer and is given by

$$Q_n(x)=P_n(x)\left\{A_n+B_n\int\frac{dx}{(1-x^2)[P_n(x)]^2}\right.$$

where A_n and B_n are constants.

4. **Rodrigue's formula for Legendre's polynomials** is given by

$$P_n(x)=\frac{1}{2^n(n)!}\frac{d^n}{dx^n}[(x^2-1)^n].$$

5. **Orthogonality property** of the Legendre's polynomials $P_n(x)$ is

$$\int_{-1}^{1}P_n(x)P_m(x)\,dx=\begin{cases}0 & , \text{ if } m\neq n\\ \dfrac{2}{2n+1}, & \text{ if } m=n\end{cases}$$

6. The function on the left side of

$$\frac{1}{\sqrt{1-2xt+t^2}}=P_0(x)+tP_1(x)+t^2P_2(x)+\cdots+t^nP_n(x)+\cdots |x|<1$$

is called the **generating function** of the Legendre polynomials.

7. Some of the **recurrence relations** for Legendre polynomials are

$$P'_{n+1}(x)=(2n+1)P_n(x)+P'_{n-1}(x)$$
$$P'_{n+1}(x)=xP'_n(x)+(n+1)P_n(x)$$
$$nP_n(x)=xP'_n(x)-P'_{n-1}(x)$$
$$(n+1)P_{n+1}(x)=(2n+1)xP_n(x)-nP_{n-1}(x)$$
$$(1-x^2)P'_n(x)=n[P_{n-1}(x)-xP_n(x)]$$
$$(1-x^2)P_n(x)=(n+1)[xP_n(x)-P_{n+1}(x)]$$

8. The **Hermite polynomials** $H_n(x)$ are polynomial solutions to **Hermite's equation** $y''-2xy'+2ny=0$, where n is a constant, and are given by

$$H_n(x)=\sum_{k=0}^{[n/2]}\frac{(-1)^k n!}{k!(n-2k)!}(2x)^{n-2k}$$

where $[n/2]$ is the greatest integer $\leq n/2$.

9. The function on the left side of equation $e^{(2xt-t^2)}=\sum_{n=0}^{\infty}H_n(x)\frac{t^n}{n!}$, for all finite x and t is the **generating function** of the Hermite polynomials $H_n(x)$.
10. Some of the **recurrence** relations of Hermite polynomials are

$$\mathbf{x\,H'_n(x) = n\ H'_{n-1}(x) + n\ H_n(x)}$$
$$\mathbf{H'_n(x) = 2n\ H_{n-1}(x)}$$
$$\mathbf{H_n(x) = 2x\ H_{n-1}(x) - H_{n-1}(x)}$$
$$\mathbf{H_n(x) = 2x\ H_{n-1}(x) - 2(n-1)\ H_{n-2}(x)}$$

11. The **Rodrigues formula** for Hermite polynomials is given by
$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2})$$

12. **Orthogonality property** of the Hermite polynomials $H_n(x)$ is
$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x)\ H_m(x)\,dx = \begin{cases} 0 & , \text{ if } m \neq n \\ 2^n\, n!\sqrt{\pi}, & \text{if } m = n \end{cases}$$

13. The **Laguerre polynomials** $L_n(x)$ are polynomial solutions to **Laguerre's** equation $x\,y'' + (1-x)\,y' + ny = 0$, where n is a constant, and are given by
$$L_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k\, n!\, x^k}{(k!)^2 (n-k)!}$$

14. The function on the left hand side of the equation
$$(1-t)^{-1} e^{\frac{-xt}{1-t}} = \sum_{n=0}^{\infty} L_n(x)\, t^n,\ |t| < 1,\ \ 0 \le x < \infty$$
is the **generating function** of the Laguerre polynomials.

15. The **Rodrigues formula** for the Laguerre polynomials $L_n(x)$ is
$$L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n}(x^n e^{-x})$$

16. Some of the **recurrence** relations of Laguerre polynomials are
$$\mathbf{(n+1)\ L_{n+1}(x) + (x-1-2n)\ L_n(x) + n\,L_{n-1}(x) = 0}$$
$$\mathbf{L'_n(x) - L'_{n-1}(x) + L_{n-1}(x) = 0}$$
$$\mathbf{x\,L'_n(x) = n\,L_n(x) - n\ L_{n-1}(x)}$$

17. **Orthogonality property** of the Laguerre polynomials $L_n(x)$ is
$$\int_0^{\infty} e^{-x} L_n(x) L_m(x)\ dx = 0\,, \quad \text{if } \ m \neq n\,.$$

3.6 SOLUTIONS/ANSWERS

E1) a) Let $I = \int_{-1}^{1} f(x) P_n(x)\,dx$

$$= \frac{1}{2^n n!} \int_{-1}^{1} f(x) \frac{d^n}{dx^n}[(x^2-1)]\,dx$$

$$= \frac{1}{2^n n!}\left[f(x) \frac{d^{n-1}}{dx^{n-1}}\{(x^2-1)^n\}\Bigg|_{-1}^{1} - \int_{-1}^{1} f^{(1)}(x) \frac{d^{n-1}}{dx^{n-1}}\{(x^2-1)^n\}\,dx \right]$$

$$= \frac{(-1)}{2^n n!} \int_{-1}^{1} f^{(1)}(x) \frac{d^{n-1}}{dx^{n-1}}\{(x^2-1)^n\}\,dx$$

Proceeding like this

$$I = \frac{(-1)^n}{2^n n!} \int_{-1}^{1} f^{(n)}(x).(x^2-1)^n\,dx\,.$$

b) Here $f(x) = x^n$, $f^{(n)}(x) = n!$

$$\therefore \int_{-1}^{1} x^n P_n(x)\,dx = \frac{(-1)^n}{2^n n!}\int_{-1}^{1} n!.(x^2-1)^n\,dx$$

$$= \frac{(-1)^n}{2^n}\int_{-1}^{1}(x^2-1)^n\,dx$$

$$= \frac{2^{n+1}(n!)^2}{(2n+1)!}$$

E2) By Rodrigue's formula

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\{(x^2-1)^n\}$$

since $(x^2-1)^n$ vanishes n times at $x=1$ and $x=-1$, thus its n^{th} derivative and hence , $P_n(x)$ have all real zeroes lying between -1 and $+1$.

$\therefore$ the roots of $P_n(x)=0$ are real and lie between -1 and $+1$.

Let, if possible, the equation

$$P_n(x)=0$$

has a repeated root, say $'\alpha'$. Then

$$P_n(\alpha)=0 \quad \text{and} \quad P'_n(\alpha)=0 \tag{123}$$

since $P_n(x)$ is a solution of Legendre's equation, therefore,

$$(1-x^2)P_n^{(2)}(x) - 2xP_n^{(1)} + n(n+1)P_n(x) = 0 \tag{124}$$

Differentiating m times, by Leibnitz's theorem, we get

$$(1-x^2)P_n^{(m+2)}(x) - 2x(m+1)P_n^{(m+1)}(x) + (n-m)(n-m+1)P_n^{(m)}(x) = 0, \tag{125}$$

Putting $x=\alpha$ in Eqn.(124) and using Eqn.(123), we get

$$P_n^{(2)}(\alpha)=0$$

Putting $m=1,2,3,\ldots$ in Eqn.(125), we get

$$P_n^{(3)}(\alpha) = 0 = P_n^{(4)}(\alpha) = P_n^{(5)}(\alpha) = \cdots = P_n^{(n)}(\alpha)$$

But from definition, $P_n^{(n)}(\alpha) = \dfrac{2n}{2^n n!} \neq 0$.

Thus Eqn.(123) is not valid and hence $P_n(x)=0$ cannot have repeated roots. Hence all the roots of $P_n(x)=0$ are distinct.

E3) From Rodrigue's formula, we have

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\{(x^2-1)^n\}$$

$$\therefore \int_{-1}^{1} P_n(x)\,dx = \frac{1}{2^n n!}\int_{-1}^{1}\frac{d^n}{dx^n}(x^2-1)^n\,dx$$

$$= \frac{1}{2^n n!}\left|\frac{d^{n-1}}{dx^{n-1}}\{(x^2-1)^n\}\right|_{-1}^{+1}$$

$$= \frac{1}{2^n n!}\left|\frac{d^{n-1}}{dx^{n-1}}\left[(x-1)^n(x+1)^n\right]\right|_{-1}^{+1}$$

$$= \frac{1}{2^n n!}[0-0] = 0 \text{ if } n \neq 0$$

Now when $n=0$, then $P_0(x)=1$

$$\therefore \int_{-1}^{1} P_0(x)\,dx = \int_{-1}^{1} 1.dx = x\Big|_{-1}^{+1} = 2.$$

E4) We know that

$$(1-2xh+h^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\,h^n \tag{126}$$

Putting $x=\frac{1}{2}$ in Eqn.(126), we get

$$(1-h+h^2)^{-1/2} = \sum_{n=0}^{\infty} P_n\left(\frac{1}{2}\right) h^n \tag{127}$$

Again, putting $x=-\frac{1}{2}$ in Eqn.(126), we get

$$(1+h+h^2)^{-1/2} = \sum_{n=0}^{\infty} P_n\left(-\frac{1}{2}\right) h^n \tag{128}$$

Now put h^2 for h in Eqn.(128), we get

$$(1+h^2+h^4)^{-1/2} = \sum_{n=0}^{\infty} h^{2n} P_n\left(-\frac{1}{2}\right) \tag{129}$$

But $(1+h^2+h^4)^{-1/2} = \{(1+h^2)^2 - h^2\}^{-1/2}$

$$= (1+h+h^2)^{-1/2}(1-h+h^2)^{-1/2}$$

Using Eqns.(127), (128) and (129) in the above, we get

$$\sum_{n=0}^{\infty} h^{2n} P_n(-1/2) = \left\{\sum_{n=0}^{\infty} h^n P_n(-1/2)\right\}\left\{\sum_{n=0}^{\infty} h^n P_n(1/2)\right\}$$

$$= [P_0(-1/2) + h\,P_1(-1/2) + h^2 P_2(-1/2) + \cdots] \times [P_0(1/2) + h\,P_1(1/2) + h^2 P_2(1/2) + \cdots]$$

Now equating the coefficient of h^{2n} on both sides, we get

$$P_n\left(-\frac{1}{2}\right) = P_0\left(-\frac{1}{2}\right)P_{2n}\left(\frac{1}{2}\right) + P_1\left(-\frac{1}{2}\right)P_{2n-1}\left(\frac{1}{2}\right) + \cdots + P_{2n-1}\left(-\frac{1}{2}\right)P_1\left(\frac{1}{2}\right)$$

$$+ P_{2n}\left(-\frac{1}{2}\right)P_0\left(\frac{1}{2}\right).$$

E5) Since $P_n(x)$ satisfies Legendre's equation, we have

$$(1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0 \tag{130}$$

a) putting $x=1$ in Eqn.(130), we get

$$-2P_n'(1) + n(n+1)\,P_n(1) = 0$$

i.e., $P_n'(1) = \frac{1}{2} n(n+1)\,P_n(1)$

$$= \frac{1}{2} n\,(n+1) \qquad [\because P_n(1) = 1]$$

b) putting $x=-1$ in Eqn.(130), we get

$$2P_n'(-1) + n\,(n+1)\,P_n(-1) = 0$$

i.e., $P_n'(-1) = -\frac{1}{2} n\,(n+1)\,P_n(-1)$

$$= (-1)^{n-1}\frac{1}{2}(n)\,(n+1) \qquad [\because P_n(-1) = (-1)^n\,]$$

E6) a) Putting $x=0$ in Eqn.(27), we obtain

$$(1+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(0)\,t^n \tag{131}$$

Expending left hand side of Eqn.(131) in binomial series

$$(1+t^2)^{-1/2} = \sum_{n=0}^{\infty} \binom{-1/2}{n} t^{2n} \tag{132}$$

Comparing terms on the right in series (131) and (132), we see that series (132) has only even powers of t. Thus, $P_n(0) = 0$ for $n = 1, 3, 5, \ldots$ or, equivalently, $P_{2n+1}(0) = 0, n = 0, 1, 2, \ldots$.

b) Since all odd terms in series (131) are zero, replace n by $2n$ in the series and compare with (132), we obtain

$$P_{2n}(0) = \binom{-1/2}{n} = \frac{\frac{-1}{2}\left(\frac{-1}{2} - 1\right)\ldots\left(\frac{-1}{2} - n + 1\right)}{n!}$$

$$= \frac{(-1)^n \frac{1}{2}\left(\frac{1}{2}+1\right)\ldots\left(\frac{1}{2}+n-1\right)}{n!}$$

$$= \frac{(-1)^n \left(\frac{1}{2}+n-1\right)\left(\frac{1}{2}+n-2\right)\ldots\left(\frac{1}{2}+1\right)\frac{1}{2}}{n!}$$

$$= (-1)^n \begin{pmatrix} \frac{1}{2}+n-1 \\ n \end{pmatrix} = (-1)^n \binom{n-1/2}{n} = \frac{(-1)^n \Gamma\left(n+\frac{1}{2}\right)}{n!\Gamma\left(\frac{1}{2}\right)}$$

$$= \frac{(-1)^n (2n)!}{2^n (n!)^2}. \qquad \left[\because \Gamma\left(n+\frac{1}{2}\right) = \frac{2n!}{2^{2n} n!}\Gamma\left(\frac{1}{2}\right)\right].$$

E7) In the expressions for $P_0(x), P_1(x), P_2(x), P_3(x)$ in terms of x, as obtained in Example 3, put $x = \cos\phi$ and obtain the required results.

E8) a) Multiplying Eqn.(37) by x and then subtracting it from Eqn.(36), we get

$$(1-x^2)P_n'(x) = n\{P_{n-1}(x) - xP_n(x)\}.$$

b) Multiplying (36) by x and then subtracting it from (37), we get

$$(1-x^2)P_n'(x) = (n+1)\{xP_n(x) - P_{n+1}(x)\}.$$

E9) From relation (39), we have

$$(1-x^2)P_n'(x) = n\{P_{n-1}(x) - x\,P_n(x)\}$$

$$= (2n+1)\,x\,P_n(x) - (n+1)\,P_{n+1}(x) - nx\,P_n(x) \quad \text{[using (38)]}$$

$$= (n+1)[x\,P_n(x) - P_{n+1}(x)]$$

E10) We know that

$$P_n(x) = \frac{1.3.\ldots.(2n-1)}{n!}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \cdots\right]$$

Thus we see that $P_0(x)$ is a constant, $P_1(x)$ is a polynomial of degree 1, $P_2(x)$ is a polynomial of degree 2 and so on.

Thus we can express x^m in terms of Legendre forms $P_0(x), P_1(x), \ldots P_m(x)$, and hence a polynomial $f(x)$ of degree $k < n$ can be expressed as

$$f(x) = A_0P_0(x) + A_1P_1(x) + \cdots + A_kP_k(x)$$

$$\therefore \int_{-1}^{1} f(x)P_n(x)dx = \sum_{r=0}^{k} A_r \int_{-1}^{1} P_r(x)P_n(x)\,dx$$

$$= \sum_{r=0}^{k} A_r.0\,, \text{ as } r \neq n$$

$$= 0.$$

E11) Proceeding as in Example 4.

$$
\begin{aligned}
f(x) &= x^3 + ax^2 + bx + c \\
&= \frac{2}{3}P_3(x) + \frac{3}{5}P_1(x) + a\left[\frac{P_0(x)}{3} + \frac{2}{3}P_2(x)\right] + b\,P_1(x) + c\,P_0(x) \\
&= \frac{2}{3}P_3(x) + \frac{2}{3}a\,P_2(x) + \left(\frac{3}{5} + b\right)P_1(x) + \left(\frac{a}{3} + c\right)P_0(x).
\end{aligned}
$$

E12) From E8) (a) and (b), we have

$$(1 - x^2)\,P_n'(x) = n\,\{P_{n-1}(x) - x\,P_n(x)\}$$

and $\quad (1 - x^2)\;P_n'(x) = (n+1)\;\{x\;P_n(x) - P_{n+1}(x)\}$

Substituting for $xP_n(x)$ from the first relation in second relation, we get

$$(1 - x^2)P_n'(x) = (n+1)\left[\left\{P_{n-1}(x) - \frac{(1 - x^2)P_n'(x)}{n}\right\} - P_{n+1}(x)\right]$$

or, $\quad (1 - x^2)\left[1 + \frac{n+1}{n}\right]P_n'(x) = (n+1)[P_{n-1}(x) - P_{n+1}(x)]$

or, $\quad (2n+1)(1 - x^2)\,P_n'(x) = n\,(n+1)[P_{n-1}(x) - P_{n+1}(x)]\,.$

E13) From generating function of Legendre Polynomials, we have

$$[1 - 2xt + t^2]^{-1/2} = \sum_{m=0}^{\infty} t^m P_m(x)$$

$$
\begin{aligned}
\therefore \int_{-1}^{1} \frac{P_n(x)}{\sqrt{1 - 2xt + t^2}}\,dx &= \int_{-1}^{1} P_n(x)\left[\sum_{m=0}^{\infty} t^m P_m(x)\right] dx \\
&= t^n \int_{-1}^{1} [\,P_n(x)]^2\,dx\,, \qquad \text{[using formula (41)]} \\
&= t^n \cdot \frac{2}{2n+1}\;.
\end{aligned}
$$

E14) We know that

$$(1 - 2xt + t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(x)$$

Integrating both sides w.r.t t between the limit 0 to 1, we get

$$\int_0^1 (1 - 2xt + t^2)^{-1/2}\,dt = \sum_{n=0}^{\infty} P_n(x)\int_0^1 t^n\,dt$$

Putting $x = \cos\theta$, we get

$$\int_0^1 \frac{dt}{\sqrt{1 - 2\cos\theta . t + t^2}} = \sum_{n=0}^{\infty} P_n(\cos\theta).\left.\frac{t^{n+1}}{n+1}\right|_0^1$$

or, $\quad \displaystyle\int_0^1 \frac{dt}{\sqrt{(t - \cos\theta)^2 + \sin^2\theta}} = \sum_{n=0}^{\infty} P_n(\cos\theta).\frac{1}{n+1}$

or, $\quad \displaystyle\ln\left[(t - \cos\theta) + \sqrt{\sin^2\theta + (t - \cos\theta)^2}\;\right]\Bigg|_0^1 = \sum_{n=0}^{\infty}\frac{P_n(\cos\theta)}{n+1}$

or, $\quad \displaystyle\ln\left[\frac{(1 - \cos\theta) + \sqrt{\sin^2\theta + (1 - \cos\theta)^2}}{1 - \cos\theta}\right] = \sum_{n=0}^{\infty}\frac{P_n(\cos\theta)}{n+1}$

or, $\quad \displaystyle\ln\frac{\sqrt{2} + \sqrt{1 - \cos\theta}}{\sqrt{1 - \cos\theta}} = \sum_{n=0}^{\infty}\frac{P_n(\cos\theta)}{n+1}$

or, $$\ln \frac{\sqrt{2}+\sqrt{2\sin^2\frac{\theta}{2}}}{\sqrt{2\sin^2\frac{\theta}{2}}} = \sum_{n=0}^{\infty} \frac{P_n(\cos\theta)}{n+1}$$

or, $$\ln \frac{1+\sin\frac{\theta}{2}}{\sin\frac{\theta}{2}} = 1+\frac{1}{2}P_1(\cos\theta)+\frac{1}{3}P_2(\cos\theta)+\frac{1}{4}P_3(\cos\theta)+\cdots.$$

E15) $x=\cos\phi$, therefore, $\frac{dx}{d\phi}=-\sin\phi=-\sqrt{1-x^2}$ and $\frac{dy}{d\phi}=-\sin\phi\frac{dy}{dx}$

$$\therefore \frac{1}{\sin\phi}\frac{d}{d\phi}\left(\sin\phi\frac{dy}{d\phi}\right)+n(n+1)y = \frac{-1}{\sqrt{1-x^2}}\left[2x\sqrt{1-x^2}\frac{dy}{dx} + (1-x^2)\frac{d^2y}{dx^2}\left(-\sqrt{1-x^2}\right)\right]+n(n+1)y$$

$$=(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y\,.$$

E16) a) $y=P_n(x)$ satisfies the given equation and the boundary conditions are $y(0)=0 \Rightarrow P_n(0)=0$ and $y(1)=1 \Rightarrow P_n(1)=1$, (which is true).
Now $P_n(0)=0$, for n odd (see Example 1)
$\therefore\ y=P_n(x)$ is a solution for $n=1,3,5,\ldots$.

b) $y=P_n(x)$ satisfies the given equation and the condition $y(1)=1$ [$\because P_n(1)=1$]
$y'(0)=0 \Rightarrow P_n'(0)=0$, but $P_n'(0)=n\,P_{n-1}(0)$ [using (39)]
$P_n'(0)=0 \Rightarrow P_{n-1}(0) \Rightarrow n=0,2,4,\ldots$.
$\therefore\ y=P_n(x)$ is a solution for $n=0,2,4,\ldots$.

E17) Multiplying both sides of Eqn.(61) with $P_m(t)$ and integrating from -1 to 1 w.r.t. t, we get

$$\int_{-1}^{1}\frac{1}{x-t}P_m(t)dt=\sum_{n=0}^{\infty}Q_n(x)(2n+1)\int_{-1}^{1}P_n(t)P_m(t)dt$$

$$\therefore \int_{-1}^{1}\frac{P_m(t)}{x-t}dt = Q_n(x)\ (2n+1)\ \frac{2}{2n+1}$$

or, $$Q_n(x)=\frac{1}{2}\int_{-1}^{1}\frac{P_n(t)}{x-t}\,dt\,.$$

E18) a) Comparing the given equation with Eqn.(1), we see that given equations is a Legendre's equation with $n=0$, so its general solution is $y=C_1P_0(x)+C_2Q_0(x)$.

b) Comparing the given equation with Eqn.(1), we find that $n(n+1)=12$, therefore, $n=3,-4$. Since n is non-negative, the general solution of the given equation is $y=C_1P_3(x)+C_2Q_3(x)$.

c) $y=C_1P_1(x)+C_2Q_1(x)$.

d) $y=C_1P_5(x)+C_2Q_5(x)$.

E19) $W(P_0,Q_0)=\begin{vmatrix}P_0 & Q_0\\ P_0' & Q_0'\end{vmatrix}=P_0Q_0'-Q_0P_0'=\frac{1}{1-x^2}$.

E20) $W[P_n(x), Q_n(x)] = P_n Q_n' - Q_n P_n'$

Substituting in the above equation for Q_n and Q_n' from Eqn.(53), we get

$$W[P_n(x), Q_n(x)] = P_n P_n'\left[A_n + B_n \int \frac{dx}{(1-x^2)[P_n(x)]^2}\right] + [P_n(x)]^2 B_n \frac{1}{(1-x^2)[P_n(x)]^2}$$

$$- P_n P_n'\left[A_n + B_n \int \frac{dx}{(1-x^2)[P_n(x)]^2}\right]$$

$$= \frac{B_n}{1-x^2},\ n = 0, 1, 2, \cdots.$$

E21) Consider $y = \sum_{m=0}^{\infty} c_m x^m,\ c_0 \neq 0$. Substitute for y, y' and y'' in Eqn.(63), simplify and equate the coefficients of equal power of x to get two linearly independent solutions $y_1(x)$ and $y_2(x)$ [ref. E9) Unit 2].

E22) $$e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)t^n}{n!} = H_0(x) + H_1(x)t + \frac{H_2(x)}{2!}t^2 + \frac{H_3(x)}{3!}t^3 + \cdots$$

Now, $$e^{2tx-t^2} = 1 + (2tx - t^2) + \frac{(2tx-t^2)^2}{2!} + \frac{(2tx-t^2)^3}{3!} + \cdots$$

$$= 1 + (2x)t + (2x^2 - 1)t^2 + \left(\frac{4x^3 - 6x}{3}\right)t^3 + \cdots$$

Comparing the two series, we have

$$H_0(x) = 1,\ H_1(x) = 2x,\ H_2(x) = 4x^2 - 2,\ H_3(x) = 8x^3 - 12x.$$

E23) a) Let $w = e^{(2xt-t^2)} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n$

then, $(x - t)\frac{\partial w}{\partial x} - t\frac{\partial w}{\partial t} = 0$

or, $$\sum_{n=0}^{\infty} x\frac{H_n'(x)}{n!}t^n - \sum_{n=0}^{\infty} \frac{n H_n(x)t^n}{n!} = \sum_{n=0}^{\infty} \frac{n H_{n-1}'(x)}{n!}t^n$$

Comparing coefficients of t^n on both the sides, weget

$$x H_n'(x) - nH_n(x) = n H_{n-1}(x).$$

b) From relation (75), we get $H_n(x) = 2x H_{n-1}(x) - [2(n-1) H_{n-2}(x)]$ [using relation (74)].

c) $H_n''(x) = 2n H_{n-1}'(x)$ [using Eqn.(74)]

$= 2n[2x H_{n-1}(x) - H_n(x)]$ [using Eqn.(75)]

$= 2x H_n'(x) - 2n H_n(x)$ [using Eqn.(74)].

E24) a) We have from Eqn.(68)

$$H_{2n}(x) = \sum_{k=0}^{n} \frac{(-1)^k 2n!(2x)^{2n-2k}}{k!\,(2n-2k)!}$$

The last term in the above summation is a constant and therefore, we have

$$H_{2n}(0) = \frac{(-1)^n 2n!}{n!}.$$

b) From Eqn.(68), we have

$$H_{2n+1}(x) = \sum_{k=0}^{n} \frac{(-1)^k 2n+1!(2x)^{2n+1-2k}}{k!\,(2n+1-2k)!}$$

The last term in the above summation is a function of x, hence we get

$$H_{2n+1}(0) = 0.$$

c) Differentiating both sides of Eqn.(68), we obtain

$$H'_n(x) = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \frac{(-1)^k n!(2x)^{n-2k-1}.2}{k!(n-2k-1)!}$$

from which it follows that

$$H'_{2n}(x) = \sum_{k=0}^{n-1} \frac{(-1)^k (2n)!.2.(2x)^{2n-2k-1}}{k!\,(2n-2k-1)!}$$

Putting $x = 0$, we get

$H'_{2n}(0) = 0$

d) $$H'_{2n+1}(x) = \sum_{k=0}^{n} \frac{(-1)^k (2n+1)!.2.(2x)^{2n-2k}}{k!\,(2n-2k)!}$$

$$H'_{2n+1}(0) = \frac{(-1)^n\,(2n+1)!.2}{n!} = \frac{(-1)^n (2n+2)!}{(n+1)!}.$$

E25) From Eqn.(66), we have

$$e^{(2xt-t^2)} = \sum_{k=0}^{\infty} H_k(x)\frac{t^k}{k!}$$

$$e^{2xt} = e^{t^2}\sum_{k=0}^{\infty} H_k(x)\frac{t^k}{k!}$$

$$\therefore \sum_{n=0}^{\infty} \frac{(2x)^n t^n}{n!} = \sum_{m=0}^{\infty} \frac{t^{2m}}{m!}\sum_{k=0}^{\infty} H_k(x)\frac{t^k}{k!}$$

$$= \sum_{m=0}^{\infty}\sum_{k=0}^{[n/2]} \frac{H_{n-2k}(x)\ t^n}{k!(n-2k)!}$$ [by interchanging m and k and setting k = n − 2k].

Comparing the coefficients of t^n on both the sides, we have

$$x^n = \sum_{k=0}^{[n/2]} \frac{n!H_{n-2k}(x)}{2^n k!(n-2k)!}$$

E26) Using E22) required result is, $\frac{-3}{2}H_0 + \frac{7}{4}H_1 - \frac{3}{4}H_2 + \frac{1}{8}H_3$.

E27) We have, $L_n(x) = e^x \frac{d^n}{dx^n}(x^n e^{-x})$, then

$$L_0(x) = 1,\ L_1(x) = e^x \frac{d}{dx}(x\ e^{-x}) = 1 - x,$$

$$L_2(x) = e^x \frac{d^2}{dx^2}(x^2 e^{-x}) = 2 - 4x + x^2$$

$$L_3(x) = e^x \frac{d^3}{dx^3}(x^3 e^{-x}) = 6 - 18x + 9x^2 - x^3$$

E28) Let $w(x,t) = (1-t)^{-1}\exp\left(\frac{-xt}{1-t}\right)$. It satisfy the identity

$$(1-t)^2\frac{\partial w}{\partial t} + (x - 1 + t)\,w = 0$$

Using Eqn.(86), the above equation reduces to

$$\sum_{n=1}^{\infty} [(n+1)L_{n+1}(x) + (x - 1 - 2n)L_n(x) + n\,L_{n-1}(x)]\,t^n = 0$$

Equating the coefficients of t^n to zero, relation (89) is proved.

Similarly, using Eqn.(86) into the identity $(1-t)\frac{\partial w}{\partial x} + tw = 0$, relation (90) is obtained.

Differentiating relation (89), using relation (90) in it and then eliminating $L'_{n+1}(x)$ and $L'_{n-1}(x)$ from it relation (91) is obtained.

Differentiating relation (91) and using relation (90), relation (92) is obtained.

E29) From Laguerre's differential equation we have for any two Laguerre polynomials $L_m(x)$ and $L_n(x)$

$$x L''_m + (1-x) L'_m + mL_m = 0$$
$$x L''_n + (1-x) L'_n + n L_n = 0$$

Multiplying these equations by L_n and L_m respectively and subtracting, we obtain

$$x[L_n L''_m - L_m L''_n] + (1-x)[L_n L'_m - L_m L'_n] = (n-m) L_m L_n$$

or, $\frac{d}{dx}[L_n L'_m - L_m L'_n] + \frac{(1-x)}{x}[L_n L'_m - L_m L'_n] = \frac{(n-m)}{x} L_m L_n$

or, $\frac{d}{dx}\{x e^{-x}[L_n L'_m - L_m L'_n]\} = (n-m)e^{-x} L_m L_n$, where we have multiplied by I.F. $e^{\int (1-x)/x\, dx} = e^{\ln x - x} = x e^{-x}$

Integrating from 0 to ∞

$$(n-m)\int_0^\infty e^{-x} L_m(x) L_n(x) dx = \left\{x e^{-x}[L_n L'_m - L_m L'_n]\right\}_0^\infty = 0$$

If $m \neq n$, $\int_0^\infty e^{-x} L_m(x) L_n(x)\, dx = 0$.

E30) a) Putting $\psi_n(x) = e^{-x^2/2} H_n(x)$ and $\psi'_n(x) = e^{-x^2/2} H'_n(x) - x e^{-x^2/2} H_n(x)$, in the given equation, it reduces to

$2nH_{n-1}(x) = H'_n(x)$ which is relation (74)

b) The given equation reduces to

$$2xH_n(x) - 2n H_{n-1}(x) = H_{n+1}(x)$$

replacing n by $(n-1)$ we get relation (76).

c) Substituting for $\psi_n(x)$ and its derivatives in the given equation, relation (75) is obtained.

E31) $\int_{-\infty}^{\infty} [\psi_n(x)]^2 dx = 1 \Rightarrow \int_{-\infty}^{\infty} c_n^2 e^{-x^2} [H_n(x)]^2 dx = 1$

$\Rightarrow c_n^2 2^n n! \sqrt{\pi} = 1$ or, $c_n = \frac{1}{\left[2^n n! \sqrt{\pi}\right]^{1/2}}$ [using result (78)]

$\therefore \psi_n(x) = \left[\sqrt{\pi} 2^n n!\right]^{-\frac{1}{2}} e^{-x^2} H_n(x), \quad n = 0, 1, 2, \ldots$

—x—

UNIT 4 BESSEL FUNCTIONS

Structure **Page No.**

4.1 INTRODUCTION

In Unit 3, we studied Legendre, Hermite and Laguerre polynomials and discussed a number of their special properties. In this unit we shall study about Bessel functions. Bessel functions were first discovered in 1732 by D. Bernoulli (1700-1782), who provided a series solution (representing a Bessel function) for the oscillatory displacement of a heavy hanging chain. In 1764, Euler developed a series similar to that of Bernoulli, in connection with his work on the vibration of a circular drumhead. J.Fourier (1768-1836) also used Bessel functions in his classical treatise on heat in 1822, but it was Bessel the German astronomer who derived the differential equation bearing his name and studied the general properties of its solutions (now called Bessel functions) in his 1824 memoir. These functions occur so frequently in practice that they are undoubtedly the most important functions beyond the elementary ones.

We shall start the discussion in Sec.4.2 with the Bessel's differential equation and obtain its power series solution in terms of Bessel functions of first and second kind. We shall also discuss various properties of Bessel functions here. In Vibrational phenomena, one of the fundamental problems is to describe the wave motion of a natural or mechanical system: such as a vibrating membrane, subject to certain initial and boundary conditions. In Sec.4.4 we shall discuss this problem of vibrating membrane and illustrate the role played by Bessel functions in getting its solution. We shall also discuss here the problem concerning the buckling of vertical columns under a compressive load.

Objectives

After studying this unit you should be able to

- identify Bessel's differential equation;
- obtain the series solutions of Bessel's differential equation;
- derive Rodrigue's formula, generating function, recurrence relations and orthogonal property of Bessel functions;
- use recurrence relations and other properties of Bessel functions occurring in equations governing physical phenomenon, for finding their solutions.

4.2 BESSEL'S EQUATION

The differential equation

$$x^2 y'' + xy' + (x^2 - \upsilon^2)\, y = 0 \qquad (1)$$

where υ is a non-negative real number is known as **Bessel's equation of order** υ and its solutions are known as **Bessel functions.** The equation was studied by Friedrich Wilhelm Bessel (1784 – 1846), director of the astronomical observatory at Konigsberg, in connection with his work on planetary motion. Apart from this the equation appears in a wide range of applications such as steady and unsteady diffusion in cylindrical regions and one-dimensional wave propagation and diffusion in variable cross-section media etc. We shall now obtain the series solution of this equation.

Dividing Eqn.(1) throughout by the leading coefficient x^2, we see that it can be put in the form

$$y'' + P(x)\,y' + Q(x)\,y = 0$$

where $P(x) = \dfrac{1}{x}$ and $Q(x) = (x^2 - \upsilon^2)/x^2$. This shows that there is one singular point $x = 0$, and that it is **regular singular** point because $x\,P(x) = 1$, and $x^2 Q(x) = x^2 - \upsilon^2$, are analytic at $x = 0$. Therefore, a Frobenius series solution exists for Eqn.(1). Here we shall be discussing two cases

i) $\upsilon \neq$ integer and ii) $\upsilon =$ integer.

Case when υ is not an integer

Consider the series solution of Eqn.(1) of the form

$$y(x) = \sum_{m=0}^{\infty} c_m x^{m+k}, \quad c_0 \neq 0 \tag{2}$$

Substituting for y, y' and y'' in Eqn.(1), we obtain

$$\sum_{m=0}^{\infty}(m+k)(m+k-1)c_m x^{m+k} + \sum_{m=0}^{\infty}(m+k)c_m x^{m+k} + \sum_{m=0}^{\infty}c_m x^{m+k+2} - \upsilon^2\sum_{m=0}^{\infty}c_m x^{m+k} = 0$$

or, $$\sum_{m=0}^{\infty}\left[(m+k)(m+k-1) + (m+k) - \upsilon^2\right]c_m x^{m+k} + \sum_{m=0}^{\infty}c_m x^{m+k+2} = 0$$

or, $$\sum_{m=0}^{\infty}\left[(m+k+\upsilon)(m+k-\upsilon)\right]c_m x^{m+k} + \sum_{m=0}^{\infty}c_m x^{m+k+2} = 0$$

or, $$\sum_{m=0}^{\infty}\left[(m+k)^2 - \upsilon^2\right]c_m x^{m+k} + \sum_{m=0}^{\infty}c_m x^{m+k+2} = 0 \tag{3}$$

Equating to zero the coefficient of smallest power of x (i.e. x^k) to zero, we get the **indicial equation** as

$$c_0(k^2 - \upsilon^2) = 0 \Rightarrow k = \upsilon, -\upsilon \;(\because c_0 \neq 0) \tag{4}$$

Equating the coefficient of x^{k+1} to zero in Eqn.(3), we obtain

$$[(k+1)^2 - \upsilon^2)c_1 = 0, \quad \text{or } c_1 = 0, \quad \text{for } k = \pm\,\upsilon$$

To obtain the **recurrence** relation, we equate to zero the coefficients of x^{k+m} in Eqn.(3). This yields

$$(m+k+\upsilon)\;(m+k-\upsilon)\;c_m = -\,c_{m-2}$$

i.e, $$c_m = \frac{-1}{(m+k+\upsilon)\;(m+k-\upsilon)}\;c_{m-2} \tag{5}$$

For $\mathbf{k = \upsilon}$, relation (5) simplifies to

$$c_m = \frac{-1}{(m-2\upsilon)\,m}c_{m-2} \tag{6}$$

Since $c_1 = 0$, putting $m = 3, 5, 7, \ldots$ in relation (6), we get

$$c_3 = c_5 = c_7 = \ldots = 0 \tag{7}$$

and for $m = 2, 4, 6, \ldots$, we obtain

$$c_2 = \frac{-c_0}{2^2(1+\upsilon)},\; c_4 = \frac{-c_2}{2^3(2+\upsilon)} = \frac{c_0}{2^4(2!)(1+\upsilon)(2+\upsilon)}$$

$$c_6 = \frac{-c_4}{2^2(3)(3+\upsilon)} = \frac{-c_0}{2^6(3!)(1+\upsilon)(2+\upsilon)(3+\upsilon)^3}\cdots$$

In general, we can write

$$c_{2m} = \frac{(-1)^m c_0}{2^{2m}(m!)(1+\upsilon)(2+\upsilon)\cdots(m+\upsilon)}, \quad m = 1, 2, \ldots \tag{8}$$

Putting these values of c's in Eqn.(2), we obtain one of the linearly independent solution of the Bessel's equation as

$$y_1(x) = c_0 x^{\upsilon}\left[1 - \frac{x^2}{2^2(1+\upsilon)} + \frac{x^4}{2^4\,(2!)\,(1+\upsilon)\,(2+\upsilon)} - \cdots \right.$$
$$\left. \cdots + \frac{(-1)^m x^{2m}}{2^m\,(m!)\,(1+\upsilon)\,(2+\upsilon)\ldots(m+\upsilon)} + \cdots\right] \qquad (9)$$

If υ were an integer, then the product $(1+\upsilon)(2+\upsilon)\ldots(m+\upsilon)$ could be simplified into closed form as $(\upsilon+m)!$ But if υ is not an integer then we can simplify the right hand side of Eqn.(9) by making a suitable choice of a value for c_0.

Let the value of c_0 be chosen as

$$c_0 = \frac{1}{2^{\upsilon}\,\Gamma(\upsilon+1)} \qquad (10)$$

$\Gamma(\upsilon+1) = \upsilon\Gamma\upsilon$

where Γ is the Gamma function defined by

$$\Gamma(\upsilon) = \int_0^{\infty} e^{-t}\,t^{\upsilon-1}\,dt \qquad (11)$$

Substituting the value of c_0 from Eqn.(10) in Eqn.(8), we obtain

$$c_{2m} = \frac{(-1)^m}{2^{2m+\upsilon}\,(m!)\,\Gamma(\upsilon+1)\,[(1+\upsilon)\,(2+\upsilon)\ldots(m+\upsilon)]} = \frac{(-1)^m}{2^{2m+\upsilon}\,(m!)\Gamma(m+\upsilon+1)} \qquad (12)$$

and the solution $y(x)$ given by Eqn.(9) reduces to

$$y(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\upsilon+1)}\left(\frac{x}{2}\right)^{2m+\upsilon} \qquad (13)$$

This $y(x)$ defines the **Bessel function of the first kind of order** υ and is denoted by $J_{\upsilon}(x)$. Therefore we have one solution of Eqn.(1) as

$$J_{\upsilon}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\upsilon+1)}\left(\frac{x}{2}\right)^{2m+\upsilon}$$
$$= \left(\frac{x}{2}\right)^{\upsilon} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\upsilon+1)}\left(\frac{x}{2}\right)^{2m} \qquad (14)$$

To obtain a second linearly independent solution, we turn to the other indicial root, $k = -\upsilon$. There is no need to repeat all the steps; all we need to do is to change υ to $-\upsilon$ everywhere on the right side of Eqn.(14). Denoting the result as $J_{-\upsilon}(x)$, we have the second solution of Eqn.(1), the **Bessel function of the first kind of order** $-\boldsymbol{\upsilon}$ as

$$J_{-\upsilon}(x) = \left(\frac{x}{2}\right)^{-\upsilon} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m-\upsilon+1)}\left(\frac{x}{2}\right)^{2m} \qquad (15)$$

Using ratio test it can be shown that both the series (14) and (15) converge for all x. The leading terms of the series in (14) and (15) are constant times x^{υ} and $x^{-\upsilon}$, respectively so neither of the solutions $J_{\upsilon}(x)$ and $J_{-\upsilon}(x)$ is a scalar multiple of the other. Thus they are linearly independent, and we conclude that, when υ **is not an integer**

$$\mathbf{y(x) = A\,J_{\upsilon}(x) + B\,J_{-\upsilon}(x)} \qquad (16)$$

is the **general solution** of Bessel's Eqn.(1).

Writing Eqns.(14) and (15) in the form

$$J_{\upsilon}(x) = x^{\upsilon}\left[\frac{1}{\Gamma(\upsilon+1)\,2^{\upsilon}} - \frac{1}{\Gamma(\upsilon+2)\,2^{\upsilon+2}}\,x^2 + \cdots\right] \qquad (17)$$

$$J_{-\upsilon}(x) = x^{-\upsilon}\left[\frac{1}{\Gamma(1-\upsilon)\,2^{-\upsilon}} - \frac{1}{\Gamma(2-\upsilon)\,2^{-\upsilon+2}}\,x^2 + \cdots\right] \tag{18}$$

you may **observe** that the power series within the square brackets tend to $1/\Gamma(\upsilon+1)\,2^{\upsilon}$ and $1/\Gamma(1-\upsilon)\,2^{-\upsilon}$, respectively, as $x \to 0$. We see that $J_{\upsilon}(x) \sim \left[1/\Gamma(\upsilon+1)\,2^{\upsilon}\right]x^{\upsilon}$ and $J_{-\upsilon}(x) \sim \left[1/\Gamma(1-\upsilon)\,2^{-\upsilon}\right]x^{-\upsilon}$ as $x \to 0$. In more concise form we can write $J_{\upsilon}(x) = O(x^{\upsilon})$ and $J_{-\upsilon}(x) = O(x^{-\upsilon})$ as $x \to 0$. Thus, $J_{\upsilon}(x)'s$ tend to zero and the $J_{-\upsilon}(x)'s$ tend to infinity as $x \to 0$. In Fig.1, we have plotted $J_{1/2}(x)$ and $J_{-1/2}(x)$, for $\upsilon = 1/2$.

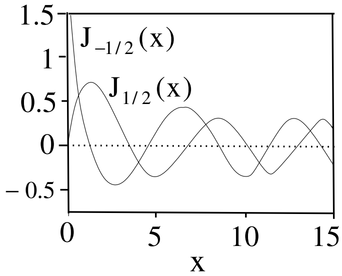

Fig.1

The cases where υ is half-integral i.e. $\upsilon = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \cdots,$ are of special interest because these Bessel functions are actually elementary functions. In particular, the case $\upsilon = \frac{1}{2}$ leads to the interesting results where

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x \quad \text{and} \quad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x$$

We are leaving it for you to verify it yourself.

Let us consider the following examples.

Example 1: Find the solution of the differential equation

$$x^2 y'' + xy + \left(x^2 - \frac{1}{4}\right)y = 0.$$

Solution: The given differential equation is the Bessel's equation with $\upsilon = \frac{1}{2}$. The general solution of the equation is

$$y(x) = A\,J_{1/2}(x) + B\,J_{-1/2}(x)$$

***

Example 2: Transform the equation

$$x^2 y'' + xy' + 4(x^4 - \upsilon^2)y = 0$$

using the substitution $x^2 = z$ and hence find the general solution of the equation.

Solution: For $x^2 = z$, we have

$$y' = \frac{dy}{dx} = \frac{dy}{dz}.\frac{dz}{dx} = 2x\frac{dy}{dz} = 2\sqrt{z}\,\frac{dy}{dz}$$

and $$y'' = \frac{d^2y}{dx^2} = 2\frac{dy}{dz} + 4x^2\frac{d^2y}{dz^2} = 2\frac{dy}{dz} + 4z\frac{d^2y}{dz^2}$$

On substituting y' and y'' from above, the given equation reduces to

$$z\left(4z\frac{d^2y}{dz^2} + 2\frac{dy}{dz}\right) + 2z\frac{dy}{dz} + 4(z^2 - \upsilon^2)\,y = 0$$

or, $$z^2\frac{d^2y}{dz^2} + z\frac{dy}{dz} + (z^2 - \upsilon^2)\,y = 0$$

which is a Bessel's differential equation of order υ. Hence the general solution is given by

$$y(z) = A\,J_{\upsilon}(z) + B\,J_{-\upsilon}(z)$$

or, $$y(x) = A\,J_{\upsilon}(x^2) + B\,J_{-\upsilon}(x^2)$$

***

You may now try the following exercises.

E1) Using Eqns.(14) and (15) and the relations $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, show that

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x, \quad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x.$$

E2) Show that the given functions are solutions of the corresponding equations

a) $y'' + k^2 xy = 0$, $\quad y = \sqrt{x}\, J_{1/3}\left(\frac{2kx^{3/2}}{3}\right)$

b) $y'' + k^2 x^2 y = 0$, $\quad y = \sqrt{x}\, J_{1/4}\left(\frac{k x^2}{2}\right)$

c) $y'' + k^2 x^4 y = 0$, $\quad y = \sqrt{x}\, J_{1/6}\left(\frac{kx^3}{3}\right)$

E3) Using the substitutions given alongside, transform the following differential equations and hence, find their general solution.

a) $4x^2 y'' + 4x y' + (x - \upsilon^2)\, y = 0$; $\quad (\sqrt{x} = z)$

b) $xy'' - y' + xy = 0$; $\quad (y = xu)$

c) $xy'' + (1 + 2k)\, y' + xy = 0$; $\quad (y = u / x^k)$

We now take up the case when υ is an integer.

Case when υ is an integer

Let $\upsilon = n$ be an integer. Since $\Gamma(\upsilon + 1) = \upsilon\Gamma(\upsilon)$, we have for $\upsilon = n$ $\Gamma(1) = 1,\ \Gamma(2) = \Gamma(1) = 1!,\ \Gamma(3) = 2\Gamma(2) = 2!, \ldots, \Gamma(n+1) = n\Gamma(n) = n!$ and the expression for c_{2m} given by Eqn.(12) can be written as

$$c_{2m} = \frac{(-1)^m}{2^{2m+n}(m!)(m+n)!}, \quad \text{and} \quad c_0 = \frac{1}{2^n\, n!}, \text{ from Eqn.(10)} \tag{19}$$

The Bessel function $J_n(x)$ is then defined as

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,(m+n)!}\left(\frac{x}{2}\right)^{2m+n} \tag{20}$$

The series given by Eqn.(20) converges for all x. Functions $J_{-\upsilon}(x)$ for $\upsilon = n$, reduces to

$$J_{-n}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m-n+1)}\left(\frac{x}{2}\right)^{2m-n} = \sum_{m=n}^{\infty} \frac{(-1)^m}{m!\Gamma(m-n+1)}\left(\frac{x}{2}\right)^{2m-n} \tag{21}$$

where we have used the fact that $\Gamma(m - n + 1)$ is undefined when its argument is zero or a negative integer, namely, for $m = 0, 1, \ldots, n-1$. To start the series at zero once again, we make the change of index $m - n = k$ in Eqn.(21), which yields

$$J_{-n}(x) = \sum_{k=0}^{\infty} \frac{(-1)^{n+k}}{(n+k)!\Gamma(k+1)}\left(\frac{x}{2}\right)^{2k+n} = \sum_{k=0}^{\infty} \frac{(-1)^{n+k}}{k!(n+k)!}\left(\frac{x}{2}\right)^{2k+n} \tag{22}$$

If $(-1)^n$ is factored out the series that remains on the right hand side of En.(22) is the same as that given in Eqn.(20), so that

$$J_{-n}(x) = (-1)^n J_n(x) \tag{23}$$

This shows that $J_n(x)$ and $J_{-n}(x)$ are linearly dependent. Thus, whereas $J_\upsilon(x)$ and $J_{-\upsilon}(x)$ are linearly independent and give the general solution (16) of Eqn.(1) when υ is not an integer, we have only one linearly independent solutions so far for the case where $\upsilon =$ integer, namely, $y_1(x) = J_n(x)$ given by Eqn.(20). The second linearly independent solution is yet to be determined which we shall take up later in Sec.4.2.2.

The Bessel functions that arise most frequently in practice are $\mathbf{J_0(x)}$ and $\mathbf{J_1(x)}$, whose series representations are

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2}\left(\frac{x}{2}\right)^{2m}$$

$$= 1 - \frac{x^2}{2^2} + \frac{x^4}{2^4(2!)^2} - \frac{x^6}{2^6(3!)^2} + \cdots \qquad (24)$$

and $$J_1(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m+1)!}\left(\frac{x}{2}\right)^{2m+1}$$

$$= \frac{x}{2} - \frac{x^3}{2^3 1!2!} + \frac{x^5}{2^5 2!3!} - \frac{x^7}{2^7 3!4!} + \cdots \qquad (25)$$

At $x = 0$, we see that $J_0(0) = 1$ while $J_1(0) = 0$ and from Eqn.(20) it follows that $J_n(0) = 0,\ n = 2, 3, 4, \ldots$. For x near zero, the $m = 0$ term of Eqn.(20) yields the asymptotic formulas

$$J_0(x) \sim 1 \ \text{ as } \ x \to 0^+ \qquad (26)$$

$$J_n(x) \sim \frac{(x/2)^n}{n!} \ \text{ as } \ x \to 0^+,\ n = 1, 2, 3, \ldots \qquad (27)$$

The graphs of $J_0(x)$, $J_1(x)$ are shown in Fig.2. **Observe** that these functions exhibit an oscillatory behaviour somewhat like that of the sinusoidal functions, except that the amplitude (maximum departure from the x-axis) of the Bessel function diminishes as x increases and the (infinitely many) zeros of these functions are not evenly spaced. The location of these zeros is of great theoretical and practical importance, but we shall not be going into the details of the theory here. However, the first five zeros of $J_0(x)$ and $J_1(x)$ are given below:

$J_0(x)$: 2.405, 5.520, 8.654, 11.792, 14.931

$J_1(x)$: 3.832, 7.016, 10.173, 13.324, 16.471

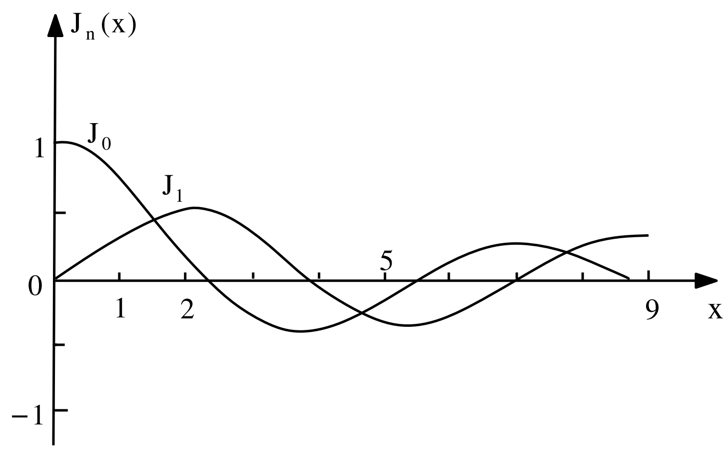

Fig.2

You may now try the following exercises.

E4) Show that $J_n(x)$ is an even function for n even and an odd function for n odd where n is an integer.

E5) Using the series expansion for $J_0(x)$ and $J_1(x)$ show that $J_0'(x) = -J_1(x)$.

We shall now discuss some of the properties of the Bessel functions.

4.2.1 Bessel Function of the First Kind

Bessel functions of first kind satisfy a number of properties that can be developed directly from their series definition. We shall now prove some of these properties.

Recurrence relations

Multiplying the Bessel function $J_\upsilon(x)$, given in Eqn.(14) by x^υ, we get

$$x^\upsilon J_\upsilon(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+2\upsilon}}{m!\Gamma(m+\upsilon+1)2^{2m+\upsilon}}$$

Differentiating this equation with respect to x, and using the relation $\Gamma(m+\upsilon+1) = (m+\upsilon)\Gamma(m+\upsilon)$, we get

$$\frac{d}{dx}\left[x^\upsilon J_\upsilon(x)\right] = \sum_{m=0}^{\infty} \frac{(-1)^m (2m+2\upsilon) x^{2m+2\upsilon-1}}{m!2^{2m+\upsilon}\Gamma(m+\upsilon+1)}$$

$$= x^\upsilon \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\upsilon)} \left(\frac{x}{2}\right)^{2m+\upsilon-1}$$

$$= x^\upsilon J_{\upsilon-1}(x)$$

Therefore, $$\frac{\mathbf{d}}{\mathbf{dx}}[\mathbf{x}^\upsilon \mathbf{J}_\upsilon(\mathbf{x})] = \mathbf{x}^\upsilon \mathbf{J}_{\upsilon-1}(\mathbf{x}) \tag{28}$$

Setting $\upsilon = 0$, we get $J_0'(x) = J_{-1}(x) = -J_1(x)$, since $J_{-n}(x) = (-1)^n J_n(x)$. Integrating Eqn.(28), we can write

$$\int \mathbf{x}^\upsilon \mathbf{J}_{\upsilon-1}(\mathbf{x})\,\mathbf{dx} = \mathbf{x}^\upsilon \mathbf{J}_\upsilon(\mathbf{x}) + \mathbf{c} \tag{29}$$

Similarly, if we multiply the Bessel function $J_\upsilon(x)$ by $x^{-\upsilon}$ and differentiate it with respect to x, we get

$$\frac{d}{dx}[x^{-\upsilon} J_\upsilon(x)] = \sum_{m=0}^{\infty} \frac{(-1)^m (2m) x^{2m-1}}{m!2^{2m+\upsilon}\Gamma(m+\upsilon+1)}$$

$$= \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m-1}}{(m-1)!2^{2m+\upsilon-1}\Gamma(m+\upsilon+1)}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^{2k+1}}{k!2^{2k+\upsilon+1}\Gamma(k+\upsilon+2)} \qquad [\text{setting } m-1=k]$$

$$= x^{-\upsilon} \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k!\Gamma(k+\upsilon+2)} \left(\frac{x}{2}\right)^{2k+\upsilon+1}$$

$$= -x^{-\upsilon} J_{\upsilon+1}(x)$$

Therefore, $$\frac{\mathbf{d}}{\mathbf{dx}}[\mathbf{x}^{-\upsilon} \mathbf{J}_\upsilon(\mathbf{x})] = -\mathbf{x}^{-\upsilon} \mathbf{J}_{\upsilon+1}(\mathbf{x}) \tag{30}$$

Integrating Eqn.(30), we obtain

$$\int \mathbf{x}^{-\upsilon} \mathbf{J}_{\upsilon+1}(\mathbf{x})\,\mathbf{dx} = -\mathbf{x}^{-\upsilon} \mathbf{J}_\upsilon(\mathbf{x}) + \mathbf{c} \tag{31}$$

If we carry out the differentiation on the left-hand sides of Eqns.(28) and (30) and divide the results by the factors $x^{\upsilon-1}$ and $x^{-\upsilon-1}$, respectively, we deduce that

$$\mathbf{x}\,\mathbf{J}_\upsilon'(\mathbf{x}) + \upsilon\,\mathbf{J}_\upsilon(\mathbf{x}) = \mathbf{x}\,\mathbf{J}_{\upsilon-1}(\mathbf{x}) \tag{32}$$

and $$\mathbf{x}\,\mathbf{J}_\upsilon'(\mathbf{x}) - \upsilon\,\mathbf{J}_\upsilon(\mathbf{x}) = -\mathbf{x}\,\mathbf{J}_{\upsilon+1}(\mathbf{x}) \tag{33}$$

When υ is an integer n, we write

$$\mathbf{x}\,\mathbf{J}_\mathbf{n}'(\mathbf{x}) = \mathbf{x}\,\mathbf{J}_{\mathbf{n}-1}(\mathbf{x}) - \mathbf{n}\,\mathbf{J}_\mathbf{n}(\mathbf{x}) \tag{34}$$

and $$\mathbf{x}\,\mathbf{J}_\mathbf{n}'(\mathbf{x}) = \mathbf{n}\,\mathbf{J}_\mathbf{n}(\mathbf{x}) - \mathbf{x}\,\mathbf{J}_{\mathbf{n}+1}(\mathbf{x}) \tag{35}$$

Finally, the sum and difference of relations (32) and (33) yields, respectively

$$\mathbf{2\,J'_{\upsilon}(x) = J_{\upsilon-1}(x) - J_{\upsilon+1}(x)} \tag{36}$$

and $$\mathbf{2\,\upsilon J_{\upsilon}(x) = x\;J_{\upsilon-1}(x) + x\;J_{\upsilon+1}(x)} \tag{37}$$

Let us look at the following examples.

Example 3: Show that $J_4(x) = \frac{8}{x}\left(\frac{6}{x^2} - 1\right) J_1 - \left(\frac{24}{x^2} - 1\right) J_0$.

Solution: Using recurrence relation (37), we obtain

$$J_{n+1}(x) = \frac{2n}{x}\, J_n(x) - J_{n-1}(x)$$

$$\therefore J_4(x) = \frac{6}{x} J_3(x) - J_2(x)$$

$$= \frac{6}{x}\left\{\left(\frac{8}{x^2} - 1\right) J_1 - \frac{4}{x} J_0\right\} - \frac{2}{x}\, J_1 + J_0$$

$$= \frac{8}{x}\left(\frac{6}{x^2} - 1\right) J_1 - \left(\frac{24}{x^2} - 1\right) J_0$$

***

Example 4: Show that

a) $[x\;J_n(x)\;J_{n+1}(x)]' = x\left[J_n^2(x) - J_{n+1}^2(x)\right]$

b) $J_1''(x) = \frac{1}{x}\, J_2(x) - J_1(x)$

c) $J_{-5/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{1}{x^2}(3 - x^2)\cos x + \frac{3}{x}\sin x\right]$

Solution:

a) $[x\,J_n(x)\;J_{n+1}(x)]' = [\{x^{-n}\,J_n(x)\}\,\{x^{n+1}\;J_{n+1}^{(x)}\}]'$

$$= x^{-n} J_n(x)[x^{n+1} J_n(x)] + x^{n+1} J_{n+1}[-x^{-n} J_{n+1}(x)]$$

[using relations (28) and (30)]

$$= x\;J_n^2(x) - x\;J_{n+1}^2\;(x)$$

b) Using relation (36) with $\upsilon = 1$, we get

$$2J_1'(x) = J_0(x) - J_2(x)$$

Differentiating it we obtain

$$2\,J_1''(x) = J_0'(x) - J_2'(x)$$

Using relation (32) with $\upsilon = 2$, and the fact that $J_0'(x) = -J_1(x)$, we get

$$2\,J_1''(x) = -J_1(x) - \left[\;J_1(x) - \frac{2}{x}\;J_2(x)\right]$$

or, $$J_1''(x) = \frac{1}{x}\;J_2(x) - J_1(x)\,.$$

c) Using relation (37) for $\upsilon = -3/2$ and $\upsilon = -1/2$, we obtain

$$-\,3\,J_{-3/2}(x) = x\;[\;J_{-5/2}(x) + J_{-1/2}(x)]$$

and $$-\,J_{-1/2}(x) = x\;[\,J_{-3/2}(x) + J_{1/2}(x)\;]$$

Eliminating $J_{-3/2}(x)$ from the above two equations we obtain

$$x^2 J_{-5/2}(x) = 3\,J_{-1/2}(x) - x^2\,J_{-1/2}(x) + 3x\;J_{1/2}(x)$$

Using $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x$ and $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x$, in the above equation we have,

$$J_{-5/2}(x)=\sqrt{\frac{2}{\pi x}}\left[\frac{1}{x^2}(3-x^2)\cos x+\frac{3}{x}\sin x\right]$$

***

Example 5: Prove the following

a) $\int J_0(x)\ J_1(x)\,dx=-\frac{1}{2}J_0^2(x)+c$

b) $\int x\,J_0^2(x)\,dx=\frac{x^2}{2}[J_0^2(x)+J_1^2(x)]+c$

c) $\int J_3(x)\,dx=-J_2(x)-\frac{2}{x}J_1(x)+c$

Solution: a) We know that

$$[J_0^2(x)]'=2J_0(x)J_0'(x)=-2J_0(x)J_1(x) \qquad [\because J_0'(x)=-J_1(x)]$$

$$\therefore \quad \int J_0(x)J_1(x)\,dx=\frac{-1}{2}J_0^2(x)+c$$

b) $\int x\,J_0^2(x)\,dx=\frac{x^2}{2}J_0^2(x)+\int x^2J_0(x)\ J_1(x)\,dx \qquad [\because J_0'(x)=-J_1(x)]$

$$=\frac{x^2}{2}J_0^2(x)+\int (x\,J_0(x))\ (x\ J_1(x))\,dx$$

$$=\frac{x^2}{2}J_0^2(x)+(x\,J_1(x))^2-\int (x\,J_1(x))(x\ J_1(x))'\,dx \quad \text{[using relation (29)]}$$

$$=\frac{x^2}{2}J_0^2(x)+(x\,J_1(x))^2-\frac{(x\,J_1(x))^2}{2}+c$$

$$=\frac{x^2}{2}\,[J_0^2(x)+J_1^2(x)]+c$$

c) $\int J_3(x)\,dx=\int x^2(x^{-2}J_3(x))\,dx$

$$=x^2(-x^{-2}J_2(x))+\int x^{-2}J_2(x)\,2x\,dx \quad \text{[using relation (31)]}$$

$$=-J_2(x)+2(-x^{-1}J_1(x))+c \qquad \text{[using relation (31)]}$$

$$=-J_2(x)-\frac{2}{x}\,J_1(x)+c$$

***

Example 6: Evaluate $\int x^3J_3(x)\,dx$.

Solution: $\int x^3J_3(x)\,dx=\int x^5[x^{-2}J_3(x)]\,dx$

$$=x^5[-x^{-2}J_2(x)]-\int[-x^{-2}J_2(x)]5x^4dx \quad \text{[using relation (31)]}$$

$$=-x^3J_2(x)+5\int x^2J_2(x)\,dx$$

Now $\int x^2J_2(x)\,dx=\int x^3[x^{-1}J_2(x)]\,dx$

$$=x^3[-x^{-1}J_1(x)]-\int[-x^{-1}J_1(x)]3x^2\,dx$$

$$=-x^2J_1(x)+3\int x\,J_1(x)\ dx$$

$$\int x\,J_1(x)\,dx=-\int x\,J_0'(x)\,dx=-[x\,J_0(x)-\int J_0(x)\,dx]$$

$$=-x\,J_0(x)+\int J_0(x)\,dx$$

Then $\int x^3J_3(x)\,dx=-x^3J_2(x)-5x^2J_1(x)-15x\,J_0(x)+15\int J_0(x)\,dx$

The integral $\int J_0(x)\,dx$ cannot be obtained in closed form.

***

You may now try the following exercises.

E6) Express the Bessel functions $J_2(x)$ and $J_3(x)$ in terms of $J_0(x)$ and $J_1(x)$.

E7) Prove the following:

a) $J_2'(x) = \left(1 - \frac{4}{x^2}\right) J_1(x) + \frac{2}{x} J_0(x)$

b) $J_n''(x) = \frac{1}{4}[J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)]$

c) $[J_\upsilon^2(x)]' = \frac{x}{2\upsilon}[J_{\upsilon-1}^2(x) - J_{\upsilon+1}^2(x)]$

d) $J_{-3/2}(x) = -\sqrt{\frac{2}{\pi x}}\left(\frac{x \sin x + \cos x}{x}\right)$

e) $J_{1/2}^2(x) + J_{-1/2}^2(x) = \frac{2}{\pi x}$.

E8) Show that

a) $4J_0'''(x) + 3J_0'(x) + J_3(x) = 0$

b) $J_0^2(x) + 2(J_1^2(x) + J_2^2(x) + J_3^2(x) + \cdots) = 1$

E9) Prove the following:

a) $\int J_{n+1}(x)\,dx = \int J_{n-1}(x)\,dx - 2J_n(x) + c$

b) $\int J_2(x)\,dx = -2J_1(x) + \int J_0(x)\,dx + c$

c) $\int J_5(x)\,dx = -J_4(x) - \frac{4}{3}J_3(x) - \frac{8}{x^2}J_2(x) + c$

d) $\int x^3 J_0(x)\,dx = x(x^2 - 4)J_1(x) + 2x^2 J_0(x) + c$

e) $\int x^5 J_0(x)\,dx = x^5 J_1(x) - 4x^4 J_2(x) + 8x^3 J_3(x) + c$

f) $\int_0^a x\,J_0(r,x)\,dx = \frac{a}{r}J_1(ar)$, $\quad a, r$ are constant.

g) $\int_0^1 (x - x^3)J_0(x)\,dx = 4J_1(1) - 2J_0(1)$

h) $\int J_1(x)\sin x\,dx = x\,J_1(x)\sin x + J_0(x)(x\cos x - \sin x) + c$.

We shall now show that the Bessel function of the first kind, $J_n(x)$ can also be obtained using a generating function.

Generating Function

We shall show that

$$e^{\frac{1}{2}x\left(t - \frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(x)\,t^n\ , \ t \neq 0. \tag{38}$$

The function on the left hand side is called the **generating function** of the Bessel function $J_n(x)$.

We have

$$e^{\frac{x}{2}\left(t - \frac{1}{t}\right)} = e^{xt/2}e^{-x/2t} = \left\{\sum_{r=0}^{\infty} \frac{(xt/2)^r}{r!}\right\}\left\{\sum_{m=0}^{\infty} \frac{(-x/2t)^m}{m!}\right\}$$

$$=\sum_{r=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m (x/2)^{r+m}\, t^{r-m}}{r!m!} \tag{39}$$

Our goal is to obtain a single series in powers of t. Thus we make the change of index $n = r - m$, and because both r and m have an infinite range of values, it follows that n varies from $-\infty$ to ∞. Then the sum on the right hand side of Eqn.(39) becomes

$$\sum_{n=-\infty}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m (x/2)^{n+2m}\, t^n}{(n+m)!m!} = \sum_{n=-\infty}^{\infty}\left\{\sum_{m=0}^{\infty}\frac{(-1)^m (x/2)^{n+2m}}{m!(n+m)!}\right\} t^n$$

$$= \sum_{n=-\infty}^{\infty} J_n(x)\, t^n.$$

which proves formula (38).

In many situations the integral representation of $J_n(x)$ is more convenient to use than its series representation. We shall now see how the generating function relation can be used to derive the integral representation of Bessel function of the first kind.

Integral Representation

We start with the generating function relation

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n,\ \ t \neq 0.$$

and set $t = e^{i\theta}$. Then

$$e^{\frac{x}{2}(e^{i\theta}-e^{-i\theta})} = e^{i x \sin\theta} = \sum_{n=-\infty}^{\infty} J_n(x) e^{in\theta} = \sum_{n=-\infty}^{\infty} J_n(x)[\cos n\theta + i \sin n\theta]$$

$$\therefore \cos(x\sin\theta) + i\sin(x\sin\theta) = \{J_0(x) + [J_{-1}(x) + J_1(x)]\cos\theta + [J_{-2}(x) + J_2(x)]\cos 2\theta + \cdots\}$$
$$+\, i\{[J_1(x) - J_{-1}(x)]\sin\theta + [J_2(x) - J_{-2}(x)]\sin 2\theta + \cdots\}$$

$$= \{J_0(x) + 2J_2(x)\cos 2\theta + \cdots\} + i\{2J_1(x)\sin\theta + 2J_3(x)\sin 3\theta + \cdots\} \tag{40}$$

where in Eqn.(40), we have used the fact that $J_{-n}(x) = (-1)^n J_n(x)$.

Equating real and imaginary parts of Eqn.(40), we obtain

$$\cos(x\sin\theta) = J_0(x) + 2\{J_2(x)\cos 2\theta + J_4(x)\cos 4\theta + \cdots\} \tag{41}$$

and $$\sin(x\sin\theta) = 2\{J_1(x)\sin\theta + J_3(x)\sin 3\theta + \cdots\} \tag{42}$$

These series are usually called the **Jacobi series.** Multiplying both sides of Eqn.(41) by $\cos n\theta$ and integrating over the interval $[0, \pi]$, we obtain

$$\frac{1}{\pi}\int_0^{\pi}\cos(x\sin\theta)\cos n\theta\, d\theta = \begin{cases} J_n(x), & n \text{ is even} \\ 0 & , \ n \text{ is odd}\end{cases} \tag{43}$$

where we have used the result $\int_0^{\pi}\cos m\theta \cos n\theta\ d\theta = \begin{cases} 0, m \neq n \\ \dfrac{\pi}{2}, m = n\end{cases}$.

Similarly, multiplying both sides of Eqn.(42) by $\sin n\theta$, integrating over the interval $[0, \pi]$ and using $\int_0^{\pi}\sin m\theta \sin n\theta\, d\theta = \begin{cases} 0, m \neq n \\ \dfrac{\pi}{2}, m = n\end{cases}$, we obtain

$$\frac{1}{\pi}\int_0^{\pi}\sin(x\sin\theta)\sin n\theta\ d\theta = \begin{cases} J_n(x), & n \text{ is odd} \\ 0 & , \ n \text{ is even}\end{cases} \tag{44}$$

Adding Eqns.(43) and (44), we obtain

$$\frac{1}{\pi}\int_0^{\pi} \cos n\theta \cos(x\sin\theta) + \sin n\theta \sin(x\sin\theta)\ d\theta = J_n(x)$$

or, $$\mathbf{\frac{1}{\pi}\int_0^{\pi} \cos(n\,\theta - x\sin\theta)\ d\theta = J_n(x)} \qquad (45)$$

for all **integral values of** n.

Eqn.(45) gives the **Bessel's integral formula.**

Let us consider the following examples.

Example 7: Show that

$$J_n(x+y) = \sum_{k=-\infty}^{\infty} J_{n-k}(x)\ J_k(y) \qquad (46)$$

Solution: Using formula (38), we can write

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}\, e^{\frac{y}{2}\left(t-\frac{1}{t}\right)} = e^{\left[\frac{x+y}{2}\left(t-\frac{1}{t}\right)\right]} = \sum_{n=-\infty}^{\infty} J_n(x+y)\ t^n$$

Now the product of two exponentials on the left is also

$$\left[\sum_{j=-\infty}^{\infty} J_j(x)\, t^j\right]\left[\sum_{k=-\infty}^{\infty} J_k(y)\ t^k\right] = \sum_{n=-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty} J_{n-k}(x)\ J_k(y)\right] t^n$$

Equating the coefficients of t^n in the above two expressions, we obtain

$$J_n(x+y) = \sum_{k=-\infty}^{\infty} J_{n-k}(x)\ J_k(y).$$

***

Example 8: Using the generating function for $J_n(x)$, prove that

$J_{n-1}(x) + J_{n+1}(x) = \dfrac{2n}{x} J_n(x)$, for integer values of n.

Solution: We know that

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n .$$

Differentiating both sides with respect to t, we have

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}\,\frac{x}{2}\left(1+\frac{1}{t^2}\right) = \sum_{n=-\infty}^{\infty} n\ J_n(x)\ t^{n-1}$$

or, $$\frac{x}{2}\left(1+\frac{1}{t^2}\right)\sum_{n=-\infty}^{\infty} J_n(x)\ t^n = \sum_{n=-\infty}^{\infty} n\ J_n(x)\ t^{n-1}$$

i.e., $$\sum_{n=-\infty}^{\infty} \frac{x}{2}\left(1+\frac{1}{t^2}\right) J_n(x)\, t^n = \sum_{n=-\infty}^{\infty} n\ J_n(x)\ t^{n-1}$$

The above equation can be written as

$$\sum_{n=-\infty}^{\infty} \frac{x}{2} J_n(x)\ t^n + \sum_{n=-\infty}^{\infty} \frac{x}{2} J_n(x)\ t^{n-2} = \sum_{n=-\infty}^{\infty} n\ J_n(x)\ t^{n-1}$$

or, $$\sum_{n=-\infty}^{\infty} \frac{x}{2} J_n(x)\ t^n + \sum_{n=-\infty}^{\infty} \frac{x}{2} J_{n+2}(x)\ t^n = \sum_{n=-\infty}^{\infty} (n+1)\ J_{n+1}(x)\ t^n$$

i.e. $$\sum_{n=-\infty}^{\infty} \left[\frac{x}{2} J_n(x) + \frac{x}{2} J_{n+2}(x)\right] t^n = \sum_{n=-\infty}^{\infty} (n+1)\ J_{n+1}(x)\ t^n$$

Equating the coefficients of t^n on both the sides, we obtain

$$\frac{x}{2} J_n(x) + \frac{x}{2} J_{n+2}(x) = (n+1) J_n(x)$$

Replacing n by $(n-1)$ in the above equation, the required result is obtained.

***

Example 9: Show that

$$\cos(x\cos\theta) = J_0 - 2J_2\cos 2\theta + 2J_4\cos 4\theta - \cdots$$

and $$\sin(x\cos\theta) = 2[J_1\cos\theta - J_3\cos 3\theta + \cdots]$$

Solution: From Eqn.(38), we have

$$\begin{aligned} e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} &= J_0(x) + J_1(x)\,t + J_2(x)\,t^2 + \cdots + J_n(x)\,t^n + \cdots \\ &\quad - J_1(x)\,t^{-1} + J_2(x)\,t^{-2} - \cdots + (-1)^n J_n(x)\,t^{-n} + \cdots \\ &= J_0(x) + \left(t - \frac{1}{t}\right) J_1(x) + \left(t^2 + \frac{1}{t^2}\right) J_2(x) + \cdots \end{aligned} \tag{47}$$

Now if we put $t = i\,e^{i\theta}$ in Eqn.(47), then

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = e^{i x\cos\theta} = \cos(x\cos\theta) + i\sin(x\cos\theta) \tag{48}$$

The right hand side of Eqn.(47) becomes

$$\begin{aligned} &J_0 + 2i\,J_1\cos\theta - 2J_2\cos 2\theta - 2i\,J_3\cos 3\theta + \cdots \\ &= [J_0 - 2J_2\cos 2\theta + 2J_4\cos 4\theta - \cdots] + 2i[J_1\cos\theta - J_3\cos 3\theta + \cdots] \end{aligned} \tag{49}$$

Comparing Eqns.(48) and (49), we obtain

$$\cos(x\cos\theta) = J_0 - 2J_2\cos 2\theta + 2J_4\cos 4\theta - \cdots \tag{50}$$

and $$\sin(x\cos\theta) = 2[J_1\cos\theta - J_3\cos 3\theta + \cdots] \tag{51}$$

***

Example 10: Show that $J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\cos\theta)\,d\theta$

Solution: Using the result obtained in Example 9 and integrating Eqn.(50) over the interval $[0, \pi]$, we obtain

$$\int_0^{\pi} \cos(x\cos\theta)\,d\theta = J_0(x)\int_0^{\pi} d\theta = \pi J_0(x).$$

since all other terms vanish.

***

It is often necessary in mathematical physics to expand a given function in terms of Bessel functions, where the particular type of expansion depends on the problem at hand. The simplest and most useful expansions of this kind are series of the form

$$f(x) = \sum_{n=1}^{\infty} a_n J_\upsilon(\lambda_n x) = a_1 J_\upsilon(\lambda_1 x) + a_2 J_\upsilon(\lambda_2 x) + \cdots \tag{52}$$

where $f(x)$ is defined on the interval $0 \le x \le 1$, and the λ_n are the positive zeros of some fixed Bessel function $J_\upsilon(x)$ with $\upsilon \ge 0$. We have chosen the interval $0 \le x \le 1$ only for the sake of simplicity. By a simple change of variable a function defined on an interval of the form $0 \le x \le R$, can be considered. Such expansions play a major role in physical problems. We shall illustrate the use of expansion (52) in solving the two-dimensional wave equation for a vibrating circular membrane in Sec.4.3. Determination of the coefficients in Eqn.(52) depend on the orthogonality property of the Bessel functions $J_\upsilon(\lambda_n x)$ which we shall discuss now.

Orthogonality of Bessel Functions

In λ_n and λ_m are the positive zeros of Bessel functions $J_\upsilon(x)$ with $\upsilon \ge 0$ then we shall show that

$$\int_0^1 x\, J_\upsilon(\lambda_m x)\, J_\upsilon(\lambda_n x)\, dx = \begin{cases} 0 & , \quad \text{if } m \neq n \\ \dfrac{1}{2} J^2_{\upsilon+1}(\lambda_n), & \text{if } m = n \end{cases} \tag{53}$$

Formula (53) shows that the functions $J_\upsilon(\lambda_n x)$ are orthogonal with respect to the weight functions x on the interval $0 \le x \le 1$.

To establish (53), we begin with the fact that $y = J_\upsilon(x)$ is a solution of

$$y'' + \frac{1}{x} y' + \left(1 - \frac{\upsilon^2}{x^2}\right) y = 0$$

If a and b are distinct positive constants, it follows that the function $u(x) = J_\upsilon(ax)$ and $v(x) = J_\upsilon(bx)$ satisfy the equations

$$u'' + \frac{1}{x} u' + \left(a^2 - \frac{\upsilon^2}{x^2}\right) u = 0 \tag{54}$$

and $$v'' + \frac{1}{x} v' + \left(b^2 - \frac{\upsilon^2}{x^2}\right) v = 0 \tag{55}$$

Multiply Eqns.(54) and (55) by v and u, respectively, and subtract the results, to obtain

$$\frac{d}{dx}(u'v - v'u) + \frac{1}{x}(u'v - v'u) = (b^2 - a^2)uv$$

After multiplication with x, the above equation becomes

$$\frac{d}{dx}[x\,(u'v - v'u)] = (b^2 - a^2)\; xuv \tag{56}$$

Integrating Eqn.(56) from $x = 0$ to $x = 1$, we get

$$(b^2 - a^2) \int_0^1 x\;u\;v\,dx = [x\,(u'v - v'u)]_0^1 \tag{57}$$

In Eqn.(57), the expression in brackets on the right hand side clearly vanishes at $x = 0$. Now since we have $u(1) = J_\upsilon(a)$ and $v(1) = J_\upsilon(b)$, it follows that if a and b are distinct positive zeros λ_m and λ_n of $J_\upsilon(x)$, then integral on the left is zero. That is, we have

$$\int_0^1 x\; J_\upsilon(\lambda_m x)\; J_\upsilon(\lambda_n x) = 0 \tag{58}$$

which is the first part of (53). Now evaluation of integral (58) has to be done when $m = n$.

Multiplying Eqn.(54) by $2x^2u'$, it becomes

$$2x^2u'\;u'' + 2x\;u'^2 + 2a^2x^2\;u\;u' - 2\upsilon^2\;u\;u' = 0$$

or, $$\frac{d}{dx}(x^2u'^2) + \frac{d}{dx}(a^2x^2u^2) - 2a^2x\;u^2 - \frac{d}{dx}(\upsilon^2u^2) = 0$$

Integrating the above equation from $x = 0$ to $x = 1$, we obtain

$$2\;a^2\int_0^1 x\,u^2\;dx = [x^2\;u'^2 + (a^2x^2 - \upsilon^2)\;u^2]_0^1 \tag{59}$$

When $x = 0$, the expression in brackets vanishes, and since $u'(1) = a\,J'_\upsilon(a)$, Eqn.(59) yields

$$\int_0^1 x\;J_\upsilon^2(ax)\;dx = \frac{1}{2}\;J'^2_\upsilon(a) + \frac{1}{2}\left(1 - \frac{\upsilon^2}{a^2}\right) J_\upsilon^2(a)$$

Putting $a = \lambda_n$, we obtain

$$\int_0^1 x\, J_\upsilon^2(\lambda_n x)\,dx = \frac{1}{2}\, J_\upsilon'^2(\lambda_n) = \frac{1}{2}\, J_{\upsilon+1}^2(\lambda_n) \tag{60}$$

where we have used the relation $J_\upsilon'(x) - \frac{\upsilon}{x} J_\upsilon(x) = J_{\upsilon+1}(x)$ in Eqn.(60).

Thus formula (53) is completely established.

As an illustration of this formula let us see how it is used in determining the coefficients of the series (52).

If an expansion of the form (52) is assumed to be possible, then multiplying Eqn.(52) throughout by $x J_\upsilon(\lambda_m x)$, integrating term by term from 0 to 1, and using formula (53), we obtain

$$\int_0^1 x f(x)\, J_\upsilon(\lambda_m x)\,dx = \frac{a_m}{2}\, J_{\upsilon+1}^2(\lambda_m)$$

and on replacing m by n we obtain the following formula for a_n:

$$a_n = \frac{2}{J_{\upsilon+1}^2(\lambda_n)} \int_0^1 x\, f(x)\, J_\upsilon(\lambda_n x)\, dx \tag{61}$$

The series (52), with its coefficients calculated by (61), is called the **Bessel series** or sometimes the **Fourier-Bessel series** of the function $f(x)$.

As an illustration consider the following example.

Example 11: Compute the Bessel series of the function $f(x) = 1$, for the interval $0 \le x \le 1$, in terms of the functions $J_\upsilon(\lambda_n x)$, where λ_n are the positive zeros of $J_0(x)$.

Solution: In this case Eqn.(61), yields

$$a_n = \frac{2}{J_1^2(\lambda_n)} \int_0^1 x\, J_0(\lambda_n x)\, dx$$

Now using the recurrence relation (29) for $\upsilon = 1$, we obtain

$$\int_0^1 x\, J_0(\lambda_n x)dx = \left[\frac{1}{\lambda_n} x\, J_1(\lambda_n x)\right]_0^1 = \frac{J_1(\lambda_n)}{\lambda_n}$$

$$\therefore \quad a_n = \frac{2}{\lambda_n J_1(\lambda_n)}$$

It follows then

$$1 = \sum_{n=1}^{\infty} \frac{2}{\lambda_n J_1(\lambda_n)}\, J_0(\lambda_n x), \quad (0 \le x \le 1)$$

is the desired Bessel series.

***

You may now try the following exercises.

E10) Show that the change of variables $y = u\,x^a$, $t = x^c$ and $z = bt$ reduces the differential equation

$$x^2 y'' + (1-2a)\,xy' + [b^2c^2x^{2c} + (a^2 - n^2c^2)]\,y = 0$$

to the Bessel's equation

$$z^2 \frac{d^2u}{dz^2} + z\frac{du}{dz} + (z^2 - n^2)\, u = 0.$$

E11) Prove the identity

$$1 = J_0^2(x) + 2J_1^2(x) + 2J_2^2(x) + \cdots$$

E12) If $f(x)$ is defined by

$$f(x)=\begin{cases}1, & 0\le x\le \frac{1}{2}\\ \frac{1}{2}, & x=\frac{1}{2}\\ 0, & \frac{1}{2}<x\le 1\end{cases}$$

show that

$$f(x)=\sum_{n=1}^{\infty}\frac{J_1(\lambda_n/2)}{\lambda_n J_1^2(\lambda_n)}J_0(\lambda_n x)$$

where the λ_n are the positive zeros of $J_0(x)$.

E13) If $f(x)=x^{\upsilon}$ for the interval $0\le x<1$, show that its Bessel series in the functions $J_\upsilon(\lambda_n x)$, where the λ_n are the positive zeros of $J_\upsilon(x)$, is

$$x^{\upsilon}=\sum_{n=1}^{\infty}\frac{2}{\lambda_n J_{\upsilon+1}(\lambda_n)}J_\upsilon(\lambda_n x).$$

We have shown in Sec.4.2 that when υ is not an integer then $J_\upsilon(x)$ and $J_{-\upsilon}(x)$ are linearly independent solutions of Eqn.(1) and the complete solution is given by

$$y(x)=A\,J_\upsilon(x)+B\,J_{-\upsilon}(x).$$

However, when $\upsilon=n$ is an integer, $J_n(x)$ and $J_{-n}(x)$ are linearly dependent since $J_{-n}(x)=(-1)^n J_n(x)$. We therefore have only one independent solution. In order to find the general solution, we now obtain the second linearly independent solution of the Bessel's equation.

4.2.2 Bessel Function of the Second Kind

For obtaining the second solution of the Bessel's equation for the case when $\upsilon=n$ is an integer, we once again make use of the power series method discussed in Unit 2. Let us illustrate the method for the case $n=0$, for Bessel's equations of order zero

$$x\,y''+y'+xy=0 \qquad (62)$$

If we substitute the series solution $y(x)=\sum_{m=0}^{\infty}c_m x^{m+k}$ in Eqn.(62), we have the case of repeated indicial roots $k=0$. If you recall Case 3, discussed in Sec.2.4.2, Unit 2, then you know that two linearly independent solutions are given by $[y(x)]_{k=0}$ and $[\partial y/\partial k]_{k=0}$. Accordingly, if $y_1(x)=[y(x)]_{k=0}$, then second solution is of the form

$$\begin{aligned}y_2(x)&=y_1(x)\ln x+[c_1x+c_2x^2+\cdots],\ x>0\\ &=J_0(x)\ln x+[c_1x+c_2x^2+\cdots]\end{aligned}$$

since $y_1(x)=J_0(x)$.

Further, we have

$$y_2'(x)=J_0'\ln x+\frac{1}{x}J_0+[c_1+2c_2x+3c_3x^2+\cdots]$$

$$y_2''(x)=J_0''\ln x+\frac{2}{x}J_0'-\frac{1}{x^2}J_0+[2c_2+6c_3x+\cdots]$$

Substituting the above values of $y_2(x)$, $y_2'(x)$ and $y_2''(x)$ in Eqn.(62), we obtain

$$\begin{aligned}&[x\,J_0''+J_0'+x\,J_0]\ln x+2J_0'+[c_1+4c_2x+(c_1+9c_3)x^2+\cdots\\ &\qquad +[c_{m-1}+(m+1)^2c_{m+1}]x^m+\cdots]=0 \qquad (63)\end{aligned}$$

The first term of Eqn(63) vanishes as $J_0(x)$ is a solution of Eqn.(62). Substituting

$$J_0'(x)=\sum_{m=1}^{\infty}\frac{(-1)^m 2m\,x^{2m-1}}{(m!)^2\,2^{2m}}=\sum_{m=1}^{\infty}\frac{(-1)^m x^{2m-1}}{m!(m-1)!2^{2m-1}}=\frac{-x}{2}+\frac{x^3}{(2!)\,2^3}-\frac{x^5}{(3!)\,(2!)\,2^5}+\cdots$$

in Eqn.(63), we obtain

$$\left[-x+\frac{x^3}{(2!)2^2}-\frac{x^5}{(3!)\,(2!)2^4}+\cdots\right]+[c_1+4c_2x+(c_1+9c_3)x^2+(c_2+16c_4)x^4+\cdots]=0$$

Equating the coefficients of even power of x to zero, we obtain

$$c_1=0, c_1+9c_3=0\Rightarrow c_3=0,\ldots,c_{2m-1}+(2m+1)^2c_{2m+1}=0, m=1,2,\ldots$$

Therefore, $c_1=0=c_3=c_5=\cdots$

Coefficient of x equated to zero yields,

$$-1+4c_2=0,\ \text{or}\ ,\ c_2=\frac{1}{4}=\frac{1}{2^2}.$$

Equating the coefficient of x^{2m+1} to zero, we obtain

$$\frac{(-1)^{m+1}}{m!(m+1)!2^{2m}}+c_{2m}+(2m+2)^2\,c_{m+2}=0,\ \ m=1,2,\ldots$$

For $m=1$, we get

$$\frac{1}{(2!)2^2}+c_2+16c_4=0,\ \text{or}\ \ c_4=\frac{-1}{16}\left[\frac{1}{(2!)2^2}+\frac{1}{2^2}\right]=\frac{-1}{2^2 4^2}\left[1+\frac{1}{2}\right]$$

For $m=2$, we have

$$\frac{-1}{(3!)\,(2!)\,2^4}+c_4+36c_6=0$$

or, $$c_6=\frac{1}{6^2}\left[\frac{1}{(3!)\,(2!)\,2^4}+\frac{1}{2^2.4^2}\left(1+\frac{1}{2}\right)\right]=\frac{1}{2^2.4^2.6^2}\left[1+\frac{1}{2}+\frac{1}{3}\right],\ldots$$

Therefore, the second linearly independent solution of Eqn.(62) is

$$y_2(x)=J_0(x)\ln x+\left[\frac{1}{2^2}x^2-\frac{1}{2^2.4^2}\left(1+\frac{1}{2}\right)x^4+\frac{1}{2^2.4^2.6^2}\left(1+\frac{1}{2}+\frac{1}{3}\right)x^6-\cdots\right]$$

and is called the **Neumann function of order zero.** However, any other linear combination of $y_1(x)$ and $y_2(x)$ can also be taken as the second linearly independent solution. In practice, it is convenient and standard to use

$$y_2^*(x)=\frac{2}{\pi}[\,y_2(x)+(\gamma-\ln 2)\,J_0\,(x)\,] \tag{64}$$

where γ is the **Euler constant** $(\gamma=0.5772157)$, as the second linearly independent solution. This solution is called the **Bessel function of the second kind of order zero**. It is denoted by $Y_0(x)$.

Hence,

$$Y_0(x)=\frac{2}{\pi}\left[J_0(x)\ln\left(\frac{x}{2}+\gamma\right)+\frac{x^2}{2^2}-\left(1+\frac{1}{2}\right)\frac{x^4}{2^4(2!)^2}+\left(1+\frac{1}{2}+\frac{1}{3}\right)\frac{x^6}{2^6(3!)^2}-\cdots\right] \tag{65}$$

It can be seen that $Y_0(x)\sim\frac{2}{\pi}J_0(x)\left(\ln\frac{x}{2}+\gamma\right)$ as $x\to 0$. However, since $J_0(x)\sim 1$ and $|\ln x|>>\gamma-\ln 2$ for small x we deduce that $Y_0(x)$ has the logarithmic behaviour $Y_0(x)\sim\frac{2}{\pi}\ln x$ as $x\to 0$ (see Fig.3).

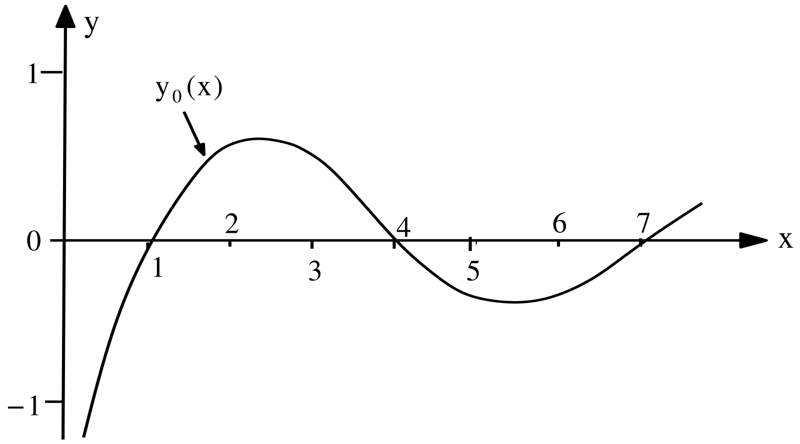

Fig.3: Bessel function of the second kind $Y_0(x)$

For $n = 1,2,\ldots$ the solution can be obtained in a similar manner. It can be seen that the indicial roots $k = \pm\, n$ differ by an integer and the second linearly independent solution always contains a logarithmic term (see Case 2, Sec.2.4.2, Unit 2).

This solution can be written as

$$Y_n(x) = J_n(x)\ln x - \frac{1}{2}\sum_{j=0}^{n-1} \frac{(n-j-1)!}{j!}\left(\frac{x}{2}\right)^{2j-n}$$
$$- \frac{1}{2}\sum_{j=0}^{\infty} \frac{(-1)^j[\phi(j)+\phi(j+n)]}{j!(j+n)!}\left(\frac{x}{2}\right)^{2j+n} \qquad (66)$$

where, $\phi(0) = 0$ and $\phi(j) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{j}$, for $j \geq 1$.

Thus, we have two different general solutions for Eqn.(1), depending on whether υ is an integer or not.

$$y(x) = A\,J_\upsilon(x) + B\,J_{-\upsilon}(x), \text{ if } \upsilon \text{ is not an integer}$$

and $$y(x) = A\,J_n(x) + B\,Y_n(x), \text{ if } \upsilon = n, \text{ an integer.}$$

Without proof we state the result and define the **Bessel function of the second kind of order $\boldsymbol{\upsilon}$** as

$$Y_\upsilon(x) = \frac{(\cos\upsilon\pi)\,J_\upsilon(x) - J_{-\upsilon}(x)}{\sin\upsilon\pi} \qquad (67)$$

and $$Y_n(x) = \lim_{\upsilon\to n} Y_\upsilon(x)$$

that is, the limit of $Y_\upsilon(x)$ as $\upsilon \to n$ $(n = 0,1,2,\ldots)$ give the same result as (66).

Further, you may observe that for $\upsilon > 0$, $J_\upsilon(x) \sim 0$ for $x \to 0$ and thus

$$Y_\upsilon(x) \sim \frac{-J_{-\upsilon}(x)}{\sin\upsilon\pi}$$
$$\sim \frac{-(x/2)^{-\upsilon}}{\Gamma(1-\upsilon)\sin\upsilon\pi}, \quad \upsilon > 0, \quad x \to 0$$

But since $\Gamma(x)\Gamma(1-x) = \pi/\sin\pi x$, we finally obtain the asymptotic formula

$$Y_\upsilon(x) \sim -\frac{\Gamma(\upsilon)}{\pi}\left(\frac{2}{x}\right)^{\upsilon}, \quad \upsilon > 0, \quad x \to 0 \qquad (68)$$

Also since $J_\upsilon(x)$ and $Y_\upsilon(x)$ are linearly independent, the **general solution of the Bessel's differential equation for all values of** υ, is

$$\mathbf{y(x) = A\,J_\upsilon(x) + B\,Y_\upsilon(x)} \qquad (69)$$

where $Y_\upsilon(x)$ is defined by Eqn.(67).

Let us look at the following examples.

Example 12: Find the solution of the differential equation

$$x^2 y'' + xy' + (x^2 - 16)\, y = 0$$

Solution: The given equation is the Bessel's equation with $\upsilon = 4$. Its general solution is thus written as

$$y(x) = A\, J_4(x) + B\, Y_4(x)\,.$$

***

Example 13: Using the substitution $z = 2\sqrt{x}$, obtain the solution of the differential equation $xy'' + y' + y = 0$.

Solution: We have for $z = 2\sqrt{x}$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x}}\frac{dy}{dz} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{1}{x}\frac{d^2y}{dz^2} - \frac{1}{2x^{3/2}}\frac{dy}{dz}$$

and the given equation reduces to

$$z\frac{d^2y}{dz^2} + \frac{dy}{dz} + zy = 0$$

This is the Bessel's equation of order zero. Its general solution is

$$y(z) = A\, J_0(z) + B\, Y_0(z)$$

or, $$y(x) = A\, J_0(2\sqrt{x}) + B\, Y_0(2\sqrt{x})$$

***

You may now try the following exercise.

E14) Using the substitutions given in E10) reduce the following equations to the Bessel's equations and hence find their solution

a) $x^2 y'' + xy' + \frac{1}{4}(x - n^2)\; y = 0$

b) $xy'' + y = 0$

c) $xy'' + y' + y/4 = 0$.

We shall now consider some applications of Bessel functions.

4.3 APPLICATIONS OF BESSEL FUNCTIONS

In this section we shall illustrate the role played by the Bessel functions in getting solutions of problems related to physical situations.

Vibrating Membrane

One of the simplest physical applications of Bessel functions occurs in Euler's theory of the vibration of a circular membrane. In this context a membrane is understood to be a uniform thin sheet of flexible material pulled taut into a state of uniform tension and clamped along a given closed curve. When this membrane is slightly displaced from its equilibrium position and then released, the restoring forces due to the deformation cause it to vibrate and the problem is to analyze this vibrational motion. The governing equation for a wide variety of applications involving vibration phenomena is the wave equation

$$\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = c^{-2}\,\frac{\partial^2 u}{\partial t^2} \tag{70}$$

Here we consider the case of a circular membrane.

The Circular Membrane

We begin by determining the small displacements u of a thin circular membrane of unit radius whose edge is rigidly fixed (see. Fig.4). The governing equation for this problem is the wave Eqn.(70), where c is a physical constant having the dimension of velocity. We shall not be going into the details of formulating Eqn.(70) here. However, we mention that several simplifying assumptions lead to the formulation of partial differential Eqn.(70), and we hope that this equation describes the motion with a reasonable degree of accuracy.

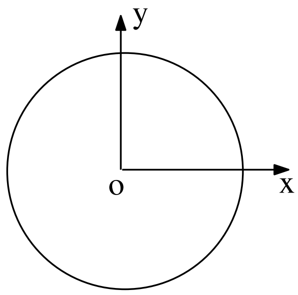

Fig.4

Since we are considering the case of circular membrane, it is natural to use polar coordinates with the origin located at the centre. If the displacement u depends only upon the radial distance r from the centre of the membrane and on time t, then Eqn.(70) reduces to the radial symmetric form of the wave equation given by

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = c^{-2}\frac{\partial^2 u}{\partial t^2}\ , \quad 0 < r < 1, \quad t > 0 \tag{71}$$

where $u = u(r,\ t)$. Since we have assumed that the membrane is rigidly fixed on the boundary, we impose the boundary condition

$$u(1,\ t) = 0\ , \quad t > 0 \tag{72}$$

If the membrane is initially deflected to the shape $f(r)$ with velocity $g(r)$, we also prescribe the initial conditions

$$u(r,0) = f(r)\ , \quad \frac{\partial u}{\partial t}(r,0) = g(r), \quad 0 < r < 1 \tag{73}$$

We now apply the method of separation of variables and look for a particular solution of the form

$$u(r,\ t) = R(r)\ T(t) \tag{74}$$

When values of u and its derivatives are inserted in Eqn.(71), we obtain

$$\frac{R'' + \left(\frac{1}{r}\right)R'}{R} = \frac{T''}{c^2 T} = -\lambda \tag{75}$$

where $-\lambda$ is the separation constant (the choice of the negative sign is conventional, not necessary). Hence Eqn.(71) is equivalent to the system of ordinary differential equations

$$rR'' + R' + \lambda r R = 0\ , \quad 0 < r < 1 \tag{76}$$

$$T'' + \lambda c^2 T = 0\ , \quad t > 0 \tag{77}$$

Under the assumption $u(r,\ t) = R(r)T(t)$, the boundary condition(72) becomes

$$u(1,t) = R(1)\,T(t) = 0$$

from which we deduce that

$$R(1) = 0 \tag{78}$$

If you look at Eqn.(76), you will notice that it is a generalised form of Bessel's equation of order zero. By writing $\lambda = k^2 > 0$, the general solution is obtained as

$$R(r) = C_1\, J_0(kr) + C_2 Y_0(kr)\ , \tag{79}$$

where C_1 and C_2 are constants.

Note that for $\lambda \le 0$, Eqn.(76) has no bounded non-trivial solutions satisfying condition (78).

To maintain finite displacements of the membrane at $r = 0$, we must set $C_2 = 0$, since Y_0 becomes unbounded when the argument is zero. The remaining solution $R(r) = C_1 J_0(kr)$ must then satisfy the boundary condition (78), i.e.

$$R(1) = C_1 J_0(k) = 0 \tag{80}$$

You know that the Bessel function J_0 has infinitely many zeros on the positive axis, but they are not evenly spaced. Thus for $C_1 \neq 0$, we can satisfy Eqn.(80) by selecting k as one of the zeros of J_0, which we denote by $k_n\,(n = 1, 2, 3, \ldots)$. With k so restricted, we set $\lambda = k_n^2\,(n = 1, 2, \ldots)$ in Eqn.(77) to obtain

$$T'' + k_n^2\, c^2\, T^2 = 0$$

which has the general solution

$$T_n(t) = a_n \cos k_n ct + b_n \sin k_n ct, \quad n = 1, 2, \ldots \tag{81}$$

where the a's and b's are arbitrary constants.
Combining our result, we have the family of solutions

$$u_n(r, t) = (a_n \cos k_n ct + b_n \sin k_n ct)\, J_0(k_n r), \; n = 1, 2, \ldots \tag{82}$$

These solutions are called **standing waves**, since each can be viewed as having fixed shape $J_0(k_n r)$ with varying amplitude $T_n(t)$. The zeros of a standing wave, i.e. curves for which $J_0(k_n r) = 0$, are referred to as **nodal lines.** Clearly the number of nodal lines depends on the value of n. For example, when $n = 1$, there is no nodal line for $0 < r < 1$. There is one nodal line when $n = 2$; there are two nodal lines when $n = 3$, and so on (see Fig.5).

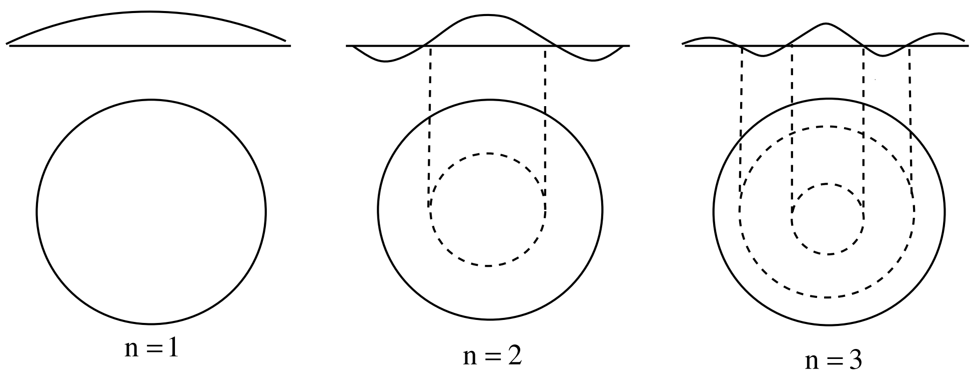

Fig.5: Nodal lines for a circular membrane

By forming a linear combination of the solution (82) through the superposition principle, we obtain

$$u(r, t) = \sum_{n=1}^{\infty} (a_n \cos k_n ct + b_n \sin k_n ct)\; J_0(k_n r) \tag{83}$$

The constants a_n and b_n are selected in such a way that the initial conditions (73) are satisfied. Therefore, we find

$$u(r, 0) = f(r) = \sum_{n=1}^{\infty} a_n J_0(k_n r)\;, \quad 0 < r < 1 \tag{84}$$

and

$$\frac{\partial u}{\partial t}(r, 0) = g(r) = \sum_{n=1}^{\infty} k_n c\, b_n\, J_0(k_n r), \;\; 0 < r < 1 \tag{85}$$

Eqns.(84) and (85) are the **Bessel series** for $f(r)$ and $g(r)$, respectively, where

$$a_n = \frac{2}{[J_1(k_n)]^2} \int_0^1 r\, f(r)\, J_0(k_n r)\; dr, \; n = 1, 2, \ldots \tag{86}$$

and

$$k_n c\, b_n = \frac{2}{[J_1(k_n)]^2} \int_0^1 r\; g(r)\, J_0(k_n r)\, dr, \quad n = 1, 2, \ldots \tag{87}$$

give the coefficients of the series (83) and the required solution is completely determined.

Before we consider another application of Bessel functions you may try the following exercises.

E15) Show that if $u = u(r, t)$, then in polar coordinate system Eqn.(70) reduces to Eqn.(71).

E16) If the initial conditions (73) are given by

$$u(r, 0) = A\ J_0(k_3 r) \qquad (A\ \text{constant})$$

$$\frac{\partial u}{\partial r}\ (r, 0) = 0$$

where $J_0(k_3) = 0$, show that the subsequent displacements of the membrane are described by

$$u(r, t) = A\, J_0(k_3 r) \cos k_3 ct\ .$$

E17) If the initial conditions (73) are given by $f(r) = 0$ and $g(r) = 1$, show that the subsequent displacement of the membrane are described by

$$u(r, t) = \frac{2}{c} \sum_{n=1}^{\infty} \frac{\sin k_n ct}{k_n^2\, J_1(k_n)}\ J_0(k_n r)$$

where, $J_0(k_n) = 0 \quad (n = 1, 2, \ldots)$.

Let us consider another application of the Bessel functions.

Buckling of a Long Column

One of the oldest engineering problems concerns the buckling of vertical columns under a compressive load. The first mathematical model to accurately predict the **critical compressive** load that a column can withstand before deformation or buckling takes place was developed by Euler (1707 – 1783).

Let us consider a long column or rod of length b that is simply supported at each end and is subject to an axial compressive load P applied at the top, as shown in Fig.6.

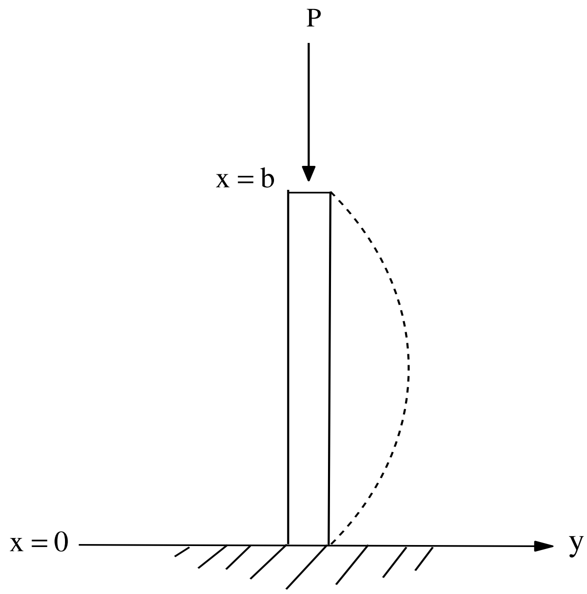

Fig.6: Buckling Column

By 'long' we mean that the length of the column is much greater than the largest dimension in its cross section. From the elementary theory of small deflections of beams and columns, the departure y from the vertical x-axis for such a beam or column is governed by the equation

$$EI\,y'' = M \tag{88}$$

Where M is the **bending moment**, E is the **modulus of elasticity**, and I is the **moment of inertia** of the cross section of the column. When the column is deflected a small amount y from the vertical due to the loading P, the bending moment is $M = -Py$. Since the column is assumed to be simply supported, there can be no displacement at the ends. Thus Eqn.(88) together with the boundary conditions takes the form

$$EI\,y'' + Py = 0\ , \quad y(0) = 0\ , \quad y(b) = 0 \tag{89}$$

The problem described by Eqn.(89), clearly has the trivial solution $y = 0$, corresponding to the column not bending away from the x-axis. However, if P is large enough, there may exist non-trivial solution of Eqn.(89). That is, if P is sufficiently large, the column may suddenly come out of its equilibrium state, called a state of **buckling**. The smallest value of P that leads to buckling is called the **Euler critical load**, and the corresponding deflection mode is called the **fundamental buckling mode.**

The classical example of a buckling column involves the case when E, I, and P are all constant, and then it can be easily shown that the Euler critical load P_1, and corresponding deflection mode are given, respectively, by

$$P_1 = \frac{\pi^2 EI}{b}\ ,\ y = C\sin\frac{\pi x}{b} \tag{90}$$

where C is an arbitrary constant. The constant C remains undetermined here as the model is linear.

We now consider the case where the column is taperred so the moment of inertia is not constant, but is given by $I(x) = \alpha x$, where $\alpha > 0$. In this case Eqn.(89) becomes

$$E\alpha x\,y'' + Py = 0\ ,\ y(0) = 0\ ,\ y(b) = 0 \tag{91}$$

We rewrite the above equation in the form

$$x^2 y'' + k^2 xy = 0 \tag{92}$$

where $k^2 = P / E\alpha$. Further it can be easily verified that first substituting $y = \sqrt{x}\,z$, and then $t = \sqrt{x}$ Eqn.(92) reduces to the form

$$t^2\frac{d^2z}{dt^2} + t\frac{dz}{dt} + (4k^2t^2 - 1)\ z = 0 \tag{93}$$

We see that Eqn.(93) is the Bessel's equation of order one and its solution can be written as

$$z(t) = C_1 J_1(2kt) + C_2 Y_1(2kt)$$

or, $$y(x) = \sqrt{x}\ [C_1 J_1(2k\sqrt{x}) + C_2 Y_1(2k\sqrt{x})] \tag{94}$$

where C_1 and C_2 are arbitrary constants.

To apply the first boundary condition $y(0) = 0$, we use the asymptotic forms for both J_1 and Y_1; hence using formulas (27) and (68), we obtain

$$\begin{aligned} y(0) &= \lim_{x\to 0} \sqrt{x}\ [C_1 J_1(2k\sqrt{x}) + C_2\ Y_1(2k\sqrt{x})] \\ &= \lim_{x\to 0}\sqrt{x}\left[C_1\,k\sqrt{x} - C_2\frac{1}{\pi k\sqrt{x}}\right] \\ &= -C_2\frac{1}{\pi k} \end{aligned} \tag{95}$$

which can vanish only if $C_2 = 0$. The second boundary condition then requires that

$$y(b) = C_1\sqrt{b}\,J_1(2k\sqrt{b}) = 0 \tag{96}$$

There are infinitely many solutions of $J_1(2k\sqrt{b}) = 0$, the first of which yields the Euler critical load P_1. That is, if we let k_1 denote the smallest value of k for which Eqn.(96) is satisfied, the corresponding Euler critical load is

$$P_1 = E\alpha k_1^2 \qquad (97)$$

leading to the fundamental deflection mode

$$y = C_1\sqrt{x}\ J_1(2k_1\sqrt{x}) \qquad (98)$$

where C_1 remains undetermined.

E18) Show that the solution of the boundary value problem (89) for constant E, I and P gives the critical load $P_1 = \pi^2 \dfrac{EI}{b^2}$ and corresponding deflection $y = C\sin\dfrac{\pi x}{b}$, where C is an arbitrary constant.

E19) Verify that on substituting $y = \sqrt{x}z$ and $t = \sqrt{x}$, Eqn.(92) reduces to Eqn.(93).

We now end this unit by giving a summary of what we have covered in it.

4.4 SUMMARY

In this unit we have learnt the following points:

1. The differential equation $x^2y'' + xy' + (x^2 - \upsilon^2)y = 0$, where υ is a non-negative real number is known as **Bessel's equation of order** υ and its solutions are known as **Bessel functions.**
2. When υ **is not an integer,** one of the linearly independent series solution of Bessel's equation is the **Bessel function of the first kind of order** υ and is denoted by $J_\upsilon(x)$ where
$$J_\upsilon(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\upsilon+1)}\left(\frac{x}{2}\right)^{2m+\upsilon}$$
3. The second linearly independent solution of Bessel's equation is $J_{-\upsilon}(x)$ and thus the **general solution of Bessel's equation when** υ **is not an integer** is $y(x) = AJ_\upsilon(x) + BJ_{-\upsilon}(x)$, where A and B are arbitrary constants.
4. When υ**= n is an integer** then $J_n(x) = \sum_{m=0}^{\infty}\dfrac{(-1)^m}{m!(m+n)!}\left(\dfrac{x}{2}\right)^{2m+n}$ is one linearly independent solution of the Bessel's equation and the second linearly independent solution is denoted by $Y_n(x)$, and given by
$$Y_n(x) = J_n(x)\ln x - \frac{1}{2}\sum_{j=0}^{n-1}\frac{(n-j-1)!}{j!}\left(\frac{x}{2}\right)^{2j-n} - \frac{1}{2}\sum_{j=0}^{\infty}(-1)^j\frac{[\phi(j)+\phi)(j+n)]}{j!(j+n)!}\left(\frac{x}{2}\right)^{2j+n}$$
where $\phi(0) = 0$ and $\phi(j) = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{1}{j}$ for $j \geq 1$.
5. The **Bessel function of the second kind of order** υ is defined as
$$Y_\upsilon(x) = \frac{(\cos\upsilon x)\ J_\upsilon(x) - J_{-\upsilon}(x)}{\sin\ \upsilon x}$$
where $Y_n(x) = \lim_{\upsilon\to n} Y_\upsilon(x)$ and the general **solution of the Bessel's differential equation for all values of** υ**, whether** υ **is an integer or not,** is $y(x) = AJ_\upsilon(x) + BY_\upsilon(x)$.
6. Some of the **recurrence relations** for Bessel functions are

$$\frac{d}{dx}[x^{\upsilon}J_{\upsilon}(x)] = x^{\upsilon}J_{\upsilon-1}(x)$$
$$\frac{d}{dx}[x^{-\upsilon}J_{\upsilon}(x)] = -x^{-\upsilon}J_{\upsilon+1}(x)$$
$$x\,J'_{\upsilon}(x) + \upsilon J_{\upsilon}(x) = x\,J_{\upsilon-1}(x)$$
$$x\,J'_{\upsilon}(x) - \upsilon J_{\upsilon}(x) = -x\,J_{\upsilon+1}(x)$$
$$2\,J'_{\upsilon}(x) = J_{\upsilon-1}(x) - J_{\upsilon+1}(x)$$
$$2\upsilon J_{\upsilon}(x) = xJ_{\upsilon-1}(x) + xJ_{\upsilon+1}(x)$$

7. The function on the left side of equation

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(x)\,t^n,\ \ t \neq 0$$

is called the **generating function** of the Bessel function $J_n(x)$.

8. **Orthogonality property** of Bessel function $J_{\upsilon}(x)$ is

$$\int_0^1 x\,J_{\upsilon}(\lambda_m x)\,J_{\upsilon}(\lambda_n x)\,dx = \begin{cases} 0 & ,\ \text{if}\ \ m \neq n \\ \frac{1}{2}J^2_{\upsilon+1}(\lambda_n), & \text{if}\ \ m = n \end{cases}$$

where λ_n and λ_m are the positive zeros of Bessel functions $J_{\upsilon}(x)$ with $\upsilon \geq 0$.

4.5 SOLUTIONS/ANSWERS

E1) $$J_{1/2}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{1/2+2r}}{r!\,\Gamma(r+3/2)} = \frac{(x/2)^{1/2}}{\Gamma(3/2)} - \frac{(x/2)^{5/2}}{1!\,\Gamma(5/2)} + \frac{(x/2)^{9/2}}{2!\,\Gamma(7/2)} - \cdots$$

$$= \frac{(x/2)^{1/2}}{(1/2)\sqrt{\pi}} - \frac{(x/2)^{5/2}}{1!(3/2)(1/2)\sqrt{\pi}} + \frac{(x/2)^{7/2}}{2!(5/2)(3/2)(1/2)\sqrt{\pi}} - \cdots$$

$$= \frac{(x/2)^{1/2}}{(1/2)\sqrt{\pi}}\left\{1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right\} = \frac{(x/2)^{1/2}}{(1/2)\sqrt{\pi}}\frac{\sin x}{x} = \sqrt{\frac{2}{\pi x}}\sin x$$

$$J_{-1/2}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{1/2+2r}}{r!\,\Gamma(r+1/2)} = \frac{(x/2)^{-1/2}}{\Gamma(1/2)} - \frac{(x/2)^{3/2}}{1!\,\Gamma(3/2)} + \frac{(x/2)^{7/2}}{2!\,\Gamma(5/2)} - \cdots$$

$$= \frac{(x/2)^{-1/2}}{\sqrt{\pi}}\left\{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right\} = \sqrt{\frac{2}{\pi x}}\cos x$$

E2) a) $y'' + k^2xy = 0$

Put $y = u\sqrt{x}$ and $z = \frac{2k}{3}x^{3/2}$

$$y' = \frac{d}{dx}\left(u\sqrt{x}\right) = \sqrt{x}\,u' + \frac{1}{2\sqrt{x}}u$$

$$y'' = \sqrt{x}\,u'' + \frac{1}{2\sqrt{x}}u' + \frac{1}{2\sqrt{x}}u' - \frac{1}{4(x)^{3/2}}\,u$$

Then, $y'' + k^2xy = 0 \Rightarrow \sqrt{x}\,u'' + \frac{1}{\sqrt{x}}u' - \frac{1}{4(x)^{3/2}}u + k^2(x)^{3/2}u = 0$

$$u' = \frac{du}{dx} = \frac{du}{dz}.\frac{dz}{dx} = \frac{2k}{3}.\frac{3}{2}.x^{1/2}.\frac{du}{dz} = kx^{1/2}\frac{du}{dz}$$

$$u'' = \frac{d}{dx}\left(kx^{1/2}\frac{du}{dz}\right) = \frac{k}{2\sqrt{x}}\frac{du}{dz} + kx^{1/2}\frac{d^2u}{dz^2}.\frac{dz}{dx} = \frac{k}{2\sqrt{x}}\frac{du}{dz} + k^2x\frac{d^2u}{dz^2}$$

The given equation reduces to

$$\sqrt{x}\left[k^2x\frac{d^2u}{dz^2} + \frac{k}{2\sqrt{x}}\frac{du}{dz}\right] + \frac{1}{\sqrt{x}}\left[k\sqrt{x}\frac{du}{dz}\right] + \left(k^2x^{3/2} - \frac{1}{4x^{3/2}}\right)u = 0$$

or, $\quad \dfrac{d^2u}{dz^2}+\dfrac{1}{z}\dfrac{du}{dz}+\left[1-\left(\dfrac{1}{3z}\right)^2\right]u=0$

or, $\quad z^2\dfrac{d^2u}{dz^2}+z\dfrac{du}{dz}+\left(z^2-\left(\dfrac{1}{3}\right)^2\right)u=0$, which is Bessel's equation of order $1/3$ and its solution is

$$u=J_{1/3}(z)$$

$$\Rightarrow y=\sqrt{x}\,u=\sqrt{x}\,J_{1/3}(z)=\sqrt{x}\;J_{1/3}\left(\frac{2k}{3}x^{3/2}\right).$$

b) Put $y=u\sqrt{x}$ and $z=\dfrac{kx^2}{2}$, then the given equation transforms to

$z^2\dfrac{d^2u}{dz^2}+z\dfrac{du}{dz}+\left(z^2-\left(\dfrac{1}{4}\right)^2\right)u=0$, which is Bessel's equation of order $\dfrac{1}{4}$.

Hence, $\qquad y=\sqrt{x}\;u=\sqrt{x}\;J_{1/4}\left(\dfrac{kx^2}{2}\right).$

c) Put $y=u\sqrt{x}$ and $z=\dfrac{kx^3}{3}$

Then given equation transforms to

$$z^2\frac{d^2u}{dz^2}+z\frac{du}{dz}+\left[z^2-\left(\frac{1}{6}\right)^2\right]u=0,$$

which is Bessel's equation of order $1/6$ and its solution is

$$y=\sqrt{x}u=\sqrt{x}\;J_{1/6}(z)=\sqrt{x}\;J_{1/6}\left(\frac{kx^3}{3}\right).$$

E3) a) $\quad z=\sqrt{x}\Rightarrow\dfrac{dy}{dx}=\dfrac{1}{2\sqrt{x}}\dfrac{dy}{dz},\;\dfrac{d^2y}{dx^2}=\dfrac{1}{4x}\dfrac{d^2y}{dz^2}-\dfrac{1}{4x^{3/2}}\dfrac{dy}{dz}$

Substituting these values in the given equation, we obtain

$$x\frac{d^2y}{dz^2}+\sqrt{x}\frac{dy}{dz}+(x-\upsilon^2)\;y=0$$

$$\therefore y=A\,J_\upsilon(\sqrt{x})+B\,J_{-\upsilon}(\sqrt{x})$$

b) $\quad y=xu\Rightarrow y'=u+xu'$ and $y''=2u'+xu''$. The given equation reduces to $x^2u''+xu'+(x^2-1)\;u=0$

Its general solution is

$u=A\,J_1(x)+B\,J_{-1}(x)$, or, $\quad y=x\,C\,J_1(x) \qquad$ (since $J_{-1}(x)=-J_1(x)$)

c) $\quad y=u/x^k,\; y'=x^{-k}u'-kx^{-k-1}u$

$$y''=x^{-k}u''-2kx^{-k-1}u'+k(k+1)\,x^{-k-2}u$$

The given equation reduces to

$$x^2u''+xu'+(x^2-k^2)\;u=0$$

Its general solution is

$$u=A\,J_k(x)+B\,J_{-k}(x)$$

or, $\quad y=x^{-k}\left[A\,J_k(x)+B\,J_{-k}(x)\right]$

E4) $\quad J_n(-x)=(-1)^nJ_n(x)=J_n(x)$, for n even

and $J_n(-x)=(-1)^nJ_n(x)=-J_n(x)$, for n odd

E5) $J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^4(2!)^2} - \frac{x^6}{2^6(3!)^2} + \cdots$

$$\therefore J_0'(x) = -\frac{x}{2} + \frac{4x^3}{2^4(2!)^2} - \frac{6x^5}{2^6(3!)^2} + \frac{8x^7}{2^8(4!)^2} - \cdots$$

$$= -\left[\frac{x}{2} - \frac{x^3}{2^3!2!} + \frac{x^5}{2^5 2!3!} - \frac{x^7}{2^7 3!4!} + \cdots\right] = -J_1(x)$$

E6) $J_2(x) = \frac{2}{x} J_1(x) - J_0(x)$, $\quad J_3(x) = \left(\frac{8}{x^2} - 1\right) J_1(x) - \frac{4}{x} J_0(x)$

E7) a) Set $\upsilon = 2$ in Eqn.(32) and $\upsilon = 1$ in Eqn.(37) and simplify.

b) Take $\upsilon = n$ and differentiate Eqn.(36). Use Eqn.(36) again.

c) $2J_\upsilon(x)J_\upsilon'(x) = J_\upsilon(x)\ (J_{\upsilon-1}(x) - J_{\upsilon+1}(x))$. Using Eqns.(32) and (33), we get

$J_\upsilon = \frac{x}{2\upsilon}(J_{\upsilon-1} + J_{\upsilon+1})$; hence $2J_\upsilon(x)J_\upsilon'(x) = \frac{x}{\upsilon}(J_{\upsilon-1}^2(x) - J_{\upsilon+1}^2(x))$

Or, $[J_\upsilon^2(x)]' = \frac{x}{2\upsilon}\ (J_{\upsilon-1}^2(x) - J_{\upsilon+1}^2(x))$

d) Using $J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$ for $n = -1/2$ and values for $J_{1/2}(x)$ and $J_{-1/2}(x)$, required solution is obtained.

e) Use the expressions for $J_{1/2}(x)$ and $J_{-1/2}(x)$.

E8) We have $J_0'(x) = -J_1(x)$. Differentiating , we get

$$J_0''(x) = -J_1'(x) = -\frac{1}{2}\ [J_0(x) - J_2(x)] \qquad \text{[using Eqn.(36)]}$$

Differentiating again, we get

$$J_0'''(x) = -\frac{1}{2}[J_0'(x) - J_2'(x)] = -\frac{1}{2}\left[J_0'(x) - \frac{1}{2}(J_1(x) - J_3(x))\right] \qquad \text{[(using Eqn.(36)]}$$

or, $4J_0'''(x) + 2J_0'(x) - J_1(x) + J_3(x) = 0$

or, $4J_0'''(x) + 3J_0'(x) + J_3(x) = 0$, since $J_1(x) = -J_0'(x)$.

E9) a) Use Eqn.(36) with $\upsilon = n$.

b) Write $J_2(x) = J_0(x) - 2J_1'(x)$ and integrate

c) Write $J_5(x) = x^4(x^{-4}\ J_5(x))$ and integrate. Use Eqn.(31) repeatedly.

d) Write $x^3 J_0 = x^2(x\ J_0)$ and use Eqn.(29). Then use Eqn.(37).

e) $\int x^5 J_0\, dx = x^5 J_1 - 4\int x^4 J_1\, dx$

$$= x^5 J_1 - 4x^4 J_2 + 8\int x^3 J_2\, dx = x^5 J_1 - 4x^4 J_2 + 8x^3 J_2 + C$$

f) Let $r\,x = t$ and use Eqn.(29).

g) Using Eqn.(29), we obtain

$$\int_0^1 (x - x^3) J_0\, dx = 2J_2(1) = 2[2J_1(1) - J_0(1)] \qquad \text{[using Eqn.(37)]}$$

h) Integrate by parts, use Eqn.(32) and again integrate by parts.

E10) Use the transformations $y = ux^a$, $t = x^c$ and $z = bt$, successively and get the required result.

E11) From Example 7, $J_n(x+y) = \sum_{k=-\infty}^{\infty} J_{n-k}(x) J_k(y)$

For $n = 0$, we have

$$J_0(x+y) = \sum_{k=-\infty}^{\infty} J_{-k}(x)\ J_k(y)$$

$$= J_0(x)J_0(y) + \sum_{k=1}^{\infty} J_{-k}(x)J_k(y) + \sum_{k=1}^{\infty} J_k(x)J_{-k}(y)$$

$$= J_0(x)J_0(y) + \sum_{k=1}^{\infty} (-1)^k [J_k(x)J_k(y) + J_k(x)J_k(y)]$$

$$= J_0(x)J_0(y) + \sum_{k=1}^{\infty} (-1)^k 2J_k(x)J_k(y)$$

or, $J_0(x+y) = J_0(x)J_0(y) - 2J_1(x)J_1(y) + 2J_2(x)J_2(y) - \cdots$

Replacing y by $-x$ and using the fact that $J_n(x)$ is even or odd according as n is even or odd, the required identity is obtained.

E12) We have, $a_n = \dfrac{2}{J_1^2(\lambda_n)} \displaystyle\int_0^1 x\, f(x)\, J_0(\lambda_n x)\, dx$

Now $\displaystyle\int_0^1 x\, f(x) J_0(\lambda_n x)\, dx = \int_0^{1/2} x\, J_0(\lambda_n x)\, dx = \frac{x}{\lambda_n} J_1(\lambda_n x)\Big|_0^{1/2} = \frac{1}{2\lambda_n} J_1(\lambda_n/2)$

$\therefore a_n = \dfrac{1}{J_1^2(\lambda_n)} \dfrac{1}{\lambda_n} J_1(\lambda_n/2)$

and it follows that, $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{J_1(\lambda_n/2)}{\lambda_n J_1^2(\lambda_n)} J_0(\lambda_n x)$

E13) We have, $a_n = \dfrac{2}{J_{\upsilon+1}^2(\lambda_n)} \displaystyle\int_0^1 x^{\upsilon+1} J_\upsilon(\lambda_n x)\, dx$

Now $\displaystyle\int_0^1 x^{\upsilon+1} J_\upsilon(\lambda_n x)\, dx = \frac{1}{\lambda_n} x^{\upsilon+1} J_{\upsilon+1}(\lambda_n x)\Big|_0^1 = \frac{1}{\lambda_n} J_{\upsilon+1}(\lambda_n)$

$\therefore a_n = \dfrac{2}{J_{\upsilon+1}^2(\lambda_n)} \dfrac{1}{\lambda_n} J_{\upsilon+1}(\lambda_n)$

and $x^\upsilon = \displaystyle\sum_{n=1}^{\infty} \frac{2}{\lambda_n J_{\upsilon+1}(\lambda_n)} J_\upsilon(\lambda_n x)$

E14) a) Comparing with E10), we get $a = 0,\ c = \dfrac{1}{2},\ b = 1$, therefore

$y = u, t = \sqrt{x}, z = t$ and $y = A\,J_n(\sqrt{x}) + B\,J_{-n}(\sqrt{x})$ or,

$A\,J_n(\sqrt{x}) + B\,Y_n(\sqrt{x})$, depending on n being a non-integer or an integer.

b) Comparing with E10), we get $a = 1/2,\ c = 1/2,\ b = 2,\ n = 1$ or

$y = u\sqrt{x},\ t = \sqrt{x},\ z = 2t,\ y = \sqrt{x}\,[A\,J_1(2\sqrt{x}) + B\,Y_1(2\sqrt{x})]$.

c) Comparing with E10), we get $y = u,\ t = \sqrt{x},\ z = t$,

$y = A\,J_0(\sqrt{x}) + B\ Y_0(\sqrt{x})$.

E15) Define $x = r\cos\theta,\ y = r\sin\theta$

Use the chain rule to show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

since u does not depend on θ, deduce that

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}$$

E16) We have the solution given by Eqn.(83) as

$u(r,t)=\sum_{n=1}^{\infty}(a_n \cos k_n ct + b_n \sin k_n ct)J_0(k_n r)$. We use the given initial conditions to obtain a_n's and b_n's.

$$u(r,0)=A J_0(k_3 r)=\sum_{n=1}^{\infty} a_n J_0(k_n r)\ ,\quad 0<r<1$$

$$\frac{\partial u}{\partial t}(r,0)=0=\sum_{n=1}^{\infty} k_n c b_n J_0(k_n r)\ ,\quad 0<r<1$$

$$\Rightarrow b_n=0,\ n=1,2,\ldots$$

and a_n's are given by

$$a_n=\frac{2}{J_1^2(k_n)}\int_0^1 r\ AJ_0(k_3 r)J_0(k_n r)dr,\quad 0<r<1$$

Now since, $\int_0^1 rJ_0(k_3 r)J_0(k_n r)\,dr=0$, if $n\neq 3$

$$=\frac{1}{2}J_1^2(k_3), \text{for } n=3$$

$\therefore\ a_n=0$ for $n\neq 3$

$$=\frac{2A}{J_1^2(k_3)}\frac{1}{2}J_1^2(k_3)=A \text{ for } n=3$$

$\therefore$ solution (83) reduces to

$$u(r,t)=AJ_0(k_3 r)\cos k_3 ct\,.$$

E17) The given initial conditions are

$u(r,0)=0$ and $\frac{\partial u}{\partial t}(r,0)=1$

Therefore, we have from Eqn.(83)

$$0=\sum_{n=1}^{\infty} a_n J_0(k_n r)\ ,\quad 0<r<1$$

$$\Rightarrow a_n=0\ ,\ n=1,2,\ldots$$

and $1=\sum_{n=1}^{\infty} k_n c b_n J_0(k_n r)\ ,\ 0<r<1$

$$\therefore k_n c b_n=\frac{2}{J_1^2(k_n)}\int_0^1 r J_o(k_n r)\ dr$$

Now $\int_0^1 r J_0(k_n r)\,dr=\frac{1}{k_n} r J_1(k_n r)\Big|_0^1=\frac{1}{k_n}J_1(k_n)$

$$\therefore b_n=\frac{2}{c}\frac{1}{k_n^2 J_1(k_n)}$$

and Eqn.(83) reduces to

$$u(r,t)=\frac{2}{c}\sum\frac{\sin k_n ct}{k_n^2 J_1(k_n)} J_0(k_n r)$$

E18) Eqn.(89) gives $y'' + \frac{P}{EI}\, y = 0$, its solution is given by

$$y(x) = B\cos\sqrt{\frac{P}{EI}}\, x + C\sin\sqrt{\frac{P}{EI}}\, x,\ B \text{ and } C \text{ arbitrary constants}$$

$y(0) = 0 \Rightarrow B = 0$, therefore $y = C\sin\sqrt{\frac{P}{EI}}\, x$

$y(b) = C\sin\sqrt{\frac{P}{EI}}\, b = 0$, if $C \neq 0$ then $\sqrt{\frac{P}{EI}}\, b = \pi$ or, $P = \frac{EI\pi^2}{b^2}$

$$\therefore\quad y(x) = C\sin\frac{\pi x}{b}$$

E19) Substituting $y = \sqrt{x}\,z$ in Eqn.(92) it reduces to

$$x^2 z'' + xz' + \left(k^2 x - \frac{1}{4} z\right) = 0$$

Putting $t = \sqrt{x}$ in the above equation, Eqn.(93) is obtained.

—x—

UNIT 5 GREEN'S FUNCTION METHOD

Structure **Page No.**

5.1 INTRODUCTION

In the theory of differential equations an important role is played by certain mathematical objects known as Green's functions. These functions, besides being of fundamental importance in many applications, serve as mathematical characterization of important physical concepts. Consequently, for introducing Green's functions we have considered, in this unit two approaches: (i) physical and (ii) mathematical.

Since the motivation for their study can be brought to sharper focus through the first approach, therefore in Sec.5.2, we shall start with an example from mechanics, whose solution turns out to be an example of Green's function. In Sec. 5.3 we shall define Green's function for a boundary value problem (bvp) and show how it can be constructed for a given bvp. We shall also give here a method of solving a bvp using its Green's function. Since the construction of Green's function for a bvp is quite an involved process we have given in Sec.5.4 an alternative procedure for the construction of Green's functions for linear, non-homogeneous, second order differential equations by establishing a relationship between the method of variation of parameters and Green's functions.

Objectives

After studying this unit you should be able to

- define and construct Green's functions for a boundary value problem;
- obtain solution of a given boundary value problem using its Green's function.

5.2 DEFLECTION OF A PERFECTLY FLEXIBLE ELASTIC STRING UNDER THE ACTION OF A CONCENTRATED LOAD

We shall now determine the deflection of a perfectly flexible elastic string under the action of a concentrated load. Before we embark upon the task of obtaining the solution to the given problem we may mention that the problem to be dealt here is a mathematical idealisation of the corresponding problem for the determination of deflection under the action of distributed load. Just as the introduction of point charges, point masses, point heat source facilitate the solution of the problems in their respective fields, in the same way, the introduction of point force/concentrated load for solving physical problems related to beams and strings has turned out to be quite fruitful. It is a mathematical fiction, which cannot be realised physically as any non-zero load concentrated at a single point implies an infinite pressure, which would immediately cut through the string.

For the problem under consideration we have a perfectly flexible elastic string stretched to a length ℓ under tension T. Further, the string $0A = \ell$, is under the action of a concentrated load L as shown in Fig 1.

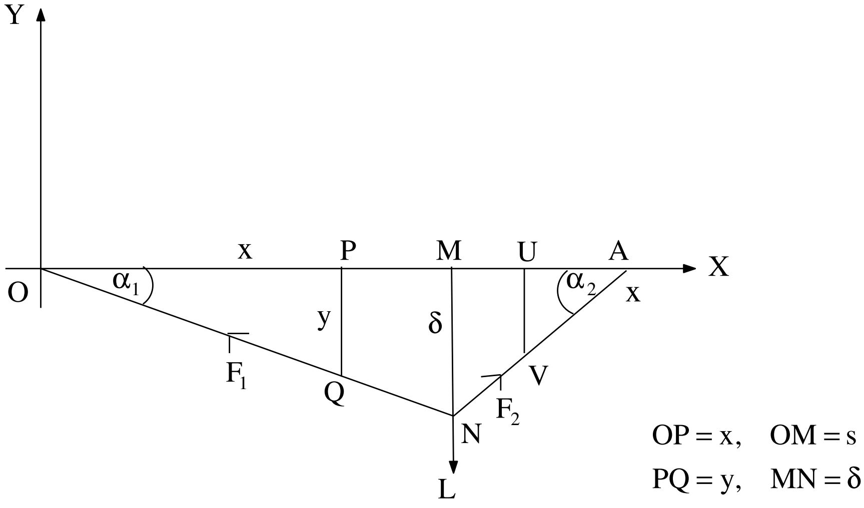

Fig.1

For determining the deflection $y(x)$ of the string at any point P at a distance x from O, we determine the governing equations, keeping in view, that the system is in equilibrium. Thus, through the balance of horizontal and vertical components of the forces, we have

$$F_1 \cos\alpha_1 = F_2 \cos\alpha_2 = T \tag{1}$$

$$F_1 \sin\alpha_1 + F_2 \sin\alpha_2 = -L \tag{2}$$

From Eqns.(1) and (2), we can write

$$\tan\alpha_1 + \tan\alpha_2 = -\frac{L}{T} \tag{3}$$

Substituting the values of $\tan\alpha_1$ and $\tan\alpha_2$ from the similar triangles OMN and AMN , we get

$$\frac{-\delta}{s} + \frac{-\delta}{\ell - s} = -\frac{L}{T}$$

or,

$$\delta = \frac{L}{T\ell}(\ell - s)\,s \tag{4}$$

In the above equation δ denotes the transverse deflection of the point of the string that was located at a distance s from the origin.

For determining the deflection $y(x)$ at any point x , we have from triangles OPQ and OMN: $\frac{y}{x} = \frac{\delta}{s}$.

Substituting the value of δ from Eqn.(4) in the above equation, we get

$$y(x) = \frac{L}{T\ell}(\ell - s)x, \quad 0 \le x \le s$$

Similarly, if the position of the point P where deflection is observed is changed to point U then from triangles AUV and AMN we have, $\frac{y}{\delta} = \frac{\ell - x}{\ell - s}$.

or, $$y = \frac{L}{T\ell}(\ell - x)s, \; s \le x \le \ell .$$

Thus $$y(x,s) = \begin{cases} \frac{L}{T\ell}(\ell - s)x, & 0 \le x \le s \quad (5a) \\ \frac{L}{T\ell}(\ell - x)s, & s \le x \le \ell \quad (5b) \end{cases}$$

It may be **noted** that in writing Eqns.(5a) and (5b) we have replaced $y(x)$ by $y(x,s)$ which indicates that the deflection y depends both on points s where the load is applied and the point x where deflection is observed.

If the roles of x and s are interchanged, that is, the load L is now at x and the deflection is measured at $s(>x)$ then the expression for deflection is obtained from Eqn.(5 b) with x and s, interchanged. Thus, we get $y(x,s)=\frac{L}{T\ell}(\ell-s)x$.

Further, when L assumes the value unity, the deflection $y(x,s)$ becomes $g(x,s)$ where $g(x,s)$ is given by

$$g(x,s)=\begin{cases}\dfrac{(\ell-s)x}{T\ell}, & 0\le x\le s \quad (6a)\\ \dfrac{(\ell-x)s}{T\ell}, & s\le x\le \ell \quad (6b)\end{cases}$$

The function $g(x,s)$ is called an **influence function**. The word influence is used in the sense that the function $g(x,s)$ describes the influence which a unit load concentrated at the point s has at any point x of the given string.

Now we shall show that the influence function **g(x,s)** associated with the stretched string with fixed ends satisfies the following properties

(i) It is symmetric, that is, $g(x,s)=g(s,x)$.
(ii) It satisfies the boundary conditions $g(0,s)=g(\ell,s)=0$, for all values of s satisfying $0\le s\le \ell$.
(iii) It is a continuous function of x on the interval $[0,\ell]$.
(iv) $g_x(x,s)$ is continuous for $0\le x<s$ and $s<x\le \ell$ but has a jump $\left(-\frac{1}{T}\right)$ at $x=s$.

We examine these properties for $g(x,s)$ defined in (6a) and (6b).

For proving property (i), we interchange the role of x and s in (6a) and (6b) and obtain $g(s,x)=g(x,s)$.

For property (ii), put $x=0$ in (6 a) and $x=\ell$ in (6 b) to get $g(0,s)=g(\ell,s)=0$.

For property (iii), we equate left handed limit = right handed limit = value of the function at $x=s$

Thus, $$\lim_{x\to s^-} g(x,s)=\lim_{x\to s^-}\frac{(\ell-s)x}{T\ell}=\frac{(\ell-s)s}{T\ell}$$

$$\lim_{x\to s^+} g(x,s)=\lim_{x\to s^+}\frac{(\ell-x)s}{T\ell}=\frac{(\ell-s)s}{T\ell}$$

At $x=s$, $g(x,s)=g(s,s)=\left(\frac{\ell-s}{T\ell}\right)s$

Hence, $g(x,s)$ is a continuous function of x on the interval $[0,\ell]$.

Regarding property (iv) we have to show that

$$\lim_{x\to s^+} g_x(x,s)-\lim_{x\to s^-} g_x(x,s)=-\frac{1}{T}$$

or,

$$\left(\frac{-s}{T\ell}\right)-\left(\frac{\ell-s}{T\ell}\right)=-\frac{1}{T}.$$

Alternatively, the above property can be expressed as $g_x(x,s)$ is continuous for $0\le x<s$ and $s<x\le \ell$ but has a jump discontinuity $\left(-\frac{1}{T}\right)$ at $x=s$.

Thus the influence function $g(x,s)$ defined by Eqns(6a) and (6b) satisfies properties (i) – (iv) listed above. With little modifications in the properties for $g(x,s)$, we can define a new class of function called the **Green's function** which we shall discuss in the next section but before that we give here another application of influence function.

Application of Influence Function

Let us take up the application of influence function $g(x,s)$ for determining the deflection of the string under any piecewise continuous distributed load. **Refer** to Fig.1 once again where instead of concentrated load L, we have the distributed load.

Let $w(x)$ be the distributed load per unit length. Through the balance of horizontal and vertical components of the forces we obtain

$$F_1 \cos\alpha_1 = F_2 \cos\alpha_2 = T \tag{7}$$

$$F_2 \sin\alpha_2 = F_1 \sin\alpha_1 - w(x)\Delta x \tag{8}$$

Combining Eqns(7) and (8), we get

$$\tan\alpha_2 = \tan\alpha_1 - \frac{w(x)\Delta x}{T} \tag{9}$$

Eqn(9) can be written in the following form

$$\frac{\left(\frac{dy}{dx}\right)_{x+\Delta x} - \left(\frac{dy}{dx}\right)_x}{\Delta x} = -\frac{w(x)}{T}$$

Taking limit as $\Delta x \to 0$, we have

$$\lim_{\Delta x \to 0} \frac{\left(\frac{dy}{dx}\right)_{x+\Delta x} - \left(\frac{dy}{dx}\right)_x}{\Delta x} = -\frac{w(x)}{T}$$

or, $$T y'' = -w(x) \tag{10}$$

Eqn.(10) is the differential equation satisfied by the deflection curve of the string.

A straight forward procedure to solve this problem is to solve Eqn.(10) under the boundary conditions that the string is fixed at both the ends.

But here we shall describe the procedure for obtaining the desired deflection in terms of the influence function $g(x,s)$, without solving the governing differential Eqn.(10).

Let us divide the interval $[\,0,\ell\,]$ into n equal parts each of length $\Delta s_i = s_i - s_{i-1}$ at the points $0 = s_0, s_1, s_2, \ldots, s_n = \ell$. Further, assume ξ_i to be an arbitrary point in Δs_i. If $w(x)$ is the distributed load per unit length then the portion of the distributed load acting on Δs_i is $w(\xi_i)\Delta s_i$, concentrated at the point ξ_i.

Keeping in view the definition of $g(x,s)$ one can write the expression for the deflection produced at the point x by the load as $w(\xi_i)\Delta s_i\, g(x,s)$. In writing down this expression we have used the formula that the deflection produced at any point x by the load $w(\xi_i)\Delta s_i$ is the product of the load and the deflection produced at x by a unit load at the point ξ_i.

For obtaining the deflection due to the distributed load we sum up all the deflections to get

$$y(\Delta s_i) = \sum_{i=1}^{n} w(\xi_i)\, g(x,\xi_i)\Delta s_i \tag{11}$$

Taking the limit $\Delta s_i \to 0$, Eqn.(11) assumes the following form:

$$y(x) = \int_0^{\ell} w(s)\, g(x,s)\, ds \tag{12}$$

Eqn.(12) reveals that the deflection of the string under any piecewise continuous distributed load can be obtained by evaluating the integral (12), once we have the knowledge of $g(x,s)$.

You may now try the following exercises.

E1) Find the deflection curve of a string of length ℓ bearing a load per unit length as given below

$$w(x) = \begin{cases} 0, & 0 \le x < \dfrac{\ell}{2} \\ -1, & \dfrac{\ell}{2} < x \le \ell \end{cases}.$$

E2) Determine the deflection curve of a string of length ℓ bearing concentrated load P at $x = \dfrac{\ell}{4}$, $-2P$ at $x = \dfrac{\ell}{2}$ and $3P$ at $x = \dfrac{3\ell}{4}$.

Let us now see how the properties satisfied by the influence function $g(x,s)$ can be modified to define the Green's functions for boundary value problems.

5.3 GREEN'S FUNCTIONS

In the previous section we derived the expression for the influence function $g(x,s)$ for determining the deflection of the stretched string of negligible mass, tension T and length ℓ due to the concentrated unit force/load. Further, the properties listed there-in are satisfied by $g(x,s)$ of the above said problem. However, with little modification in the properties for $g(x,s)$ we can enlarge the scope of the function $g(x,s)$ in the sense that we can arrive at a larger class of functions $G(x,s)$ associated with linear differential equation with constant or variable coefficients. Accordingly, we examine the following boundary value problem

$$p_0(x)\frac{d^2y}{dx^2} + p_1(x)\frac{dy}{dx} + p_2(x)\, y = 0 \tag{13}$$

$$\alpha y(a) - \beta y'(a) = 0, \quad \gamma y(b) - \delta y'(b) = 0 \tag{14}$$

where α, β, γ and δ are non-zero.

A function $G(x,\xi)$ with the following properties is termed as the **Green's function** of the boundary-value problem (13) and (14), if

1) $G(x,\xi)$ satisfies the differential equation for $a \le x < \xi$, and $\xi < x \le b$.
2) $G(x,\xi)$ satisfies the homogeneous boundary conditions that is ,
   $\alpha G(a,\xi) - \beta G'(a,\xi) = 0,\ \gamma G(b,\xi) - \delta G'(b,\xi) = 0$.
3) $G(x,\xi)$ is a continuous function of x for all x satisfying $a \le x \le b$.
4) For any ξ in [a, b], the function $G(x,\xi)$ has continuous derivative $G_x(x,\xi)$ in each of the half intervals $[\ a, \xi\ [$ and $]\xi, b]$ and the derivative $G_x(x,\xi)$ satisfies the condition
   $$G_x(\xi^+,\xi) - G_x(\xi^-,\xi) = \frac{-1}{p_0(\xi)}, \quad \text{if} \quad x = \xi.$$

Similarly, for a bvp of the form:

$$Ly = p_0(x)\frac{d^2y}{dx^2} + p_1(x)\frac{dy}{dx} + p_2(x)\, y = f(x) \tag{15}$$

$$\alpha y(a) - \beta y'(a) = 0 \ \text{and} \ \gamma y(b) - \delta y'(b) = 0 \tag{16}$$

a Green's function $G(x, \xi)$, is the solution of

$$LG = \delta(x-\xi) \tag{17}$$

satisfying properties 2) – 4) above. The only change in this case is that the source function $f(x)$ is replaced by a concentrated unit source at $x = \xi$. The function $\delta(x-\xi)$, called the Dirac-delta function, is the functional that maps a function $f(x)$ continuous at $x = \xi$, onto its value at $x = \xi$ such that $\int\limits_0^\infty \delta(x-\xi)\,dx = 1$.

$$\delta(x-\xi) = \begin{cases} \infty, & \text{if } x = \xi \\ 0, & \text{if } x \neq \xi \end{cases}$$

We shall now see how such a function $G(x, \xi)$, satisfying properties 1) – 4) can be obtained for a given bvp.

5.3.1 Construction of Green's Functions for boundary Value Problems

We shall illustrate through examples the construction of Green's functions for bvps.

Example 1: Using the definition of Green's function, construct Green's function for the boundary-value problem

$$\frac{d^2y}{dx} = 0, \quad y(0) = y'(1); \quad y'(0) = y(1)$$

Solution: We have to obtain $G(x,\xi)$ satisfying properties 1) - 4) listed above. Since $G(x,\xi)$ satisfies the homogeneous differential equation $\frac{d^2G}{dx^2} = 0$, therefore its solution can be written as

$$G(x,\xi) = \begin{cases} Ax + B, & 0 \le x \le \xi \quad (18a) \\ Cx + D, & \xi \le x \le 1. \quad (18b) \end{cases}$$

$G(x,\xi)$ satisfies the homogeneous boundary conditions, that is,

$$G(0,\xi) = G'(1,\xi); \qquad G'(0,\xi) = G(1,\xi)$$

Using the above conditions, we get,

$$B = C \text{ and } A = C + D \tag{19}$$

Further at $x = \xi$, $\quad A\xi + B = C\xi + D$ (20)

From the jump condition $\frac{\partial G}{\partial x}(1,\xi) - \frac{\partial G}{\partial x}(0,\xi) = -1$, we get

$$C - A = -1 \tag{21}$$

Hence we have four equation for the four unknowns A, B, C and D. From Eqn.(20)

$$(C - A)\xi = B - D \tag{22}$$

Combining Eqns.(19) and (21), we get $D = 1$.

Further, from Eqns.(21), (22) and (19), we get $B = C = (1-\xi)$.

$\therefore A = C + 1 = -(\xi - 2)$. Hence,

$$G(x,\xi) = -\begin{cases} (\xi-2)x + (\xi-1), & 0 \le x \le \xi \\ (\xi-1)x + (-1), & \xi \le x \le 1. \end{cases}$$

***

Example 2: Construct Green's function for the differential equation $x\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$, under the conditions $y(x)$ is bounded as $x \to 0$ and $y(1) = \alpha y'(1), \ \alpha \neq 0$.

Solution: Since $G(x,\xi)$ satisfies the homogeneous differential equation we have

$$x\frac{d^2G}{dx^2}+\frac{dG}{dx}=0$$

or, we can write $x^2\dfrac{d^2G}{dx^2}+x\dfrac{dG}{dx}=0$ (23)

Treating Eqn.(24) as Euler's equation, its solution can be written by transforming it into a differential equation with constant coefficient viz.,

$$D(D-1)G+DG=0,\quad \text{where } x=e^z,\quad D=\frac{d}{dz}$$

or,

$$D^2G=0 \tag{24}$$

Solving Eqn.(24) we can write

$$G(x,\xi)=\begin{cases}A+Bz=A+B\ln x\ , & 0<x\le\xi\\ C+Dz=C+D\ln x\ , & \xi\le x\le 1\end{cases}$$

Using the condition of continuity at $x=\xi$, we get

$$A+B\ln\xi=C+D\ln\xi$$

or,

$$(A-C)+(B-D)\ln\xi=0 \tag{25}$$

Using the jump condition, we get

$$\lim_{x\to\xi^+}G_x(x,\xi)-\lim_{x\to\xi^-}G_x(x,\xi)=-\lim_{x\to\xi}\left(\frac{1}{x}\right)$$

or, $\dfrac{D}{\xi}-\dfrac{B}{\xi}=\dfrac{-1}{\xi}\Rightarrow D-B=-1$ (26)

Further, boundary condition that $G(x,\xi)$ is bounded as $x\to 0$ gives

$$B=0, \tag{27}$$

also $G(1,\xi)=\alpha G'(1,\xi)$ yields, $C=\alpha D$ (28)

Since $B=0$; $D=-1$ gives $C=-\alpha$, using these values of B, C, D in Eqn(25), we obtain

$$A=-\ (\alpha+\ln\xi) \tag{29}$$

Hence, $G(x,\xi)=-\begin{cases}\alpha+\ln\xi\ , & 0<x\le\xi\\ \alpha+\ln x\ , & \xi\le x\le 1\end{cases}$

***

Example 3: Construct Green's function for the boundary value problem

$$\frac{d^2y}{dx^2}-y=x\ ,\qquad y(0)=y(1)=0$$

Solution: Keeping in view that $G(x,\xi)$ satisfies the homogeneous differential equation $\dfrac{d^2G}{dx^2}-G(x,\xi)=0$, $G(x,\xi)$ can be written as follows:

$$G(x,\xi)=\begin{cases}Ae^x+Be^{-x}\ , & 0\le x\le\xi \quad (30a)\\ Ce^x+De^{-x}\ , & \xi\le x\le 1 \quad (30b)\end{cases}$$

Since $G(x,\xi)$ satisfies the homogeneous boundary conditions

$$\therefore\quad G(0,\xi)=A+B=0\Rightarrow B=-A \tag{31}$$

and $\quad G(1,\xi)=Ce+De^{-1}=0,\Rightarrow D=-Ce^2$ (32)

$$\therefore \quad G(x,\xi) = \begin{cases} A(e^x - e^{-x}) \Rightarrow 2A\sinh x, & 0 \le x \le \xi \\ C(e^x - e^{2-x}), & \xi \le x \le 1 \end{cases}$$

Using the condition of continuity of $G(x,\xi)$ at $x = \xi$, we get

$$A(e^{\xi} - e^{-\xi}) = C\{e^{\xi} - e^{2-\xi}\}$$

or, $$(C - A)e^{\xi} + (A - Ce^2)e^{-\xi} = 0 \tag{33}$$

Using the jump condition

$$\lim_{x\to\xi^+} G_x(x,\xi) - \lim_{x\to\xi^-} G_x(x,\xi) = -1 \tag{34}$$

Substituting the value of G_x in Eqn.(34) and then taking the indicated limit, we get

$$(C - A)e^{\xi} - e^{-\xi}(A - Ce^2) = -1 \tag{35}$$

Solving Eqns.(33) and (35), we get

$$C - A = -\frac{1}{2}e^{-\xi} \tag{36}$$

$$A - Ce^2 = \frac{1}{2}e^{\xi} \tag{37}$$

Combining Eqns.(36) and (37), we get

$$A = \frac{1}{2(1-e^2)}(e^{\xi} - e^{-\xi+2}) \; ; \qquad C = \frac{1}{(1-e^2)}\sinh\xi \tag{38}$$

Now $$\sinh 1 = \frac{e - e^{-1}}{2} = \frac{e^2 - 1}{2e} \quad \text{or}, \quad (1 - e^2) = -2e\sinh 1 \tag{39}$$

Thus, from Eqns. (38) and (39), we can write

$$A = \frac{1}{-4e\sinh 1}(e^{\xi} - e^{-\xi+2}) = -\frac{1}{2\sinh 1}\left(\frac{e^{\xi-1} - e^{-(\xi-1)}}{2}\right)$$

$$\therefore \quad A = -\frac{1}{2\sinh 1}\sinh(\xi - 1) \tag{40}$$

and $$C = -\frac{1}{2e\sinh 1}\sinh\xi \tag{41}$$

Thus, $$G(x,\xi) = \begin{cases} -\dfrac{1}{\sinh 1}\sinh(\xi-1)\sinh x, & 0 \le x \le \xi \\ -\dfrac{1}{2e\sinh 1}\sinh\xi.\ (e^x - e^{2-x}), & \xi \le x \le 1 \end{cases}$$

or, $$G(x,\xi) = -\begin{cases} \dfrac{1}{\sinh 1}\sinh(\xi-1)\sinh x, & 0 \le x \le \xi \\ \dfrac{1}{\sinh 1}\sinh\xi\sinh(x-1), & \xi \le x \le 1 \end{cases} \tag{42}$$

***

Example 4: Construct Green's function for the following boundary value problem

$$\frac{d^2y}{dx^2} + k^2y = 0, \qquad y(0) = y(1) = 0$$

Solution: Keeping in view that $G(x,\xi)$ satisfies the homogeneous differential equation $\dfrac{d^2G}{dx^2} + k^2G(x,\xi) = 0$, for $0 \le x \le \xi$, $\xi \le x \le 1$, we can write

$$G(x,\xi)=\begin{cases} A\cos kx + B\sin kx \quad , & 0\le x\le \xi \\ C\cos kx + D\sin kx \quad , & \xi\le x\le 1 \end{cases} \tag{43}$$

$G(x,\xi)$ satisfies the conditions $G(0,\xi)=0=G(1,\xi)$ (44)

Using conditions (44), we get

$$A=0, \text{ and } C\cos k + D\sin k = 0 \tag{45}$$

Hence, $G(x,\xi)$ assumes the following form

$$G(x,\xi)=\begin{cases} B\sin kx \quad , & 0\le x\le \xi \\ -C\dfrac{\sin k(x-1)}{\sin k} \quad , & \xi\le x\le 1 \end{cases} \tag{46}$$

Using the continuity condition at $x=\xi$, we get

$$B\sin k\xi = \frac{C}{\sin k}\ \sin\ k(1-\xi) = -\frac{C}{\sin k}\{\sin k\xi\cos k - \cos k\xi \sin k\}$$

$$\therefore \quad B = -\frac{C}{\sin k\sin k\xi}\ \sin k(\xi-1)$$

$$\text{Hence, } G(x,\xi)=\begin{cases} \dfrac{C\sin k(1-\xi)\sin kx}{\sin k\sin k\xi} \quad , & 0\le x\le \xi \\ \dfrac{C\sin k(1-x)\sin k\xi}{\sin k\sin k\xi} \quad , & \xi\le x\le 1 \end{cases} \tag{47}$$

The jump discontinuity condition to be satisfied is

$$\lim_{x\to\xi^+} G_x(x,\xi) - \lim_{x\to\xi^-}(x)\ G_x(x,\xi) = -1 \quad (\text{as } p_0(x)=1) \tag{48}$$

Eqn.(48) yields

$$\frac{C}{\sin k\sin k\xi}\cos k(1-\xi)(-1)k\sin k\xi - C\frac{\sin k(1-\xi)}{\sin k\sin k\xi}k\cos k\xi = -1 \tag{49}$$

simplifying Eqn.(49), we get

$$C=\frac{1}{k}\ \frac{\sin k\sin k\xi}{\sin k\xi\cos k(1-\xi)+\cos k\xi\sin k(1-\xi)} = \frac{\sin k\xi}{k} \tag{50}$$

$$\therefore G(x,\xi)=\begin{cases} \dfrac{\sin k(1-\xi)\sin kx}{k\sin k} \quad , & 0\le x\le \xi \\ \dfrac{\sin k\xi\sin k(1-x)}{k\sin k} \quad , & \xi\le x\le 1 \end{cases}$$

***

Example 5: Construct Green's function for the boundary value problem

$$\frac{d^3y}{dx^3}=0\ , \quad y(0)=y(1)=0\ , \quad y'(0)+y'(1)=0$$

Solution: We have to obtain $G(x,\xi)$ satisfying $\dfrac{d^3G}{dx^3}=0$, $G(0,\xi)=0$, $G(1,\xi)=0$, $G_x(0,\xi)+G_x(1,\xi)=0$, $G(\xi_-,\xi)=G(\xi_+,\xi)=G(\xi,\xi)$

Also, $\dfrac{\partial G}{\partial x}(1,\xi)-\dfrac{\partial G}{\partial x}(0,\xi)=1$

$$\frac{d^3G}{dx^3}=0 \Rightarrow G(x,\xi)=\begin{cases} Ax^2+Bx+C\ , & 0\le x\le \xi \\ Dx^2+Ex+F\ , & \xi\le x\le 1 \end{cases} \tag{51}$$

Where A, B, C, D, E and F are to be determined using the above conditions:

$$G(0,\xi)=0 \Rightarrow C=0, \quad (52)$$

$$G(1,\xi)=0 \Rightarrow D+E+F=0 \quad (53)$$

Further, $\dfrac{\partial G(0,\xi)}{\partial x}+\dfrac{\partial G(1,\xi)}{\partial x}=0 \Rightarrow (2Ax+B)_{x=0}+(2Dx+E)_{x=1}=0$

or, $\quad B+2D+E=0 \quad (54)$

Further, continuity condition yields $A\xi^2+B\xi+C=D\xi^2+E\xi+F$

or, $\quad (A-D)\xi^2+(B-E)\xi+(C-F)=0 \quad (55)$

Finite jump condition gives $\dfrac{\partial G(1,\xi)}{\partial x}-\dfrac{\partial G(0,\xi)}{\partial x}=-1=2D+E-B \quad (56)$

Combining Eqns.(54) and (56) we get, $B=\dfrac{1}{2}$

Also, from Eqn.(52), $C=0$. Further, $2D+E=\dfrac{1}{2} \Rightarrow D-F=\dfrac{1}{2}$. It is not possible to solve explicitly for the remaining arbitrary constants A, D, E, F. In short, we remark that Green's function does not exist for the given bvp.

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Example 5 shows that construction of Green's function is not always possible for a given boundary value problem. But then the question arises: Is there a way to establish the existence of Green's function without going into the detailed working as done in Example 5? The **theorem on the existence of Green's function** guarantees the existence of Green's function for a given bvp. The application of this theorem to a given problem results in establishing the existence or non-existence of Green's function without going into the detailed working. We shall now state this theorem.

Theorem 1: Suppose we have a differential equation of order n given by

$$L(y)=p_0(x)\,y^n+p_1(x)\,y^{n-1}+\cdots+p_n(x)\,y=f(x), \quad (57)$$

where the functions $p_0(x), p_1(x), \ldots, p_n(x)$ and $f(x)$ are continuous on the closed interval $[a,b], p_0(x)\neq 0$ on $[a,b]$ and the boundary conditions are

$$B_k(y)=\alpha_k^0\,y(a)+\alpha_k^1\,y'(a)+\alpha_k^2\,y''(a)+\cdots+\alpha_k^{n-1}\,y^{n-1}(a)$$
$$+B_k^0\,y(b)+B_k^1\,y'(b)+\cdots+B_k^{n-1}\,y^{n-1}(b)=0 \quad (58)$$
$$k=1,2,\ldots,n$$

where the linear forms $B_1, B_2, \ldots, B_n$, in $y(a), y'(a), \ldots y^{n-1}(a), y(b), y'(b), \ldots, y^{n-1}(b)$ are linearly independent. If the boundary value problem (57) – (58) has only the trivial solution $y(x)=0$, then for the operator L, Green's function exists and is unique.

We shall not be going into the details of this theorem here. To illustrate the theorem let us once again consider the bvp considered in Example 5.

We are given

$$\frac{d^3y}{dx^3}=0, \text{ and } \quad y(0)=y(1)=0, \quad y'(0)+y'(1)=0$$

$\dfrac{d^3y}{dx^3}=0 \Rightarrow y(x)=Ax^2+Bx^2+C$ and applying the boundary conditions we get

$$y(0)=C=0 \quad (59)$$

$$y(1)=A+B=0 \quad (60)$$

$$y'(x)=2Ax+B \quad ; \quad y'(0)=B \quad , \quad y'(1)=2A+B$$

Thus, the third boundary condition $y'(0)+y'(1)=0$, yields

$$B+2A+B=2(A+B)=0 \tag{61}$$

Eqn.(61) is same as Eqn.(60). Thus, one of the boundary condition is redundant. Hence we have infinite number of solutions. While for the existence of Green's function we should have a trivial solution. Hence Green's function does not exist for the given bvp.

You may now try the following exercise.

E3) Check whether Green's function exists for the following boundary value problems and if it exist, construct it

a) $xy''+y'=0$, $y(0)$ is bounded, $y(\ell)=0$.

b) $xy''+y'-\frac{1}{x}y=0$; $y(0)$ is finite, $y(1)=0$.

c) $y'''(x)=0$, $y(0)=y(1)=0$, $y'(0)+y'(1)=0$.

d) $y''(x)=0$, $y(0)=y(1)$, $y'(o)=y'(1)$.

e) $y''-k^2y=0$, $(k\neq 0)$; $y(0)=y(1)=0$.

5.3.2 Solution of Boundary Value Problems via Green's Functions

Up till now we have been concerned with the development of procedure for the construction of Green's function for boundary value problem. We shall now state a theorem that helps in writing down the solution of the boundary value problem using its Green's function.

Theorem 2: If $G(x,\xi)$ is the Green's function of the homogeneous boundary-value problem

$$Ly(x)=0 \quad , \quad B_1(y)=0 \quad , \quad B_2(y)=0,\ldots,B_n(y)=0$$

where L and B_k are as defined in Theorem 1, then the solution of the boundary value problem

$$Ly=f(x); \quad B_k(y)=0, \quad (k=1,2,3,\ldots,n)$$

is given by

$$y(x)=\int_a^b G(x,\xi)\, f(\xi)\,d\xi \tag{62}$$

We shall now illustrate this theorem through an example.

Example 6: Using Green's function, solve the boundary-value problem

$$-\left(\frac{d^2y}{dx^2}+y\right)=x \quad , \qquad y(0)=y\left(\frac{\pi}{2}\right)=0$$

Solution: Since we have to solve the bvp via Green's function, we therefore, first prove the existence of Green's function for the boundary-value problem.
Solution of the homogeneous problem is given by

$$y(x)=A\cos x+B\sin x$$

Using boundary conditions, we obtain $y(0)=A=0, \quad y\left(\frac{\pi}{2}\right)=B=0$

Therefore, the homogeneous bvp possesses trivial solution $y(x)=0$. Hence, the Green's function for the problem exists and can be expressed as

$$G(x,\xi)=\begin{cases} A\cos x+B\sin x \ , & 0\le x\le \xi \\ C\cos x+D\sin x \ , & \xi\le x\le \dfrac{\pi}{2} \end{cases} \tag{63}$$

For obtaining A, B, C and D we use the properties of Green's function.

$$G(0,\xi)=A=0 \ , \ G\left(\frac{\pi}{2},\xi\right)=D=0$$

Hence, $G(x,\xi)=\begin{cases} B\sin x \quad , & 0\le x\le \xi \\ C\cos x \quad , & \xi \le x\le \dfrac{\pi}{2}\end{cases}$

For continuity at $x=\xi$; $B\sin\xi = C\cos\xi$ (64)

Also, $\left(\dfrac{\partial G}{\partial x}\right)_{x=\frac{\pi}{2}} - \left(\dfrac{\partial G}{\partial x}\right)_{x=0} = \dfrac{-1}{p_0(x)} = 1$, yields

$$-C\sin\xi - B\cos\xi = .1 \tag{65}$$

Combining Eqns.(64) and (65) we get,

$$B=-\cos\xi \quad , \qquad C=-\sin\xi \tag{66}$$

Thus, $G(x,\xi)=\begin{cases} -\cos\xi\sin x \quad , & 0\le x\le \xi \\ -\sin\xi\cos x \quad , & \xi\le x\le \dfrac{\pi}{2}\end{cases}$ (67)

We now make use of Eqn.(62) to obtain the solution $y(x)$.

Hence, $y(x)=-\int_0^x (\cos x\sin\xi)\,\xi\, d\xi + \int_x^{\frac{\pi}{2}} (-\sin x\cos\xi)\xi\, d\xi$ (68)

$$=\cos x\,[\,(\xi\cos\xi)_0^x - \int_0^x \cos\xi\, d\xi\,] - \sin x\,[\,(\xi\sin\xi)_x^{\frac{\pi}{2}} - \int_x^{\frac{\pi}{2}} \sin\xi\, d\xi\,]$$

$$=\cos x\,(x\cos x - 0) - \cos x\sin x - \sin x\left(\frac{\pi}{2} - x\sin x\right) + \sin x\,(-\cos\frac{\pi}{2} + \cos x)$$

$$= x(\cos^2 x + \sin^2 x) - \frac{\pi}{2}\sin x$$

$$= x - \frac{\pi}{2}\sin x.$$

***

You may now try the following exercise.

E4) Using Green's function, solve the boundary value problem

$$-\left(\frac{d^2y}{dx^2} - y\right) = x \ , \quad y(0)=y(1)=0$$

You must have realised that the procedure discussed above for the construction of Green's function for a boundary value problem is quite an involved process. Consequently, in the next section, we look for an alternative procedure to construct Green's function by establishing a relationship between the method of variation of parameters and the Green's function approach for linear, non-homogeneous differential equation of second order with variable or constant coefficients.

5.4 METHOD OF VARIATION OF PARAMETERS AND GREEN'S FUNCTIONS

Consider the general form of second order, linear, non-homogeneous differential equation

$$p_0(x)\frac{d^2y}{dx^2} + p_1(x)\frac{dy}{dx} + p_2(x)y = f(x) \tag{69}$$

where the functions $p_0(x), p_1(x), p_2(x)$ are continuous on the closed interval $[a,b], p_0(x)\neq 0$ on $[a,b]$ and the boundary conditions are $y(a)=0=y(b)$.

From your knowledge of the method of variation of parameters (Ref. Unit 7, Block-2, MTE-08) you know that if $y_1(x)$ and $y_2(x)$ are two linearly independent solutions of the associated homogeneous equation to Eqn.(69), then

$$y(x) = u_1(x)\,y_1(x) + u_2(x)\,y_2(x) \tag{70}$$

is a solution of the non-homogeneous Eqn.(69) provided

$$\frac{du_1}{dx} = -\frac{y_2(x)\,f(x)}{p_0(x)\,W(x)} \quad , \quad \frac{du_2}{dx} = \frac{y_1(x)\,f(x)}{p_0(x)\,W(x)} \tag{71}$$

Further, $W(x)$ is the Wronskian of the two solutions y_1 and y_2 and is given by

$$W(x) = y_1(x)\,y_2'(x) - y_1'(x)\,y_2(x) \tag{72}$$

For obtaining y as defined in Eqn.(70) we have to obtain the respective integrals of $\frac{du_1}{dx}$ and $\frac{du_2}{dx}$ between a to x and x to b. Thus,

$$u_1(x) = -\int_a^x \frac{y_2(t)\,f(t)}{W(t)\,p_0(t)}\,dt, \quad u_2(x) = -\int_x^b \frac{y_1(t)\,f(t)}{W(t)\,p_0(t)}\,dt$$

Hence,

$$y(x) = -y_1(x)\int_a^x \frac{y_2(t)\,f(t)}{W(t)\,p_0(t)}\,dt - y_2(x)\int_x^b \frac{y_1(t)\,f(t)}{W(t)\,p_0(t)}\,dt \tag{73}$$

Solution (73) has to satisfy the conditions $y(a) = 0$, and $y(b) = 0$. Thus, for $y(a)$ to be zero, the first integral on the rhs of Eqn.(73) is zero and if for arbitrary values of function $f(x), y_2(x)$ is zero when $x = a$ then $y(a) = 0$. Similarly, for $y(b)$ to be zero the second integral on the rhs of Eqn.(73) is zero and the first integral vanishes for all arbitrary values of $f(x)$, if for $x = b,\ y_1(b) = 0$.

Assuming that $y_1(x)$ and $y_2(x)$ are so chosen that these solutions satisfy the conditions $y_2(a) = 0$, and $y_1(b) = 0$, we can write

$$y(x) = -\left[\int_a^x \frac{y_1(x)\,y_2(t)}{W(t)\,p_0(t)} f(t)\,dt + \int_x^b \frac{y_2(x)\,y_1(t)}{W(t)\,p_0(t)} f(t)\,dt\right] \tag{74}$$

Eqn.(74) can be replaced by

$$y(x) = -\int_a^b G(x,t)\,f(t)\,dt \tag{75}$$

where, $$G(x,t) = \begin{cases} \dfrac{y_1(t)\,y_2(x)}{W(t)\,p_0(t)} \quad , & x \le t \le b, \text{ i.e. } a \le x \le t \\ \dfrac{y_1(x)\,y_2(t)}{W(t)\,p_0(t)} \quad , & a \le t \le x, \text{ i.e. } t \le x \le b \end{cases} \tag{76}$$

The function $G(x,t)$ as defined in (76) satisfies all the properties of the Green's function. It is continuous at $x = t$, i.e., $\lim_{t \to t^-} G(t^-, t) = \lim_{t \to t^+} G(t^+, t) = G(t,t)$. Also $G(x,t)$ satisfies the boundary conditions, that is,

$$G(a,t) = \frac{y_1(t)\,y_2(a)}{p_0(t)\,W(t)} = 0 \text{ as } y_2(a) = 0,$$

similarly, $$G(b,t) = \frac{y_1(b)\,y_2(t)}{W(t)\,p_0(t)} = 0 \text{ as } y_1(b) = 0$$

Further, $G(x,t)$ satisfies the homogeneous differential equation $p_0(x)\,y''(x) + p_1(x)\,y'(x) + p_2(x)\,y(x) = 0$, as shown through method of variation of parameters.

To verify the jump condition $\left(\dfrac{\partial G}{\partial x}\right)_{x=t^+} - \left(\dfrac{\partial G}{\partial x}\right)_{t^{-1}} = \dfrac{-1}{p_0(t)}$

$$\text{We have, } G_x(x,t) = \begin{cases} \dfrac{y_1(t)\,y_2'(x)}{p_0(t)\,W(t)}, & a \le x \le t \\ \dfrac{y_1'(x)\,y_2(t)}{W(t)\,p_0(t)}, & t \le x \le b \end{cases} \tag{77}$$

$$\text{Hence, } \left(\frac{\partial G}{\partial x}\right)_{x=t^+} - \left(\frac{\partial G}{\partial x}\right)_{x=t^-} = \frac{y_1'(t)\,y_2(t) - y_1(t)\,y_2'(t)}{p_0(t)\,W(t)} = \frac{-1}{p_0(t)} \tag{78}$$

Also, $G(x,t) = G(t,x)$.

Thus, $G(x,t)$ defined by Eqn.(76)satisfies all the conditions that are necessary for it to be termed as Green's function.

Let us consider the following example.

Example 7: Determine the appropriate Green's function by using the method of variation of parameters for the boundary value problem

$$-\frac{d^2y}{dx^2} = f(x), \quad y'(0) = 0, \quad y(1) = 0,$$

and express the solution as a definite integral.

Solution: We have to determine $y_1(x)$ and $y_2(x)$ as two linearly independent solutions of the associated homogeneous differential equation $\dfrac{d^2y}{dx^2} = 0$. So that $y_1'(0) = 0$, $y_2(1) = 0$.

We take, $y_1(x) = 1$ and $y_2(x) = 1 - x$

The Wronskian of these two functions is

$$W(x) = \begin{vmatrix} 1 & 1-x \\ 0 & -1 \end{vmatrix} = -1 \ne 0.$$

The Green's function can now be constructed by using Eqn.(76). We obtain

$$G(x,\xi) = \begin{cases} x-1, & 0 \le x \le \xi \\ \xi - 1, & \xi \le x \le 1 \end{cases} \tag{79}$$

Thus solution of a given bvp as a definite integral is given by

$$y(x) = \int_0^1 G(x,\xi)\, f(\xi)\, d\xi. \tag{80}$$

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There are certain **limitations** of formula (76) for obtaining Green's function. This method can be used only if we know the solution of the corresponding homogeneous differential equation, which we know is not always possible. Consequently, we give an alternative approach for developing Green's function expansion in a series of suitably chosen orthogonal functions. We take up an example that illustrate the use of eigen function expansion for obtaining Green's function for ODEs.

Example 8: Construct Green's function for the non-homogeneous problem

$$\frac{d^2y}{dx^2} = f(x), \; y(0) = y(\ell) = 0 \tag{81}$$

Solution: As per definition $G(x,\xi)$ satisfies

$$\frac{d^2G}{dx^2} = \delta(x-\xi) \tag{82}$$

$$G(0,\xi) = G(\ell,\xi) = 0 \tag{83}$$

Since $G(x,\xi)$ vanishes at both the ends of the interval $(0,\ell)$ therefore it can be expanded in terms of a series of suitably chosen orthogonal functions. Herein, we choose Fourier sine series for $G(x,\xi)$, that is

$$G(x,\xi)=\sum_{n=1}^{\infty} \gamma_n(\xi)\sin\frac{n\pi x}{\ell}, \tag{84}$$

where the coefficients γ_n depend on the parameter ξ.
From Eqn.(84), we get

$$\frac{\partial G}{\partial x}=\sum_{n=1}^{\infty}\left(\frac{n\pi}{\ell}\right)\gamma_n(\xi)\cos\frac{n\pi}{\ell}x.$$

$$\frac{\partial^2 G}{\partial x^2}=\sum_{n=1}^{\infty}-\left(\frac{n^2\pi^2}{\ell^2}\right)\gamma_n(\xi)\sin\frac{n\pi}{\ell}x \tag{85}$$

The r.h.s of Eqn.(82) can also be expressed as

$$\delta(x-\xi)=\sum_{n=1}^{\infty}\Delta_n\sin\frac{n\pi}{\ell}x \tag{86}$$

where the Fourier coefficients Δ_n are given by

$$\Delta_n(\xi)=\frac{2}{\ell}\int_0^{\ell}\delta(x-\xi)\sin\frac{n\pi}{\ell}x\;dx=\frac{2}{\ell}\sin\frac{n\pi}{\ell}\xi \tag{87}$$

Combining Eqns.(82), (85) and (86), we get

$$\sum_{n=1}^{\infty}-\left(\frac{n^2\pi^2}{\ell^2}\right)\gamma_n(\xi)\sin\frac{n\pi}{\ell}x=\sum_{n=1}^{\infty}\frac{2}{\ell}\sin\frac{n\pi}{\ell}\xi\,\sin\frac{n\pi}{\ell}x$$

or, $\quad \gamma_n(\xi)=-\dfrac{2\ell}{n^2\pi^2}\sin\dfrac{n\pi}{\ell}\xi.$

Hence, $$G(x,\xi)=-\frac{2\ell}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi\xi}{\ell}\sin\frac{n\pi}{\ell}x \tag{88}$$

Through the explicit formula (76) the Green's function for the given problems turns out to be

$$G(x,\xi)=\begin{cases}\dfrac{x(\ell-\xi)}{\ell} & (0\le x\le\xi)\\[2mm] \dfrac{\xi(\ell-x)}{\ell} & (\xi\le x\le\ell)\end{cases} \tag{89}$$

Though the two expressions given by Eqns.(88) and (89) for the Green's function appear to be different but they are same. You can verify it by expanding Eqn.(89) in a Fourier sine series. In order to obtain the solution of the non-homogeneous bvp (82) we use the formula

$$y(x)=\int_0^{\ell}G(x,\xi)f(\xi)\,d\xi. \tag{90}$$

On using the value of $G(x,\xi)$ from Eqn.(88), we get

$$y(x)=\frac{-2\ell}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi x}{\ell}\int_0^{\ell}f(\xi)\sin\frac{n\pi\xi}{\ell}\,d\xi \tag{91}$$

On using the substitution

$$b_n=\frac{2}{\ell}\int_0^{\ell}f(\xi)\sin\frac{n\pi}{\ell}\xi\,d\xi$$

we get,

$$y(x)=-\frac{\ell^2}{\pi^2}\sum_{n=1}^{\infty}\frac{b_n}{n^2}\sin\frac{n\pi}{\ell}x. \tag{92}$$

as the required solution of bvp (82).

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You may now try the following exercises.

E5) Solve each of the following boundary value problem by determining the appropriate Green's function by using the method of variation of parameters and expressing the solution as a definite integral.

a) $-y'' = f(x)$, $y(0) = 0$, $y(1) + y'(1) = 0$.

b) $-(y'' + y) = f(x)$, $y'(0) = 0$, $y(1) = 0$.

c) $-y'' = f(x)$, $y(0) = 0$, $y'(1) = 0$.

d) $-(y'' + y) = f(x)$, $y(0) = 0$, $y(1) = 0$.

We now end this unit by giving a summary of what we have learnt in it.

5.5 SUMMARY

In this unit, we have covered the following:

1. Introduced the concept of Green's function through physical consideration that has eventually led to the definition of influence function.
2. Employed the influence function for obtaining the deflection of the stretched string of negligible mass under piecewise continuous distributed load.
3. Enlarged the scope of the influence function by defining Green's function in the context of boundary value problems associated with linear ODE with constant or variable coefficients.
4. Constructed Green's functions for bvps and obtained the solution of bvps via Green's functions.
5. Outline a procedure for constructing Green's function via the method of variation of parameters.

5.6 SOLUTIONS/ANSWERS

E1) $$y(x) = \begin{cases} -\dfrac{\ell x}{8T} & , \quad 0 \le x \le \ell \\ \dfrac{4x^2 - 5\ell x + \ell^2}{8T}, & \dfrac{\ell}{2} \le x \le \ell \end{cases}$$

E2) $$y(x) = \begin{cases} \dfrac{Px}{2T} & , \quad 0 \le x \le \dfrac{\ell}{4} \\ \dfrac{P(\ell - 2x)}{4T}, & \dfrac{\ell}{4} \le x \le \dfrac{\ell}{2} \\ \dfrac{3P(\ell - x)}{2T}, & \dfrac{3\ell}{4} \le x \le \ell \end{cases}$$

E3) a) $$G(x, \xi) = \begin{cases} \ln \dfrac{\xi}{\ell} & , \quad 0 \le x \le \xi \\ \ln \dfrac{x}{\ell} & , \quad \xi \le x \le \ell \end{cases}$$

b) $$G(x, \xi) = \begin{cases} \dfrac{x}{2}\left(\xi - \dfrac{1}{\xi}\right) & , \quad 0 \le x \le \xi \\ \dfrac{\xi}{2}\left(x - \dfrac{1}{x}\right) & , \quad \xi \le x \le 1 \end{cases}$$

c) Green's function doesn't exist.

d) Green's function does not exist, bvp has infinite number of solutions.

e) $$G(x,\xi)=\begin{cases}\dfrac{\sinh k(\xi-1)\sinh x}{k\sinh k}, & 0\le x\le\xi\\ \dfrac{\sinh k\xi\sinh k(x-1)}{k\sinh k}, & \xi\le x\le 1\end{cases}$$

E4) Solution of given bvp can be written in the form

$$y(x)=\int_0^1 G(x,\xi)\,\xi\,d\xi$$

where we have from Example 3

$$G(x,\xi)=-\begin{cases}\dfrac{\sinh x\sinh(\xi-1)}{\sinh 1}, & 0\le x\le\xi\\ \dfrac{\sinh\xi\sinh(x-1)}{\sinh 1}, & \xi\le x\le 1\end{cases}$$

$$\therefore\ -y(x)=\int_0^x \frac{\xi\sinh\xi\sinh\,(x-1)}{\sinh 1}\,d\xi+\int_x^1\frac{\xi\sinh x\sinh\,(\xi-1)}{\sinh 1}\,d\xi$$

$$=\frac{\sinh\,(x-1)}{\sinh 1}\int_0^x \xi\sinh\xi+\frac{\sinh x}{\sinh 1}\int_x^1 \xi\sinh\,(\xi-1)\,d\xi\,.$$

Now,

$$\int_0^x \xi\sinh\xi\,d\xi=(\xi\cosh\xi)_0^x-\int_0^x\cosh\xi\,d\xi=x\cosh x-\sinh x$$

and

$$\int_x^1 \xi\sinh\,(\xi-1)\,d\xi=1-x\cosh\,(x-1)+\sinh\,(x-1)$$

Thus

$$-y(x)=\frac{1}{\sinh x}\begin{cases}\sinh\,(x-1)\,[x\cosh x-\sinh x]+\\ \sinh x\,[1-x\cosh(x-1)+\sinh\,(x-1)]\end{cases}$$

$$=\frac{\sinh x}{\sinh 1}-x\,.$$

E5) Solution in all cases is given by $y(x)=\int_0^1 G(x,\xi)\,f(\xi)\,d\xi$.

a) We take $y_1(x)=x,\ y_2(x)=2-x$ which satisfy

$$y_1(0)=0,\ y_2(1)=2-1=1,\ y_2'(1)=-1.$$

$$\therefore\ \ y_2(1)+y_2'(1)=0$$

$$W(\xi)=\begin{vmatrix}\xi & 2-\xi\\ 1 & -1\end{vmatrix}=-\xi-2+\xi=-2\neq 0$$

Hence, $$G(x,\xi)=\begin{cases}\dfrac{\xi(2-x)}{(-2)}, & 0\le x\le\xi\\ \dfrac{x(2-\xi)}{(-2)}, & \xi\le x\le 1\end{cases}$$

b) $y_1(x)=\cos x\ ,\ \ y_1'(0)=\sin 0=0$

$y_2(x)=\sin\,(x-1),\ y_2(1)=0$

$W(\xi)=-\cos\xi\cos(-\xi+1)+\sin(1-\xi)\sin\xi=\cos(\xi-\xi+1)=\cos 1\neq 0$

$$\text{Thus,}\quad G(x,\xi)=\begin{cases}\dfrac{\cos\xi\sin(x-1)}{\cos 1}, & 0\le x\le\xi\\[2mm] \dfrac{\cos x\sin(\xi-1)}{\cos 1}, & \xi\le x\le 1\end{cases}$$

c) $y_1(x)=x,\ y_2(x)=1$

$W(\xi)=\xi\times 0-1.1=-1\neq 0$

$$\text{Hence,}\quad G(x,\xi)=\begin{cases}\dfrac{\xi}{-1}, & 0\le x\le\xi\\[2mm] \dfrac{x}{-1}, & \xi\le x\le 1\end{cases}$$

d) $G(x,\xi)$ satisfies the homogeneous differential equation
$y''+y=0;\ y_1(x)=\sin x\ ;\ y_1(0)=0$
$y_2(x)=\sin(x-1),\ y_2(1)=0$

$$W(\xi)=\begin{vmatrix}\sin\xi & \sin(\xi-1)\\ \cos\xi & \cos(\xi-1)\end{vmatrix}=\sin\xi\cos(\xi-1)-\cos\xi\sin(\xi-1)$$

$$=\sin(\xi-\xi+1)=\sin 1\neq 0$$

$$\therefore G(x,\xi)=\begin{cases}\dfrac{\sin(x-1)\sin\xi}{\sin 1}, & 0\le x\le\xi\\[2mm] \dfrac{\sin x\sin(\xi-1)}{\sin 1}, & \xi\le x\le 1\end{cases}.$$

—x—