UNIT 8 SINGLESTEP METHODS FOR SOLVING INITIAL VALUE PROBLEMS

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8.1 INTRODUCTION

Many physical problems in science and engineering are modelled in the form of ordinary differential equations. The solution of the differential equation is the function that satisfies the differential equation and the given initial or boundary conditions. Such a solution is called the exact or the analytical solution. The methods for finding the exact solution are limited to certain special form of equations, mostly linear, homogeneous equations with constant coefficients or, to certain non-homogeneous equations with non-homogeneous terms restricted to some standard functions. However, numerical methods do not have any such limitations.

In this unit we shall discuss **singlestep** numerical methods for solving the **initial value problems** of first and higher orders. We shall start the unit with some preliminaries in Sec.8.2, which are required for further discussion in this unit. In Sec.8.3, we shall discuss Taylor series methods of first and higher order for solving single or system of initial value problems. Runge-Kutta methods of first and higher order will be discussed in Sec.8.4 both for single as well as system of equations. Finally, we shall discuss the stability of these singlestep methods in Sec.8.5.

Objectives

After studying this unit you should be able to

- explain singlestep and multistep methods for solving initial value problems;
- use Taylor series and Runge-Kutta methods for solving initial value problems;
- obtain the interval of absolute stability for a given singlestep method.

8.2 PRELIMINARIES

We start with giving some of the basic definitions in this section.
Consider the nth-order initial value problem (IVP)

$$y^{(n)}(x) = f(x, y, y', \ldots, y^{(n-1)})$$
$$y(x_0) = y_0, y'(x_0) = y'_0, \ldots, y^{(n-1)}(x_0) = y_0^{(n-1)} \qquad (1)$$

where x_0 is the initial point.

The nth order IVP (1) is equivalent to the following system of n first order initial value problems. We write $y = y_1$ and

$$\begin{aligned}
y' = y'_1 &= y_2 &&, \quad y_1(x_0) = y_0 \\
y'_2 &= y_3 &&, \quad y_2(x_0) = y'_0 \\
&\vdots && \quad \vdots \\
y'_{n-1} &= y_n &&, \quad y_{n-1}(x_0) = y_0^{(n-2)} \\
y'_n &= f(x, y_1, y_2, \ldots, y_n) &&, \quad y_n(x_0) = y_0^{(n-1)}
\end{aligned} \qquad (2)$$

In vector notation, this system can be written as

$$y'(x) = f(x, y), \; y(x_0) = \alpha \tag{3}$$

where

$$y = [y_1', y_2', \ldots, y_n']^T,$$
$$f(x, y) = [y_2, y_3, \ldots, f(x, y_1, y_2, \ldots, y_n)]^T,$$
$$\alpha = [y_0, y_0', \ldots, y_0^{(n-1)}]^T = [y_1(x_0), y_2(x_0), \ldots, y_n(x_0)]^T.$$

Hence, it is sufficient to study numerical methods for solving the single first order IVP

$$y'(x) = f(x, y), \; y(x_0) = y_0 \tag{4}$$

and then extend these methods for solving the system of first order IVP's (3) or, the nth order IVP(1).

We assume that the solution of the IVP(4) exists and is unique and also $f(x, y)$ has continuous partial derivatives with respect to x and y of as high order as required for all x in the interval $[x_0, b]$, in which the solution of the IVP(4) is to be obtained.

We divide the interval $[x_0, b]$ into N subintervals by the points

$$x_0 < x_1 < x_2 \ldots < x_N = b$$

The points $x_0, x_1, \ldots, x_N$ are called the **nodal point, grid points** or the **mesh-points**. We assume that the nodal points are equispaced with the spacing or the **step length**

$$h = \frac{x_N - x_0}{N}, \text{ where } x_N = b \tag{5}$$

We obtain

$$x_i = x_0 + ih \quad , \; i = 1, 2, \ldots, N$$

In numerical methods, we determine y_i which is an approximate value of the exact solution y(x) at the nodal point x_i, that is

$$y_i \approx y(x_i) \tag{6}$$

The numerical methods for solving the initial value problems can be broadly classified into two categories

(i) singlestep methods, (ii) multistep methods.

In singlestep methods, the solution value y_{i+1} at the nodal point x_{i+1}, is obtained by using the solution value y_i at only the previous point x_i. Such a method is called an **explicit singlestep method** and is written as

$$y_{i+1} = y_i + h\phi(x_i, y_i, h), \quad i = 0, 1, \ldots \tag{7}$$

where h is the step size and ϕ is called the **increment function**. If the method also uses the solution value y_{i+1}, then the method is called an **implicit singlestep method**. In multistep methods, the solution value y_{i+1} at the nodal point x_{i+1} is obtained by using the previously calculated k-values y_{i-m} at the previous k nodal points x_{i-m}, $m = 0, 1, \ldots, (k-1)$.

If the error in the numerical method (truncation error)

$$TE = y(x_{i+1}) - y_{i+1}$$

can be written as

$$TE = C h^{p+1} y^{(p+1)}(\xi), \quad x_{i-m} < \xi < x_{i+1} \tag{8}$$

where C is a constant independent of h, then the method is said to be of **order** p. With this background we shall now discuss singlestep methods of various orders. In this unit we shall confine our discussion only to **explicit** singlestep methods.

8.3 TAYLOR SERIES METHODS

Consider the IVP

$$y' = f(x, y), \quad y(x_0) = y_0. \tag{9}$$

We assume that y(x) can be expanded in Taylor series about any point x_i, that is,

$$y(x) = y(x_i) + (x - x_i)\,y'(x_i) + \frac{1}{2!}(x - x_i)^2\, y''(x_i) + \cdots$$
$$+ \frac{1}{p!}(x - x_i)^p\, y^{(p)}(x_i) + \frac{1}{(p+1)!}(x - x_i)^{p+1}\, y^{(p+1)}(x_i) + \cdots \tag{10}$$

Substituting $x = x_{i+1}$ and $x_{i+1} - x_i = h$, we get

$$y(x_{i+1}) = y(x_i) + h\,y'(x_i) + \frac{h^2}{2!} y''(x_i) + \cdots$$
$$+ \frac{h^p}{p!} y^{(p)}(x_i) + \frac{h^{p+1}}{(p+1)!} y^{(p+1)}(x_i) + \cdots \tag{11}$$

which is an infinite series. If we retain terms upto pth powers of h, we obtain

$$y_{i+1} = y_i + h\,y_i' + \frac{h^2}{2!} y_i'' + \cdots + \frac{h^p}{p!} y_i^{(p)} \tag{12}$$

where y_i is an approximation to $y(x_i)$.
The error term is given by

$$\text{error} = y(x_{i+1}) - y_{i+1}$$
$$= \frac{h^{p+1}}{(p+1)!} y^{(p+1)}(x_i) + \frac{h^{p+2}}{(p+2)!} y^{(p+2)}(x_i) + \cdots$$

which is also an infinite series.
The principal error term, called the **truncation error** (TE) is obtained as

$$TE = \frac{h^{p+1}}{(p+1)} y^{(p+1)}(\xi) \quad , \qquad x_i < \xi < x_{i+1}$$

Therefore, the method (12) is of order p.

The method (11) is called the Taylor's series method of order p. For different values of p, we obtain different order methods.

p = 1: first order Taylor series method

$$y_{i+1} = y_i + h\,y_i'$$

or,
$$y_{i+1} = y_i + h\,f(x_i, y_i)\,, \quad i = 0, 1, \cdots \tag{13}$$
$$TE = \frac{h^2}{2!} y''(\xi), \quad x_i < \xi < x_{i+1}$$

This method is also called the **Euler method**.

p = 2: second order Taylor series method

$$y_{i+1} = y_i + h\,y_i' + \frac{h^2}{2!} y_i'' \tag{14}$$
$$TE = \frac{h^3}{3!} y'''(\xi)\,, \quad x_i < \xi < x_{i+1}$$

and so on.
The higher order derivatives $y_i^{(m)}, m = 1, 2, \ldots$ in Eqn.(12) can be obtained by differentiating the given differential equation repeatedly and evaluating them at the nodal point x_i.

System of first order IVP's

For the system of first order IVP's (3)

$$\mathbf{y}'(x) = \mathbf{f}(x, \mathbf{y}), \quad \mathbf{y}(x_0) = \boldsymbol{\alpha}$$

we write the method (12) as

$$(\mathbf{y})_{x_{i+1}} = (\mathbf{y})_{x_i} + h(\mathbf{y}')_{x_i} + \frac{h^2}{2!}(\mathbf{y}'')_{x_i} + \cdots + \frac{h^p}{p!}(\mathbf{y}^{(p)})_{x_i} \tag{15}$$

where, $(\mathbf{y})_{x_i} = [y_{1i}, y_{2i}, \ldots, y_{ni}]^T$

$$(y^{(m)})_{x_i} = \left[y_{1i}^{(m)}, y_{2i}^{(m)}, \ldots, y_{ni}^{(m)}\right]^T, \quad m = 1, 2, \ldots, p$$

and $$y_{ji}^{(m)} = \frac{d^{m-1}}{dx^{m-1}} f_j(x_i, y_{1i}, y_{2i}, \ldots, y_{ni}), \quad j = 1, 2, \ldots, n$$

Note that each term in Eqn.(15) is a column vector having the same number of elements as the number of first order differential equations.

The **truncation error** associated with the method (15) is given by

$$TE = \frac{h^{p+1}}{(p+1)!} y^{(p+1)}(\xi), \quad x_i < \xi < x_{i+1}$$

and hence the method (15) is of order p.

Example 1: Obtain the approximate value of $y(1.4)$ for the initial value problem

$$y' = -2xy^2, \; y(1) = 1$$

using .

(a) Taylor series method of first order (Euler method)

(b) Taylor series method of second order with (i) $h = 0.1$, (ii) $h = 0.2$. Compare the results with the exact solution $y(x) = \frac{1}{x^2}$.

Solution:

(a) Taylor series method of **first order** is given by

$$y_{i+1} = y_i + h y_i', \; i = 0, 1, \ldots$$

(i) **For h = 0.1,** we obtain for

i = 0 : $x_0 = 1, y_0 = 1, y_0' = -2x_0 y_0^2 = -2$

$y_1 \approx y(1.1) = y_0 + h y_0' = 1 + 0.1(-2) = 0.8$

i = 1 : $x_1 = 1.1, y_1 = 0.8, y_1' = -2x_1 y_1^2 = -1.408$

$y_2 \approx y(1.2) = y_1 + h y_1' = 0.8 + (0.1)(-1.408) = 0.6592$

i = 2 : $x_2 = 1.2, y_2 = 0.6592, y_2' = -2x_2 y_2^2 = -1.042907$

$y_3 \approx y(1.3) = y_2 + h y_2' = 0.544909$

i = 3 : $x_3 = 1.3, y_3 = 0.544909, y_3' = -2x_3 y_3^2 = -0.772007$

$y_4 \approx y(1.4) = y_3 + h y_3' = 0.467708$

We also have from the exact solution

$$y(1.4) = 1/(1.4)^2 = 0.510204$$

Actual error at x = 1.4 = $y(1.4) - y_4 = 0.0425$.

(ii) **For h = 0.2,** we obtain

i = 0 : $x_0 = 1, y_0 = 1, y_0' = -2x_0 y_0^2 = -2$

$y_1 \approx y(1.2) = y_0 + h y_0' = 1 + 0.2(-2) = 0.6$

i = 1 : $x_1 = 1.2, y_1 = 0.6, y_1' = -2x_1 y_1^2 = -0.864$

$y_2 \approx y(1.4) = y_1 + h y_1' = 0.4272$

Actual error at x = 1.4 = $y(1.4) - y_2 = 0.0830$.

(b) Taylor series method of **second order** is given by

$$y_{i+1} = y_i + h y_i' + \frac{h^2}{2} y_i'', \; i = 0, 1, \ldots$$

We have ,

$$y' = -2xy^2, \qquad y_i' = -2x_i y_i^2$$

$$y'' = -2y^2 - 4xyy', \qquad y_i'' = -2y_i^2 - 4x_i y_i y_i'$$

(i) **For h = 0.1**, we obtain

i = 0 : $x_0 = 1, y_0 = 1, y'_0 = -2, y''_0 = 6$

$$y_1 \approx y(1.1) = y_0 + h\, y'_0 + \frac{h^2}{2} y''_0 = 0.83$$

i = 1 : $x_1 = 1.1, y_1 = 0.83, y'_1 = -1.51558, y''_1 = 4.157098$

$$y_2 \approx y(1.2) = y_1 + h\, y'_1 + \frac{h^2}{2} y''_1 = 0.699227$$

i = 2 : $x_2 = 1.2, y_2 = 0.699227, y'_2 = -1.173404, y''_2 = 2.960447$

$$y_3 \approx y(1.3) = y_2 + h\, y'_2 + \frac{h^2}{2} y''_2 = 0.596689$$

i = 3 : $x_3 = 1.3, y_3 = 0.596689, y'_3 = -0.925698, y''_3 = 2.160163$

$$y_4 \approx y(1.4) = y_3 + h\, y'_3 + \frac{h^2}{2} y''_3 = 0.514920$$

Actual error at $x = 1.4$ $= y(1.4) - y_4 = -0.0047$.

(ii) **For h = 0.2**, we obtain

i = 0 : $x_0 = 1, y_0 = 1, y'_0 = -2, y''_0 = 6$

$$y_1 \approx y_0 + h\, y'_0 + \frac{h^2}{2} y''_0 = 0.72$$

i = 1 : $x_1 = 1.2, y_1 = 0.72, y'_1 = -1.24416, y''_1 = 3.263017$

$$y_2 \approx y(1.4) = y_1 + h\, y'_1 + \frac{h^2}{2} y''_1 = 0.536428$$

Actual error at $\mathbf{x = 1.4} =$ $y(1.4) - y_4 = -0.0262$.

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Remark: You may notice that the actual error in magnitude decreases when (i) h is reduced or (ii) higher order method is used.

Example 2: Solve the initial value problem $y' = x^2 + y^2, y(0) = 1$ upto $x = 0.4$ using fourth order Taylor series method with $h = 0.2$.

Solution: The **fourth order** Taylor series method is given by

$$y_{i+1} = y_i + h\, y'_i + \frac{h^2}{2} y''_i + \frac{h^3}{6} y'''_i + \frac{h^4}{24} y_i^{iv}. \qquad i = 0, 1, \ldots$$

We have ,

$$\begin{aligned} y' &= x^2 + y^2, & y'_i &= x_i^2 + y_i^2 \\ y'' &= 2x + 2y\,y', & y''_i &= 2x_i + 2y_i\, y'_i \\ y''' &= 2 + 2(y')^2 + 2y\,y'', & y'''_i &= 2 + 2(y'_i)^2 + 2y_i\, y''_i \\ y^{iv} &= 6y'\,y'' + 2y\,y''', & y_i^{iv} &= 6y'_i\, y''_i + 2y_i\, y'''_i \end{aligned}$$

For h = 0.2, we obtain

i = 0 : $x_0 = 0, y_0 = 1, y'_0 = 1, y''_0 = 2, y'''_0 = 8, y_0^{iv} = 28$

$$\begin{aligned} y_1 \approx y(0.2) &= y_0 + h\, y'_0 + \frac{h^2}{2} y''_0 + \frac{h^3}{6} y'''_0 + \frac{h^4}{24} y_0^{iv} \\ &= 1 + 0.2(1) + \frac{0.04}{2}(2) + \frac{0.008}{6}(8) + \frac{0.0016}{24}(28) \\ &= 1.252533 \end{aligned}$$

i = 1 : $x_1 = 0.2, y_1 = 1.252533, y'_1 = 1.608839, y''_1 = 4.430248,$
$y'''_1 = 18.274789, y_1^{iv} = 88.544887$

$$y_2 \approx y(0.4) = y_1 + h y_1' + \frac{h^2}{2} y_1'' + \frac{h^3}{6} y_1''' + \frac{h^4}{24} y_1^{iv}$$
$$= 1.252533 + 0.2(1.608839) + \frac{0.04}{2}(4.430248)$$
$$+ \frac{0.008}{6}(18.274789) + \frac{0.0016}{24}(88.544887)$$
$$= 1.693175$$

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Example 3: Solve the system of equations

$$y_1' = -3y_1 + 2y_2 \quad , \quad y_1(0) = 0$$
$$y_2' = 3y_1 - 4y_2 \quad , \quad y_2(0) = 1/2$$

on the interval [0, 0.4] using Taylor series second order method with h = 0.2.

Solution: The **second order** Taylor series method is given by

$$(\mathbf{y})_{x_{i+1}} = (\mathbf{y})_{x_i} + h(\mathbf{y}')_{x_i} + \frac{h^2}{2}(\mathbf{y}'')_{x_i}, \qquad i = 0, 1, \ldots$$

We have

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \mathbf{y}' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -3y_1 + 2y_2 \\ 3y_1 - 4y_2 \end{bmatrix},$$

$$\mathbf{y}'' = \begin{bmatrix} y_1'' \\ y_2'' \end{bmatrix} = \begin{bmatrix} -3y_1' + 2y_2' \\ 3y_1' - 4y_2' \end{bmatrix}$$

For h = 0.2, we obtain

$$\mathbf{i = 0}: \qquad x_0 = 0, \; (\mathbf{y})_{x_0} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}_{x_0} = \begin{bmatrix} 0 \\ 1/2 \end{bmatrix},$$

$$(\mathbf{y}')_{x_0} = \begin{bmatrix} -3y_1 + 2y_2 \\ 3y_1 - 4y_2 \end{bmatrix}_{x_0} = \begin{bmatrix} -3(0) + 2(1/2) \\ 3(0) - 4(1/2) \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$(\mathbf{y}'')_{x_0} = \begin{bmatrix} -3y_1' + 2y_2' \\ 3y_1' - 4y_2' \end{bmatrix}_{x_0} = \begin{bmatrix} -3(1) + 2(-2) \\ 3(1) - 4(-2) \end{bmatrix} = \begin{bmatrix} -7 \\ 11 \end{bmatrix}$$

$$(\mathbf{y})_{x_1} \approx \mathbf{y}(0.2) = (\mathbf{y})_{x_0} + h(\mathbf{y}')_{x_0} + \frac{h^2}{2}(\mathbf{y}'')_{x_0}$$
$$= \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} + 0.2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \frac{(0.2)^2}{2} \begin{bmatrix} -7 \\ 11 \end{bmatrix} = \begin{bmatrix} 0.06 \\ 0.32 \end{bmatrix}$$

$$\mathbf{i = 1}: \qquad x_1 = 0.2, (\mathbf{y})_{x_1} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}_{x_1} = \begin{bmatrix} 0.06 \\ 0.32 \end{bmatrix}$$

$$(\mathbf{y}')_{x_1} = \begin{bmatrix} -3y_1 + 2y_2 \\ 3y_1 - 4y_2 \end{bmatrix}_{x_1} = \begin{bmatrix} -3(0.06) + 2(0.32) \\ 3(0.06) - 4(0.32) \end{bmatrix} = \begin{bmatrix} 0.46 \\ -1.10 \end{bmatrix}$$

$$(\mathbf{y}'')_{x_1} = \begin{bmatrix} -3y_1' + 2y_2' \\ 3y_1' - 4y_2' \end{bmatrix}_{x_1} = \begin{bmatrix} -3(0.46) + 2(-1.10) \\ 3(0.46) - 4(-1.10) \end{bmatrix} = \begin{bmatrix} -3.58 \\ 5.78 \end{bmatrix}$$

$$(\mathbf{y})_{x_2} = \mathbf{y}(0.4) = (\mathbf{y})_{x_1} + (\mathbf{y}')_{x_1} + \frac{h^2}{2}(\mathbf{y}'')_{x_1}$$
$$= \begin{bmatrix} 0.06 \\ 0.32 \end{bmatrix} + 0.2 \begin{bmatrix} 0.46 \\ -1.10 \end{bmatrix} + \frac{0.04}{2} \begin{bmatrix} -3.58 \\ 5.78 \end{bmatrix}$$
$$= \begin{bmatrix} 0.0804 \\ 0.2156 \end{bmatrix}$$

Hence, $y_1(0.4) \approx 0.0804$, $y_2(0.4) \approx 0.2156$.

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Example 4: Reduce the second order initial value problem

$$y'' = y' + 1$$
$$y(0) = 1,\ y'(0) = 1$$

to a system of first order initial value problems. Hence, find an approximate value of $y(0.4)$ and $y'(0.4)$ using Taylor series second order method with $h = 0.2$.

Solution: Set $y = y_1$, we get

$$y_1' = y_2 \quad , \quad y_1(0) = 1$$
$$y_2' = 1 + y_2 \ , \quad y_2(0) = 1$$

we have

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \mathbf{y}' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} y_2 \\ 1 + y_2 \end{bmatrix}$$

$$\mathbf{y}'' = \begin{bmatrix} y_1'' \\ y_2'' \end{bmatrix} = \begin{bmatrix} y_2' \\ y_2' \end{bmatrix} = \begin{bmatrix} 1 + y_2 \\ 1 + y_2 \end{bmatrix}$$

The **second order** Taylor series method is given by

$$(\mathbf{y})_{x_{i+1}} = (\mathbf{y})_{x_i} + h(\mathbf{y})_{x_i} + \frac{h^2}{2}(\mathbf{y}'')_{x_i}, \quad i = 0, 1, \ldots$$

For h = 0.2, we get

$\mathbf{i = 0:}$

$$x_0 = 0, (\mathbf{y})_{x_0} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}_{x_0} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, (\mathbf{y}')_{x_0} = \begin{bmatrix} y_2 \\ 1 + y_2 \end{bmatrix}_{x_0} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$(\mathbf{y}'')_{x_0} = \begin{bmatrix} 1 + y_2 \\ 1 + y_2 \end{bmatrix}_{x_0} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$(\mathbf{y})_{x_1} \approx \mathbf{y}(0.2) = (\mathbf{y})_{x_0} + h(\mathbf{y}')_{x_0} + \frac{h^2}{2}(\mathbf{y}'')_{x_0}$$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0.2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{0.04}{2} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1.24 \\ 1.44 \end{bmatrix}$$

$\mathbf{i = 1:}$

$$x_1 = 0.2, (\mathbf{y})_{x_1} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}_{x_1} = \begin{bmatrix} 1.24 \\ 1.44 \end{bmatrix}.$$

$$(\mathbf{y}')_{x_1} = \begin{bmatrix} y_2 \\ 1 + y_2 \end{bmatrix}_{x_1} = \begin{bmatrix} 1.44 \\ 2.44 \end{bmatrix}, (\mathbf{y}'')_{x_1} = \begin{bmatrix} 1 + y_2 \\ 1 + y_2 \end{bmatrix}_{x_1} = \begin{bmatrix} 2.44 \\ 2.44 \end{bmatrix}$$

$$(\mathbf{y})_{x_2} \approx \mathbf{y}(0.4) = (\mathbf{y})_{x_1} + h(\mathbf{y}')_{x_1} + \frac{h^2}{2}(\mathbf{y}'')_{x_1}$$

$$= \begin{bmatrix} 1.24 \\ 1.44 \end{bmatrix} + 0.2 \begin{bmatrix} 1.44 \\ 2.44 \end{bmatrix} + \frac{0.04}{2} \begin{bmatrix} 2.44 \\ 2.44 \end{bmatrix}$$

$$= \begin{bmatrix} 1.5768 \\ 1.9768 \end{bmatrix}$$

Hence, $y(0.4) \approx 1.5768$ and $y'(0.4) \approx 1.9768$.

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You may now try the following exercises.

E1) Obtain the approximate value $y(1.2)$ for the initial value problem

$$y' = -y^2, \quad y(1) = 1$$

using Euler method with $h = 0.1$.

E2) Solve the initial value problem

$$y' = 2x + 3y, \ y(0) = 1$$

in the interval [0, 0.4] using (a) Taylor series second order method, (b) Taylor series fourth order method. Take h = 0.2 in each case.

E3) Solve the initial value problem

$$y' = x + \sin y,\ y(1) = 1$$

in the interval [1, 1.2] using (a) Taylor series second order method, (b) Taylor series fourth order method. Take h = 0.1 in each case.

E4) Find the approximate value of y (0.4) and u (0.4) for the system of equations

$$y' = 2y + u \quad , \quad y(0) = 1$$
$$u' = 3y + 4u \quad , \quad u(0) = 1$$

using Taylor series second order method with h = 0.2

E5) Write the second order initial value problem

$$y'' = -2y\,y',\ y(0) = 1,\ y'(1) = 1$$

as a system of two first order initial value problems. Hence, obtain approximate value of $y(1.4)$ and $y'(1.4)$ using second order Taylor series method with $h = 0.2$.

Taylor series method of any arbitrary order for solving the initial value problem (4) can be easily obtained. However, from application point of view, Taylor series method has a serious disadvantage. The method requires evaluation of derivatives of higher order for the function $f(x, y)$ of two variables x and y. These higher order derivatives have to be obtained manually for each problem. Therefore, we need to develop methods, which do not require the evaluation of higher order derivatives and still compare with the Taylor series methods. One such class of methods is the Runge-Kutta methods which we shall discuss in the next section.

8.4 RUNGE-KUTTA METHODS

To illustrate the idea behind Runge-Kutta methods, we integrate $y' = f(x, y)$ in the interval $[x_i, x_{i+1}]$ and obtain

$$\int_{x_i}^{x_{i+1}} \left(\frac{dy}{dx}\right) dx = \int_{x_i}^{x_{i+1}} f(x, y)\, dx$$

or, $$y(x_{i+1}) = y(x_i) + \int_{x_i}^{x_{i+1}} f(x, y)\, dx \tag{16}$$

We note that $f(x, y)$ is the slope of the solution curve and it changes continuously in $[x_i, x_{i+1}]$. Let the continuously varying slope $f(x, y)$ in this interval be approximated by the fixed slope at $x = x_i$, that is, $f(x, y) \approx f(x_i, y_i)$ in $[x_i, x_{i+1}]$. Then from Eqn.(16), we obtain

$$y_{i+1} = y_i + f(x_i, y_i) \int_{x_i}^{x_{i+1}} dx$$

or, $$y_{i+1} = y_i + h\, f(x_i, y_i) \tag{17}$$

which is the **Euler method** (13).

Now, approximate $f(x, y)$ in $[x_i, x_{i+1}]$ by the slope at the point $x = x_{i+1}$, that is, $f(x, y) \approx f(x_{i+1}, y_{i+1})$.

Then, we obtain from Eqn.(16)

$$y_{i+1} = y_i + f(x_{i+1}, y_{i+1}) \int_{x_i}^{x_{i+1}} dx$$

or, $$y_{i+1} = y_i + h\, f(x_{i+1}, y_{i+1}) \tag{18}$$

which is the **backward Euler method**. This method is an implicit method. The non-linear equation in y_{i+1} for non-linear problems can be solved using the Newton-Raphson method.

Now, approximate $f(x,y)$ in $[x_i, x_{i+1}]$ by the mean of the slopes at $x = x_i$ and $x = x_{i+1}$, that is, $f(x,y) = \frac{1}{2}[f(x_i, y_i) + f(x_{i+1}, y_{i+1})]$. Then we obtain the implicit method.

$$y_{i+1} = y_i + \frac{h}{2}[f(x_i, y_i) + f(x_{i+1}, y_{i+1})] \tag{19}$$

which is called the **trapezoidal method**.

If we approximate y_{i+1} in the right side of Eqn.(19) by the Euler method (17), then we obtain the explicit method

$$y_{i+1} = y_i + \frac{h}{2}[f(x_i, y_i) + f(x_i + h, y_i + h f_i)]$$

where $f_i = f(x_i, y_i)$. This method is called the **Heun's method** or **Euler Cauchy method**. We can re-write this method in the form

$$\begin{aligned} y_{i+1} &= y_i + \frac{1}{2}(K_1 + K_2) \\ \text{where, } K_1 &= h f(x_i, y_i) \\ K_2 &= h f(x_i + h, y_i + K_1) \end{aligned} \tag{20}$$

The philosophy behind the Runge-Kutta methods is to consider a weighted average of slopes or approximate slopes at a number of points in $[x_i, x_{i+1}]$. If we use ν slopes, then the method is written as

$$y_{i+1} = y_i + w_1 K_1 + w_2 K_2 + \cdots + w_\nu K_\nu, \tag{21}$$

where,

$$\begin{aligned} K_1 &= h f(x_i, y_i) \\ K_2 &= h f(x_i + \alpha_2 h, y_i + \beta_{21} K_1) \\ &\vdots \\ K_\nu &= h f\left(x_i + \alpha_\nu h, y_i + \sum_{j=1}^{\nu-1} \beta_{\nu j} K_j\right) \end{aligned}$$

w_i's, α_i's and β_{ij}'s are parameters to be determined such that the method (21) agrees with the Taylor series method of certain order.

We shall only discuss the cases $\nu = 2$ and $\nu = 4$ which provide the methods of orders 2 and 4 respectively.

Second order Runge-Kutta method (two stage method)

Consider the following two-stage method

$$y_{i+1} = y_i + w_1 K_1 + w_2 K_2 \tag{22}$$

where

$$\begin{aligned} K_1 &= h f(x_i, y_i) \\ K_2 &= h f(x_i + \alpha_2 h, y_i + \beta_{21} K_1) \end{aligned}$$

Expanding K_1 and K_2 in Taylor series about the point x_i, we get

$$\begin{aligned} K_1 &= h f(x_i, y_i) = h f_i \\ K_2 &= h f(x_i + \alpha_2 h, y_i + \beta_{21} h f_i) \\ &= h\left[f_i + \left(\alpha_2 h \frac{\partial f}{\partial x} + \beta_{21} h f_i \frac{\partial f}{\partial y}\right)\right. \\ &\quad \left. + \frac{1}{2}\left(\alpha_2^2 h^2 \frac{\partial^2 f}{\partial x^2} + 2\alpha_2 \beta_{21} h^2 f_i \frac{\partial^2 f}{\partial x \partial y} + \beta_{21}^2 h^2 f_i^2 \frac{\partial^2 f}{\partial y^2}\right) + \cdots\right] \end{aligned}$$

$$= h f_i + h^2\left(\alpha_2 \frac{\partial f}{\partial x} + \beta_{21} f_i \frac{\partial f}{\partial y}\right) + \frac{h^3}{2}\left(\alpha_2^2 \frac{\partial^2 f}{\partial x^2} + 2\alpha_2\beta_{21} f_i \frac{\partial^2 f}{\partial x \partial y} + \beta_{21}^2 f_i^2 \frac{\partial^2 f}{\partial y^2}\right) + \cdots$$

where the partial derivatives are evaluated at the point (x_i, y_i). Substituting the expressions for K_1 and K_2 in Eqn.(20), we get

$$\begin{aligned} y_{i+1} &= y_i + w_1 h f_i + w_2\Bigg[h f_i + h^2\left(\alpha_2 \frac{\partial f}{\partial x} + \beta_{21} f_i \frac{\partial f}{\partial y}\right) \\ &\quad + \frac{h^3}{2}\left(\alpha_2^2 \frac{\partial^2 f}{\partial x^2} + 2\alpha_2\beta_{21} f_i \frac{\partial^2 f}{\partial x \partial y} + \beta_{21}^2 f_i^2 \frac{\partial^2 f}{\partial y^2}\right) + \cdots\Bigg] \\ &= y_i + (w_1 + w_2) h f_i + h^2 w_2\left(\alpha_2 \frac{\partial f}{\partial x} + \beta_{21} f_i \frac{\partial f}{\partial y}\right) \\ &\quad + \frac{1}{2} h^3 w_2\left(\alpha_2^2 \frac{\partial^2 f}{\partial x^2} + 2\alpha_2\beta_{21} f_1 \frac{\partial^2 f}{\partial x \partial y} + \beta_{21}^2 f_i^2 \frac{\partial^2 f}{\partial y^2}\right) + \cdots \end{aligned} \tag{23}$$

From the given differential equation, we get

$$y' = f(x, y)$$

$$y'' = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = \frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y}$$

$$\begin{aligned} y''' &= \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x \partial y}\frac{dy}{dx}\right) + f\left(\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2}\frac{dy}{dx}\right) + \frac{\partial f}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}\right) \\ &= \left(\frac{\partial^2 f}{\partial x^2} + f\frac{\partial^2 f}{\partial x \partial y}\right) + f\left(\frac{\partial^2 f}{\partial x \partial y} + f\frac{\partial^2 f}{\partial y^2}\right) + \frac{\partial f}{\partial y}\left(\frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y}\right) \\ &= \left(\frac{\partial^2 f}{\partial x^2} + 2f\frac{\partial^2 f}{\partial x \partial y} + f^2\frac{\partial^2 f}{\partial y^2}\right) + \frac{\partial f}{\partial y}\left(\frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y}\right) \end{aligned}$$

Now, Taylor series method for solving the given differential equation is given by

$$\begin{aligned} y(x_{i+1}) &= y(x_i) + h\,y'(x_i) + \frac{h^2}{2!}y''(x_i) + \frac{h^3}{3!}y'''(x_i) + \cdots \\ &= y_i + h f_i + \frac{h^2}{2}\left(\frac{\partial f}{\partial x} + f_i\frac{\partial f}{\partial y}\right) \\ &\quad + \frac{h^3}{6}\left[\frac{\partial^2 f}{\partial x^2} + 2f_i\frac{\partial^2 f}{\partial x \partial y} + f_i^2\frac{\partial^2 f}{\partial y^2} + \frac{\partial f}{\partial y}\left(\frac{\partial f}{\partial x} + f_i\frac{\partial f}{\partial y}\right)\right] + \cdots \end{aligned} \tag{24}$$

where partial derivatives are evaluated at the point (x_i, y_i).

Comparing the coefficients of h and h^2 in Eqns.(23) and (24) we obtain

$$\begin{aligned} w_1 + w_2 &= 1 \\ w_2\,\alpha_2 &= \frac{1}{2} \\ w_2\,\beta_{21} &= \frac{1}{2} \end{aligned} \tag{25}$$

It is not possible to compare the coefficient of h^3, as there are three terms in Eqn.(23) and five terms in Eqn.(24).

The solution of the system of Eqn.(25) is obtained as

$$\beta_{21} = \alpha_2, w_2 = \frac{1}{2\alpha_2}, w_1 = 1 - \frac{1}{2\alpha_2}$$

and $\alpha_2 \neq 0$ is arbitrary. The method (22) becomes

$$y_{i+1} = y_i + \left(1 - \frac{1}{2\alpha_2}\right)K_1 + \frac{1}{2\alpha_2}K_2 \tag{26}$$

where

$$K_1 = h\,f(x_i, y_i)$$
$$K_2 = h\,f(x_i + \alpha_2 h, y_i + \alpha_2 K_1)$$

The truncation error associated with the method is given by

$$\begin{aligned} TE &= y(x_{i+1}) - y_{i+1} \\ &= h^3\left[\left(\frac{1}{6} - \frac{1}{2}w_2\alpha_2^2\right)\left(\frac{\partial^2 f}{\partial x^2} + 2f_i\frac{\partial^2 f}{\partial x \partial y} + f_i^2\frac{\partial^2 f}{\partial y^2}\right)\right. \\ &\quad \left. + \frac{1}{6}\frac{\partial f}{\partial y}\left(\frac{\partial f}{\partial x} + f_i\frac{\partial f}{\partial y}\right)\right] + O(h^4) \end{aligned}$$

Therefore, the two stage Runge-Kutta method (26) is a second order method for all values of α_2. For different values of α_2 we obtain different second order methods.

For $\alpha_2 = 1$, we get, $\beta_{21} = 1, w_1 = w_2 = \frac{1}{2}$ and the method is

$$y_{i+1} = y_i + \frac{1}{2}(K_1 + K_2) \tag{27}$$
$$K_1 = h\,f(x_i, y_i)$$
$$K_2 = h\,f(x_i + h, y_i + K_1)$$

which is the **Euler-Cauchy method** or the **Heun method** (18)

For $\alpha_2 = \frac{1}{2}$, we get $\beta_{21} = \frac{1}{2}, w_1 = 0, w_2 = 1$ and the method is

$$y_{i+1} = y_i + K_2 \tag{28}$$
$$K_1 = h\,f(x_i, y_i)$$
$$K_2 = h\,f\left(x_i + \frac{h}{2}, y_i + \frac{1}{2}K_1\right)$$

which is also called the **modified Euler-Cauchy method**.

Fourth order Runge-Kutta method (Four-stage method)

Without derivation, we list one of the four stage Runge-Kutta methods

$$y_{i+1} = y_i + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) \tag{29}$$
$$K_1 = h\,f(x_i, y_i)$$
$$K_2 = h\,f\left(x_i + \frac{h}{2}, y_i + \frac{1}{2}K_1\right)$$
$$K_3 = h\,f\left(x_i + \frac{h}{2}, y_i + \frac{1}{2}K_2\right)$$
$$K_4 = h\,f(x_i + h, y_i + K_3)$$

which is called the **classical Runge-Kutta method** and is of fourth order.

Remarks

1. Both Taylor series method and Runge-Kutta methods are singlestep methods.
2. We require two function evaluations per step for a second order Runge-Kutta method and four function evaluations per step for a fourth order Runge-Kutta method.

System of first order IVP's

For the system of n first order IVP's (3)

$$\mathbf{y}' = \mathbf{f}(x, \mathbf{y}), \mathbf{y}(x_0) = \boldsymbol{\alpha}$$

we write the Runge-Kutta methods as follows:

Second order method (27)

$$\mathbf{y}_{i+1} = \mathbf{y}_i + \frac{1}{2}(\mathbf{K}_1 + \mathbf{K}_2) \tag{30}$$

where

$$\mathbf{K}_1 = \begin{bmatrix} K_{11} \\ K_{21} \\ \vdots \\ K_{n1} \end{bmatrix}, \quad \mathbf{K}_2 = \begin{bmatrix} K_{12} \\ K_{22} \\ \vdots \\ K_{n2} \end{bmatrix}$$

and

$$K_{j1} = h f_j(x_i, y_{1i}, y_{2i}, \ldots, y_{ni})$$
$$K_{j2} = h f_j(x_i + h, y_{1i} + K_{11}, y_{2i} + K_{21}, \ldots, y_{ni} + K_{n1})$$
$$j = 1, 2, \ldots, n.$$

Fourth order method (29)

$$\mathbf{y}_{i+1} = \mathbf{y}_i + \frac{1}{6}(\mathbf{K}_1 + 2\mathbf{K}_2 + 2\mathbf{K}_3 + 2\mathbf{K}_4) \tag{31}$$

where, $\mathbf{K}_1 = \begin{bmatrix} K_{11} \\ K_{21} \\ \vdots \\ K_{n1} \end{bmatrix}, \mathbf{K}_2 = \begin{bmatrix} K_{12} \\ K_{22} \\ \vdots \\ K_{n2} \end{bmatrix}, \mathbf{K}_3 = \begin{bmatrix} K_{13} \\ K_{23} \\ \vdots \\ K_{n3} \end{bmatrix}, \mathbf{K}_4 = \begin{bmatrix} K_{14} \\ K_{24} \\ \vdots \\ K_{n4} \end{bmatrix}$

and

$$K_{j1} = h f_j(x_i, y_{1i}, y_{2i}, \ldots, y_{ni})$$
$$K_{j2} = h f_j\left(x_i + \frac{h}{2}, y_{1i} + \frac{1}{2}K_{11}, y_{2i} + \frac{1}{2}K_{21}, \ldots, y_{ni} + \frac{1}{2}K_{n1}\right)$$
$$K_{j3} = h f_j\left(x_i + \frac{h}{2}, y_{1i} + \frac{1}{2}K_{12}, y_{2i} + \frac{1}{2}K_{22}, \ldots, y_{ni} + \frac{1}{2}K_{n2}\right)$$
$$K_{j4} = h f_j(x_i + h, y_{1i} + K_{13}, y_{2i} + K_{23}, \ldots, y_{ni} + K_{n3})$$
$$j = 1, 2, \ldots, n.$$

We shall now illustrate the methods discussed above through examples.

Example 5: Solve the initial value problem

$$y' = x^2 + y^2, y(1) = 2$$

on the interval [1, 1.4] using the second order Runge-Kutta method (27) with h = 0.2.

Solution: We are given that $f(x, y) = x^2 + y^2$ and $h = 0.2$.

We obtain from Eqn.(27) for

$\mathbf{i = 0}$: $x_0 = 1, y_0 = 2$

$$K_1 = h f(x_0, y_0) = h f(1, 2) = 0.2[(1)^2 + (2)^2] = 1$$
$$K_2 = h f(x_0 + h, y_0 + K_1) = h f(1.2, 3) = 0.2[(1.2)^2 + 3^2] = 2.088$$
$$y_1 \approx y(1.2) = y_0 + \frac{1}{2}(K_1 + K_2) = 3.544$$

$\mathbf{i = 1}$: $x_1 = 1.2, y_1 = 3.544$

$$K_1 = h f(x_1, y_1) = h f(1.2,\ 3.544)$$
$$= 0.2[(1.2)^2 + (3.544)^2] = 2.799987$$
$$K_2 = h f(x_1 + h, y_1 + K_1) = h f(1.4,\ 6.343987)$$
$$= 0.2[(1.4)^2 + (6.343987)^2] = 8.441235$$
$$y_2 \approx y(1.4) = y_1 + \frac{1}{2}(K_1 + K_2) = 9.164611.$$

***

Example 6: Find an approximate value of y(0.8) for the initial value problem.

$$y' = \sqrt{x+y}, \quad y(0.4) = 0.41$$

using the second order Runge-Kutta method (26) with h = 0.2.

Solution: We are given that $f(x,y) = \sqrt{x+y}$ and $h = 0.2$.

We obtain from Eqn.(26), for

i = 0: $x_0 = 0.4, \ y_0 = 0.41$

$$K_1 = hf(x_0, y_0) = hf(0.4, 0.41) = 0.2\sqrt{0.4 + 0.41} = 0.18$$

$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}K_1\right) = hf(0.5, 0.5) = 0.2\sqrt{0.5 + 0.5} = 0.2$$

$$y_1 \approx y(0.6) = y_0 + K_2 = 0.41 + 0.2 = 0.61$$

i = 1: $x_1 = 0.6, \ y_1 = 0.61$

$$K_1 = hf(x_1, y_1) = hf(0.6, 0.61) = 0.2\sqrt{0.6 + 0.61} = 0.22$$

$$K_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{1}{2}K_1\right) = hf(0.7, 0.72)$$

$$= 0.2\sqrt{0.7 + 0.72} \qquad = 0.2383$$

$$y_2 \approx y(0.8) = y_1 + K_2 = 0.61 + 0.2383 = 0.8483$$

***

Example 7: Find an approximate value of y(0.2) for the initial value problem

$$y' = 1 + y^2, \quad y(0) = 1$$

using the classical Runge-Kutta fourth order method (29) with h = 0.1.

Solution: We are given that $f(x,y) = 1 + y^2$ and $h = 0.1$.

We obtain from Eqn.(29) for

i = 0: $x_0 = 0, y_0 = 1$

$$K_1 = hf(x_0, y_0) = hf(0,1) = 0.1\left[1 + (1)^2\right] = 0.2$$

$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right) = hf(0.05, 1.1) = 0.1\left[1 + (1.1)^2\right] = 0.221$$

$$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right) = hf(0.05, 1.1105)$$

$$= 0.1\left[1 + (1.1105)^2\right] = 0.223321$$

$$K_4 = hf(x_0 + h, y_0 + K_3) = hf(0.1, 1.223321)$$

$$= 0.1\left[1 + (1.223321)^2\right] = 0.249651$$

$$y_1 \approx y(0.1) = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$$= 1.223049$$

i = 1: $x_1 = 0.1, \ y_1 = 1.223049$

$$K_1 = hf(x_1, y_1) = hf(0.1, 1.223049) = 0.1\left[1 + (1.223049)^2\right] = 0.249585$$

$$K_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{K_1}{2}\right) = hf(0.15, 1.347841)$$

$$= 0.1\left[1 + (1.347841)^2\right] = 0.281668$$

$$K_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{1}{2}K_2\right) = hf(0.15, 1.363883)$$

$$= 0.1\left[1 + (1.363883)^2\right] = 0.286018$$

$$K_4 = h\,f(x_1 + h, y_1 + K_3) = h\,f(0.2, 1.509067)$$
$$= 0.1\left[1 + (1.509067)^2\right] = 0.327728$$

$$y_2 \approx y(0.2) = y_1 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$
$$= 1.508497$$

***

Example 8: Solve the system of IVP's

$$y' = 2y + z,\; y(0) = 1$$
$$z' = 3y + 4z,\; z(0) = 1$$

on the interval [0, 0.4] using second order Runge-Kutta method (30) with $h = 0.2$.

Solution: We are given that $f_1(x, y, z) = 2y + z$, $f_2(x, y, z) = 3y + 4z$ and $h = 0.2$. We obtain from Eqn.(30), for

i = 0: $x_0 = 0, y_0 = 1, z_0 = 1$

$$K_{11} = h\,f_1(x_0, y_0, z_0) = h\,f_1(0, 1, 1) = 0.6$$
$$K_{21} = h\,f_2(x_0, y_0, z_0) = h\,f_2(0, 1, 1) = 1.4$$
$$K_{12} = h\,f_1(x_0 + h, y_0 + K_{11}, z_0 + K_{21}) = h\,f_1(0.2, 1.6, 2.4) = 1.12$$
$$K_{22} = h\,f_2(x_0 + h, y_0 + K_{11}, z_0 + K_{21}) = h\,f_2(0.2, 1.6, 2.4) = 2.88$$
$$y_1 \approx y(0.2) = y_0 + \frac{1}{2}[K_{11} + K_{12}] = 1 + \frac{1}{2}[0.6 + 1.12] = 1.86$$
$$z_1 \approx z(0.2) = z_0 + \frac{1}{2}[K_{21} + K_{22}] = 1 + \frac{1}{2}[1.4 + 2.88] = 3.14$$

i = 1: $x_1 = 0.2, y_1 = 1.86, z_1 = 3.14$

$$K_{11} = h\,f_1(x_1, y_1, z_1) = h\,f_1(0.2, 1.86, 3.14) = 1.372$$
$$K_{21} = h\,f_2(x_1, y_1, z_1) = h\,f_2(0.2, 1.86, 3.14) = 3.628$$
$$K_{12} = h\,f_1(x_1 + h, y_1 + K_{11}, z_1 + K_{21}) = h\,f_1(0.4, 3.232, 6.768) = 2.6464$$
$$K_{22} = h\,f_2(x_1 + h, y_1 + K_{11}, z_1 + K_{21}) = h\,f(0.4, 3.232, 6.768) = 7.3536$$
$$y_2 \approx y(0.4) = y_1 + \frac{1}{2}[K_{11} + K_{12}] = 1.86 + \frac{1}{2}[1.372 + 2.6464] = 3.8692$$
$$z_2 \approx z(0.4) = z_1 + \frac{1}{2}[K_{21} + K_{22}] = 3.14 + \frac{1}{2}[3.628 + 7.3536] = 8.6308$$

* * *

And now some exercises for you.

E6) Solve the IVP

$$y' = x + y^2,\; y(0) = 1$$

on the interval [0, 0.4] using Runge-Kutta second order method (27) with $h = 0.2$.

E7) Find an approximate value of $y(1.2)$ for the initial value problem

$$y' = \frac{y - x}{y + x}, \qquad y(1) = 2$$

using Runge-Kutta second order method (28) with $h = 0.1$.

E8) Find an approximate value of $y(2.4)$ for the initial value problem

$$y' = x(y - x),\; y(2) = 3$$

using classical fourth order Runge-Kutta method with $h = 0.2$.

E9) Solve the initial value problem

$$y' = x + \cos y,\; y(1) = 1$$

in the interval [1, 1.2] using classical fourth order Runge-Kutta method with $h = 0.1$.

E10) Solve the system of IVP's

$$y' = xz + 1, \; y(0) = 0$$
$$z' = xy \;, \quad z(0) = 1$$

on the interval [0, 0.4] using second order Runge-Kutta method (30) with $h = 0.2$.

E11) Write the second order IVP $y'' = \frac{3}{2}y^2, y(0) = 1, y'(0) = 1$, as a system of two first order IVP's. Obtain approximate value of $y(0.4)$ and $y'(0.4)$ by solving this system using Rung-Kutta fourth order method (31) with $h = 0.2$.

We shall now find the interval of stability for various methods discussed above.

8.5 STATBILITY OF SINGLE STEP METHODS

The numerical solution y_i is an approximation of $y(x_i)$ and contains errors (both truncation error and round off error). The error

$$\in_i = y(x_i) - y_i$$

is called the **numerical error** at the nodal point x_i. If the error remains bounded as $i \to \infty$ (or $h \to 0$), then the numerical method is said to be **stable**.

We discuss the stability of the single step methods by applying the method to the **test equation.**

$$y' = \lambda y, \quad y(x_0) = y_0 \tag{32}$$

where λ is a positive or a negative constant. The exact solution of Eqn.(30) is obtained as

$$y(x) = y(x_0)e^{\lambda(x-x_0)}$$

Substituting $x = x_i$ and $x = x_{i+1}$, we get

$$y(x_i) = y(x_0)e^{\lambda(x_i-x_0)} = y(x_0)e^{\lambda ih}$$
$$y(x_{i+1}) = y(x_0)e^{\lambda(x_{i+1}-x_0)} = y(x_0)e^{\lambda(i+1)h}$$

Dividing the above two equations, we get

$$\frac{y(x_{i+1})}{y(x_i)} = \frac{y(x_0)e^{\lambda(i+1)h}}{y(x_0)e^{\lambda ih}} = e^{\lambda h}$$

Hence, we obtain

$$y(x_{i+1}) = e^{\lambda h}\, y(x_i) \tag{33}$$

and h is the step size.

The exact solution of Eqn.(32) grows (when $\lambda > 0$) or decays (when $\lambda < 0$) by the **growth-factor** $e^{\lambda h}$.

When we apply a numerical method to solve Eqn.(32), we obtain a relation of the form

$$y_{i+1} = E(\lambda h)\, y_i \tag{34}$$

where $E(\lambda h)$ is some approximation to $e^{\lambda h}$. The numerical solution grows or decays by the growth factor $E(\lambda h)$. The growth factor $E(\lambda h)$ should behave similar to $e^{\lambda h}$ for meaningful results.

Substituting $y_i = y(x_i) + \in_i$ and $y_{i+1} = y(x_{i+1}) + \in_{i+1}$, we get

$$y(x_{i+1}) + \in_{i+1} = E(\lambda h)\left[y(x_i) + \in_i\right]$$

Since $y(x_{i+1}) = y(x_i)\, e^{\lambda h}$, we get

$$y(x_i)\, e^{\lambda h} + \in_{i+1} = E(\lambda h)\, \left[y(x_i) + \in_i\right]$$

or,

$$\epsilon_{i+1} = \left[E(\lambda h) - e^{\lambda h}\right] y(x_i) + E(\lambda h)\epsilon_i \qquad (35)$$

Thus the error ϵ_{i+1} at the nodal point x_{i+1} consists of two parts.

The first part $E(\lambda h) - e^{\lambda h}$ is the **local truncation error** and can be made as small as we like by increasing the order of the numerical method or by reducing the step size h. We find that this error $\to 0$ as $h \to 0$.

The second part $E(\lambda h)\epsilon_i$ is called the **propagation error**. We consider the following two cases

Case 1: $\lambda < 0$

In this case the exact solution decreases. For meaningful numerical results, the growth factor $E(\lambda h)$ of the numerical solution should decay at least as fast as $e^{\lambda h}$, the growth factor of the exact solution. Since $e^{\lambda h} < 1$ for $\lambda < 0$, we obtain the condition

$$|E(\lambda h)| < 1, \ \lambda < 0 \qquad (36)$$

and the method in this case is called **absolutely stable**. In this case $y_i \to 0$ as $i \to \infty$. The set of values of λh which satisfy Eqn.(36) give the **interval of absolute stability** $(\lambda h, 0)$ for the given method. If the interval of absolute stability is $]-\infty, 0[$, then the method is called **A-stable**.

Case 2: $\lambda > 0$

In this case the exact solution increases. For meaningful numerical results, the growth factor $E(\lambda h)$ of the numerical solution should not increase faster than the growth factor $e^{\lambda h}$ of the exact solution. Therefore, we obtain the condition

$$E(\lambda h) < e^{\lambda h}, \ \lambda > 0 \qquad (37)$$

and the method is called **relatively stable.**

The set of the values of λh which satisfy Eqn.(37) give the **interval of relative stability** $(0, \lambda h)$.

Remarks

1) If $\lambda < 0$, we consider absolute stability and if $\lambda > 0$, we consider relative stability
2) The test Eqn.(32) is the linearised form of $y' = f(x, y)$.
3) For the general initial value problem $y' = f(x, y), y(x_0) = y_0$ we generally take $\lambda = (\partial f / \partial y)_{x_0}$.

Now, we obtain the intervals of absolute stability for the Taylor series and Runge-Kutta methods discussed earlier.

Taylor series method

For the IVP

$$y' = \lambda y, \ y(x_0) = y_0 \qquad (38)$$

we have

$$y'' = \lambda y' = \lambda^2 y$$
$$y''' = \lambda y'' = \lambda^3 y$$
$$\vdots$$
$$y^{(p)} = \lambda y^{(p-1)} = \lambda^p y$$

Applying the p th order Taylor series method

$$y_{i+1} = y_i + h y_i' + \frac{h^2}{2!} y_i'' + \cdots + \frac{h^p}{p!} y_i^{(p)}$$

to the test equation (36), we get

$$y_{i+1} = y_i + \lambda h y_i + \frac{\lambda^2 h^2}{2!} y_i + \cdots + \frac{\lambda^p h^p}{p!} y_i$$

$$= \left[1 + \lambda h + \frac{\lambda^2 h^2}{2!} + \cdots + \frac{\lambda^p h^p}{p!}\right] y_i$$

$$= E(\lambda h)\ y_i$$

where

$$E(\lambda h) = 1 + \lambda h + \frac{\lambda^2 h^2}{2!} + \cdots + \frac{\lambda^p h^p}{p!} \qquad (39)$$

For $\lambda > 0$, $E(\lambda h) < e^{\lambda h}$ for all λh and therefore, the Taylor series method is always relatively stable.

For $\lambda < 0$, we obtain the intervals of absolute stability for different values of p.

p = 1: $\quad |E(\lambda h)| < |1 + \lambda h| < 1$

We get,

$$-1 < 1 + \lambda h < 1 \text{ or, } -2 < \lambda h < 0$$

Hence, the interval of absolute stability is $]-2,\ 0[$

p = 2: $\quad |E(\lambda h)| = \left|1 + \lambda h + \frac{\lambda^2 h^2}{2}\right| < 1$

We get,

$$-1 < 1 + \lambda h + \frac{\lambda^2 h^2}{2} < 1 \text{ or, } -1 < \frac{1}{2}\left[(1 + \lambda h)^2 + 1\right] < 1$$

or, $\quad -2 < (1 + \lambda h)^2 + 1 < 2$

The left inequality is always satisfied. From the right inequality, we get

$$(1 + \lambda h)^2 + 1 < 2 \text{ or, } (1 + \lambda h)^2 < 1$$

or, $\quad |1 + \lambda h| < 1$ or, $-1 < 1 + \lambda h < 1$

or, $\quad -2 < 1 + \lambda h < 0$

Hence, the interval of absolute stability is $]-2, 0[$

p = 3: $\quad |E(\lambda h)| = \left|1 + \lambda h + \frac{\lambda^2 h^2}{2} + \frac{\lambda^3 h^3}{6}\right| < 1$

It can be verified that $\lambda h \in (-2.5, 0)$

p = 4: $\quad |E(\lambda h)| = \left|1 + \lambda h + \frac{\lambda^2 h^2}{2} + \frac{\lambda^3 h^3}{6} + \frac{\lambda^4 h^4}{24}\right| < 1$

It can be verified that $\lambda h \in]-2.78, 0[$

RUNGE-KUTTA METHODS

Second order method

Applying the method (22) to the test equation

$$y' = \lambda y,\ y(x_0) = y_0$$

We get,

$$K_1 = h f(x_i, y_i) = \lambda h\ y_i$$

$$K_2 = h f(x_i + \alpha_2 h, y_i + \beta_{21} K_1) = \lambda h\ (y_i + \beta_{21} K_1)$$

$$= \lambda h\ (y_i + \beta_{21} \lambda^2 h^2\ y_i)$$

$$= (\lambda h + \beta_{21} \lambda^2 h^2)\ y_i$$

and

$$y_{i+1} = y_i + w_1 K_1 + w_2 K_2$$

$$= y_i + w_1 \lambda h\, y_i + w_2 \left[\lambda h + \beta_{21} \lambda^2 h^2\right] y_i$$

$$= \left[1 + (w_1 + w_2)\lambda h + w_2 \beta_{21} \lambda^2 h^2\right] y_i$$

Using the conditions $w_1 + w_2 = 1$ and $w_2 \beta_{21} = \frac{1}{2}$ (see Eqn.(25)), we get

$$y_{i+1} = \left(1 + \lambda h + \frac{\lambda^2 h^2}{2}\right) y_i = E(\lambda h)\, y_i$$

where $E(\lambda h) = 1 + \lambda h + \frac{\lambda^2 h^2}{2}$

For $\lambda > 0$, $E(\lambda h) < e^{\lambda h}$ for all λh. Therefore, this method is relatively stable for all λh.

For $\lambda < 0$, $|E(\lambda h)| < 1$ gives $\lambda h \in]-2, 0[$. Hence, the interval of absolute stability for second order Runge-Kutta method is $]-2, 0[$.

Remark: The interval of absolute stability of all second order Runge-Kutta methods is $]-2, 0[$.

Fourth order method

Applying the method (29) to the test equation

$$y' = \lambda y,\ y(x_0) = y_o$$

we get,

$$K_1 = h f(x_i, y_i) = \lambda h\, y_i$$

$$K_2 = h f\left(x_i + \frac{h}{2}, y_i + \frac{1}{2}K_i\right) = \lambda h\left[y_i + \frac{1}{2}K_i\right]$$

$$= \lambda h\left[y_i + \frac{1}{2}\lambda h\, y_i\right] = \left[\lambda h + \frac{\lambda^2 h^2}{2}\right] y_i$$

$$K_3 = h f\left(x_i + \frac{h}{2}, y_i + \frac{1}{2}K_2\right) = \lambda h\left[y_i + \frac{1}{2}K_2\right]$$

$$= \lambda h\left[y_i + \frac{1}{2}\left(\lambda h + \frac{\lambda^2 h^2}{2}\right) y_i\right] = \left(\lambda h + \frac{\lambda^2 h^2}{2} + \frac{\lambda^3 h^3}{4}\right) y_i$$

$$K_4 = h f(x_i + h, y_i + K_3) = \lambda h (y_i + K_3)$$

$$= \lambda h\left[y_i + \left(\lambda h + \frac{\lambda^2 h^2}{2} + \frac{\lambda^3 h^3}{4}\right) y_i\right]$$

$$= \left(\lambda h + \lambda^2 h^2 + \frac{\lambda^3 h^3}{2} + \frac{\lambda^3 h^3}{4}\right) y_i$$

and

$$y_{i+1} = y_i + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$$= \left[1 + \frac{1}{6}(\lambda h) + \frac{2}{6}\left(\lambda h + \frac{\lambda^2 h^2}{2}\right) + \frac{2}{6}\left(\lambda h + \frac{\lambda^2 h^2}{2} + \frac{\lambda^3 h^3}{4}\right)\right.$$

$$\left. + \frac{1}{6}\left(\lambda h + \lambda^2 h^2 + \frac{\lambda^3 h^3}{2} + \frac{\lambda^3 h^3}{4}\right)\right] y_i$$

$$= \left[1 + \lambda h + \frac{\lambda^2 h^2}{2} + \frac{\lambda^3 h^3}{6} + \frac{\lambda^4 h^4}{24}\right] y_i$$

$$= E(\lambda h)\, y_i$$

where $E(\lambda h) = 1 + \lambda h + \frac{\lambda^2 h^2}{2} + \frac{\lambda^3 h^3}{6} + \frac{\lambda^4 h^4}{24}$

Since $E(\lambda h) < e^{\lambda h}$, $\lambda > 0$, this method is relatively stable for all λh.

For $\lambda < 0$, $|E(\lambda h)| < 1$ gives the interval of stability as $]-2.78, 0[$.

Example 9: Find the interval of absolute stability for the second order Runge-Kutta method (27).

Solution: Applying the method (27) to the test equation

$$y' = \lambda y,\ y(x_0) = y_0,\quad \lambda < 0$$

we get

$$K_1 = hf(x_i, y_i) = \lambda h\ y_i$$

$$K_2 = hf(x_i + h,\ y_i + K_1) = \lambda h\ (y_i + K_1)$$

$$= \lambda h (y_i + \lambda h\ y_i)$$

$$= (\lambda h + \lambda^2 h^2)\ y_i$$

and

$$y_{i+1} = y_i + \frac{1}{2}[K_1 + K_2]$$

$$= y_i + \frac{1}{2}\lambda h\ y_i + \frac{1}{2}(\lambda h + \lambda^2 h^2)\ y_i$$

$$= \left[1 + \lambda h + \frac{\lambda^2 h^2}{2}\right] y_i$$

$$= E(\lambda h)\ y_i$$

where $E(\lambda h) = 1 + \lambda h + \frac{\lambda^2 h^2}{2}$

The condition $|E(\lambda h)| < 1$ gives the interval of absolute stability as $]-2, 0[$.

***

Example 10: The second order Runge-Kutta method is applied to the IVP

$$y' = -100y,\ y(0) = 1$$

Find the values of h so that the method produces stable results.

Solution: Comparing the given problem with the test equation $y' = \lambda y$, we get $\lambda = -100$.

Since for stability : $|\lambda h| < 2$, we get $100h < 2$ or, $h < \frac{1}{50}$.

Hence, for $0 < h < \frac{1}{50}$, the method will produce stable results.

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And now some exercises for you.

E12) Find the interval of absolute stability of the two stage Runge-Kutta method given by

$$y_{i+1} = y_i + \frac{1}{4}(K_1 + 3K_2)$$

where $K_1 = hf(x_i, y_i)$

$$K_2 = hf\left(x_i + \frac{2h}{3},\ y_i + \frac{2}{3}K_1\right).$$

E13) Fourth order classical Runge-Kutta method is used to solve the initial value problem

$$y' = -200y,\quad y(0) = 1$$

Determine the value of h so that the method produces stable results.

We now end this unit by giving a summary of what we have covered in it.

8.6 SUMMARY

In this unit we have discussed the following points

1. In **singlestep** methods for solving IVP(4) viz.,

$$y'(x) = f(x, y), \quad y(x_0) = y_0, \quad x \in [x_0, b]$$

the solution value y_{i+1} at the nodal point x_{i+1} is obtained by using the solution value y_i at only the previous point x_i. Such a method is called an **explicit singlestep** method and is written as

$$y_{i+1} = y_i + h\phi(x_i, y_i, h), \quad i = 0, 1, \ldots$$

where h is the **step size** and ϕ is called the **increment function.** If the method also uses the solution value y_{i+1}, then the method is called an **implicit singlestep** method.

2. Taylor series method of order p for the solution of the IVP(4) is given by

$$y_{i+1} = y_i + h y_i' + \frac{h^2}{2!} y_i'' + \cdots + \frac{h^p}{p!} y_i^{(p)}$$

where, $x_i = x_0 + ih$, $i = 0, 1, 2 \ldots, N$, $x_N = b$ and y_i is an approximation to $y(x_i)$. The error of approximation is given by

$$TE = \frac{h^{p+1}}{(p+1)!} y^{(p+1)}(\xi), \quad x_i < \xi < x_{i+1}$$

3. Euler's method is the Taylor series method of order one.
4. Taylor series method of order p for the system of first order IVP's(3) is given by

$$(\mathbf{y})_{x_{i+1}} = (\mathbf{y})_{x_i} + h(\mathbf{y}')_{x_i} + \frac{h^2}{2!}(\mathbf{y}'')_{x_i} + \cdots + \frac{h^p}{p!}(\mathbf{y}^{(p)})_{x_i}$$

where $(\mathbf{y})_{x_i} = [y_{1i}, y_{2i}, \ldots, y_{ni}]^T$

$$(\mathbf{y}^{(m)})_{x_i} = [y_{1i}^{(m)}, y_{2i}^{(m)}, \ldots, y_{ni}^{(m)}]^T, \quad m = 1, 2, \ldots, p$$

and $y_{ji}^{(m)} = \dfrac{d^{m-1}}{d_{x_{m-1}}} f_j(x_i, y_{1i}, y_{2i}, \ldots, y_{ni}), \quad j = 1, 2, \ldots, n.$

5. A v-stage Runge-Kutta method for the solution of IVP(4)

$$y' = f(x, y), \; y'(x_0) = y_0, \; x \in [x_0, b]$$

where the mesh points are $x_i = x_0 + ih$, $i = 0, 1, \ldots, n$, $x_n = b = x_0 + nh$, is obtained by writing

$$y_{i+1} = y_i + h\,(\text{weighted sum of the slopes})$$

$$= y_i + \sum_{i=1}^{v} w_i K_i$$

where v slopes are used. These slopes are defined by

$$K_i = f\left[x_i + \alpha_i h, \; y_i + \sum_{j=1}^{i-1} \beta_{ij} K_j\right], \quad i = 1, 2, \ldots, v, \quad \alpha_i \neq 0$$

The unknowns α_i, β_{ij} and w_i are then obtained by expanding K_i's and y_{i+1} in Taylor series about the point (x_i, y_i) and comparing the coefficients of different powers of h.

6. Unlike Taylor series methods, Runge-Kutta methods do not need calculation of higher order derivatives of $f(x, y)$ but need only the evaluation of $f(x, y)$ at the off-step points.
7. The numerical solution y_i which is an approximation of $y(x_i)$ contains errors. The error at the nodal point x_i is given by

$$\epsilon_i = y(x_i) - y_i$$

and is called **numerical error**. If this error remains bounded as $i \to \infty$ (or $h \to 0$), then the numerical method is said to be **stable**.

8. When a numerical method is applied to the following linearised form of IVP(4), (test equation)

$$y' = \lambda y,\ y(x_0) = y_0,\ x \in [x_0, b]$$

with nodal points $x_i = x_0 + ih,\ i = 0,1,\ldots,N$ and $x_n = b = x_0 + Nh$, a relation of the form

$$y_{i+1} = E(\lambda h)\, y_i$$

is obtained where $E(\lambda h)$ is some approximation to $e^{\lambda h}$. The error ϵ_{i+1} at the nodal point x_{i+1} consists of two parts and is given by

$$\epsilon_{i+1} = [E(\lambda h) - e^{\lambda h}]\, y(x_i) + E(\lambda h)\epsilon_i$$

The first part $E(\lambda h) - e^{\lambda h}$ is the **local truncation error** and the second part $E(\lambda h)\epsilon_i$ is called the **propagation error** and the following results hold

i) numerical method is called **absolutely stable** if

$$|E(\lambda h)| < 1,\quad \lambda < 0$$

In this case $y_i \to 0$ as $i \to \infty$. The set of values of λh which satisfy the above condition give the **interval of absolute stability** $]\lambda h, 0[$ for the given method. If this interval is $]-\infty, 0[$, then the method is called **A-stable.**

ii) numerical method is called **relatively stable** if

$$E(\lambda h) < e^{\lambda h},\ \lambda > 0$$

and the set of values of λh which satisfy the above condition give the **interval of relative stability.**

8.7 SOLUTIONS/ANSWERS

E1) $x_0 = 1,\ y_0 = 1,\ y_0' = -1$

$y_1 \approx y(1.1) = y_0 + h\,y_0' = 1 + (0.1)\,(-1) = 0.9$

$x_1 = 1.1,\ y_1 = 0.9,\ y_1' = -0.81$

$y_2 \approx y(1.2) = y_1 + h\,y_1' = 0.9 + (0.1)\,(-0.81) = 0.819.$

E2) a) $x_0 = 0, y_0 = 1, y_0' = 3, y_0'' = 11;\ y_1 \approx y(0.2) = 1.82.$

$x_1 = 0.2, y_1 = 1.82, y_1' = 5.86, y_1'' = 19.58,\ y_2 \approx y(0.4) = 3.3836.$

b) $x_0 = 0,\ y_0 = 1,\ y_0' = 3,\ y_0'' = 11,\ y_0''' = 33, y_0^{iv} = 99,$

$y_1 \approx y(0.2) = 1.8706.$

$x_1 = 0.2,\ y_1 = 1.8706,\ y_1' = 6.0118,\ y_1'' = 20.0354,\ y_1''' = 60.1062,$

$y_1^{iv} = 180.3186,\ y_2 \approx y(0.4) = 3.5658.$

E3) a) $x_0 = 1, y_0 = 1, y_0' = 1.841471, y_0'' = 1.540302,\ y_1 \approx y(1.1) = 1.191849.$

$x_1 = 1.\ y_1 = 1.191849,\ y_1' = 2.029055,\ y_1'' = 1.369943,$

$y_2 \approx y(1.2) = 1.401604$

b) $x_0 = 1,\ y_0 = 1,\ y_0' = 1.841471,\ y_0'' = 1.540302,\ y_0''' = -.841471,$

$y_0^{iv} = -0.540302,\ y_1 \approx y(1.1) = 1.191706.$

$x_1 = 1.1,\ y_1 = 1.191706,\ y_1' = 2.029002,\ y_1'' = 1.370076,$

$y_1''' = -0.929002,\ y_1^{iv} = -0.370076,\ y_2 \approx y(1.2) = 1.401300.$

E4) $x_0 = 0,\ \mathbf{y}_0 = [1,\ 1]^T,\ \mathbf{y}'_0 = [3,\ 7]^T,\ \mathbf{y}''_0 = [13,\ 37]^T$

$\mathbf{y}_1 \approx [y(0.2),\ u(0.2)]^T = [y_1,\ u_1]^T = [1.86,\ 3.14]^T$

$x_1 = 0.2,\ \mathbf{y}_1 = [1.86,\ 3.14]^T,\ \mathbf{y}'_1 = [6.86,\ 18.14]^T,$

$\mathbf{y}''_1 = [31.86,\ 93.14]^T$

$\mathbf{y}_2 \approx [y(0.4),\ u(0.4)]^T = [y_2,\ u_2]^T = [3.8692,\ 8.6308]^T$

E5) $y'_1 = y_2,\ y_1(1) = 1;\ y'_2 = -2y_1y_2,\ y_2(1) = 1$

$x_0 = 1,\ \mathbf{y}_0 = [1,\ 1]^T,\ \mathbf{y}'_0 = [1,\ -2]^T,\ \mathbf{y}''_0 = [-2,\ 2]^T$

$\mathbf{y}_1 \approx [y_1(1.2),\ y_2(1.2)]^T = [y_{11},\ y_{21}]^T = [1.16,\ 0.64]^T$

$x_1 = 1.2,\ \mathbf{y}_1 = [1.16,\ 0.64]^T,\ \mathbf{y}'_1 = [0.64,\ -1.4848]^T,$

$\mathbf{y}''_1 = [-1.4848,\ 2.6255]^T,$

$\mathbf{y}_2 \approx [y_1(1.4),\ y_2(1.4)]^T = [y_{12},\ y_{22}]^T = [1.2583,\ 0.3956]^T.$

E6) $x_0 = 0,\ y_0 = 1,\ K_1 = 0.2,\ K_2 = 0.328,\ y_1 \approx y(0.2) = 1.264.$

$x_1 = 0.2,\ y_1 = 1.264,\ K_1 = 0.359539,\ K_2 = 0.607176$

$y_2 \approx y(0.4) = 1.747338.$

E7) $x_0 = 1,\ y_0 = 2,\ K_1 = 0.033333,\ K_2 = 0.031522,$

$y_1 \approx y(1.1) = 2.031522.$

$x_1 = 1.1,\ y_1 = 2.031522,\ K_1 = 0.029747,\ K_2 = 0.028044,$

$y_2 \approx y(1.2) = 2.059566.$

E8) $x_0 = 2,\ y_0 = 3,\ K_1 = 0.4,\ K_2 = 0.462,\ K_3 = 0.47502,$

$K_4 = 0.5{,}16009,\ y_1 \approx y(2.2) = 3.472508.$

$x_1 = 2.2,\ y_1 = 3.472508,\ K_1 = 0.559904,\ K_2 = 0.668132,$

$K_3 = 0.693024,\ K_4 = 0.847455,\ y_2 \approx y(2.4) = 4.160787.$

E9) $x_0 = 1,\ y_0 = 1,\ K_1 = 0.154030,\ K_2 = 0.152396,\ K_3 = 0.148639,$

$K_4 = 0.150623,\ y_1 \approx y(1.1) = 1.152397.$

$x_1 = 1.1,\ y_1 = 1.152397,\ K_1 = 0.150630,\ K_2 = 0.148639,$

$K_3 = 0.148733,\ K_4 = 0.146641,\ y_2 \approx y(1.2) = 1.301066.$

E10) $x_0 = 0,\ y_0 = 0,\ z_0 = 1,$

$$\mathbf{K}_1 = \begin{bmatrix} K_{11} \\ K_{21} \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0 \end{bmatrix},\ \mathbf{K}_2 = \begin{bmatrix} K_{12} \\ K_{22} \end{bmatrix} = \begin{bmatrix} 0.24 \\ 0.008 \end{bmatrix},\ \begin{bmatrix} y_1 \\ z_1 \end{bmatrix} \approx \begin{bmatrix} y(0.2) \\ z(0.2) \end{bmatrix} = \begin{bmatrix} 0.22 \\ 1.004 \end{bmatrix}$$

$x_1 = 0.2,\ y_1 = 0.22,\ z_1 = 1.004$

$$\mathbf{K}_1 = \begin{bmatrix} K_{11} \\ K_{21} \end{bmatrix} = \begin{bmatrix} 0.24016 \\ 0.0088 \end{bmatrix},\ \mathbf{K}_2 = \begin{bmatrix} K_{12} \\ K_{22} \end{bmatrix} = \begin{bmatrix} 0.281024 \\ 0.036813 \end{bmatrix}$$

$$\begin{bmatrix} y_2 \\ z_2 \end{bmatrix} \approx \begin{bmatrix} y(0.4) \\ z(0.4) \end{bmatrix} = \begin{bmatrix} 0.480592 \\ 1.026807 \end{bmatrix}.$$

E11) $y'_1 = y_2,\ y_1(0) = 1;\ y'_2 = \frac{3}{2}y_1^2,\ y_2(0) = 1.$

$x_0 = 0,\ y_{10} = 1,\ y_{20} = 1$

$\mathbf{K}_1 = [0.2,\ 0.3]^T,\ \mathbf{K}_2 = [0.23,\ 0.363]^T,$

$\mathbf{K}_3 = [0.2363,\ 0.374838]^T,\ \mathbf{K}_4 = [0.274968,\ 0.458531]^T,$

$[y_{11}, y_{21}]^T \approx [y_1(0.2), y_2(0.2)]^T = [1.234595, 1.372368]^T$.

$x_1 = 0.2,\ y_{11} = 1.234595,\ y_{21} = 1.372368,$

$K_1 = [0.274474, 0.457267]^T,\ K_2 = [0.320200, 0.564577]^T,$

$K_3 = [0.330931, 0.583552]^T,\ K_4 = [0.391184, 0.735261]^T,$

$[y_{12}, y_{22}]^T \approx [y_1(0.4), y_2(0.4)]^T = [1.562582, 1.953832]^T$.

E12) Apply the method to the test equation
$y' = \lambda y,\ \lambda < 0,$ we get

$$K_1 = \lambda h y_i,\ K_2 = \lambda h\left(1 + \frac{2\lambda h}{3}\right) y_i$$

$$y_{i+1} = \left(1 + \lambda h + \frac{\lambda^2 h^2}{2}\right) y_i = E(\lambda h) y_i$$

From $|E(\lambda h)| < 1$, we get $-2 < \lambda h < 0$.

E13) Comparing the given problem with the test equation $y' = \lambda y$, we get $\lambda = -200$.
For stability, we require $|\lambda h| < 2.78$. Therefore, $|-200h| < 2.78$ or,
$h < 0.0139$.

—x—

PRACTICAL ASSIGNMENTS

Session 1:

1. Write a program using Euler's method to solve the initial value problem $y' = f(x, y)$, $y(x_0) = y_0$. Input the step length h and the number of steps of integration. Test your program on exercise E1).

2. Write a program using Taylor series method of second order to solve the initial value problem $y' = f(x, y)$, $y(x_0) = y_0$. User shall give the expression for y''. Input the step length h and the number of steps of integration. Test your program on exercises E2) and E3).

3. Write a program using Runge-Kutta second order method to solve the initial value problem $y' = f(x, y)$, $y(x_0) = y_0$. Input the step length h and the number of steps of integration. Test your program on exercises E6) and E7).

4. Write a program using fourth order classical Runge-Kutta method to solve the initial value problem $y' = f(x, y)$, $y(x_0) = y_0$. Input the step length h and the number of steps of integration. Test your program on exercises E8) and E9).

UNIT 9 MULTISTEP AND PREDICTOR – CORRECTOR METHODS FOR SOLVING INITIAL VALUE PROBLEMS

Structure **Page No.**

9.1 INTRODUCTION

In Unit 8 you have studied singlestep (Taylor series and Runge-Kutta) methods for solving the initial value problem (IVP)

$$y' = f(x,y), \quad y(x_0) = y_0, \tag{1}$$

These methods require only the immediate previous value y_i at the nodal point x_i to calculate y_{i+1} at the nodal point x_{i+1}. The order of the highest derivative used in Taylor series methods and the number of function evaluations per step in the Runge-Kutta method increases with the order of the method. Thus very high order singlestep methods are not used for solving the initial value problems.

There is another class of methods for solving IVP's which require the k solution values y_{i-m} at previous k nodal points x_{i-m}, $m = 0,1,\ldots,(k-1)$ to obtain the value of y_{i+1} at the nodal point x_{i+1}. Such methods are called **multistep** or **k-step** methods. These methods are classified as **explicit** or **implicit** according to whether they **do not use** or **use** y_{i+1} in addition to the previously calculated k values. These methods required only one extra function evaluation per step irrespective of the order of the method. In this unit we shall discuss both explicit and implicit multistep methods.

In Sec.9.2 we have used two approaches to discuss the derivation and implementation of various explicit and implicit multistep methods. The explicit and implicit methods are combined to define a new class of methods called the **predictor-corrector** methods which we shall discuss in Sec.9.3. Finally, in Sec.9.4 we have discussed the stability of the multistep methods.

Objectives

After reading this unit you should be able to

- explain the multistep methods;
- explain the predictor-corrector methods;
- derive the explicit and implicit multistep methods;
- obtain the solution of initial value problems using multistep and predictor-corrector methods;
- obtain the interval of absolute stability of multistep methods.

9.2 MULTISTEP METHODS

A numerical method for solving the IVP(1) viz.,

$$y' = f(x,y), \; y(x_0) = y_0$$

is called a multistep or a k-step method if it uses the values of $y(x)$ and $f(x, y)$ at the k nodal points x_{i-m}, $m = 0, 1, \ldots, (k-1)$ to determine the solution value y_{i+1} at the nodal point x_{i+1}. Such a method is called an **explicit** or a **predictor method.** If the method also uses the values of $y(x)$ and $f(x, y)$ at the nodal point x_{i+1}, then the method is called an **implicit** or a **corrector method.** The value y_i is the initial value and the values $y_{i-1}, y_{i-2}, \ldots, y_{i-k+1}$ are called the starting values. The starting values are obtained using some singlestep method before using the multistep method. Generally, Taylor series methods and Runge-Kutta methods are used for starting a multistep method. Thus multistep methods are not self-starting methods. A predictor method is used to predict a value of y_{i+1} and a corrector method is used to improve upon this value. In this section we shall use two approaches for the derivation of multistep methods.

9.2.1 Interpolation Approach

Integrating $y' = f(x, y)$ in the interval $[x_i, x_{i+1}]$, we obtain

$$\int_{x_i}^{x_{i+1}} y' \, dx = \int_{x_i}^{x_{i+1}} f(x, y) \, dx$$

or

$$y(x_{i+1}) = y(x_i) + \int_{x_i}^{x_{i+1}} f(x, y) \, dx \tag{2}$$

To derive multistep methods, we replace $f(x, y)$ in the integral in Eqn.(2) by some suitable interpolating polynomial. We shall now discuss some of these methods.

Adams-Bashforth methods

These are explicit methods. Through the k data values $(x_i, f_i), (x_{i-1}, f_{i-1}), \ldots, (x_{i-k+1}, f_{i-k+1})$, we fit the Newton's backward difference interpolating polynomial of degree $k-1$ as

$$\begin{aligned} P_{k-1}(x) = f_i &+ \frac{(x-x_i)}{h} \nabla f_i + \frac{(x-x_i)(x-x_{i-1})}{2!\, h^2} \nabla^2 f_i + \ldots \\ &+ \frac{(x-x_i)(x-x_{i-1}) \ldots (x-x_{i-k+2})}{(k-1)!\, h^{k-1}} \nabla^{k-1} f_i. \end{aligned} \tag{3}$$

The error of interpolation is given by

$$TE = \frac{(x-x_i)(x-x_{i-1}) \ldots (x-x_{i-k+1})}{k!} f^{(k)}(\xi), \tag{4}$$

where ξ lies in some interval containing the points $x_i, x_{i-1}, \ldots, x_{i-k+1}$ and x. Replacing $f(x,y)$ by $P_{k-1}(x)$ in Eqn.(2), we get

$$y_{i+1} = y_i + \int_{x_i}^{x_{i+1}} \left[f_i + \frac{(x-x_i)}{h} \nabla f_i + \frac{(x-x_i)(x-x_{i-1})}{2!h^2} \nabla^2 f_i + \cdots \right] dx.$$

Set $x - x_i = hs$. Then $x - x_m = x - x_i + x_i - x_m = (s + i - m)h$. Hence, we get

$$y_{i+1} = y_i + \int_0^1 \left[f_i + s\nabla f + \frac{1}{2} s(s+1)\nabla^2 f_i + \cdots \right] ds$$

$$= y_i + h\left[f_i + \frac{1}{2}\nabla f_i + \frac{5}{12}\nabla^2 f_i + \cdots \right]. \tag{5}$$

Error term can be obtained by integrating Eqn.(4) in the interval $[x_i, x_{i+1}]$ or, directly by using Taylor series. For different values of k, we obtain different methods.

k = 1: we get a singlestep method $y_{i+1} = y_i + h f_i$ (6)

which is the **Euler method.** The error term is given by

$$TE = y(x_{i+1}) - y_{i+1} = y(x_{i+1}) - [y_i + h f_i]$$

$$= y(x_{i+1}) - y_i - h y_i'$$

where $f_i = y'_i$.

Expanding each term in Taylor series about the point x_i, we obtain

$$TE = \frac{h^2}{2} y''(\xi), \quad x_i < \xi < x_{i+1}.$$

This method is of order 1.

k = 2:
$$y_{i+1} = y_i + h\left[f_i + \frac{1}{2}\nabla f_i\right]$$

$$= y_i + h\left[f_i + \frac{1}{2}(f_i - f_{i-1})\right] = y_i + \frac{h}{2}(3f_i - f_{i-1}) \qquad (7)$$

The error term is given by

$$TE = y(x_{i+1}) - y_{i+1}$$
$$= y(x_i + h) - [y_i + h(3y'_i - y'_{i-1})]$$
$$= \frac{5}{12} h^3 y'''(\xi), \quad x_{i-1} < \xi < x_{i+1}$$

This method is of order 2.

k = 3:
$$y_{i+1} = y_i + h\left[f_i + \frac{1}{2}\nabla f_i + \frac{5}{12}\nabla^2 f_i\right]$$

$$= y_i + h\left[f_i + \frac{1}{2}(f_i - f_{i-1}) + \frac{5}{12}(f_i - 2f_{i-1} + f_{i-2})\right]$$

$$= y_i + \frac{h}{12}[23f_i - 16f_{i-1} + 5f_{i-2}] \qquad (8)$$

The error term is given by

$$TE = y(x_i + h) - \left[y_i + \frac{h}{12}(23y'_i - 16y'_{i-1} + 5y'_{i-2})\right]$$

$$= \frac{3}{8} h^4 y^{(4)}(\xi), \quad x_{i-2} < \xi < x_{i+1}$$

This method is of order 3.

In general a k-step method of the form (5) gives a method of order k. We require one function evaluation per step for the application of the multistep method.

Example 1: Find the approximate value of $y(1.0)$ using the multistep method (Adams-Bashforth second order method)

$$y_{i+1} = y_i + \frac{h}{2}(3f_i - f_{i-1})$$

with $h = 0.2$ for the initial value problem

$$y' = -2xy^2, \; y(0) = 1$$

Calculate the starting value using the Runge-Kutta second order method

$$y_{i+1} = y_i + \frac{1}{2}(K_1 + K_2)$$
$$K_1 = hf(x_i, y_i)$$
$$K_2 = h\,f(x_i + h, y_i + K_1)$$

with the same step size h.

Solution : For the given multistep method

$$y_{i+1} = y_i + \frac{h}{2}(3f_i - f_{i-1})$$

or,

$$y_{i+2} = y_{i+1} + \frac{h}{2}(3f_{i+1} - f_i) \qquad (9)$$

we get for $i = 0$

$$y_2 = y_1 + \frac{h}{2}(3f_1 - f_0)$$

Now y_0 is the given initial value and y_1 is the starting value. We calculate the starting value using the given second order Runge-Kutta method. We obtain from

$$f(x\ \ y) = -2x\ y^2,\ x_0 = 0,\ y_0 = 1,\ h = 0.2$$

$$K_1 = hf(x_0, y_0) = h\ f(0, 1) = 0$$

$$K_2 = hf(x_0 + h, y_0 + K_1) = h\ f(0.2, 1) = -0.08$$

$$y_1 \quad y(0.2) = y_0 + \frac{1}{2}(K_1 + K_2) = 0.96$$

After the starting value has been determined, we use the multistep method (9). We obtain for

i = 0: $\quad x_0 = 0, y_0 = 1,\ f_0 = -2x_0 y_0^2 = 0;$

$$x_1 = 0.2, y_1 = 0.96,\ f_1 = -2x_1\ y_1^2 = -0.36864$$

$$y_2 \approx y(0.4) = y_1 + \frac{h}{2}(3f_1 - f_0) = 0.849408.$$

i = 1: $\quad x_2 = 0.4, y_2 = 0.849408,\ f_2 = -2x_2\ y_2^2 = -0.577195$

$$y_3 \approx y(0.6) = y_2 + \frac{h}{2}(3f_2 - f_1) = 0.713114.$$

i = 2: $\quad x_3 = 0.6, y_3 = 0.713114,\ f_3 = -2x_3\ y_3^2 = -0.610237$

$$y_4 \approx y(0.8) = y_3 + \frac{h}{2}(3f_3 - f_2) = 0.587762.$$

i = 3: $\quad x_4 = 0.8, y_4 = 0.5887762,\ f_4 = -2x_4\ y_4^2 = -0.552743$

$$y_5 \approx y(1.0) = y_4 + \frac{h}{2}(3f_4 - f_3) = 0.482963.$$

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Example 2: Solve the initial value problem

$$y' = x + y^2,\ y(0) = 1$$

in the interval $[0,1]$ with step length $h = 0.2$ using the multistep method

$$y_{i+1} = y_i + \frac{h}{3}(23f_i - 16f_{i-1} + 5f_{i-2})$$

(Adams-Bashforth third order method).
Compute the starting values using third order Taylor series method with the same step size h.

Solution: From the given multistep method

$$y_{i+1} = y_i + \frac{h}{3}(23f_i - 16f_{i-1} + 5f_{i-2})$$

or,

$$y_{i+3} = y_{i+2} + \frac{h}{3}(23f_{i+2} - 16f_{i+1} + 5f_i) \tag{10}$$

we obtain for $i = 0$

$$y_3 = y_2 + \frac{h}{3}(23f_2 - 16f_1 + 5f_0)$$

Now y_0 is the given initial value, whereas y_1 and y_2 are the starting values. We compute the starting values using the third order Taylor series method

$$y_{j+1} = y_j + h\,y_j' + \frac{h^2}{2}y_j'' + \frac{h^3}{6}y_j''',\quad j = 0,1 \tag{11}$$

We have

$$y' = x + y^2,\quad y'' = 1 + 2y\,y',\quad y''' = 2y\,y'' + 2(y')^2$$

We obtain from Eqn.(11) for $h = 0.2$ and

j = 0: $\quad x_0 = 0, y_0 = 1, y_0' = 1,\ y_0'' = 3, y_0''' = 8$

$$y_1 \approx y(0.2) = y_0 + h\, y_0' + \frac{h^2}{2} y_0'' + \frac{h^3}{6} y_0'''$$
$$= 1 + 0.2(1) + \frac{(0.2)^2}{2}(3) + \frac{(0.2)^3}{6}(8)$$
$$= 1.270667.$$

$j = 1$: $x_1 = 0.2, y_1 = 1.270667, y_1' = 1.814595,$
$y_1'' = 5.611492,\ y_1''' = 20.846185$ and

$$y_2 \approx y(0.4) = y_1 + h\, y_1' + \frac{h^2}{2} y_1'' + \frac{h^3}{6} y_1'''$$
$$= 1.270667 + 0.2(1.814595)$$
$$= \frac{(0.2)^2}{2}(5.611492) + \frac{(0.2)^3}{6}(20.846185)$$
$$= 1.773611.$$

After determining the starting values, we use the multistep method (10). We have $h = 0.2$ and

$$x_0 = 0,\ y_0 = 1,\ f_0 = x_0 + y_0^2 = 1,$$
$$x_1 = 0.2, y_1 = 1.270667,\ f_1 = x_1 + y_1^2 = 1.814595,$$
$$x_2 = 0.4, y_2 = 1.773611, f_2 = x_2 + y_2^2 = 3.545696$$

We obtain from Eqn.(10) for

$i = 0$: $y_3 = y(0.6) = y_2 + \frac{h}{12}(23 f_2 - 16 f_1 + 5 f_0)$

$$= 1.773611 + \frac{0.2}{12}\left[23(3.545696) - 16(1.814595) + 5(1)\right]$$
$$= 2.732236.$$

$i = 1$: $x_3 = 0.6, y_3 = 2.732236, f_3 = x_3 + y_3^2 = 8.065112$

$$y_4 \approx y(0.8) = y_3 + \frac{h}{12}(23 f_3 - 16 f_2 + 5 f_1)$$
$$= 2.732236 + \frac{0.2}{12}\left[23(8.065112) - 16(3.545696) + 5(1.814595)\right]$$
$$= 5.029560.$$

$i = 2$: $x_4 = 0.8, y_4 = 5.029560, f_4 = x_4 + y_4^2 = 26.096470$

$$y_5 \approx y(1.0) = y_4 + \frac{h}{12}\left[23 f_4 - 16 f_3 + 5 f_2\right]$$
$$= 5.029560 + \frac{0.2}{12}\left[23(26.096470) - 16(8.065112) + 5(3.545696)\right]$$
$$= 13.177985.$$

Adams-Moulton methods

These are implicit methods. Through the $k + 1$ data values $(x_{i+1}, f_{i+1}), (x_i, f_i), \ldots, (x_{i-k+1}, f_{i-k+1})$, we fit the Newton's backward difference interpolating polynomial of degree k as

$$P_k(x) = f_{i+1} + \frac{(x - x_{i+1})}{h} \nabla f_{i+1} + \frac{(x - x_{i+1})\,(x - x_i)}{2!h^2} \nabla^2 f_{i+1} + \cdots + \frac{(x - x_{i+1})(x - x_i)\ldots(x - x_{i-k+2})}{k!h^k} \nabla^k f_{i+1}. \tag{12}$$

Note that we are using the current data point (x_{i+1}, f_{i+1}) also.

The error of interpolation is given by

$$TE = \frac{1}{(k+1)!}(x - x_{i+1})(x - x_i)\ldots(x - x_{i-k+1})f^{(k+1)}(\xi) \qquad (13)$$

where ξ lies in some interval containing the points $x_{i+1}, x_i, \ldots, x_{i-k+1}$ and x. Replacing $f(x, y)$ by $P_k(x)$ in Eqn.(2), we get

$$y_{i+1} = y_i + \int_{x_i}^{x_{i+1}} \left[f_{i+1} + \frac{(x - x_{i+1})}{h}\nabla f_{i+1} + \frac{(x - x_{i+1})(x - x_i)}{2!h^2}\nabla^2 f_{i+1} + \cdots \right] dx$$

Set $x - x_i = hs$. Then $x - x_m = (s + i - m)\,h$. Hence, we obtain

$$y_{i+1} = y_i + h\int_0^1 \left[f_{i+1} + (s-1)\nabla f_{i+1} + \frac{(s-1)s}{2!}\nabla^2 f_{i+1} + \cdots \right] ds$$

$$= y_i + h\left[f_{i+1} - \frac{1}{2}\nabla f_{i+1} - \frac{1}{12}\nabla^2 f_{i+1} - \cdots \right] \qquad (14)$$

The error term can be obtained by integrating (13) in the interval $[x_i, x_{i+1}]$ or, directly by using the Taylor series. For different values of k, we obtain different methods.

$k = 0$: We get a singlestep method. Retaining one term inside the brackets in Eqn.(14), we get the method

$$y_{i+1} = y_i + h f_{i+1} \qquad (15)$$

which is the **backward Euler method**. The error term is given by

$$TE = y'(x_{i+1}) = y_{i+1} = y(x_i + h) - [y_i + h y'_{i+1}]$$

where $f_{i+1} = y'_{i+1}$. Expanding each term in Taylor series about x_i, we get

$$TE = -\frac{h^2}{2}y''(\xi), \qquad x_i < \xi < x_{i+1}$$

This method is of order 1.

$k = 1$: We again get a singlestep method. Retaining two terms inside the brackets in Eqn.(14), we get the method

$$y_{i+1} = y_i + h\left[f_{i+1} - \frac{1}{2}\nabla f_{i+1} \right]$$

$$= y_i + h\left[f_{i+1} - \frac{1}{2}(f_{i+1} - f_i) \right] = y_i + \frac{h}{2}[f_i + f_{i+1}] \qquad (16)$$

which is called the **trapezoidal method.**

The error term is given by

$$TE = (x_{i+1}) - y_{i+1}$$

$$= y(x_i + h) - \left[y_i + \frac{h}{2}(y'_i + y'_{i+1}) \right]$$

$$= -\frac{h^3}{12}y'''(\xi), \qquad x_i < \xi < x_{i+1}$$

This method is of order 2.

$k = 2$: Retaining three terms inside the brackets in Eqn.(14), we get the method

$$y_{i+1} = y_i + h\left[f_{i+1} - \frac{1}{2}\nabla f_{i+1} - \frac{1}{12}\nabla^2 f_{i+1} \right]$$

$$= y_i + h\left[f_{i+1} - \frac{1}{2}(f_{i+1} - f_i) - \frac{1}{12}(f_{i+1} - 2f_i + f_{i-1}) \right]$$

$$= y_i + \frac{h}{12}[5f_{i+1} + 8f_i - f_{i-1}] \qquad (17)$$

The error term is obtained as

$$TE = y(x_{i+1}) - y_{i+1}$$

$$= y(x_i + h) - y_i - \frac{h}{12}[5y'_{i+1} + 8y'_i - y'_{i-1}]$$

$$= \frac{h^4}{24} y^{(iv)}(\xi), \qquad x_{i-1} < \xi < x_{i+1}$$

This method is of order 3.

In general, the k-step method of the form (14) is of order $(k+1)$.

Since the methods are implicit, we obtain a non-linear equation in y_{i+1} for non-linear initial value problem. The solution can be obtained by using Newton-Raphson method or, any other iterative method. Generally we take the initial approximation $y_{i+1} = y_i$. For a linear initial value problem, we can obtain y_{i+1} directly without iteration.

Milne-Simpson methods

These are implicit methods. If we integrate $y' = f(x, y)$ in the interval $[x_{i-1}, x_{i+1}]$, we get

$$y(x_{i+1}) = y(x_{i-1}) + \int_{x_{i-1}}^{x_{i+1}} f(x, y)\,dx \tag{18}$$

Now we replace $f(x, y)$ in Eqn.(18) by the interpolating polynomial $P_k(x)$ given by Eqn.(12) based on the data values $(x_{i+1}, f_{i+1}), (x_i, f_i), \ldots, (x_{i-k+1}, f_{i-k+1})$. Using the substitution $x - x_i = hs$, we obtain as in the case of Adams-Moulton methods

$$y_{i+1} = y_{i-1} + h \int_{-1}^{1} \left[f_{i+1} + (s-1)\nabla f_{i+1} + \frac{(s-1)s}{2!} \nabla^2 f_{i+1} + \cdots \right] ds$$

$$= y_{i-1} + h\left[2f_{i+1} - 2\nabla f_{i+1} + \frac{1}{3}\nabla^2 f_{i+1} + O\nabla^3 f_{i+1} + \cdots \right] \tag{19}$$

By choosing different values of k, we get different methods.

k = 0 : Retaining one term inside the brackets in Eqn.(19), we get the method

$$y_{i+1} = y_{i-1} + 2h f_{i+1} \tag{20}$$

The error term is given by

$$TE = y(x_{i+1}) - y_{i+1}$$
$$= y(x_i + h) - [\, y_{i-1} + 2h\, y'_{i+1} \,]$$
$$= -2h^2\, y''(\xi), \qquad x_{i-1} < \xi < x_{i+1}$$

This method is of order 1.

k = 1 : Retaining two terms inside the brackets in Eqn.(19), we get the method

$$y_{i+1} = y_{i-1} + h[\, 2f_{i+1} - 2\nabla f_{i+1} \,]$$
$$= y_{i-1} + h[\, 2f_{i+1} - 2(f_{i+1} - f_i) \,]$$
$$= y_{i-1} + 2h f_i \tag{21}$$

This is an explicit method. The error term is given by

$$TE = y(x_{i+1}) - y_{i+1}$$
$$= y(x_i + h) - [\, y_{i-1} + 2h\, y'_i \,]$$
$$= \frac{h^3}{3} y'''(\xi), \qquad x_{i-1} < \xi < x_{i+1}$$

This method is of order 2.

k = 2 : Retaining three terms inside the brackets in Eqn.(19), we get the method

$$y_{i+1} = y_{i-1} + h\left[2f_{i+1} - 2\nabla f_{i+1} + \frac{1}{3}\nabla^2 f_{i+1} \right]$$

$$= y_{i-1} + h\left[2f_{i+1} - 2(f_{i+1} - f_i) + \frac{1}{3}(f_{i+1} - 2f_i + f_{i-1}) \right]$$

$$= y_{i-1} + \frac{h}{3}[\, f_{i+1} + 4f_i + f_{i-1} \,] \tag{22}$$

The error term is given by

$$TE = y(x_{i+1}) - y_{i+1}$$

$$= y(x_i + h) - \left[y_{i-1} + \frac{h}{3}(y'_{i+1} + 4y'_i + y'_{i-1}) \right]$$

$$= -\frac{h^5}{90} y^{(5)}(\xi), \quad x_{i-1} < \xi < x_{i+1}$$

Thus the method is of order 4. The method (22) is called the **Milne-Simpson method of order 4**.
For-$\mathbf{k} = \mathbf{3}$, we obtain the same method as (22). Hence, for $k = 2$ and $k = 3$, we obtain the same method.

Example 3: Find the approximate value of $y(0.5)$ for the initial value problem

$$y' = x + y, \ y(0) = 1,$$

using the multistep method

$$y_{i+1} = y_{i-1} + \frac{h}{3}(f_{i+1} + 4f_i + f_{i-1})$$

(Milne-Simpson fourth order method) with $h = 0.1$
Compute the starting value using classical Runge-Kutta fourth order method with the same step length h.

Solution: The given multistep method is an implicit method. Since the given initial value problem is linear we do not require any iteration. We can compute y_{i+1} directly. From the given multistep method

$$y_{i+1} = y_{i-1} + \frac{h}{3}(f_{i+1} + 4f_i + f_{i-1})$$

or,

$$y_{i+2} = y_i + \frac{h}{3}(f_{i+2} + 4f_{i+1} + f_i) \tag{23}$$

we obtain for $i = 0$

$$y_2 = y_1 + \frac{h}{3}(f_2 + 4f_1 + f_0)$$

Now y_0 is the given initial value, whereas y_1 is the starting value. We compute y_1 using classical fourth order Runge-Kutta method.
We have $f(x,y) = x + y$, $x_0 = 0$, $y_0 = 1$ and $h = 0.1$. We obtain

$$K_1 = h\ f(x_0, y_0) = h\ f(0, 1) = 0.1$$

$$K_2 = h\ f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right) = h\, f(0.05, 1.05) = 0.11$$

$$K_3 = h\ f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right) = h\, f(0.05, 1.055) = 0.1105$$

$$K_4 = h\ f(x_0 + h, y_0 + K_3) = h\, f(0.1, 1.1105) = 0.12105$$

$$y_1 \approx y(0.1) = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) = 1.110342$$

After determining the starting value, we use the multistep method (23). From Eqn.(23), we obtain

$$y_{i+2} = y_i + \frac{h}{3}[(x_{i+2} + y_{i+2}) + 4(x_{i+1} + y_{i+1}) + (x_i + y_i)]$$

or,

$$\left(1 - \frac{h}{3}\right) y_{i+2} = y_i + \frac{h}{3}[4y_{i+1} + y_i + x_{i+2} + 4x_{i+1} + x_i]$$

Using $x_{i+2} = x_i + 2h$, $x_{i+1} = x_i + h$, we get

$$\left(1 - \frac{h}{3}\right) y_{i+2} = y_i + \frac{h}{3}[4y_{i+1} + y_i + 6x_i + 6h]$$

$$= \frac{4h}{3} y_{i+1} + \left(1 + \frac{h}{3}\right) y_i + \frac{h}{3}(6x_i + 6h) \tag{24}$$

Using $x_0 = 0$, $y_0 = 1$, $x_1 = 0.1$, $y_1 = 1.110342$, $h = 0.1$, we obtain from Eqn.(24) for

i = 0: $\frac{29}{30} y_2 = \frac{4}{30} y_1 + \frac{31}{30} y_0 + \frac{1}{30}\left(6x_0 + \frac{6}{10}\right)$

$= 1.201379$

or, $y_2 \approx y(0.2) = 1.242806$, $x_2 = 0.2$

i = 1: $\frac{29}{30} y_3 = \frac{4}{30} y_2 + \frac{31}{30} y_1 + \frac{1}{30}\left(6x_1 + \frac{6}{10}\right)$

$= 1.353061$

or, $y_3 \approx y(0.3) = 1.399718$, $x_3 = 0.3$

i = 2: $\frac{29}{30} y_4 = \frac{4}{30} y_3 + \frac{31}{30} y_2 + \frac{1}{30}\left(6x_2 + \frac{6}{10}\right)$

$= 1.530862$

or, $y_4 \approx y(0.4) = 1.583650$, $x_4 = 0.4$

i = 3: $\frac{29}{30} y_5 = \frac{4}{30} y_4 + \frac{31}{30} y_3 + \frac{1}{30}\left(6x_3 + \frac{6}{10}\right)$

$= 1.737529$

or, $y_5 \approx y(0.5) = 1.797443$

***

Example 4: Find an approximate value of $y(1.2)$ for the initial value problem

$$y' = x^2 + y^2,\ y(1) = 2$$

using the Adams-Moulton third order method

$$y_{i+1} = y_i + \frac{h}{12}[\,5f_{i+1} + 8f_i - f_{i-1}]$$

with $h = 0.1$. Calculate the starting value using third order Taylor series method with $h = 0.1$.

Solution: The Adams-Moulton third order method can be written as

$$y_{i+2} = y_{i+1} + \frac{h}{12}[\,5f_{i+2} + 8f_{i+1} - f_i], \qquad i = 0, 1, \ldots \tag{25}$$

For $i = 0$, we get

$$y_2 = y_i + \frac{h}{12}[\,5f_2 + 8f_1 - f_0]$$

Now, y_0 is the initial value and y_1 is the starting value. We calculate y_1 using Taylor series third order method

$$y_1 = y_0 + h\,y_0' + \frac{h^2}{2} y_0'' + \frac{h^3}{6} y_0''' \tag{26}$$

with $h = 0.1$. We have

$$y_0 = 2,\ y_0' = x_0^2 + y_0^2 = 5\ \ y_0'' = 2x_0 + 2y_0\,y_0' = 22,$$

$$y_0''' = 2 + 2y_0\,y_0'' + 2(y_0')^2 = 140$$

Therefore, we obtain from Eqn.(26)

$$y(1.1) \approx y_1 = 2 + (0.1)\,(2) + \frac{(0.1)^2}{2}\,(22) + \frac{(0.1)^3}{6}(140)$$

$$= 2.633333$$

After determining the starting value, we use the multistep method (25). From Eqn.(25), we obtain for $i = 0$.

$$y(1.2) \approx y_2 = y_1 + \frac{h}{12}\left[\,5(x_2^2 + y_2^2) + 8(x_1^2 + y_1^2) - (x_0^2 + y_0^2)\right]$$

Using $y_0 = 2$, $y_1 = 2.633333$, $x_0 = 1$, $x_1 = 1.1$, $x_2 = 1 + 2h = 1.2$ and simplifying, we get

$y_2 = 0.041667\ y_2^2 + 3.194629$, which is a nonlinear equation in y_2.

We use Newton-Raphson method to determine y_2.

Let $F(y_2) = 0.041667y_2^2 - y_2 + 3.194629$.

We get, $F'(y_2) = 0.083334y_2 - 1$

The Newton-Raphson method is given by

$$y_2^{(s+1)} = y_2^{(s)} - \frac{F(y_2^{(s)})}{F'(y_2^{(s)})},\ s = 0,1,\ldots$$

Starting with $y_2^{(0)} = y_1 = 2.633333$, we get

$$y_2^{(1)} = y_2^{(0)} - \frac{F(y_2^{(0)})}{F'(y_2^{(0)})} = 2.633333 - \frac{0.850233}{(-0.780554)} = 3.722602$$

$$y_2^{(2)} = y_2^{(1)} - \frac{F(y_2^{(1)})}{F'(y_2^{(1)})} = 3.722602 - \frac{0.049439}{(-0.689781)} = 3.794275$$

$$y_2^{(3)} = y_2^{(2)} - \frac{F(y_2^{(2)})}{F'(y_2^{(2)})} = 3.794275 - \frac{0.000214}{(-0.683808)} = 3.794588$$

Since $F(y_2^{(3)}) = 0.0000001$, we take $y(1.2) \approx y_2 = 3.794588$

***

Note that we have to use iteration at every step.

And now some exercises for you.

E1) Find the truncation error and the order of the method

$$y_{i+3} = y_i + \frac{3h}{8}[\ y_i' + 3y_{i+1}' + 3y_{i+2}' + y_{i+3}']$$

for solving the initial value problem $y' = f(x,y),\ y(x_0) = y_0$.

E2) Find an approximate value of $y(0.6)$ for the initial value problem

$$y' = x - y^2,\ y(0) = 1$$

using the multistep method

$$y_{i+1} = y_i + \frac{h}{2}(3f_i - f_{i-1})$$

with the step length $h = 0.2$. Compute the starting value using Taylor series second order method with the same steplength h.

E3) Find an approximate value of $y(1.5)$ for the initial value problem

$$y' = x^2 + y^2,\ y(1) = 0$$

using the multistep method

$$y_{i+1} = y_i + \frac{h}{12}(23f_i - 16f_{i-1} + 5f_{i-2})$$

with $h = 0.1$. Calculate the starting values using third order Taylor series method with the same steplength h.

E4) Find an approximate value of $y(2)$ for the initial value problem

$$y' = y + y^2,\ y(1) = 1$$

using the multistep method

$$y_{i+1} = y_{i-1} + \frac{h}{3}[\ 7f_i - 2f_{i-1} + f_{i-2}]$$

with $h = 0.2$. Compute the starting values using the Heun's method with the same steplength h.

E5) Solve the initial value problem

$$y' = 2x + 3y,\ y(1) = 2$$

on the interval [1, 1.4] using the multistep method

$$y_{i+1} = y_{i-1} + \frac{h}{3}(y'_{i+1} + 4y'_i + y'_{i-1})$$

with $h = 0.1$. Calculate the starting value using classical fourth order Runge-Kutta method, with the same steplength h.

Let us now discuss the second approach for the derivation of multistep methods.

9.2.2 Method of Undetermined Parameters

The general multistep or k-step method for the solution of the initial value problem (1) can be written as

$$y_{i+1} = a_1 y_i + a_2 y_{i-1} + \cdots + a_k y_{i-k+1}$$
$$+ h[b_0 y'_{i+1} + b_1 y'_i + \cdots + b_k y'_{i-k+1}]$$

or, $$y_{i+1} = \sum_{m=1}^{k} a_m y_{i-m+1} + h\sum_{m=0}^{k} b_m y'_{i-m+1} \quad (27)$$

Changing i to $i + k - 1$ in Eqn.(24), we can also write the method as

$$y_{i+k} = \sum_{m=1}^{k} a_m y_{i+k-m} + h\sum_{m=0}^{k} b_m y'_{i+k-m} \quad (28)$$

In method (27), y_i is the initial value and $y_{i-1}, \ldots, y_{i-k+1}$ are the starting values.
If $b_0 = 0$, the method (27) defines an explicit method and if $b_0 \neq 0$, the method (27) defines an implicit method.
In method (27) a'_ms and b'_ms are unknowns which are to be determined such that the method is of a particular order. We write the error term as

$$TE = y(x_{i+1}) - y_{i+1}$$
$$= y(x_i + h) - \sum_{m=1}^{k} a_m y(x_{i-m+1}) - h\sum_{m=0}^{k} b_m y'(x_{i-m+1}) \quad (29)$$

Expanding each term in (29) in Taylor series about the point x_i and collecting the terms of various powers of h, we can write

$$TE = C_0 y(x_i) + C_1 h\, y'(x_i) + \cdots + C_p h^p y^{(p)}(x_i)$$
$$+ C_{p+1} h^{p+1} y^{(p+1)}(x_i) + \cdots \quad (30)$$

where C'_ms are independent of h and depend on a'_ms and b'_ms. The multistep method (27) is said to be of order p, if

$$C_0 = C_1 = \cdots = C_p = 0 \text{ and } C_{p+1} \neq 0$$

The truncation error in this case is given by

$$TE = C_{p+1} h^{p+1} y^{(p+1)}(x_i) + O(h^{p+2})$$

Solving the system of equations obtained from

$$C_0 = C_1 = \cdots = C_p = 0$$

we determine the constants a'_ms and b'_ms and hence the method. The first non-zero term in Eqn.(30) gives the error term and the order of the method.
A method is said to be **consistent** if $p \geq 1$.

Note that we have $(2k + 1)$ arbitrary unknowns in the method (27) and we need $2k + 1$ equations to determine these unknowns. Hence, a method of the form (27) can be of maximum order 2k. However, the stability conditions restrict the order of the method to k for an explicit method. For an implicit method, the order is restricted to k+1, when k is odd; and k+2, when k is even. We shall discuss the stability of these methods in Sec.9.4.

Adams-Bashforth method

The form of a few important class of methods is

$$y_{i+1} = y_i + h\sum_{m=1}^{k} b_m y'_{i-m+1}$$

This is an explicit method and is of order k.

Nystrom method

$$y_{i+1} = y_{i-1} + h\sum_{m=1}^{k} b_m y'_{i-m+1}$$

This is an explicit method and is of order k.

Adams-Moulton method

$$y_{i+1} = y_i + h\sum_{m=0}^{k} b_m y'_{i-m+1}$$

This is an implicit method and is of order k+1.

Milne-Simpson method

$$y_{i+1} = y_{i-1} + h\sum_{m=0}^{k} b_m y'_{i-m+1}$$

This is an implicit method and is of order $k+1$, when k is odd and of order $k+2$, when k is even.

We illustrate the derivation of multistep methods by taking some examples.

Example 5: Obtain the constants a_1, b_1 and b_2 in the explicit multistep method

$$y_{i+1} = a_1 y_i + h[\, b_1 y'_i + b_2 y'_{i-1}]$$

Determine the truncation error and the order of the method.

Solution: We write

$$\begin{aligned} TE &= y(x_{i+1}) - y_{i+1} \\ &= y(x_{i+1}) - [\, a_1 y(x_i) + h\{b_1 y'(x_i) + b_2 y'(x_{i-1})\}] \\ &= y(x_i + h) - a_1 y(x_i) - h[\, b_1 y'(x_i) + b_2 y'(x_i - h)] \end{aligned}$$

Expanding each term in Taylor series about the point x_i and collecting the coefficients of various power of h, we get

$$\begin{aligned} TE &= (1 - a_1) y(x_i) + \{1 - (b_1 + b_2)\} h\ y'(x_i) \\ &\quad + \left(\frac{1}{2} + b_2\right) h^2\ y''(x_i) + \left(\frac{1}{6} - \frac{1}{2} b_2\right) h^3\ y'''(x_i) + \cdots \end{aligned} \tag{31}$$

We have three unknowns and we need three equations to determine these unknowns. Setting the constant term and the coefficients of h, h^2 in Eqn.(31) to zero, we obtain the system of equations

$$\begin{aligned} &1 - a_1 = 0, &&\text{or,}\quad a_1 = 1 \\ &1 - (b_1 + b_2) = 0, &&\text{or,}\quad b_1 + b_2 = 1 \\ &\frac{1}{2} + b_2 = 0, &&\text{or,}\quad b_2 = -\frac{1}{2} \end{aligned}$$

whose solution is $a_1 = 1,\ b_1 = \frac{3}{2},\ b_2 = -\frac{1}{2}$. Thus we obtain the method as

$$y_{i+1} = y_i + \frac{h}{2}[\, 3y'_i - y'_{i-1}] \tag{32}$$

and from Eqn.(31), we obtain

$$\begin{aligned} TE &= \left(\frac{1}{6} - \frac{1}{2} b_2\right) h^3\ y'''(x_i) + O(h^4) \\ &= \frac{5}{12} h^3\ y'''(x_i) + O(h^4) \end{aligned}$$

Therefore, the method is of order 2.

The method (32) is the Adams-Bashforth method of order 2.

***

Example 6: Derive the constants in the explicit method

$$y_{i+1} = a_1 y_i + h\{b_1 y'_i + b_2 y'_{i-1} + b_3 y'_{i-2}\}$$

Determine also the truncation error and the order of the method.

Solution: We write

$$\begin{aligned}TE &= y(x_{i+1}) - y_{i+1}\\ &= y(x_{i+1}) - [a_1 y(x_i) + h\{b_1 y'(x_i) + b_2 y'(x_{i-1}) + b_3 y'(x_{i-2})\}]\\ &= y(x_i + h) - a_1 y(x_i) - h[b_1 y'(x_i) + b_2 y'(x_i - h) + b_3 y'(x_i - 2h)]\end{aligned}$$

Expanding each term in Taylor series about the point x_i and collecting the coefficients of various powers of h, we get

$$\begin{aligned}TE &= (1 - a_1) y(x_i) + \{1 - (b_1 + b_2 + b_3)\} h y'(x_i)\\ &+ \left\{\frac{1}{2} + (b_2 + 2b_3)\right\} h^2 y''(x_i) + \left\{\frac{1}{6} - \frac{1}{2}(b_2 + 4b_3)\right\} h^3 y'''(x_i)\\ &+ \left\{\frac{1}{24} + \frac{1}{6}(b_2 + 8b_3)\right\} h^4 y^{iv}(x_i) + \cdots \qquad (33)\end{aligned}$$

We have four unknowns and we need four equations to determine these unknowns. Equating the constant terms and the coefficients of h, h^2 and h^3 in Eqn.(33) to zero, we obtain the system of equations.

$$\begin{aligned}&1 - a_1 = 0 &&\text{or,}\quad a_1 = 1\\ &1 - (b_1 + b_2 + b_3) = 0 &&\text{or,}\quad b_1 + b_2 + b_3 = 1\\ &\frac{1}{2} + (b_2 + 2b_3) = 0 &&\text{or,}\quad b_2 + 2b_3 = -\frac{1}{2}\\ &\frac{1}{6} - \frac{1}{2}(b_2 + 4b_3) = 0 &&\text{or,}\quad b_2 + 4b_3 = \frac{1}{3}\end{aligned}$$

Solving this system of equations, we obtain

$$a_1 = 1,\ b_1 = \frac{23}{12},\ b_2 = -\frac{16}{12},\ b_3 = \frac{5}{12}$$

and the method is given by

$$y_{i+1} = y_i + \frac{h}{12}[23\, y'_i - 16 y'_{i-1} + 5 y'_{i-2}] \qquad (34)$$

which is Adams-Bashforth third order method. From Eqn.(33), we get

$$\begin{aligned}TE &= \left[\frac{1}{24} + \frac{1}{6}(b_2 + 8b_3)\right] h^4\, y^{iv}(x_i) + O(h^5)\\ &= \frac{3}{8} h^4\, y^{iv}(x_i) + O(h^5).\end{aligned}$$

***

Example 7: Derive the method

$$y_{i+1} = a_1 y_i + a_2 y_{i-2} + h(b_0\, y'_{i+1} + b_1\, y'_i + b_2\, y'_{i-1})$$

Determine the truncation error and order of the method.

Solution: We write

$$\begin{aligned}TE &= y(x_{i+1}) - y_{i+1}\\ &= y(x_{i+1}) - [a_1\, y(x_i) + a_2\, y(x_{i-2}) + h\{b_0\, y'(x_{i+1}) + b_1 y'(x_i) + b_2 y'(x_{i-1})\}]\\ &= y(x_i + h) - [a_1\, y(x_i) + a_2\, y(x_i - 2h) + h\{b_0\, y'(x_i + h)\\ &\qquad + b_1\, y'(x_i) + b_2\, y'(x_i - h)\}]\end{aligned}$$

Expanding each term in Taylor series about the point x_i and collecting the coefficients of various power of h, we get

$$TE = (1 - a_1 - a_2)\, y(x_i) + [\,1 + 2a_2 - (b_0 + b_1 + b_2)]\, h\, y'(x_i)$$
$$+ \left[\frac{1}{2} - 2a_2 - (b_0 - b_2)\right] h^2 y''(x_i) + \left[\frac{1}{6} + \frac{4}{3}a_2 - \frac{1}{2}(b_0 + b_2)\right] h^3 y'''(x_i)$$
$$+ \left[\frac{1}{24} - \frac{2}{3}a_2 - \frac{1}{6}(b_0 - b_2)\right] h^4 y^{iv}(x_i) + \left[\frac{1}{120} + \frac{4}{15}a_2 - \frac{1}{24}(b_0 + b_2)\right] h^5 y^{v}(x_i)$$
$$+ \cdots \tag{35}$$

We have five unknowns, and we need five equations to determine these unknowns. Setting the constant term and the coefficients of h, h^2, h^3 and h^4 in Eqn.(35) to zero, we obtain the system of equations

$$1 - a_1 - a_2 = 0 \;, \qquad \text{or} \qquad a_1 + a_2 = 1$$
$$1 + 2a_2 - (b_0 + b_1 + b_2) = 0, \qquad \text{or} \qquad b_0 + b_1 + b_2 - 2a_2 = 1$$
$$\frac{1}{2} - 2a_2 - (b_0 - b_2) = 0 \;, \qquad \text{or} \qquad b_0 - b_2 + 2a_2 = \frac{1}{2}$$
$$\frac{1}{6} + \frac{4}{3}a_2 - \frac{1}{2}(b_0 + b_2) = 0\;, \qquad \text{or} \qquad b_0 + b_2 - \frac{8}{3}a_2 = \frac{1}{3}$$
$$\frac{1}{24} - \frac{2}{3}a_2 - \frac{1}{6}(b_0 - b_2) = 0\;, \qquad \text{or} \qquad b_0 - b_2 + 4a_2 = \frac{1}{4}$$

Solving these equations, we obtain

$$a_1 = \frac{9}{8}, \quad a_2 = -\frac{1}{8}, \quad b_0 = \frac{3}{8}, \quad b_1 = \frac{6}{8}, \quad b_2 = -\frac{3}{8}$$

and the method is given by

$$y_{i+1} = \frac{1}{8}(9y_i - y_{i-2}) + \frac{h}{8}[\,3y'_{i+1} + 6y'_i - 3y'_{i-1}] \tag{36}$$

We obtain from (35)

$$TE = \left[\frac{1}{120} + \frac{4}{15}a_2 - \frac{1}{24}(b_0 + b_2)\right] h^5 y^{v}(x_i) + O(h^6)$$
$$= -\frac{1}{40}h^5 y^{v}(x_i) + O(h^6)$$

Hence, the method (36) is implicit and is of order 4.

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And now some exercises for you.

E6) Derive the method
$$y_{i+1} = a_1 y_{i-1} + h[\,b_1 y'_i + b_2 y'_{i-1} + b_3 y'_{i-2}]$$
for solving the initial value problem $y' = f(x, y),\ y(x_0) = y_0$. Find the truncation error and the order of the method.

E7) Derive the method
$$y_{i+1} = a_1 y_i + a_2 y_{i-1} + h[\,b_1 y'_i + b_2 y'_{i-1} + b_3 y'_{i-2}]$$
for solving the initial value problem $y' = f(x, y),\ y(x_0) = y_0$. Find the truncation error and the order of the method.

E8) Derive the method
$$y_{i+1} = a_1 y_i + h[\,b_0 y'_{i+1} + b_1 y'_i + b_2 y'_{i-1}]$$
for solving the initial value problem $y' = f(x, y),\ y(x_0) = y_0$. Find the truncation error and order of the method.

E9) Derive the method
$$y_{i+1} = a_1 y_i + a_2 y'_{i-1} + h\, b_0 y'_{i+1}$$
for solving the initial value problem $y' = f(x, y),\ y(x_0) = y_0$. Find the truncation error and the order of the method.

So far we have developed a number of explicit and implicit multistep methods to solve an initial value problem $y' = f(x, y),\quad y(x_0) = y_0$. Based on the stability properties of the method which we shall be discussing in Sec.9.4, it is seen that all the explicit methods converge only if the step size h is sufficiently small. Hence, if the solution of the initial value problem is required in a large interval, then we may require a large number of integration steps, which besides increasing the round-off error, is not economical. On the other hand, most implicit methods have strong stability properties, that is one can choose sufficiently large value of h for integration. However, we need to solve a non-linear equation by iteration at each step, which is also expensive. Hence, we combine the explicit and implicit methods to define a new class of methods called the **predictor-corrector methods** which we shall discuss now.

9.3 PREDICTOR-CORRECTOR METHODS

The explicit methods are called the **predictors** which predict or give an initial approximation to $y(x_{i+1})$. This initial approximation is denoted by $y_{i+1}^{(p)}$. The implicit methods are called the **correctors** which use $y_{i+1}^{(p)}$ to obtain an improved value of y_{i+1}. This corrected value is denoted by $y_{i+1}^{(c)}$. The corrector method is used till convergence is obtained. We define predictor-corrector methods (P-C sets) as follows:

P : Predict an approximate value of $y(x_{i+1})$. This approximate value is denoted by $y_{i+1}^{(p)}$ or $y_{i+1}^{(0)}$.

C : Correct this value using a corrector method to obtain $y_{i+1}^{(c)}$.
We take $y(x_{i+1}) = y_{i+1}^{(c)}$.

When the predictor and corrector methods are of the same order, we need only one or two corrector iterations per step. However, if predictor is of lower order, we may require more number of corrector iterations for convergence. The predictor-corrector methods (P-C methods) are used as sets. Some examples of P-C sets are

1. P: $y_{i+1}^{(0)} = y_i + h f(x_i, y_i)$ (Euler method)

   C: $y_{i+1}^{(s+1)} = y_i + h f(x_{i+1}, y_{i+1}^{(s)})$ (backward Euler method)

   $s = 0, 1, \ldots$

The order of both the methods is one.

2. P: $y_{i+1}^{(0)} = y_i + h f(x_i, y_i)$ (Euler method)

   C: $y_{i+1}^{(s+1)} = y_i + \frac{h}{2}\left[f(x_i, y_i) + f(x_{i+1}, y_{i+1}^{(s)})\right]$ (Trapezoidal method)

   $s = 0, 1, \ldots$

The predictor is of first order and the corrector is of second order.

3. P: $y_{i+1}^{(0)} = y_{i-3} + \frac{4h}{3}(2f_i - f_{i-1} + 2f_{i-2})$

   C: $y_{i+1}^{(s+1)} = y_{i-1} + \frac{h}{3}\left[f_{i-1} + 4f_i + f(x_{i+1}, y_{i+1}^{(s)})\right]$

   $s = 0, 1, \ldots$

Both the methods are of fourth order. This P-C set is also called **Milne's P-C set.** The starting values are obtained using some singlestep explicit method like Taylor series or Runge-Kutta method. The order of the singlestep method may be same or less than that of the corrector method.

Example 8: Obtain the approximate value of $y(0.3)$ for the initial value problem

$$y' = x^2 + y^2, \quad y(0) = 1$$

using the method:

P = predictor: $y_{i+1}^{(p)} = y_i + h f_i$

C = corrector: $y_{i+1}^{(c)} = y_i + \frac{h}{2}[f_i + f(x_{i+1}, y_{i+1})]$ (37)

with the step length h = 0.1. Perform two corrector iterations per step.

Solution: Since both the predictor and corrector methods are singlestep methods, we do not need any starting value. We write the methods (37) as

$$P: \quad y_{i+1}^{(0)} = y_i + h f(x_i, y_i) = y_i + h\left[x_i^2 + y_i^2\right]$$

$$C: \quad y_{i+1}^{(s+1)} = y_i + \frac{h}{2}\left[f(x_i, y_i) + f(x_{i+1}, y_{i+1}^{(s)})\right]$$

$$= y_i + \frac{h}{2}\left[x_i^2 + y_i^2 + x_{i+1}^2 + \left\{y_{i+1}^{(s)}\right\}^2\right], \quad s = 0, 1,$$

$i = 0, 1, 2.$

since $f(x, y) = x^2 + y^2$.

Using the given predictor method for h = 0.1, we get

i = 0: $x_0 = 0, \quad y_0 = 1,$

$$P: \quad y_1^{(0)} = y_0 + h(x_0^2 + y_0^2) = 1.1$$

$$C: \quad y_1^{(s+1)} = y_0 + \frac{h}{2}\left[f(x_i, y_i) + f(x_{i+1}, y_{i+1}^{(s)})\right]$$

$$s = 0: y_1^{(1)} = y_0 + \frac{h}{2}\left[x_0^2 + y_0^2 + x_1^2 + (y_1^{(0)})^2\right] = 1.111$$

$$s = 1: y_1^{(2)} = y_0 + \frac{h}{2}\left[x_0^2 + y_0^2 + x_1^2 + (y_1^{(1)})^2\right] = 1.112216$$

Therefore, after two corrector iterations, we get

$$y_1 \approx y(0.1) = 1.112216$$

i = 1: $x_1 = 0.1, \quad y_1 = 1.112216$

$$P: \quad y_2^{(0)} = y_1 + h(x_1^2 + y_1^2) = 1.236918$$

$$C: \quad y_2^{(s+1)} = y_1 + \frac{h}{2}\left[x_1^2 + y_1^2 + x_2^2 + (y_2^{(s)})^2\right]$$

$$s = 0: y_2^{(1)} = y_1 + \frac{h}{2}\left[x_1^2 + y_1^2 + x_2^2 + (y_2^{(0)})^2\right] = 1.253066$$

$$s = 1: y_2^{(2)} = y_1 + \frac{h}{2}\left[x_1^2 + y_1^2 + x_2^2 + (y_2^{(1)})^2\right] = 1.255076$$

Therefore, after two corrector iterations, we obtain.

$$y_2 \approx y(0.2) = 1.255076$$

i = 2: $x_2 = 0.2, \quad y_2 = 1.255076$

$$P: \quad y_3^{(0)} = y_2 + h(x_2^2 + y_2^2) = 1.416598$$

$$C: \quad y_3^{(s+1)} = y_2 + \frac{h}{2}\left[x_2^2 + y_2^2 + x_3^2 + (y_3^{(s)})^2\right]$$

$$s = 0: y_3^{(1)} = y_2 + \frac{h}{2}\left[x_2^2 + y_2^2 + x_3^2 + (y_3^{(0)})^2\right] = 1.440674$$

$$s = 1: y_3^{(2)} = y_2 + \frac{h}{2}\left[x_2^2 + y_2^2 + x_3^2 + (y_3^{(1)})^2\right] = 1.444114$$

Therefore, after two corrector iterations, we obtain

$$y_3 \approx y(0.3) = 1.444114$$

***

Example 9: Solve the initial value problem

$$y' = x^2 + y^3, \quad y(1) = 0$$

on the interval [1, 1.6] using the predictor-corrector method

$$\text{P: } y_{i+1} = y_i + \frac{h}{2}(3f_i - f_{i-1})$$

$$\text{C: } y_{i+1} = y_i + \frac{h}{12}(5f_{i+1} + 8f_i - f_{i-1}) \tag{38}$$

with the step length $h = 0.2$. Perform three corrector iterations per step. Compute the starting value using Taylor series second order method with the same step length h.

Solution: We have $f(x, y) = x^2 + y^3$. We write the method (38) as

$$\text{P:} \quad y_{i+1}^{(0)} = y_i + \frac{h}{2}(3f_i - f_{i-1})$$

$$\text{or,} \quad y_{i+2}^{(0)} = y_{i+1} + \frac{h}{2}(3f_{i+1} - f_i) \tag{39}$$

$$\text{C:} \quad y_{i+1}^{(s+1)} = y_i + \frac{h}{12}(5f(x_{i+1}, y_{i+1}^{(s)}) + 8f_i - f_{i-1})$$

$$\text{or,} \quad y_{i+2}^{(s+1)} = y_{i+1} + \frac{h}{12}\left[5f(x_{i+2}, y_{i+2}^{(s)}) + 8f_{i+1} - f_i \right] \tag{40}$$

where $s = 0, 1, 2$ and $i = 0, 1, \ldots$

For $i = 0$, we get from Eqns.(39) and (40)

$$\text{P:} \quad y_2^{(0)} = y_1 + \frac{h}{2}(3f_1 - f_0)$$

$$\text{C:} \quad y_2^{(s+1)} = y_1 + \frac{h}{12}\left[5f(x_2, y_2^{(s)}) + 8f_1 - f_0 \right], \; s = 0, 1, \ldots$$

Now y_0 is the given initial value, whereas y_1 is the starting value. We compute the starting value using the Taylor series second order method

$$y_1 = y_0 + h y_0' + \frac{h^2}{2} y_0''$$

with $h = 0.2$. We have

$$y' = x^2 + y^3, \qquad y'' = 2x + 3y^2 y'$$

Therefore, for $x_0 = 1,\ y_0 = 0$, we obtain $y_0' = 1, \quad y_0'' = 2$

and $y_1 \approx y(1.2) = 0 + 0.2(1) + \dfrac{(0.2)^2}{2}(2) = 0.24$

After determining the starting value, we use the predictor-corrector method given by Eqns.(39) and (40). We obtain for

$\mathbf{i = 0}$: $\quad x_0 = 1, \quad y_0 = 0, \quad x_1 = 1.2, \quad y_1 = 0.24$

$$f_0 = x_0^2 + y_0^3 = 1, \; f_1 = x_1^2 + y_1^3 = 1.453824$$

$$\text{P:} \quad y_2^{(0)} = y_1 + \frac{h}{2}[\, 3f_1 - f_0] = 0.576147$$

Now $x_2 = 1.4,\ f(x_2,\ y_2^{(0)}) = x_2^2 + (y_2^{(0)})^3 = 2.151250$

$$\text{C:} \quad s = 0: y_2^{(1)} = y_1 + \frac{h}{12}\left[5f(x_2, y_2^{(0)}) + 8f_1 - f_0)\right] = 0.596447$$

and $f(x_2,\ y_2^{(1)}) = x_2^2 + (y_2^{(1)})^3 = 2.172185$

$$s = 1: y_2^{(2)} = y_1 + \frac{h}{12}\left[5f(x_2, y_2^{(1)}) + 8f_1 - f_0 \right] = 0.598192$$

and $f(x_2, y_2^{(2)}) = x_2^2 + (y_2^{(2)})^3 = 2.174053$

$$s = 2: y_2^{(3)} = y_1 + \frac{h}{12}\left[5f(x_2, y_2^{(2)}) + 8f_1 - f_0 \right] = 0.598348$$

Therefore, after three corrector iterations

$$y_2 \approx y(1.4) = 0.598348$$

i = 1: $x_1 = 1.2,\ y_1 = 0.24,\ x_2 = 1.4,\ y_2 = 0.598348$

$f_1 = 1.453824,\ f_2 = x_2^2 + y_2^3 = 2.174220$

$$P:\ y_3^{(0)} = y_2 + \frac{h}{2}[3f_2 - f_1] = 1.105232$$

Now $x_3 = 1.6,\ f(x_3, y_3^{(0)}) = x_3^2 + (y_3^{(0)})^3 = 3.910083$

$$C:\ s = 0: y_3^{(1)} = y_2 + \frac{h}{12}\left[5f(x_3, y_3^{(0)}) + 8f_2 - f_1\right] = 1.189854$$

and $f(x_3, y_3^{(1)}) = x_3^2 + (y_3^{(1)})^3 = 4.244539$

$$s = 1: y_3^{(2)} = y_2 + \frac{h}{12}\left[5f(x_3, y_3^{(1)}) + 8f_2 - f_1\right] = 1.217725$$

and $f(x_3, y_3^{(2)}) = x_3^2 + (y_3^{(2)})^3 = 4.365709$

$$s = 2: y_3^{(3)} = y_2 + \frac{h}{12}\left[5f(x_3, y_3^{(2)}) + 8f_2 - f_1\right] = 1.227823$$

Hence, after three corrector iterations

$y_3 \approx y(1.6) = 1.227823$

***

E10) Solve the initial value problem

$$y' = y + \sin y,\quad y(1) = 1$$

in the interval $[1, 1.6]$ using the predictor-corrector method

$$P:\ y_{i+1} = y_i + \frac{h}{2}(3f_i - f_{i-1})$$

$$C:\ y_{i+1} = y_i + \frac{h}{2}(f_{i+1} + f_i)$$

with $h = 0.2$. Calculate the starting value using second order Taylor series method with the same steplength. Perform two corrector iterations per step.

E11) Obtain the approximate value of $y(1.0)$ for the initial value problem

$$y' = x^2 + y^2,\quad y(0) = 1$$

using the predictor-corrector method

$$P:\ y_{i+1} = y_{i-1} + \frac{4h}{3}(2f_i - f_{i-1} + 2f_{i-2})$$

$$C:\ y_{i+1} = y_{i-1} + \frac{h}{3}(f_{i+1} + 4f_i + f_{i-1})$$

with $h = 0.2$. Calculate the starting values using the Euler method with the same steplength. Perform two corrector iterations per step.

Let us now discuss the stability of multistep methods.

9.4 STABILITY OF MULTISTEP METHODS

If we apply the multistep method

$$y_{i+1} = a_1 y_i + a_2 y_{i-1} + \cdots + a_k y_{i-k+1} + h\left[b_0 y'_{i+1} + b_1 y'_i + \cdots + b_k y'_{i-k+1}\right] \qquad (41)$$

to the test equation

$$y' = \lambda y,\quad y(x_0) = y_0$$

we get

$$y_{i+1} = a_1 y_i + a_2 y_{i-1} + \cdots + a_k y_{i-k+1} + \lambda h\left[b_0 y_{i+1} + b_1 y_i + \cdots + b_k y_{i-k+1}\right]$$

or,
$$y_{i+1} - a_1 y_i - a_2 y_{i-1} - \cdots - a_k y_{i-k+1} - \lambda h\left[b_0 y_{i+1} + b_1 y_i + \cdots + b_k y_{i-k+1}\right] = 0 \qquad (42)$$

The exact solution of Eqn.(42) is given by
$$y(x) = y(x_0)\, e^{(x-x_0)\lambda}$$
Therefore, we get
$$y(x_i) = y(x_0)\, e^{(x_i - x_0)\lambda} = y(x_0)\, e^{\lambda hi}$$
and $\quad y(x_{i+1}) = y(x_0)\, e^{(x_{i+1}-x_0)\lambda} = y(x_0)\, e^{\lambda h(i+1)}$

Hence, we obtain
$$\frac{y(x_{i+1})}{y(x_i)} = e^{\lambda h} \quad \text{or,} \quad y(x_{i+1}) = y(x_i)\, e^{\lambda h}$$
We find that the exact solution of Eqn.(41) grows $(\lambda > 0)$ or, decays $(\lambda < 0)$ by the factor $e^{\lambda h}$.

Substituting $y_i = y(x_0)\, e^{\lambda hi}$ in Eqn.(42), we get
$$\left\{e^{\lambda h(i+1)} - a_1 e^{\lambda hi} - \cdots - a_k e^{\lambda h(i-k+1)}\right\} y(x_0) - \lambda h\left\{ b_0 e^{\lambda h(i+1)} + b_1 e^{\lambda hi} + \cdots + b_k e^{\lambda h(i-k+1)}\right\} y(x_0) = 0$$
Dividing by $e^{\lambda h(i-k+1)}$, we get
$$[(e^{\lambda hk} - a_1 e^{\lambda h(k-1)} - \cdots - a_k) - \lambda h(b_0 e^{\lambda hk} + b_1 e^{\lambda h(k-1)} + \cdots + b_k)] = 0 \qquad (43)$$
Substituting $e^{\lambda h} = \xi$ in Eqn.(43), we get
$$(\xi^k - a_1 \xi^{k-1} - \cdots - a_k) - \lambda h (b_0 \xi^k + b_1 \xi^{k-1} + \cdots + b_k) = 0$$
or,
$$\rho(\xi) - \lambda h\, \sigma(\xi) = 0 \qquad (44)$$
where,
$$\rho(\xi) = \xi^k - a_1 \xi^k - \cdots - a_k$$
$$\sigma(\xi) = b_0 \xi^k + b_1 \xi^k + \cdots + b_k$$
We note that $\sigma(\xi)$ is a polynomial of degree k for an implicit method $(b_0 \neq 0)$ and a polynomial of degree $k-1$ for an explicit method $(b_0 = 0)$.

The polynomial equation $\rho(\xi) - \lambda h\, \sigma(\xi) = 0$ is called the **characteristic equation** associated with the multistep method (41).

The polynomial $\rho(\xi) = 0$ is called the **reduced characteristic equation**.

Let $\xi_1, \xi_2, \ldots, \xi_k$ be the roots of $\rho(\xi) = 0$ and $\xi_{1h}, \xi_{2h}, \ldots, \xi_{kh}$ be the roots of $\rho(\xi) - \lambda h\, \sigma(\xi) = 0$.

We note that for a consistent method, $\xi = 1$ is always a root of $\rho(\xi) = 0$.

The multistep method (41) is said to be

stable : if $\quad |\xi_i| < 1, \; i \neq 1$ and $\xi_1 = 1$

unstable: if $\quad |\xi_i| > 1$ for some i or, if there is a multiple root of $\rho(\xi) = 0$ of modulus unity.

conditionally stable (weakly stable): if ξ_i' s are simple and if more than one of these roots have modulus unity.

For a stable method, we further define

absolutely stable : if $\lambda < 0$ and $|\xi_{ih}| < 1$ for all i

relatively stable : if $\lambda > 0$ and $|\xi_{ih}| < e^{\lambda h}$ for all i

interval of absolute stability : The set of values of λh for which the method is absolutely stable, that is the interval $(\lambda h, 0)$, $\lambda < 0$ and $|\xi_{ih}| < 1$ for all i

Similarly, we define the interval of relative stability.

We generally use the **Routh-Hurwitz criterion** to obtain the interval of absolute stability. To use this criterion, we substitute
$$\xi = \frac{1+z}{1-z} \qquad (45)$$

in the characteristic equation $\rho(\xi) - \lambda h\ \sigma(\xi) = 0$ and obtain the **transformed equation**.

$$P(z) = v_0 z^k + v_1 z^{k-1} + \cdots + v_{k-1} z + v_k = 0 \tag{46}$$

where v_i's depend on λh.

Let

$$D = \begin{vmatrix} v_1 & v_3 & v_5 & \cdots & v_{2k-1} \\ v_0 & v_2 & v_4 & \cdots & v_{2k-2} \\ 0 & v_1 & v_3 & \cdots & v_{2k-3} \\ 0 & v_0 & v_2 & \cdots & v_{2k-4} \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & v_k \end{vmatrix}$$

then the method is absolutely stable if

(i) $v_i > 0$ for all i and

(ii) leading principal minors of D are positive.

We obtain the condition of absolute stability as

$\mathbf{k=1:}\quad v_0 > 0,\quad v_1 > 0$

$\mathbf{k=2:}\quad v_0 > 0,\quad v_1 > 0,\quad v_2 > 0$

$\mathbf{k=3:}\quad v_0 > 0,\quad v_1 > 0,\quad v_2 > 0\quad$ and $\quad v_1 v_2 - v_0 v_3 > 0$

Note that if any of v_i's is zero and other v_i's are positive, then it indicates that a root lies on the unit circle $|\xi| = 1$. If any of v_i's is negative, then there is at least one root for which $|\xi_i| > 1$.

In both the cases, the method is unstable.

Without giving proof we state the following result.

Theorem 1 The order of a stable explicit k-step method of the form (24) cannot exceed k and the order of a stable implicit k-step method of the form (24) cannot exceed $k+1$ if k is odd and $k+2$ if k is even.

We now take up a few examples

Example 10: Find the interval of absolute stability of the Adams-Bashforth methods

(i) $\quad y_{i+1} = y_i + \dfrac{h}{2}(3y_i' - y_{i-1}')$

(ii) $\quad y_{i+1} = y_i + \dfrac{h}{12}(23\,y_i' - 16\,y_{i-1}' + 5\,y_{i-2}')$

Solution: We apply the method to the test equation $y' = \lambda y$.

(i) we get the characteristic equation $(k = 2)$ as

$$\xi^2 - \xi - \frac{\lambda h}{2}(3\xi - 1) = 0$$

or

$$\xi^2 - (1 + \frac{3\lambda h}{2})\,\xi + \frac{\lambda h}{2} = 0$$

Substituting $\xi = \dfrac{1+z}{1-z}$, we get

$$\left(\frac{1+z}{1-z}\right)^2 - \left(1 + \frac{3\lambda h}{2}\right)\left(\frac{1+z}{1-z}\right) + \frac{\lambda h}{2} = 0$$

or, $$(1+z)^2 - \left(1 + \frac{3\lambda h}{2}\right)(1 - z^2) + \frac{\lambda h}{2}(1-z)^2 = 0$$

or, $$(2 + 2\lambda\,h)\,z^2 + (2 - \lambda\,h)\,z + [-\lambda\,h] = 0$$

or, $$v_0\,z^2 + v_1\,z + v_2 = 0.$$

where, $v_0 = 2 + 2\lambda h$, $\quad v_1 = 2 - \lambda h$, $\quad v_2 = -\lambda h$.

We require that

$$\nu_0 > 0, \quad \nu_1 > 0 \quad \nu_2 > 0 \quad \text{when} \quad \lambda h < 0$$

We find that ν_0 and ν_1 are always positive.

Now $\nu_2 > 0 \Rightarrow 2(1+\lambda h) > 0$ or, $\lambda h > -1$

Hence, interval of absolute stability is $]-1, 0[$

(ii) we obtain the characteristic equation $(k = 3)$ as

$$(\xi^3 - \xi^2) - \frac{\lambda h}{12}(23\,\xi^3 - 16\,\xi + 5) = 0$$

or, $$\xi^3 - \left(1 + \frac{23H}{12}\right)\xi^2 + \frac{16H}{12}\,\xi - \frac{5H}{12} = 0$$

where, $H = \lambda h$.

Substituting $\xi = \dfrac{1+z}{1-z}$, we get

$$(1+z)^3 - \left(1 + \frac{23}{12}H\right)(1+z)^2\,(1-z) + \frac{16H}{12}(1+z)\,(1-z)^2 - \frac{5H}{12}(1-z)^3 = 0$$

or, $$(1+3z+3z^2+z^3) - \left(1+\frac{23}{12}H\right)(1+z-z^2-z^3) + \frac{16H}{12}(1-z-z^2+z^3) - \frac{5H}{12}(1-3z+3z^2-z^3) = 0$$

or, $$\left(2+\frac{44H}{12}\right)z^3 + \left(4-\frac{2H}{3}\right)z^2 + (2-2H)\,z - H = 0$$

Comparing with $\nu_0 z^3 + \nu_1 z^2 + \nu_2 z + \nu_3 = 0$, we get

$\nu_0 = \frac{1}{3}(6+11H)$, $\nu_1 = \frac{2}{3}(6-H)$, $\nu_2 = 2(1-H)$, $\nu_3 = -H$. Since $\lambda h < 0$, we have $H < 0$. We find that ν_1, ν_2, ν_3 are always positive. From $\nu_0 > 0$, we get the condition $H > -6/11$. We also find that $\nu_1\,\nu_2 - \nu_0\,\nu_3 = 5H^2 - (22/3)H + 8$, which is positive for all $H < 0$. Hence, the interval of absolute stability is $]-6/11, 0[$.

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Example 11: Find the interval of absolute stability of the methods.

(i) $y'_{i+1} = y_i + \frac{h}{2}[\,y'_{i+1} + y'_i]$

(ii) $y'_{i+1} = y_i + \frac{h}{12}[\,5y'_{i+1} + 8y'_i - y'_{i-1}]$

Solution:

(i) We have $k = 1$. The characteristic equation is obtain as

$$(\xi - 1) - \frac{\lambda h}{2}(\xi + 1) = 0$$

Solving for ξ, we get

$$\xi = \frac{1+\lambda h/2}{1-\lambda h/2}$$

For $\lambda h < 0$, $|\,\xi\,| < 1$ for all λh. Hence, the method is absolutely stable for all h. The interval of absolute stability is $]-\infty, 0[$ and therefore the method is A-stable.

(ii) We have $k = 2$. The characteristic equation is given by

$$\xi^2 - \xi = \frac{\lambda h}{12}(5\,\xi^2 + 8\,\xi - 1)$$

or, $$\left(1-\frac{5H}{12}\right)\xi^2-\left(1+\frac{2H}{3}\right)\xi+\frac{H}{12}=0$$

where $H=\lambda h$.

Substituting $\xi=\dfrac{1+z}{1-z}$, we get

$$\left(1-\frac{5H}{12}\right)(1+z)^2-\left(1+\frac{2H}{3}\right)(1+z)(1-z)+\frac{H}{12}(1-z)^2=0$$

or, $$\left(1-\frac{5H}{12}\right)(1+2z+z^2)-\left(1+\frac{2H}{3}\right)(1-z^2)+\frac{H}{12}(1-2z+z^2)=0$$

or, $$\left(2+\frac{H}{3}\right)z^2+(2-H)z-H=0$$

Comparing with $\nu_0 z^2+\nu_1 z+\nu_2=0$, we get

$$\nu_0=2+\frac{H}{3},\quad \nu_1=2-H,\quad \nu_2=-H$$

Since $\lambda<0$, we find that $H<0$. Now ν_1 an ν_2 are both positive for $H<0$. $\nu_0>0$ if $H>-6$. Hence, the interval of absolute stability is $]-6, 0[$.

***

Example 12: Show that the Milne-Simpson method

$$y_{n+1}=y_{n-1}+\frac{h}{3}(y'_{n+1}+4y'_n+y'_{n-1})$$

is not absolutely stable for any h.

Solution: Here $k=2$. The characteristic equations is obtained as

$$\xi^2-1-\frac{\lambda h}{3}(\xi^2+4\xi+1)=0$$

or, $$\left(1-\frac{H}{3}\right)\xi^2-\frac{4H}{3}\xi-\left(1+\frac{H}{3}\right)=0,\quad H=\lambda h$$

Note that the roots of $\rho(\xi)=\xi^2-1=0$ are ±1. Since two roots of $\rho(\xi)=0$ are simple and of modulus 1, the method is conditionally stable.

Alternatively

Substitute $\xi=\dfrac{1+z}{1-z}$ in the characteristic equation and obtain

$$\left(1-\frac{H}{3}\right)(1+z)^2-\frac{4H}{3}(1+z)(1-z)-\left(1+\frac{H}{3}\right)(1-z)^2=0$$

$$\left(1-\frac{H}{3}\right)(1+2z+z^2)-\frac{4H}{3}(1-z^2)-\left(1+\frac{H}{3}\right)(1-2z+z^2)=0$$

or, $$\frac{2H}{3}z^2+4z-\frac{2H}{3}=0$$

Comparing with $\nu_0 z^2+\nu_1 z+\nu_2=0$, we get

$$\nu_0=\frac{2H}{3},\quad \nu_1=4,\quad \nu_2=-\frac{2H}{3}.$$

Since $\lambda<0$, $H<0$. For $H<0$, ν_1 and ν_2 are always positive. However, ν_0 is not positive for any $H<0$. Hence, the method is not absolutely stable for any $H<0$.

***

And now some exercises for you.

E12) Determine the interval of absolute stability for the method

$$y_{i+1}=y_{i-1}+\frac{h}{3}(7y'_i-2y'_{i-1}+y'_{i-2})$$

when applied to the test equation $y'=\lambda y,\ \lambda<0$.

E13) Show that the method

$$y_{i+1} = 8y_i + 9y_{i-1} + \frac{h}{3}(17y'_i + 14y'_{i-1} - y'_{i-2})$$

is not stable when applied to the test equation $y' = \lambda y$, $\lambda < 0$.

E14) Determine the interval of absolute stability for the method

$$y_{i+1} = \frac{9}{8}y_i - \frac{1}{8}y_{i-2} + \frac{3h}{8}(y'_{i+1} + 2y'_i - y'_{i-1})$$

when applied to the test equation $y' = \lambda y$, $\lambda < 0$.

E15) Show that the method

$$y_{i+1} = \frac{4}{3}y_i - \frac{1}{3}y_{i-1} + \frac{2h}{3}y'_{i+1}$$

is A-stable when applied to the test equation $y' = \lambda y$, $\lambda < 0$.

We now end the unit by giving a summary of what we have covered in it.

9.5 SUMMARY

In this unit, we have learnt the following points:

1. In **multistep** methods for solving the IVP(1)

   $$y' = f(x, y), \quad y(x_0) = y_0$$

   the solution value y_{i+1} at the nodal point x_{i+1} is obtained by using previously, calculated k values y_{i-m} at the nodal points $x_{i-m}, m = 0, 1, \ldots, k-1$. Such a method is called an **explicit multistep method.** If the method also uses the solution value y_{i+1}, then the method is called **implicit multistep method.**

2. The multistep methods are not self starting. y_i is called the initial value and $y_{i-1}, y_{i-2}, \ldots, y_{i-k+1}$ are called the starting values. The starting values are to be calculated using some singlestep method before the multistep method can be used.

3. A general multistep method or a **k-step multistep method** can be written in the form

   $$y_{i+1} = a_1 y_i + a_2 y_{i-1} + \cdots + a_k y_{i-k+1} + h(b_0 y'_{i+1} + b_i y'_i + \cdots + b_k y'_{i-k+1})$$

   The method is explicit if $b_0 = 0$ and implicit if $b_0 \neq 0$. The unknowns a_i's and b_i's are obtained using (i) interpolation methods (ii) method of undetermined parameters.

   We have the following form of a few important class of methods

   (a) **Adams-Bashforth method:** $a_1 = 1, a_2 = a_3 = \cdots a_k = 0, b_0 = 0$. These methods are explicit methods and are of order k.

   (b) **Nystrom method:** $a_2 = 1, a_1 = a_3 = \cdots = a_k = 0, b_0 = 0$. These methods are explicit methods and are of order k.

   (c) **Adams-moulton methods:** $a_1 = 1, a_2 = a_3 = \cdots = a_k = 0, b_0 \neq 0$. These methods are implicit methods and are of order $k+1$.

   (d) **Milne-Simpson methods:** $a_2 = 1, a_1 = a_3 = \cdots = a_k = 0, b_0 \neq 0$. These methods are implicit methods and are of order $k+1$ if k is odd and of order $k+2$ if k is even.

4. An explicit method is called a **predictor** and an implicit method is called a **corrector.** We use the explicit method to predict the value $y_{i+1}^{(p)}$ for y_{i+1} and use the corrector, which uses $y_{i+1}^{(p)}$ iteratively to obtain y_{i+1}, the improved value of y_{i+1}. After the required number of corrector iterations, we take $y_{i+1} = y_{i+1}^{(c)}$.

5. If we apply the multistep method (44) to the test equation $y' = \lambda y$, we obtain the **characteristic equation**

$$\rho(\xi) - \lambda h\,\sigma(\xi) = 0$$

where

$$\rho(\xi) = \xi^k - a_1\xi^{k-1} - \cdots - a_k$$

$$\sigma(\xi) = b_0\xi^k + b_1\xi^{k-1} + \cdots + b_k$$

Let $\xi_1, \xi_2, \ldots, \xi_k$ be the roots of the **reduced characteristic equation** $\rho(\xi) = 0$ and $\xi_{1h}, \xi_{2h}, \ldots, \xi_{kh}$ be the roots of the characteristic equation. Then, the method is

(i) **stable:** if $|\xi_i| < 1,\ i \neq 1$ and $\xi_1 = 1$.

(ii) **unstable:** if $|\xi_i| > 1$ for some i, or if there is a multiple root of $\rho(\xi) = 0$ of modulus one.

(iii) **conditionally stable (weakly stable):** if ξ_i's are simple and if more than one of these roots have modulus one.

The method is called

absolutely stable: if $\lambda < 0$ and $|\xi_{ih}| < 1$ for all i. The set of values of λh which satisfy this condition give the **interval of absolute stability.** If this interval is $]-\infty,\ 0[$, the method is called **A-stable.**

relatively stable: if $\lambda > 0$ and $|\xi_{ih}| < e^{\lambda h}$ for all i. The set of values of λh which satisfy this condition give the **interval of relative stability.**

The order of a stable k-step explicit method cannot exceed k and the order of a stable k-step implicit method cannot exceed $k + 1$ if k is odd and $k + 2$ if k is even.

9.6 SOLUTIONS/ANSWERS

E1) $TE = -\dfrac{3}{80}h^5 y^{v}(x_i) + O(h^6)$; order: 4

E2) $x_0 = 0,\ y_0 = 1,\ y_0' = -1,\ y_0'' = 3$; starting value: $y_1 \approx y(0.2) = 0.86$;
We obtain from the multistep method
$f_0 = y_0' = -1,\ f_1 = y_1' = -0.5396,\ y_2 \approx y(0.4) = 0.79812$;
$f_2 = y_2' = -0.236996,\ y_3 \approx y(0.6) = 0.780981$.

E3) $x_0 = 1,\ y_0 = 0,\ y_0' = 1,\ y_0'' = 2,\ y_0''' = 4$; Starting value:
$y_1 \approx y(1.1) = 0.110667$
$x_1 = 1.1,\ y_1 = 0.110667,\ y_1' = 1.222247,\ y_1'' = 2.861296,$
$y_1''' = 5.621078$; starting value $y_2 \approx y(1.2) = 0.248135$.
From the multistep method we get
$f_0 = 1,\ f_1 = 1.222247,\ f_2 = 1.501571;\ y_3 \approx y(1.3) = 0.414637$;
$f_3 = 1.861924;\ y_4 \approx y(1.4) = 0.622223$;
$f_4 = 2.347162;\ y_5 \approx y(1.5) = 0.886405$.

E4) Starting values:
$x_0 = 1,\ y_0 = 1,\ K_1 = 0.4,\ K_2 = 0.672,\ y_1 \approx y(1.2) = 1.536$;
$x_1 = 1.2,\ y_1 = 1.536,\ K_1 = 0.779059,\ K_2 = 1.534912,$
$y_2 \approx (1.4) = 2.692985$.
From the multistep method, we get
$f_0 = 2,\ f_1 = 3.895296,\ f_2 = 9.945153,\ y_3 \approx y(1.6) = 5.791032$;
$f_3 = 39.327084,\ y \approx y(1.8) = 19.979290$;
$f_4 = 419.151320,\ y_5 \approx y(2.0) = 196.814380$.

E5) Starting value:
$x_0 = 1,\ y_0 = 2,\ K_1 = 0.8,\ K_2 = 0.93,\ K_3 = 0.9495,$
$K_4 = 1.10485,\ y_1 \approx y(1.1) = 2.943975.$
From the multistep method, we get
$$(1-h)y_{i+2} = y_i + \frac{h}{3}(2x_{i+2} + 4f_{i+1} + f_i),\ i = 0,1,2$$
We obtain
$x_0 = 1,\ y_0 = 2,\ f_0 = 8,\ x_1 = 1.1,$
$y_1 = 2.943975,\ f_1 = 11.031925,\ x_2 = 1.2$
$y_2 \approx y(1.2) = 4.241767;\ f_2 = 15.125301,\ x_3 = 1.3,$
$y_3 \approx y(1.3) = 6.016755;\ f_3 = 20.650264,\ x_4 = 1.4,$
$y_4 \approx y(1.4) = 8.436273.$

E6) $a_1 = 1,\ b_1 = \frac{7}{3},\ b_2 = -\frac{2}{3},\ b_3 = \frac{1}{3}.$

$TE = \frac{1}{3}h^4 y^{iv}(x_i) + O(h^5);$ order : 3.

E7) $a_1 = -8,\ a_2 = 9,\ b_1 = \frac{17}{3},\ b_2 = \frac{14}{3},\ b_3 = -\frac{1}{3}.$

$TE = \frac{1}{9}h^5 y^{v}(x_i) + O(h^6);$ order : 4.

E8) $a_1 = 1,\ b_0 = \frac{5}{12},\ b_1 = \frac{8}{12},\ b_2 = -\frac{1}{12}.$

$TE = -\frac{h^4}{24} y^{iv}(x_i) + O(h^5);$ order : 3

E9) $a_1 = \frac{4}{3},\ a_2 = -\frac{1}{3},\ b_0 = \frac{2}{3}.$

$TE = -\frac{1}{9}h^3 y'''(x_i) + O(h^4);$ order : 2

E10) Starting value:
$x_0 = 1,\ y_0 = 1,\ y_0' = 1.841471,\ y_0'' = 2.381773,$
$y_1 \approx y(1.2) = 1.415930.$
From the given P-C set, we get
$f_0 = 1.841471,\ x_1 = 1.2,\ f_1 = 2.403962,\ x_2 = 1.4,$
$y_2^{(0)} = 1.952972;\ f(x_2, y_2^{(0)}) = 2.880828$
$y_2^{(1)} = 1.944409;\ f(x_2, y_2^{(1)}) = 2.875424$
$y_2^{(2)} \approx y(1.4) = y_2 = 1.943869,\ f_2 = 2.875081;\ x_3 = 1.6$
$y_3^{(0)} = 2.565997;\ f(x_3, y_3^{(0)}) = 3.110332,$
$y_3^{(1)} = 2.542410;\ f(x_3, y_3^{(1)}) = 3.106377,$
$y_3^{(2)} \approx y(1.6) = y_3 = 2.542015.$

E11) Starting values
$y_1 \approx y(0.2) = 1.2,\quad y_2 \approx y(0.4) = 1.496,$
$y_3 \approx y(0.6) = 1.9756.$
Using the P-C set, we get
$y_4^{(0)} = 3.4235,\quad y_4^{(1)} = 3.6167,\quad y_4^{(2)} = 3.7074$
Therefore, $y_4 \approx y(0.8) = 3.7074.$

$y_5^{(0)} = 9.0140,\ \ y_5^{(1)} = 11.5792,\ \ y_5^{(2)} = 15.1009$

Therefore, $y_5 \approx y(1.0) = 15.1009$

E12) Characteristic equation is obtained as

$$\xi^3 - \frac{7H}{3}\xi^2 - \left(1 - \frac{2H}{3}\right)\xi - \frac{H}{3} = 0, \quad H = \lambda h < 0.$$

Substituting $\xi = (1+z)/(1-z)$, we obtain

$$v_0 z^3 + v_1 z^2 + v_2 z + v_3 = 0$$

where

$$v_0 = \frac{10H}{3},\ v_1 = 4 + \frac{2H}{3},\ v_2 = 4 - 2H,\ v_3 = -2H$$

Since $v_0 < 0$ for all $H < 0$, method is not stable for any H.

E13) The roots of the reduced characteristic equation

$$\rho(\xi) = \xi^2 + 8\xi - 9 = 0$$

are $\xi_1 = 1,\ \xi_2 = -9$. Since $|\xi_2| > |$, the method is unstable.

E14) Characteristic equation is obtained as

$$\left(1 - \frac{3H}{8}\right)\xi^3 - \left(\frac{9}{8} + \frac{3H}{4}\right)\xi^2 + \frac{3H}{8}\xi + \frac{1}{8} = 0, \quad H = \lambda h < 0.$$

Substituting $\xi = (1+z)/(1-z)$, we get

$$v_0 z^3 + v_1 z^2 + v_2 z + v_3 = 0$$

where

$$v_0 = \frac{1}{4}(8+3H),\ v_1 = \frac{1}{4}(18-3H),\ v_2 = \frac{3}{4}(2-3H),\ v_3 = -\frac{3H}{4}$$

and $D = v_1 v_2 - v_0 v_3 = \frac{3}{4}[9 - 13H + 3H^2]$. Interval of absolute stability is $\left(-\frac{8}{3}, 0\right)$.

E15) The characteristic equation is obtained as

$$\left(1 - \frac{2H}{3}\right)\xi^2 - \frac{4}{3}\xi + \frac{1}{3} = 0, \quad H = \lambda h < 0.$$

Substituting $\xi = (1+z)/(1-z)$, we get

$$v_0 z^2 + v_1 z + v_2 = 0$$

where

$$v_0 = \frac{4}{3} - \frac{H}{3},\ v_1 = \frac{2}{3} - \frac{2H}{3},\ v_2 = -\frac{H}{3}$$

since v_0, v_1, v_2 are all positive for all $H < 0$, the method is A-stable.

—x—

PRACTICAL ASSIGNMENTS

Session 2:

1. Write a program to solve the initial value problem $y' = f(x, y)$, $y(x_0) = y_0$, using Euler method as predictor and Backward Euler method as corrector. Use the corrector three times. Input the step length h and the number of steps of integration.

2. Write a program to solve the initial value problem $y' = f(x, y)$, $y(x_0) = y_0$, using Euler method as predictor and Trapezoidal method as corrector. Use the corrector three times. Input the step length h and the number of steps of integration.

3. Write a program to solve the initial value problem $y' = f(x, y)$, $y(x_0) = y_0$, using Milne's predictor-corrector method. Required starting values are to be computed using the Euler method.

4. Write a program to solve the initial value problem $y' = f(x, y)$, $y(x_0) = y_0$, using Milne's predictor-corrector method. Required starting values are to be computed using the Taylor series method of second order. Test your program on Example 9.

UNIT 10 SECOND ORDER BOUNDARY VALUE PROBLEMS

Structure **Page No.**

10.1 INTRODUCTION

In Units 8 and 9, you have studied numerical methods for solving initial value problems where the conditions are given at a point called the initial point. The solution is obtained in some given interval in a step-by-step manner, that is, the solution is obtained at one nodal point at a time.

In many physical problems, the conditions are given at two points say $x = a$ and $x = b$. The differential equation together with these conditions is called a boundary value problem. The points $x = a$ and $x = b$ are called boundary points and the solution is obtained at nodal points inside the interval $[a, b]$. In this unit we shall consider second order boundary value problems and discuss two numerical methods viz; finite difference and shooting methods for solving them.

In Sec.10.2 we shall start with introducing three kinds of boundary value problems and then discuss the **finite difference methods** of solving them. **Shooting method** in which the given boundary value problem is converted to an initial value problem and then solution is obtained in a step-by-step manner, will be discussed in Sec.10.3.

Objectives

After studying this unit you should be able to

- identify a second order boundary value problem of first, second and third kind;
- obtain the solution of a given boundary value problem using finite difference methods;
- obtain the solution of a given boundary value problem using shooting method.

10.2 FINITE DIFFERENCE METHODS

You may recall that in Unit 1 we introduced you to the boundary conditions associated with a differential equation and boundary value problems. Here we give you three kinds of boundary conditions for a general second order ODE.

Consider the second order differential equation

$$y'' = f(x, y, y') \tag{1}$$

For Eqn.(1) we can have the following three types of boundary conditions:

Boundary conditions of the first kind

$$y(a) = r_1, \quad y(b) = r_2 \tag{2}$$

where r_1 and r_2 are given constants.

Boundary conditions of the second kind

$$y'(a) = r_1, \quad y'(b) = r_2 \tag{3}$$

where r_1 and r_2 are given constants.

Boundary conditions of the third kind (mixed conditions)

$$\begin{aligned} a_0\, y(a) - a_1\, y'(a) &= r_1 \\ b_0\, y(b) + b_1\, y'(b) &= r_2 \end{aligned} \tag{4}$$

where a_0, a_1, b_0, b_1, r_1 and r_2 are given constants such that

$$a_0 a_1 \geq 0, \qquad |a_0| + |a_1| \neq 0$$
$$b_0 b_1 \geq 0, \qquad |b_0| + |b_1| \neq 0$$

The differential Eqn.(1) together with the boundary conditions (2) or (3) or (4) is called a **boundary value problem (bvp)** of the second order.

The points $x = a$ and $x = b$ are the end points of some interval in which the solution is to be obtained.

We shall now discuss finite difference methods of solving these bvp's.

We divide the interval $[a, b]$ into $N+1$ subintervals of equal length h by the points

$$a = x_0 < x_1 < \ldots < x_{N+1} = b$$

where

$$x_i = x_0 + ih, \quad i = 0, 1, \ldots, N+1$$

and

$$h = \frac{b-a}{N+1}.$$

The points $x_0, x_1, \ldots, x_{N+1}$ are called nodal points and we determine the solution values $y_i \approx y(x_i)$ at these points. For a boundary value problem of the first kind, we obtain solution values at the interior points $x_1, x_2, \ldots, x_N$. Whereas, for the boundary value problem of the second or the third kind we determine the solution values at the interior points as well as at the boundary points x_0 and x_{N+1}.

In finite difference method, we have the following three steps:

Step 1: Discretize the differential equation at each nodal point, that is

$$y_i'' = f(x_i, y_i, y_i'), \; i = 1, 2, \ldots, N \tag{5}$$

For boundary value problem of second or third kind, we discretize the differential equations at x_0 and / or x_{N+1} also.

Step 2: Replace the derivatives y' and y'' in the differential equation and the boundary conditions by the difference approximations. We know that

$$y''(x_i) = \frac{1}{h^2}[y(x_{i-1}) - 2y(x_i) + y(x_{i+1})] - \frac{h^2}{12} y^{iv}(\xi), \; x_{i-1} < \xi < x_{i+1}. \tag{6}$$

$$y'(x_i) = \frac{1}{h}[y(x_{i+1}) - y(x_i)] - \frac{h}{2} y''(\xi), \quad x_i < \xi < x_{i+1}. \tag{7i}$$

$$y'(x_i) = \frac{1}{h}[y(x_i) - y(x_{i-1})] + \frac{h}{2} y''(\xi), \quad x_{i-1} < \xi < x_i. \tag{7ii}$$

$$y'(x_i) = \frac{1}{2h}[y(x_{i+1}) - y(x_{i-1})] - \frac{h^2}{6} y'''(\xi), \quad x_{i-1} < \xi < x_{i+1}. \tag{7iii}$$

The approximations (6) and (7iii) are of second order, whereas the approximations (7i) and (7ii) are of first order. Thus we can replace y'' and y' in the differential equations and the boundary conditions by the difference approximations

$$y_i'' = \frac{1}{h^2}(y_{i-1} - 2y_i + y_{i+1}) \tag{8}$$

$$y_i' = \frac{1}{h}(y_{i+1} - y_i) \tag{9i}$$

$$y_i' = \frac{1}{h}(y_i - y_{i-1}) \tag{9ii}$$

$$y_i' = \frac{1}{2h}(y_{i+1} - y_{i-1}) \tag{9iii}$$

When we use the approximations (8) and (9iii), we obtain a second order finite difference method. When we use the approximations (8) and (9i) or (9ii), we obtain a first order finite difference method.

Note that the lower of the orders of the difference approximations used for y' and y'' gives the order of the finite difference method.

For boundary value problems of the first kind, we obtain a system of N algebraic equations in N unknowns $y_1, y_2, \ldots, y_N$, since $y_0 = r_1$ and $y_{N+1} = r_2$ are known. For boundary value problems of the second and third kinds, we obtain a system of $N+2$ algebraic equations in $N+2$ unknowns $y_0, y_1, \ldots, y_{N+1}$.

If the function $f(x, y, y')$ is linear in y and y' we obtain a system of linear equations. If $f(x, y, y')$ is non-linear in y and / or y', we obtain a system of non-linear equations.

Step 3: Solve the resulting system of algebraic equations by direct or iterative methods. For system of non-linear equations, we generally use Newton-Raphson method.

We illustrate the method with the help of examples.

Example 1: Solve the boundary value problem

$$y'' - 4y' + 3y = 0$$
$$y(0) = 1, \quad y(1) = 0$$

using second order finite difference method with (i) $h = \frac{1}{2}$, (ii) $h = \frac{1}{3}$. Compare the numerical results with the exact solution $y(x) = \frac{1}{e^2 - 1}\left[e^{x+2} - e^{3x}\right]$

Solution: (i) For $h = \frac{1}{2}$, we get three nodal points $x_0 = 0$, $x_1 = \frac{1}{2}$ and $x_2 = 1$. The boundary conditions give $y_0 = 1$ and $y_2 = 0$. We need the solution value $y_1 \approx y(x_1)$. From the given differential equations, we get

$$y_i'' - 4y_i' + 3y_i = 0, \qquad i = 1 \tag{10}$$

Using the approximations (8) and (9iii) for $i = 1$, we obtain from Eqn.(10)

$$\frac{1}{h^2}(y_0 - 2y_1 + y_2) - \frac{4}{2h}(y_2 - y_0) + 3y_1 = 0$$

Using $h = \frac{1}{2}$, $y_0 = 1$ and $y_2 = 0$, we get

$$4\,(1 - 2y_1 + 0) - 4(0 - 1) + 3y_1 = 0$$

or, $\quad -5y_1 + 8 = 0$

or, $\quad y_1 \approx y\left(\frac{1}{2}\right) = 1.6$

The exact solution is $y\left(\frac{1}{2}\right) = 1.20531$

(ii) For $h = \frac{1}{3}$, we get four nodal points $x_0 = 0$, $x_1 = \frac{1}{3}$, $x_2 = \frac{2}{3}$ and $x_3 = 1$. The boundary conditions give $y_0 = 1$ and $y_3 = 0$. We need the solution values $y_1 \approx y(x_1)$ and $y_2 \approx y(x_2)$. We discretize the differential equation at the points x_1 and x_2. We get

$$y_i'' - 4y_i' + 3y_i = 0, \quad i = 1, 2$$

using the approximations (8) and (9iii), we get

$$\frac{1}{h^2}(y_{i-1} - 2y_i + y_{i+1}) - \frac{4}{2h}(y_{i+1} - y_{i-1}) + 3y_i = 0, \; i = 1, 2$$

Using $h = \frac{1}{3}$, we get

$$9(y_{i-1} - 2y_i + y_{i+1}) - 6(y_{i+1} - y_{i-1}) + 3y_i = 0, \; i = 1, 2 \tag{11}$$

From (11), we obtain

$i = 1:\quad 9(y_0 - 2y_1 + y_2) - 6(y_2 - y_0) + 3y_1 = 0$

$i = 2:\quad 9(y_1 - 2y_2 + y_3) - 6(y_3 - y_1) + 3y_2 = 0$

Using the boundary conditions $y_0 = 1$ and $y_3 = 0$, we get the system of equations

$$9(1 - 2y_1 + y_2) - 6(y_2 - 1) + 3y_1 = 0$$
$$9(y_1 - 2y_2 + 0) - 6(0 - y_1) + 3y_2 = 0$$

or,
$$-15y_1 + 3y_2 + 15 = 0$$
$$15y_1 - 15y_2 = 0$$

whose solution is $y_1 = y_2 = 1.25$

Exact solution is $y\left(\frac{1}{3}\right) = 1.188591,\ y\left(\frac{2}{3}\right) = 1.096071$.

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Example 2: Solve the boundary value problem

$$(1 + x^2)y'' + 4x\,y' + 2y = 2$$
$$y(0) = 0, \qquad y(1) = \frac{1}{2}$$

using the

(i) first order finite difference method (approximations (8) and (9i))

(ii) second order finite difference method (approximations (8) and (9iii)) with $h = \frac{1}{3}$ *in each case.*

Solution: For $h = \frac{1}{3}$, we have four nodal points $x_0 = 0,\ x_1 = \frac{1}{3},\ x_2 = \frac{2}{3}$ and $x_3 = 1$. The boundary conditions give $y_0 = 0,\ y_3 = \frac{1}{2}$. We need the solution values y_1 and y_2. From the given differential equation we get

$$(1 + x_i^2)y_i'' + 4x_i\,y_i' + 2y_i = 2,\quad i = 1, 2 \tag{12}$$

(i) Using the approximations (8) and (9i) in (12), we get

$$\frac{1}{h^2}(1 + x_i^2)\,(y_{i-1} - 2y_i + y_{i+1}) + \frac{4x_i}{h}(y_{i+1} - y_i) + 2y_i = 2,\ i = 1, 2$$

Using $h = \frac{1}{3}$, we obtain

$$9(1 + x_i^2)\,(y_{i-1} - 2y_i + y_{i+1}) + 12x_i\,(y_{i+1} - y_i) + 2y_i = 2$$

For $i = 1$ and $i = 2$, we get

$i = 1:\quad 9(1 + x_1^2)\,(y_0 - 2y_1 + y_2) + 12x_1(y_2 - y_1) + 2y_1 = 2$

$i = 2:\quad 9(1 + x_2^2)\,(y_1 - 2y_2 + y_3) + 12x_2(y_3 - y_2) + 2y_2 = 2$

Using $x_1 = \frac{1}{3},\ x_2 = \frac{2}{3},\ y_0 = 0,\ y_3 = \frac{1}{2}$, we get

$$9\left(1 + \frac{1}{9}\right)(0 - 2y_1 + y_2) + 12\left(\frac{1}{3}\right)(y_2 - y_1) + 2y_1 = 2$$
$$9\left(1 + \frac{4}{9}\right)\left(y_1 - 2y_2 + \frac{1}{2}\right) + 12\left(\frac{2}{3}\right)\left(\frac{1}{2} - y_2\right) + 2y_2 = 2$$

or,
$$-22y_1 + 14y_2 = 2$$
$$13y_1 - 32y_2 = -\frac{17}{2}$$

whose solution is

$$y_1 \approx y\left(\frac{1}{3}\right) = 0.105364,\quad y_2 \approx y\left(\frac{2}{3}\right) = 0.308429.$$

(ii) Using the approximations (8) and (9iii), we obtain from Eqn.(12)

$$\frac{1}{h^2}(1+x_i^2)(y_{i-1}-2y_i+y_{i+1})+\frac{4x_i}{2h}(y_{i+1}-y_{i-1})+2y_i=2,\ i=1,2.$$

Using $h=\frac{1}{3}$, we get

$$9(1+x_i^2)(y_{i-1}-2y_i+y_{i+1})+6x_i(y_{i+1}-y_{i-1})+2y_i=2,\ i=1,2$$

For $i=1$ and $i=2$, we get

$\mathbf{i=1}$: $\quad 9(1+x_1^2)(y_0-2y_1+y_2)+6x_1(y_2-y_0)+2y_1=2$

$\mathbf{i=2}$: $\quad 9(1+x_2^2)(y_1-2y_2+y_3)+6x_2(y_3-y_1)+2y_2=2$

Using $x_1=\frac{1}{3},\ x_2=\frac{2}{3},\ y_0=0,\ y_3=\frac{1}{2}$, we get

$$9\left(1+\frac{1}{9}\right)(0-2y_1+y_2)+6\left(\frac{1}{3}\right)(y_2-0)+2y_1=2$$

$$9\left(1+\frac{4}{9}\right)\left(y_1-2y_2+\frac{1}{2}\right)+6\left(\frac{2}{3}\right)\left(\frac{1}{2}-y_1\right)+2y_2=2$$

or, $\quad -18y_1+12y_2=2$

$$9y_1-24y_2=-\frac{13}{2}$$

whose solution is $y_1\approx y\left(\frac{1}{3}\right)=0.092593,\ y_2\approx y\left(\frac{2}{3}\right)=0.305556$.

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Example 3: Solve the boundary value problem

$$y''=xy+1$$

$$y(0)+y'(0)=1\ ,\qquad y(1)=1$$

using the second order finite difference method with $h=\frac{1}{2}$.

Solution: For $h=\frac{1}{2}$, we have the nodal points $x_0=0,\ x_1=\frac{1}{2}$ and $x_2=1$. From the boundary conditions we get $y_2=1$. We need to determine y_0 and y_1. From the given differential equation we get

$$y_i''=x_i\,y_i+1\ ,\qquad i=0,1$$

Using the approximation (8), we obtain

$$\frac{1}{h^2}(y_{i-1}-2y_i+y_{i+1})=x_i\,y_i+1,\ \ i=0,1$$

Using $h=\frac{1}{2}$, we get

$\mathbf{i=0}$: $\quad 4(y_{-1}-2y_0+y_1)=x_0\,y_0+1$

$\mathbf{i=1}$: $\quad 4(y_0-2y_1+y_2)=x_1\,y_1+1$

Using $x_0=0,\ x_1=\frac{1}{2}$ and $y_2=1$, we get

$$4(y_{-1}-2y_0+y_1)=1$$

$$4(y_0-2y_1+1)=\frac{1}{2}y_1+1$$

or, $\quad 4y_{-1}-8y_0+4y_1=1$

$$4y_0-\frac{17}{2}y_1=-3 \qquad (13)$$

Using the approximations (9iii) with $i=0$, we obtain from the first boundary condition

$$y_0+\frac{1}{2h}(y_1-y_{-1})=1$$

for $h = \frac{1}{2}$, we get

$$y_0 + y_1 - y_{-1} = 1$$

or, $$y_{-1} = y_0 + y_1 - 1 \tag{14}$$

Eliminating y_{-1} from Eqn.(13), we get the system of equation

$$-4y_0 + 8y_1 = 5$$

$$4y_0 - \frac{17}{2}y_1 = -3$$

whose solution is $y_0 \approx y(0) = -9.25$, $y_1 \approx y(1/2) = -4$.

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Example 4: Solve the boundary value problem

$$y'' - 3y' + 2y = 2$$

$$y(0) - y'(0) = -1 \;, \qquad y(1) + y'(1) = 1$$

using the second order finite difference method with $h = \frac{1}{2}$.

Solution: For $h = \frac{1}{2}$, we have the nodal points $x_0 = 0$, $x_1 = \frac{1}{2}$ and $x_2 = 1$. Since the boundary value problem is of third kind, we need to determine the solution values y_0, y_1 and y_2.

From the given differential equation, we get

$$y_i'' - 3y_i' + 2y_i = 2 \quad , \qquad i = 0,1,2$$

Using the approximations (8) and (9iii), we obtain

$$\frac{1}{h^2}(y_{i-1} - 2y_i + y_{i+1}) - \frac{3}{2h}(y_{i+1} - y_{i-1}) + 2y_i = 2$$

For $h = \frac{1}{2}$, we get

$$4(y_{i-1} - 2y_i + y_{i+1}) - 3(y_{i+1} - y_{i-1}) + 2y_i = 2$$

or, $$7y_{i-1} - 6y_i + y_{i+1} = 2$$

We obtain for $i = 0, 1$ and 2

$i = 0$: $\quad 7y_{-1} - 6y_0 + y_1 = 2$

$i = 1$: $\quad 7y_0 - 6y_1 + y_2 = 2$

$i = 2$: $\quad 7y_1 - 6y_2 + y_3 = 2$ (15)

Using the approximation (9iii), we obtain from the boundary conditions

$i = 0$: $\quad y_0 - \frac{1}{2h}(y_1 - y_{-1}) = -1$

$i = 2$: $\quad y_2 + \frac{1}{2h}(y_3 - y_1) = 1$

For $h = \frac{1}{2}$, we get

$$y_0 - y_1 + y_{-1} = -1$$

$$y_2 + y_3 - y_1 = 1$$

or, $$y_{-1} = -1 - y_0 + y_1$$

and $$y_3 = 1 + y_1 - y_2$$

Eliminating y_{-1} and y_3 from Eqn.(15), we obtain the system of equations.

$$7(-1 - y_0 + y_1) - 6y_0 + y_1 = 2$$

$$7y_0 - 6y_1 + y_2 = 2$$

$$7y_1 - 6y_2 + (1 + y_1 - y_2) = 2$$

or, $$-13y_0 + 8y_1 = 9$$

$$7y_0 - 6y_1 + y_2 = 2$$
$$8y_1 - 7y_2 = 1$$

whose solution can be obtained by using Gauss-elimination method. We can write

$$-y_0 + 0.615385\ y_1 = 0.692308$$
$$y_0 - 0.857143\ \ y_1 + 0.142857 = 0.285714$$
$$8y_1 - 7y_2 = 1$$

Eliminating y_0, we obtain

$$-.241758y_1 + .142857y_2 = .978022$$
$$8y_1 - 7y_2 = 1$$

Solving the above system of equations we obtain the solution

$$y_2 \approx y(1) = -14.680011,\ \ y_1 \approx y(1/2) = -12.720010,\ \ y_0 \approx y(0) = -8.520006.$$

***

Example 5: Using second order finite difference method solve the boundary value problem

$$y'' = 2y^3\ ,\ y(1) = 1\ ,\ y(2) = \frac{1}{2}$$

with (i) $h = \frac{1}{2}$ and (ii) $h = \frac{1}{3}$. The exact solution is $y(x) = \frac{1}{x}$.

Solution: (i) For $h = \frac{1}{2}$, we have the nodal points $x_0 = 1$, $x_1 = \frac{3}{2}$ and $x_2 = 2$. From the boundary conditions we get $y_0 = 1$ and $y_2 = \frac{1}{2}$. We need to determine y_1. From the given differential equation we get

$$y_i'' = 2y_i^3 \quad , \quad i = 1$$

Using the approximation (8), we get

$$\frac{1}{h^2}(y_{i-1} - 2y_i + y_{i+1}) = 2y_i^3$$

For $h = \frac{1}{2}$ and $i = 1$, we obtain

$$4(y_0 - 2y_1 + y_2) = 2y_1^3$$

or, $$4\left(y_1 - 2y_1 + \frac{1}{2}\right) = 2y_1^3$$

or, $$F(y_1) = 2y_1^3 + 8y_1 - 6 = 0$$

which is a non-linear equation in y_1. Using the Newton-Raphson method

$$y_1^{(m+1)} = y_1^{(m)} - \frac{F(y_1^{(m)})}{F'(y_1^{(m)})},\ m = 0, 1, \ldots$$
$$= y_1^{(m)} - \frac{2(y_1^{(m)})^3 + 8y_1^{(m)} - 6}{6(y_1^{(m)})^2 + 8}$$

Since $0.5 < y$ we may choose $y_1 \approx 0.75$.

With $y_1^{(0)} = 0.75$, we obtain

$$y_1^{(1)} = 0.75 - \frac{0.84375}{11.375} = 0.675824$$
$$y_1^{(2)} = 0.675824 - \frac{0.023941}{10.740428} = 0.673595$$
$$y_1^{(3)} = 0.673595 - \frac{0.000021}{10.722381} = 0.673593$$

Hence, $y_1 \approx y\left(\frac{3}{2}\right) = 0.673593$.

The exact solution is $y\left(\frac{3}{2}\right) = 0.666667$.

(ii) For $h = \frac{1}{3}$, we have the nodal points $x_0 = 1, x_1 = \frac{4}{3}, x_2 = \frac{5}{3}$ and $x_3 = 2$. From the boundary conditions, we get $y_0 = 1$ and $y_3 = \frac{1}{2}$. We need to determine y_1 and y_2.

From the given differential equation we have

$$y_i'' = 2y_i^3 \quad , \quad i = 1,2$$

Using the approximation (8), we write

$$\frac{1}{h^2}(y_{i-1} - 2y_i + y_{i+1}) = 2y_i^3 \quad i = 1,2$$

For $h = \frac{1}{2}$, $i = 1$ and 2, we obtain

$i = 1$: $\quad 4(y_0 - 2y_1 + y_2) = 2y_1^3$

$i = 2$: $\quad 4(y_1 - 2y_2 + y_3) = 2y_2^3$

Using the boundary conditions $y_0 = 0$ and $y_3 = \frac{1}{2}$, we get

$$4(1 - 2y_1 + y_2) = 2y_1^3$$

$$4\left(y_1 - 2y_2 + \frac{1}{2}\right) = 2y_2^3$$

Hence, we obtain a system of two non-linear equations

$$f_1(y_1, y_2) = 2y_1^3 + 8y_1 - 4y_2 - 4 = 0$$

$$f_2(y_1, y_2) = 2y_2^3 - 4y_1 + 8y_2 - 2 = 0$$

we use the Newton-Raphson method to solve this system. If you are not familiar with this method then you can refer to Appendix given at the end of the unit.

We have

$$\begin{bmatrix} y_{1,m+1} \\ y_{2,m+1} \end{bmatrix} = \begin{bmatrix} y_{1m} \\ y_{2m} \end{bmatrix} - \mathbf{J}_m^{-1} \begin{bmatrix} f_1(y_{1m}, y_{2m}) \\ f_2(y_{1m}, y_{2m}) \end{bmatrix}, \quad m = 0, 1, \ldots$$

where

$$\mathbf{J}_m = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{bmatrix}_m = \begin{bmatrix} 6y_{1m}^2 + 8 & -4 \\ -4 & 6y_{2m}^2 + 8 \end{bmatrix}$$

$$\mathbf{J}_m^{-1} = \frac{1}{D_m} \begin{bmatrix} 6y_{2m}^2 + 8 & 4 \\ 4 & 6y_{1m}^2 + 8 \end{bmatrix}$$

and

$$D_m = (6y_{1m}^2 + 8)\,(6y_{2m}^2 + 8) - 16$$

Since $\frac{1}{2} < y_1, y_2 < 1$, we may choose $y_{10} = 0.8$ and $y_{20} = 0.6$. We get

$$\begin{bmatrix} y_{11} \\ y_{21} \end{bmatrix} = \begin{bmatrix} 0.8 \\ 0.6 \end{bmatrix} - \frac{1}{104.2944} \begin{bmatrix} 10.16 & 4 \\ 4 & 11.84 \end{bmatrix} \begin{bmatrix} 1.024 \\ 0.032 \end{bmatrix} = \begin{bmatrix} 0.699018 \\ 0.557094 \end{bmatrix}$$

$$\begin{bmatrix} y_{12} \\ y_{22} \end{bmatrix} = \begin{bmatrix} 0.699018 \\ 0.557094 \end{bmatrix} - \frac{1}{91.810321} \begin{bmatrix} 9.862122 & 4 \\ 4 & 10.931575 \end{bmatrix} \begin{bmatrix} 0.046885 \\ 0.006472 \end{bmatrix}$$

$$= \begin{bmatrix} 0.693702 \\ 0.554281 \end{bmatrix}$$

$$\begin{bmatrix} y_{13} \\ y_{23} \end{bmatrix} = \begin{bmatrix} 0.693702 \\ 0.554281 \end{bmatrix} - \frac{1}{91.168012}\begin{bmatrix} 9.843365 & 4 \\ 4 & 10.887335 \end{bmatrix}\begin{bmatrix} .000142 \\ 0.000021 \end{bmatrix}$$
$$= \begin{bmatrix} 0.693686 \\ 0.554272 \end{bmatrix}$$

Hence, we get $y_1 \approx y\left(\frac{4}{3}\right) = 0.693686,\ y_2 \approx y\left(\frac{5}{3}\right) = 0.554272$.

Exact solution is $y\left(\frac{4}{3}\right) = 0.75,\ y\left(\frac{5}{3}\right) = 0.6$

***

You may now try the following exercises.

E1) Using second order finite difference method solve the boundary value problem
$$y'' = y + x\ ,\ y(0) = 0\ ,\ y(1) = 0$$
with (a) $h = \frac{1}{2}$, (b) $h = \frac{1}{3}$ and $h = \frac{1}{4}$.

E2) Using second order finite difference method solve the boundary value problem
$$x^2 y'' = 2y - x\ ,\ y(2) = 0\ ,\ y(3) = 0$$
with (a) $h = \frac{1}{2}$ and (b) $h = \frac{1}{3}$.

E3) Solve the boundary value problem
$$y'' - 3y' + 2y = 0$$
$$2y(0) - y'(0) = 1\ ,\quad y(1) + y'(1) = 2e + 3e^2$$
using second order finite difference method with $h = \frac{1}{2}$.

E4) Solve the boundary value problem
$$y'' = 2yy'\ ,\ y(0) = \frac{1}{2}\ ,\ y(1) = 1$$
using second order finite difference method with (a) $h = \frac{1}{2}$ and (b) $h = \frac{1}{3}$.

E5) Solve the boundary value problem
$$y'' = \frac{3}{2}y^2\ ,\ y(0) = 4\ ,\ y(1) = 1$$
using second order finite difference method with $h = \frac{1}{2}$ and $h = \frac{1}{3}$.

In finite difference methods for solving boundary value problems, we solve a system of linear or non-linear algebraic equations. For obtaining accurate results, we keep the step size h small and we have to solve a large system.

There is another class of methods called shooting methods, in which we convert the given boundary value problem to an initial value problem and obtain the solution in step-by-step manner using some singlestep method. In this method we do not have to solve a system of equations.

10.3 SHOOTING METHOD

In this method, we convert the given boundary value problem to an initial value problem by adding sufficient number of conditions at one end and adjust these conditions until the given conditions are satisfied at the other end. We may write the second order initial value problem as a system of two first order initial value problems and solve this system by using any method given in Unit-8 or solve the second order initial value problem directly. For this reason, this method is also called **initial value problem method.** We consider linear and non-linear second order boundary value problems separately.

Linear second order boundary value problems

First we give you the broad idea used in the method for any kind of boundary conditions.
Consider the linear second order differential equation

$$y'' + p(x)\,y' + q(x)\,y = r(x) \tag{16}$$

subject to given boundary conditions which could be of first, second or third kind.
We write the solution of Eqn.(16) as

$$y(x) = \lambda\, y_1(x) + (1-\lambda)\, y_2(x) \tag{17}$$

where $y_1(x)$ and $y_2(x)$ are respectively the solutions of the differential Eqn.(16) i.e.,

$$y_1'' + p(x)\,y_1' + q(x)\,y_1 = r(x) \tag{18i}$$

and

$$y_2'' + p(x)\,y_2' + q(x)\,y_2 = r(x) \tag{18ii}$$

with suitable initial conditions. We determine λ so that the boundary condition at one end is satisfied. Since the initial value problem is of second order, we require two linearly independent conditions at one end.

We now outline the method for each of the three kinds of boundary conditions.

Boundary conditions of the first kind

Here, we are given the boundary conditions as $y(a) = r_1$ and $y(b) = r_2$. Now we have one condition at $x = a$. We assume the other condition $y'(a) = s$, where s is arbitrarily chosen. Normally we choose $s = 0$ and $s = 1$. We now solve the two initial value problems

(i) $$y_1'' + p(x)\,y_1' + q(x)\,y_1 = r(x)$$

subject to $$y_1(a) = r_1\ ,\quad y_1'(a) = 0 \tag{19i}$$

(ii) $$y_2'' + p(x)\,y_2' + q(x)\,y_2 = r(x)$$

subject to $$y_2(a) = r_1\ ,\quad y_2'(a) = 1 \tag{19ii}$$

upto $x = b$, that is, we obtain $y_1(b)$ and $y_2(b)$.
From Eqn.(17), we get

$$y(b) = r_2 = \lambda\, y_1(b) + (1-\lambda)\, y_2(b)$$

which gives

$$\lambda = \frac{r_2 - y_2(b)}{y_1(b) - y_2(b)}\ ,\quad y_1(b) \neq y_2(b) \tag{20}$$

Once the value of λ is obtained, Eqn.(17) gives the solution at various nodal points.

Boundary condition of the second kind

Here, we are given $y'(a) = r_1$ and $y'(b) = r_2$. We have one condition at $x = a$. We assume the other condition $y(a) = s$ where s is arbitrarily chosen. Normally we choose $s = 0$ and $s = 1$. We now solve the two initial value problems

(i) $$y_1'' + p(x)\,y_1' + q(x)\,y_1 = r(x)$$
$$y_1(a) = 0\ ,\quad y_1'(a) = r_1 \tag{21i}$$

(ii) $$y_2'' + p(x)\,y_2' + q(x)\,y_2 = r(x)$$
$$y_2(a) = 1\ ,\quad y_2'(a) = r_1 \tag{21ii}$$

upto $x = b$, that is we obtain $y_1(b)$, $y_1'(b)$ and $y_2(b)$, $y_2'(b)$.
From Eqn.(17) we get

$$y'(x) = \lambda\, y_1'(x) + (1-\lambda)\, y_2'(x) \tag{21iii}$$

Therefore, we obtain

$$y'(b) = r_2 = \lambda\, y_1'(b) + (1-\lambda)\, y_2'(b)$$

which gives

$$\lambda = \frac{r_2 - y_2^*(b)}{y_1'(b) - y_2'(b)}, \quad y_1'(b) \neq y_2'(b) \tag{22}$$

We then obtain the solution values from Eqn.(17) and derivative values from Eqn.(21iii) at various nodal points.

Boundary conditions of the third kind

Here, we are given $a_0\, y(a) - a_1\, y'(a) = r_1$ and $b_0\, y(b) + b_1\, y'(b) = r_2$. We assume the value of $y(a)$ or $y'(a)$. If we choose $y(a) = s$, then we get $y'(a) = (a_0\, s - r_1)/a_1$, where s is arbitrarily chosen. Normally, we choose $s = 0$ and $s = 1$. We now solve the two initial value problems

(i) $\quad y_1'' + p(x)y_1' + q(x)y_1 = r(x)$

$$y_1(a) = 0, \quad y_1'(a) = -r_1/a_1 \tag{23i}$$

(ii) $\quad y_2'' + p(x)y_2' + q(x)y_2 = r(x)$

$$y_2(a) = 1, \quad y_2'(a) = (a_0 - r_1)/a_1 \tag{23ii}$$

upto $x = b$, that is we obtain $y_1(b),\ y_1'(b)$ and $y_2(b),\ y_2'(b)$.

From Eqn.(17), we have

$$y(b) = \lambda\, y_1(b) + (1-\lambda)\, y_2(b)$$

and

$$y'(b) = \lambda\, y_1'(b) + (1-\lambda)\, y_2'(b)$$

Therefore,

$$b_0 y(b) + b_1 y'(b) = r_2 = b_0 \left[\lambda\, y_1(b) + (1-\lambda)\, y_2(b)\right] + b_1 \left[\lambda\, y_1'(b) + (1-\lambda)\, y_2'(b)\right]$$

which gives

$$\lambda = \frac{r_2 - [b_0\, y_2(b) + b_1\, y_2'(b)]}{[b_0\, y_1(b) + b_1\, y_1'(b)] - [b_0\, y_2(b) + b_1\, y_2'(b)]}. \tag{24}$$

We obtain the solution values from Eqn.(17) for various nodal points. We can sum up the above discussion in the form of following theorem.

Theorem 1: For a linear second order boundary value problem of the form (16), it is sufficient to solve two initial value problems with two linearly independent sets of assumed initial conditions.

Solution of second order initial value problems

As we have mentioned earlier the second order IVP can be solved either directly or, by resolving it to the system of two first order IVP's. Accordingly, we give here two methods for solving the second order initial value problem

$$y'' = f(x, y, y'), \quad y(x_0) = y_0, \quad y'(x_0) = y_0' \tag{25}$$

where $f(x, y, y')$ can be a linear or a non-linear function of y and/or y'.

Method 1

We can write Eqn.(25) as a system of two first order initial value problems

$$u' = v, \qquad u(x_0) = y(x_0)$$

$$v' = f(x, u, v), \qquad v(x_0) = y'(x_0)$$

where $y(x) = u(x)$. We can solve this system of first order initial value problems using Taylor series method or Runge-Kutta method as discussed in Unit 7.

Method 2

We can solve the initial value problem (25) directly by using the m^{th} order Taylor series method.

$$y_{i+1} = y_i + h\, y_i' + \frac{h^2}{2!} y_i'' + \cdots + \frac{h^m}{m!} y_i^{(m)}$$

$$y_{i+1}' = y_i' + h\, y_i'' + \cdots + \frac{h^m}{m!} y_i^{(m+1)} \tag{26}$$

The truncation error in both y and y' is of order h^{m+1}. If we use Taylor series method of order p to determine y_{i+1} and that of order q to determine y'_{i+1}, then the final order of the Taylor series method is smaller of p and q. Higher order derivatives are obtained by differentiating the given differential equation.

We now illustrate these methods with the help of examples.

Example 6: Using shooting method, solve the boundary value problem

$$y'' = y + 1, \quad y(0) = 0, \quad y(1) = e - 1$$

Use the Taylor series method

$$y_{i+1} = y_i + h y'_i + \frac{h^2}{2} y''_i$$
$$y'_{i+1} = y'_i + h y''_i$$

with $h = 0.25$ to solve the resulting second order initial value problems. Compare this numerical solution with the exact solution $y(x) = e^x - 1$.

Solution: We solve the two initial value problems

$$y''_1 = y_1 + 1, \quad y_1(0) = 0, \quad y'_1(0) = 0$$
$$y''_2 = y_2 + 1, \quad y_2(0) = 0, \quad y'_2(0) = 1$$

The given Taylor series method gives

$$y_{1,\,i+1} = y_{1\,i} + h\, y'_{1\,i} + \frac{h^2}{2} y''_{1\,i}$$
$$= y_{1\,i} + h\, y'_{1\,i} + \frac{h^2}{2}(y_{1\,i} + 1)$$
$$= \left(1 + \frac{h^2}{2}\right) y_{1\,i} + h\, y'_{1\,i} + \frac{h^2}{2}$$
$$y'_{1,\,i+1} = y'_{1i} + h\, y''_{1i}$$
$$= y'_{1i} + h(y_{1\,i} + 1)$$
$$= h\, y_{1\,i} + y'_{1\,i} + h\,, \qquad i = 0, 1, 2, 3$$

Similarly we obtain

$$y_{2,\,i+1} = y_{2\,i} + h\, y'_{2\,i} + \frac{h^2}{2} y''_{2\,i}$$
$$= y_{2\,i} + h\, y'_{2\,i} + \frac{h^2}{2}(y_{2\,i} + 1)$$
$$= \left(1 + \frac{h^2}{2}\right) y_{2\,i} + h\, y'_{2\,i} + \frac{h^2}{2}$$
$$y'_{2,\,i+1} = y'_{2\,i} + h\, y''_{2\,i}$$
$$= y'_{2\,i} + h(y_{2\,i} + 1)$$
$$= h\, y_{2\,i} + y'_{2\,i} + h$$

For $h = \frac{1}{4}$, we obtain

$$y_{1,\,i+1} = 1.03125\ y_{1\,i} + 0.25\ y'_{1\,i} + .03125$$
$$y'_{1,\,i+1} = 0.25\ y_{1\,i} + y'_{1\,i} + 0.25 \tag{27}$$

and

$$y_{2,\,i+1} = 1.03125\ y_{2i} + 0.25\ y'_{2\,i} + 0.03125$$
$$y'_{2,\,i+1} = 0.25\ y_{2\,i} + y'_{2\,i} + 0.25 \tag{28}$$

$i = 0, 1, 2, 3$.

Using the conditions $y_{1\,0} = 0$, $y'_{1\,0} = 0$, we get from Eqn.(27)

$$x = 0.25, \quad y_{11} = 0.03125, \quad y'_{11} = 0.25$$
$$x = 0.50, \quad y_{12} = 0.125977, \quad y'_{12} = 0.507813$$
$$x = 0.75, \quad y_{13} = 0.288117, \quad y'_{13} = 0.789307$$
$$x = 1.0, \quad y_{14} = 0.525697, \quad y'_{14} = 1.111336$$

Using the conditions $y_{20} = 0, \; y'_{20} = 1$, we get from Eqn.(28)

$$x = 0.25, \quad y_{21} = 0.28125, \quad y'_{21} = 1.25$$
$$x = 0.50, \quad y_{22} = 0.633789, \quad y'_{22} = 1.570313$$
$$x = 0.75, \quad y_{23} = 1.077423, \quad y'_{23} = 1.978760$$
$$x = 1.0, \quad y_{24} = 1.637032, \quad y'_{24} = 2.498116$$

From Eqn.(20), we obtain

$$\lambda = \frac{(e-1) - y_{24}}{y_{14} - y_{24}} = -0.073110$$

Hence, we obtain

$$y(x) = -0.073110 \; y_1(x) + 1.073110 \; y_2(x) \tag{29}$$

Therefore, we get from Eqn.(29)

Numerical solution	**Exact solution**
$y(0.25) \approx 0.299528,$	$y(0.25) = 0.284025$
$y(0.50) \approx 0.670915,$	$y(0.5) = 0.648721$
$y(0.75) \approx 1.135129,$	$y(0.75) = 1.117000$
$y(1.00) \approx 1.718282,$	$y(1.0) = 1.718282$

***

Example 7: Solve the boundary value problem

$$y'' - 3y' + 2y = 0$$
$$2y(0) - y'(0) = 1, \quad y(1) + y'(1) = 2e + 3e^2$$

using shooting method. Use the Taylor series method

$$y_{i+1} = y_i + h y'_i + \frac{h^2}{2} y''_i + \frac{h^3}{6} y'''_i$$
$$y'_{i+1} = y'_i + h y''_i + \frac{h^2}{2} y'''_i \tag{30}$$

with $h = 0.25$ to solve the resulting initial value problems.

Solution: We have

$$y'' = 3y' - 2y$$

and

$$y''' = 3y'' - 2y' = 3(3y' - 2y) - 2y' = -6y + 7y'$$

From the given Taylor series method, we get

$$y_{i+1} = y_i + h y'_i + \frac{h^2}{2}(3y' - 2y) + \frac{h^3}{6}(-6y + 7y')$$
$$= (1 - h^2 - h^3) y_i + \left(h + \frac{3h^2}{2} + \frac{7h^3}{6}\right) y'_i$$
$$y'_{i+1} = y'_i + h(3y' - 2y) + \frac{h^2}{2}(-6y + 7y')$$
$$= (-2h - 3h^2) \, y_i + \left(1 + 3h + \frac{7}{2}h^2\right) y'_i$$

For $h = \frac{1}{4}$, we get

$$y_{i+1} = 0.921875\, y_i + 0.361979\, y'_i$$
$$y'_{i+1} = -0.6875\, y_i + 1.96875\, y'_i$$

or,

$$\begin{bmatrix} y_{i+1} \\ y'_{i+1} \end{bmatrix} = \begin{bmatrix} 0.921875 & 0.361979 \\ -0.6875 & 1.96875 \end{bmatrix} \begin{bmatrix} y_i \\ y'_i \end{bmatrix} \tag{31}$$

Now, we solve the two initial value problems

$$y''_1 = 3y'_1 + 2y_1 = 0$$
$$y_1(0) = 0, \quad y'_1(0) = -1$$

and

$$y''_2 - 3y''_2 + 2y_2 = 0$$
$$y_2(0) = 1, \quad y'_2(0) = 1$$

using the method given in Eqn.(30).

Using $y_1(0) = 0,\ y'_1(0) = -1$ and

$$\begin{bmatrix} y_{1,\,i+1} \\ y'_{1,\,i+1} \end{bmatrix} = \begin{bmatrix} 0.921875 & 0.361979 \\ -0.6875 & 1.96875 \end{bmatrix} \begin{bmatrix} y_{1\,i} \\ y'_{1\,i} \end{bmatrix}, \; i = 0,1,2,3$$

we get

$$\begin{bmatrix} y_{11} \\ y'_{11} \end{bmatrix} = \begin{bmatrix} -0.361975 \\ -1.96875 \end{bmatrix}, \quad \begin{bmatrix} y_{12} \\ y'_{12} \end{bmatrix} = \begin{bmatrix} -1.046342 \\ -3.627119 \end{bmatrix}$$

$$\begin{bmatrix} y_{13} \\ y'_{13} \end{bmatrix} = \begin{bmatrix} -2.277537 \\ -6.421530 \end{bmatrix}, \quad \begin{bmatrix} y_{14} \\ y'_{14} \end{bmatrix} = \begin{bmatrix} -4.424063 \\ -11.076581 \end{bmatrix}$$

Using $y_2(0) = 1,\ y'_2(0) = 1$

and

$$\begin{bmatrix} y_{2,\,i+1} \\ y'_{2,\,i+1} \end{bmatrix} = \begin{bmatrix} 0.921875 & 0.361979 \\ -0.6875 & 1.96875 \end{bmatrix} \begin{bmatrix} y_{2\,i} \\ y'_{2\,i} \end{bmatrix}, \; i = 0,1,2,3$$

we get

$$\begin{bmatrix} y_{21} \\ y'_{21} \end{bmatrix} = \begin{bmatrix} 1.283854 \\ 1.28125 \end{bmatrix}, \quad \begin{bmatrix} y_{22} \\ y'_{22} \end{bmatrix} = \begin{bmatrix} 1.647339 \\ 1.639811 \end{bmatrix}$$

$$\begin{bmatrix} y_{23} \\ y'_{23} \end{bmatrix} = \begin{bmatrix} 2.112218 \\ 2.095832 \end{bmatrix}, \quad \begin{bmatrix} y_{24} \\ y'_{24} \end{bmatrix} = \begin{bmatrix} 2.705848 \\ 2.674019 \end{bmatrix}$$

From Eqn.(24), we get for $b_0 = 1$, $b_1 = 1$, and $r_2 = 2e + 3e^2$

$$\lambda = \frac{2e + 3e^2 - [y_{24} + y'_{24}]}{(y_{14} + y'_{14}) - (y_{24} + y'_{24})} = \frac{22.223865}{-20.880511}$$
$$= -1.064335$$

Hence, we get

$$y(x) = -1.064335\, y_1(x) + 2.064335\, y_2(x)$$

and

$$y(0.25) \approx 3.035567, \qquad y(0.50) \approx 4.514318$$
$$y(0.75) \approx 6.784388, \qquad y(1.00) \approx 10.294462$$

***

Non-linear second order boundary value problem

We now consider the non-linear second order differential equation

$$y'' = f(x, y, y')$$

subject to given boundary conditions. Since the boundary value problem is non-linear, we cannot write the solution in the form of Eqn.(17). In this case we proceed as follows:

Boundary conditions of the first kind

Here, we are given $y(a) = r_1$ and $y(b) = r_2$. We assume $y'(a) = s$ where s is arbitrary, and solve the initial value problem

$$y'' = f(x, y, y')$$
$$y(a) = r_1 \;,\; y'(a) = s \tag{32}$$

upto $x = b$ using any numerical method. The solution of the initial value problem (32) at $x = b$ denoted by $y(b, s)$ should satisfy the boundary condition at $x = b$. Let

$$\phi(s) = y(b, s) - r_2 \tag{33}$$

Hence, the problem is to find s, such that $\phi(s) = 0$.

Boundary conditions of the second kind

Here, we are given $y'(a) = r_1$, $y'(b) = r_2$. We assume $y(a) = s$, where s is arbitrary, and solve the initial value problem

$$y'' = f(x, y, y')$$
$$y(a) = s \;,\quad y'(a) = r_1 \tag{34}$$

upto $x = b$ using any numerical method. The solution of the initial value problem (34) at $x = b$ denoted by $y(b, s)$ should satisfy the boundary condition at $x = b$. Let

$$\phi(s) = y'(b, s) - r_2 \tag{35}$$

Hence, the problem is to find s, such that $\phi(s) = 0$.

Boundary condition of the third kind

Here, we are given $a_0 y(a) - a_1 y'(a) = r_1$ and $b_0\, y(b) + b_1\, y'(b) = r_2$ we assume the value of $y(a)$ or $y'(a)$. If we take $y'(a) = s$, we get $y(a) = (a_1 s + r)/a_0$. We now solve the initial value problem

$$y'' = f(x, y, y')$$
$$y(a) = (a_1 s + r_1)/a_0, \quad y'(a) = s \tag{36}$$

upto $x = b$ using any numerical method. The solution of the initial value problem (36) at $x = b$ denoted by $y(b, s)$ should satisfy the boundary condition at $x = b$. Let

$$\phi(s) = b_0\, y(b, s) + b_1 y'(b, s) - r_2 \tag{37}$$

Hence, the problem is to find s, such that $\phi(s) = 0$.

The function $\phi(s)$ in Eqn(33) or Eqn.(35) or Eqn.(37) is a non-linear function in s. We solve the equation

$$\phi(s) = 0 \tag{38}$$

by using any iterative method to get the required solution.

One of the iterative methods, the **secant method** of solving Eqn.(38) is given by

$$s^{(k+1)} = s^{(k)} - \left[\frac{s^{(k)} - s^{(k-1)}}{\phi(s^{(k)}) - \phi(s^{(k-1)})}\right] \phi(s^{(k)}),\ k = 1, 2, \ldots \tag{39}$$

where $s^{(0)}$ and $s^{(1)}$ are two initial approximations to s. To start the application of the secant method, we solve the initial value problem (32) or (34) or (36) for two values of s, that is $s^{(0)}$ and $s^{(1)}$. The iterations are stopped when $\left|\, \phi\,(s^{(k+1)})\, \right| < \in$ where $\in$ is the given error tolerance.

We illustrate the method with the help of an example.

Example 8: Using the shooting method solve the boundary value problem

$$y'' = 2\,y\,y' \;,\; y(0) = \frac{1}{2} \;,\; y(1) = 1$$

Use the Taylor series method

$$y_{i+1} = y_i + h\,y'_i + \frac{h^2}{2}\, y''_i + \frac{h^3}{6}\, y'''_i$$

$$y'_{i+1} = y'_i + h y''_i + \frac{h^2}{2} y'''_i$$

with $h = 0.25$ to solve the resulting second order initial value problem and the secant method for iteration. Take $s^{(0)} = \frac{1}{2}$, $s^{(1)} = 0.1$ and perform two iterations of the secant method. Compare the numerical solution with the exact solution $y(x) = 1/(2-x)$.

Solution: We solve the initial value problem

$$y'' = 2yy'$$

$$y(0) = \frac{1}{2}, \quad y'(0) = s^{(0)} = 0.5$$

using the given Taylor series method with $h = 0.25$. We get

$$y''' = 2(y')^2 + 2y\, y''$$

i = 0: $x = 0,\ y_0 = 0.5,\ y'_0 = 0.5,\ y''_0 = 0.5,\ y'''_0 = 1$

$$y_1 = y_0 + 0.25 y'_0 + \frac{(0.25)^2}{2} y''_0 + \frac{(0.25)^3}{6} y'''_0 = 0.643229$$

$$y'_1 = y'_0 + 0.25 y''_0 + \frac{(0.25)^2}{2} y'''_0 = 0.62625$$

i = 1: $x_1 = 0.25,\ y_1 = 0.643229,\ y'_1 = 0.65625,\ y''_1 = 0.844238,\ y'''_1 = 1.947405$

$$y_2 = y_1 + 0.25 y'_1 + \frac{(0.25)^2}{2} y''_1 + \frac{(0.25)^3}{6} y'''_1 = 0.838736$$

$$y'_2 = y'_1 + 0.25\ y''_1 + \frac{(0.25)^2}{2} y'''_1 = 0.928166$$

i = 2: $x_2 = 0.50,\ y_2 = 0.838736,\ y'_2 = 0.928166,\ y''_2 = 1.556972,\ y'''_2 = 4.334761$

$$y_3 = y_2 + 0.25\ y'_2 + \frac{(0.25)^2}{2} y''_2 + \frac{(0.25)^3}{6} y'''_2 = 1.130721$$

$$y'_3 = y'_2 + 0.25 y''_2 + \frac{(0.25)^2}{2} y'''_2 = 1.452870$$

i = 3: $x_3 = 0.75,\ y_3 = 1.130721,\ y'_3 = 1.452870,\ y''_3 = 3.285581,\ y'''_3 = 11.651813$

$$y_4 = y_3 + (0.25) y'_3 + \frac{(0.25)^2}{2} y''_3 + \frac{(0.25)^3}{6} y'''_3 = 1.626956$$

$$y'_4 = y'_3 + (0.25)\ y''_3 + \frac{(0.25)^2}{2} y'''_3 = 2.638384.$$

From Eqn.(33), we get

$$\phi(s^{(0)}) = y(1, s^{(0)}) - 1 = 0.626956$$

We now solve the initial value problem

$$y'' = 2yy'$$

$$y(0) = \frac{1}{2}, \quad y'(0) = s^{(1)} = 0.1$$

using the given Taylor series method with $h = 0.25$. We get

i = 0: $x = 0,\ y_0 = 0.5,\ y'_0 = 0.1,\ y''_0 = 0.1,\ y'''_0 = 0.12$

$$y_1 = y_0 + (0.25) y'_0 + \frac{(0.25)^2}{2} y''_0 + \frac{(0.25)^3}{6} y'''_0 = 0.528438$$

$$y'_1 = y'_0 + (0.25) y''_0 + \frac{(0.25)^2}{2} y'''_0 = 0.12875.$$

i = 1: $x_1 = 0.25,\ y_1 = 0.528438,\ y'_1 = 0.12875,\ y''_1 = 0.136073,\ y'''_1 = 0.176965$

$$y_2 = y_1 + (0.25) y'_1 + \frac{(0.25)^2}{2} y''_1 + \frac{(0.25)^3}{6} y'''_1 = 0.565339$$

$$y'_2 = y'_1 + (0.25)\, y''_1 + \frac{(0.25)^2}{2}\, y'''_1 = 0.168298.$$

i = 2: $x_2 = 0.5,\ y_2 = 0.565339,\ y'_2 = 0.168298,\ y''_2 = 0.190291,\ y'''_2 = 0.271806$

$$y_3 = y_2 + (0.25)\, y'_2 + \frac{(0.25)^2}{2} y''_2 + \frac{(0.25)^3}{6}\, y'''_2 = 0.614068$$

$$y'_3 = y'_2 + (0.25)\, y''_2 + \frac{(0.25)^2}{2}\, y'''_2 = 0.224365$$

i = 3: $x_3 = 0.75,\ y_3 = 0.614068,\ y'_3 = 0.224365,\ y''_3 = 0.275550,\ y'''_3 = 0.439092$

$$y_4 = y_3 + 0.25\, y'_3 + \frac{(0.25)^2}{2} y''_3 + \frac{(0.25)^3}{6}\, y'''_3 = 0.679914$$

$$y'_4 = y'_3 + 0.25\, y''_3 + \frac{(0.25)^2}{2}\, y'''_3 = 0.306974.$$

From Eqn.(33), we get

$$\phi(s^{(1)}) = y(1, s^{(1)}) - 1 = -0.320086$$

From Eqn.(39), we get

$$s^{(2)} = s^{(1)} - \left[\frac{s^{(1)} - s^{(0)}}{\phi(s^{(1)}) - \phi(s^{(0)})}\right] \phi(s^{(1)}) = 0.235194$$

Now we solve the initial value problem

$$y'' = 2y\, y'$$

$$y(0) = \frac{1}{2},\ \ y'(0) = s^{(2)} = 0.235194$$

using the given Taylor series method with $h = 0.25$. We obtain

i = 0: $y_0 = \frac{1}{2},\ y'_0 = .235194,\ y''_0 = 0.235194,\ y'''_0 = 0.345826$

$y_1 = 0.567049, y'_1 = 0.304800.$

i = 1: $y_1 = 0.567049,\ y'_1 = 0.304800,\ y''_1 = 0.345673,\ y'''_1 = 0.577833$

$y_2 = 0.655556,\ y'_2 = 0.409276.$

i = 2: $x_2 = 0.5,\ y_2 = 0.655556,\ y'_2 = 0.409276,\ y''_2 = 0.536607,\ y'''_2 = 1.038566$

$y_3 = 0.777349,\ \ y'_3 = 0.575883$

i = 3: $x_3 = 0.75,\ y_3 = 0.777349,\ y'_3 = 0.575883,\ y''_3 = 0.895324,\ y'''_3 = 2.055241$

$y_4 = 0.954651,\ y'_4 = 0.863940$

From Eqn.(33), we get

$$\phi(s^{(2)}) = y(1, s^{(2)}) - 1 = -0.045349$$

from Eqn(39), we get

$$s^{(3)} = s^{(2)} - \left[\frac{s^{(2)} - s^{(1)}}{\phi(s^{(2)}) - \phi(s^{(1)})}\right] \phi(s^{(2)}) = 0.257510$$

Now, we solve the initial value problem

$$y'' = 2y\, y'$$

$$y(0) = \frac{1}{2},\ \ y'(0) = s^{(3)} = 0.257510$$

Using the given Taylor series method with $h = 0.25$. We have

i = 0: $x_0 = 0,\ y_0 = \frac{1}{2},\ y'_0 = .257510,\ y''_0 = 0.25751,\ y'''_1 = 0.390133$

$y_1 \approx y(0.25) = 0.573441, y'_1 \approx y'(0{:}25) = 0.334079.$

i = 1: $x_1 = 0.25,\ y_1 = 0.573441,\ y'_1 = 0.334079,\ y''_1 = 0.383149,\ y'''_1 = 0.662645$

$y_2 \approx y(0.50) = 0.670660,\ y'_2 \approx y'(0.50) = 0.450574.$

$i = 2:$ $\quad x_2 = 0.50,\ y_2 = 0.670660,\ y'_2 = 0.450574,\ y''_2 = 0.604364,\ y'''_2 = 1.216679$

$y_3 \approx y(0.75) = 0.805358 \ y'_3 \approx y'(0.75) = 0.639686$

$i = 3:$ $\quad x_3 = 0.75,\ y_3 = 0.805358,\ y'_3 = 0.639686,\ y''_3 = 1.030353,\ y'''_3 = 2.478002$

$y_4 \approx y(1.0) = 1.003931,\ y'_4 \approx y'(1.0) = 0.974712.$

Exact solution is given by

$$y(x) = \frac{1}{2-x} \text{ and}$$

$$y(0.25) = 0.571429,\ y(0.50) = 1$$

$$y(0.75) = 0.8, \quad y(1.0) = 1$$

The exact value of $s = y'(0)$ is 0.25.

You may now try the following exercises.

E6) Find the solution of the boundary value problem

$$x^2 y'' - 2y + x = 0$$

$$y(2) = 0, \quad y(3) = 0$$

using the shooting method. Use the Taylor series method

$$y_{i+1} = y_i + h y'_i + \frac{h^2}{2} y''_i$$

$$y'_{i+1} = y'_i + h\ y''_i$$

with $h = 0.25$ to solve the resulting initial value problems.

E7) Find the solution of the boundary value problem

$$y'' = 2y - y'$$

$$y(1) = 2e + e^{-2}, \quad y(2) = 2e^2 + e^{-4}$$

using shooting method. Use the Taylor series method

$$y_{i+1} = y_i + h y'_i + \frac{h^2}{2} y''_i + \frac{h^3}{6} . y'''_i$$

$$y'_{i+1} = y'_i + h y''_i + \frac{h^2}{2} y'''_i$$

with $h = \frac{1}{3}$ to solve the resulting initial value problems.

E8) Using shooting method solve the boundary value problem

$$y'' = y \ , \ y'(0) = 3 \ , \ y'(1) = e + 2/e$$

Use the Taylor series method

$$y_{i+1} = y_i + h y'_i + \frac{h^2}{2} y''_i + \frac{h^3}{6} y'''_i$$

$$y'_{i+1} = y'_i + h y''_i + \frac{h^2}{2} y'''_i$$

with $h = \frac{1}{4}$ to solve the resulting initial value problems.

E9) Solve the boundary value problem

$$y'' = xy + 1$$

$$y(0) + y'(0) = 1, \quad y(1) = 1$$

using shooting method. Use the Taylor series method

$$y_{i+1} = y_i + h y'_i + \frac{h^2}{2} y''_i + \frac{h^3}{6} y'''_i$$

$$y'_{i+1} = y'_i + h y''_i + \frac{h^2}{2} y'''_i$$

with $h = 0.25$ to solve the resulting initial value problems.

E10) Use shooting method to solve the boundary value problem

$$y'' = 6y^2 \ , \ y(0) = 1 \ , \ y(0.3) = \frac{100}{169}$$

use the Taylor series method

$$y_{i+1} = y_i + h\, y_i' + \frac{h^2}{2} y_i'' + \frac{h^3}{6} y_i'''$$

$$y_{i+1}' = y_i' + h\, y_i'' + \frac{h^2}{2} y_i'''$$

with $h = 0.1$ to solve the resulting initial value problems and the secant method for iteration. Take $s^{(0)} = -1.8$, $s^{(1)} = -1.9$ and perform two iterations of the secant method. Compare the numerical results with the exact solutions

$$y(x) = \frac{1}{(1+x)^2}.$$

We now end this unit by giving a summary of what we have covered in it.

10.4 SUMMARY

In this unit we have discussed the following points:

1. A second order differential equation

$$y'' = f(x, y, y')$$

with one of the three kinds of boundary conditions

(i) first kind : $y(a) = r_1$, $y(b) = r_2$

(ii) second kind : $y'(a) = r_1$, $y'(b) = r_2$

(iii) third kind (mixed kind): $a_0 y(a) - a_1 y'(a) = r_1$

$$b_0 y(b) + b_1 y'(b) = r_2$$

is called a boundary value problem (bvp) of first kind, or second kind, or third kind respectively. The points $x = a$ and $x = b$ are called boundary points.

2. The bvp is linear if f is linear in y and y'. The bvp is called non-linear if it is non-linear in y and/or y'

3. We divide the interval $[a, b]$ into $N+1$ subintervals of equal length $h = \frac{b-a}{N+1}$.

We obtain the nodal points $x_0, x_1, \ldots, x_{N+1}$, where

$$x_0 = a$$

$$x_i = x_0 + ih \ , \quad i = 1, 2, \ldots, N \ , \ x_{N+1} = b$$

The solution of the bvp is obtained at these nodal points.

4. In finite difference methods of solving a bvp, we replace y' and y'' by some difference approximations at each nodal point. In corporating the boundary conditions, we obtain a system of N equations for bvp of first kind and a system of $N+2$ equations for bvp's of second and third kind. The system of equations is linear for a linear bvp and non-linear for a non-linear bvp. We solve this system of equations using some direct or iterative method.

5. In shooting method of solving a bvp, we convert the given bvp to an IVP by assuming one of the following conditions:

(i) $y'(a) = s$, for bvp of first kind

(ii) $y(a) = s$, for bvp of second kind

(iii) $y(a) = s$, or, $y'(a) = s$, for bvp of third kind

we generally take $s = 0$ or $s = 1$

For linear bvp's, we solve two IVP's

$$y_1'' + p(x)\, y_1' + q(x)\, y_1 = r(x)$$

and

$$y_2'' + p(x)\, y_2' + q(x)\, y_2 = r(x)$$

with the associated initial conditions

(i) bvp of first kind : $y_1(a) = r_1$, $y_1'(a) = 0$

$y_2(a) = r_1$, $y_2'(a) = 1$

(ii) bvp of second kind : $y_1(a) = 0$, $y_1'(a) = r_1$

$y_2(a) = 1$, $y_2'(a) = r_1$

(iii) bvp of third kind : $y_1(a) = 0$, $y_1'(a) = -r_1 / a_1$

$y_2(a) = 1$, $y_2'(a) = (a_0 - r_1)/a_1$

using any singlestep method.

We write the solution of the bvp as

$$y(x) = \lambda y_1(x) + (1-\lambda) y_2(x)$$

and determine λ, so that the boundary condition at $x = b$ is satisfied. We then determine the solutions values from the above equation for $y(x)$.

For non-linear bvp's, we again assume $y(a) = s$ or $y'(a) = s$ and solve the given differential equation upto $x = b$ using two different values of s, say $s^{(0)}$ and $s^{(1)}$. We use secant method iteratively to determine $s^{(k)}$, $k = 2,3,\ldots$, so that the boundary conditions at $x = b$ is satisfied to the required accuracy.

10.5 SOLUTIONS/ANSWERS

E1) We obtain from the differential equation

$$y_{i+1} - (2+h^2)y_i + y_{i-1} = h^2 x_i,\ i = 1,2,\ldots$$

a) $h = \frac{1}{2}$, $x_0 = 0$, $x_1 = \frac{1}{2}$, $x_2 = 1$, $y_0 = 0$, $y_2 = 0$, $i = 1$;

$y_1 \approx y(1/2) = -1/18 = -0.055556$.

b) $h = 1/3$, $x_0 = 0$, $x_1 = 1/3$, $x_2 = 2/3$, $x_3 = 1$, $y_0 = 0$, $y_3 = 0$, $i = 1$ and $i = 2$;

$y_1 \approx y(1/3) = -0.044048$, $y_2 \approx y(2/3) = -0.055952$

c) $h = 1/4$, $x_0 = 0$, $x_1 = 1/4$, $x_2 = 2/4$, $x_3 = 3/4$, $x_4 = 1$, $y_0 = 0$, $y_4 = 0$, $i = 1,2,3$;

$y_1 \approx y(1/4) = -0.034885$, $y_2 \approx y(2/4) = -0.056326$,

$y_3 \approx y(3/4) = -0.050037$.

E2) We obtain from the differential equation

$$x_i^2 y_{i+1} - 2(x_i^2 + h^2) y_i + x_i^2 y_{i-1} = -x_i h^2,\ i = 1,2,\ldots$$

a) $h = 1/2, x_0 = 2, x_1 = 5/2, x_2 = 2, y_0 = 0, y_2 = 0, i = 1$;

$y_1 \approx y(5/2) = 1/8 = 0.125$.

b) $h = 1/3, x_0 = 2, x_1 = 7/3, x_2 = 8/3, x_3 = 3, y_0 = 0, y_3 = 0, i = 1,2$;

$y_1 \approx y(1/3) = 0.140777$, $y_2 \approx y(2/3) = 0.165049$.

E3) We obtain from the differential equation

$$y_{i+1} - 2y_i + y_{i-1} - \frac{3h}{2}(y_{i+1} - y_{i-1}) + 2h^2 y_i = 0,\ i = 0,1,2,\ldots \qquad (40)$$

For $h = 1/2$, we get $x_0 = 0, x_1 = 1/2, x_2 = 1$. We need to determine y_0, y_1, y_2. From boundary conditions, we get $y_{-1} = 1 - 2y_0 + y_1$, $y_3 = y_1 - y_2 + 2e + 3e^2$. We obtain on eliminating y_{-1} and y_3 from the equations obtained from Eqn.(40) for $i = 0,1,2$ and solving the resulting system of equations

$y_0 \approx y(0) = 1.593159$, $y_1 \approx y(1/2) = 3.107898, y_2 \approx y(1) = 7.495274$.

E4) We obtain from the differential equation

$$(y_{i+1} - 2y_i + y_{i-1}) - h\, y_i (y_{i+1} - y_{i-1}) = 0,\ i = 1,2\ldots$$

a) For $h = 1/2$, we get $x_0 = 0$, $x_1 = 1/2$, $x_2 = 1, y_0 = y(0) = 1/2$,

$y_2 = y(1) = 1, i = 1$; we obtain $y_1 \approx y(1/2) = 2/3$.

b) For $h = 1/3$, we get $x_0 = 0, x_1 = 1/3, x_2 = 2/3, x_3 = 1$,
$y_0 = y(0) = 1/2, y_3 = y(1) = 1, i = 1, 2$. We obtain the system of non-linear equations

$$f_1(y_1, y_2) = 2y_1y_2 + 11y_1 - 6y_2 - 3 = 0$$
$$f_2(y_1, y_2) = y_1y_2 + 3y_1 - 7y_2 + 3 = 0$$

Using Newton-Raphson method with $y_1^{(0)} = 0.6, y_2^{(0)} = 0.8$, we obtain

$$y_1^{(1)} = 0.6,\ y_2^{(1)} = 0.75,\ y_1^{(2)} = 0.6,\ y_2^{(2)} = 0.75$$

Hence,

$$y_1 \approx y(1/3) = 0.6,\ y_2 \approx y(2/3) = 0.75.$$

E5) We obtain from the differential equation

$$y_{i+1} - 2y_i + y_{i-1} = \frac{3h^2}{2} y_i^2,\ i = 1, 2, \ldots$$

a) For $h = \frac{1}{2}$, we get $x_0 = 0,\ x_1 = \frac{1}{2},\ x_2 = 1$,
$y_0 = y(0) = 4,\ y_2 = y(1) = 1,\ i = 1$. We obtain the non-linear equation

$$f(y_1) = \frac{3}{2} y_1^2 + 8y_1 - 20 = 0$$

Using Newton-Raphson method with $y_1^{(0)} = 2$, we get

$$y_1^{(1)} = 1.857143,\ y_1^{(2)} = 1.854887,\ y_1^{(3)} = 1.854887$$

Hence $y_1 \approx y(1/2) = 1.854887$.

b) For $h = \frac{1}{3}$, we get $x_0 = 0, x_1 = \frac{1}{3},\ x_2 = \frac{2}{3},\ x_3 = 1$
$y_0 = y(0) = 4,\ y_3 = y(1) = 1,\ i = 1, 2$. We obtain the system of non-linear equations

$$f_1(y_1, y_2) = y_1^2 + 12y_1 - 6y_2 - 24 = 0$$
$$f_2(y_1, y_2) = y_2^2 - 6y_1 + 12y_2 - 6 = 0$$

Using Newton-Raphson method with $y_1^{(0)} = 2.5,\ y_2^{(0)} = 1.5$, we obtain

$$y_1^{(1)} = 2.302455\ ,\quad y_2^{(1)} = 1.469866;$$
$$y_1^{(2)} = 2.295044\ ,\quad y_2^{(2)} = 1.467949;$$
$$y_1^{(3)} = 2.295040\ ,\quad y_2^{(3)} = 1.467947.$$

Hence, $y_1 \approx y\left(\frac{1}{3}\right) = 2.295040,\quad y_2 \approx y\left(\frac{2}{3}\right) = 1.467947.$

E6) We solve the two initial value problems

(i) $y_1'' = \frac{2}{x^2} y_1 - x\ ,\quad y_1(2) = 0,\ \ y_1'(2) = 0$

(ii) $y_2'' = \frac{2}{x^2} y_2 - x\ ,\quad y_2(2) = 0,\ \ y_2'(2) = 1$

using the given method with $h = 0.25$. We obtain

$$y_1(2.25) \approx -0.0625\ ,\quad y_1'(2.25) \approx -0.5$$
$$y_1(2.50) \approx -0.258584\ ,\quad y_1'(2.5) \approx -1.068673$$
$$y_1(2.75) \approx -0.606463\ ,\quad y_1'(2.75) \approx -1.714360$$
$$y_1(3.0) \approx -1.126003\ ,\quad y_1'(3.0) = -2.441957$$

and

$$y_2(2.25) \approx 0.1875\ ,\quad y_2'(2.25) \approx 0.5$$
$$y_2(2.50) \approx 0.244502\ ,\quad y_2'(2.50) \approx -0.043982$$

$y_2(2.75) \approx 0.157827 \quad , \quad y_2'(2.75) \approx -0.404920$

$y_2(3.0) \approx -0.028036 \quad , \quad y_2'(3.0) \approx -1.081985$

From $y(3) = 0 = \lambda y_1(3) + (1-\lambda) y_2(3)$, we obtain $\lambda = -0.025534$.

From $y(x) = \lambda y_1(x) + (1-\lambda) y_2(x)$, we obtain

$y(2.25) \approx 0.193884 \quad , \quad y(2.50) \approx 0.257384$,

$y(2.75) \approx 0.177342 \quad , \quad y(3.0) \approx 0.$

E7) We solve the two initial value problems

(i) $y_1'' + y_1' - 2y_1 = 0 \quad , \quad y_1(1) = 2e + e^{-2} \quad , \quad y_1'(1) = 0$

(ii) $y_2'' + y_2' - 2y_2 = 0 \quad , \quad y_2(1) = 2e + e^{-2} \quad , \quad y_2'(1) = 1$

using the given method with $h = \frac{1}{3}$. We obtain

$y_1(4/3) \approx 6.122210 \quad , \quad y_1'(4/3) \approx 3.095499$

$y_1(5/3) \approx 7.644058 \quad , \quad y_1'(5/3) \approx 5.980810$

$y_1(2) \approx 10.171119 \quad , \quad y_1'(2) \approx 9.230707$

and

$y_2(4/3) \approx 6.418506 \quad , \quad y_2'(4/3) = 3.928833$

$y_2(5/3) \approx 8.216531 \quad , \quad y_2'(5/3) = 6.839864$

$y_2(2) \approx 11.054667 \quad , \quad y_2'(2) = 10.264626$

From Eqn.(20), with $r_2 = 2e^2 + e^{-4}$, we obtain $\lambda = -4.23927$. Using $y(x) = \lambda y_1(x) + (1-\lambda) y_2(x)$ we obtain

$y(4/3) \approx 7.673298, \; y(5/3) \approx 10.640912, \; y(2) \approx 14.796428$.

E8) We solve the two initial value problems

(i) $y_1'' = y_1 \quad , \quad y_1(0) = 0 \quad , \quad y_1'(0) = 3$

(ii) $y_2'' = y_2 \quad , \quad y_2(0) = 1 \quad , \quad y_2'(0) = 3$

using the given method with $h = \frac{1}{4}$. We obtain

$y_1(0.25) \approx .757813 \quad , \quad y_1'(0.25) \approx 3.09375$

$y_1(0.50) \approx 1.562989 \quad , \quad y_1'(0.50) \approx 3.379883$

$y_1(0.75) \approx 2.465605, \quad y_1'(0.75) \approx 3.876252$

$y_1(1.0) \approx 3.521813 \quad , \quad y_1'(1.0) \approx 4.613796$

and

$y_2(0.25) \approx 1.789063 \quad , \quad y_2'(0.25) \approx 3.34375$

$y_2(0.50) \approx 2.689616 \quad , \quad y_2'(0.50) \approx 3.895508$

$y_2(0.75) \approx 3.757688 \quad , \quad y_2'(0.75) \approx 4.689647$

$y_2(1.0) \approx 5.059740 \quad , \quad y_2'(1.0) \approx 5.775620$

Using Eqn.(22) with $r_2 = e + 2/e = 3.454041$, we get

$$\lambda = \frac{r_2 - y_2'(1.0)}{y_1'(1.0) - y_2'(1.0)} = 1.998219$$

Hence, from $y(x) = \lambda y_1(x) + (1-\lambda) y_2(x)$, we obtain

$y(0.25) \approx -0.271600 \quad , \quad y(0.50) \approx 0.438369$

$y(0.75) \approx 1.175823 \quad , \quad y(1.0) \approx 1.986625$

E9) We solve the two initial value problems

(i) $y_1'' = x y_1 + 1 \quad , \quad y_1(0) = 0 \quad , \quad y_1'(0) = 1$

(ii) $y_2'' = x y_2 + 1 \quad , \quad y_2(0) = 1 \quad , \quad y_2'(0) = 0$

using the given method with $h = 0.25$. We obtain

$y_1(0.25) \approx 0.28125 \quad , \quad y_1'(0.25) \approx 1.25$

$y_1(0.50) \approx 0.628745$, $y_1'(0.50) \approx 1.536148$
$y_1(0.75) \approx 1.057494$, $y_1'(0.75) \approx 1.908392$
$y_1(1.0) \approx 1.597108$, $y_1'(1.0) \approx 2.434447$

and

$y_2(0.25) \approx 1.033854$, $y_2'(0.25) \approx 0.28125$
$y_2(0.50) \approx 1.146369$, $y_2'(0.50) \approx 0.630371$
$y_2(0.75) \approx 1.356930$, $y_2'(0.75) \approx 1.069341$
$y_2(1.0) \approx 1.692941$, $y_2'(1.0) \approx 1.641232$

Using Eqn.(24) with $b_0 = 1$, $b_1 = 0$ and $r_2 = 1$, we get

$$\lambda = \frac{r_2 - y_2(1)}{y_1(1) - y_2(1)} = 7.230714$$

Hence, from $y(x) = \lambda\, y_1(x) + (1-\lambda)\, y_2(x)$, we get

$y(0.25) \approx -4.408010$, $y(0.50) \approx -2.596422$
$y(0.75) \approx -0.808206$, $y(1.0) \approx 1$

E10) We solve the two initial value problems

(i) $y_1'' = 6y_1^2$, $y_1(0) = 1$, $y_1'(0) = s^{(0)} = -1.8$

(ii) $y_2'' = 6y_2^2$, $y_2(0) = 1$, $y_2'(0) = s^{(1)} = -1.9$

using the given method with $h = 0.1$. We obtain

$y_1(0.1) \approx 0.8464$, $y_1'(0.1) \approx -1.308$
$y_1(0.2) \approx 0.734878$, $y_1'(0.2) \approx -0.94459$
$y_1(0.3) \approx 0.655232$, $y_1'(0.3) \approx -0.578913$

$$\phi(s^{(0)}) = y_1(0.3) - \frac{100}{169} = 0.063516$$

and

$y_2(0.1) \approx 0.8362$, $y_2'(0.1) \approx -1.414$
$y_2(0.2) \approx 0.713412$, $y_2'(0.2) \approx -1.065405$
$y_2(0.3) \approx 0.620620$, $y_2'(0.3) \approx -0.805635$

$$\phi(s^{(1)}) = y_2(0.3) - \frac{100}{169} = 0.028904$$

From Eqn.(39), we obtain $s^{(2)} = -1.983504$.
We now solve the initial value problem

$$y_3'' = 6y_3^2 \quad , \quad y_3(0) = 1 \quad , \quad y_3'(0) = s^{(2)}$$

and obtain

$y_3(0.1) \approx 0.835616$, $y_3'(0.1) \approx -1.502520$
$y_3(0.2) \approx 0.703801$, $y_3'(0.2) \approx -1.158899$
$y_3(0.3) \approx 0.601140$, $y_3'(0.3) \approx -0.910636$

$$\phi(s^{(2)}) = y_3(0.3) - \frac{100}{169} = 0.009424$$

From Eqn.(39) we obtain $s^{(3)} = -2.023909$
Exact value of $y'(0) = -2$.
Finally, we solve the initial value problem

$$y'' = 6y^2 \quad , \quad y(0) = 1 \quad , \quad y'(0) = -2.023909$$

and obtain

$y(0.1) \approx 0.823561$, $y(0.2) \approx 0.686829$, $y(0.3) \approx 0.577837$.

APPENDIX

Newton-Raphson Method for System of Nonlinear Equations

We now extend the Newton-Raphson method derived for the single eqution $f(x)=0$ to a system of nonlinear equations. We first consider a system of two nonlinear equations in two unknowns as

$$\begin{aligned} &f(x,y)=0 \\ &g(x,y)=0 \end{aligned} \tag{A1}$$

Let (x_k, y_k) be suitable approximation to the root (ξ,η) of the system (A1). Let Δx be an increment in x_k and Δy be an increment in y_k such that $(x_k+\Delta x, y_k+\Delta y)$ is the exact solution, that is

$$f(x_k+\Delta x,\ y_k+\Delta y)\equiv 0$$

$$g(x_k+\Delta x,\ y_k+\Delta y)\equiv 0$$

Expanding in Taylor series about the point (x_k, y_k), we get

$$f(x_k,y_k)+\left[\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right]f(x_k,y_k)+\frac{1}{2!}\left[\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right]^2 f(x_k,y_k)+\cdots=0$$

$$g(x_k,y_k)+\left[\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right]g(x_k,y_k)+\frac{1}{2!}\left[\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right]^2 g(x_k,y_k)+\cdots=0$$

Neglecting second and higher powers of Δx and Δy, we obtain

$$\begin{aligned} &f(x_k,y_k)+\Delta x\ f_x(x_k,y_k)+\Delta y\ f_y(x_k,y_k)=0 \\ &g(x_k,y_k)+\Delta x\ g_x(x_k,y_k)+\Delta y\ g_y(x_k,y_k)=0 \end{aligned} \tag{A2}$$

where suffixes with respect to x and y represent partial derivatives. Solving Eqn.(A2) for Δx and Δy, we get

$$\Delta x=-\frac{1}{D_k}\left[f(x_k,y_k)\ g_y(x_k,y_k)-g(x_k,y_k)\ f_y(x_k,y_k)\right]$$

$$\Delta y=-\frac{1}{D_k}\left[g(x_k,y_k)\ f_x(x_k,y_k)-f(x_k,y_k)\ g_x(x_k,y_k)\right]$$

where $D_k=f_x(x_k,y_k)g_y(x_k,y_k)-g_x(x_k,y_k)\ f_y(x_k,y_k)$.

We obtain $x_{k+1}=x_k+\Delta x$, and $y_{k+1}=y_k+\Delta y$.

Writing the Eqns.(A2) in matrix form, we get

$$\begin{bmatrix} f_x(x_k,y_k) & f_y(x_k,y_k) \\ g_x(x_k,y_k) & g_y(x_k,y_k) \end{bmatrix}\begin{bmatrix}\Delta x\\ \Delta y\end{bmatrix}=-\begin{bmatrix} f(x_k,y_x) \\ g(x_k,y_k)\end{bmatrix}$$

or, $$\mathbf{J}_k\ \mathbf{\Delta x}=-\mathbf{F}(x_k,y_k) \tag{A3}$$

where $\mathbf{J}_k=\begin{bmatrix} f_x & f_y \\ g_x & g_y\end{bmatrix}_{(x_k,y_k)}$, $\mathbf{F}=\begin{bmatrix} f\\ g\end{bmatrix}$, $\mathbf{\Delta x}=\begin{bmatrix}\Delta x\\ \Delta y\end{bmatrix}$

The solution of the system (A3) is

$$\mathbf{\Delta x}=-\mathbf{J}_k^{-1}\ \mathbf{F}(x_k,y_k)$$

where $\mathbf{J}_k^{-1}=\begin{bmatrix} f_x & f_y \\ g_x & g_y\end{bmatrix}^{-1}_{(x_k,y_k)}=\frac{1}{\mathbf{D}_k}\begin{bmatrix} g_y & -f_y \\ -g_x & f_x\end{bmatrix}_{(x_k,y_k)}$

Therefore, we can write

$$\begin{bmatrix}\Delta x\\ \Delta y\end{bmatrix}=-\mathbf{J}_k^{-1}\begin{bmatrix} f(x_k,y_k) \\ g(x_k,y_k)\end{bmatrix}$$

and $$\begin{bmatrix} x_{k+1}\\ y_{k+1}\end{bmatrix}=\begin{bmatrix} x_k\\ y_k\end{bmatrix}+\begin{bmatrix}\Delta x\\ \Delta y\end{bmatrix}=\begin{bmatrix} x_k\\ y_k\end{bmatrix}-\mathbf{J}_k^{-1}\begin{bmatrix} f(x_k,y_k) \\ g(x_k,y_k)\end{bmatrix},\quad k=0,1,\ldots$$

or, $$\mathbf{x}^{(k+1)}=\mathbf{x}^{(k)}-\mathbf{J}_k^{-1}\ \mathbf{F}(\mathbf{x}^{(k)}) \tag{A4}$$

where $\mathbf{x}^{(k)} = [x^{(k)}, y^{(k)}]^T$, $\mathbf{F}(\mathbf{x}^{(k)}) = [f(x_k, y_k), \quad g(x_k, y_k)]^T$

This method can be easily generalized for solving a system of n equations in n unknowns

$$\begin{aligned} &f_1(x_1, x_2, \ldots, x_n) = 0 \\ &f_2(x_1, x_2, \ldots, x_n) = 0 \\ &\ldots \quad \ldots \\ &f_n(x_1, x_2, \ldots, x_n) = 0 \end{aligned} \tag{A5}$$

or, $$\mathbf{F}(\mathbf{x}) = \mathbf{0}$$

where $\mathbf{x} = [x_1, x_2, \ldots, x_n]^T, \mathbf{F} = [f_1, f_2, \ldots, f_n]^T$

If $\mathbf{x}^{(0)} = [x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}]^T$ is an initial approximation to the solution vector $\mathbf{x}$, then we can write the method as

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - \mathbf{J}_k^{-1}\,\mathbf{F}(\mathbf{x}^{(k)}), \quad k = 0, 1, \ldots \tag{A6}$$

where

$$\mathbf{J}_k = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 & \cdots & \partial f_1/\partial x_n \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 & \cdots & \partial f_2/\partial x_n \\ \vdots & & \cdots & \\ \partial f_n/\partial x_1 & \partial f_n/\partial x_2 & \cdots & \partial f_n/\partial x_n \end{bmatrix}_{(\mathbf{x}^{(k)})}$$

is the Jacobian matrix of the functions $f_1, f_2, \ldots, f_n$ evaluated at $\mathbf{x}^{(k)}$.

Note that the matrix $\mathbf{J}_k^{-1}$ is to be evaluated for each iteration. We can also write the method as

$$\mathbf{J}_k(\mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}) = -\mathbf{F}(\mathbf{x}^{(k)}) \tag{A7}$$

or, $$\mathbf{J}_k\,\boldsymbol{\in}^{(k)} = -\mathbf{F}(\mathbf{x}^{(k)}) \tag{A8}$$

where, $\boldsymbol{\in}^{(k)} = \mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}$ is the error vector.

We solve it as linear system of equations (by a direct method if the system is small and by an iterative method if the system is large) for each iteration.

—x—

PRACTICAL ASSIGNMENTS

Session 3:

1. Write a program to solve the boundary value problem $y'' + p(x)y = r(x)$, $y(a) = \alpha$, $y(b) = \beta$, $a \le x \le b$, using a second order finite difference method. Input h.

2. Write a program to solve the boundary value problem $a(x)y'' + b(x)y' + c(x)y = r(x)$, $y(a) = \alpha$, $y(b) = \beta$, $a \le x \le b$, using second order central difference approximations to both y'' and y'. Input h. Test your program on exercises E4) and E5).

3. Write a program to solve the boundary value problem $a(x)y'' + b(x)y' + c(x)y = r(x)$, $y(a) = \alpha$, $y(b) = \beta$, $a \le x \le b$, using shooting method. Use Taylor series method of second order to solve the corresponding initial value problems. Test your program on Example 6 and E6).