UNIT 4 SINGLE SPECIES POPULATION MODELS

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4.1 INTRODUCTION

Ecological studies have assumed greater significance in view of human concern for environmental degradation witnessed during the last four decades or so. Ecological modelling has been revived with great interest in the recent years due to the following:

i) Measurements in ecosystems are becoming more accurate and precise due to instrumental facilities.

ii) Satellite data has the potential to provide estimates of vegetation in the water body.

iii) Present capability of models to deal with a large system of non-linear equation due to improving computing facilities.

Mathematical ecology can be described as the study of inter dependence of several species in a variable eco-system. The approach based on modelling pre-supposes certain degree of idealization so that mathematical techniques can be brought into play. Although the so-called idealized models might not capture the full diversity of dynamic environmental landscape, they promise to provide an insight into growth pattern of various populations. This knowledge has been exploited fruitfully in management of renewable resources, ecological control of pests, evolution of pesticide resistant strains, etc. The prospective application of studies on bacteria and viruses to various bio-medical sciences has attracted many researchers to this area.

In this unit, we shall discuss some discrete and continuous models of single species population growth. In Sec. 4.2, we give some basic concepts of modelling population growth. Many species have no overlap between successive generations and so population growth for such species is in discrete steps. Some discrete population models are discussed in Sec. 4.3. For continuous population growth the delay model incorporating the effect of time delay in growth and reproduction of individuals in the population is discussed in Sec.4.4. For understanding the discussion in this unit the knowledge of the equilibrium points of a difference equation and their stability analysis is essential. We have given these details in an appendix at the end of the unit. You must go through the appendix while reading this unit.

Objectives

After studying this unit, you should be able to:

- apply the knowledge of calculus, analysis, differential equations and difference equations in building mathematical models of population dynamics both for discrete as well as continuous population growth;
- analyse the models both quantitatively and qualitatively;
- carry out the stability analysis of models of population growth.

4.2 BASIC CONCEPTS OF MATHEMATICAL MODELLING IN POPULATION DYNAMICS

A mathematical model in population dynamics represents the pattern of growth of a given population in the presence of various environmental factors. Real life eco-system consists of many species, which interact with each other in many different ways. In addition, various environmental factors like migration, territorial behaviour and climatic fluctuations also play important role. But all these factors cannot be accommodated in one model since otherwise the model would become too complex for solution by known mathematical techniques. The crucial decision, while modelling a given eco-system, lies in the choice of most relevant variables. The simplest approach is to include one factor at a time and subsequently modify the model by adding another factor.

Thus, a good mathematical model in population dynamics relies on thorough understanding and appreciation of biological problems, mathematical description of the relevant biological phenomena and subsequent derivation and interpretation of mathematical results for the use of biologists. It acts like a bridge in the description of theoretical details of a population and their mathematical abstractization.

We shall now discuss in brief some of the concepts which we shall be using for the study in this unit.

Classification

The mathematical models in population dynamics can be classified based on various factors like nature of growth rate, its fluctuation, environmental factors, size of the population and so on.

On the basis of growth rate, the population models can be further classified as follows:

1) **The deterministic models** which presuppose a fairly large population size and ignore random fluctuation in the environmental factors with time.
2) **The stochastic models** which are more appropriate when populations are small or when there is significant random fluctuation in the parameters.

The deterministic models are further subdivided into two subclasses:

a) **Continuous deterministic models** which are employed when population is very large and its growth rate is also fast so that variables can be modelled by continuous functions and the growth rate by their derivatives. These models lead to differential equations. (Their typical examples are those dealing with insect population, grass and zooplanktons).

b) **Discrete deterministic models** which are applicable to the cases when moderately large populations such as humans, large animals like lions, elephant are modelled. In such models the variations in population are studied by means of difference equations.

Analysis of Models

The models in population dynamics can be either quantitative or qualitative. Quantitative analysis of models involves consideration of actual parameters and prediction in terms of numbers. On the other hand, qualitative analysis concentrates on study of the nature and pattern of growth and their dependency on parameters relevant to the model. Thus while a quantitative study may require the knowledge of numerical analysis, the qualitative study of a model needs comprehensive knowledge of stability analysis techniques of difference/differential equations.

Stability Analysis

Consider a system involving a number of variables $x_1, x_2, x_3, \ldots, x_n$ denoting the densities of species composing the system. The state of the system can then be represented by a point in n-dimensional phase space. To each point in this space we can attach a vector or an arrow indicating how the system moves. These vectors can be joined to form trajectories which show the long term behaviour of the system.

A **stationary point** or an **equilibrium point** of a system is one which has associated with it a vector of zero length. That is, if a system is at a stationary point at an instant it will remain at that point at the next instant.

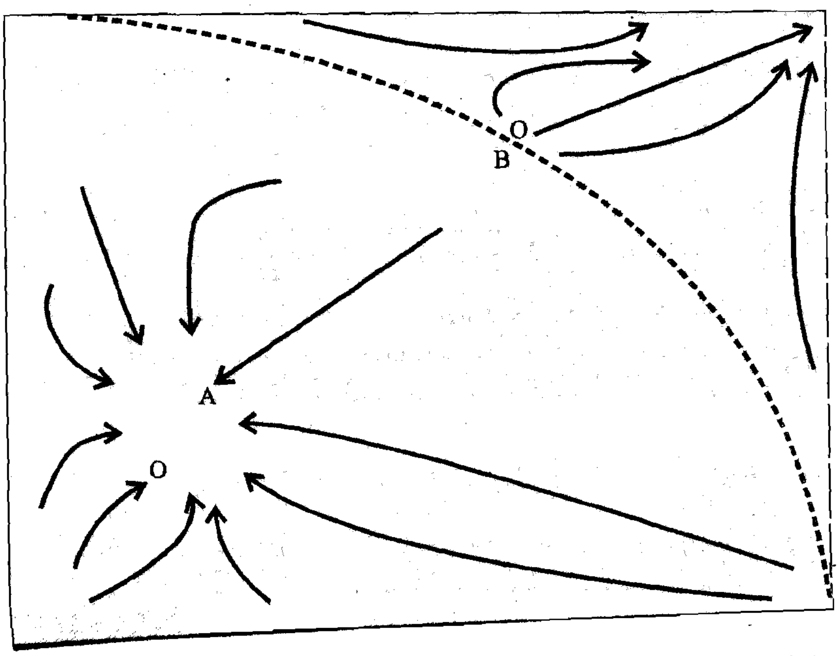

Fig. 1: A System with A as stable and B as unstable equilibrium points.

The equilibrium point may be stable or unstable. If a system slightly disturbed from equilibrium point returns to that point, then, the equilibrium point is said to be **stable**. More precisely, it is said to have neighbourhood stability. On the other hand when a system disturbed slightly from an equilibrium point continues to move further away from that point, then the equilibrium point is said to be **unstable**. Fig. 1 on previous page shows the trajectories and equilibrium points A and B of a system.

If a system moves to a particular equilibrium point irrespective of the point from where it started that equilibrium point is said to have **global stability**. You may note that in Fig. 1 the equilibrium point A has neighborhood stability but not global stability.

The stability of equilibrium point can well be understood by the analogy of a landscape in which troughs represent stable points and peaks represent unstable ones and the behaviour of the system is represented by a rolling ball. Mathematically speaking the neighborhood stability analysis is the easiest one. By considering only small displacements from an equilibrium point, it is possible to linearize the equations and discuss their behaviour. We shall be using this technique for the discussion of the models considered in this unit.

In the next section we shall discuss various discrete population models of single species.

4.3 DISCRETE POPULATION MODELS

In many species, the populations do not overlap. There is a fixed interval of time between successive generations and so population grows in discrete steps. The size of steps can vary from species to species. In general the population of a given generation depends on that of the preceding generations. Thus if we scale the time step to be 1 and denote the initial population by x_0 and the population of nth generation at time n by x_n then population of next generation x_{n+1} at time $(n+1)$ can be written in the form of following difference equation:

$$x_{n+1} = x_n \; F(x_n) = f(x_n) \tag{1}$$

where $f(x_n)$ is, in general, a non-linear function of x_n. Thus, study of discrete population models is equivalent to the study of difference equations of the form given by Eqn. (1).

The efficiency of a model representing a specific population growth lies in determining the appropriate form of $F(x_n)$ to reflect the factual situation of the species in question. The function $F(x_n)$ is called **recruitment function**. Taking various forms for $F(x_n)$, we can construct different models of population growth. We shall now discuss them one-by-one.

4.3.1 Exponential/Constant Growth Model

In the simplest of the cases, we consider the species for which the birth and death rates are constant. For example, many insects and annual desert plants reproduce once and then they fade away. The surviving offspring forms the basis for the next generation. Thus, the population increases or decreases by the same amount each year. In such a situation we can take the recruitment function as constant and arrive at linear or exponential model of discrete population. This model was first propounded by Thomas R. Malthus (1766-1834), an English clergyman and political economist and hence is also called **Malthusian growth model.**

Formulation

Let us assume that the population is closed i.e., it changes only by births and deaths and there is no migration into or out of the region. We further suppose that the birth rate b and death rate d are constants. This means that the birth and death of individuals in a given population are proportional to the population size.
Then

$$x_{n+1} - x_n = (b-d)\ x_n$$

or

$$x_{n+1} = x_n + (b-d)\ x_n = (1+b-d)\ x_n \tag{2}$$

Substituting $r = (1+b-d)$ in Eqn. (2) we get the following linear homogeneous difference equation

$$x_{n+1} = r\ x_n \tag{3}$$

where constant r represents the **net growth rate**, also called **net reproductive rate**. Eqn. (3) together with the prescribed initial population size x_0 determines the population size in each generation.

Solution and Interpretation

By a solution of the difference equation with initial value x_0, we mean a sequence $\{x_n\}$ such that $x_{n+1} = rx_n$ for $n = 1, 2, \ldots$ with x_0 as prescribed. The difference Eqn. (3) can be solved iteratively by substituting

$$x_n = r\ x_{n-1},\ x_{n-1} = r\ x_{n-2}, \ldots, x_1 = r\ x_0$$

Thus unique solution is given by

$$x_n = r^n\ x_0,\ n = 1, 2, \ldots$$

In general, we can say that if $|r| < 1$ then $x_n \to 0$ as $n \to \infty$ implying that the population is ultimately driven to extinction and for $|r| > 1$, x_n grows unbounded as $n \to \infty$. In the present situation the case $r < 0$ is ruled out. Therefore, if $0 \le r < 1$ then x_n decreases monotonically to zero and if $r > 1$, then x_n increases to $+\infty$.

If $r > 1$, the growth of population over each time interval of length n occurs by the same rate but not by the same amount as shown in Fig. 2.

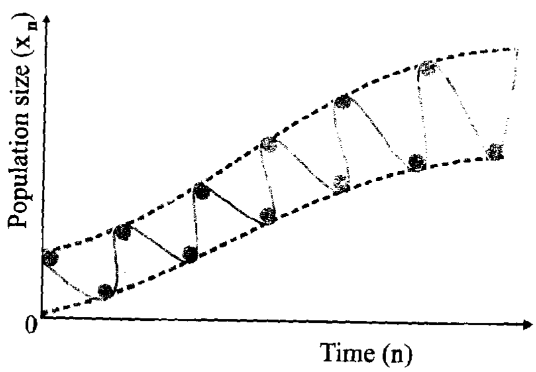

Fig. 2: Discrete Population Model

In case of species which reproduce annually and death occurs throughout the year, the population growth curve resembles a jagged saw blade with a sharp increase, resulting from births followed by gradual decrease from death during the rest of the year. The overall curve will rise exponentially, because the growth rate is positive. The size of each tooth in the growth curve will increase year after year because same proportional increase will add more individuals to a large population. If the time interval between reproductive periods reduces, the occurrence of teeth on the graph gets closer. Finally if the time interval is infinitesimally small the curve is no longer jagged but smooth and resembles the corresponding curve for exponential model of continuous growth.

This simple model of population growth is not very realistic for most populations nor for long times but, even so, it has been used successfully with some justification for the early stages of growth of certain bacteria. Another variation of the exponential growth model can be considered by incorporating a constant migration rate m per generation which we shall consider.

Formulation

Let the positive value of m represents immigration and negative value denotes emigration. Then the difference Eqn. (3) becomes

$$x_{n+1} = r\,x_n + m \tag{4}$$

Solution and Interpretation

Eqn. (4) can be solved iteratively by considering

$$x_1 = r\,x_0 + m$$

$$x_2 = r\,x_1 + m$$

$$= r(r\,x_0 + m) + m = r^2\,x_0 + r\,m + m$$

$$x_3 = r\,x_2 + m$$

$$= r(r^2\,x_0 + r\,m + m) + m = r^3\,x_0 + r^2\,m + r\,m + m$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\vdots \qquad \vdots \qquad \vdots \text{ and so on;}$$

Then by induction method we can prove that,

$$x_n = r^n x_0 + m(r^{n-1} + r^{n-2} + \cdots + r + 1)$$
$$= r^n x_0 + m\left(\frac{1-r^n}{1-r}\right)$$
$$= \left(x_0 - \frac{m}{1-r}\right) r^n + \frac{m}{1-r}$$

Now, if $r > 1$, then x_n grows unbounded for $m > (1-r)\, x_0$ but x_n reaches zero if $m < (1-r)\, x_0$; thus sufficiently large emigration will wipe out a population that would otherwise grow unbounded. If $0 < r < 1$, then as $n \to \infty$, x_n tends to the limit $m/(1-r) > 0$ for $m > 0$, while x_n tends to zero for $m < 0$. Thus, immigration may help survival of a population that would otherwise become extinct. Even this model is not very realistic. We now list the limitations of this model.

Limitations

- The model has constant per capita birth and death rates and generates limitless growth. This is highly unrealistic.
- The model ignores lags. The growth rate does not depend upon the past. Moreover the population responds instantaneously to change in the current population size.
- There is no consideration of temporal variability.

In the next sub-subsection we shall discuss a model which is an improvement over this model but let us first consider some applications of this model.

Example 1 (Fibonacci Sequence): Fibonacci, in the 18^{th} century, set a modelling exercise involving an hypothetical growing rabbit population. Start with a pair (male and female) of immature rabbits which after one reproductive season produce two pairs of male and female immature rabbits after which the parents stop reproducing. Their offspring pairs then do exactly the same and so on. The question is to determine the number of pairs of rabbits at each reproductive period. If we denote the number of pairs of rabbits by x_n at the n^{th} reproductive period then we have the model equation as

$$x_{n+2} = x_{n+1} + x_n, \; n = 1, 2, \ldots$$

with $x_0 = 0, \; x_1 = 1$

what is known as the Fibonacci sequence, namely

1, 1, 2, 3, 5, 8, 13, ...

To find the number of pairs of rabbits at the n^{th} reproductive period consider the equation

$$x_{n+2} = x_{n+1} + x_n$$

The equation can be written as

$$(E^2 - E - 1)\, x_n = 0 \text{ where, } E^p x_k = x_{k+p},$$

whose roots are given by

$$m^2 - m - 1 = 0.$$

Solving we obtain

$$m = \left(\frac{1 \pm \sqrt{5}}{2}\right).$$

The solution of the above model can be written as

$$x_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

After applying the conditions $x_0 = 0,\ x_1 = 1$ we get

$$x_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

which gives the number of pairs of rabbits at the n^{th} reproductive period.

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Example 2: Suppose that a population of yeast satisfying exponential growth model increases by 10% in an hour. If the initial population of yeast is 100,000 then find the population of yeast after four hours. How much time is required by the population to grow to double of its initial size?

Solution: The population of yeast satisfies the equation

$$P_{n+1} = (1 + 0.1) P_n \text{ with } P_0 = 100{,}000.$$

The population after one hour is $P_1 = 1.1\ P_0 = 110,\ 000$. After two hours, $P_2 = 1.1\ P_1 = (1.1)^2 P_0 = 121,\ 000$. Thus, after 4 hours, $P_4 = (1.1)^4 P_0 = 146,\ 410$.

For the population to double, it must reach $2P_0 = 200,\ 000$. Thus, we must solve $2P_0 = (1.1)^n P_0$ or $2 = (1.1)^n$.

By taking the logarithms of both sides we have $\ln(2) = \ln(1.1)^n = n\ \ln(1.1)$ or $n = \ln(2)/\ln(1.1) = 7.27$ hours as the required time.

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You may **notice** that the discrete Malthusian growth model is closely related to **compound interest problems**. If interest is compounded annually, then the amount of principal in any year n satisfies the discrete Malthusian growth model. The general formula for determining the amount of principal when interest rate is r (annual), which is compounded k times a year for n years, given an initial amount of P_0 satisfies:

$$P_n = (1 + r/k)^{kn} P_0,$$

where P_n is the amount of principal after n years.

In population studies, one can use this concept to examine growth rates for a population growing according to the Malthusian growth model for differing periods of time.

You may now try the following exercise.

E1) Suppose that a business is started with constructing a cow shed for 85 cows and a decision is taken that there will be an addition of 38 cows every year. Now, if mortality rate of cows is 5 percent per year then what is the population size of the cow shed after 15 years.

Before proceeding further, we give you a **graphical method** called **cobwebbing** of solving the models of discrete population growth.

Cobwebbing

Consider the difference Eqn. (1) viz.,

$$x_{n+1} = x_n F(x_n) = f(x_n).$$

Generally, in a population when the population size becomes very large, there is lack of food, space and other resources and pollution due to overcrowding. We expect $f(x_n)$ to have some maximum at x_m say, with f as a function of x_n decreasing for $x_n > x_m$. A typical growth form of f is as shown in Fig. 3.

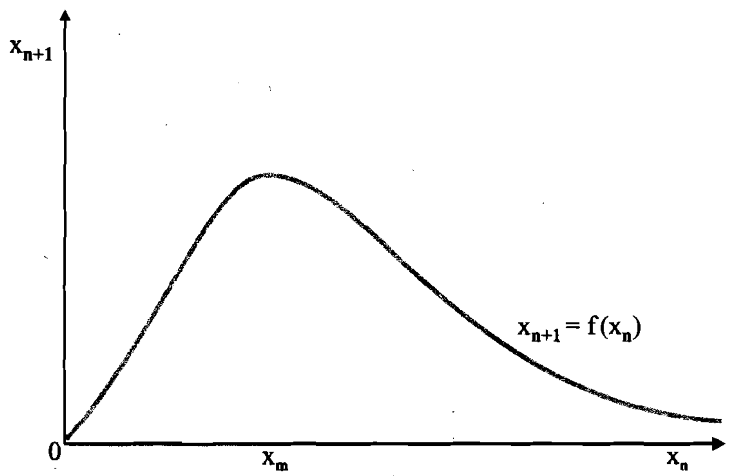

Fig. 3: Typical growth form in the model $x_{n+1} = f(x_n)$.

The steady states of Eqn. (1) are solutions x^* of

$$x^* = f(x^*) = x^* F(x^*)$$

so that $x = x^*$ is a constant solution of Eqn. (1).
Thus, the steady states are given by $x^* = 0$ or $F(x^*) = 1$.

Graphically the steady states are points of intersections of the curve $x_{n+1} = f(x_n)$ and the straight-line $x_{n+1} = x_n$ as shown in Fig. 4.

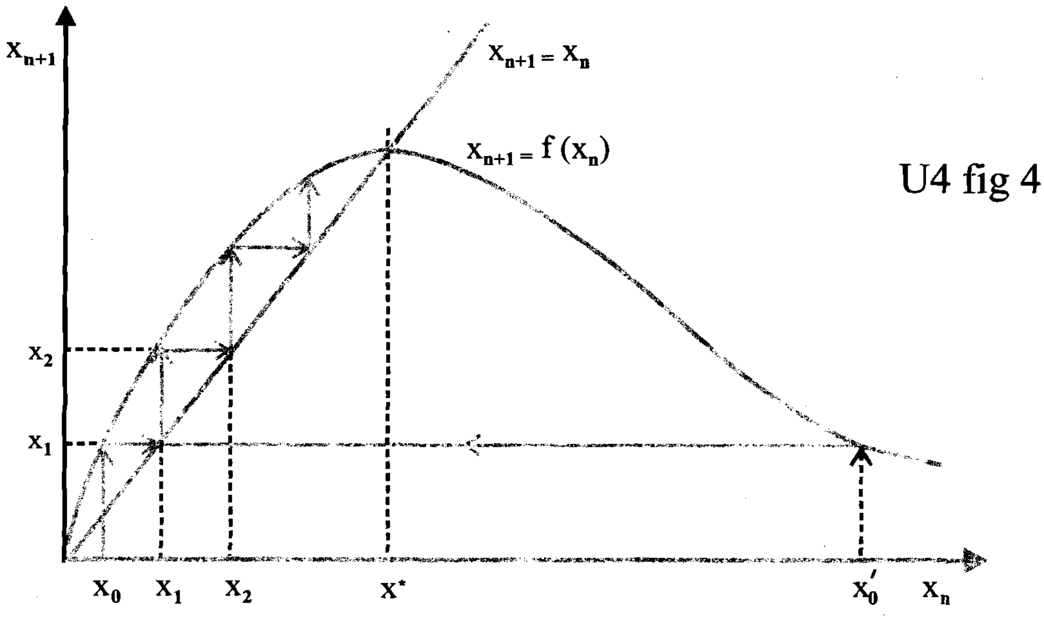

Fig. 4: Graphical determination of the steady-states of $x_{n+1} = f(x_n)$

We begin by drawing the point x_0 as shown in Fig. (4). Then x_1 is given by simply moving vertically to the curve $x_{n+1} = f(x_n)$. Then we go horizontally to the line $x_{n+1} = x_n$. We can then employ x_1 to get x_2 in similar fashion and proceed to arrive at points $x_3, x_4, \ldots$ and so on. The arrows show the path sequence. The path is simply a series of reflections in the line $x_{n+1} = x_n$. We observe that $x_n \to x^*$ monotonically as $n \to \infty$ as illustrated in Fig. (5). This behaviour has already been obtained analytically for the case $f(x_n) = rx_n$.

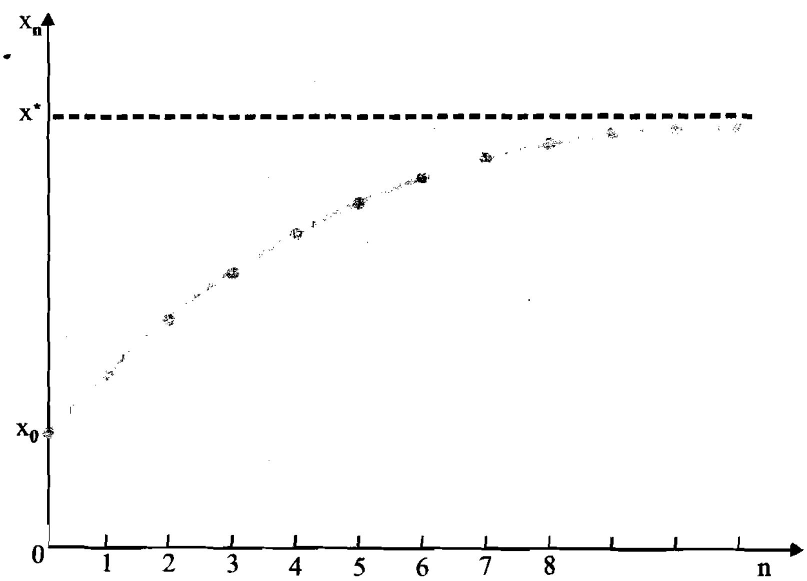

Fig. 5: Continuous curve showing populations at different time-steps for more clarity.

The Cobwebbing method can be applied to any difference equation of the form $x_{n+1} = F(x_n)$. It gives information about the behaviour of the solution and is particularly useful for difference equations whose analytic solutions are complicated.

You may now try the following exercise.

E2) A discrete model for a population N_t consists of

$$N_{t+1} = \frac{r\,N_t}{1+b\,N_t^2} = f(N_t)$$

where t is the discrete time and r and b are positive parameters. What do r and b represent in this model? Show, with the help of a cobweb, that after a long time the population N_t is bounded by

$$N_{min.} = \frac{2r^2}{(4+r^2)\sqrt{b}} \le N_t \le \frac{r}{2\sqrt{b}}.$$

Prove that, for any r, the population will become extinct if $b > 4$.

In the long run there must be some adjustment to such exponential growth. P. F. Verhulst in 1838 proposed that a self-limiting process should operate when a population becomes too large and suggested a model called logistic growth model which we shall discuss now.

4.3.2 Logistic Growth Model

As we have already mentioned the exponential growth model is applicable for early phases of population growth. As the population grows it faces crowding effects due to factors like toxic buildup, self-regulation or space and resource limitations, etc. In order to incorporate such factors, a negative quadratic term is added to the recruitment function reflecting consequent decrease in the growth of the population. This leads to the Logistic Growth Model.

Formulation

Assuming that the population size is large and the growth rate decreases with the increase in population on account of crowding effects, a difference equation model can be formulated as

$$x_{n+1} = r\,x_n\left(1-\frac{x_n}{K}\right) \tag{5}$$

where $r > 0$ represents **intrinsic growth** rate i.e., growth rate free from environmental constraints and K denotes the **carrying capacity** of the population which physically means the maximum sustainable population size that a particular environment can support over a long period of time. Eqn. (5) is called the **logistic difference equation.**

Let us analyze the model qualitatively. For the background needed for the qualitative analysis of the given model refer to the **appendix** given at the end of this unit.

Stability Analysis

The logistic growth model given by Eqn. (5) can be rescaled by substituting $u_n = \frac{x_n}{K}$, so that the carrying capacity is 1. The Eqn. (5) thus takes the form

$$u_{n+1} = r\, u_n (1 - u_n), \quad r > 0 \tag{6}$$

where we assume that $0 < u_0 < 1$ and we are interested in solutions $u_n \geq 0$. The steady states of Eqn. (6) are given by

$$u^* = f(u^*) = u^* F(u^*) \Rightarrow u^* = 0 \text{ or } F(u^*) = 1$$

Here $F(u^*) = r(1 - u^*)$ so $F(u^*) = 1 \Rightarrow u^* = \dfrac{r-1}{r}$

thus

$$u_1^* = 0 \text{ and } u_2^* = \frac{(r-1)}{r} \text{ are the two steady states of Eqn. (6).}$$

Further, $u_1^* = 0$ gives $\lambda_1 = f'(0) = r - 2ru^*\big|_{u^*=0} = r$

and $u_2^* = \dfrac{r-1}{r}$ gives $\lambda_2 = f'\left(\dfrac{r-1}{r}\right) = 2 - r$.

Thus, the eigen values corresponding to u_1^* and u_2^* are given by $\lambda_1 = r$ and $\lambda_2 = 2 - r$ respectively. If $0 < r < 1$, the steady state u_1^* is stable since $0 < \lambda_1 < 1$. As r increases and crosses the value 1, the steady state u_1^* becomes unstable while the positive steady state u_2^* becomes stable as long as $-1 < \lambda_2 < 1$. Hence the first bifurcation occurs at $r = 1$. The second bifurcation occurs at $r = 3$, where the positive steady state u_2^* undergoes a qualitative change. If r lies between 2 and 3 the equilibrium u_2^* is stable and as soon as r exceeds 3, it becomes unstable. Fig. 6 shows a schematic diagram of stable solutions of Eqn. (6) as r passes through bifurcation values. At each bifurcation the preceding steady state becomes unstable and has been represented by the dashed lines.

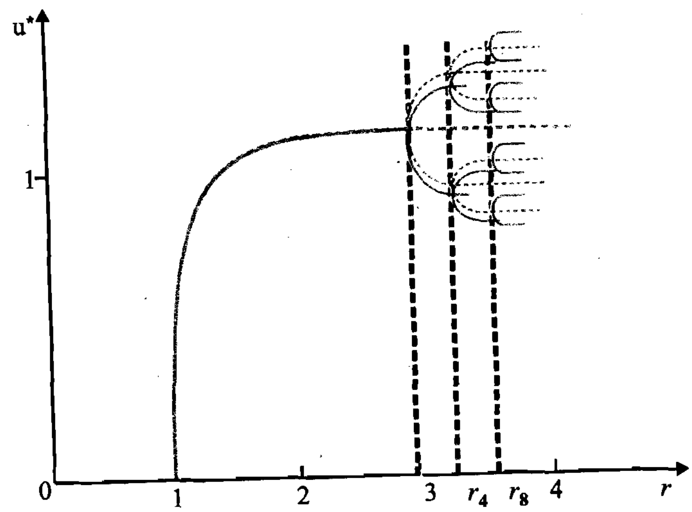

Fig. 6: Schematic diagram for the stability of Eqn. (6).

To see what is happening when r passes through the bifurcation value $r = 3$, we introduce the iterative procedure as follows:

$$u_1 = f(u_0)$$
$$u_2 = f(f(u_0)) = f^2(u_0)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$u_n = f^n(u_0)$$

Thus for Eqn. (6) the first iterative is simply Eqn. (6) while the second iterative is

$$u_{n+2} = f^2(u_n, r) = f(f(u_n, r))$$
$$= r[r\,u_n(1-u_n)]\,[1 - r\,u_n(1-u_n)]$$

It can be easily shown that there are two more equilibria u_3^* and u_4^* for the second iterative $u_{n+2} = f^2(u_n, r)$ when r exceeds 3 and that both these steady states are stable. This means that there is a stable equilibrium of the second iterative which results into a stable periodic solution of period 2 of Eqn. (6).

As r continues to increase, the eigen values at u_3^* and u_4^* again undergo a qualitative change and so these 2-period solutions become unstable. At this stage we consider the steady states of fourth iterative of Eqn. (6) and a 4-cycle periodic solution is obtained. Fig. 7 shows a 4-cycle periodic solution schematically.

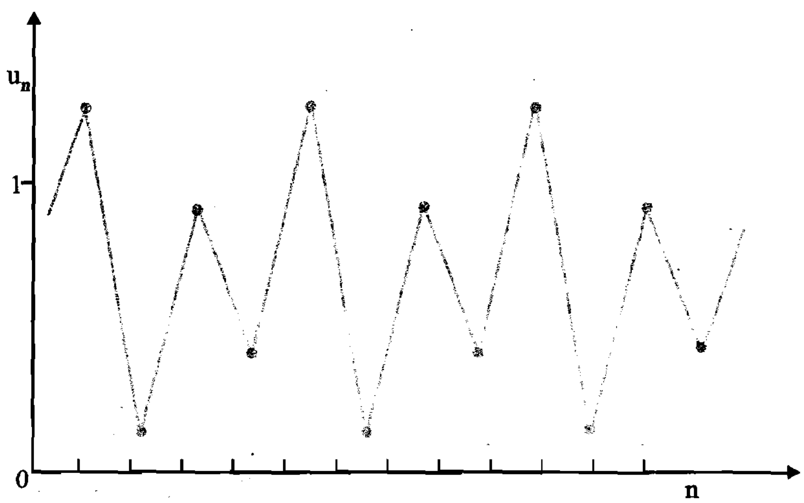

Fig. 7: A 4-cycle periodic solution

Thus as r increases the solution passes through a series of bifurcation, doubling the period of periodic solution each time. There is a limiting value r_c at which instability sets in for all periodic solutions and a chaotic or aperiodic solution occurs. This critical value in the present model as given by $r_c \approx 3.828$ [ref. 'Mathematical Biology' by J. D. Murray]. The solutions in this case oscillate randomly. The fluctuations in a chaotic solution do not arise by chance factor or randomness. Once the parameters are specified, the same erratic population track will be obtained. The main property of chaotic solutions is that a very small change in initial conditions can lead to very different population behaviour. There is widespread interest among scientists regarding chaotic behaviour. The research in this direction has resulted in many different applications of chaos.

The discrete logistic model is very significant from biological as well as mathematical point of view, because of the fact that such a simple model is capable of predicting an apparently unpredictable behaviour. However, there are limitations of this model too which we list below.

Limitations

Sigmoids are tilted S-shaped curves that resemble trends in the life-cycle of many living organisms and phenomenon.

- The logistic model is not suitable for a population of small size.
- Although many populations exhibit the sigmoid pattern of growth still they do not increase according to the logistic equation eventually.
- The carrying capacity of the population has been taken as constant whereas in view of the changing environmental factors, it keeps varying.
- The population has been considered as a homogeneous group of individuals where death and birth take place simultaneously. But in real populations, there may be some delay on account of development time, breeding seasons and other environmental factors. Further, the intensity of these processes is different for different age groups.

Let us now consider some applications of this model.

Example 3: Consider the model given by Verhulst

$$x_{n+1} = \frac{rx_n}{x_n + A}, \; r > 0, \; A > 0.$$

In this case we have

$$f(x_n) = \frac{rx_n}{x_n + A} \text{ and } F(x_n) = \frac{r}{x_n + A}.$$

Hence the equilibrium points are given by $x_n = 0$ and $x_n = r - A$. Thus, if $r < A$ the only equilibrium corresponding to a non negative population size is $x_n = 0$. Since $f'(0) = r/A < 1$, at this point system is **asymptotically stable** and every solution tends to zero (see Appendix). If $r > A$ there are two steady states at $x_n = 0$ and $x_n = r - A$. Since, $f'(0) = r/A > 1$, the equilibrium at $x_n = 0$ is **unstable** and since $f'(r-A) = \frac{A}{r} < 1$ the equilibrium $x_n = r - A$ is **asymptotically stable.**

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Example 4: Consider an example of the Ricker model given by the equation

$$N_{t+1} = N_t \exp\left[r\left(1 - \frac{N_t}{K}\right)\right]. \tag{7}$$

The dynamics of this model is similar to the continuous time logistic model if population growth rate is small $(0 < r < 0.5)$. However, if the population growth rate is high, then the model may exhibit more complex dynamics including damping oscillations, cycles or chaos. It can be shown that Ricker's model is stable if $0 < r < 2$. When r assumes the value 2 the first bifurcation occurs leading to a 2-cycle periodic solution. If r further increases these solution become unstable and at $r = 2.6$, a 4-cycle periodic solution appears. Thus with increasing values of r the solution passes through a series of bifurcations, each time doubling the period. When r reaches near the critical value 3 then an aperiodic or chaotic solution exists. There are simulations of population dynamics using Ricker's method with different values of r, as shown in Fig. 8.

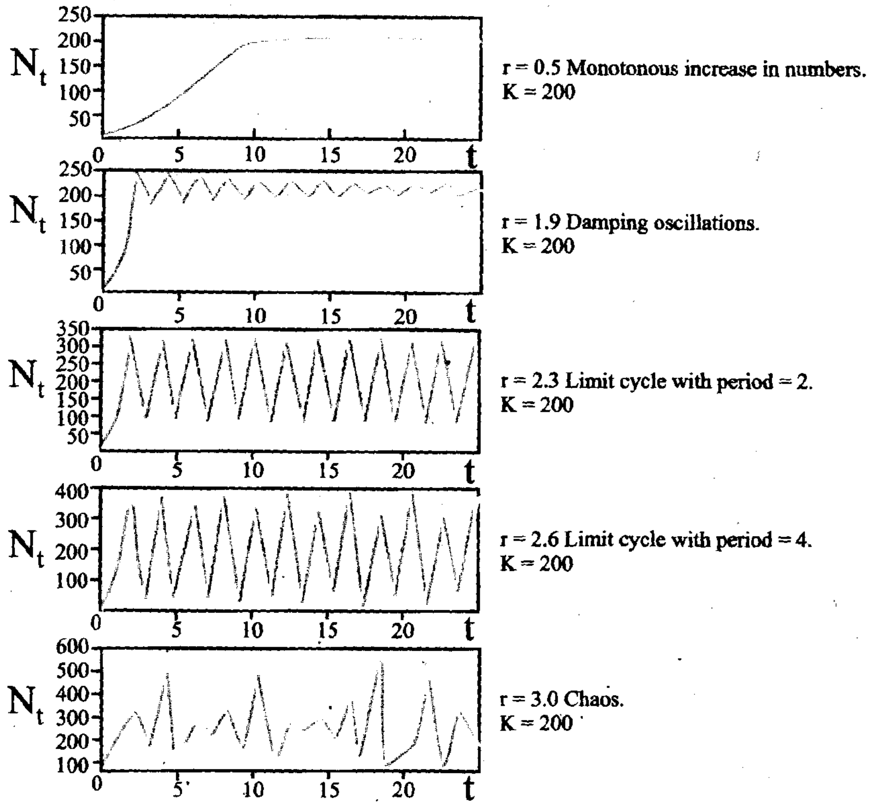

Fig. 8: Solutions N_t of the Model (7) for various r.

In the upper two figures the model has a stable equilibrium, only the patterns of approaching the equilibrium are different. In the lower three figures there is no stable equilibrium. Non-equilibrium dynamics may be of 2 types: a limit cycle when the trajectory repeats itself, and chaotic when the trajectory does not repeat itself.

Chaos dynamics looks like stochastic noise, however the model is absolutely deterministic. Chaotic models are widely used for random number generation in computers.

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You may now try the following exercises.

The removal of members of a population at a specified rate is termed as **harvesting**.

E3) Find the two equilibria u_3^* and u_4^* of the second iterative given by Eqn. (8) and verify that they are stable.

E4) Consider a population of peacock modeled by the equation $P_{i+1} = P_i + r^* h^* P_i^* (1 - P_i / K) - h^* H$, $i = 0, 1, 2, 3, \ldots$ where H is the harvesting rate.

Let $P_0 = 100$, $t_0 = 0$, $r = 0.0987$, $h = 1.0$ year and $K = 194.6$; all populations are expressed in thousands.

In general the harvesting rate depends on time and is represented by the function $H(t)$. If $H(t)$ is constant then harvesting is called **constant yield harvesting**. If $H(t)$ is a linear function of population size then it is called **proportional** or **constant effort harvesting**.

a) Determine and graph the populations of peacocks for a period of 50 years using the harvesting rates $H = 0, 2, 4$, and 6. For each value of H describe the trend of the peacock population. Does it appear to approach a stable state? If so, what is that value?

b) What is the first value of H (to the nearest tenth) that does not produce a steady state population?

E5) Determine the non negative steady state and discuss their linear stability of the following discrete time population models:

a) $N_{t=1} = N_t\left[1 + r\left(1 - \frac{N_t}{K}\right)\right].$

b) $N_{t=1} = \frac{r\,N_t}{(1 + a\,N_t)^b}.$

In earlier models of discrete population growth, a common assumption was that all the members of a generation contribute to the growth. This is not true in general because for most of species the individuals reproduce only after attaining a certain degree of maturity in age. We shall now discuss a model that incorporates the effect of this delay.

4.3.3 Delay Model

For most of the animals and other species there is a substantial maturation time to sexual maturity. Such a delay can also be caused by delayed response to environmental changes. If the delay is observed to be m time steps then the basic difference equation of the model takes the form

$$x_{n+1} = f(x_n, x_{n-m}) \tag{8}$$

Now we study such a model.

Formulation

We assume that the growth rate of a population reduces with the increase in population due to crowding effects. Further, unlike the logistic model, the population x_{n+1} at $(n+1)$th generation remains positive. Thus the model can be formulated as

$$x_{n+1} = x_n \exp r\left(1 - \frac{x_{n-m}}{K}\right) \tag{9}$$

where $r > 0$ is the intrinsic growth rate and $K > 0$ is the carrying capacity. Rescaling the model by substituting $u_n = x_n / K$, it can be written as

$$u_{n+1} = u_n \exp\left[r(1 - u_{n-m})\right],\ r > 0 \tag{10}$$

As a simple case we consider the delay to be of one time step. Then Eqn. (10) can be written as

$$u_{n+1} = u_n \exp\left[r\,(1 - u_{n-1})\right] \tag{11}$$

The steady states of the difference Eqn. (11) are given by

$u_1^* = 0$ and $u_2^* = 1$.

We now discuss the stability analysis of the steady states.

Stability Analysis

Let us discuss the stability of the equilibrium states $u_1^* = 0$ and $u_2^* = 1$.
Since $|f'(u_1^*)| = e^r > 0$.
The steady state $u_1^* = 0$ turns out to be **unstable**.
Next we linearize the equation about the steady state $u_2^* = 1$ by substituting $u_n = 1 + \varepsilon_n$, where $|\varepsilon_n| << 1$.

We obtain

$$1 + \varepsilon_{n+1} = (1 + \varepsilon_n) \exp(-r\varepsilon_{n-1})$$

Omitting higher powers of ε_n because $|\varepsilon_n| << 1$.
We have

$$1 + \varepsilon_{n+1} = (1 + \varepsilon_n)(1 - r\varepsilon_n)$$

or, $1 + \varepsilon_{n+1} = 1 + \varepsilon_n - r\,\varepsilon_{n-1}$
Hence the difference equation to determine the steady state is given by

$$\varepsilon_{n+1} - \varepsilon_n + r\,\varepsilon_{n-1} = 0 \tag{12}$$

Substituting $\varepsilon_n = z^n$ Eqn. (12) takes the form

$$z^2 - z + r = 0 \tag{13}$$

If $r < 1/4$ then the roots of Eqn. (13) are real and are given by

$$z_1 = 1/2\,[1 + (1 - 4r)^{1/2}]$$

and $z_2 = 1/2\,[1 - (1 - 4r)^{1/2}]$
if $r > 1/4$ then the roots of Eqn. (13) are imaginary and are given by

$$z_1 = 1/2\,[1 + i\,(4r - 1)^{1/2}]$$

$$z_2 = 1/2\,[1 - i\,(4r - 1)^{1/2}]$$

The complete solution of Eqn. (13) is given by

$$\varepsilon_n = c_1 z_1^n + c_2 z_2^n \tag{14}$$

where c_1 and c_2 are arbitrary constants.

If $0 < r < 1/4$ then both z_1 and z_2 lies between 0 and 1 and hence $\varepsilon_n \to 0$ as $n \to \infty$. Therefore the state $u^* = 1$ is linearly **stable equilibrium** state.

If $r > \frac{1}{4}$ then the two complex roots of quadratic Eqn. (13) can be written as

$$z_1 = \rho e^{i\theta} \quad z_2 = \rho e^{-i\theta} \text{ with } z_2 = \bar{z}_1$$

where $\rho = r^{1/2}$ and $\theta = \tan^{-1}(4r - 1)^{1/2}$

and $z_1 z_2 = |z_1|^2 = \rho^2 = r$.
Thus for $\frac{1}{4} < r < 1$, $|z_1||z_2| < 1$.

In this case solution (14) is

$$\varepsilon_n = c_1 z_1^n + c_2 \bar{z}_1^n \tag{15}$$

and since it is real we must have $c_2 = \bar{c}_1$.

If we now let $c_1 = \alpha e^{i\gamma}$ then $c_2 = \alpha e^{-i\gamma}$ then for complex roots Eqn. (15) can be written as

$$\varepsilon_n = \alpha\,[\rho^n e^{i(n\theta+\gamma)} + \rho^n e^{-i(n\theta+\gamma)}]$$

or, $\varepsilon_n = 2\alpha\rho^n \cos(n\theta + \gamma)$ where $\gamma = \tan^{-1}\frac{c_1}{c_2}$ and $\theta = \tan^{-1}(4r-1)^{1/2}$.

Hence $\varepsilon_n \to 0$ as $n \to \infty$ and the state $u^* = 1$ is **stable**.

As $r \to 1,\ \theta \to \tan^{-1}\sqrt{3} = \frac{\pi}{3}$.

As r passes through the critical $r_c = 1$ i.e. $r > 1$ then $|z_1| > 1$ and ε_n given by Eqn. (15) grows unboundedly with $n \to \infty$ and $u^* = 1$ is then unstable. Since $\theta \approx \pi/3$ for $r \approx 1$ and $\varepsilon_n \approx 2\alpha\cos(n\pi/3 + \gamma)$, which has a period of 6 we expect the solution of Eqn. (11), at least for r just greater than $r_c (=1)$ to exhibit 6 cycle periodic solution.

Limitations

- In the model discussed above, the discrete delay due to development time of the species has been incorporated. But this model does not accurately model populations with continuous growth and time lags.
- Sometimes the delay can be caused by the factors limiting the species. This is generally the case when the limiting factor itself is a species subject to delay due to prolonged development.

Let us consider the following example.

Example 5: An animal may feed on the number of plants which is annual, so that the total food available depends on the number of the plants in the previous year and thus in turn is a function of the number of animals present in the previous years. The logistic equation can then be written as a time delay equation of the form.

$$\frac{n(T_{m+1}) - n(T_m)}{T} = n(T_m)\left(\frac{1 - n(T_{m-1})}{N}\right)$$

with a delay of one period, equal to T units of time.

We can easily find out that the equilibrium value of $n(T_m)$ in terms of N. To find out how $n(T_m)$ is going to behave near the value N we will use the expression given below in the place of above equation for small y,

$$y(T_{m+1}) - y(T_m) \approx y(T_{m-1})\,T/\tau \tag{16}$$

where we have neglected the quadratic term in y.

The general solution of an equation of the form (16), which is called a recurrence relation, is

$$y(T_m) = Az_1^m + Bz_2^m \tag{17}$$

where A and B are constants which can be determined by using the initial conditions and z_1 and z_2 are the roots of the quadratic equation

$$z^2 - z + \frac{T}{\tau} = 0. \qquad (18)$$

Solving Eqn. (18) we obtain

$$z_1 = \frac{1}{2}\left(1 + \sqrt{(1 - 4T/\tau)}\right) \text{ and } z_2 = \frac{1}{2}\left(1 - \sqrt{1 - 4T/\tau}\right)$$

z_1, z_2 are both real if $\frac{T}{\tau} \le \frac{1}{4}$ and then the usual logistic behaviour occurs.

When $\frac{T}{\tau} > \frac{1}{4}$, the two roots z_1, z_2 are complex conjugate and can be written in the form $re^{\pm i\theta}$ where, $r = \sqrt{\frac{T}{\tau}}$, $\cos\theta = \sqrt{\frac{\tau}{4T}}$. $\left(0 \le \theta \le \frac{\pi}{2}\right)$.

The solution of Eqn. (18) can then be written as

$$y(T_m) = C\, r^m \sin(\theta m + D)$$

where C and D are constants determined by the initial conditions in the same way as A and B. In these conditions $n(T_m)$ oscillate about its equilibrium value with a period $2\pi T/\theta$. When $T/\tau > 1$ the oscillation get larger with m, so that Eqn. (17) is not valid in this case. When $T/\tau < 1$ the oscillation gets smaller, so that $n(T_m)$ converges to its equilibrium value N. This shows that a time delay makes the population less stable.

***

You may now try the following exercise.

E6) The population $n(m)$ of a certain species after the m^{th} breeding season is related to $n(m-1)$ by

$$n(m) = N\,\alpha\, n(m-1)/(N + n(m-1))$$

where α is a number greater than one, and N is a constant. Find the equilibrium value of n and show that n approaches this value monotonically.

In the next section we shall consider various models of continuous population growth.

4.4 CONTINUOUS POPULATION MODELS

As mentioned earlier, if the population size is very large and births take place continuously we may assume the population growth to be continuous.

The continuous models of exponential growth or Malthusian model and logistic model which are analogues of discrete models given by Eqn. (3) and Eqn. (5) respectively, have been dealt exhaustively in Block 3, MTE-14 of mathematical modelling course of our Bachelors Degree programme. We shall assume that you are familiar with these models. In case you are not, then you can go through Unit 8, Block 3 of MTE-14.

You may **observe** that both the logistic and exponential models of continuous growth, share the assumption of absence of no migration, genetic variation or age structure in a given population. In addition logistic model assumes constant carrying capacity, limited resources and density dependent growth. This means that when an individual is added to the population the per capita growth rate decreases immediately. But this is not the case in general. In many population the density dependent response assumes some time lags. Individuals do not immediately adjust into their growth and reproduction. Seasonal availability of resources and age structure can also cause time lag in population growth. These delays can affect population dynamics of a given species significantly. We shall now study the effect of such delays.

Delay Model

Consider the logistic growth model which is given by the following equation

$$\frac{du}{dt} = ru(t)\left(1 - \frac{u(t)}{K}\right) \tag{19}$$

where $u(t)$ is the population at time t and the positive constants r and K denote the intrinsic growth rate and the carrying capacity of the population respectively. In this model we now incorporate the effect of time delay in growth and reproduction of individuals in the population and formulate the delay model.

Formulation

Let us assume that there is a time lag of length $T > 0$ between the change in population size and its effect on population growth rate. Consequently, the growth rate $\frac{du}{dt}$ of the population at time t, depends on the size of population at time $(t-T)$. Thus, in the logistic growth model incorporating the effect of delay, Eqn. (19) takes the form

$$\frac{du}{dt} = ru(t)\left(1 - \frac{u(t-T)}{K}\right) \tag{20}$$

where r, K and T are positive constants. Eqn. (20) is the **differential delay equation.** The behaviour of Eqn. (20) depends on the length of time lag T as well as the response time which is inversely proportional to the growth rate r [Gotelli, 1995].

The Analytical solution of Eqn. (20) cannot be obtained in general so we have to analyze it qualitatively.

Stability Analysis: Periodic Solutions

The steady states or the equilibrium points of Eqn. (20) are given by

$$u^* = 0 \text{ and } u^* = K$$

For computational convenience we rescale the model by substituting
$N(t) = \frac{u(t)}{K}$, $t^* = rt$, $T^* = rt$. Thus the model (20) becomes

$$\frac{dN}{dt^*} = N(t^*)\ [1 - N(t^* - T^*)]$$

For the sake of simplicity we drop the asterisks from the above model and write the delay model in the form

$$\frac{dN}{dt} = N(t)\ [1 - N(t - T)] \tag{21}$$

The equilibrium points of Eqn. (21) are $N_1 = 0$ and $N_2 = 1$.

Let us first consider the linearization about equilibrium point $N_1 = 0$.

We assume

$$N = N_1 + n(t), \text{ where } |n(t)| << 1. \tag{22}$$

Using Eqn. (22), Eqn. (21) reduces to

$$\frac{dn(t)}{dt} = (N_1 + n(t))\ [1 - (N_1 + n(t - T))]$$

or, $$\frac{dn(t)}{dt} = n(t)\,[1 - n(t - T)] \qquad [\because\ N_1 = 0]$$

Omitting higher powers of $n(t)$ in the above equation we get

$$\frac{dn(t)}{dt} = n(t) \tag{23}$$

which on integration yields

$$n(t) = A\ e^t$$

where A is a constant. This implies that $n(t) \to \infty$ as $t \to \infty$.

Hence, the steady state $N_1 = 0$ is **unstable** with **exponential growth.**

Next we consider, perturbation about the steady state $N_2 = 1$.

Using Eqn. (22), Eqn. (21) in this case reduces to

$$\frac{dn(t)}{dt} = (1 + n(t))\ [1 - (1 + n(t - T))] \qquad [N_2 = 1]$$

or, $$\frac{dn(t)}{dt} = (1 + n(t))\,[-n(t - T)]$$

Omitting the higher powers of $n(t)$, we obtain

$$\frac{dn(t)}{dt} = -n\ (t - T) \tag{24}$$

We now look for solutions of Eqn. (24) in the form

$$n(t) = c\ e^{\lambda t} \tag{25}$$

where c is a constant.

Substituting the value of $n(t)$ from Eqn. (25) into Eqn. (24) we obtain

$$c\lambda\ e^{\lambda t} = -c\ e^{\lambda(t-T)}$$

or, $\lambda = -e^{-\lambda T}$ (26)

You can thus see that the eigen values λ of $n(t)$ are the solution of Eqn. (26) which is a transcendental equation and it is not easy to solve it analytically.

However, from a stability point of view we are interested to know whether there are any solutions with $\text{Re}\lambda > 0$ for which Eqn. (25) implies instability since $n(t)$ grows exponentially with time. That is, if we set $\lambda = \mu + i\omega$ then does there exists a real number μ_0 such that all solution λ of Eqn. (26) satisfy $\text{Re}\lambda < \mu_0$. To see this, we have from Eqn. (26)

$$|\lambda| = |-e^{-\lambda T}| = |e^{-(\mu+i\omega)T}| = |e^{-\mu T}||e^{-i\omega T}| = e^{-\mu T}$$

Thus, if $|\lambda| \to \infty$ then $e^{-\mu T} \to \infty$ provided $\mu \to -\infty$.
Thus there must exist a real number μ_0 so that $\text{Re}\lambda$ i.e., μ is bounded above by μ_0.

If we now introduce $z = 1/\lambda$ and $\omega(z) = 1 + z e^{-T/z}$ then $z = 0$ is an essential singularity of $\omega(z)$. By Picard's theorem $\omega(z) = 0$ will then have infinitely many complex roots in the neighborhood of $z = 0$. This means there are infinitely many roots λ (ref. Sec. 65, Page-232, of Brown and Churchill).

Let us now consider the real and imaginary parts of Eqn. (26), namely

$$\mu = -e^{-\mu T} \cos \omega T \quad (27)$$

$$\omega = e^{-\mu T} \sin \omega T \quad (28)$$

We want to determine the range of T such that $\mu < 0$. That is, we want to find the conditions such that the upper limit μ_0 on μ is negative. Here we consider two cases viz., $\omega = 0$ and $\omega \neq 0$.

Case I: $\omega = 0$

When $\omega = 0$, λ becomes real. You can see that for $\omega = 0$, Eqn. (28) is satisfied and Eqn. (27) becomes

$$\mu = -e^{-\mu T} \quad (29)$$

which has no positive roots $\mu > 0$. Since $e^{-\mu T} > 0$ for all μT.

Case II: $\omega \neq 0$.

If ω is a solution of Eqns. (27) and (28) then $-\omega$ also satisfies these equations. Thus, without any loss of generality, we can consider $\omega > 0$. From Eqn. (27) you may observe that

$$\mu < 0 \text{ requires } \omega T < \pi/2 \text{ since } -e^{-\mu T} < 0 \ \forall \mu T.$$

Since Eqns. (27) and (26) defines $\mu(T)$ and $\omega(T)$, we are interested in finding that value of T when $\mu(T)$ first crosses from $\mu < 0$ to $\mu > 0$.

As T increases from zero then μ first becomes zero only when $\omega T = \pi/2$. Now we see, that for $\mu = 0$ Eqn. (28) has the only relevant solution $\omega = 1$ occurring at $T = \pi/2$. Since this is the first zero of μ as T increases this gives the bifurcating value $T = T_c = \pi/2$.

Alternately, we can use the argument that

$$e^{-\mu T} \sin \omega T = \omega$$
$$\Rightarrow \quad T\, e^{-\mu T} \sin \omega T = \omega T < \pi/2$$
$$\Rightarrow \quad T\, e^{-\mu T} \sin \omega T < \pi/2$$
$$\Rightarrow \quad 0 < T < \pi/2 \quad (\text{Since } \sin \omega T < 1 \text{ and } e^{-\mu T} > 1 \text{ if } \mu < 0).$$

which gives the condition on T for the stability of $N_2 = 1$.

Returning to our Eqn. (20) we can thus say that the steady state $u^*(t) = K$ is **stable** if $0 < rT < \pi/2$ and **unstable** if $rT > \pi/2$.

In the latter case we expect the solution to exhibit stable limit cycle behaviour. The critical value $rT = \pi/2$ is the bifurcation value, that is the value of the parameter, rT here, where the character of the solution of Eqn. (20) changes abruptly or bifurcates from stable steady state to a time varying solution. The effect of delay in models is usually to increase potential for instability. As T is increased beyond the bifurcation value $T_c = \pi/2r$, the steady state becomes unstable.

Let us now obtain the first estimate of the period of the bifurcating oscillatory solution. Consider Eqn. (21) and let

$$T = T_c + \varepsilon = \frac{\pi}{2} + \varepsilon, \quad 0 < \varepsilon << 1 \tag{30}$$

For $T = \frac{\pi}{2}$, $\operatorname{Re}\lambda$ is largest and the solution $\lambda = \mu + i\omega$ of Eqns. (27) and (28) is $\mu = 0,\ \omega = 1$. We would then expect that for ε small, μ and ω also differ from $\mu = 0$ and $\omega = 1$ by small quantities. Accordingly, we let

$$\mu = \alpha,\ \omega = 1 + \beta, \quad 0 < \alpha << 1, \quad |\beta| << 1 \tag{31}$$

where α and β are to be determined. Substituting these values of μ and ω in Eqn. (28) and expanding for small α, β and ω, we get

$$1 + \beta = \exp\left[-\alpha\left(\frac{\pi}{2} + \varepsilon\right)\right] \sin\left[(1+\beta)\left(\frac{\pi}{2} + \varepsilon\right)\right] \Rightarrow \beta \approx \frac{-\pi\alpha}{2} \tag{32}$$

to the first order of α, β and ε. Similarly Eqn. (27) gives

$$\alpha = -\exp\left[-\alpha\left(\frac{\pi}{2} + \varepsilon\right)\right] \cos\left[(1+\beta)\left(\frac{\pi}{2} + \varepsilon\right)\right] \Rightarrow \alpha \approx \varepsilon + \frac{\pi\beta}{2}. \tag{33}$$

Thus on solving Eqns. (32) and (33) we obtain

$$\alpha \approx \frac{\varepsilon}{1 + \frac{\pi^2}{4}}, \quad \beta \approx \frac{-\varepsilon\pi}{2\left(1 + \frac{\pi^2}{4}\right)} \tag{34}$$

and hence, near the bifurcation, using Eqn. (25), $N(t) = 1 + n(t)$ reduces to

$$N(t) = 1 + \mathrm{Re}\{c \exp[\alpha t + i(1+\beta)t]\}$$

$$\approx 1 + \mathrm{Re}\left\{c \exp\left[\frac{\varepsilon t}{1+\pi^2/4}\right] \exp\left[it\left\{1 - \frac{\varepsilon\pi}{2(1+\pi^2/4)}\right\}\right]\right\}$$

This shows that the instability is by growing oscillations with period

$$\frac{2\pi}{1 - \dfrac{\varepsilon\pi}{2(1+\pi^2/4)}} \approx 2\pi$$

for small ε. Now since $rT = \pi/2$, the period of oscillation is then $4T$.

The model discussed above is also not perfect as it has its limitations which we are stating below.

Limitations

- This model incorporates the delay due to maturation period but ignores the age structure. Further the sex ratio also plays an important role in determining the birth rate which is not considered. The sex ratio may be different in different species.
- In addition to the term corresponding to the crowding effect the absolute death rate should be considered. The age structure also affects the death rate..

There are models with age distribution which incorporates the effect of age structure of the population for both discrete and continuous growth but we shall not be discussing these models here.

Let us consider the following application of the delay model.

Example 6: Consider a laboratory population of the blowfly Lucilia Cupriva kept in a cage and given a limited supply of food. Along with the adult blowflies the cage contains larvae also, which are supplied with unlimited food. Let $x(t)$ be the number of adult blowflies at time t, c be the constant mortality rate per unit time and τ be the time taken by the egg to develop into an adult. Assuming that the laying of eggs be proportional to the initial population. The growth equation of blowflies can be written as

$$\frac{dx}{dt} = -cx(t) + \frac{1}{2} mksx(t-\tau) \tag{35}$$

where k is the constant of proportionality and s is the probability that an egg will grow into an adult and m is the mass of the blowflies.

Putting $-c = a$ and $\frac{msk}{2} = b$ Eqn. (35) can be reduced to the form

$$\frac{dx}{dt} = ax(t) = bx(t-\tau) \tag{36}$$

where, $a < 0$ and $b > 0$ are real constants.

Model (36) is known as the Nicolson's model and its solution is to be obtained under the initial conditions $x(t) = 0$ for $t < 0$ and $x(0) = \tau$.

Taking the Laplace transform of both sides of Eqn. (36) and using the given initial conditions we obtain

$$sX(s) - \tau = aX(s) + b\,e^{-s}\,X(s).$$

Hence,

$$x(t) = \frac{\tau}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{st}}{s-a-be^{-s}}\,ds.$$

The only singularity of the integrand are at the zeros of $s-a-be^{-s}$ and the solution is the sum of the residues of the integrand at these poles. To calculate the zeros, we put $s = \alpha + i\beta$ where α and β are real. This gives

$$\alpha + i\beta = a + be^{-(\alpha+i\beta)} \tag{37}$$

Equating real and imaginary parts of Eqn. (37), we obtain the simultaneous equations

$$\alpha = a + be^{-\alpha}\cos\beta \tag{38}$$

$$\beta = -be^{-\alpha}\sin\beta \tag{39}$$

Clearly, one solution of Eqn. (39) is $\beta = 0$, and then Eqn. (38) becomes

$$\alpha = a + be^{-\alpha}. \tag{40}$$

When $b \geq 0$, Eqn. (40) has just one real root which is positive if $a+b>0$ and negative otherwise. If α_0 is a real root of Eqn. (40) then the corresponding contribution to $x(t)$ from the residue has a form $\exp(\alpha^0 t)$. Hence the condition for exponential divergence are $a+b>0$ or $a>\tau$ and $b>-e^{a-1}$.

If $\beta \neq 0$ then the stability boundaries for the oscillatory solutions are given parametrically by

$$a = \beta\cot\beta,\ b = -\frac{\beta}{\sin\beta},\ \beta > 0.$$

***

You may now try the following exercises.

E7) Consider the budworm population dynamics to be governed by the equation

$$\frac{dN}{dt} = r_B N\left(1-\frac{N}{K_B}\right) - p(N) \tag{41}$$

where r_B is the linear birth rate of the budworm and K_B is the carrying capacity. The $p(N)$ term is predation generally by birds. Find out the steady states and do the stability analysis of Eqn. (41).

E8) An animal may feed on the number of plant which is annual, so that the total food available depends on the number of the plants in the previous year and thus in term is a function of the number of animals present in the previous years. Thus the logistic equation can be written as a time delay equation of the form

$$\frac{dn(t)}{dt} = \frac{n(t)}{\tau}\left(1 - \frac{n(t-T)}{N}\right).$$

Discuss the steady state and carry out the perturbation analysis.

E9) A continuous time model for the baleen whale is the delay equation

$$\frac{dN}{dt} = -\mu N(t) + \mu N(t-T)\,[1 + q\{1 - [N(t-T)/K]^z\}]:$$

Here $\mu > 0$ is a measure of mortality and $q > 0$ is the maximum increase in fecundity the population is capable of, K is the unharvested carrying capacity, T is the time of sexual maturity and $z > 0$ is the measure of density of the population. Discuss the steady states and carry out the perturbation analysis.

We now end this unit by giving a summary of what we have covered in it.

4.5 SUMMARY

In this unit, we have covered the following:

1. Mathematical ecology can be described as the study of inter-dependence of several species in a variable eco-system employing mathematical modeling.
2. A mathematical model in population dynamics represents quantitatively the pattern of growth of a given population in the presence of various environmental factors.
3. The mathematical models in population dynamics can be classified on account of various factors like nature of growth rate, its fluctuation and environmental factors, size of the population and so on.
4. On the basis of growth rate, the population models can be classified as
   i) Deterministic models
   ii) Stochastic models
5. The models can be analyzed in two ways i.e., quantitatively and qualitatively. The quantitative analysis of models involves consideration of actual parameter and prediction in terms of numbers. On the other hand qualitative analysis concentrates on study of the nature and pattern of growth and their dependency on parameters relevant to the model.
6. The discrete deterministic models are applicable to the cases when moderately large populations such as humans, large animals like lions, elephant are modelled. In such models the variations in population are studied by means of difference equations.
7. Continuous deterministic models are employed when populations are very large and their growth rate is also fast so that variables can be modelled by continuous functions and their growth rate by their derivatives. These models lead to differential equations.

4.6 SOLUTIONS/ANSWERS

E1) **Hint:** Proceed as in Example 2. Apply formula for compound interest.

E2) For the model

$$N_{t+1} = \frac{r\,N_t}{1+bN_t^2} = f(N_t).$$

Let us take $N_{t+1} = N_t = N^*$ then for stability we have

$$N^* = \frac{r\,N^*}{1+bN^{*2}} = f(N^*)$$

The steady states are

$$N_1^* = 0,\ N_2^* = \sqrt{\frac{r-1}{b}}$$

$$f'(N^*) = \frac{(1+bN^{*2})\,r - 2\,rbN^{*2}}{(1+bN^{*2})}.$$

$$f'(N_1^*) = r \text{ and } f'(N_2^*) = \frac{2-r}{r} = \frac{2}{r} - 1.$$

For $r < 1$, N_1^* is stable.

For $r > 1$, N_1^* becomes unstable and N_2^* is stable.

E3) With little manipulation Eqn. (7) can be written as

$$u^*\,[ru^* - (r-1)]\ [r^2u^{*2} - r(r+1)u^* + (r+1)] = 0$$

which has solutions

$$u^* = 0 \text{ or } u^* = \frac{r-1}{r} > 0 \text{ if } r > 1$$

$$u^* = \frac{(r+1) \pm [(r+1)\,(r-3)]^{1/2}}{2r} > 0 \text{ if } r > 3$$

$$\therefore\ u_3^* = \frac{(r+1) + [(r+1)\,(r-3)]^{1/2}}{2r} \text{ and } u_4^* = \frac{(r+1) - [(r+1)\,(r-3)]^{1/2}}{2r}.$$

$$[f^2(u, r)]' = r^2(1-2u)\,(1-2ru+2ru^2)$$

$$\therefore\ [f^2(u, r)]'\Big|_{u=u_3^*} = -(r^2+2r+2) < 1 \text{ for } r > 3$$

$\Rightarrow$ u_3^* is a stable steady state.

Similarly show that u_4^* is stable steady state.

E4) Apply the same method as in Example 3 and draw the results as in Rickers model in Example 4.

E5) a) The steady states of the given model are
$N_1 = 0$, $N_2 = K$ and the corresponding eigenvalues are
$\lambda_1 = 1 + r$, $\lambda_2 = 1 - r$.
For $0 < r < 1$, the only equilibrium corresponding to non-negative population sizes is $N_1 = 0$, which is unstable.
For $r = 1$, $\lambda_1 = \lambda_2$.

For $r > 1$, N_1 is unstable and N_2 is stable.

b) Proceed as in a) above.

E6) Proceed as in Example 5 and get the solution.

E7) In the given model put right hand side equal to zero to find out equilibrium state i.e.

$$r_B N\left(1-\frac{N}{K_B}\right)-p(N)=0.$$

Then to do the stability analysis use perturbation i.e.

$N = N^* + n_1$ and $\frac{dN}{dt} = \frac{dn_1}{dt}$ then

$$\frac{dn_1}{dt} = r_B\,(N^* + n_1)\left(1-\frac{N^*+n_1}{K_B}\right)-p(N^*+n_1)$$

expanding the above expression and applying Taylor's series we get

$$= r_B\,N^* - \frac{r_B N^{*2}}{K_B} - \frac{r_B N^* n_1}{K_B} + r_B n_1$$
$$-\left\{p(N^*)+\frac{n_1}{1!}p'(N^*)+\frac{n_1^2}{2!}p''(N^*)+\cdots\right\}$$

Neglecting the higher order terms and rearranging we get

$$\frac{dn_1}{dt} = \left[1\frac{r_B N^*}{K_B} + r_B - p(N^*)\right] n_1 .$$

After integrating we get

$$n_1 = e^{-At+c} \text{ where } A = -\frac{r_B N^*}{K_B} + r_B - p(N^*)$$

if $t \to \infty$ then $n_1 \to$ zero.

So we can draw the conclusion that when time tends to infinity at that time species n_1 tends to zero i.e., it will go to extinct.

E8) This problem can be converted into delay logistic equation by substituting $N^* = \frac{n}{N}$, $t^* = \frac{t}{\tau}$, $T^* = \frac{t}{\tau}$ and then solved.

E9) First of all we will find out steady states for that we will do the perturbation $n(t)$ about the positive equilibrium and we will get

$$\frac{dn(t)}{dt} \approx -\mu n(t) - \mu(qz-1)\, n(t-T)$$

and hence that the stability of the equilibriums is determined by $\mathrm{Re}\,\lambda$.

$$\lambda = -\mu - \mu(qz-1)\, e^{-\lambda T}$$

After this we can perform the stability analysis and find out its stability condition and oscillating conditions.

—x—

APPENDIX

The stability of a population relates to its persistence for a large number of generations. For this, we wish to know if our model possesses any stable steady state or equilibrium. You already know that steady-state or an equilibrium of a difference equation of the form

$$x_{n+1} = f(x_n) \tag{1}$$

is a value x^* such that $x^* = f(x^*)$ so that $x = x^*$ is a constant solution of the difference equation. In other words, $x = x^*$ is a fixed point of the mapping $y = f(x^*)$. Knowing the equilibrium points of Eqn. (1) the next step is to investigate the nature of these equilibrium points.

Stability Analysis

The behaviour of solution near an equilibrium can be studied by using the process of linearisation analogous to that used for differential equations.

To investigate the linear stability of x^*, we write

$$x_n = x^* + \varepsilon_n, \quad |\varepsilon_n| << 1$$

Substituting this in Eqn. (1) and expanding for small ε_n, using Taylor expansion, we get

$$\begin{aligned} x^* + \varepsilon_{n+1} &= f(x^* + \varepsilon_n) \\ &= f(x^*) + \varepsilon_n \; f'(x^*) + O(\varepsilon_n^2) \end{aligned}$$

Since $x^* = f(x^*)$, the equation for determining the linear stability of x^* is given by

$$\varepsilon_{n+1} = \varepsilon_n \; f'(x^*) + \cdots, \quad n = 0, 1, 2, \ldots \tag{2}$$

Let $\lambda = f'(x^*)$ be the **eigenvalue** of the first iterate at the steady-state point x^*. Then Eqn. (2) can be written as

Eigenvalues are the scalar values for which the nontrivial solution of the system exists.

$$\varepsilon_{n+1} = \lambda \; \varepsilon_n$$

and its solution is given by

$$\varepsilon_n = \lambda^n \varepsilon_0$$

which gives $\varepsilon_n \to 0$ as $n \to \infty$ if $|\lambda| < 1$ and $\varepsilon_n \to \pm\infty$ as $n \to \infty$ if $|\lambda| > 1$.

Thus x^* is **stable if $-1 < f'(x^*) < 1$** and **unstable if $|f'(x^*)| > 1$**.

The steady state x^* is stable if any small perturbation from this equilibrium state decays to zero monotonically when $0 < f'(x^*) < 1$ and with decreasing oscillations when $-1 < f'(x^*) < 0$. On the other hand x^* is unstable if any perturbation grows monotonically when $f'(x^*) > 1$ and by growing oscillations $f'(x^*) < -1$. This argument can be derived by graphical method also.

If $|f'(x^*)| < 1$ then the equilibrium has the property that every solution of Eqn. (1) with initial value x_0 close enough to x^*, remains close to it and as n tends to infinity, the solution approaches to x^*. This property is called **asymptotic stability** of the equilibrium. In biological applications the asymptotic stability is more important than stability, because an asymptotically stable equilibrium is not significantly affected by perturbations.

Bifurcation and Chaos

The population models of single species involve at least one parameter say r. As this parameter varies the solution of the general model

$$x_{n+1} = f(x_n, r)$$

usually undergoes some qualitative changes at specific values of r resulting in so called **bifurcation**. Such bifurcation can lead to periodic solutions. The frequency of periods increases with the increasing value of the parameter r and when the value of r exceeds some critical value say r_c, chaotic solutions are obtained; the term reflecting their random oscillations. From the graphical analysis by Cobwebbing method, it can be concluded that bifurcation occurs when the eigenvalues of the system pass through $\lambda = 1$, 0 and $\lambda = -1$.

UNIT 5 MODELLING ENVIRONMENT

Structure

5.1 INTRODUCTION

Modelling has become one of the most powerful tools for the environmental scientists to have a better understanding of the interactions between the environment, ecosystems and the populations of human and other animals. This understanding is increasingly important in environmental stewardship (monitoring and management) and the development of increasingly sustainable means of human dependency on environmental systems.

Environment is a collection of all the biotic and abiotic factors that affect an individual organism at any one point in its life cycle.

Environmental systems are studied by physicists, chemists and biologists also but, the level of abstraction of the environmental scientists is very different. A physicist might study the behaviour of gases, liquids or solids under controlled conditions of temperature or pressure. A chemist might study the interaction of molecules in aqueous solution while a biologist must integrate what we know from these sciences to understand how a cell or plant or an animal lives and functions. The environmental scientist or geographer or ecologist approaches his science at a much greater level of abstraction in which physical and chemical 'laws' provide the rule base for understanding the interaction between living organisms and their nonliving environments. Integrated environmental systems are different in many ways from the isolated objects of study in physics and chemistry. But the integrated study of the environment cannot take place without the building blocks provided by research in physics and chemistry.

Ecosystem consists of a biotic community and its abiotic environment.

Ecology is the science of the inter-relations between living organisms and their environment.

Species are organisms forming a natural population or group of populations that transmit specific characteristics from parent to offspring.

The organisms in the environmental system are constantly exposed to some two million natural and synthetic chemicals. Therefore, the probability of toxic effect on population is large. Mathematical modelling is a mechanism that has proved to be useful in the integration and synthesis of chemical and ecological informations.

A chemical or artificial pollutant that adversely affects the organisms like death, illness and change in metabolic activities is referred to as **toxicant**.

In this unit, in Sec. 5.2 we have discussed the models which study the effects of toxicants on single and two species interacting populations. We have assumed the concentration of the toxicant in the environment as a constant to begin with and then assumed it to be a function of time. In Sec. 5.3, we have discussed spatial ecological systems using reaction-diffusion and reaction-diffusion-advection equations in one space dimension for single and two species interacting populations. At the end of this unit we have given two appendices. Appendix-I, reffered throughout the unit, states and illustrates, some of the

Diffusion is a phenomenon by which the particle groups (organisms in this case) spread as a whole according to the irregular motion of each particle.

Advection is the horizontal directed movement of the organisms.

concepts, definitions and theorems used frequently in the discussion. The details related to the stability criterion for a system of two or more differential equations are given in Appendix-II. You must go through these appendices carefully as they will provide you with the necessary background and help you understand the stability analysis of the various models discussed in this unit.

Objectives

After studying this unit, you should be able to:

- write down the model equation to study the effect of toxicants on **single population** when the concentration of the toxicant in the environment is assumed to be **constant**, do the stability analysis of the formulated equations and interpret the results obtained;
- write down the model equation to study the effect of toxicants on **single population** when the concentration of the toxicant in the environment is assumed to be a **function of time**, do the stability analysis of the formulated equations and interpret the results obtained;
- write down the model equation to study the effect of toxicant on **two interacting species** (prey and predator) population assuming that the toxicant is directly/indirectly affecting the prey and predator populations, do the stability analysis of the formulated equations and interpret the results obtained;
- write down the model equations to discuss the population dynamics in spatially heterogeneous environment using reaction-diffusion and reaction-diffusion-advection equations in one space dimension for **single species**, do the stability analysis of the formulated equations and interpret the results obtained;
- write down the model equations to discuss the population dynamics in spatially heterogeneous environment using reaction-diffusion and reaction-diffusion-advection equations in one space dimension for **two interacting species**, do the stability analysis of the formulated equations and interpret the results obtained.

5.2 MODELLING THE EFFECT OF TOXICANTS ON POPULATION

There are essentially two modes through which a toxicant can enter organisms. One mode is through direct absorption from the **environment** and the another mode is through **food-chain**. First we will discuss the models for the effect of toxicants on population considering that the prime source of the toxicant is the environment and the food chain accumulation is negligible.

Let us first consider the mathematical formulation of the model for the effect of toxicant on **single population**.

Formulation 1: The population under consideration is assumed to follow logistic growth (for the logistic growth model ref. Block 3 of MTE-14), in the absence of the toxicant. The toxicant and population equations are coupled by a population dose-response involving birth and mortality rates.

Let $N = N(t)$ be the density of the population at time t, $C_0 = C_0(t)$ be the concentration of toxicant in the individuals of the population and P is the concentration of the toxicant in the environment and assumed to be constant. The logistic equation for the growth of the population with density $N(t)$ is given by the following ordinary differential equation

$$\frac{dN}{dt} = N(b - d - sN), \quad 0 \le t < \infty \tag{1}$$

where, b, d and s are positive constants and are the birth rate, death rate and intraspecific coefficient of competition, respectively.

In the formulation of the equation for the dynamics of C_0, it is assumed that C_0 satisfies first order kinetics, i.e. first order chemical reaction, and hence the equation satisfied by C_0 is given by

$$\frac{dC_0}{dt} = kP - gC_0 - mC_0 \tag{2}$$

where k, g, m are all positive constants, representing uptake rate, loss rate and depuration rate, respectively.

In Eqn. (2), $kP - gC_0$ represents net uptake of toxicant by the individuals of the population from the environment and mC_0 represents depuration of toxicant due to metabolic processing and other losses.

A coupling between toxicant and population is achieved by assuming that the **intrinsic (natural) growth rate** of population $b - d$, is a linear function of C_0 which is given by the expression:

$$b - d = r_0 - r_1 C_0 \tag{3}$$

where, r_0 and r_1 are positive constants.

Thus, the **model for the effect of toxicant on single population** when the environmental concentration of the toxicant is constant and the mode of transfer of toxicant in the population is only through environment and not through food chain is written as

$$\frac{dN}{dt} = N(r_0 - r_1 C_0 - sN) \tag{4}$$

$$\frac{dC_0}{dt} = kP - gC_0 - mC_0 \tag{5}$$

with initial conditions

$$N(0) = N_0 > 0, \; C_0(0) = 0. \tag{6}$$

We shall now do the stability analysis of the model given by Eqns. (4) and (5).

Stability Analysis

The equilibrium points (or critical points) E_0 and E_1 of the system of

Eqns. (4) and (5) obtained by putting $\frac{dN}{dt} = 0$ and $\frac{dC_0}{dt} = 0$ are

$$E_0 : N^* = 0, C_0^* = \frac{kP}{(g+m)} \tag{7}$$

and

$$E_1 : \overline{N} = \frac{r_0 - r_1 C_0^*}{s}, \quad C_0^* = \frac{kP}{(g+m)}$$

where, $\overline{N} > 0$ if $r_0 > r_1 \ C_0^*$ i.e., if $r_0(g+m) > r_1 \ kP$. (8)

The **variational matrix** (ref. Appnedix-I) of the system of Eqns. (4) and (5) is given by

$$A = \begin{bmatrix} r_0 - 2sN - r_1 \ C_0 & -r_1 N \\ 0 & -(g+m) \end{bmatrix} \tag{9}$$

For the first equilibrium point E_0 given by Eqn. (7) matrix A reduces to

$$A|_{E_0} = \begin{bmatrix} r_0 - r_1 \ C_0^* & 0 \\ 0 & -(g+m) \end{bmatrix}. \tag{10}$$

The eigenvalues of the above matrix are $\lambda_1 = r_0 - r_1 \ C_0^*$ and $\lambda_2 = -(g+m)$. We find that $\lambda_1 < 0$ if,

$$r_0 < r_1 \ C_0^* \text{ or, } r_0(g+m) < r_1 \ kP. \tag{11}$$

Thus, if Eqn. (11) holds, then both λ_1 and λ_2 are real and $\lambda_1 < \lambda_2 < 0$. Hence, the steady-state solution $\mathbf{E_0}$ **is asymptotically stable**. The result relating stability and eigenvalues can be seen in Appendix-II. Now since the equilibrium value of the population is zero, the asymptotic stability of E_0 would mean that the population would get extinct if the natural growth rate of population is less than a certain amount of toxicant entering into the environment of the population.

For the second equilibrium point E_1 given by Eqn. (8) the matrix A reduces to

$$A|_{E_1} = \begin{bmatrix} -s\overline{N} & -r_1 \ \overline{N} \\ 0 & -(g+m) \end{bmatrix}. \tag{12}$$

The eigenvalues of the above matrix are $\lambda_1 = -s \ \overline{N}$ and $\lambda_2 = -(g+m)$. If Eqn. (8) holds then both the eigenvalues will be negative and hence, the equilibrium point $\mathbf{E_1}$ **will be asymptotically stable.**

Interpretation of the Solution

The asymptotic stability of E_1 shows that the population will survive but at lower equilibrium level if the natural growth rate of the population exceeds the amount of toxicant entering into the environment of the population because then the equilibrium value of the population is positive. By lower equilibrium level we mean the decrease in the equilibrium value from its natural value i.e., the equilibrium value of the population without any effect of toxicant.

We now give the limitations of this model.

Limitations

In the formulation of the model given by Eqns. (4) and (5), it has been assumed that all individuals in the population under toxicant stress are identical. Their age or size factor is not considered. The concentration of the toxicants in the environment is assumed to be constant. Also, the accumulation of toxicant in the individuals through food chain pathways, that is, biomagnifications factor, is not considered.

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As a little modification to Formulation 1, we shall now consider a model in which the environmental concentration of the toxicant P is considered to be a function of time t.

Formulation 2: Let us consider that $P = P(t)$ changes with respect to time. In this case, we supplement the two Eqns. (4) and (5) with a separate equation for $P(t)$ which is given as follows:

$$\frac{dP}{dt} = -k_1 \; PN + g_1 C_0 N - hP + Q \tag{13}$$

where k_1, g_1, h and Q are all positive rate constants. Q represents the exogenous rate of input of toxicant into the environment i.e., the rate of toxicant entering into the environment.

In Eqn. (13), the first two terms represent toxicant uptake by the population as determined by the first order kinetics. k_1PN is the uptake of $P(t)$ by the population N(t) and g_1C_0N is the input to the environment from the endogeneous contribution of toxicant from the population i.e., the amount of toxicant discharged by the organisms present within the system. The term hP represents the totality of losses from the system environment including processes such as biological transformation, chemical hydrolysis, volatilization, microbial degradation and photosynthetic degradation. .

Thus, the **complete model in this case is** given by Eqns. (4), (5) and (13) i.e.,

$$\frac{dN}{dt} = N(r_0 - r_1 \; C_0 - sN)$$

$$\frac{dC_0}{dt} = kP - g \; C_0 - m \; C_0$$

$$\frac{dP}{dt} = -k_1 PN + g_1 \; C_0 N - hP + Q$$

together with the initial conditions

$$N(0) = N_0 > 0, C_0(0) = 0, P(0) = P_0 > 0. \tag{14}$$

Let us now do the stability analysis of this model.

Stability Analysis

The steady state solution of the system of Eqns. (4), (5) and (13) is obtained by putting $\frac{dN}{dt}=0$, $\frac{dC_0}{dt}=0$ and $\frac{dP}{dt}=0$. The equilibrium points E_0 and E_1 of the system are given by

$$E_0 : N^* = 0,\ C_0^* = \frac{kQ}{h(g+m)},\ P^* = Q/h \tag{15}$$

$$E_1 : \overline{N} = \frac{r_0 - r_1\,\overline{C}_0}{s},\ \overline{C}_0 = \frac{k\overline{P}}{(g+m)},\ \overline{P} = \frac{Q + g_1\overline{C}_0\overline{N}}{h + k_1\overline{N}} \tag{16}$$

where $\overline{N} > 0$ provided, $r_0 > r_1\ \overline{C}_0$. (17)

The variational matrix of the system is given by

$$A = \begin{bmatrix} r_0 - 2sN - r_1\,C_0 & -r_1N & 0 \\ 0 & -(g+m) & k \\ -k_1P + g_1\,C_0 & g_1N & -h - k_1N \end{bmatrix}. \tag{18}$$

For the first equilibrium point E_0 given by Eqn. (21), the matrix A reduces to

$$A\big|_{E_0} = \begin{bmatrix} r_0 - r_1\ C_0^* & 0 & 0 \\ 0 & -(g+m) & k \\ -k_1P^* + g_1C_0^* & 0 & -h \end{bmatrix}. \tag{19}$$

The eigenvalues of the above matrix are

$$\left.\begin{aligned} \lambda_1 &= r_0 - r_1\ C_0^* \\ \lambda_2 &= -(g+m) \\ \lambda_3 &= -h \end{aligned}\right\}. \tag{20}$$

From the above, we find that

$$\lambda_1 < 0 \text{ if } r_0 < \frac{r_1\,k\,Q}{h(g+m)}, \text{ i.e., if } r_0\,h(g+m) < r_1\,k\,Q\,. \tag{21}$$

Hence, we derive that the equilibrium solution E_0 is **asymptotically stable** under the condition given by Eqn. (21). If Eqn. (21) does not hold good then E_0 is **unstable.**

Now, for the second equilibrium solution E_1 given by Eqn. (16), the matrix A reduces to

$$A\big|_{E_1} = \begin{bmatrix} -s\overline{N} & -r_1\overline{N} & 0 \\ 0 & -(g+m) & k \\ -k_1\overline{P} + g_1\ \overline{C}_0 & g_1\overline{N} & -h - k_1\,\overline{N} \end{bmatrix}. \tag{22}$$

The characteristic equation of the above matrix is given by

$$\lambda^3 + (g+m+h+k_1\,\overline{N}+s\overline{N})\,\lambda^2 +$$

$$[(g+m)\,(h+k_1\,\overline{N})+s\overline{N}(g+m+h+k_1\,\overline{N})-kg_1\,\overline{N}]\,\lambda$$

$$+[s\,\overline{N}(g+m)\,(h+k_1\,\overline{N})+r_1\,\overline{N}\,kg_1\,\overline{C}_0 - s\,\overline{N}^2\,kg_1 - r_1k\,k_1\overline{N}\,\overline{P}]=0. \qquad (23)$$

Using **Hurwitz's criteria** (ref. Appendix-I) in Eqn. (23) it can be shown that the characteristic Eqn. (23) has eigenvalues with negative real part if the following conditions are satisfied

$$g_1\,\overline{C}_0 > k_1\,\overline{P},\ (g+m)\,(h+k_1\,\overline{N}) > k\,g_1\,\overline{N} \qquad (24)$$

and

$$s\overline{N}(g+m+h+k_1\overline{N}) > r_1kg_1\,\overline{C}_0. \qquad (25)$$

We are leaving this for you to verify yourself.

E1) Verify that the eigenvalues of the matrix given by Eqn. (22) have negative real parts if conditions (24) and (25) are satisfied.

We can thus deduce that the equilibrium point E_1 is **asymptotically stable** provided Eqns. (24) and (25) are satisfied.

Interpretation of the Solution

From the stability of E_0 it can be concluded that the population will tend to extinction if $r_0h(g+m) < r_1kQ$, i.e., if the product of the growth rate of the population and the depletion rate of the environmental toxicant is less than the amount of toxicant entering into the environment of the population. From the stability of E_1 it is concluded that the population will survive at lower equilibrium level due to effect of toxicant.

Just as in the case of Formulation 1, this model also has certain limitations which we state below.

Limitations

In the formulation of the model given by Eqns. (4), (5) and (13), it is assumed that all individuals in the population under toxicant stress are identical. Their age or size factor is not considered. The accumulation of toxicant in the individuals through food chain is not taken into account. Moreover, the toxicant input rate Q which taken to be constant here can be time dependent also.

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So far, in Formulations 1 and 2, we considered the effect of toxicant on single population. We shall now discuss the effect of toxicant on the population consisting of two species. In particular, we shall now consider the prey predator interacting system under the effect of toxicant, when the concentration of the toxicant in the environment is assumed to be constant. (For the prey-predator population model you may refer to Unit 9, Block 3 of MTE-14.) The model which we are going to formulate is a straight forward extension of Formulation 1.

Formulation 3: Let us assume that toxicant is directly affecting the prey and predator populations and food chain accumulation of toxicant is negligible. Also, the concentration of toxicant in the environment is taken to be constant. Under these assumptions the system of equations describing the system takes the following form:

$$\frac{dN_1}{dt} = r_0 N_1 - r_1 C_0 N_1 - b\, N_1 N_2 \tag{26}$$

$$\frac{dN_2}{dt} = -d_0 N_2 - d_1 V_0 N_2 + \beta_0 b N_1 N_2 \tag{27}$$

$$\frac{dC_0}{dt} = k_1 P - g_1 C_0 - m_1 C_0 \tag{28}$$

$$\frac{dV_0}{dt} = k_2 P - g_2 V_0 - m_2 V_0 . \tag{29}$$

The variables and parameter notations in the system of Eqns. (26)-(29) are given below. Here r_0, d_0, b, β_0, m_1, m_2, P, k_1, k_2, g_1, g_2, r_1 and d_1 are all positive constants.

$N_1(t) =$ Density of prey population

$N_2(t) =$ Density of predator population

$C_0(t) =$ Concentration of the toxicant in the individuals of the prey population

$V_0(t) =$ Concentration of the toxicant in the individuals of the predator population

$r_0 =$ growth rate or birth rate

$d_0 =$ death rate

$b =$ Predation rate

$\beta_0 =$ Conversion coefficient

$m_1, m_2 =$ Depuration rates

$k_1, k_2 =$ Uptake rates

$g_1, g_2 =$ Loss rates

$r_1 =$ Death rate due to C_0

$d_1 =$ Death rate due to V_0

$P =$ Constant environmental toxicant concentration.

We now do the stability analysis of system of Eqns. (26)-(29) under the initial conditions

$$N_1(0) = N_{10}\ \ N_2(0) = N_{20},\ C_0(0) = 0,\ V_0(0) = 0 .$$

Stability Analysis

The variational matrix for the system given by Eqns. (26)-(29) can be written as

$$A = \begin{bmatrix} r_0 - r_1 C_0 - bN_2 & -bN_1 & -r_1 N_1 & 0 \\ \beta_0 b N_2 & -d_0 - d_1 V_0 + \beta_0 b N_1 & 0 & -d_1 N_2 \\ 0 & 0 & -(g_1 + m_1) & 0 \\ 0 & 0 & 0 & -(g_2 + m_2) \end{bmatrix}. \tag{30}$$

The equilibrium points E_0 and E_1 of the system of Eqns. (26)-(29) are given by

$$E_0 : N_1^* = 0,\ N_2^* = 0,\ C_0^* = \frac{k_1 P}{(g_1 + m_1)} \text{ and } V_0^* = \frac{k_2 P}{(g_2 + m_2)} \tag{31}$$

$$E_1 : \overline{N}_1 = \frac{d_0 + d_1 \overline{V}_0}{\beta_0 b},\ \overline{N}_2 = \frac{r_0 - r_1 \overline{C}_0}{b},\ \overline{C}_0 = \frac{k_1 P}{(g_1 + m_1)},\ \overline{V}_0 = \frac{k_2 P}{(g_2 + m_2)}. \tag{32}$$

$\overline{N}_2 > 0$ provided, $r_0 > r_1 \overline{C}_0$.

The matrix A for the equilibrium point E_0 given by Eqn. (31) reduces to

$$A|_{E_0} = \begin{bmatrix} r_0 - r_1 C_0^* & 0 & 0 & 0 \\ 0 & -d_0 - d_1 V_0^* & 0 & 0 \\ 0 & 0 & -(g_1 + m_1) & 0 \\ 0 & 0 & 0 & -(g_2 + m_2) \end{bmatrix}. \tag{33}$$

The eigenvalues of the above matrix are $(r_0 - r_1 C_0^*), -(d_0 + d_1 V_0^*), -(g_1 + m_1)$ and $-(g_2 + m_2)$. Hence, it can be deduced that the equilibrium solution E_0 is **asymptotically stable** if

$$r_0 < r_1 C_0^* \text{ or, } r_0(g_1 + m_1) < r_1 k_1 P. \tag{34}$$

Let us now consider the second equilibrium point E_1 given by Eqn. (32). The matrix A for the equilibrium point E_1 reduces to

$$A|_{E_1} = \begin{bmatrix} 0 & -b\overline{N}_1 & -r_1 \overline{N}_1 & 0 \\ \beta_0 b \overline{N}_2 & 0 & 0 & -d_1 \overline{N}_2 \\ 0 & 0 & -(g_1 + m_1) & 0 \\ 0 & 0 & 0 & -(g_2 + m_2) \end{bmatrix}. \tag{35}$$

The characteristic equation associated with the above matrix is

$$(\lambda + g_1 + m_1)(\lambda + g_2 + m_2)(\lambda^2 + b^2 \beta_0 \overline{N}_1 \overline{N}_2) = 0. \tag{36}$$

Clearly, the two eigenvalues of Eqn. (36) are negative and the other two are imaginary. Hence the equilibrium solution E_1 is **stable**.

Interpretation of the Solution

From the stability analysis of E_0 it may be concluded that both the prey and predator population will tend to extinction if the organismal concentration of the toxicant in the individuals of the prey population increases. The number of prey population will decrease leading to extinction on account of the presence of toxicant in their body and hence the predator population will also tend to extinction because the predator population depdends on prey population for their survival. The stability criteria for E_1 shows that both the prey and predator population will coexist and will oscillate about the positive equilibrium level with the reduction in the predator population at equilibrium because of the effect of toxicant which can be noted from the value of $\overline{N}_2$.

Limitations

In this formulation, all the individuals in the prey (predator) population under toxicant stress are assumed to be identical. Their age or size factor is not considered. Also the concentration of the toxicant in the environment is assumed to be constant, and the accumulation of the toxicant in the individuals through food chain pathways is not considered.

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We shall now consider a model in which the predator population is indirectly affected by the toxicant accumulated in the population through food chain pathways.

Formulation 4: In Formulation 3 neglecting the direct effect of toxicant on the predator population i.e., ignoring V_0 this model can be described by the system of Eqns. (26)-(28), i.e.,

$$\frac{dN_1}{dt} = r_0N_1 - r_1\,C_0N_1 - bN_1N_2$$

$$\frac{dN_2}{dt} = -d_0N_2 + \beta\,(C_0)\,bN_1N_2$$

$$\frac{dC_0}{dt} = k_1\,P - g_1\,C_0 - m_1\,C_0$$

along with the initial conditions:

$$N_1(0) = N_{10}, N_2(0) = N_{20}, C_0(0) = 0 \text{ and } \beta\,(C_0) = \beta_0 - \beta_1\,C_0,$$

where $\beta\,(C_0)$ is the conversion coefficient depending upon C_0.

Stability Analysis

The variational matrix of the above system of equations can be written as

$$A = \begin{bmatrix} r_0 - r_1C_0 - bN_2 & -bN_1 & -r_1N_1 \\ \beta(C_0)\,bN_2 & -d_0 + \beta(C_0)bN_1 & -\beta_1bN_1N_2 \\ 0 & 0 & -(g_1 + m_1) \end{bmatrix}. \tag{37}$$

The equilibrium points E_0 and E_1 of the above system are given by

$$E_0 : N_1^* = 0,\ N_2^* = 0,\ C_0^* = \frac{k_1P}{(g_1 + m_1)}. \tag{38}$$

$$E_1 : \overline{N}_1 = \frac{d_0}{\beta(\overline{C}_0)\,b},\ \overline{N}_2 = \frac{r_0 - r_1\,\overline{C}_0}{b},\ \overline{C}_0 = \frac{k_1P}{(g_1 + m_1)}. \tag{39}$$

The matrix A for the equilibrium point E_0 reduces to

$$A|_{E_0} = \begin{bmatrix} r_0 - r_1\,C_0^* & 0 & 0 \\ 0 & -d_0 & 0 \\ 0 & 0 & -(g_1 + m_1) \end{bmatrix} \tag{40}$$

and from the eigenvalues of the above matrix, it may be deduced that the equilibrium solution E_0 is **asymptotically stable** if $r_0 < r_1 C_0^*$, that is, if

$$r_0(g_1 + m_1) < r_1 k_1 P \tag{41}$$

Further, the matrix A for the equilibrium point E_1 becomes

$$A|_{E_1} = \begin{bmatrix} 0 & -b\overline{N}_1 & -r_1\overline{N}_1 \\ \beta(\overline{C}_0)b\overline{N}_2 & 0 & -\beta_1 b\overline{N}_1\overline{N}_2 \\ 0 & 0 & -(g_1+m_1) \end{bmatrix}. \tag{42}$$

The characteristic equation associated with the above matrix is

$$(\lambda + g_1 + m_1)(\lambda^2 + b^2\beta(\overline{C}_0)\overline{N}_1\overline{N}_2) = 0. \tag{43}$$

From the characteristic Eqn. (43) it is obvious that one of its eigenvalue is negative and the other two eigenvalues are imaginary. Hence the equilibrium solution E_1 is **stable**.

Interpretation of the Solution

From the stability of E_0 it is concluded that both the prey and predator populations will tend to extinction under the effect of toxicant. The stability criteria for E_1 shows that both the prey and predator population will coexist. The equilibrium level of prey may increase and that of predator may decrease because of the lowering of conversion of prey into predator.

Limitations

In this formulation all the individuals in the prey (predator) population under toxicant stress are assumed to be identical. Their age or size factor is not considered. Also the concentration of the toxicant in the environment is assumed to be constant.

***

It is now time for you to try some exercises. All the variables and parameters notations in Exercises E2)-E4) given below have the same meaning as defined for various models considered in this section.

E2) Do the stability analysis of the following model which is formulated to study the effect of toxicant on a competing species.

$$\frac{dN_1}{dt} = r_1 N_1 - \alpha_1 N_1 N_2 - d_1 C_0 N_1$$

$$\frac{dN_2}{dt} = r_2 N_2 - \alpha_2 N_1 N_2$$

$$\frac{dC_0}{dt} = k_1 P - g_1 C_0 - m_1 C_0.$$

where $N_1(0) = N_{10}$, $N_2(0) = N_{20}(0)$, $C_0(0) = 0$.

E3) Do the stability analysis of the following prey-predator model under toxicant stress in which it is assumed that the predators are not affected by the toxicant because they are generally strong.

$$\frac{dN_1}{dt} = r_0 N_1 - r_1 C_0 N_1 - bN_1N_2$$

$$\frac{dN_2}{dt} = -d_0N_2 + \beta_0 b N_1 N_2$$

$$\frac{dC_0}{dt} = k_1 P - g_1 C_0 - m_1 C_0.$$

where $N_1(0) = N_{10}$, $N_2(0) = N_{20}(0)$, $C_0(0) = 0$.

E4) Do the stability analysis of the following prey-predator model under toxicant stress in which predators are not affected by the toxicant and environmental toxicant concentration is taken to be a variable depending upon time.

$$\frac{dN_1}{dt} = r_0N_1 - r_1C_0N_1 - bN_1N_2$$

$$\frac{dN_2}{dt} = -d_0N_2 + \beta_0 bN_1N_2$$

$$\frac{dC_0}{dt} = k_1P - g_1 C_0 - m_1C_0$$

$$\frac{dP}{dt} = Q - hP - kPN_1 + gC_0N_1.$$

where $N_1(0) = N_{10}$, $N_2(0) = N_{20}(0)$, $C_0(0) = 0$, $P(0) = P_0 > 0$.

We shall now discuss in the next section some spatial ecological systems.

5.3 MODELLING SPATIAL ECOLOGICAL SYSTEMS

Models such as the logistic and the Lotka-volterra (Ref. Block 3, MTE-14) have played a key role in the development of ecological theory and have been termed 'classical ecological models'. Typically, classical ecological models assume that ecological interactions take place in a spatially homogeneous environment. The reasons behind the adoption of this view are many. The aim is to keep theoretical and experimental studies relatively simple. The classical approach to model ecological systems ignores space by assuming, at least implicitly, that every individual is equally accessible to every other individual. The resulting models usually take the form of a series of difference or differential equations. However, ecologists have shown that space may alter the dynamics of non spatial (or spatially implicit) ecological models. For example, **species** diversity within an area may be related to **habitat** heterogeneity. The dynamics of population and predator-prey interactions may be more stable or persistent in patchy environments. The spread of contagious disturbances such as fire or pathogen outbreak is altered by patchiness and dispersal.

Habitat is the sum of the environment conditions where an organism, population or community lives.

Community is a collection of different types of populations.

Three broad 'frameworks' for the introduction of space into ecological models are available. The first of these is based on reaction-diffusion systems, which may be either discrete or continuous in both time and space. A second approach is to consider changes in the occupation of different patches overtime. Such models may be termed patch-occupancy or meta-population models. A third and spatially explicit approach is to use cellular automata or grid-based models; these may be individual-based or more aggregated. Here in this unit we shall concentrate only on reaction-diffusion systems and reaction diffusion-advection systems.

5.3.1 Reaction-Diffusion Systems

Let us consider continuous-time and continuous-space models that are known as reaction-diffusion equations/systems.

We shall begin with a **single-species** reaction-diffusion model in **one spatial** dimension.

Let $N(x, t)\Delta x =$ number of individuals in the interval $(x, x+\Delta x)$ at time t, where x denotes the space variable. (44)

If we consider the rate of change of the number of individuals in a given interval of space, we may write

$$\frac{\partial}{\partial t}[N(x, t)\Delta x] = \text{growth rate of individuals in } (x, x+\Delta x) + \text{rate of individuals entering at } x - \text{rate of individuals leaving at } x+\Delta x \quad (45)$$

or,

$$\frac{\partial N}{\partial t}\Delta x = f(x, t)\,\Delta x + J(x, t) - J(x+\Delta x, t) \quad (46)$$

where, $f(x, t)$ is the growth rate of the population per unit length and $J(x, t)$ is the positive (left to right) 'flux' of individuals at position x and time t.

Dividing by Δx, Eqn. (46) becomes

$$\frac{\partial N}{\partial t} = f(x, t) - \left[\frac{J(x+\Delta x, t) - J(x, t)}{\Delta x}\right]. \quad (47)$$

Taking the limit as Δx tends to zero in Eqn. (47), we get the conservative law for the density of individuals as

$$\frac{\partial N}{\partial t} = f(x, t) - \frac{\partial J}{\partial x}. \quad (48)$$

We now use **Fick's law** given by a German physiologist Adolf Fick in 1855 for explaining the phenomenon of diffusion. According to Fick's law the mass of the solute (in this case it is the individuals in the population) crossing a unit area per unit time in any direction is proportional to the negative gradient of the concentration of solute (population in this case) in that direction. That means J in Eqn. (48) can be written as

$$J = -D\frac{\partial N}{\partial x}. \quad (49)$$

where D is the diffusion coefficient and $\frac{\partial N}{\partial x}$ is the gradient of the population in the x-direction.

Using Eqn. (49) in Eqn. (48), we get

$$\frac{\partial N}{\partial t} = f(x, t) - \frac{\partial}{\partial x}\left(-D\frac{\partial N}{\partial x}\right) \tag{50}$$

or,

$$\frac{\partial N}{\partial t} = f(x, t) + \frac{\partial}{\partial x}\left(D\frac{\partial N}{\partial x}\right). \tag{51}$$

If the diffusion coefficient D is constant then Eqn. (51) can be written as

$$\frac{\partial N}{\partial t} = f(x, t) + D\frac{\partial^2 N}{\partial x^2}. \tag{52}$$

Eqns. (51) and (52) are the required **reaction-diffusion model** in one space-dimension for single species. Different forms of the growth function $f(x, t)$ for the population may be considered. Here we shall consider the following two forms of the function $f(x, t)$.

1. **Exponential growth:** When

$$f(x, t) = r\,N(x, t) \tag{53}$$

where, r = rate of population increase.

2. **Logistic growth:** when

$$f(x, t) = r\,N(x, t)\left[1 - \frac{N(x, t)}{K}\right] \tag{54}$$

where, r = rate of population increase, and

K = carrying capacity of the environment.

Let us first discuss the model for the exponential growth function.

Formulation 1: The continuous form of a reaction-diffusion equation model in which the function $f(x, t)$ follow **exponential growth** is given by the following equation

$$\frac{\partial N}{\partial t} = r\,N + D\frac{\partial^2 N}{\partial x^2}, \qquad 0 \le x \le L. \tag{55}$$

The initial and boundary conditions associated with Eqn. (55) are as follows:

Initial conditions:

$$N(x,0) = f(x) > 0 \text{ for all } x : 0 \le x \le L. \tag{56}$$

No flux boundary conditions:

$$\frac{\partial N}{\partial x} = 0 \text{ at } x = 0, \forall\, t \tag{57}$$

and $$\frac{\partial N}{\partial x} = 0 \text{ at } x = L, \forall\, t. \tag{58}$$

Let us now do the stability analysis of Eqn. (55) under the conditions given by Eqns. (56)-(58).

Stability Analysis

The uniform steady-state solution of Eqn. (55) is $N^* = 0$, which is obtained by taking $D=0$ and putting $\frac{\partial N}{\partial t}=0$ in Eqn. (55).

To do the stability analysis of the steady-state $N^*=0$, we take the small perturbation from the steady-state $N^*=0$ as

$$N(x,\ t)=N^*+p\ (x,\ t) \tag{59}$$

or, $$N(x,\ t)=p(x,\ t). \qquad \text{[because } N^*=0\text{]} \tag{60}$$

Using Eqn. (60) in Eqn. (55), we get

$$\frac{\partial p}{\partial t}=r\,p+D\frac{\partial^2 p}{\partial x^2}. \tag{61}$$

In order to study the stability property of the steady-state solution $N^*=0$, we consider the following **positive definite function** (ref. Appendix-I).

$$V(t)=\frac{1}{2}\int_0^L p^2\,dx. \tag{62}$$

Then, $$\frac{dV}{dt}=\int_0^L p\,\frac{\partial p}{\partial t}\,dx. \tag{63}$$

Using Eqn. (61) in Eqn. (63), we get

$$\frac{dV}{dt}=\int_0^L p\left[r\,p+D\frac{\partial^2 p}{\partial x^2}\right]dx \tag{64}$$

or, $$\frac{dV}{dt}=r\int_0^L p^2\,dx+D\int_0^L p\,\frac{\partial^2 p}{\partial x^2}\,dx$$

or, $$\frac{dV}{dt}=r\int_0^L p^2\,dx+D\,p\,\frac{\partial p}{\partial x}\Bigg|_{x=0}^{x=L}-D\int_0^L\left(\frac{\partial p}{\partial x}\right)^2dx$$

or, $$\frac{dV}{dt}=r\int_0^L p^2\,dx-D\int_0^L\left(\frac{\partial p}{\partial x}\right)^2dx \tag{65}$$

$$\left[\because\ \frac{\partial p}{\partial x}=0 \text{ at } x=0 \text{ and } \frac{\partial p}{\partial x}=0 \text{ at } x=L\right].$$

Using Poincare's inequality in Eqn. (65) we get

Poincare's Inequality:

$$\int_0^L\left(\frac{\partial p}{\partial x}\right)^2dx\ge\frac{\pi^2}{L^2}\int_0^L p^2dx$$

$$\frac{dV}{dt}\le r\int_0^L p^2dx-D\frac{\pi^2}{L^2}\int_0^L p^2\,dx$$

or, $$\frac{dV}{dt}\le-\left(D\frac{\pi^2}{L^2}-r\right)\int_0^L p^2\,dx \tag{66}$$

Thus, we find from the r.h.s. expression of Eqn. (66) that $\frac{dV}{dt}$ is **negative**

definite if $D\frac{\pi^2}{L^2} > r$ and hence from **Liapunov's theorem** (ref. Appendix-I) we deduce that the steady-state solution $N^* = 0$ is **asymptotically stable** if the condition $D\frac{\pi^2}{L^2} > r$ is satisfied.

It may be **noted** here that the unstable steady-state solution $N^* = 0$ can be made stable if the spatial diffusion is introduced in the exponential growth equation.

Interpretation of the Solution

From the above analysis we may conclude that the exponential growing population may get extinct if dispersal (diffusion) is allowed.

Limitations

D is assumed to be constant. Intra-specific competition for resource is not considered. The population dynamics are deterministic. Although, individuals move randomly throughout their life time but at the population level movement is deterministic.

***

After analysing the spatial exponential growth model let us, now consider the spatial **logistic growth model.**

Formulation 2: Model (52) in which $f(x, t)$ is taken as $f(x, t) = r\,N\left(1-\frac{N}{K}\right)$ reduces to

$$\frac{\partial N}{\partial t} = r\,N\left(1-\frac{N}{K}\right) + D\frac{\partial^2 N}{\partial x^2},\quad 0 \le x \le L. \tag{67}$$

Eqn. (67) is called the **Fisher equation**.

The initial and boundary conditions associated with Eqn. (67) are as follows:

Initial conditions:

$$N(x, 0) = f(x) > 0 \text{ for all } x : 0 \le x \le L. \tag{68}$$

No Flux boundary conditions:

$$\frac{\partial N}{\partial x} = 0 \text{ at } x = 0, \forall\, t \tag{69}$$

and $$\frac{\partial N}{\partial x} = 0 \text{ at } x = L, \forall\, t. \tag{70}$$

Let us now carry out the stability analysis of Eqn. (67) under the conditions given by Eqns. (68)-(70).

Stability Analysis

The uniform steady-state solutions of Eqn. (67) are $N^* = 0$ and $\overline{N} = K$ which are obtained by taking $D = 0$ and putting $\frac{\partial N}{\partial t} = 0$ in Eqn. (67).

To do the linear stability analysis of the steady-state $\overline{N} = K$ we take the small perturbation from the steady-state $\overline{N} = K$ as

$$N(x, t) = \overline{N} + p(x, t) \tag{71}$$

$$= K + p(x, t) \tag{72}$$

where, $|p(x, t)| << 1$.

Using Eqn. (72) in Eqn. (67), we get

$$\frac{\partial p(x,t)}{\partial t} = D\frac{\partial^2 p(x,t)}{\partial x^2} + r(K + p(x,t))\left[1 - \frac{K + p(x, t)}{K}\right]$$

$$= D\frac{\partial^2 p(x, t)}{\partial x^2} + r(K + p(x, t))\left[-\frac{p(x, t)}{K}\right]$$

$$= D\frac{\partial^2 p}{\partial x^2} - r\,p - \frac{r}{K}p^2 \tag{73}$$

Neglecting the term $\frac{r}{K}p^2$ in Eqn. (73), we get the following linear differential equation about K

$$\frac{\partial p}{\partial t} = D\frac{\partial^2 p}{\partial x^2} - r\,p. \tag{74}$$

In order to study the stability properties of the steady-state solution $\overline{N} = K$, we consider the following positive definite function

$$V(t) = \frac{1}{2}\int_0^L p^2\,dx. \tag{75}$$

Then, $$\frac{dV}{dt} = \int_0^L p\frac{\partial p}{\partial t}\,dx. \tag{76}$$

Using Eqn. (74) in Eqn. (76), we get

$$\frac{dV}{dt} = \int_0^L \left[D\,p\frac{\partial^2 p}{\partial x^2} - r\,p^2\right]dx$$

$$= D\int_0^L p\frac{\partial^2 p}{\partial x^2} - r\int_0^L p^2\,dx. \tag{77}$$

Integrating the r.h.s. of Eqn. (77) by parts and using no flux boundary conditions given by Eqns. (69) and (70), we get

$$\frac{dV}{dt} = -D\int_0^L \left(\frac{\partial p}{\partial x}\right)^2 dx - r\int_0^L p^2\, dx \qquad \left[\because \frac{\partial p}{\partial x}(0,\, t) = 0 = \frac{\partial p}{\partial x}(L,\, t)\right]$$

$$= -\int_0^L \left[D\left(\frac{\partial p}{\partial x}\right)^2 + r\, p^2\right] dx\,. \tag{78}$$

Thus, we find from the r.h.s. expression of the Eqn. (78) that $\frac{dV}{dt}$ is negative definite and hence from Liapunov's theorem we derive that the steady-state solution $\overline{N} = K$ is **asymptotically stable.**

Similarly, it can be proved that the trivial steady-state solution $N^* = 0$ is unstable and can be made stable with the help of diffusion (dispersal). We are leaving it for you to prove it yourself.

E5) Prove that otherwise unstable steady-state solution $N^* = 0$ of Eqn. (67) can be made stable with the help of diffusion.

Interpretation of the Solution

From the above analysis, we conclude that the population will survive and the dispersal or diffusion of the population may reduce crowding effect (intra-specific competition) which may further strengthen the existence of the population. It is also concluded that after long-time interval the population may reach its carrying capacity.

Limitations

The model is of deterministic nature and the dispersal or diffusion of the population is taken in only one direction considering constant dispersal (or diffusion) coefficient.

***

Let us now consider **two-species** reaction-diffusion model in **one spatial** dimension. We consider an example of prey-predator population.

Formulation 3: Let us now consider interaction and diffusion of both prey and predator populations. Following the reaction-diffusion model given by Eqn. (52) we construct the model as given below:

$$\frac{\partial N_1}{\partial t} = a_1 N_1 - b_1 N_1 N_2 + D_1 \frac{\partial^2 N_1}{\partial x^2} \tag{79}$$

$$\frac{\partial N_2}{\partial t} = -d_1 N_2 + C_1 N_1 N_2 + D_2 \frac{\partial^2 N_2}{\partial x^2} \tag{80}$$

where, $0 \le x \le L$.

The above system of Eqns. (79) and (80) is associated with the following initial conditions and no flux boundary conditions

$$\left.\begin{aligned} N_1(x, 0) = f_1(x) > 0 \\ N_2(x, 0) = f_2(x) > 0, \\ \text{for } 0 \le x \le L \end{aligned}\right\} \tag{81}$$

$$\left.\begin{aligned} \frac{\partial N_1}{\partial x} = 0 \text{ at } x = 0 \text{ and } x = L \ \forall\, t \\ \frac{\partial N_2}{\partial x} = 0 \text{ at } x = 0 \text{ and } x = L \ \forall\, t \end{aligned}\right\}. \tag{82}$$

The variables and parameters of the system given by Eqns. (79)-(80) are as follows:

N_1 = Density of prey population
N_2 = Density of predator population
a_1 = Growth rate
d_1 = Death rate
b_1 = Predation rate
C_1 = Conversion rate
D_1, D_2 = Diffusion coefficients.

where, a_1, d_1, b_1, C_1, D_1 and D_2 are all positive constants.

Let us now do the stability analysis of Eqns. (79) and (80) along with the initial and boundary conditions given by Eqns. (81) and (82).

Stability Analysis

The uniform steady-state solution of two interacting species population is obtained by putting $D_1 = 0, D_2 = 0$ and then solving $\frac{\partial N_1}{\partial t} = 0$ and $\frac{\partial N_2}{\partial t} = 0$ in Eqns. (79) and (80).

The equilibrium solutions E_0 and E_1 of the system of Eqns. (79) and (80) are

$$E_0 : N_1^* = 0,\ N_2^* = 0 \tag{83}$$

$$E_1 : \overline{N}_1 = d_1 / C_1,\ \overline{N}_2 = a_1 / b_1 \tag{84}$$

To do the stability analysis of the equilibrium point E_0, we take the small perturbation from the steady-state $N_1^* = 0$ and $N_2^* = 0$, as follows.

$$N_1 = N_1^* + p_1(x, t) \tag{85}$$

$$N_2 = N_2^* + p_2(x, t) \tag{86}$$

where $|p_1(x, t)| << 1$ and $|p_2(x, t)| << 1$.

Using Eqns. (85) and (86) in Eqns.(79) and (80) respectively, we get

$$\frac{\partial p_1}{\partial t} = a_1 p_1 + D_1 \frac{\partial^2 p_1}{\partial x^2}. \tag{87}$$

$$\frac{\partial p_2}{\partial t} = -d_1 p_2 + D_2 \frac{\partial^2 p_2}{\partial x^2}. \tag{88}$$

Let us consider the following positive definite function:

$$V(t) = \frac{1}{2}\int_0^L (p_1^2 + p_2^2)\, dx. \tag{89}$$

Then, $$\frac{dV}{dt} = \int_0^L \left(p_1 \frac{\partial p_1}{\partial t} + p_2 \frac{\partial p_2}{\partial t} \right) dx. \tag{90}$$

Using Eqns. (87)-(88) in Eqn. (90), we get

$$\frac{dV}{dt} = \int_0^L \left[a_1 p_1^2 + D_1 p_1 \frac{\partial^2 p_1}{\partial x^2} - d_1 p_2^2 + D_2 p_2 \frac{\partial^2 p_2}{\partial x^2} \right] dx. \tag{91}$$

After integrating by parts and using boundary conditions, we get

$$\frac{dV}{dt} = \int_0^L \left[a_1 p_1^2 - D_1 \left(\frac{\partial p_1}{\partial x} \right)^2 - d_1 p_2^2 - D_2 \left(\frac{\partial p_2}{\partial x} \right)^2 \right] dx. \tag{92}$$

Using Poincare's inequality in Eqn. (92), we obtain

$$\frac{dV}{dt} \le -\int_0^L \left[\left(D_1 \frac{\pi^2}{L^2} - a_1 \right) p_1^2 + \left(D_2 \frac{\pi^2}{L^2} + d_1 \right) p_2^2 \right] dx. \tag{93}$$

Hence, $\frac{dV}{dt}$ is negative definite if $D_1 \frac{\pi^2}{L^2} > a_1$. Thus, we derive from Liapunov's theorem that the trivial equilibrium solution E_0 is **asymptotically stable**. This shows that the otherwise unstable equilibrium solution can be made stable by introducing the diffusion in the system.

In the same way the stability analysis of equilibrium point E_1 can be done. We are leaving it to you to do it yourself.

E6) For the system of Eqns. (79) and (80), do the stability analysis of the equilibrium point E_1 given by Eqn. (84). Also interpret the solution obtained.

Interpretation of the Solution

From the stability of E_0 we derive that both the populations of prey and predator would tend to extinct if their random movement (diffusion) is taken into consideration.

Limitations

In this model we have not considered intra-specific competition and also the diffusion coefficients are not considered to be depending upon densities of the populations.

***

In the next sub-section we shall discuss models in which along with diffusion, we also take into consideration the effect of advection, i.e., the horizontal directed movement of organisms.

5.3.2 Advection-Diffusion-Reaction Systems

Let us assume that the organisms move randomly and also have a horizontal directed movement towards a favourable region in the x-direction. The flux J, given by Eqn. (49) in this case can then be written as

$$J_A = v\ N(x, t) - D\frac{\partial N}{\partial x} \tag{94}$$

where, v is the advection velocity in x-direction and D is the diffusion coefficient (or diffusivity). Using Eqn. (94) in Eqn. (48), we get the following advection-diffusion-reaction equation in **one spatial dimension** for **single species**.

$$\frac{\partial N}{\partial t} = f(x, t) - \frac{\partial}{\partial x}[J_A] \tag{95}$$

or, $$\frac{\partial N}{\partial t} + \frac{\partial}{\partial x}(v\ N(x,t)) = f(x, t) + \frac{\partial}{\partial x}\left(D\frac{\partial N}{\partial x}\right). \tag{96}$$

If v and D are constants then Eqn. (96) can be written as

$$\frac{\partial N}{\partial t} + v\frac{\partial N}{\partial x} = f(x, t) + D\frac{\partial^2 N}{\partial x^2}. \tag{97}$$

We shall now be discussing the model given by Eqn. (97) for the two forms of the growth rate function $f(x, t)$, the exponential growth rate and the logistic growth rate.

Formulation 1: Let us first study the advection-diffusion-reaction model for single population given by Eqn. (97), for the **exponential growth** rate function. Eqn. (97) with exponential growth rate function becomes

$$\frac{\partial N}{\partial t} = r\ N + D\frac{\partial^2 N}{\partial x^2} - v\frac{\partial N}{\partial x}, \quad 0 \le x \le L \tag{98}$$

The initial and boundary conditions associated with Eqn. (98) are as follows.

.Initial condition:

$$N(x, 0) = f(x) > 0;\ 0 \le x \le L \tag{99}$$

Hostile boundary conditions:

$$N = N^* \text{ at } x = 0,\ \forall\ t \tag{100}$$

and $$N = N^* \text{ at } x = L,\ \forall\ t \tag{101}$$

where N^* is the equilibrium point of the given Eqn. (98).

Note that here we have taken hostile boundary conditions which means that the population will remain at its equilibrium level on the boundaries of the habitat. In the case when we take advection in the model then the equilibrium points of the model will become stable only if we consider hostile boundary conditions and not the no-flux boundary conditions.

Let us now do the stability analysis of Eqn. (98) under the conditions given by Eqns. (99)-(101).

Stability Analysis

The uniform steady-state solution of Eqn. (98) is $N^* = 0$, which is obtained by putting $\frac{\partial N}{\partial t} = 0$ and considering $D = 0,\ v = 0$ in Eqn. (98).

To do the stability analysis of the steady-state $N^* = 0$, we take the small perturbation from the steady-state $N^* = 0$ as

$$N(x, t) = N^* + p(x, t) \tag{102}$$

$$N(x, t) = p(x, t) \qquad [\because\ N^* = 0] \tag{103}$$

where $|p(x, t)| << 1$.

Using Eqn. (103) in Eqn. (98), we get

$$\frac{\partial p}{\partial t} = r\,p + D\,\frac{\partial^2 p}{\partial x^2} - v\,\frac{\partial p}{\partial x}. \tag{104}$$

In order to study the stability property of the steady-state solution $N^* = 0$, we consider the following positive definite function

$$V(t) = \frac{1}{2}\int_0^L p^2\, dx \tag{105}$$

$$\frac{dV}{dt} = \int_0^L p\,\frac{\partial p}{\partial t}\, dx. \tag{106}$$

Using Eqn. (104) in Eqn. (106), we get

$$\frac{dV}{dt} = \int_0^L p\left[r\,p + D\frac{\partial^2 p}{\partial x^2} - v\frac{\partial p}{dx}\right] dx \tag{107}$$

or, $$\frac{dV}{dt} = r\int_0^L p^2 dx + D\int_0^L p\,\frac{\partial^2 p}{\partial x^2}\, dx - v\int_0^L p\,\frac{\partial p}{\partial x}\, dx \tag{108}$$

or, $$\frac{dV}{dt} = r\int_0^L p^2\, dx - D\int_0^L \left(\frac{\partial p}{\partial x}\right)^2 dx \qquad \left[\because\ p(0, t) = p(L, t) = 0\right]. \tag{109}$$

Using Poincare's Inequality in Eqn. (109), we get

$$\frac{dV}{dt} \le -\left(D\frac{\pi^2}{L^2} - r\right)\int_0^L p^2\, dx. \tag{110}$$

Thus, we find from the r.h.s. of Eqn. (110) that $\frac{dV}{dt}$ is negative definite if

$D\frac{\pi^2}{L^2} > r$. Hence, from Liapunov's theorem, we conclude that the steady-state solution $N^* = 0$ is **asymptotically stable** if the condition $D\frac{\pi^2}{L^2} > r$ is satisfied. It may be **noted** here that under the hostile boundary conditions, the advection does not play any role regarding the stability of the equilibrium point.

Interpretation of the Solution

From the analysis it may be concluded that the exponentially growing population may get extinct if diffusion (random movement) and advection (directed movement) are considered under the hostile boundary condition.

Limitation

In the model formulation the diffusion coefficient and the advection velocity are both taken to be constants which, however, may depend on space variables and population density. Also, the hostile boundary condition is used instead of no-flux boundary condition.

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Now we will consider the **advection-diffusion-reaction** equation with **logistic growth function**.

Formulation 2: Let us consider the function $f(x, t)$ in Eqn. (97) to be the Logistic growth function. Eqn. (97) then reduces to the following form:

$$\frac{\partial N}{\partial t} = r\,N\left(1-\frac{N}{K}\right) + D\frac{\partial^2 N}{\partial x^2} - v\frac{\partial N}{\partial x}, \quad 0 \le x \le L. \tag{111}$$

The initial and boundary conditions associated with Eqn. (111) are as follows:

$$N(x, 0) = f(x) > 0 \ \forall\ x; \quad 0 \le x \le L \tag{112}$$

$$N(0, t) = N^* \text{ or } \overline{N}\ \forall\ t \tag{113}$$

$$N(L, t) = N^* \text{ or } \overline{N}\ \forall\ t \tag{114}$$

Let us carry out the stability analysis of Eqn. (111) under the conditions (112)-(114).

Stability Analysis

The uniform steady-state solutions of Eqn. (111) are $N^* = 0$ and $\overline{N} = K$, which are obtained by taking $D = 0,\ v = 0$ in Eqn. (111).

To do the stability analysis of the steady-state $\overline{N} = K$, we take the small perturbation from the steady-state $\overline{N} = K$ in the form

$$N(x, t) = \overline{N} + p(x, t) \tag{115}$$

or,

$$N(x, t) = K + p(x, t) \tag{116}$$

where, $|\,p(x, t)\,| << 1$.

Using Eqn. (116) in Eqn. (111), we get

$$\frac{\partial p}{\partial t} = D\frac{\partial^2 p}{\partial x^2} - v\frac{\partial p}{\partial x} + r\,(K+p)\left[1-\frac{K+p}{K}\right]$$

$$= D\frac{\partial^2 p}{\partial x^2} - v\frac{\partial p}{\partial x} + r\,(K+p)\left(-\frac{p}{K}\right)$$

$$= D\frac{\partial^2 p}{\partial x^2} - v\frac{\partial p}{\partial x} - r\,p \qquad \text{[Neglecting the term } -\frac{r}{K}p^2\text{]}. \tag{117}$$

In order to study the stability properties of the steady-state solution $\overline{N} = K$, we consider the following positive definite function

$$V(t) = \frac{1}{2}\int_0^L p^2\, dx\,. \tag{118}$$

Then, $$\frac{dV}{dt} = \int_0^L p\,\frac{\partial p}{\partial t}\, dx\,. \tag{119}$$

Using Eqn. (117) in Eqn. (119), we get

$$\frac{dV}{dt} = \int_0^L p\left[D\frac{\partial^2 p}{\partial x^2} - v\frac{\partial p}{\partial x} - r\,p\right] dx \tag{120}$$

$$= D\int_0^L p\,\frac{\partial^2 p}{\partial x^2} dx - v\int_0^L p\,\frac{\partial p}{\partial x}\, dx - r\int_0^L p^2\, dx\,. \tag{121}$$

Integrating the r.h.s. of Eqn. (121) by parts and using the boundary conditions (113) and (114), we get

$$\frac{dV}{dt} = -D\int_0^L \left(\frac{\partial p}{\partial x}\right)^2 dx - r\int_0^L p^2\, dx$$

or, $$\frac{dV}{dt} = -\int_0^L \left[D\left(\frac{\partial p}{\partial x}\right)^2 + r\,p^2\right] dx\,. \tag{122}$$

Thus, we find from the r.h.s. expression of Eqn. (122) that $\frac{dV}{dt}$ is negative definite. Hence from Liapunov's theorem we derive that the steady-state solution $\overline{N} = K$ is **asymptotically stable**.

In the same way, it can be shown that the trival steady-state solution $N^* = 0$ is unstable and can be made stable with the help of diffusion in the presence of advection. We are leaving it for you to show it yourself.

E7) Show that otherwise unstable trivial steady-state solution $N^* = 0$ of Eqn. (111) can be made stable with the help of diffusion in the presence of advection.

Interpretation of the Solution

From the analysis, it may be concluded that the population will survive and the random (diffusion) and directed (advection) movements will increase the chances of survival.

Limitations

In the above model formulation, the diffusion coefficient and the advection velocity are assumed to be constant. The habitat is also considered one dimensional instead of two dimensional. Moreover, the stability result may change if no-flux boundary conditions are considered instead of hostile boundary conditions.

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The **two-species reaction-diffusion-advection** system in **one spatial dimension,** can also be discussed the way we have discussed two-species reaction-diffusion system. We are leaving this as an exercise for you to do it yourself.

You may now try the following exercises.

E8) Do the stability analysis of the following competing species system of equations with diffusion.

$$\frac{\partial N_1}{\partial t} = r_1 N_1 - \alpha_1 N_1 N_2 + D_1 \frac{\partial^2 N_1}{\partial x^2}$$

$$\frac{\partial N_2}{\partial t} = r_2 N_2 - \alpha_2 N_1 N_2 + D_2 \frac{\partial^2 N_2}{\partial x^2}; \quad 0 \le x \le L$$

where, N_1 and N_2 are population densities, D_1 and D_2 are diffusion coefficients, r_1 and r_2 are growth rates, α_1 and α_2 are interspecific competition coefficients. $r_1, r_2, \alpha_1, \alpha_2, D_1$ and D_2 are all positive constants. The initial and boundary conditions are:

$$N_i(x, 0) = f_i(x) > 0; \; 0 \le x \le L; \; i = 1, 2$$

$$\frac{\partial N_i}{\partial x} = 0 \text{ at } x = 0 \text{ and } x = L, \forall \, t; \; i = 1, 2.$$

E9) Do the stability analysis of the following competing species system with diffusion and advection

$$\frac{\partial N_1}{\partial t} = r_1 N_1 - \alpha_1 N_1 N_2 + D_1 \frac{\partial^2 N_1}{\partial x^2} - v_1 \frac{\partial N_1}{\partial x}$$

$$\frac{\partial N_2}{\partial t} = r_2 N_2 - \alpha_2 N_1 N_2 + D_2 \frac{\partial^2 N_2}{\partial x^2} - v_2 \frac{\partial N_2}{\partial x}, \quad 0 \le x \le L.$$

All the variables and constants in the above model are same as defined for the model considered in E8). v_1 and v_2 are constant advection velocities with $v_1 > 0$ and $v_2 > 0$. The initial and boundary conditions are:

$$N_i(x, 0) = f_i(x) > 0; \; 0 \le x \le L, \; i = 1, 2$$

$$N_i = \overline{N}_i \text{ at } x = 0 \text{ and } x = L \;\; \forall \, t, \quad i = 1, 2$$

where $\overline{N}_i$ are the equilibrium solutions of the given system of equations.

We now end this unit by giving a summary of what we have covered in it.

5.4 SUMMARY

In this unit, we have covered the following:

1. **Environment** is a collection of all the biotic (living) and abiotic (nonliving) factors that affect an individual organism at any one point in its life cycle.
2. Mathematical modelling of the environment can be used for **understanding** the processes that form the environment around us and it helps in **evaluating** the impacts of environmental deterioration on the ecosystem due to human activities.
3. The organisms in the environmental system are constantly exposed to natural and synthetic chemicals called **toxicants** which adversely affect the organisms like death; illness and change in their metabolic activities.
4. Mathematical models to study the effect of toxicants on one and two interating populations have been formulated in terms of **single/system of ordinary differential equations**.
5. The stability analysis of these single/system of ordinary differential equations gives the **criterion** for the extinction, survival or coexistence of populations under the **effect of toxicants**.
6. Mathematical models to study **spatio-temporal behaviour** of one and two interacting populations in one-dimensional linear habitat based on **reaction-diffusion systems** (in case of random movement) and **reaction-diffusion-advection system** (in case of random as well as directed movement) have been formulated in terms of **single/system of partial differential equations**.
7. The stability analysis of these single/system of PDEs gives the **criterion** for the survival, extinction or coexistence of the population under **diffusion** and **advection**.

5.5 SOLUTIONS/ANSWERS

E1) Hurwitz's conditions for the polynomial $\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3$ for the negativity of the real parts of the roots are:

$a_1 > 0,\ a_2 > 0,\ a_3 > 0$ and $a_1a_2 - a_3 > 0$. In the polynomial given by Eqn. (23);

$a_1 = g + m + h + k_1\overline{N} + s\overline{N} > 0$ because g, m, h, k_1, s and $\overline{N}$ are positive.

$a_2 = (g+m)\ (h + k_1\overline{N}) + s\overline{N}(g + m + h + k_1\overline{N}) - kg_1\overline{N} > 0$ if

$(g+m)\ (h + k_1\overline{N}) > kg_1\overline{N}$ because g, m, h, k_1, s, $\overline{N}$, k, g_1 are positive.

$a_3 = s\overline{N}\,[(g+m)\ (h + k_1\overline{N}) - kg_1\overline{N}] + r_1\overline{N}k\,[g_1\overline{C}_0 - k_1\overline{P}] > 0$ if

$g_1\overline{C}_0 > k_1\overline{P}$ and $(g+m)(h+k_1\overline{N}) > kg_1\overline{N}$ because

$s, \overline{N}, g, m, h, k_1, r_1, \overline{C}_0, \overline{P}, g_1, k$ are positive.

$$a_1a_2 - a_3 = (g+m+h+k_1\overline{N})\,[(g+m)(h+k_1\overline{N}) - kg_1\overline{N}]$$
$$+ s\overline{N}(g+m+h+k_1\overline{N})^2 + r_1kk_1\overline{N}\,\overline{P}$$
$$+ \overline{N}[s\overline{N}(g+m+h+k_1\overline{N}) - r_1kg_1\overline{C}_0] > 0.$$

If $s\overline{N}(g+m+h+k_1\overline{N}) > r_1kg_1\overline{C}_0$.

E2) The equilibrium points are

$$E_0: \; N_1^* = 0, N_2^* = 0, C_0^* = \frac{k_1P}{(g_1+m_1)}$$

$$E_1: \; \overline{N}_1 = r_2/\alpha_2, \overline{N}_2 = (r_1 - d_1\,\overline{C}_0)/\alpha_1$$
$$\overline{C}_0 = \frac{k_1P}{(g_1+m_1)}.$$

The variational matrix about E_0 is

$$A\big|_{E_0} = \begin{bmatrix} r_1 - d_1\,C_0^* & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & -(g_1+m_1) \end{bmatrix}$$

The equilibrium point E_0 is not stable, that is, unstable because one eigenvalue is positive.

The varitional matrix about E_1 is

$$A\big|_{E_1} = \begin{bmatrix} 0 & -\alpha_1\overline{N}_1 & -d_1\overline{N}_1 \\ -\alpha_2\,\overline{N}_2 & 0 & 0 \\ 0 & 0 & -(g_1+m_1) \end{bmatrix}.$$

The characteristic equation related to A_1 is

$(\lambda + g_1 + m_1)(\lambda^2 - \alpha_1\alpha_2\,\overline{N}_1\overline{N}_2) = 0$.

Since one root is positive and two roots are negative, E_1 is unstable.

E3) The equilibrium points are

$$E_0: \; N_1^* = 0, N_2^* = 0, C_0^* = \frac{k_1P}{(g_1+m_1)}$$

$$E_1: \; \overline{N}_1 = \frac{d_0}{\beta_0 b}, \overline{N}_2 = \frac{r_0 - r_1\,C_0}{b}$$
$$\overline{C}_0 = \frac{k_1P}{(g_1+m_1)}$$

The variational matrix about E_0 is

$$A\big|_{E_0} = \begin{bmatrix} r_0 - r_1\,C_0^* & 0 & 0 \\ 0 & -d_0 & 0 \\ 0 & 0 & -(g_1+m_1) \end{bmatrix}$$

E_0 is asymptotically stable if $r_0 < r_1\ C_0^*$.

The characteristic equation about E_1 is

$$\lambda^3 + \lambda^2(g_1 + m_1) + b^2\beta_0\ \overline{N}_1\overline{N}_2\lambda + (g_1 + m_1)\ b^2\beta_0\ \overline{N}_1\overline{N}_2 = 0.$$

E_1 is unstable because the Hurwitz's criteria is not being satisfied.

E4) Equlibrium points are

$$E_0:\ N_1^* = 0,\ N_2^* = 0,\ C_0^* = \frac{k_1 P^*}{(g_1 + m_1)} \text{ and } P^* = Q/h.$$

E_0 is linearly asymptotically stable if $r_0 < r_1\ C_0^*$.

$$E_1:\ \overline{N}_1 = \frac{d_0}{\beta_0 b},\ \overline{N}_2 = \frac{r_0 - r_1\ \overline{C}_0}{b},$$

$$\overline{C}_0 = \frac{k_1\overline{P}}{(g_1 + m_1)} \text{ and } \overline{P} = \frac{Q + g\ \overline{C}_0\overline{N}_1}{h + k\ \overline{N}_1}$$

where, $\overline{N}_2 > 0$ provided $r_0 > r_1\ \overline{C}_0$.

The characteristic equation for E_1 is

$$(\lambda + h)\ (\lambda + g_1 + m_1)\ (\lambda^2 + b^2\overline{N}_1\beta_0\overline{N}_2) + r_1\overline{N}_1 k_1(k\,\overline{P} - g\,\overline{C}_0)\lambda = 0$$

Using Hurwitz's criteria in the above polynomial you can find about the stability properties of E_1.

E5) Proceed exactly as we did for doing the stability analysis of the steady-state $\overline{N} = K$.

E6) Let us take the small perturbation from the steady-state $\overline{N}_1$ and $\overline{N}_2$ as follows:

$$N_1 = \overline{N}_1 + p_1(x,\ t)$$

$$N_2 = \overline{N}_2 + p_2(x,\ t)$$

where $|p_1(x,\ t)| << 1,\ |p_2(x,\ t)| << 1$.

Using the above equations in the given system

$$\left.\begin{aligned} \frac{\partial p_1}{\partial t} &= -\frac{b_1 d_1}{C_1} p_2 + D_1\ \frac{\partial^2 p_1}{\partial x^2} \\ \frac{\partial p_2}{\partial t} &= \frac{C_1 a_1}{b_1} p_1 + D_2\ \frac{\partial^2 p_2}{\partial x^2} \end{aligned}\right\}.$$

Consider the following positive definite function

$$V = \frac{1}{2}\int_0^L (p_1^2 + A\,p_2^2)\ dx$$

where A is arbitrary positive constant.

Then, $\dfrac{dV}{dt} = \int_0^L \left(p_1 \dfrac{\partial p_1}{\partial t} + A\, p_2^2 \dfrac{\partial p_2}{\partial t} \right) dx$

or,

$$\frac{dV}{dt} = \int_0^L \left(-\frac{b_1 d_1}{C_1} p_1 p_2 + D_1 \frac{\partial^2 p_1}{\partial x^2} p_1 + A \frac{C_1 a_1}{b_1} p_1 p_2 + D_2 \frac{\partial^2 p_2}{\partial x^2} p_2 \right) dx$$

choosing $A = \dfrac{b_1^2 d_1}{C_1^2 a_1}$ in the r.h.s. of the above equation, we get

$$\frac{dV}{dt} = \int_0^L \left[D_1 \left(p_1 \frac{\partial^2 p_1}{\partial x^2} \right) + D_2 \left(p_2 \frac{\partial^2 p_2}{\partial x^2} \right) \right] dx .$$

Integrating by parts and using the boundary conditions, $\dfrac{dV}{dt}$ reduces to

$$\frac{dV}{dt} = -\int_0^L \left[D_1 \left(\frac{\partial p_1}{\partial x} \right)^2 + D_2 \left(\frac{\partial p_2}{\partial x} \right)^2 \right] dx .$$

Hence, $\dfrac{dV}{dt}$ is negative definite and from the Liapunov's theorem we prove that the equilibrium solution E_1 is **asymptotically stable.** From the stability of E_1 we conclude that both the populations of prey and predator would coexist.

E7) You can do the stability analysis of $N^* = 0$ exactly the way we did for $N^* = K$.

E8) The equilibrium points are:

$E_0 : N_1^* = 0,\ N_2^* = 0$

$E_1 : \overline{N}_1 = r_2 / \alpha_2,\ \overline{N}_2 = r_1 / \alpha_1$

Both the above equilibrium points are unstable in the absence of diffusion. But they can be made stable with the help of diffusion. The given system about E_0 is

$$\frac{\partial p_1}{\partial t} = r_1 p_1 + D_1 \frac{\partial^2 p_1}{\partial x^2}$$

and $\dfrac{\partial p_2}{\partial t} = r_2 p_2 + D_2 \dfrac{\partial^2 p_2}{\partial x^2}$

where $N_i(x, t) = N_i^* + p_i(x,t);\ i = 1, 2$ where $| p_i(x, t) | << 1$.

Consider the following positive definite function

$$V(t) = \frac{1}{2} \int_0^L (p_1^2 + p_2^2)\, dx .$$

Then, $\dfrac{dV}{dt} = \int_0^L \left(p_1 \dfrac{\partial p_1}{\partial t} + p_2 \dfrac{\partial p_2}{\partial t} \right) dx$ ..

Using the given system and associated boundary conditions in $\frac{dV}{dt}$ above, we get

$$\frac{dV}{dt} = \int_0^L \left[r_1 p_1^2 - D_1\left(\frac{\partial p_1}{\partial x}\right)^2 + r_2 p_2^2 - D_2\left(\frac{\partial p_2}{\partial x}\right)^2 \right] dx .$$

Using Poincare's inequality in the r.h.s. of the above expression, we get

$$\frac{dV}{dt} \le -\int_0^L \left[\left(D_1 \frac{\pi^2}{L^2} - r_1 \right) p_1^2 + \left(D_2 \frac{\pi^2}{L^2} - r_2 \right) p_2^2 \right] dx .$$

Thus, we obtain that $\frac{dV}{dt}$ is negative definite if $D_1 \frac{\pi^2}{L^2} > r_1$ and $D_2 \frac{\pi^2}{L^2} > r_2$. Hence, E_0 is asymptotically stable under the above conditions. Now, the given system about E_1 is

$$\frac{\partial p_1}{\partial t} = -\alpha_1 \overline{N}_1 p_2 + D_1 \frac{\partial^2 p_1}{\partial x^2}$$

and $\frac{\partial p_2}{\partial t} = -\alpha_2 \overline{N}_2 p_1 + D_2 \frac{\partial^2 p_2}{\partial x^2}$

where, $N_i(x, t) = \overline{N}_i + p_i(x, t), i = 1, 2$. Taking the same positive definite function, we get

$$\frac{dV}{dt} = \int_0^L \left[-\alpha_1 \overline{N}_1 p_1 p_2 + D_1 p_1 \frac{\partial^2 p_1}{\partial x^2} - \alpha_2 \overline{N}_2 p_1 p_2 + D_2 p_2 \frac{\partial^2 p_2}{\partial x^2} \right] dx .$$

Using the given boundary conditions, we get

$$\frac{dV}{dt} = \int_0^L \left[-(\alpha_1 \overline{N}_1 + \alpha_2 \overline{N}_2) p_1 p_2 - D_1\left(\frac{\partial p_1}{\partial x}\right)^2 - D_2\left(\frac{\partial p_2}{\partial x}\right)^2 \right] dx .$$

Using Poincare's inequality and the inequality $a^2 + b^2 \ge \pm\, 2ab$ in the r.h.s. of the above expression, we get

$$\frac{dV}{dt} \le -\int_0^L \left[\left(D_1 \frac{\pi^2}{L^2} - \frac{(\alpha_1 \overline{N}_1 + \alpha_2 \overline{N}_2)}{2} \right) p_1^2 + \right.$$
$$\left. \left(D_2 \frac{\pi^2}{L^2} - \frac{(\alpha_1 \overline{N}_1 + \alpha_2 \overline{N}_2)}{2} \right) p_2^2 \right] dx .$$

Hence, E_1 is linearly asymptotically stable if

$$D_1 \frac{\pi^2}{L^2} > \frac{1}{2} (\alpha_1 \overline{N}_1 + \alpha_2 \overline{N}_2)$$

and $D_2 \frac{\pi^2}{L^2} > \frac{1}{2} (\alpha_1 \overline{N}_1 + \alpha_2 \overline{N}_2)$.

E9) Like E8), the results can be obtained for E9) also. Here you may note that

$$v_1 \int_0^L p_1 \frac{\partial p_1}{\partial x} dx = v_1 \int_0^L \frac{\partial \frac{1}{2} p_1^2}{\partial x} dx = \frac{1}{2} v_1 p_1^2 \Big|_0^L = 0$$

and

$$v_2 \int_0^L p_2 \frac{\partial p_2}{\partial x} dx = v_2 \int_0^L \frac{\partial \frac{1}{2} p_2^2}{\partial x} dx = \frac{1}{2} v_2 p_2^2 \Big|_0^L = 0 .$$

—x—

APPENDIX-I

Variational matrix: Consider the system of two differential equations:

$$\frac{dN_1}{dt} = F(N_1, N_2)$$

$$\frac{dN_2}{dt} = G(N_1, N_2) \qquad (1)$$

The Jacobian or the **variational matrix** of the system of Eqn. (1) is defined as

$$J = \begin{bmatrix} \frac{\partial F}{\partial N_1} & \frac{\partial F}{\partial N_2} \\ \frac{\partial G}{\partial N_1} & \frac{\partial G}{\partial N_2} \end{bmatrix}$$

Positive definite function: Suppose, $E(N_1, N_2)$ is continuous and has continuous first order partial derivatives in some region containing the origin.

If $E(0, 0) = 0$, then $E(N_1, N_2)$ is said to be positive definite if $E(N_1, N_2) > 0$ for $(N_1, N_2) \neq (0, 0)$.

Example: $E(N_1, N_2) = \frac{1}{2}(N_1^2 + N_2^2)$

Negative definite function: The function $E(N_1, N_2)$ is said to be negative definite if $E(0, 0) = 0$ and $E(N_1, N_2) < 0$ for $(N_1, N_2) \neq (0, 0)$.

Example: $E(N_1, N_2) = -\frac{1}{2}(N_1^2 + N_2^2)$.

Positive semi-definite function: The function $E(N_1, N_2)$ is said to be positive semi-definite if $E(0, 0) = 0$ and $E(N_1, N_2) \geq 0$ for $(N_1, N_2) \neq (0, 0)$.

Example: $E(N_1, N_2) = \frac{1}{2}(N_1 - N_2)^2$.

Negative semi-definite function: The function $E(N_1, N_2)$ is said to be negative semi-definite if $E(0, 0) = 0$, and $E(N_1, N_2) \leq 0$ for $(N_1, N_2) \neq (0, 0)$.

Example: $E(N_1, N_2) = -\frac{1}{2}(N_1 - N_2)^2$.

Stability by Liapunov's direct method

$$\frac{dE(N_1, N_2)}{dt} = \frac{\partial E}{\partial N_1}\frac{dN_1}{dt} + \frac{\partial E}{\partial N_2}\frac{dN_2}{dt}$$

$$= \frac{\partial E}{\partial N_1}F + \frac{\partial E}{\partial N_2}G \qquad (2)$$

$\left[\because \frac{dN_1}{dt} = F \text{ and } \frac{dN_2}{dt} = G \text{ from the system of Eqn. (1)}\right]$

Liapunov Function: A positive definite function $E(N_1, N_2)$ with the property that $\frac{\partial E}{\partial N_1}F + \frac{\partial E}{\partial N_2}G$ is negative semi-definite is called a Liapunov function.

Theorem: If there exists a Liapunov function $E(N_1, N_2)$ for the system of Eqn. (1) then the critical point (or equilibrium point) (N_1^*, N_2^*) of the system is stable. Further, if this function $E(N_1, N_2)$ has the additional property that $\frac{dE}{dt}$ given by the Equation by Eqn. (2) is negative definite, then the equilibrium point (or critically point) (N_1^*, N_2^*) is asymptotically stable.

Variational matrix or Jacobian of a system of n differential equation

Consider the system of n differential equations

$$\frac{dN_i}{dt} = f_i(N_1, N_2, \ldots, N_n)$$

where $i = 1, 2, \ldots, n$.

Variational matrix or Jacobian of the above system is

$$A = [a_{ij}]_{n \times n},$$

where, $a_{ij} = \frac{\partial f_i}{\partial N_j}$, $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$.

The characteristic equation is then given by $|A - \lambda I| = 0$ i.e., $\det(A - \lambda I) = 0$ where I is the $(n \times n)$ identity matrix.

Hurwitz's Criteria: A **necessary** and **sufficient** condition for the negativity of the real parts of all the roots of the polynomial $\lambda^n + a_1\lambda^{n-1} + \cdots + a_n$ with real coefficients is the positivity of all the principal diagonals of the minors of the Hurwitz matrix.

$$H_n = \begin{bmatrix} a_1 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 \\ a_3 & a_2 & a_1 & 1 & 0 & 0 & 0 & \ldots & 0 \\ a_5 & a_4 & a_3 & a_2 & a_1 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & a_n \end{bmatrix}$$

For example, we apply the Hurwitz's criteria for the second, third and fourth degree polynomials.

(i) $\lambda^2 + a_1\lambda + a_2$
Hurwitz's condition reduces to
$a_1 > 0$ and $a_2 > 0$

(ii) $\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3$
Hurwitz's condition reduces to
$a_1 > 0, a_2 > 0, a_3 > 0$ and $a_1 a_2 - a_3 > 0$.

(iii) $\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4$
Hurwitz's condition reduces to $a_1 > 0, a_2 > 0, a_3 > 0, a_4 > 0$ and
$a_1 a_2 a_3 - a_3^2 - a_1^2 a_4 > 0$.

Equilibrium point or critical point: The equilibrium point of the system of Eqns. (1) can be obtained by solving the following system of equations.

$$F(N_1, N_2) = 0$$

$$G(N_1, N_2) = 0$$

Therefore the solution of the above system, suppose, $N_1 = N_1^*$ and $N_2 = N_2^*$ is equilibrium point if

$$F(N_1^*, N_2^*) = 0 \text{ and } G(N_1^*, N_2^*) = 0$$

—x—

APPENDIX-II

Consider a system of two linear first order equations of the form

$$\frac{dN_1}{dt} = F(N_1, N_2) \qquad (1)$$

$$\frac{dN_2}{dt} = G(N_1, N_2)$$

System (1) is said to be **autonomous** when the independent variable t does not appear explicitly. We assume that the functions F and G are continuously differentiable in some region R in the xy-plane, which is called the **phase plane** for the system (1). Then according to the existence and uniqueness theorem (ref. Unit 1, Block 1 of MTE-08), given t_0 and any point (N_{10}, N_{20}) of R, there exists a unique solution $N_1 = N_1(t), N_2 = N_2(t)$ of (1) that is defined on some open interval $a < t_0 < b$ and satisfies the initial conditions

$$N_1(t_0) = N_{10}, N_2(t_0) = N_{20} \qquad (2)$$

The equation $N_1 = N_1(t), N_2 = N_2(t)$ then describes a solution in the phase plane. Any such solution curve is called a **trajectory** or the **orbit** of the system (1) and precisely one trajectory passes through each point of R.

A **critical point** of the system (1) is a point (N_1^*, N_2^*) obtained by setting $\frac{dN_1}{dt} = 0, \frac{dN_2}{dt} = 0$ and such that

$$F(N_1^*, N_2^*) = G(N_1^*, N_2^*) = 0 \qquad (3)$$

Conversely, if (N_1^*, N_2^*) is a critical point of the system, then the constant valued functions

$$N_1(t) = N_1^*, N_2(t) = N_2^* \qquad (4)$$

satisfy system (1) and are called **equilibrium** solutions of the system.

Note that the trajectory of the equilibrium solution in Eqn. (3) consists of the single point (N_1^*, N_2^*).

In practical situations, the equilibrium solutions and trajectories are of most interest. For example, suppose that the system $N_1' = F(N_1, N_2), N_2' = G(N_1, N_2)$ models two populations $N_1(t)$ and $N_2(t)$ of animals that cohabit the same environment, and compete for the same food or prey on one another. $N_1(t)$ might denote the number of foxes and $N_2(t)$ the number of rabbits present at time t. Then a critical point (N_1^*, N_2^*) of the system specifies a constant population N_1^* of foxes and a constant population N_2^* of rabbits that can coexists with one another in the environment. If (N_1^*, N_2^*) is not a critical point of the system, it is not possible for constant populations of N_1^* foxes and N_2^* rabbits to coexist; one or both must change with time.

As you know, it is not always possible to obtain the analytical solution of the system of the form (1). We thus make the qualitative study of system (1) to learn as much as we can about the system. Let us assume that the system (1) is of the form

$$\frac{dN_1}{dt} = a_1N_1 + b_1N_2$$

$$\frac{dN_2}{dt} = a_2N_1 + b_2N_2 \tag{5}$$

or in matrix form as

$$\frac{d}{dt}\begin{pmatrix} N_1 \\ N_2 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} N_1 \\ N_2 \end{pmatrix} \tag{6}$$

where a_1, b_1, a_2, b_2 are constants.

The main purpose of our qualitative study of this two dimensional linear system (1) is to know about the system by classifying the critical points of the system with respect to their nature and stability. But then what do we mean by the stability of the critical point?

A critical point (N_1^*, N_2^*) of the linear system (6) is said to be **stable** provided that if the initial point (N_{10}, N_{20}) is sufficiently close to (N_1^*, N_2^*) then the point $(N_1(t), N_2(t))$ remains close to (N_1^*, N_2^*) $\forall\ t > 0$. Formally, if $N(t) = (N_1(t), N_2(t))$ $N_0 = (N_{10}, N_{20})$ and $N^* = (N_1^*, N_2^*)$ then the critical point N^* is **stable** provided for $\varepsilon = 0$, $\exists\ \delta > 0$, s.t.

$$|N_0 - N^*| < \delta \Rightarrow |N(t) - N^*| < \varepsilon\ \forall\ t > 0 \tag{7}$$

The critical point (N_1^*, N_2^*) is **unstable** if it is not stable. Further, the critical point (N_1^*, N_2^*) is **asymptotically stable** if it is stable and every trajectory that begins sufficiently close to (N_1^*, N_2^*) also approaches (N_1^*, N_2^*) as $t \to +\infty$. That is, for $\delta > 0$,

$$|N_0 - N^*| < \delta \Rightarrow \lim_{t\to\infty} N(t) = N^*. \tag{8}$$

Let us now discuss the stability analysis of the critical points of linear system of the form (5), i.e.,

$$\frac{dN_1}{dt} = a_1N_1 + b_1N_2$$

$$\frac{dN_2}{dt} = a_2N_1 + b_2N_2$$

System (5) has origin $(0, 0)$ as an obvious critical point. We assume that

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$$

so that $(0, 0)$ is the only critical point of the system (5).

In order to find the nontrivial solution of the system (5) we write it in the form

$$\frac{d}{dt}\begin{pmatrix} N_1 \\ N_2 \end{pmatrix} = A \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}$$

where the coefficients matrix $A = \begin{pmatrix} a_1 & b_2 \\ a_2 & b_2 \end{pmatrix}$. We then obtain the eigenvalues of A and their corresponding eigenvectors. Eigenvalues of A are the roots of the equation

$$|A - \lambda I| = 0$$

or, $$\lambda^2 - (a_1 + b_2)\lambda + (a_1 b_2 - a_2 b_1) = 0 \tag{9}$$

which is called the auxiliary equations of the system.

Now if λ_1 and λ_2 are the eigenvalues and $\begin{pmatrix} A_1 \\ B_1 \end{pmatrix}, \begin{pmatrix} A_2 \\ B_2 \end{pmatrix}$ are the eigenvectors corresponding to them then the general solution of (5) can be written in the form

$$N_1 = c_1 A_1 e^{\lambda_1 t} + c_2 A_2 e^{\lambda_2 t}$$

$$N_2 = c_1 B_1 e^{\lambda_1 t} + c_2 B_2 e^{\lambda_2 t} \tag{10}$$

where c's are arbitrary constants.

The nature of the critical point $(0, 0)$ of the system (5) is determined by the nature of the numbers λ_1 and λ_2. We are now stating a theorem which gives the criteria for the stability of a critical point.

Theorem 1: The critical point $(0, 0)$ of the linear system (5) is

1) asymptotically stable if the eigenvalues λ_1, λ_2 are real and negative or have negative real part;

2) stable, but not asymptotically stable, if λ_1 and λ_2 are pure imaginary;

3) unstable if λ_1 and λ_2 are real and either is positive, or if they have positive real part.

We will accept the results of Theorem 1 without proof. These results can be summed up in the form as given in Table 1 below.

Table 1

Nature of λ_1, λ_2	**Type of stability of system (5)**
$\lambda_1 > \lambda_2 > 0$	Unstable
$\lambda_1 < \lambda_2 < 0$	Asymptotically stable
$\lambda_2 < 0 < \lambda_1$	Unstable
$\lambda_1 = \lambda_2 > 0$	Unstable
$\lambda_1 = \lambda_2 < 0$	Asymptotically stable
$\lambda_1, \lambda_2 = \lambda \pm i\mu$ $\lambda > 0$ $\lambda < 0$	 Unstable Asymptotically stable
$\lambda_1 = i\mu,\ \lambda_2 = -i\mu$	Stable

The results we have discussed for a system of two equations can be generalised for a system of n equations also.

Consider the linear system of n equations

$$\frac{dN}{dt} = A\,N \tag{11}$$

where, $N = (N_1, N_2, \ldots, N_n)$, $A = [a_{ij}]$; A is $n \times n$ matrix with constant elements and A can be a variational matrix about the equilibrium point. Then following two theorems can be used to determine the stability of the equilibrium solution (or critical point) by knowing the nature of the characteristic roots (eigenvalues).

Theorem 2: If all the characteristic roots of **A have negative real parts**, then every solution of

$$\frac{dN}{dt} = A\,N$$

where $A = [a_{ij}]$ is constant matrix, is **asymptotically stable.**

Theorem 3: If all the characteristic roots of A with **multiplicity greater than one** have **negative real parts** and all its roots with **multiplicity one have non-positive real parts,** then all the solutions of Eqn. (1) are bounded and hence **stable.**

Note: If any one of the characteristic roots (eigenvalues) is **positive** then the equilibrium solution (critical point) of the linear system (11) is **unstable.**

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UNIT 6 MODELLING IN MEDICINE

Structure

6.1 INTRODUCTION

Over the past few decades there has been a considerable increase in the application of quantitative methods to the study of physiological systems. This is because new techniques for making physiological measurements are being constantly developed and there is more classification of clinical data. Also, there has been a corresponding increase in the mathematical/statistical methods available for the analysis and interpretation of such data.

Mathematical modelling is increasingly being applied to the field of medicine as well, especially i) to interpret and predict the dynamics and control of diseases in order to improve global health and ii) to improve the diagnosis and decide on the dosage of medicines. Specific examples include predicting the impact of vaccination strategies against common infections such as measles and rubella and determining optimal control strategies against HIV and vector-borne diseases. There is tremendous potential for using computer based numerical techniques to study the effects of the drug on the patient by comparing the growth of blood supply to a tumour and its invasion into the normal tissue. A mathematical model could predict what would happen to the tumour, whether it would respond to treatment or not.

All physiological systems are characterized by their complexity. It is important that we should understand this complexity, since a mathematical model that we create will be a simplification, an approximation of that complex reality. By understanding something of this complexity we shall be in a better position to make the simplifying assumptions that correspond to the particular model formulation that we shall adopt. In essence, the model that we develop needs to have taken into account both this inherent complexity that we have simplified and the availability of measurement or clinical data which will be used in estimating the parameters of our model.

In this unit, we have illustrated the application of mathematical modelling in the field of medicine by considering two examples where we have formulated simple models for tumour growth and diabetes. Before formulating these models we have given briefly the complexities associated with the corresponding physiological system for each model. In Sec. 6.2, we have started with describing the process of progression of solid tumours and then discussed models for the growth of solid tumours. Different stages in the growth of a tumour and models for avascular tumours are also discussed here.

In Sec. 6.3, after giving a brief physiological detail of blood glucose metabolism, we have discussed a model for the detection of diabetes mellitus which can be used to diagnose diabetes on the basis of the data obtained by the glucose tolerance test of a patient. As a matter of interest, we have given the complete pathophysiology of diabetes mellitus in an Appendix at the end of the unit.

Objectives

After studying this unit, you should be able to:

- state the essential factors to be considered in the modelling of growth development in solid tumours;
- formulate a model for the growth of solid tumour and use it to find the density of the cancer cells in the tumour's surface area at any point of time;
- describe different stages in the growth of a tumour;
- formulate a model for avascular tumour and estimate the growth of tumour in terms of Malthus/logistic equations;
- describe the blood glucose regulatory system;
- use the model for the detection of diabetes to diagnose mild diabetes on the basis of data obtained from glucose tolerance test of a patient.

6.2 TUMOUR MODELS

Tumour is the unbounded proliferation of cells. Specifically, it is the uncontrolled growth of a host's own "self" cells. Much of the mechanism of tumour development and proliferation is still unknown. But, it is known that abnormal cell growth occurs because of malfunctioning in the mechanisms that control cell growth and differentiation.

Neoplasia literally means "new growth", and the new growth is a "Neoplasm". The term "tumour" was originally applied to the swelling casued by inflammation. Neoplasm also may induce swellings, and the term tumour is now equated with neoplasm. Oncology is the study of tumours of neoplasm. Cancer is the common term for all malignant tumours. Tumour growth modelling has been the subject of much recent literature, but the existing mathematical models appear to neglect the local interaction between cancer and tissue cells. This interaction between cancer and tissue cells affects the cancer cells reproduction rate the way that it becomes dependent on their local concentration. Therefore, tumour growth is mainly determined by the behaviour of cancer cells adjacent to the tumour surface. This process, in its turn, is largely dependent on the tumour's local curvature.

A mathematical model should be viewed as an attempt to understand the growth dynamics of a single vascularized solid tumour growing within the confines of an organ, such as a primary lung or breast. A model can be formulated by stressing the competition within the environment provided by that organ. Therefore, the model may give rise to useful tools that oncologists can use to help decide the proper course of treatment for specific patients.

You may try the following exercise.

E1) What do your mean by malignant tumours? Also, which study deals with tumours?

6.2.1 Growth of Solid Tumour

One can try to investigate the macroscopic effects of tumour growth in a homogeneous tissue. The effects of local interaction can be most simply described in terms of reproduction function. Fig. 1 shows hypothetical non-monotone curves 1-3 describing the reproduction rate of cancer cells i.e., $\dot{c}$ vs local concentration i.e., c.

The curves represent the interaction of carcinogenic factors with the immune system, the availability of nutrients and the local cellular interaction.

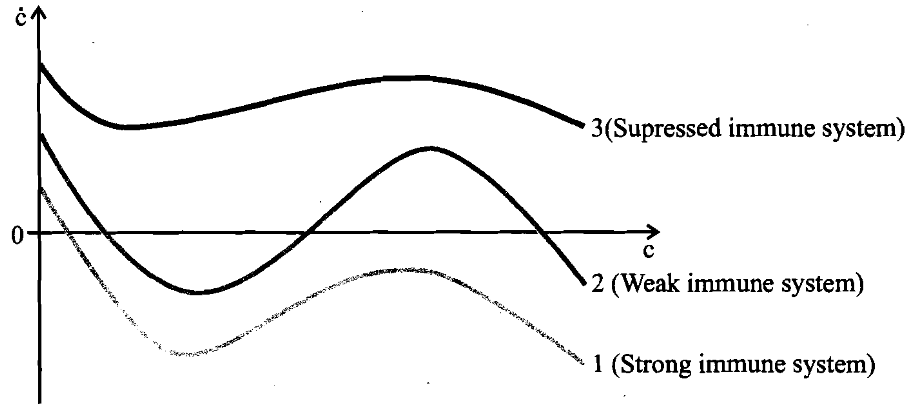

Fig. 1: Reproduction rates of cancer cells as a function of their local concentration c (curves 1-3).

Furthermore, as the concentration of cancer cells increases, the suppression of their reproduction through local interaction between cancer and tissue cells is first slowed, then completely stopped. Finally, it gives way to a reverse process in wich cancer cells stimulate the pathological division of the tissue cells. In Fig. 1, this process corresponds to the interval of increase. A still higher concentration of cancer cells causes a deficit of nutrients which results in the inhibition of cancer cells' reproduction rate and finally to their death. In Fig. 1, this process corresponds to the second interval of decrease. Curve 1 describes the case when tumour onset in the tissue is fully suppressed. Curve 2 represents the case when the immune system cannot provide complete suppression and tumours can spontaneously arise and develop. This case is the subject of our study. Curve 3 images a suppressed immune system when the tumour spreads throughout the organism.

Formulation

For the sake of simplicity, we shall model this problem in two dimensions which corresponds to the behaviour of tumour sections. The difference between the three- and two-dimensional formulations is of minor importance as we consider the processes only qualitatively. Assume that the reproduction function is given by $\phi(c)$. Let R be the radius of a spherical tumour, and r the radius of cancer and tissue cells. $c = x/(x+y)$, where x and y are the number of cancer and tissue cells, respectively, in the considered tissue area. (For simplicity and without loss of generality, cancer and tissue cells are assumed hereafter to be of the same size). The density c of cancer cells in the vicinity of the surface is defined as $\frac{S_c}{S_c + S_t}$, where $S_c = 4\pi r(R - r)$ is the area of a layer of tumour cells that contact the surface from inside, and $S_t = 4\pi r(R + r)$ is the area of a layer of tissue cells that contact the surface from outside. This implies

$$c = \frac{R - r}{2R} \tag{1}$$

R and $\dot{R}$ are found from Eqn. (1) as

$$R = \frac{r}{1 - 2c} \tag{2}$$

$$\therefore \ \dot{R} = \frac{2r\dot{c}}{(1 - 2c)^2}. \tag{3}$$

Let a tumour occupy the area $A = \pi R^2$. Then,

$$\dot{A} = 2\pi R\dot{R}. \tag{4}$$

On the other hand, $\dot{A}$ is the area of cancer cells produced per unit time in the area $S_c + S_t$. Therefore,

$$\dot{A} = (S_c + S_t)\ \varphi(c) = 8\pi r R\varphi(c) \tag{5}$$

From Eqns. (4) and (5), we obtain

$$\dot{R} = 4r\varphi(c) \tag{6}$$

Substituting Eqn. (6) in Eqn. (3) we obtain the differential equation for the density of cancer cells in the tumour's surface layer:

$$\dot{c} = 2\varphi(c)\ (1 - 2c)^2 = \psi(c) \tag{7}$$

Fig. 2 shows the plot of $\psi(c)$ (blue curve) compared with the plot of $\varphi(c)$ (red curve).

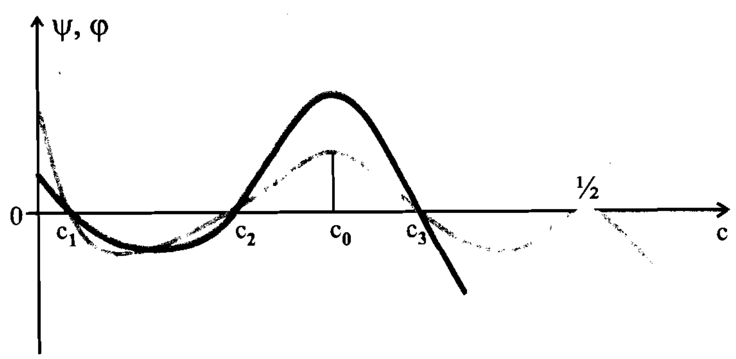

Fig. 2: Derivative of cancer cells density at the tumour surface $\psi(c)$ (blue curve), and initial reproduction function $\varphi(c)$ (red curve).

Example 1: A spherical tumour is occupying the space in an organ with outer and inner radii 0.2×10^{-6} mm and 0.001×10^{-9} mm, respectively, for tumour and cancer cells. Find the density of cancer cells in the vicinity of the surface.

Solution: Let $R = 0.2\times10^{-6}$mm and $r = 0.001\times10^{-9}$mm denote these radii. We know that the density c of the cancer cells in the vicinity of the surface is given by $c = \dfrac{S_c}{S_c + S_t}$, where $S_c = 4\pi r(R - r)$ and

$$S_t = 4\pi r(R + r).$$

Thus, $c = \dfrac{S_c}{S_c + S_t} = \dfrac{1.999\times10^{-7}}{10^{-7}\times(2.001+1.999)} = 0.49975\text{mm}.$

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You may now try the following exercises.

E2) Find the reproduction rate of the tumour cells proliferation within the tissue of an animal brain with the density of cancer and local cells 0.3×10^2 and 3×10^{15} cells respectively.

E3) The reproduction function of the cancer cells within a spherical tumour is given by $\varphi(c) = \dfrac{3c+1}{(1-2c)^2}$; $c \neq \dfrac{1}{2}$ with initial conditions given by $c = c_0$ at $t = 0$. Find the density of the cancer cells in the tumour's surface area at $t = 30$ days.

6.2.2 Avascular Tumour

There are several different stages in the growth of a tumour before it becomes so large that it causes the patient to die or reduces permanently their quality of life. There is a lot of controversy over how exactly cancer is initiated, but it is a generally accepted view that it requires several gene mutations to turn a normal cell into a cancer cell. The factors that trigger these mutations are largely unknown, but are thought to include both environmental and hereditary effects. One of the outcomes of this series of mutations is an increase in the proliferation rate and a decrease in the death rate of the cells, giving rise to a clump of tumour cells growing faster than the host cells. However, even a fast growing clump of tumour cells cannot grow beyond a certain size, since there is a balance between cells inside the clump consuming nutrients and nutrient diffusion into the clump. Therefore, one of the most important steps in malignant tumour growth is angiogenesis, which is the process by which tumours develop their own blood supply. For this reason novel drugs are being developed specifically to target tumour blood vessels. Once the tumours have acquired their own blood supply, the tumour cells can escape the primary tumour via the circulatory system (metastasis) and set up secondary tumours elsewhere in the body. After angiogenesis and metastasis, the patient is left with multiple tumours in different parts of the body that are very difficult to detect and even more difficult to treat. Because there are three distinct stages (avascular, vascular, and metastatic) to cancer development, researchers often concentrate their efforts on answering specific questions on each of these stages. **Avascular tumours are tumours without blood vessels.** This may not be the most important aspect of tumour growth. On the contrary, from a clinical point of view angiogenesis and vascular tumour growth together with metastasis are what cause the patient to die, and modelling and understanding these is crucial for cancer therapy. Nevertheless, when attempting to model any complex system it is wise to try and understand each of the components as well as possible, before they are all put together in a model. Avascular tumour growth is much simpler to model mathematically, and yet contains many of the phenomena which are needed to address in a general model of vascular tumour growth. Moreover, the ease and reproducibility of experiments with avascular tumours means that the quality and quantity of experimental evidence exceeds that for vascular tumours, for which it is often difficult to isolate individual effects.

You may now try this exercise to test your understanding of whatever we have discussed above.

E4) Write a short note on angiogenesis.

Different regions of both avascular and vascular tumours are shown in Fig. 3. Avascular tumour modelling can be of use when making predictions and designing experiments on vascular and metastatic tumours, which are much more time consuming and difficult.

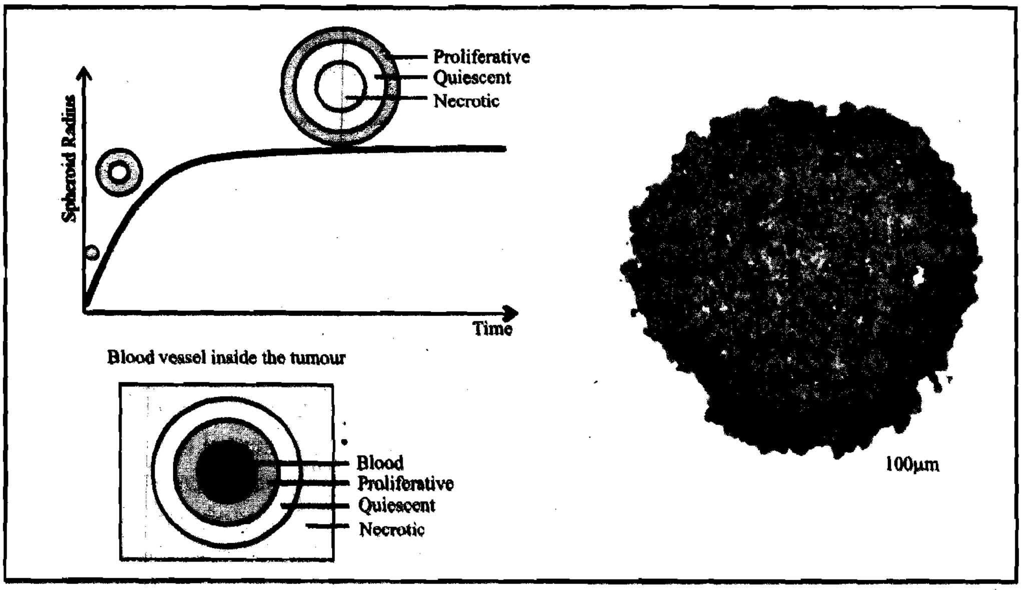

Fig. 3: Schematic illustration showing tumour spheroid growth.

The main finding of in vitro tumour spheroid experiments so far has been that the spheroids grow until they reach a critical size, when growth stops. This critical size is determined by a balance between cell proliferation and cell death inside the spheroid. The main experimental observations have been that when cancer cells are in a high nutrient environment they proliferate, in low nutrient levels the tumour cells trigger cell death (apoptosis), and in intermediate nutrient levels the tumour cells stay quiescent. This translates into the schematic growth curve shown in the above figure. The different nutrient levels inside the tumour spheroid are determined by the movement and consumption of nutrients within the tumour.

The fact that in vitro experiments clearly showed that nutrient (in particular oxygen) diffusion limits tumour spheroid growth paved the way for the angiogenesis hypothesis. This hypothesis is that, in order to grow large, tumours need to obtain their own blood vessels and, therefore, must recruit vessels from the host vasculature through angiogenesis.

Tumour cells consume nutrients. Nutrients diffuse into the tumour tissue from the surrounding tissue. Therefore, if the tumour is very large, the nutrients cannot reach all parts of the tumour tissue. This leads to a decrease in tumour cell proliferation and eventual cell death in regions lacking nutrients. The steady size of the tumour spheroid is reached when the cell proliferation in regions rich in nutrients balances cell death in regions poor in nutrients.

Formulation

The dynamics of solid tumours which are large enough to be directly observable, at least prior to therapeutic or experimental treatment, e.g., are roughly 1 to 3 mm or more in diameter "initially". Modelling such comparatively late stages is important to carcinogenesis models when comparing to observed outcomes. Models of tumour growth and treatment based on a small number of ordinary differential equations have a very long history, dating back to the equation of exponential growth,

$$\frac{dc}{dt} = \lambda c, \quad \lambda = \text{constant}. \tag{8}$$

Here, $c(t)$ is the number of cells in a tumour, regarded as so large that small-number ("demographic") fluctuations are negligible and c can be treated as a continuous, deterministic function of the time t.

Mathematically speaking, models of this kind, using one or several ordinary differential equations, are naive and oversimplified compared to various other kinds of models. Reaction-diffusion partial differential equations are one example among many used in current mathematical and computer biology and medicine. However, a mathematician who looks in the literature for the models used routinely by experimental biologists and clinicians will see that it is the simplest ordinary differential equation models which form the foundation of applied biological modelling in practice. The most important case in point is the equation of exponential growth itself. Discussions of the value of the Malthusian growth parameter λ under various conditions still form a major portion of practical tissue culture and tumour modelling. More generally, it is usually the specification of parameter values for simple ordinary differential equation models that is the crux of the discussion. Such models aim to capture key features using a small number of adjustable parameters and, equally important, aim to neglect peripheral features judiciously.

Often a mathematically quite sophisticated model is basically a marginal elaboration of some simple ordinary differential equation model.

Example 2: Suppose that the tumour cell population is 20,000 cells, growth and decay rate is 300 and 150 cells per day. Then the rate of increase in the tumour growth can be calculated by

$$r = \frac{300-150}{20000} = 0.0075\%$$

where r is called the intrinsic rate of increase.

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You may try these exercises.

E5) In Eqn. (8), if the tumour cells in a particular organ of a human body are 5×10^3, their growth increases up to 7.2×10^5 within five days. Find the value of λ.

E6) Consider the Malthusian model. If the number of tumour cells in a particular organ is 2×10^5 and $\lambda=6$, what will be the growth of the tumour after fifteen days?

The tumour growth has further been estimated in terms of logistic equations. This model governs the growth of tumour cells in a bounded environment. We have classified the growth of malignant cells into two categories:

Tumour Growth in a Closed Region

We define $c(t)$ the population of tumour cells. Let $G(c)$ and $D(c)$ respectively denote the growth rate and decay rate of parenchyma cells in the tissues.

Then, clearly,

$$\frac{d}{dt}(c(t)) = G(c) - D(c)$$

let $G(c)=\lambda c$ and $D(c)=\mu c^2$ then we can write

$$\frac{d}{dt}(c(t)) = \lambda c - \mu c^2 = c(\lambda-\mu c) \tag{9}$$

where λ and μ are positive constants representing the growth and decay control of the tumour.

The initial condition associated with Eqn. (9) is

$c=c_0$ at $t=0$.

Solution

Eqn. (9) can be written in the form

$$\frac{dc}{c(\lambda-\mu c)} = dt.$$

By using partial fraction, we can integrate the equation, and get

$\ln c - \ln(\lambda-\mu c) = \lambda t + \ln a$ where a is integration constant.

or, $$\ln\frac{c}{a(\lambda-\mu c)} = \lambda t$$

or, $$\frac{c}{a(\lambda-\mu c)} = e^{\lambda t}. \tag{10}$$

Using the given initial condition, we obtain

$$\frac{c_0}{(\lambda-\mu c_0)} = a. \tag{11}$$

Therefore, combining Eqns. (10) and (11), we get

$$c(t) = \frac{\dfrac{c_0}{(\lambda-\mu c_0)}\lambda e^{\lambda t}}{\dfrac{c_0}{(\lambda-\mu c_0)}\mu e^{\lambda t}-1}$$

$$\text{or,}\quad c(t) = \frac{\dfrac{c_0}{(\lambda-\mu c_0)}\lambda}{\dfrac{c_0}{(\lambda-\mu c_0)}\mu - e^{-\lambda t}} = \frac{c_0\lambda}{c_0\mu-(\lambda-\mu c_0)\,e^{-\lambda t}}. \tag{12}$$

From Eqn. (12) it follows that the growth of tumour cells $c(t) \to \frac{\lambda}{\mu}$ as $t \to \infty$.

From Eqn. (12), if $c_0 < \lambda/\mu$, $c(t)$ simply increases monotonically to λ/μ, while if $c_0 > \lambda/\mu$ it decreases monotonically to λ/μ. In former case there is a quantitative difference depending on whether $c_0 > (\lambda/\mu)/2$ or $c_0 < (\lambda/\mu)/2$; with $c_0 < (\lambda/\mu)/2$ the form has a typical sigmoid character, which is commonly observed.

The main point about the model is that it is a particularly convenient form to take when seeking qualitative dynamic behaviour in tumour cell populations in which $c = 0$ is an unstable state and $c(t)$ tends to finite positive stable steady state.

It is instructive to try to understand why this form was accepted since it highlights an important point in modelling in the biomedical sciences. The growth form in Eqn. (12) has three parameters, c_0, λ and μ with which to assign to compare with actual data.

You may try this exercise now.

E7) For $\lambda = 20$, $\mu = 15$ and initial tumour cell concentration is $c_0 = (2\times10^{12})$ in the above model, what will be the growth of tumour cells in five days?

Tumour Growth in the Open Region

The equation for this category is defined as

$$\frac{d}{dt}(c(t)) = \lambda c - \mu c \pm \gamma c = c(\lambda - \mu \pm \gamma) \tag{13}$$

where γ is the migration and emigration of cells from the tumour region due to vascularization.

Solution

The solution of the Eqn. (13) is to be obtained under the initial condition given earlier. Solving Eqn. (13), we obtain

$$\ln c(t) = (\lambda - \mu \pm \gamma)\, t + \ln d \quad \text{where d is integration constant}$$

or, $$\ln\left(\frac{c(t)}{d}\right) = (\lambda - \mu \pm \gamma)\, t$$

or, $$\left(\frac{c(t)}{d}\right) = e^{(\lambda-\mu\pm\gamma)t} \text{ or } c(t) = de^{(\lambda-\mu\pm\gamma)t}$$

using initial conditions $t = 0$, $c = c_0$, we get

$$c(t) = c_0 e^{(\lambda-\mu\pm\gamma)t} \qquad (14)$$

Eqn. (14) gives the proliferation of tumour cells in the open region.

Limitations

We may caution that the models presented here were not designed for direct clinical applications. Much more specific information and rigorous comparisons between model results and actual tumours is required before they are applied to treatment decisions in any way.

You may now try this exercise.

E8) The control parameters of growth and decay of a tumour are respectively 1500 and 800 per day. Also, the damaged cells migrate due to vascularization of blood at a rate of 300 cells per day. Use logistic model to find the ratio of the growth of tumour after 20 days with the initial tumour.

We shall now discuss the model for the detection of diabetes.

6.3 A MODEL FOR DETECTING DIABETES

Diabetes mellitus, often referred to as diabetes, is a syndrome of disordered metabolism, usually due to a combination of hereditary and environmental causes, resulting in abnormally high blood sugar/glucose levels. If glucose concentration level in blood is constantly out of the range (70-110 mg/dl) then the person is considered to have blood glucose problem known as **hyperglycemia**. It mainly develops due to insufficient secretion of insulin by the β cells of pancreas. Before formulating a mathematical model for the detection of diabetes, we give you, in brief, the physiological details of blood glucose metabolism.

The blood glucose regulating system is complicated. However, we are giving here only the essential details to understand the rationale behind the model we shall be presenting.

Carbohydrates in our food are absorbed by the digestive system of the body and converted into glucose. This glucose is oxidized and is used as an important source of energy for the functioning of the body. The **liver** helps to regulate the blood glucose concentration. It acts as a major storehouse for glucose. When the blood glucose concentration is high, the excess is stored in the liver as glycogen. When the concentration is low i.e., in between meals or when we are fasting, the glycogen is split into glucose to maintain the glucose concentration in the bloodstream. A wide variety of hormones like insulin, glucagon, epinephrine, glococorticoids, thyroxin help regulate the blood glucose cycle. The **hormone insulin**, secreted by the β cells of the pancreas, predominates to maintain the blood glucose level in the body. Whenever carbohydrates are eaten, signal is sent by the digestive tract to the pancreas to secrete more insulin. Insulin facilitates the absorption of glucose by the tissues to prdocue energy and it also enhances the storehouse effect of the liver. We can thus say that insulin regulates the blood glucose concentration by

i) increasing the rate of glucose metabolism (glycolysis).

ii) increasing glycogen store in liver and muscle mass.

Fig. 4 shows the glucose-insulin interaction loop.

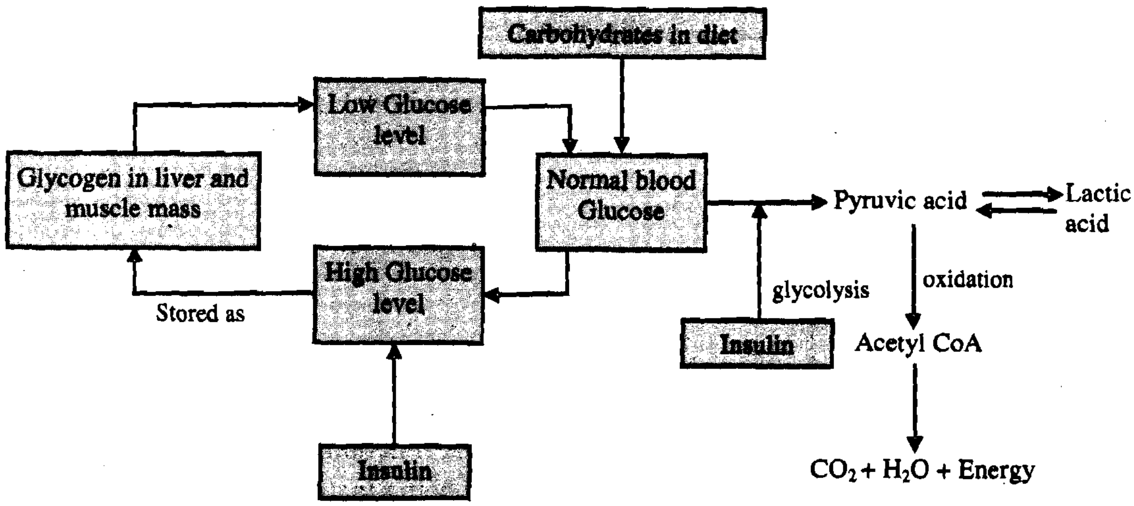

Fig. 4: Glucose-insulin interaction loop.

In diabetes mellitus, due to lack of insulin and excess of blood glucose, glucose starts excreting through urine leading to excessive thirst, hunger and weight loss which are the main symptoms of diabetes. It is, therefore, necessary to maintain the blood glucose concentration within normal range, to remain healthy. If you are interested in knowing the complete pathophysiology of diabetes mellitus then you may refer to **Appendix** at the end of the unit.

Usually, the **glucose tolerance test** (GTT) is used to diagnose diabetes. In this test, the patient who has fasted overnight, is given a large dose of glucose. During the next 2-3 hours, several measurements of the patient's blood glucose concentration are made. These measurements are then used in the diagnosis of diabetes. There is no universally accepted criterion available for interpreting the results of GTT. The criterion which we shall use in our model was given by Dr. Rosevear and Dr. Molnar, of the Mayo Clinic (USA) and Dr. Ackerman and Dr. Gatewood of the University of Minnesota (USA). They developed a simple model to obtain a criteria for distinguishing normal people from prediabetics and mild diabetics from a few blood samples during a GTT. This simplified model centers attention on two concentrations, that of glucose in the blood and net hormonal concentration which represents the cumulative effect of all the pertinent hormones. This model can still provide an accurate description of the blood glucose regulatory system due to two reasons. **Firstly**, under normal conditions, the interaction of hormone insulin, with blood glucose is so predominant that a simple 'lumped parameter' model is quite adequate. **Secondly**, normoglycemia does not depend, necessarily, on the normalcy of each kinetic mechanism of the blood glucose regulatory system. It depends on the overall performance of the blood glucose regulatory system, and this system is dominated by insulin-glucose interactions.

Lumped parameters are simplification in a mathematical model where variables that are spatially distributed fields are represented as single scalars instead. For example, a temperature field is replaced by average temperatures.

Let us now consider the formulation of this model.

Formulation

Consider the concentration of the following two substances:

G = concentration of blood glucose

H = net hormonal concentration

Those hormones, such as insulin, that decrease the blood glucose concentration are found to increase H while hormones like cortisol which increase the blood glucose concentration decrease H. The basic model is described by the equations

$$\frac{dG}{dt} = F_1(G,H) + I(t) \tag{15}$$

$$\frac{dH}{dt} = F_2(G,H) \tag{16}$$

You may **note** that the changes in G and H are determined by both G and H as specified by the functions F_1 and F_2. The function $I(t)$ represents the rate at which the blood glucose concentration is being increased externally. We assume that G and H have assumed their steady-state values G_0 and H_0 by the time the fasting patient reaches the hospital. This means that

$$F_1(G_0, H_0) = 0 = F_2(G_0, H_0). \tag{17}$$

Thus, Eqns. (15) and (16) give the formulated model. Let us now obtain the solution of this model.

Solution

We are interested here in considering only the small deviations of G and H from their steady-state values because the model is constructed to detect prediabetics and mild diabetics. Thus, we make the substitution

$$g = G - G_0, \quad h = H - H_0 \tag{18}$$

Rewriting Eqns. (15) and (16) in terms of these deviations, we get

$$\frac{dg}{dt} = F_1(G_0 + g, H_0 + h) + I(t) \tag{19}$$

$$\frac{dh}{dt} = F_2(G_0 + g, H_0 + h) \tag{20}$$

Expanding F_1 and F_2 in Eqns. (19) and (20) using Taylor series expansion and neglecting the second and higher order terms in the expansion, we can rewrite Eqns. (5) and (6) in the form

$$\frac{dg}{dt} = g\left.\frac{\partial F_1}{\partial G}\right|_{(G_0,\, H_0)} + h\left.\frac{\partial F_1}{\partial H}\right|_{(G_0,\, H_0)} + I(t) \tag{21}$$

$$\frac{dh}{dt} = g\left.\frac{\partial F_2}{\partial G}\right|_{(G_0 H_0)} + h\left.\frac{\partial F_2}{\partial H}\right|_{(G_0 H_0)} \tag{22}$$

There is no way of determining theoretically the partial derivatives appearing in Eqns. (21) and (22) since the functions F_1 and F_2 are unknown. However, using the properties of the blood glucose regulatory system, we can determine their signs.

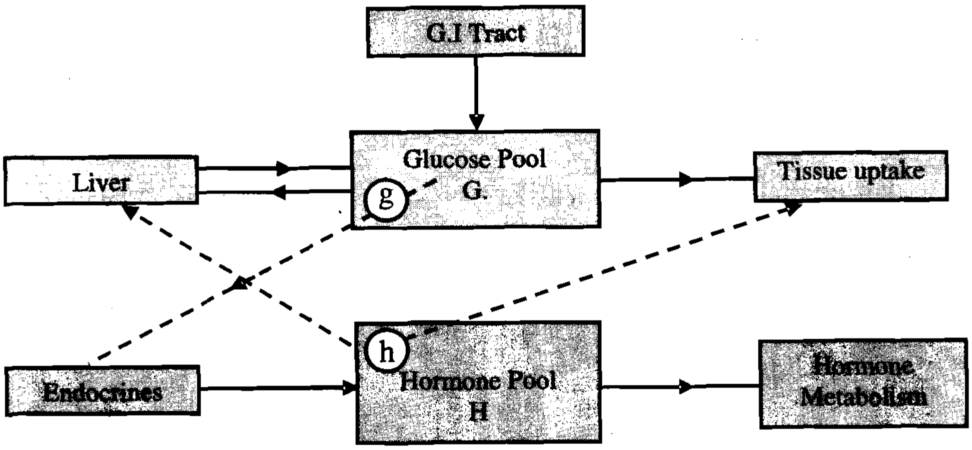

Fig. 5: Blood glucose regulatory system.

Referring to Fig. 5, we see that if $g > 0$ and $h = 0$ then the blood glucose concentration will decrease due to tissue uptake of glucose and storage of excess glucose by the liver as glycogen. Hence $\frac{dg}{dt} < 0$. Consequently, from Eqn. (21) we conclude that $\left.\frac{\partial F_1}{\partial G}\right|_{(G_0,\, H_0)}$ must be negative. Similarly, $\left.\frac{\partial F_1}{\partial H}\right|_{(G_0,\, H_0)}$ is negative, since a positive value of h tends to decrease the blood glucose concentration by facilitating tissue uptake of glucose and also increasing the conversion rate of glucose into glycogen. Further, a positive value of g causes the endocrine glands to secrete those hormones which tend to increase H and so $\frac{dh}{dt} > 0$ implying that $\left.\frac{\partial F_2}{\partial G}\right|_{(G_0,\, H_0)} > 0$. Finally, $\left.\frac{\partial F_2}{\partial H}\right|_{(G_0,\, H_0)} < 0$, since the concentration of hormones in the blood decrease through hormone metabolism. Thus, Eqns. (21) and (22) can be written in the form

$$\frac{dg}{dt} = -ag - bh + I(t) \tag{23}$$

$$\frac{dh}{dt} = -ch + dg \tag{24}$$

where a, b, c and d are positive constants. Since we only measure the concentration of glucose in the blood, we will eliminate the variable h from Eqns. (23) and (24). Differentiating Eqn. (23) with respect to t and substituting for $\frac{dh}{dt}$ from Eqn. (24), we get

$$\frac{d^2g}{dt^2} = -a\frac{dg}{dt} + bch - bdg + \frac{dI}{dt} \tag{25}$$

Substituting $bh = -\frac{dg}{dt} - ag + I(t)$ as obtained from Eqn. (23) into Eqn. (25), we obtain

$$\frac{d^2g}{dt^2} + 2A\frac{dg}{dt} + B^2 g = E(t) \tag{26}$$

where $A = \frac{a+c}{2}, B^2 = ac + bd$ and $E(t) = cI + \frac{dI}{dt}$.

You may **note** that $E(t)$ is identically zero except for short interval of time when the glucose is being ingested. Let $t = 0$ be the time when the glucose load is completely ingested. Then $E(t) = 0$ for $t \geq 0$ and Eqn. (26) reduces to

$$\frac{d^2g}{dt^2} + 2A\frac{dg}{dt} + B^2 g = 0 \tag{27}$$

Eqn. (27) is a second order ordinary differential equation with constant (positive) coefficients. You must have studied such equations in your differential equations course at the undergraduate level. You know that the solutions of this equation are of three different types, depending upon whether $A^2 - B^2$ is positive, negative or zero. These three types correspond to the overdamped, underdamped and critically damped cases, respectively. We shall discuss here the case $A^2 - B^2 < 0$ (the other cases you may deal in a similar manner). The characteristic equation corresponding to this case has complex roots and solution of the form

$$g(t) = C_1 e^{-At} \cos(B_1 t - \theta) \tag{28}$$

where $B_1^2 = B^2 - A^2$. Hence, the complete solution is

$$G(t) = G_0 + C_1 e^{-At} \cos(B_1 t - \theta) \tag{29}$$

Eqn. (29) has five unknown constants G_0, A, B_1, C_1, θ. One way of finding them is as follows.

The patient's blood glucose concentration before the glucose load is ingested is G_0. Hence, we can determine G_0 by measuring the patient's blood glucose level immediately upon his arrival at the hospital. Next, we measure the patient's blood glucose concentrations G_1, G_2, G_3 and G_4 at times t_1, t_2, t_3 and t_4 respectively. These concentrations satisfy the equations

$$G_j = G_0 + C_1 e^{-At_j} \cos(B_1 t_j - \theta) \qquad (j = 1,2,3,4)$$

The constants A, C_1, B_1, θ can be determined by solving these equations and then $B = \left(B_1^2 + A^2\right)^{1/2}$

A second way of determining the five constants is by using least squares. The patient's blood glucose concentrations $G_1, G_2, \cdots, G_n$ are measured at times $t_1, t_2, \ldots, t_n$ respectively. The optimal values of G, A, C_1, B_1 and θ are found by minimizing

$$M = \sum_{j=1}^{n}[G_j - G_0 - C_1 e^{-At_j} \cos(B_1 t_j - \theta)]^2 .$$

This method is preferable to the first method, since Eqn. (29) is only an approximation to $G(t)$. It is possible to find the values of the five constants satisfying Eqn. (29) exactly at time t_1, t_2, t_3 and t_4 but gives a poor fit to the data at other times. The method of least squares gives a better fit to the data on the entire time interval, since more measurements are involved. Observations have shown that the parameter B could be used as the basic discriminator in the GTT, since the value of the parameter B was relatively insensitive to experimental errors in G. The results obtained on the basis of data from many sources indicate that if the period $T_0 = \frac{2\pi}{B}$ is less than 4 hrs, the patient is normal, while the value of T_0 more than 4 hrs indicates mild diabetes.

Limitations of the Model

This model can only be used to diagnose mild diabetes or prediabetes and it sometimes yields a poor fit to the data 2-3 hours after the ingestion of the glucose load. The first difficulty is due to the fact that the deviation of G from G_0 must be small. The second difficulty arises during the recovery phase of the GTT response, when the glucose levels may be lowered below the fasting level. The model is a lumped-parameter model where we have not considered the effect of other hormones like epinephrine and glucose separately but lumped them together with insulin. This model also ignores the effects of diet and exercise.

You may now try the following exercises.

E9) Derive Eqn. (28)

E10) A patient arrives at the hospital after an overnight fast with a blood glucose concentration of 70 mg/100 ml blood. His blood glucose concentrations 1 hour, 2 hours and 3 hours after fully absorbing a large amount of glucose are 95, 65 and 75 mg glucose/100 ml blood, respectively. Show that the patient is of normal health.

We now end this unit by giving a summary of what we have covered in it.

6.4 SUMMARY

In this unit, we have covered the following:

1. Description of the tumour and different stages in its growth.

2. Mathematical study of the tumour growth inside the organ of any animal body.
3. Study of avascular tumors and their description in terms of mathematical formulation.
4. Tumour development within closed and open regions.
5. Physiological details of the blood glucose metabolism.
6. Mathematical model for the detection of diabetes mellitus to diagnose mild diabetes in a patient.

6.5 SOLUTIONS/ANSWERS

E1) A malignant tumour is a growth of tissue that forms an abnormal mass. Malignant tumors generally provide no useful function and grow at the expense of healthy tissues.

In general, a *malignant tumour* is caused by abnormal regulation of cell division. Typically, the division of cells in the body is strictly controlled. New cells are created to replace older ones or to perform new functions. Cells which are damaged or no longer needed die to make room for healthy replacements.

If the balance of cell division and death is disturbed, a malignant tumour may form. Malignant tumours are classified as either benign (slow-growing and often harmless depending on the location) or malignant (faster-growing and likely to spread to other parts of the body and cause problems). Malignant tumours are what we call cancer.

Abnormalities of the immune system, which usually detects and blocks aberrant growth, can lead to malignant tumours. Other causes include radiation, genetic abnormalities, certain viruses, sunlight, tobacco, benzene, certain poisonous mushrooms, and aflatoxins (a poison produced by an organism which sometimes grows on peanut plants). Tobacco causes more deaths from cancer than any other environmental agent.

A malignant tumour may be more common in one sex than the other, some are more common among children or the elderly, and some vary with diet, environment, and genetic risk factors. Oncology is the study of cancerous tumours.

E2) The density of cancer cells is $0.3\times 10^2 = 30 = x$, say.
Also, the density of healthy local cells is $3\times 10^{15} = y$, say
The reproduction rate of tumour cell proliferation is given by the ratio

$$c = \frac{x}{x+y} = \frac{30}{(30+3\times 10^{15})} = \frac{1}{10^{14}+1}$$

Thus, c is the required reproduction of the tumour cell proliferation.

E3) Given, $\varphi(c) = \frac{3c+1}{(1-2c)^2}$; $c \neq \frac{1}{2}$ and $c = c_0$ at $t = 0$

Therefore, from Eqn. (7) it follows that

$$\psi(c) = \dot{c} = 2\left(\frac{3c+1}{(1-2c)^2}\right)(1-2c)^2$$

or, $\dfrac{dc(t)}{(3c+1)} = 2dt$

Integrating the above equation, we get

$$\frac{1}{3}\ln(3c+1) = 2t + \frac{1}{3}\ln a$$

$$\Rightarrow \quad \frac{1}{3}\ln\frac{(3c+1)}{a} = 2t \text{ or } \frac{3c+1}{a} = e^{6t}.$$

Using the initial conditions, we get $a = 3c_0 + 1$.

Therefore, $c(t) = \dfrac{(3c_0+1)e^{6t} - 1}{3}$.

The value of $c(t)$ at $t = 30$, is given by

$$c(30) = \frac{(3c_0+1)e^{180} - 1}{3}.$$

E4) One of the most important steps in malignant tumour growth is angiogenesis, which is the process by which tumours develop their own blood supply. For this reason, novel drugs are being developed specifically to target tumour blood vessels. Once the tumours have acquired their own blood supply, the tumour cells can escape the primary tumour via the circulatory system (metastasis) and set up secondary tumours elsewhere in the body. After angiogenesis and metastasis, the patient is left with multiple tumours in different parts of the body that are very difficult to detect and even more difficult to treat.

E5) The initial growth of the tumor is 5×10^3, and the size after five days is 7.2×10^5. From Eqn. (8), we get

$$c = c_0 e^{\lambda t}$$

Therefore, $7.2\times10^5 = (5\times10^3)e^{5\lambda}$

or, $\quad e^{5\lambda} = \dfrac{7.2\times10^5}{5\times10^3} = 144$

or, $\quad \lambda = \dfrac{1}{5}\ln(144) = 0.994$.

E6) Given the initial population of tumor cells $c_0 = 2\times10^5$.

Then $c = c_0 e^{\lambda t} = (2\times10^5)\, e^{6t}$

Therefore, $c = (2\times10^5)\, e^{90} = 2.4408$

is the required growth of tumor cells in fifteen days.

E7) Using Eqn. (12) the growth of tumour cell is given by

$$c(t) = \frac{c_0\lambda}{c_0\mu - (\lambda - \mu c_0)e^{-\lambda t}}$$

Given the initial size $c_0 = 2\times10^{12}$, $\lambda = 20$, and $\mu = 15$, we have,

$$c(t) = \frac{(2\times10^{12})\times20}{(2\times10^{12})\times15-(20-[30\times10^{12}])e^{-20t}} \approx \frac{1.34e^{20t}}{e^{20t}-1}.$$

E8) Given $\lambda = 1500, \mu = 800$ and $\gamma = 300$.
From Eqn. (14), the net growth of the tumour in 20 days is

$$c(t) = c_0 e^{(1500-800+300)20} = c_0 e^{20000}$$

Thus, the ratio of the growth of tumour after 20 days with initial tumour is given by

$$\frac{c(t)}{c_0} = e^{20000}.$$

E9) Given equation is $\frac{d^2g}{dt^2} + 2A\frac{dg}{dt} + B^2 g = 0$
The roots of the corresponding auxiliary equation are

$$r_1, r_2 = -A \pm \sqrt{A^2 - B^2}$$

When $A^2 - B^2 < 0$ then solution of given equation can be written as

$$g(t) = e^{-At}[C_2 \cos\sqrt{B^2 - A^2}\, t + C_3 \sin\sqrt{B^2 - A^2}\, t]$$

or, $g(t) = e^{-At}[C_2 \cos B_1 t + C_3 \sin B_1 t]$
where C_1, C_2 are constants and $B_1^2 = B^2 - A^2$.
Further $g(t)$ can be written as

$$g(t) = C_1 e^{-At}\cos(B_1 t - \theta)$$

where $C_1 = \sqrt{C_2^2 + C_3^2}$ and $\tan\theta = \frac{C_2}{C_3}$.

E10) The concentration of glucose satisfy the equations

$$G_j = G_0 + C_1 e^{-At_j}\cos(B_1 t_j - \theta) \qquad (j = 1, 2, 3)$$

For $j = 1, 2, 3$, we get

$$25 = C_1 e^{-A}\cos(B_1 - \theta)$$
$$-5 = C_1 e^{-2A}\cos(2B_1 - \theta)$$
$$-5 = C_1 e^{-3A}\cos(3B_1 - \theta)$$

Solve these equations and obtain C_1, A, B, and θ and hence get the value of $T = \frac{2\pi}{B}$ where $B = (B_1^2 + A^2)^{1/2}$.

—x—

APPENDIX

The complete pathophysiology of diabetes mellitus is depicted in Fig. 1 below.

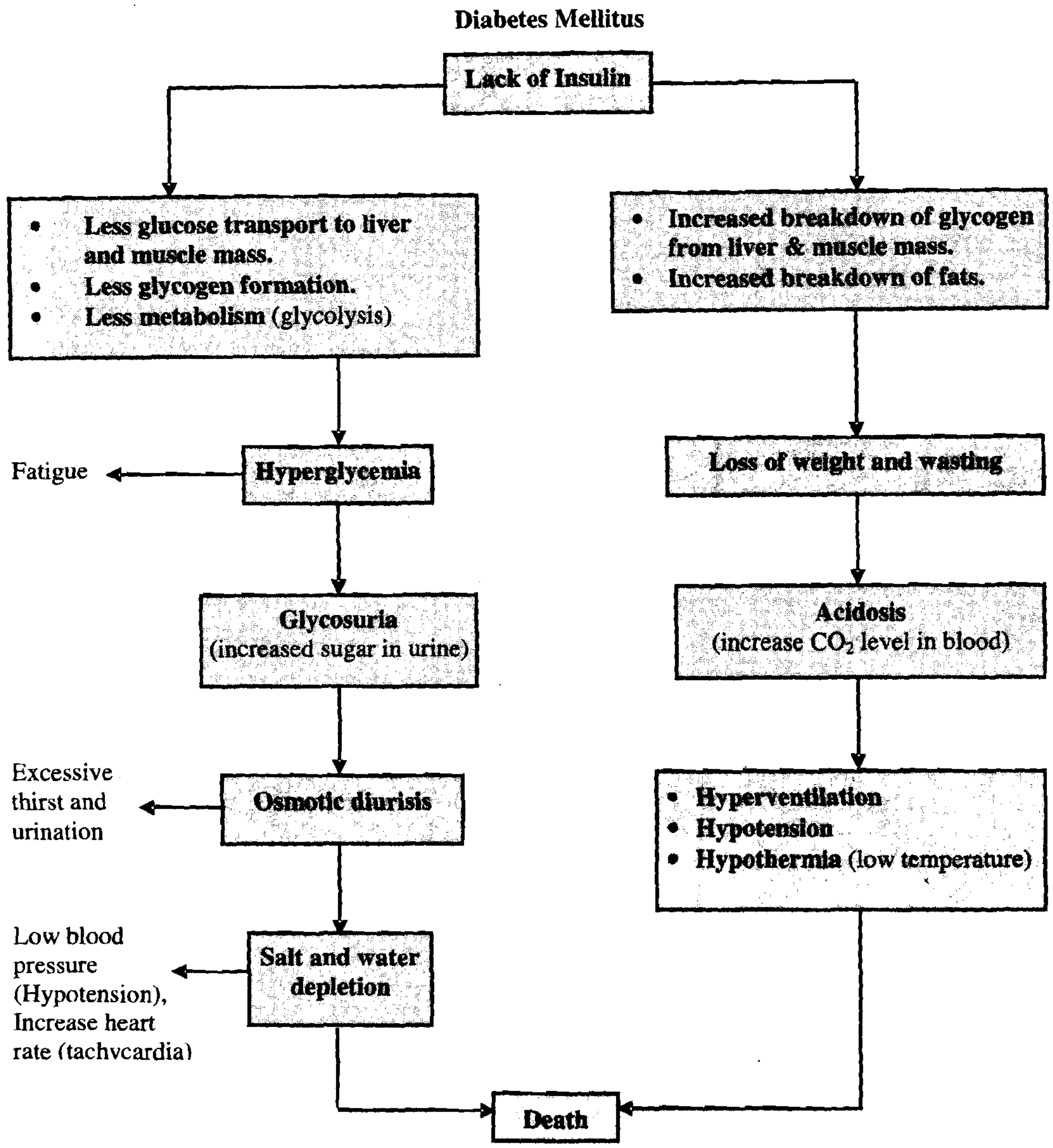

Fig. 1: Pathophysiology of diabetes mellitus.

UNIT 7 SOCIO-ECONOMIC MODELS

Structure **Page No**

7.1 INTRODUCTION

The present unit develops some models of optimization in problems related to socio-economic context. **Problems of optimization** will be taken up first to be followed by **stochastic process models.** Since the basic theory of linear programming is already known to some extent, the present study can carry the learning forward by taking up **integer programming**, a topic very much relevant in economic-industrial-technological situations; this will pave the way for taking up studies on multi-decision programming (also known as goal programming), dynamic programming and aspects of non-linear programming, in future. We shall, however, restrict ourselves to discuss integer programming only in some details in Section 7.2. This will be followed by describing the procedure for transportation problems. It will be observed that transportation problem is a variant of linear programming problem; its solution will be discussed through numerous examples in Section 7.3. Examples will in general be chosen from economics, social, engineering and technological fields.

Two queueing models with single server are discussed in Section 7.4. In the present study material some queueing techniques with multi-channel facility will be introduced. Mathematical formulations will be presented along with concrete problems and their solutions. You are required to do computer practical exercises on this unit, which are given at the end of the unit.

Objectives

After studying this unit, you should be able to

- define and discuss integer linear programming model and its usefulness specially in the areas of economics and industry;
- discuss the exact procedure for solving integer linear programming problem through examples, by two distinctly separate methods, viz., **cutting plane algorithm and branch and bound method**;
- define and discuss **transportation model** with examples;

- define and discuss **two stochastic multi-channel queueing models** with the help of examples.

7.2 INTEGER PROGRAMMING MODEL

In decision-making in business, industry and economic planning it becomes imperative to think of *integer values* of the optimal variables in a simplex-type linear programming problem (LPP). In capital-budgeting, manufacturing, distribution, etc., the optimal decision variables are, naturally, to be integers. The number of projects to be approved for capital-budgeting, the number of batches of a certain commodity to be manufactured, the number of trucks or aircrafts to be readied for shipment, etc., are all integers. The usual linear programming may, in general, yield fractional values of some or all of these variables. Rounding off may lead to a value whose difference from the value of the objective function derived through usual linear programming technique, is too high. Hence there arises the need to derive some independent method for finding the optimal integral solution. Not only that, the constraints of the given LPP may be violated as a consequence of rounding off operation for the values of the basic variables. This gives rise to Integer Linear Programming (ILP). Since all the basic variables are restricted to be integers the problem is known as a pure integer linear programming problem. In cases where only some specific variables are restricted to be integers the problem comes to be known as a mixed integer programming problem. When the basic variables can take up only zero and one as values then the problem is known as a zero-one problem. In what follows we will discuss only the pure integer linear programming problem. We will first elucidate the phenomenon by citing an example.

Example 1: A firm manufactures two products A and B. Time required to manufacture one unit of product A is 3 hours and that for the product B is 4 hours. However the manufacturer is restricted to work for not more than16 hours a week. The profit for one unit of product A is Rs.5 and that for product B is Rs.2. However the profit earned on account of product B should go to charity. How many units of A and B should be produced to maximize profit?

The modelling procedure consists in formulating the problem into a linear programming problem subject to the constraints involved. Next, its optimal solution, by applying usual simplex procedure is to be found out. This solution may not give integer solution. Therefore the next step should be to find procedure for finding the optimal integer solution. The procedure is discussed below.

Let x_1 be the number of product A manufactured in a week and x_2 be the number of product B manufactured in a week.

Maximize $Z = 5x_1 - 2x_2$ subject to the constraint $3x_1 + 4x_2 \leq 16$ where $x_1, x_2 \geq 0$, are integers. By graphical method it can be shown that

Max $Z = 26\frac{2}{3}$ at $\left(x_1 = 5\frac{1}{3}, x_2 = 0\right)$. If we take $(x_1 = 5, x_2 = 0)$, then

Max $Z = 25$ and the difference is $1\frac{2}{3}$. Taking the next integer value solution at $(x_1 = 6, x_2 = 0)$, one finds that Max $Z = 30$ which clearly is too high as also the constraint stands violated. It can be noted that (4, 1) is an integer solution. But, is it an optimal solution? We do not know. Hence there arises the need for deriving an independent method for finding a pure integer optimal solution to a given linear programming problem. This independent method to be developed may therefore be construed as deriving a **new model** to describe a new variety of problems and defining their solutions. It would therefore be prudent to precisely define the Integer Linear Programming Problem and then explain the same in details. The **definition** is as follows:

Definition 1 (Integer Linear Programming Problem): Maximize $Z = c^r x$, $(c, x) \in R^n$, subject to the constraints: $Ax = b$, $x \geq 0$, $b \in R^m$ and all $x_j \in x$ are integers, where A is an $m \times n$ real matrix, is defined to be a pure integer-linear programming problem (ILPP).

If some $x_j \in x$ are integers then the problem is a mixed integer programming problem. We will restrict ourselves to discussing pure integer linear programming only.

Before we go on to explain the model theoretically we cite an interesting problem given by H. A. Taha (Operations Research: An Introduction, 7th Edition). This will give insight into the type of real life problems and their solutions by integer programming.

Let us consider the following problem. Five projects are being evaluated over a three-year planning horizon. Table 7.1 gives the expected returns for each project and the associated yearly expenditures.

Table 7.1: Expenditures (Rs. in crores) per year

Project	1	2	3	Returns (Rs. in Crores)
1	5	1	8	20
2	4	7	10	40
3	3	9	2	20
4	7	4	1	15
5	8	6	10	30
Available fund (Rs. in crores)	25	25	25	

Which projects should be selected over the three year horizon?

First of all the problem is rewritten as a linear programming problem as follows:

Let $x_1, x_2, \ldots, x_5$ denote the value 1 or 0 according as the corresponding project is selected or not selected. Then

$$\text{Max } Z = 20x_1 + 40x_2 + 20x_3 + 15x_4 + 30x_5$$

$$5x_1 + 4x_2 + 3x_3 + 7x_4 + 8x_5 \le 25$$
$$x_1 + 7x_2 + 9x_3 + 4x_4 + 6x_5 \le 25$$
$$8x_1 + 10x_2 + 2x_3 + x_4 + 10x_5 \le 25$$

and, $x_i' = (0, 1), i = 1, \ldots, 5$.

This means that $x_i = 1$ if the project is selected otherwise it assumes the value 0. Using integer programming it has been shown that $x_1 = x_2 = x_3 = x_4 = 1, x_5 = 0$, with $Z = 95$ (Rs. in crores). This solution indicates that all except the project 5 are to be selected. This is an integer solution. However, the usual continuous linear programming will lead to a set of fractional-integer optimal solution in $(0, 1)$ given by:

$x_1 = .5789, x_2 = x_3 = x_4 = 1, x_5 = .7368, Z = 108.68$ (Rs. in crores).

Obviously, in the present context, fractional values are meaningless. On the other hand, applying rounding off fractions rule and taking all $x_1 = 1$ would violate the constraints. Thus finding some methods of solving linear programming problems for which mainly the yes-no or zero-one solutions are relevant becomes imperative.

7.2.1 Gomory's Integer Programme

R.E. Gomory (1958) described a method to do away with the fractional part of the basic variables and obtain an optimal integer solution of the problem. The problem is first solved as a linear programming problem by simplex method as usual and an optimal solution is obtained. Then this optimal solution is modified by inserting a new constraint in the table with a view to eliminate the non-integer fractional value of a basic variable. This modified problem can be solved by using the dual simplex method to obtain the optimal integer solution. This procedure to introduce a new constraint can be re-enacted to eliminate some other non-integer basic variables in the optimal table. The additional constraint to be introduced is known as Gomory's constraint.

How to Introduce Gomory's Constraint?

Let us start with a model optimal simplex table for a maximizing linear programming problem. This is represented in Table 7.2.

Table 7.2: Simplex Table

		$c \rightarrow$	c_1	c_2	c_3	c_4
c_B	y_B	x_B	y_1	y_2	y_3	y_4
c_2	y_2	y_{10}	y_{11}	y_{12}	y_{13}	y_{14}
c_3	y_3	y_{20}	y_{21}	y_{22}	y_{23}	y_{24}
		y_{00}	y_{01}	y_{02}	y_{03}	y_{04}

From the Table it is clear that max. $Z = y_{00}$ and that y_{10} and y_{20} represent the optimal basic feasible solutions. Let us **assume** that both these solutions are fractional. Since the non-basic variables $x_1 = 0, x_4 = 0$, therefore $y_{12} = 1$ and $y_{13} = 0$. Hence the first optimal constraint equation is

$$y_{10} = y_{11}x_1 + y_{12}x_2 + y_{13}x_3 + y_{14}x_4.$$

This equation can now be re-written in the form

$$x_2 = y_{10} - y_{11}x_1 - y_{14}x_4 \text{ (since } y_{12} = 1, y_{13} = 0).$$

Re-writing we get $f_{10} + f'_{10} = (f_{11} + f'_{11})x_1 + x_2 + (f_{14} + f'_{14})x_4$ where f_{1j}, f'_{1j} for $j = 0, 1, 2, 3, 4$ are integral and fractional parts of y_{1j} respectively. Thus $f'_{10} - f'_{11}x_1 - f'_{14}x_4 = x_2 + f_{11}x_1 + f_{14}x_4 - f_{10}$.

The right hand side being an integer, the left hand side must be so. This requires that

$$f'_{10} - f'_{11}x_1 - f'_{14} \le 0.$$

This is known as **Gomory's cut**. It represents a necessary condition for obtaining an integer solution. This is also referred to as a **fractional cut** as all the coefficients are fractions in this case.

Note: Let it be possible that $f'_{10} - f'_{11}x_1 - f'_{14}x_4 = h$, where $h > 0$ is an integer. Then $f'_{10} = h + f'_{11}x_1 + f'_{14}x_4$, which is clearly a number greater that one. This is impossible as f'_{10} is a fraction less than one.

Again in the optimal LPP, $x_1 = 0, x_4 = 0$. This yields $f'_{10} \le 0$. This violates the cut. Hence introduction of this cut to the optimal table will result in achieving an integer extreme point solution.

Thus, Gomory's constraint in this case is represent by $f'_{11}x_1 + f'_{14}x_4 \ge f'_{10}$ or $-f'_{11}x_1 - f'_{14}x_4 \le -f'_{10}$ and finally, $-f'_{11}x_1 - f'_{14}x_4 + s_1 = -f'_{10}$, where s_1 is the appropriate slack variable added to the Gomory first constraint.

The First Gomory Table: With this additional constraint the first Gomory optimum simplex table is given in Table 7.3.

Table 7.3: First Gomory optimum simplex table

		$c \rightarrow$	c_1	c_2	0	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	s_1
c_2	y_2	y_{10}	y_{11}	y_{12}	y_{13}	y_{14}	0
c_3	y_3	y_{20}	y_{21}	y_{22}	y_{23}	y_{24}	0
0	s_1	$-f'_{10}$	$-f'_{11}$	0	0	$-f'_{14}$	1
		y_{00}	y_{01}	y_{02}	y_{03}	y_{04}	y_{05}

This is an infeasible optimum solution as $-f'_{10} \leq 0$. Therefore the **dual simplex** method is now applied to achieve an optimum feasible solution. Having obtained an optimum solution we still have to take care of the other non-integer solution y_{20}. Now with a view to achieving an all integer solution, a second Gomory constraint has to be introduced following the method shown above. The **cutting plane algorithm** is given as below:

Pure Integer Cutting Plane Algorithm

Step 1: Obtain the optimum solution of the given LPP by employing the standard simple algorithm.

Step 2: If the solution is all-integer then the problem has already been solved. If not, follow the next step.

Step 3: Let the constraint equations be represented by

$$\sum_{j=0}^{n} y_{ij}x_j = b_i \text{ where } i = 0, 1, 2, 3, \ldots, m.$$

Let the largest fraction of b_i's be f_{k0}.

Step 4: Write down the fractional cut constraint equation as described by Gomarian procedure after adding the appropriate positive integer slack variable S_1, in the form

$$S_1 = -f_{k0} + \sum_{j=0}^{n} f_{kj}x_j.$$

Then introduce this constraint in the optimum table.

Step 5: Apply dual simplex method to complete the next table, taking S_1 as the departing variable.

Step 6: If the integer solution is arrived at then the solution is complete. Otherwise repeat from Step 3 until the optimum integer solution is arrived at.

Now let us cite the following examples to understand the procedure.

Example 2: A television company produces two models T_1 and T_2 which have profit contributions 11 (Rs. in hundred) and 4 (Rs. in hundred) per unit production respectively. Each type of television requires a certain amount of time for the manufacture of components, assembling and packing. One unit of model A requires one hour for manufacturing, five hours for assembling and two hours for packing. The corresponding figures for one unit of model B are 2, 2, 1. The company will be able to make available 4 hours for manufacturing, 16 hours for assembling and 4 hours for packing time. According to some business deal the company will be able to receive free time for manufacturing of units of model A and for packing of units of model B. Obtain the optimal production schedule for the company.

Solution: The problem described above is first of all formulated as a linear programming problem as follows. The three constraints relate to manufacturing, assembling and packing. The constraint "$x_1, x_2 \geq 0$ and are integers" characterizes the problem as an integer linear programming problem, where x_1 and x_2 are the member of products produced for the models T_1 and T_2 respectively. This problem can be construed as constructing a mathematical model.

Maximize $Z = 11x_1 + 4x_2$, subject to the constraints
$-x_1 + 2x_2 \le 4,\ 5x_1 + 2x_2 \le 16,\ 2x_1 - x_2 \le 4,\ (x_1, x_2) \ge 0$ and are integers.

First the optimum basic feasible solution ignoring the integrality condition is obtained using usual simplex algorithm which is given in Table 7.4. The Gomory cut is then applied to arrive at the pure integer solution.

Table 7.4

		$c \rightarrow$	11	4	0	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5
0	y_3	$x_3 = 4$	-1	2	1	0	0
0	y_4	$x_4 = 16$	5	2	0	1	0
0	y_5	$x_5 = 4$	2^*	-1	0	0	1
		$Z = 5$	-11	-4	0	0	0
0	y_3	$x_3 = 6$	0	$\frac{2}{3}$	1	0	$\frac{1}{2}$
0	y_4	$x_4 = 6$	0	$\frac{9^*}{2}$	0	1	$-\frac{5}{2}$
11	y_1	$x_1 = 2$	1	$-\frac{1}{2}$	0	0	$\frac{1}{2}$
		$Z = 22$	0	$-\frac{19}{2}$	0	0	$\frac{11}{2}$
0	y_3	$x_3 = 4$	0	0	1	$-\frac{1}{3}$	$\frac{4}{3}$
4	y_4	$x_2 = \frac{4}{3}$	.0	1	0	$\frac{2}{9}$	$-\frac{5}{9}$
11	y_1	$x_1 = \frac{8}{3}$	1	0	0	$\frac{1}{9}$	$\frac{2}{9}$
		$Z = \frac{104}{3}$	0	0	0	$\frac{19}{9}$	$\frac{2}{9}$

The optimum basic feasible solution with $\max. Z = \frac{104}{3} = 34\frac{2}{3}$ and fractional values of the basic variables $x_1 = \frac{8}{3}, x_2 = \frac{4}{3}$ has been obtained. We now introduce the new variable s_1 defined under the **Gomory cutting constraint** with respect to the third row of the table. This **cut** is represented by the constraint equation.

$$S_1 - \frac{1}{9}x_4 - \frac{2}{9}x_5 = -\frac{2}{3}.$$

This leads to $x_4 + 2x_5 \geq 6$.

Taking help of the second and third given constraints, we finally get $x_1 \leq 2$.

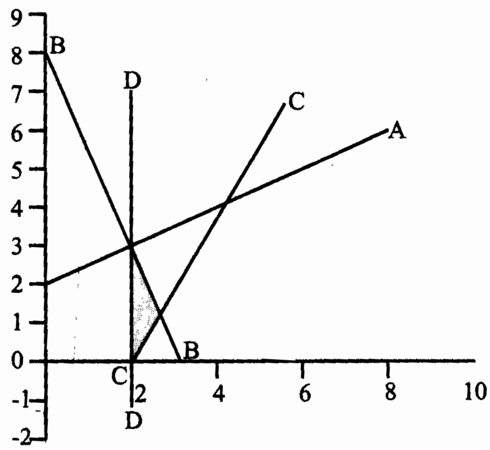

Fig. 7.1: Gomory Plane Cut

The Gomory plane cut is represented in Fig. 7.1, by the line DD ($x_1 \leq 2$) which eliminates the area included between the lines BB and DD from the feasible space bounded by the lines AA, BB, CC. The next table is given in Table 7.5.

Table 7.5: Gomory optimum Simplex table

		$c \rightarrow$	11	4	0	0	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	S_1
0	y_3	$x_3 = 4$	0	0	1	$-\frac{1}{3}$	$\frac{4}{3}$	0
4	y_2	$x_2 = \frac{4}{3}$	0	1	0	$\frac{2}{9}$	$-\frac{5}{9}$	0
11	y_1	$x_1 = \frac{8}{3}$	1	0	0	$\frac{1}{9}$	$\frac{2}{9}$	0
0	S_1	$s_1 = -\frac{2}{3}$	0	0	0	$-\frac{1}{9}$	$-\frac{2^*}{9}$	1
		$Z = \frac{104}{3}$	0	0	0	$\frac{19}{9}$	$\frac{2}{9}$	0

This is an optimum but infeasible solution. We decided to remove the infeasible solution $s_1 = -\frac{2}{3}$ and search for a new entering vector by using dual simplex method. We observe that the

$$\max.\left\{\frac{19}{9} \div \frac{-1}{9} = -19, \frac{2}{9} \div \frac{-2}{9} = -1\right\} = -1.$$

Hence, y_5 is the entering vector in Table 7.6.

Table 7.6: Gomory optimum Simplex table

		$c \to c$	11	4	0	0	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	S_1
0	y_3	$x_3 = 0$	0	0	1	-1	0	-6
4	y_2	$x_2 = 3$	0	1	0	$\frac{13}{81}$	0	$-\frac{5}{2}$
11	y_1	$x_1 = 2$	1	0	0	$\frac{11}{81}$	0	1
0	y_5	$x_5 = 3$	0	0	0	$\frac{1}{2}$	1	$-\frac{9}{2}$
		$Z = 34$	0	0	0	$\frac{173}{83}$	0	1

Hence the optimum feasible integer solution $x_1 = 2$, $x_2 = 3$, $\max.Z = 34$ is obtained. The value of $\max.Z$ diminishes by $\frac{2}{3}$ from that obtained from the non-integer basic feasible solution.

Example 3: Maximize $Z = 5x_1 + 7x_2$, subject to the constraints $-2x_1 + 3x_2 \leq 6$, $6x_1 + x_2 \leq 30$, $(x_1, x_2) \geq 0$ and are integers.

Solution: The simplex tables obtaining the optimum basic feasible solution without taking into account the conditions for integer is given in Table 7.7.

Table 7.7

		$c \to$	5	7	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4
0	y_3	$x_3 = 6$	-2	3^*	1	0
0	y_4	$x_4 = 30$	6	1	0	1
		$Z = 0$	-5	-7	0	0
7	y_2	$x_2 = 2$	$-\frac{2}{3}$	1	$\frac{1}{3}$	0
0	y_4	$x_4 = 28$	$\frac{20^*}{3}$	0	$-\frac{1}{3}$	1
		$Z = 14$	$-\frac{29}{3}$	0	$\frac{7}{3}$	0
7	y_2	$x_2 = \frac{24}{5}$	0	1	$\frac{3}{10}$	$\frac{1}{10}$
11	y_1	$x_1 = \frac{21}{5}$	1	0	$-\frac{1}{20}$	$\frac{3}{20}$
		$Z = \frac{273}{5}$	0	0	$\frac{37}{20}$	$\frac{29}{20}$

The optimal basic feasible but non-integer solution has been obtained. Towards achieving optimum integer solution we apply the Gomorian cut, arbitrarily choosing the x_2 – row for the purpose. The cut defined by

$$-\frac{4}{5}=-\frac{3}{10}x_3-\frac{1}{10}x_4+S_1$$

is now introduced in the Table 7.8.

The interpretation of incorporating this mathematical innovation is simple. The purpose is obviously to narrow down the non-integer region of the feasible space. This is evident from the result obtained from Table 7.7. The variable x_2 has already attained the integer value. The introduction of another cut will enable x_1 also to attain what may be termed as 'integerhood'. In fact, the final cut actually generates the desired all-integer solutions.

Table 7.8

c_B	y_B	x_B	y_1	y_2	y_3	y_4	S_1
7	y_2	$x_2=\frac{24}{5}$	0	1	$\frac{3}{10}$	$\frac{1}{10}$	0
5	y_1	$x_1=\frac{21}{5}$	1	0	$-\frac{1}{20}$	$\frac{3}{20}$	0
0	S_1	$S_1=-\frac{4}{5}$	0	0	$-\frac{3^*}{10}$	$-\frac{1}{10}$	1
		$Z=\frac{273}{5}$	0	0	$\frac{37}{20}$	$\frac{29}{20}$	0
7	y_2	$x_2=4$	0	1	0	0	1
5	y_1	$x_1=\frac{13}{3}$	1	0	0	$\frac{1}{6}$	$-\frac{1}{6}$
0	y_3	$x_3=\frac{8}{3}$	0	0	1	$\frac{1}{3}$	$-\frac{10}{3}$
		$Z=\frac{149}{3}$	0	0	0	$\frac{5}{6}$	$\frac{37}{6}$

The integer solution has still not been achieved in Table 7.8.

Table 7.9

		$c \rightarrow$	5	7	0	0	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	S_1	S_2
7	y_2	$x_2=4$	0	1	0	0	1	0
5	y_1	$x_1=\frac{13}{3}$	1	0	0	$\frac{1}{6}$	$-\frac{1}{6}$	0
0	y_3	$x_3=\frac{8}{3}$	0	0	1	$\frac{1}{3}$	$-\frac{10}{3}$	0
0	S_2	$S_2=-\frac{2}{3}$	0	0	0	$-\frac{1^*}{3}$	$-\frac{2}{3}$	1
		$Z=\frac{149}{3}$	0	0	0	$\frac{5}{6}$	$\frac{37}{6}$	0

We therefore introduce another cut arbitrarily choosing x_3 – row for the purpose. The row constraint equation becomes

$\frac{8}{3}=2+\frac{2}{3}=x_3+\frac{1}{3}x_4-\left(4-\frac{2}{3}\right)s_1$, and the Gomorian cut is represented by

$-\frac{2}{3}=-\frac{1}{3}x_4-\frac{2}{3}s_1+S_2$. The results are given in Table 7.9.

The last row is to be removed. Also by dual simplex procedure the entering vector is determined to be y_4. Hence the next table becomes table 7.10.

Table 7.10

		$c \rightarrow$	5	7	0	0	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	S_1	S_2
7	y_2	$x_2=4$	0	1	0	0	1	0
5	y_1	$x_1=4$	1	0	0	0	$-\frac{1}{2}$	$\frac{1}{2}$
0	y_3	$x_3=2$	0	0	1	0	-4	1
0	y_4	$x_4=2$	0	0	0	1	2	-3
		$Z=48$	0	0	0	0	$\frac{9}{2}$	$\frac{5}{2}$

Therefore, the optimal integer solution is represented by $x_1=4,\ x_2=4,\ \max.Z=48$. This solution satisfies the given constraints but the optimal integer value of Z is less by 6.6 from its non-integer optimal value which is 54.6. On the other hand, rounding off the non-integer optimal solution will give $\max.\ Z=55$, a good approximation, but then this solution will violate the first constraint.

Now, let us discuss the another procedure to find the integral solution of LPP, known as Branch and Bound Procedure.

7.2.2 Branch-and Bound (B&B) Procedure for Integer Programming

The first step in integer programming is to obtain the optimal solution, ignoring integer-constraints, of the linear programming problem (we designate this problem as LP-0) by usual simplex method. This solution does not necessarily give the desired integer solution. Therefore the next step is to **branch** or **partition** LP-0 in two separate problems termed as LP-1 and LP-2, after arbitrarily selecting one of the non-integer variables from amongst the optimal basic variables. This variable becomes the **branching variable**. The concept of partition is as follows. Suppose x_1 is chosen as the branching variable. Then x_1 cannot have any integer value between the integers

$[x_1]$ and $[x_1]+1$.

Thus the partitioned spaces are described by

$x_1 \le [x_1]$ and $x_1 \ge [x_1]+1$.

Solutions of these two problems in these spaces may give an integer solution and a non-integer solution, two non-integer solutions or two integer solutions. The non-integer solution or one of the two non-integer solutions may again give rise to branching or partitioning. The best of the resulting integer solutions, taking into consideration the entire set of solutions, as also the entire set of constraints, must be taken as the desired optimal integer solution. The procedure is explained below by solving a particular problem.

Example 4: By using Branch and Bound Algorithm find out the optimal solution to the linear programming problem given in Example 3, which is rewritten as

LP-0: Maximize $Z = 5x_1 + 7x_2$, subject to constraints
$-2x_1 + 3x_2 \leq 6,\ 6x_1 + x_2 \leq 30,\ x_1, x_2 \geq 0$ are integers.

The solution of the problem LP-0, obtained by usual simplex method, ignoring integrality constraint, is: $x_1 = 4\frac{1}{5},\ x_2 = 4\frac{4}{5},\ \max.Z = 54\frac{3}{5}$.

This is an non-integer solution showing that the optimal integer value of Z to be found out cannot exceed 54. The given problem is therefore partitioned into two mutually exclusive spaces after arbitrarily selecting x_1 as the branching variable. The two new problems arise corresponding to spaces $x_1 \leq 4$ and $x_1 \geq 5$. This is so because there is no integer value of x_1 in $4 \leq x_1 \leq 5$.

LP-1: Maximize $Z = 5x_1 + 7x_2$, subject to the constraints
$-2x_1 + 3x_2 \leq 6,\ 6x_1 + x_2 \leq 30,\ 0 \leq x_1 \leq 4,\ x_1, x_2 \geq 0$ are integers.

LP-2: Maximize $Z = 5x_1 + 7x_2$, subject to the constraints
$-2x_1 + 3x_2 \leq 6,\ 6x_1 + x_2 \leq 30,\ x_1 \geq 5,\ x_1, x_2 \geq 0$ are integers.

The solution of LP-1, obtained by simplex procedure is
$x_1 = 4,\ x_2 = 4\frac{2}{3},\ \max Z = 52\frac{2}{3}$.

This is a non-integer solution. However, this shows that the optimal integer value of Z to be found out cannot exceed 52. This problem is therefore branched after selecting x_2 as the branching variable. The two new problems defined in the two mutually exclusive spaces $x_2 \leq 4$ and $x_2 \geq 5$ are as follows.

LP-3: Maximize $Z = 5x_1 + 7x_2$, subject to the constraints
$-2x_1 + 3x_2 \leq 6,\ 6x_1 + x_2 \leq 30,\ 0 \leq x_1 \leq 4,\ x_1, x_2 \geq 0$ are integers.

LP-4: Maximize $Z = 5x_1 + 7x_2$, subject to the constraints
$-2x_1 + 3x_2 \leq 6,\ 6x_1 + x_2 \leq 30,\ 0 \leq x_1 \leq 4,\ x_2 \geq 5,\ x_1, x_2 \geq 0$ are integers.

Now the solution of LP-2 is represented by $x_1 = 5,\ x_2 = 0,\ \max Z = 25$.

This is the first integer solution. Therefore LP-2 is said to has been fathomed that is it need not be pursued further.

The solution of LP-3 is obtained as $x_1 = 4$, $x_2 = 4$, $\max Z = 48$.

This is the second integer solution. For LP-4, no optimal basic feasible solution exists. Hence the best of the two integer solutions is the desired optimal integer solution of the given LP-0. The required optimal integer solution is therefore $x_1 = 4$, $x_2 = 4$, $\max Z = 48$.

Also LP-1 has been fathomed.

It may be pointed out at this stage that the whole purpose is to block off some non-integer sub-space from the feasible space and move towards integer optimality, step by step, that is, iteration by iteration, improving the result at every succeeding step. This same problem may also be solved by plane cut algorithm to arrive at the same optimality. This exercise is left for the students.

In Fig. 7.2, LP-0 has the feasible space bounded by the axes and lines AA and BB in the first quadrant. Now the portion between the lines CC and DD of the feasible space LP-0 has been eliminated. It is to be noted that performing this branching operation does not lead to elimination of a single point from the original feasible space. Therefore the two branch-spaces are given by

$$\text{LP-1 Space} = \text{LP-0 Space} + x_1 \le 4$$

and $$\text{LP-1 Space} = \text{LP-0 Space} + x_2 \ge 5.$$

The problem LP-0 however has not yet been examined for optimal integer solution, taking x_2 as the branching variable. This gives rise to two problems LP-5 and LP-6.

LP-5: Maximize $Z = 5x_1 + 7x_2$, subject to the constraints

$$-2x_1 + 3x_2 \le 6,\ 6x_1 + x_2 \le 30,\ x_2 \le 4,\ x_1, x_2 \ge 0 \text{ are integers.}$$

This LP-5 leads to an optimal non-integer solution represented by

$$\text{Max. } Z = 49\frac{2}{3},\ x_1 = 4\frac{1}{3},\ x_2 = 4.$$

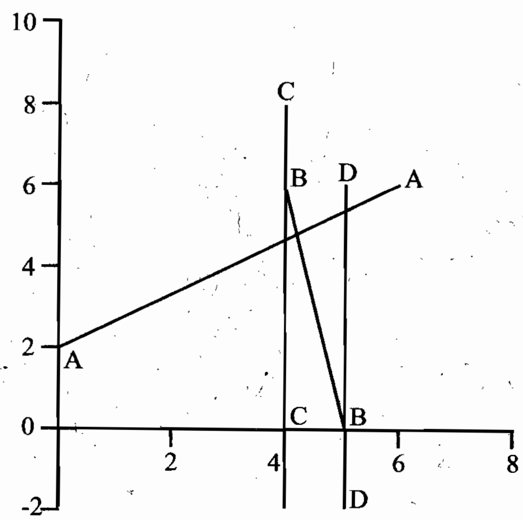

Fig. 7.2: B & B Method: Solutions Spaces LP-1 & LP-2

LP-6: Maximize $Z = 5x_1 + 7x_2$, subject to the constraints
$-2x_1 + 3x_2 \le 6,\ 6x_1 + x_2 \le 30,\ x_2 \ge 5,\ x_1, x_2 \ge 0$ are integers.

But by the first constraint $x_2 \ge 0$ leads to $x_1 \ge 5$. This violates the second constraint. This problem is therefore discarded as untenable.
From LP-5, taking x_1 as the branching variable, LP-7 and LP-8 are obtained:

LP-7: Maximize $Z = 5x_1 + 7x_2$, subject to the constraints
$-2x_1 + 3x_2 \le 6,\ 6x_1 + x_2 \le 30,\ x_1 \le 4,\ xx \le 4,\ x_1, x_2 \ge 0$ are integers.
This is the same as LP-3 which has yielded an integer solution.

LP-8: Maximize $Z = 5x_1 + 7x_2$, subject to the constraints
$-2x_1 + 3x_2 \le 6,\ 6x_1 + x_2 \le 30, x_1 \ge 5,\ x_2 \le 4,\ x_1, x_2 \ge 0$ are integers.

This violates the second constraint except when $x_1 = 5,\ x_2 = 0$, Max $Z = 25$.
LP-0 has been fathomed.

The required optimal integer solution is therefore $x_1 = 4,\ x_2 = 4$, $\max z = 48$.
This completes the solution.

Example 5:

LP-0: Maximize $Z = 5x_1 + 7x_2$, subject to $2x_1 + x_2 \le 13,\ 5x_1 + 9x_2 \le 41,\ x_1, x_2 \le 0$ are integers.   Solution: Max. $Z = 38\frac{5}{13},\ x_1 = 5\frac{11}{13},\ x_2 = 1\frac{4}{13}$.

LP-1: $x_1 \le 5,\ x_1, x_2 \ge 0$ are integers. **Solution:** Max. $Z = 37\frac{4}{9},\ x_1 = 5,\ x_2 = 1\frac{7}{9}$	LP-2: $x_1 \ge 6,\ x_1, x_2 \ge 0$ are integers. **Solution:** Max. $Z = 37,\ x_1 = 6,\ x_2 = 1$. This is the desired optimum integer solution of LP-0.

LP-3: $x_1 \le 5,\ x_2 \le 1,\ x_1, x_2 \ge 0$ are integers. **Solution:** Max. $Z = 32,\ x_1 = 5,\ x_2 = 1$.	LP-4: $x_1 \le 5,\ x_2 \ge 2,\ x_1, x_2 \ge 0$ are integers The constraints do not permit improvement of the integer solution of LP-2. Hence LP-4 need not be examined further.

Example 6: Minimize $Z = 9x_1 + 10x_2$, subject to the constraints

$$4x_1 + 3x_2 \ge 40$$
$$x_1 \le 9,\ x_2 \le 8$$
$$x_1, x_2 \ge 0 \text{ and are integers.}$$

Solution: The initial simplex table is given in Table 7.11.

Table 7.11

		$c \rightarrow$	-9	-10	0	0	0	M
c_B	y_b	x_B	y_1	y_2	y_3	y_4	y_5	y_6
$-M$	y_6	$x_6 = 40$	4	3	-1	0	0	1
0	y_4	$x_4 = 9$	1*	0	0	1	0	$0 \rightarrow$
0	y_5	$x_5 = 8$	$0\uparrow$	1	0	0	1	0
		$Z = 40M$	$-4M+9$	$-3M+10$	M	0	0	0

The optimal non-integer solution obtained by usual simplex method is:

$$\left(x_1 = 9,\ x_2 = \frac{4}{3} \max.Z = -\frac{-283}{3} \right).$$

This solution generates two sub-problems I and II.

Sub-problem I

Maximize $Z = -9x_1 - 10x_2$

Subject to $4x_1 + 3x_2 \geq 40$

$x_1 \leq 9,\ x_2 \leq 1$

$x_1, x_2 \geq 0$ and are integers.

Sub-problem II

Maximize $Z = -9x_1 - 10x_2$

Subject to $4x_1 + 3x_2 \geq 40$

$x_1 \leq 9,\ x_2 \geq 2$

$x_1, x_2 \geq 0$ and are integers.

Sub-Problem II gives the optimal non-integer solution

$$\left(x_1 = \frac{17}{2},\ x_2 = 2,\ \max.Z = -\frac{193}{2} \right).$$

The solution generates two Sub-problems III and IV represented by

Sub-problem III

Maximize $Z = -9x_1 - 10x_2$

Subject to $4x_1 + 3x_2 \geq 40$

$x_1 \leq 8,\ x_2 \geq 2$

$x_1, x_2 \geq 0$ and are integers.

Sub-problem IV

Maximize $Z = -9x_1 - 10x_2$

Subject to $4x_1 + 3x_2 \geq 40$

$x_1 \geq 9,\ x_2 \geq 2$

$x_1, x_2 \geq 0$ and are integers.

Sub-problem IV gives finally the optimal integer solution represented by

$(x_1 = 9,\ x_2 = 2,\ \min.Z = -(-101) = 101)$.

Now, let us discuss advantages and limitations of both the methods.

Advantages and Limitations of Cutting-Plane and Branch and Bound Methods

Limitations of Gomory's Cutting Plane method and Branch and Bound method consist in carrying out lengthy iterations: finding successive cutting planes by introducing new variables in the first case and deriving solutions to sub-problems originating from successive branching and fathoming them out one by one, in the second case.

The integer linear programming methods are called into play under special circumstances in economic, industrial and technological planning. The ILP serves to answer yes-no questions for favour of decision making against large number of alternative choices. The ILP is employed when solution in fractions is irrelevant. The problems discussed above will explain the point unambiguously. The solutions obtained by ILP are exact and mathematically consistent. Geometrical diagrams, in the case of two-variable problems, can effectively demonstrate the implications of the successive steps in achieving the optimal integer solutions. In picking a candidate sub-problem from the list of unfathomed candidate sub-problems, the last of the unfathomed candidate subproblem or the unfathomed candidate sub-problem whose corresponding upper bound (maximization problem) is the largest, may be selected.

Let us try some exercises.

E1) A refrigeration and air-conditioning company has been awarded a contract for the air-conditioning of a new computer installation. The company has to make a choice between two alternatives:

a) hire one or more refrigeration technicians for six hours a day or

b) hire one or more part-time refrigeration apprentice technicians for four hours a day.

The rate of wages of refrigeration technician is Rs.20 per hour, while the corresponding rate of refrigeration technician is Rs.8 per hour. The company wants to engage technicians for work not more than 25 man hours per day and also limit the charges to technicians to Rs.440. The company estimates that the productivity of a refrigeration technician is eight units and that of a part-time apprentice technician is three units. Formulate the integer programming problem to enable the company to select the optimum number of technicians and apprentices.

E2) A manufacturer of baby-dolls makes two types of dolls: Doll X and Doll Y. Processing of these two dolls is done on two machines A and B. Doll X requires two hours on machine A and six hours on machine B. Doll Y requires five hours on machine A and also five hours on machine B. There are sixteen hours of time available per day for machine A and thirty hours on machine B. The profit gained on both the dolls is same, i.e., one rupee per doll. What should be the daily production of each of the two dolls?

a) Set up and solve the linear programming problem.

b) If the optimal solution is not integer-valued, use Gomory technique to drive the optimal solution.

E3) Use Branch and Bound Model to Maximize $Z = 3x_1 + 3x_2 + 13x_3$, subject to the constraints

$$-3x_1 + 6x_2 + 7x_3 \le 8$$

$$6x_1 - 3x_2 + 7x_3 \le 8$$

$x_1, x_2, x_3 \ge 0$ and are integers.

In the following section, we shall discuss transportation models.

7.3 TRANSPORTATION OPTIMIZATION MODEL

The transportation problem is generally associated with industries. Suppose there are m origins O_i, $i = 1, 2, 3, \ldots, m$, where various amounts of a commodity are produced or stored for transportation to n destinations D_j, $j = 1, 2, 3, \ldots, n$. The i^{th} origin can supply exactly a_i unit of the commodity and the j^{th} destination can accept exactly b_j unit. The cost of transportation of one unit from O_i to D_j is c_{ij}. The object is to transport all units of the commodity from all O_i, exhaustibly, to all D_j, exactly satisfying all their requirements, in such a way that the total transportation cost becomes minimum. This transportation problem therefore is essentially a minimization problem. The transportation problem (TP) is mathematically formulated as follows:

Let x_{ij} = number of units transported from O_i to D_j. Then our object is to

$$\text{Minimize } Z = \sum_{i=1}^{i=m}\sum_{j=1}^{j=n} c_{ij}\, x_{ij} \tag{1}$$

subject to

$$\sum_{j=1}^{n} x_{ij} = a_i, \; i = 1, 2, 3, \ldots, m \tag{2}$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \; j = 1, 2, 3, \ldots, n \tag{3}$$

$$x_{ij} \geq 0, \text{ for all } i, j \tag{4}$$

$$a_i, b_j > 0, \text{ for all } i, j.$$

The TP defined above is a minimization linear programming problem. Assuming the existence of a basic feasible solution, one gets from Eqns. (2) and (3), that

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j.$$

Hence, the problem to be consistent for the existence of a feasible solution.

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j. \tag{5}$$

The TP is termed as **balance**. The condition given in Eqn. (5) represents an idealized industrial market. In reality, $\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j$ and the TP is termed as **unbalanced**. An unbalanced TP, however, can be converted into a balanced one after introducing a fictitious or **dummy constraint** with zero costs and with appropriate units of supply or demand as the case may be. Taking $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j = \lambda$ (say), one gets $x_{ij} = \dfrac{a_i b_j}{\lambda} \geq 0$. This is also a feasible solution as it satisfies the constraint Eqns. (2) and (3):

$$\sum_{i=1}^{m} x_{ij} = \sum_{i=1}^{m} \frac{a_i b_j}{\lambda} = \frac{b_j}{\lambda} \sum_{i=1}^{m} a_i = b_j$$

$$\sum_{j=1}^{n} x_{ij} = \sum_{j=1}^{n} \frac{a_i b_j}{\lambda} = \frac{a_i}{\lambda} \sum_{j=1}^{n} b_j = a_i .$$

Hence the condition (5) confirms the existence of a feasible solution of the TP described by Eqns. (2), (3) and (4). It is also clear from the constraint equations that for every feasible solution $0 \le x_{ij} \le \min.(a_i, b_j)$. Thus the feasible region is closed, bounded and non-empty and the TP must possess an optimal solution.

Matrix Formulation of the TP: The TP can be presented in the matrix form as follows:

Minimize $Z = c^T x$, $(c, x) \in R^{mn}$, subject to the constraints

$$Ax = b,\ x \ge 0,\ b \in R^{m+n},$$

where $x = [x_{11}, x_{12}, \ldots x_{1n}, x_{21}, \ldots, x_{2n}, \ldots, x_{m1}, \ldots, x_{mn}]$

$c = c_{ij}, i = 1, \ldots, m, j = 1, \ldots, n$; c is the cost vector,

$b = [a_1, \ldots, a_m, b_1, \ldots, b_n]$

$$A = \begin{bmatrix} e_{mn}^1 & e_{mn}^2 & \cdots & e_{mn}^m \\ I_n & I_n & \cdots & I_n \end{bmatrix}.$$

Here e_{mn}^j is a $m \times n$ matrix having a row of unit elements in its j^{th} row and 0 as elements in all other rows; I_n is the identity matrix of order n.

For example, if $m = 2, n = 3$,

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} e_{23}^1 & e_{23}^2 \\ I_3 & I_3 \end{bmatrix}.$$

When $m = 2, n = 4$, the equations $Ax = b$ becomes

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{21} \\ x_{22} \\ x_{23} \\ x_{24} \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}.$$

The construction of the matrix A shows that the TP is a special type of linear programming problem. Therefore some special algorithms will be developed to efficiently solve the TP.

Number of basic feasible solutions of the TP is $m+n-1$.
From the $m+n$ constraints (2) and (3) of the TP, it follows that

$$\sum_{j=1}^{n-1}\sum_{i=1}^{m} x_{ij} = \sum_{j=1}^{n} b_j \text{ and } \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{m} a_i .$$

Subtracting, $\sum_{i=1}^{m}\left(\sum_{j=1}^{n-1} x_{ij} - \sum_{j=1}^{n} x_{ij}\right) = \sum_{j=1}^{n-1} b_j - \sum_{j=1}^{n} b_j$, since $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$.

Hence $\sum_{i=1}^{m} x_{in} = b_n$.

But this is the last constraint equation of the TP. This proves that out of $m+n$ constraint equations only $m+n-1$ are linearly independent equations. Therefore, a basic feasible solution will consist of not more than $m+n-1$ variables. If the number of basic variables is less than $m+n-1$, then the solution is termed a degenerate b.f.s., as in the case of linear programming problem.

Next we prove the optimality theorem.

A Balanced TP Possesses at least One Optimal solution

It is clear from the constraints that the feasible solution is bounded by $0 \le x_{ij} \le \min.(a_i, b_j)$. The solution region is therefore closed, bounded and non-empty. This confirms the existence of an optimal solution.

The TP in the Tabular Form

It is easy to see that the $m+n$ constraints $\sum x_{ij} = a_i, \sum x_{ij} = b_j$ can be arranged in a tabular form shown below. Here $c_{ij} \to$ cost for transporting one unit commodity from the i^{th} origin O_i to the j^{th} destination D_j, $a_i \to$ number of units available for supply in the origin O_i, $b_j \to$ number of units required or demanded by the destination D_j, $x_{ij} \to$ number of units (not necessarily the optimal units) allocated to the ij^{th} cell and inserted in the upper left corner of the cell.

Table 7.12

Destination → Origin ↓	D_1	D_2	………	D_{n-1}	D_n	Supply
O_1	c_{11}	c_{12}	………	c_{1n-1}	c_{1n}	a_1
O_2	c_{21}	c_{22}	………	c_{2n-1}	c_{2n}	a_2
… …	… …	… …	……… ………	… …	… …	… …
O_m	c_{m1}	c_{m2}	………	c_{mn-1}	c_{mn}	a_m
Demand →	b_1	b_2	………	b_{n-1}	b_n	

An initial basic feasible solution will now be developed using this tabular representation of the transportation problem as given in Table 7.12. Following the simplex procedure this solution will be improved, if necessary, to arrive at the optimal solution.

An Initial Basic Feasible Solution

There exist several procedures, known as **Stepping Stone Procedures**, to install an initial basic feasible solution in the transportation table. It is important to assert at the outset that an initial solution is actually the starting solution. This starting solution is to be improved step by step, through the iterative procedure to be described below. A starting solution may be deduced by arbitrarily selecting any of the methods described below. A starting solution obtained by a particular method may however be closer to the optimal solution. This aspect of selecting the method for finding the initial basic feasible solution will be discussed in details. Students, however, should be acquainted with all the available models.

We first adopt the **North-West Corner Rule**. We start with making the first allocation $x_{11} = \min.(a_1, b_1)$ in the cell $(1, 1)$. Any residual resources from the origin O_1 can be distributed in the cell $(1, 2)$. This procedure may be continued along the first row till the resources or availability at O_1 is exhausted. Allocations in the successive rows can be made following the same procedure. This is best explained through an example. Let us consider an example given in Table 7.13.

Table 7.13

	D_1	D_2	D_3	D_4	D_5	a_i
O_1	40 1	10 2	30 6	2	3	80
O_2	3	4	40 5	20 8	1	60
O_3	3	1	1	10 2	10 6	20
O_4	4	7	3	5	40 4	40
b_j	40	10	70	30	50	$\sum a_i = \sum b_j = 200$

$x_{11} = \min.(80, 40) = 40$, $x_{12} = \min.(80 - 40 = 40, 10) = 10$,

$x_{13} = \min.(80 - 40 - 10 = 30, 70) = 30$,

$x_{23} = \min.(60, 70 - 30 = 40) = 40$, $x_{24} = \min.(60 - 40 = 20, 30) = 20$,

$x_{34} = \min.(20, 30 - 20 = 10) = 10$, $x_{35} = \min.(20 - 10 = 10, 50) = 10$,

$x_{45} = \min.(40, 50 - 10 = 40) = 40$.

Transportation Cost

$= 40 \times 1 + 10 \times 2 + 30 \times 6 + 40 \times 5 + 20 \times 8 + 10 \times 2 + 10 \times 6 + 40 \times 4 = 840$.

This solution is to be examined for optimality.

An initial basic feasible solution for the same problem is obtained by **matrix minima method** and is given in Table 7.14. In this case the successive allocations are made in the cells showing the minimum cost.

Table 7.14

	D_1	D_2	D_3	D_4	D_5	a_i
O_1	40 / 1	2	10 / 6	30 / 2	3	80
O_2	3	4	10 / 5	8	50 / 1	60
O_3	3	10 / 1	10 / 1	2	6	20
O_4	4	7	40 / 3	5	4	40
b_j	40	10	70	30	50	$\sum a_i = \sum b_j = 200$

The successive allocation is: $x_{11} = 40, x_{25} = 50, x_{32} = 10, x_{33} = 10, x_{43} = 40,$
$x_{14} = 30, x_{23} = 10, x_{13} = 10.$

Transportation Cost
$= 40 \times 1 + 10 \times 6 + 30 \times 2 + 10 \times 5 + 50 \times 1 + 10 \times 1 + 10 \times 1 + 40 \times 3 = 400.$

The solution obtained by **Row Minima Method** is given in Table 7.15.

Table 7.15

	D_1	D_2	D_3	D_4	D_5	a_i
O_1	40 / 1	10 / 2	6	30 / 2	3	80
O_2	3	4	10 / 5	8	50 / 1	60
O_3	3	1	20 / 1	2	6	20
O_4	4	7	40 / 3	5	4	40
b_j	40	10	70	30	50	$\sum a_i = \sum b_j = 200$

The successive allocations are:
$x_{11} = 40, x_{12} = 10, x_{14} = 30, x_{25} = 50, x_{23} = 10, x_{33} = 20, x_{43} = 40.$

Transportation Cost
$= 40 \times 1 + 10 \times 2 + 30 \times 2 + 10 \times 5 + 50 \times 1 + 20 \times 1 + 40 \times 3 = 360.$

The solution by using **Column Minima Method** can be similarly obtained. However, the most widely used method of finding an initial basic feasible solution is **Vogel's Approximation Method (VAM)**. This is described in Table 7.16.

Table 7.16

	D_1	D_2	D_3	D_4	D_5	a_i
O_1	1	2	6	30 2	3	$80(2-1=1)$
O_2	3	4	5	8	1	$60(3-1=2)$
O_3	3	1	1	2	6	$20(2-1=1)$
O_4	4	7	3	5	4	$40(4-3=1)$
b_j	$40(3-1=2)$	$10(2-1=1)$	$70(3-1=2)$	$30(5-2=3)$	$50(3-1=2)$	

The difference between the lowest and next to lowest costs in each row and each column are computed and recorded in parenthesis by the sides of corresponding a_i and b_j. The maximum possible allocation is made in the cell showing minimum cost and belonging to the row or column for which the difference is maximum. Thus the first allocation is $x_{14} = 30$ and column 4 is blocked out or eliminated from Table 7.17.

Table 7.17

	D_1	D_2	D_3	D_5	
O_1	1	2	6	3	$50(2-1=1)$
O_2	3	4	5	50 1	$60(3-1=2)$
O_3	3	1	1	6	$20(2-1=1)$
O_4	4	7	3	4	$40(4-3=1)$
	$40(3-1=2)$	$10(2-1=1)$	$70(3-1=2)$	$50(3-1=2)$	

Column 5 is deleted from Table 7.17 and the reduced table is given in Table 7.18.

Table 7.18

	D_1	D_2	D_3	a_i
O_1	40 1	2	6	$50(2-1=1)$
O_2	3	4	5	$10(4-3=1)$
O_3	3	1	1	$20(3-1=1)$
O_4	4	7	3	$40(4-3=1)$
b_j	$40(3-1=2)$	$10(2-1=1)$	$70(3-1=2)$	

Column 1 is deleted from Table 7.18.

Table 7.19

	D_2	D_3	a_i
O_1	2	6	$10(6-2=4)$
O_2	4	5	$10(5-4=1)$
O_3	1	1	$20(1-1=0)$
O_4	7	40 3	$40(7-3=4)$
	$10(2-1=1)$	$70(3-1=2)$	

Row 4 is deleted from Table 7.19.

Table 7.20

	D_2	D_3	a_i
O_1	2	6	$10(6-2=4)$
O_2	4	5	$10(5-4=1)$
O_3	1	20 1	$20(1-1=0)$
b_j	$10(2-1=1)$	$30(5-1=4)$	

Row 3 is deleted from Table 7.20.

Table 7.21

	D_2	D_3	a_i
O_1	10 2	6	$10(6-2=4)$
O_2	4	10 5	$10(5-4=1)$
b_j	$10(4-2=2)$	$10(6-5=1)$	

The final table is given in Table 7.22.

Table 7.22

	D_1	D_2	D_3	D_4	D_5	a_i
O_1	40 1	10 2	6	30 2	3	80
O_2	3	4	10 5	8	50 1	60
O_3	3	1	20 1	2	6	20
O_4	4	7	40 3	5	4	40
b_j	40	10	70	30	50	

Transportation Cost
$= 40\times1+10\times2+30\times2+30\times2+10\times5+50\times1+20\times1+40\times3=360$.

Test for Optimality

The constraints of the dual problem corresponding to the primal one represented by (1)-(4) can be written

$$u_i + v_j \leq c_{ij}$$

$$u_i,\ v_j \text{ unrestricted, } i = 1, 2, \ldots m;\ n = 1, 2, \ldots, n.$$

Then the net evaluations for the non-basic cells are

$$z_{ij} - c_{ij} = u_i + v_j - c_{ij}. \tag{6}$$

For the $m + n - 1$ basic cells the net evaluations vanish and we have

$$u_r + v_s = c_{rs}.$$

These equations can be solved after taking one of the $m + n$ variables to be zero. Thus the net evaluations for the non-basic cells can be computed with the help of Eqn.(6). If all net evaluations $z_{ij} - c_{ij}$ are non-positive then the current solution is an optimum solution. If at least one $z_{ij} - c_{ij}$ is positive then the optimality has not been achieved and new basic cell has to be found out leaving out one from the present set of basic variables. The variable x_{rs} having the largest positive net evaluation will enter the new basis set; an arbitrary allocation $\theta > 0$ is inserted in the cell (rs). A loop, beginning and ending with this cell and connecting some other basic cells is formed. The number of cells forming the loop is even. (The positions of cells in a basic feasible solution cannot form a loop.) Supply and demand quantities now actually determine the value of θ for the new basic cell. This solution is to be tested for optimality.

Example 7: Goods have to be transported from factories O_1, O_2, O_3 to warehouses D_1, D_2, D_3, D_4. The transportation cost per unit from

O_1 to D_1, D_2, D_3, D_4 are 10, 2, 20,11

O_2 to D_1, D_2, D_3, D_4 are 12, 7, 9, 20

O_3 to D_1, D_2, D_3, D_4 are 4, 14, 16, 18 respectively.

Capacities of the factories O_1, O_2, O_3 are 15, 25, 10 units respectively and the requirements of the warehouses D_1, D_2, D_3, D_4 are 5, 15, 15, 15 units respectively. Determine how the units can be transported to warehouses keeping the transportation cost minimum.

Solution: Firstly, the transportation problem is restated in the tabular form in Table 7.23.

Table 7.23

	D_1	D_2	D_3	D_4	a_i
O_1	10	2	20	11	15
O_2	12	7	9	20	25
O_3	4	14	16	18	10
b_j	40	10	70	30	

An initial basic feasible solution is then to be obtained by VAM is shown in Table 7.24.

Table 7.24

	D_1	D_2	D_3	D_4	a_i
O_1	10	2	20	11	15(8)
O_2	12	7	9	20	25(2)
O_3	5 4	14	16	18	10(10)
b_j	5(6)	15(5)	15(7)	15(9)	

The next table is Table 7.25.

Table 7.25

	D_2	D_3	D_4	
O_1	15 2	20	0 11	15(9)
O_2	7	9	20	25(2)
O_3	14	16	18	5(4)
	15(5)	15(7)	15(7)	

Column 2 and row 1 simultaneously stand to be blocked off in Table 7.25. This will ultimately lead to a degenerate basic feasible solution. Hence, row 1 is retained with a zero allocation in the cell (1, 4).

Table 7.26

	D_3	D_4	
O_1	20	11	0(9)
O_2	15 9	10 20	25(11)
O_3	16	5 18	5(4)
	15(7)	15(7)	

The final table showing the initial basic feasible solution Table 7.27.

Table 7.27

	D_1	D_2	D_3	D_4	a_i
O_1	10	15 2	20	0 11	15
O_2	12	7	15 9	10 20	25
O_3	5 4	14	16	5 18	10
b_j	5	15	15	15	

The Transportation Cost $= 5\times4+15\times2+0\times11+15\times9+10\times20+5\times18=475$.
We test for optimality of this solution. Let $u_i+v_j=c_{ij}$ for the basic cells.
Hence
$u_1+v_2=2,\ u_1+v_4=11,\ u_2+v_3=9,\ u_2+v_4=20,\ u_3+v_1=4,\ u_3+v_4=18$.
Taking $u_1=0$, one finds $u_2=9,\ u_3=7,\ v_1=-3,\ v_2=2,\ v_3=0,\ v_4=11$.

The net evaluations for the non-basic cells are:

$$Z_{11}=z_{11}-c_{11}=-3-10=-13,\ Z_{12}=-20,\ Z_{21}=9-3-12=-6$$

$$Z_{22}=9+2-7=4,\ Z_{32}=7+2-14=-5,\ Z_{33}=7-16=-9.$$

An amount $\theta>0$ is inserted in the cell (2, 2) having a positive net evaluation. A loop is formed with the cells (2, 2), (2, 4), (1, 4), (1, 2) with revised allocations $\theta, 10-\theta, 0+\theta, 15-\theta$ respectively in these cells. Obviously $\theta=10$ is the minimum value of θ satisfying the supply and demand limits. The cell (2, 2) enters the basis and the cell (1, 4) leaves the basis. This solution, shown in Table 7.28, is tested for optimality as before.

Table 7.28

	D_1	D_2	D_3	D_4	a_i
O_1	10	15 / 2	20	10 / 11	15
O_2	12	10 / 7	15 / 9	20	25
O_3	5 / 4	14	16	5 / 18	10
b_j	5	15	15	15	

$u_1+v_1=2,\ u_2+v_2=7,\ u_2+v_3=9,\ u_2+v_4=20,\ u_3+v_4,\ u_3+v_4=18$.

$u_1=0,\ v_2=2,\ u_2=5,\ v_3=4,\ v_4=15,\ u_3=3,\ v_1=1$.

$Z_{11}=-9,\ Z_{13}=-16,\ Z_{21}=-6,\ Z_{32}=-9,\ Z_{33}=-9$.

Since all net evaluations are non-positive, therefore optimal solution has been achieved. The optimal Transportation Cost
$=10\times2+10\times11+10\times7+15\times9+5\times4+5\times18=435$.

Example 8: Obtain the optimal solution of the following transportation problem given in Table 7.29.

Table 7.29

	D_1	D_2	D_3	a_i
O_1	7	3	4	2
O_2	2	1	3	3
O_3	3	4	6	5
b_j	4	1	5	

(Assume that this is formulation of a real life word problem in which the a_i's and b_j's represent supplies and requirements in a real situation and the elements of the matrix represent the corresponding costs.

Solution: The initial basic feasible solution is obtained by VAM is in Table 7.30.

Table 7.30

	D_1	D_2	D_3	a_i		D_1	D_3	
O_1	7	3	4	2(1)		7	2 4	2(3)
O_2	2	1 1	3	3(1)	→	2	2 3	2(1)
O_3	3	4	6	5(1)		4 3	1 6	5(3)
b_j	4(1)	1(2)	5(1)			4(1)	5(1)	

The initial feasible solution is shown in Table 7.31, the transportation cost being 33. This solution is now to be tested for optimality.

Table 7.31

	D_1	D_2	D_3	a_i
O_1	7	3	2 4	2
O_2	2	1 1	2 3	3
O_3	4 3	4	1 6	5
b_j	4	1	5	

$u_1 + v_3 = 4, u_2 + v_2 = 1,\ u_2 + v_3 = 3,\ u_3 + v_1 = 3,\ u_3 + v_3 = 6.$

$u_1 = 0, v_3 = 4,\ u_3 = 2,\ u_2 = -1,\ v_1 = 1,\ v_3 = 4,\ v_2 = 2.$

$Z_{11} = -6,\ Z_{12} = -1,\ Z_{21} = -2,\ Z_{32} = 0.$

Since all $Z_{ij} \geq 0$, therefore an optimal solution has been achieved. Optimal transportation cost is 33 and the optimal solution is $x_{13} = 2,\ x_{22} = 1,\ x_{23} = 2,\ x_{31} = 4,\ x_{33} = 1.$

Example 9: Show that the North-West Corner Rule gives a degenerate basic feasible solution to the problem given in Example 8, Optimize this solution.

Solution

Table 7.32

	D_1	D_2	D_3	a_i		D_1	D_2	D_3	
O_1	2 7	3	4	2		2 7	3	4	2
O_2	2 2	1 1	3	3	→	2 2	1 1	ε 3	3
O_3	3	4	5 6	5		3	4	5 6	5
b_j	4	1	5			4	1	5	

The number of allocations in this solution is 4 i.e., one short of $m+n-1=3+3-1=5$. Thus it is a degenerate basic feasible solution. To eliminate degeneracy, a small positive allocation ε is inserted in the cell $(2, 3)$. It may be observed that these allocations cannot form a loop and the solution remains a basic one. Hence the equations $u_1+v_1=7, u_2+v_1=2, u_2+v_2=1, u_2+v_3=3, u_3+v_3=6$, give $u_1=0, v_1=7, u_2=-5, v_2=6, v_3=8, u_2=-2$. Accordingly, $Z_{12}=3, Z_{13}=4, Z_{31}=2, Z_{32}=0$. Since all net evaluations are non-negative, the solution is not optimal. Hence $\theta>0$ is allocated in the cell $(1, 3)$ and allocations in the cells $(1, 1), (2, 1), (2, 3)$ forming a loop with $(1, 3)$ are readjusted. Clearly $\theta=\varepsilon$ removes the cell $(2, 3)$ from the basis. Hence the equations $u_1+v_1=7, u_1+v_3=4, u_2+v_1=2, u_2+v_2=1, u_3+v_3=6$, give $u_1=0, v_1=7, v_3=4, u_3=2, u_2=-5, v_2=6$. Accordingly, $Z_{12}=3, Z_{23}=-4; Z_{31}=6, Z_{32}=4$.

Again $\theta>0$ is allocated in the cell $(3, 1)$ and allocations in the cells $(3,3), (1, 3), (1, 1)$ along with the cell $(3, 1)$ are readjusted.

Table 7.33

	D_1	D_2	D_3	a_i		D_1	D_2	D_3
O_1	$2-\theta$ 7	 3	θ 4	2		$2-\varepsilon$ 7	 3	ε 4
O_2	$2+\theta$ 2	1 1	$\varepsilon-\theta$ 3	3	$\theta=\varepsilon$ $\rightarrow$	$2+\varepsilon$ 2	1 1	 3
O_3	 3	 4	5 6	5		 3	 4	5 6
b_j	4	1	5					

The succeeding tables are shown in Table 7.34 and Table 7.35.

Table 7.34

	D_1	D_2	D_3	a_i		D_1	D_2	D_3	a_i
O_1	$2-\varepsilon-\theta$ 7	 3	$\varepsilon+\theta$ 4	2		 7	 3	2 4	2
O_2	$2+\varepsilon$ 2	1 1	 3	3	$\theta=2$ $\rightarrow$ $\varepsilon=0$	2 2	1 1	 3	3
O_3	θ 3	 4	$5-\theta$ 6	5		2 3	 4	3 6	5
b_j	4	1	5			4	1	5	

The equations $u_1+v_3=4, u_2+v_1=2, u_2+v_2=1, u_3+v_1=3, u_3+v_3=6$, give $u_1=0, v_3=4, u_3=2, v_1=1, u_2=1, v_2=0$ and hence $Z_{11}=-6, Z_{12}=-3, Z_{23}=2, Z_{32}=-2$. Since $Z_{23}>0$, therefore optimality has not been achieved yet. Again $\theta>0$ is allocated in the cell $(2, 3)$ and the allocations in the cells (2, 1), (3, 1), (3, 3) forming the loop along with the cell (2, 3) are readjusted.

	D_1	D_2	D_3	a_i		D_1	D_2	D_3
O_1	7	3	2 4	2		7	3	2 4
O_2	$2-\theta$ 2	1 1	ε 3	3	$\theta = 2$ $\rightarrow$	2	1 1	2 3
O_3	$2+\theta$ 3	4	$3-\theta$ 6	5		4 3	4	1 6
b_j	4	1	5			4	1	5

Table 7.35

The equations $u_1 + v_3 = 4, u_2 + v_2 = 1, u_2 + v_3 = 3, u_3 + v_1 = 3, u_3 + v_3 = 6$ give $u_1 = 0, v_3 = 4, u_3 = 2, v_1 = 1, u_2 = -1, v_2 = 2$ and hence $Z_{11} = -6, Z_{12} = -1, Z_{21} = -2, Z_{32} = 0$.

Therefore the optimal solution $x_{13} = 2, x_{22} = 1, x_{23} = 2, x_{31} = 4, x_{33} = 1$ is obtained and the optimal transportation cost is 33.

Example 10: Determine an optimal transportation program so that the transportation cost of 340 tons of a certain type of material from three factories F_1, F_2, F_3 to five stores $S_1, \ldots, S_5$ is minimized. The five stores must receive 40 tons, 50 tons, 70 tons, 90 tons and 90 tons respectively. The availability of the material at F_1, F_2, F_3 is 100 tons, 120 tons, 120 tons respectively. The transportation costs per ton from factories to stores are given in Table 7.36.

Table 7.36

	S_1	S_2	S_3	S_4	S_5	a_i
F_1	4	1	2	6	9	100
F_2	6	4	3	5	7	120
F_3	5	2	6	4	8	120
b_j	40	50	70	90	90	

Solution: VAM is used to obtain an initial basic feasible solution. The steps are given in Tables 7.37, 7.38, 7.39 and 7.40.

Table 7.37

	S_1	S_2	S_3	S_4	S_5	a_i
F_1	4	1	2	6	9	100(1)
F_2	6	4	3	5	7	120(1)
F_3	5	50 2	6	4	8	120(2)
b_j	40(1)	50(1)	70(1)	90(1)	90(1)	

Table 7.38

	S_1	S_3	S_4	S_5	a_i
F_1	4	70 2	6	9	100(2)
F_2	6	3	5	7	120(2)
F_3	5	6	4	8	70(1)
b_j	40(1)	70(1)	90(1)	90(1)	

Table 7.39

	S_1	S_4	S_5	a_i
F_1	30 4	6	9	30(2)
F_2	6	5	7	120(1)
F_3	5	4	8	70(1)
b_j	40(1)	90(1)	90(1)	

Table 7.40

	S_1	S_4	S_5	a_i
F_2	10 6	20 5	90 7	120(1)
F_3	5	70 4	8	70(1)
b_j	10(1)	90(1)	90(1)	

The final VAM table describing the model for initial distribution is Table 7.41.

Table 7.41

	S_1	S_2	S_3	S_4	S_5	a_i
F_1	30 4	1	70 2	6	9	100
F_2	10 6	4	3	20 5	90 7	120
F_3	5	50 2	6	70 4	8	120
b_j	40	50	70	90	90	

The initial transportation cost is 1430. We proceed to test the optimality of the solution. The equations

$$u_1 + v_1 = 4, u_1 + v_3 = 2,$$

$$u_2 + v_1 = 6, u_2 + v_4 = 5,$$

$$u_2 + v_5 = 7, u_3 + v_2 = 2,$$

$$u_3 + v_4 = 4$$

give $u_1 = 0$, $v_1 = 4$, $v_3 = 2$, $u_2 = 2$, $v_4 = 3$, $v_5 = 5$, $u_3 = 1$, $v_2 = 1$ and hence $Z_{12} = 0$, $Z_{14} = -3$, $Z_{15} = -4$, $Z_{22} = -1$, $Z_{23} = 1$, $Z_{31} = 0$, $Z_{33} = -3$, $Z_{35} = -2$.

Since $Z_{23} > 0$, therefore this solution is not optimal. An allocation $\theta > 0$ is inserted in the cell (2, 3) and allocations in the cells (1, 3), (1, 1), (2, 1) forming the loop along with the cell (2, 3) are readjusted.

Table 7.42

	S_1	S_2	S_3	S_4	S_5	a_i
F_1	$30+\theta$ 4	 1	$70-\theta$ 2	 6	 9	100
F_2	$10-\theta$ 6	 4	$\theta\uparrow$ 3	20 5	90 7	120
F_3	 5	50 2	 6	70 4	 8	120
b_j	40	50	70	90	90	

This gives $\theta = 10$. The resulting solution is tested for optimality.

Table 7.43

	S_1	S_2	S_3	S_4	S_5	a_i
F_1	40 4	 1	60 2	 6	 9	100
F_2	 6	 4	10 3	20 5	90 7	120
F_3	 5	50 2	 6	70 4	 8	120
b_j	40	50	70	90	90	

Solving
$u_1 + v_1 = 4, u_1 + v_3 = 2, u_2 + v_3 = 3, u_2 + v_4 = 5, u_2 + v_5 = 7, u_3 + v_2 = 2, u_3 + v_4 = 4$
one gets $u_1 = 0,\ v_1 = 4, v_3 = 2, u_2 = 1, v_4 = 4, v_5 = 6, u_3 = 0, v_2 = 2$ and hence
$Z_{12} = 1,\ Z_{14} = -2,\ Z_{15} = -3,\ Z_{21} = -1, Z_{22} = -1,\ Z_{31} = -1,\ Z_{33} = -4,\ Z_{35} = -2$.
Since $Z_{12} > 0$, an allocation $\theta > 0$ is made in the cell (1, 2). The cells (1, 2), (3, 2), (3, 4), (2, 4), (2, 3), (1, 3) form a loop. The allocations in these cells are readjusted and Table 7.44 is obtained.

Table 7.44

	S_1	S_2	S_3	S_4	S_5	a_i
F_1	40 4	$\theta\downarrow$ 1	$60-\theta$ 2	 6	 9	100
F_2	 6	 4	$10+\theta$ 3	$20-\theta$ 5	90 7	120
F_3	 5	$50-\theta$ 2	 6	$70+\theta$ 4	 8	120
b_j	40	50	70	90	90	

Clearly $\theta = 20$ appropriately satisfies the supply-demand criteria and the resulting table is Table 7.45.

Table 7.45

	S_1	S_2	S_3	S_4	S_5	a_i
F_1	40 4	20 1	40 2	6	9	100
F_2	6	4	30 3	5	90 7	120
F_3	5	30 2	6	90 4	8	120
b_j	40	50	70	90	90	

Solving

$u_1 + v_1 = 4, u_1 + v_2 = 1, u_1 + v_3 = 2, u_2 + v_3 = 3, u_2 + v_5 = 7, u_3 + v_2 = 2, u_3 + v_4 = 4$ one gets $u_1 = 0$, $v_1 = 4$, $v_3 = 2$, $u_2 = 1$, $v_4 = 3$, $v_5 = 6$, $u_3 = 1$, $v_2 = 1$ and hence $Z_{14} = -3$, $Z_{15} = -3$, $Z_{21} = -1$, $Z_{22} = -2$, $Z_{24} = -1$, $Z_{31} = 0$, $Z_{33} = -3$, $Z_{35} = -1$. Since all net evaluations are non-positive therefore this solution $x_{11} = 40$, $x_{12} = 20$, $x_{13} = 40$, $x_{23} = 30$, $x_{25} = 90$, $x_{32} = 30$, $x_{34} = 90$ is optimal and the optimal transportation cost is 1400.

Let us try some exercises.

E4) Determine an optimal transportation program so that the transportation cost of 34 tons of a certain type of material from three factories A, B, C to five stores I, II, III, IV, V is minimized. The five stores must receive 4 tons, 5 tons, 2 tons, 9 tons and 12 tons respectively. The availability of the material at A, B and C is 1 tons, 12 tons and 12 tons respectively. The transportation cost per ton from factories to stores is given in Table 7.46.

Table 7.46

	I	II	III	IV	V
A	4	1	2	6	9
B	6	4	3	5	7
C	5	2	6	4	8

E5) A private company has manufacturing plants located in four cities A, B, C and D. Its market demands necessitate warehouses in five cities P, Q, R, S and T. Let the maximum capacities of the plants and the demand requirements of the warehouses be as shown in Table 7.47. In Table 7.46, the number in each cell represents the cost of transporting a unit from the particular plant to the given warehouse. Find the least cost transportation program so that all the demands can be met.

Table 7.47

Plant	P	Q	R	S	T	Plant capacity
A	1	2	6	2	3	800
B	3	4	5	8	1	600
C	3	1	1	2	6	200
D	4	7	3	5	4	400
Warehouse Demand	400	100	700	300	500	

E6) A farmer transfers truckloads of paddy from three different farms (F_1, F_2, F_3) to four mills (M_1, M_2, M_3, M_4). The supply and demand (both in number of truckloads) and the transportation costs per truckload are summarized in Table 7.48. Obtain the minimum transportation schedule.

Table 7.48

	M_1	M_2	M_3	M_4	Supply ↓
F_1	10	2	20	11	15
F_2	12	7	9	20	25
F_3	4	14	16	18	10
Demand →	5	15	15	15	

E7) A construction company is interested in taking loans from banks for its project named U, V, W, X, Y. The rates of interest and the lending capacities differ from bank to bank. The details are to be found in the table. How the company should take loads in order that the total interest payable may be the least? Are there any alternate optimal solutions? If so, indicate one such solution.

Table 7.49

	Interest rate in % for project					Max. credit (in thousands)
Bank	U	V	W	X	Y	
Private bank	20	18	18	17	17	Any amount
Nationalized bank	16	16	16	15	16	400
Co-operative bank	15	15	15	13	14	250
Amount required (in thousand)	200	150	200	125	75	

So far, we have discussed integer programming and transportation optimization model. In the following section, we shall focus on stochastic models with special reference to queueing models.

7.4 STOCHASTIC MODELS: QUEUEING MODELS

We are already accustomed to queueing models. Queueing theory is the mathematical study of queues or waiting lines encountered in scientific, technological, industrial or economic fields as also in our day-to-day life. It is essentially based on **stochastic process**, a time-oriented physical process that is controlled by a random mechanism. The basic concepts pertaining to queueing were discussed in an earlier course (MTE-14, No.4) The Bernoulli Process, Poisson Process, Markov Process and Markov Chain and the related topics were defined and explained with illustrations. The stochastic processes were applied to develop some queueing models such as $(M/M/1):(GD/\infty/\infty)$ and

$(M/M/)1):(GD/K/\infty)$. These are single server models based on Poisson (Markov) Process. In what follows we proceed to introduce a multi-channel queueing model. The model is explained with illustrations.

It may be worthwhile to recapitulate some of the basic concepts, definitions and examples underlying the procedure to model some of the queue phenomena.

Stochastic Process: A stochastic process is a family of random variables $\{X(t)/t \in T\}$, defined on a given probability space, indexed by the parameter t, where t varies over an index set T. The range space of $X(t) = X_i$ (say), may be discrete or continuous. The variables $X_1, X_2, X_3, \ldots$ might represent the number of customers awaiting service at a ticket booth at times 1 minute, 2 minutes, 3 minutes and so on, after the booth opens.

The values assumed by the random variable $X(t)$ are called **states** and the set of all possible values forms the **state space** of the process.

Markov Process: If $\{X(t)/t \in T\}$ is a stochastic process such that, given the value $X(s)$, the values of $X(t), t > s$, do not depend on the values of $X(u), u < s$, then the process is said to be a Markov process.

Mathematically this means, that if, for

$$t_1 < t_2 < t_3 < \ldots < t_n < t,$$

$$P\{a \le X(t) \le b / X(t) = x_1, \ldots, X(t) = x_n\}$$

$$= P\{a \le X(t) \le b / X(t_n) = x_n\}.$$

The process $\{X(t)/t \in T\}$ is a Markov process. A discrete-state Markov Process is known as a Markov Chain.

Markov Chain: Suppose, given the present state, the past states have no influences on the future. Then this property is said to be Markov Property. The systems satisfying this property are called Markov Chains. Mathematically, a stochastic process $X(t)(t = 0, 1, 2, 3, \ldots)$ with countable state space $I = (0, 1, 2, 3, \ldots)$ is said to be Markov Chain if for all states $i_0, i_1, i_2, \ldots, i_{n-1}, i, j \in I, n > 0$,

$$P\{X_{n+1} = j / X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0\}$$

$$= P\{X_{n+1} = j / X_n = i\}.$$

The sequence $\{X_n\}$ is said to possess Markov property. From the definition of Markov Chain, it is clear that the conditional distribution of the future state as a function of past and present states $i_0, i_1, i_2, \ldots, i(=i_n)$, depends only on the present state $i = i_n$.

Bernoulli Process: Suppose X_n, $(n \ge 1)$ is an outcome of the experiment of throwing an unbiased die. Then the sequence $\{X_n, n \ge 1\}$ represents a family of random variables for $n = 0, 1, 2, 3, \ldots$, and constitutes a stochastic process, the probability of each outcome x_n being $\frac{1}{6}$ in this case. This is said to be Bernoulli Process. The set of values of the random variable X_n is the state space of the stochastic process $\{X_n, n \geq 1\}$.

Poisson Process: Suppose $N(t)$ is the total number of incoming telephone calls at a switchboard in an interval of time t (that is from $t = 0$ to $t = t$). Then $N(t)$ is a stochastic process in continuous time with discrete state space; $N(t)$ follows Poisson distribution with mean λt, the Poisson law of probability being $p_n(t) = p\{N(t) = n\} = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$, $n = 0, 1, 2, 3, \ldots$ Poisson Process is Markovian. Discrete state Markov process constitutes a Markov chain. This means that $N(t)$ is independent of the number of occurrences of the event in an interval prior to the interval $(0, t)$. That is future changes of $N(t)$ are independent of the past changes.

The family or random variables $\{X(t), t \in T, T = [0, \infty]\}$ is a Poisson process. Then $X(1)$ is a random variable and $X(1) =$ number of telephone calls coming in an interval of unit time. The phenomenon of arrivals of customers for service at a counter is a stochastic process and a Poisson process at that.

Thus the Poisson process is a continuous-time, discrete-state stochastic process that is an excellent model for many practical situations. λ is called the arrival rate of the Poisson Process. The Poisson process plays the pivotal role in queueing theory.

7.4.1 Multi-channel Queueing Models

The single-channel models discussed earlier may cause queue length increase abnormally and customers may leave without taking the required service. Waiting time for the customers in the system will increase. This problem may be solved by increasing service facilities. Increasing the number of service centres may lessen the queue-length and the waiting time. Two multi-channel queueing formulations are discussed below.

Let us consider the multi-channel queueing model denoted symbolically by $(M/M/C):(GD/\infty/\infty)$. We assume the 'general-discipline'-(GD), to be FCFS-type, that is 'First-Come-First-Serve'-type. The number of service stations is C. The C stations are set in parallel and each customer in the waiting line can be served by any one server. Each service station is to offer identical service facility to the customer. The new arrival, coming from an infinite calling population, selects any station arbitrarily without any pressure or bias. Whenever there are more than C customers waiting for service the excess customers form a queue and wait until their turn at one of the C servers. The arrival rate of customers and service rate of individual channel are taken to be λ and μ respectively. These are, as we already know, the mean values from Poisson distribution and exponential distribution respectively. The customers are taken to a service station from a single queue. When a customer gets off from the station after receiving the service, the next customer in the line is to step in the state seeking service. A schematic diagram of the model is shown below.

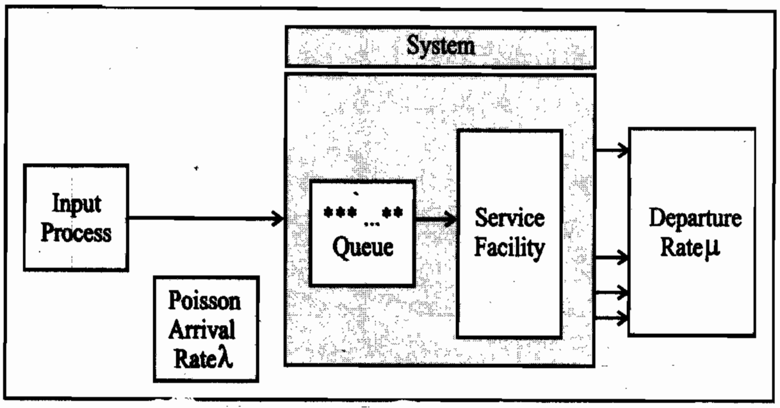

Fig. 7.3: Schematic Diagram: Multi-Channel Queue Model

Derivation of Probabilities, Computing Queue-length and Waiting-time

Let $\quad n =$ number of customers in the system

$p_n =$ probability of n customers in the system

$C =$ number of parallel service channels where $C > 1$.

Let $n < C$. In this case there is no queue as the service stations outnumber the customers. Therefore the rate of servicing will be $n\mu$, as only n channels are busy, each at the rate of μ. Also $(C-n)$ stations will remain empty. If $n = C$, then all channels will remain busy and there is no queue and the rate of service is $C\mu$. If $n < C$, then all channels will remain busy and there will be queue of $(n-C)$ customers. In this case the rate of service is $C\mu$.

It is now necessary to evaluate $p_n(t) =$ probability of n customers in the system at time t.

For $n = 0$, it can be shown that

$$p_0(t+dt) = p_0(t).(1-\lambda dt) + p_1(t).(1-\lambda dt).(\mu dt)$$
$$= p_0(t) - \lambda p_0(t)dt + \mu p_1(t)dt.$$

Hence $\dfrac{p_0(t+dt)-p_0(t)}{dt} = \mu p_1(t) - \lambda p_0(t)$.

Making $dt \to 0$, one gets $\dfrac{d}{dt}[p_0(t)] = \mu p_1(t) - \lambda p_0(t)$.

In the steady state system as $t \to \infty$, $p_i(t) \to p_i$, we have $p_1 = \dfrac{\lambda}{\mu} p_0$. (7)

For $1 \le n \le C-1$, one gets

$$p_n(t+dt) = p_n(t).(1-\lambda dt)\,(1-\mu dt) + p_{n-1}(t).\lambda dt.[1-(n-1)\mu dt]$$
$$+ p_{n+1}(t).(1-\lambda dt).[(n+1)\mu dt]$$

Hence, neglecting $(dt)^2$ and making $dt \to 0$ as before, one gets for the steady state system

$$(n+1)\mu p_{n+1} - (\lambda + n\mu)p_n + \lambda p_{n-1} = 0. \quad (8)$$

Taking $n=1$, and using Eqn.(7), one gets on simplification,

$$p_2 = \frac{1}{2!}\left(\frac{\lambda}{\mu}\right)^2 p_0 .$$

Proceeding in a similar manner it can be established that, in general,

$$p_n = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n p_0 . \tag{9}$$

Finally the case **for** $n \geq C$ is considered. Substituting $n = C-1$ in Eqn.(8), one derives

$$p_C = \frac{1}{C\mu}[\lambda + (C-1)\mu]\, p_{C-1} - \frac{\lambda}{C\mu} p_{C-2} . \tag{10}$$

From Eqn.(9), $p_{C-1} = \frac{1}{(C-1)!}\left(\frac{\lambda}{\mu}\right)^{C-1} p_0$, $p_{C-2} = \frac{1}{(C-2)!}\left(\frac{\lambda}{\mu}\right)^{C-2}$.

Using these relations one gets from Eqn.(9), after some simplification,

$$p_C = \frac{1}{C!}\left(\frac{\lambda}{\mu}\right)^C p_0 . \tag{11}$$

Substituting $n = C+1$ in Eqn.(8) and proceeding as before, one can show that

$$p_{C+1} = \frac{1}{C.C!}\left(\frac{\lambda}{\mu}\right)^{C+1} p_0 ,$$

$$p_{C+2} = \frac{1}{C^2}\frac{1}{C!}\left(\frac{\lambda}{\mu}\right)^2 . p_0 \text{, etc.,}$$

and in general, $p_n = \frac{1}{C^{n-c}.C!}\left(\frac{\lambda}{\mu}\right)^n p_0,\ n \geq C$. (12)

Next, p_0 need to be determined using the relation $\sum_{n=0}^{n=\infty} p_n = 1$. Rewriting one gets

$$\left(\sum_{n=0}^{n=C-1} p_n + \sum_{n=C}^{n=\infty} p_n\right) = 1 .$$

Hence $p_0 \cdot \left[\sum_0^{C-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{C^C}{C!}\sum_c^{\infty}\left(\frac{\lambda}{C\mu}\right)^n\right] = 1$

$$p_0 \cdot \left[\sum_0^{C-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{C^C}{C!}\left(\frac{\lambda}{C\mu}\right)^C \left\{1 + \frac{\lambda}{C\mu} + \left(\frac{\lambda}{C\mu}\right)^2 + \left(\frac{\lambda}{C\mu}\right)^3 + \cdots\infty\right\}\right] = 1$$

using $1 + \frac{\lambda}{C\mu} + \left(\frac{\lambda}{C\mu}\right)^2 + \left(\frac{\lambda}{C\mu}\right)^3 + \cdots\infty = \left(1 - \frac{\lambda}{C\mu}\right)^{-1}$ and simplifying, one gets

ultimately $$p_0 = \left[\left\{\sum_{n=0}^{n=C-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right\} + \frac{\mu}{(C-1)!\,(C\mu - \lambda)}\left(\frac{\lambda}{\mu}\right)^C\right]. \tag{13}$$

The various parameters of the C-multi-channel model can now be written using results given in Eqn.(7) to Eqn.(13) in the following way.

$$L_s = \frac{\lambda\mu}{(C-1)!(C\mu-\lambda)^2}\left(\frac{\lambda}{\mu}\right)^C \cdot p_0 + \frac{\lambda}{\mu} = \text{expected number of customers in the}$$

system,

where

λ = arrival rate of customers,

μ = service rate of individual channel,

C = number of parallel service channels, $C > 1$

n = number of customers in the systems

p_0 = probability of having no customer in the system,

$$L_q = L_s - \text{average number of customers being served} = L_s - \frac{\lambda}{\mu}$$

$= \text{expected number of customers waiting in the queue,}$

$$W_s = \frac{L_s}{\lambda} = \text{average time a customer spends in the system,}$$

$$W_q = \frac{L_q}{\lambda} = \text{expected waiting time of a customer in the queue,}$$

$$p(n \geq C) = \frac{\mu}{(C-1)!\,(C\mu-\lambda)}\left(\frac{\lambda}{\mu}\right)^c \cdot p_0$$

$= \text{probability that a customer has to wait,}$

$1 - p(n \geq C) =$ probability that a customer enters the service without waiting

Average number of servers remaining idle $= C -$ (average number of customers served), and $\frac{\lambda}{C\mu} =$ utilization rate,

Efficiency of the model = ratio of (average number of customers served) to (total number of customers).

These results will now enable us to solve some realistic problems described below.

Example 11: A telephone exchange has two long distance operators. The telephone company observes that during the peak load, long distance calls arrive in a Poisson fashion at an average rate of 15 per hour. The length of service on these calls is approximately exponentially distributed with mean length 5 minutes.

i) What is the probability that a subscriber will have to wait for his long distance call during the peak hours of the day?

ii) If the subscriber will wait and serviced in turn, what is the expected waiting time?

Solution: Here $\lambda = 15$ per hour, $\mu = \frac{1}{5} \times 60 = 12$ per hour, $C = 2$. Hence $\frac{\lambda}{\mu} = \frac{5}{4}$.

$$\therefore\; p_0 = \left[1 + \frac{\lambda}{\mu} + \frac{1}{2}\left(\frac{\lambda}{\mu}\right)^2 \frac{C\mu}{C\mu - \lambda}\right]^{-1} = \left[1 + \frac{5}{4} + \frac{1}{2}\cdot\frac{25}{16}\cdot\frac{24}{24-15}\right]^{-1} = \frac{3}{13}.$$

Thus, (i) Probability that a subscriber has to wait

$$= \frac{\mu}{(C-1)!\,(C\mu-\lambda)}\left(\frac{\lambda}{\mu}\right)^C . p_0 = \frac{12}{24-15}.\frac{25}{16}.\frac{3}{13} = 0.48\,.$$

(ii) Expected waiting time of the subscriber in the queue

$$= \frac{\mu}{(C-1)!\,(C\mu-\lambda)^2}.\left(\frac{\lambda}{\mu}\right)^2 . p_0$$

$$= \frac{12}{(24-15)^2}.\frac{25}{16}.\frac{25}{468}\text{ hours}$$

$= 3.2$ minutes.

Example 12: Four counters are being run on the frontier of a country to check the passports and necessary papers of the tourists. The tourists choose a counter at random. If arrivals are Poisson at the rate λ and the service time is exponential with parameter $\frac{1}{2}\lambda$, what is the steady state average queue at each counter?

Solution: Here $C = 4,\ \lambda = \lambda,\ \mu = \frac{1}{2}\lambda,\ \frac{\lambda}{\mu} = 2$.

$$p_0 = \left[1+\frac{\lambda}{\mu}+\frac{1}{2}\left(\frac{\lambda}{\mu}\right)^2+\frac{1}{6}\left(\frac{\lambda}{\mu}\right)^3+\frac{C\mu}{C!\,(C\mu-\lambda)}\left(\frac{\lambda}{\mu}\right)^C\right]^{-1}$$

$$= \left[1+2+2+\frac{4}{3}+\frac{2\lambda}{24(2\lambda-\lambda)}.16\right]^{-1}$$

$$= \frac{3}{23}.$$

Therefore, expected number of customers waiting in the queue,

$$L_q = \frac{\lambda\mu}{(C-1)!\,(C\mu-\lambda)^2}\left(\frac{\lambda}{\mu}\right)^C . p_0 = \frac{\frac{\lambda^2}{2}}{6\,(2\lambda-\lambda)^2}.16.\frac{3}{23}$$

$$= \frac{4}{23}.$$

Example 13: Patients arrive at the outpatient department of a hospital in accordance with a Poisson process at the mean rate of 12 per hour and the distribution of time for medical examination by an attending physician is exponential with a mean of 10 minutes. What is the minimum number of physicians to be posted for ensuring a steady-state distribution? For this number, calculate (i) the expected waiting time of a patient prior to being examined, (ii) the expected number of patients in the out-patient department and (iii) the expected number of physicians remaining idle. Also find the average time a patient has to spend in the out-patient department.

Solution: Here $\lambda = 12$ hour, $\mu = 6$ hour and hence $\frac{\lambda}{\mu} = 2$. Let C be the minimum number of physicians satisfying the steady state condition $C\mu - \lambda > 0$. This leads to $C > 2$. Hence the minimum value of $C = 3$.

i) The expected waiting time of a

$$\text{patient} = W_q = \frac{\mu}{(C-1)!(C\mu-\lambda)^2}.\left(\frac{\lambda}{\mu}\right)^C .p_0.$$

$$p_0 = \left[1+\frac{\lambda}{\mu}+\frac{1}{2}\left(\frac{\lambda}{\mu}\right)^2+\frac{C\mu}{C!(C\mu-\lambda)}.\left(\frac{\lambda}{\mu}\right)^C\right]^{-1}$$

$$= \left[1+2+2+\frac{18}{6.6}.8\right]^{-1} = \frac{1}{9}.$$

$$\therefore\ W_q = \frac{6}{2.36}.8.\frac{2}{27}\text{hour} = 4.4\,\text{minutes}.$$

ii) Expected number of patients in the system

$$= L_s = \frac{\lambda\mu}{(C-1)!\,(C\mu-\lambda)^2}.\left(\frac{\lambda}{\mu}\right)^2 .p_0 + \frac{\lambda}{\mu}$$

$$= \frac{12.6}{2.36}.8.\frac{1}{9}+2 = \frac{26}{9}.$$

iii) Expected number of physicians remaining idle

$$= 3p_0 + 2p_1 + p_1$$

$$= 3.\frac{1}{9}+2.\frac{2}{9}+1.\frac{1}{2}.\frac{4}{9} = 1,\ \text{since } p_n = \frac{1}{n!}.\left(\frac{\lambda}{\mu}\right)^n .p_0,\ n < C.$$

Again, expected time a patient has to spend in the system

$$W_s = \frac{\mu}{(C-1)!\,(C\mu-\lambda)^2}.\left(\frac{\lambda}{\mu}\right)^C .p_0 + \frac{1}{\mu}$$

$$= \frac{6}{2.36}.8.\frac{1}{9}+\frac{1}{6} = \frac{13}{54}\ \text{hour} = 14.4\ \text{minutes}.$$

Example 14: A community is served by two cab companies. Each company owns two cabs and the two companies are known to have equal shares of the market. This is evident by the fact that calls arrive at each company's dispatching office at the rate of eight per hour. The average time per ride is twelve minutes. Calls arrive according to a Poisson distribution and the ride time is exponential. The two companies were recently bought an investor who is interested in consolidating them into a single dispatching office to provide better service to customers.

Comment on the new owner's proposal.

Solution: From the standpoint of queueing theory, the cabs are the servers and the cab ride is the service. Therefore the original situation presents a $(M/M/2):(GD/\infty/\infty)$ queueing model. In this case

$$\lambda = 8 \text{ calls per hour},\ \mu = \frac{60}{12} = 5 \text{ rides per cab per hour},\ \frac{\lambda}{\mu} = \frac{8}{5} \text{ and } C = 2.$$

Therefore, average waiting time of a customer in the queue

$$W_q = \frac{\mu}{r(C-1)!\,(C\mu-\lambda)^2}.\left(\frac{\lambda}{\mu}\right)^C .p_0$$

where $p_0 = \left[1+\frac{\lambda}{\mu}+\frac{C\mu}{C!\,(C\mu-\lambda)}.\left(\frac{\lambda}{\mu}\right)^C\right]^{-1} = \left[1+\frac{8}{5}+\frac{1}{2}.\frac{10}{2}.\frac{64}{25}\right]^{-1} = \frac{1}{9}$ hours.

$\therefore\ W_q = \frac{5}{4}.\frac{64}{25}.\frac{1}{9} = 0.356$ hours $= 21.3$ minutes.

In the consolidated case, the problem is converted into a $(M/M/4):(GD/\infty/\infty)$ queueing model where $\lambda = 16$ calls per hour, $\mu = 5$ rides per cab per hour and $C = 4$. Hence

$$p_0 = \left[1+\frac{\lambda}{\mu}+\frac{1}{2}\left(\frac{\lambda}{\mu}\right)^2+\frac{1}{6}\left(\frac{\lambda}{\mu}\right)^3+\frac{C\mu}{C!\,(C\mu-\lambda)}\left(\frac{\lambda}{\mu}\right)^C\right]^{-1}$$

$$= \left[1+\frac{16}{5}+\frac{1}{2}.\frac{256}{25}+\frac{1}{6}.\frac{4096}{125}+\frac{20}{24.4}.\frac{16^4}{5^4}\right]^{-1}$$

$$= .0273.$$

$\therefore\ W_q = \frac{5}{6.16}.\frac{16^4}{5^4}.(.0273) = .149$ hours $= 9$ minutes.

The consolidation remarkably lowers the waiting time.

Let us now try the following exercises.

E8) A tax consulting firm has 3 counters in its office to receive people who have problems concerning their incomes, wealth and sales taxes. On the average 48 persons arrive in an 8-hour day. Each tax adviser spends 15 minutes on an average on an arrival. If the arrivals are Poissonly distributed and service times are according to exponential distribution, find

i) average number of customers in the system;

ii) average number of customers waiting to be served;

iii) average time a customer spends in the system;

iv) average waiting time for a customer;

v) the number of hours each week a tax adviser spends performing his job;

vi) the probability that a customer has to wait before he gets service;

vii) the expected number of idle tax advisers at any specified time.

E9) A supermarket has two girls ringing up sales at the counters. If the service time for each customer is exponential with mean 4 minutes and if people arrive in a Poisson fashion at the rate of 10 per hour, find

i) the probability of having to wait for the service;

ii) the expected percentage of idle time for each girl;

iii) the average queue length and the average number of units in the system.

7.4.2 Multiple Server Queueing Model with System Limit: $(M/M/C):(GD/N/\infty), C \le N$

Here the system limit is finite and is equal to N, the calling population remaining infinite. Hence the queue size, in this case, is limited to $N-C$; the arrival rate and service rate are λ and μ respectively. Hence comparing with generalized queueing model, one gets

$$\lambda_n = \lambda,\ 0 \le n \le N \text{ and } \lambda_n = 0,\ n \ge N;$$

$$\mu_n = n\mu,\ 0 \le n < C \text{ and } \mu_n = C\mu,\ C \le n \le N.$$

Writing $\rho \frac{\lambda}{\mu}$, it can be shown that

$$p_n = \frac{\rho^n}{n!} \cdot p_0,\ 0 \le n < C$$

$$= \frac{\rho^n}{C!\, C^{n-C}} \cdot p_0,\ C \le n \le N$$

$$\text{where } p_0 = \left[\sum_{n=0}^{n=C-1} \frac{\rho^n}{n!} + \frac{\rho^C \left(1 - \left(\frac{\rho}{C}\right)^{N-C+1}\right)}{C!\left(1 - \frac{\rho}{C}\right)} \right]^{-1},\ \rho \ne C,$$

$$= \left[\sum_{n=0}^{n=C-1} \frac{\rho^n}{n!} + \frac{\rho^n}{C!}(N-C+1) \right]^{-1},\ \rho = C$$

Hence L_q = expected or average number of customers waiting in the queue

$$= \frac{\rho^{C+1}}{(C-1)!\,(C-\rho)^2} \left[1 - \left(\frac{\rho}{C}\right)^{(N-C+1)} - (N-C+1)\left(1-\frac{\rho}{C}\right)\left(\frac{\rho}{C}\right)^{N-C} \right] p_0,\ \rho \ne C$$

For $\rho = C$, $L_q = \frac{\rho^C (N-C)(N-C+1)}{2C!} \cdot p_0$

Now considering that, in this case, the effective arrival rate λ_{eff} is less than λ, because of the system limit N, one derives that $\lambda_{lost} = \lambda \rho_N$. Then λ_{eff} is represented by

$$\lambda_{eff} = \lambda - \lambda_{lost}$$

$$= \lambda(1 - \rho_N).$$

Therefore, expected or average number of customers in the system

$$= L_s = L_q + (1 - \rho_N)\frac{\lambda}{\mu}.$$

Similarly, expected or average waiting time of a customer in the queue.

$$= W_q = \frac{L_q}{\lambda(1-\rho_N)}.$$

Also, expected or average time a customer spends in the system

$$= W_s = \frac{L_s}{\lambda(1-\rho_N)}.$$

This multi-server model with system limit N, will now be used to solve some typical queueing problems.

Examples 15: Let there be an automobile inspection situation with three inspection stalls. Assume that cars wait in such a way that when a stall becomes vacant, the car at the head of the line pulls upto it. The station can accommodate at most four cars waiting (seven in the station) at one time. The arrival pattern is Poisson with a mean of one car every minute during the peak hours. The service time is exponential with mean six minutes. Find the average number of customers in the system during peak hours, the average waiting time and the average number per hour that cannot enter the station because of full capacity.

Solution: Here $\lambda = 1$ minute, $\mu = \frac{1}{6}$ minute, $C = 3, N = 7$. Also $\rho = \frac{\lambda}{\mu} = 6$.

$$\therefore\ p_0 = \left[\sum_{n=0}^{n=C-1} \frac{1}{n!}.\rho^n + \frac{\left\{1-\left(\frac{\rho}{C}\right)^{N-C+1}\right\}}{C!\left(1-\frac{\rho}{C}\right)}.\rho_C\right]^{-1}, \frac{\rho}{C} \neq 1$$

$$= \left[1+\rho+\frac{1}{2}\rho^2 + \frac{1-2^5}{3!\,(1-2)}.6^3\right]^{-1} = [1+6+18+36\times 31]^{-1} = \frac{1}{1141}.$$

Expected or average number of customers in the system

$$= L_s = L_q + (1-\rho_N)\frac{\lambda}{\mu}$$

where

$$L_q = \frac{\rho^{C+1}}{(C-1)!\,(C-\rho)^2}\left[1-\left(\frac{\rho}{C}\right)^{(N-C+1)} - (N-C+1)\left(1-\frac{\rho}{C}\right)\left(\frac{\rho}{C}\right)^{N-C}\right]\rho_0$$

$$= \frac{6^4}{2!\,3^2}\{1-2^5-5(-1)2^4\}.\frac{1}{1141} = \frac{3528}{1141}.$$

$$\therefore\ L_s = L_q + (1-\rho_N)\frac{\lambda}{\mu} = \frac{3528}{1141} + \left(1-\frac{1}{3^4\times 6}\times 6^7 \times \frac{1}{1141}\right)\times 6$$

$$= 3.0920 + 2.9712 = 6.0632.$$

And the average time a customer spends in the system

$$= W_s = \frac{L_s}{\lambda(1-\rho_N)} = \frac{6.0632}{\left(1-\frac{576}{1141}\right)} = 12.2444 \text{ minutes.}$$

The expected number of cars per hour that cannot enter the station (that is customer loss) $= 60\lambda\rho_N = 60\times 1\times .5048 = 30.2892$ cars per hour.

Example 16: Let us again consider the Example 15 discussed above. Let the dispatching office inform the new customers of potential excessive delay once the waiting list reaches six customers. This move is certain to get new customers to seek service elsewhere but will reduce the waiting time for those on the waiting list. Compute the new waiting time.

Solution: Here $N = 6+4 = 10$ is the system limit and the new situation is represented by the Queueing Model: $(M/M/4):(GD/10/\infty)$. Also $\lambda = 16$ customers per hour, $\mu = 5$ rides per hour, $C = 4$, $\rho = \frac{\lambda}{\mu} = 3.2$, $\frac{\rho}{C} = .8$.

Then average waiting time of a customer in the queue

$$= W_q = \frac{L_q}{\lambda(1-\rho_N)} \quad W_q = \frac{36.98705038}{32.453681815.42814844} = 0.074811889$$

where L_q = expected or average number of customers waiting in the queue

$$= \frac{\rho^{C+1}}{(c-1)!(C-\rho)^2}\left[1-\left(\frac{\rho}{C}\right)^{(N-C+1)} - (N-C+1)\left(1-\frac{\rho}{C}\right)\left(\frac{\rho}{C}\right)^{N-C}\right]\rho_0,\ \rho \neq C$$

$$= \frac{3.2^5}{6\times .8^2}[1-.8^7-7(1-.8)\times .8^6]\,\rho_0 = 36.98705038\,\rho_0$$

$$\rho_0 = \left[1+3.2+\frac{1}{2}\times 3.2^2+\frac{1}{6}\times 3.2^3+\frac{3.2^4}{24\times .2}\{1-.8^7)\right]^{-1} = \frac{1}{32.04536818},$$

and $$\lambda_{eff} = \lambda(1-\rho_{10}) = 16\left[1-\frac{3.2^{10}}{24\times 4^6}\times\frac{1}{32.04536818}\right] = 15.42814844.$$

Therefore,

$$W_q = \frac{L_q}{\lambda(1-\rho_N)} = \frac{36.98705038}{32.04536818\times 15.42814844} = 0.074811889 \text{ hours}$$

$$= 4.5 \text{ minutes (approximately).}$$

This shows that (comparing with the results of Example 4 above) the waiting time for the customers in the queue have really been reduced to half by incorporating the system limit.

It has been observed that different queueing models will have to be adopted to tackle the various types of waiting line situations in the real life. A single model cannot address the peculiarities appearing in varieties of situations. However, the models are derived by suitably altering the parameters of the basic structure of the model defined by $(a/b/c):(d/e/f)$. Thus, a good number of models really come into the picture to address special type of situations. Besides these, some special models have to be derived under different probability distributions to explain various real life phenomena.

In multi-server models an effort is made to reduce waiting time in the queue and thereby attracting a larger number of customers only by agreeing to maintain a larger number of service-counters. The service-providers have to calculate if enhanced cost on account of maintaining additional service-centres can be absorbed by larger inflow of customers. Further, percentage utilization of service centres must be maintained as high as possible in order to maximize profit.

Let us try some exercises.

E10) A barber shop has two barbers and three chairs for customers. Assume that customers arrive in a Poisson fashion at a rate of 5 per hour and that each barber services customers according to an exponential distribution with mean of 15 minutes. Further, if a customer arrives and there are no empty chairs in the shop he will leave. (i) Find the steady state probabilities. (ii) What is the probability that the shop is empty? (iii) What is the expected number of customers in the shop?

E11) There are three docks in a harbour. The arrival rate of ships follow Poisson distribution and it is 36 ships per month. The service rate (loading and unloading of containers) also follows Poisson distribution and it is 13 ships per month. The average space in the harbour can accommodate a maximum of 10 ships. Find: (i) average waiting number of ships in the queue and in the system and (ii) average waiting time for a ship in the queue and in the system.

E12) Telephone users arrive at a booth following a Poisson distribution with an average time of 5 minutes between one arrival and the next. The time taken for a telephone call is, on an average, 3 minutes and it follows an exponential distribution. What is the probability that the telephone booth is busy? How many more booths should be established to reduce the system waiting time to less than or equal to half the present waiting time?

E13) An insurance company has three claim adjusters in its branch office. People with claim against the company are found to arrive in a Poisson fashion, at an average rate of 0 per 8-hour day. The amount of time that an adjuster takes is found to have exponential distribution with mean service time 40 minutes. Claimants are processed in order of their appearance.

i) How many hours a week can an adjuster expect to spend with claimants?

ii) How much time, on the average, does a claimant spend in the branch office?

We now end this unit by giving a summary of what we have covered in it.

7.5 SUMMARY

In this unit, we have covered the following:

1. The "Gomary Plane Cut Method" is an iterative method. Constraints of equation-type or $\leq$ and or $\geq$ type do not pose any special problems in the procedure for realizing optimal integer solution. The simplex procedure to obtain a non-integer optimal solution under various types of constraints is already known to us. Every iterative step in Gomory method defines and introduces a new variable making the computation table larger and larger.

2. The Branch and Bound Method is also an iterative method. The main difference in between the two methods is that the branching operations introduce new additional restrictions on the original variables only. Thus the branching operations are less cumbrous and appear to be more elegant and straightforward. The optimal integer point however can be determined unambiguously by both the methods.
3. The various methods of finding an initial basic feasible solution of a transportation model are discussed, and then optimum solutions are found. It is not predictable if the VAM or least cost method will enable one to attain optimality by going through lesser number of iterations. Sometimes the most short-cut method of finding an initial basic feasible solution may have to be adopted.
4. Also, primarily the multi-channel queueing formulations under different systems have been developed and discussed. The Markovian process has been defined in details as all the formulations are based on Markov process. The probability distributions controlling arrival and departure phenomena have been explained. Real life situations have been modeled into queues and the appropriate formulations have been selected for solution of the problems. The essence of mathematizing procedure followed by construction and solution of a model has been emphasized.

7.6 SOLUTIONS/ANSWERS

E1) Let x_1 = number of technicians to be hired
and x_2 = number of apprentices to be hired.
Productivity of a technician = 8 units per hour
Productivity of an apprentice = 3 units per hour
The objective is to maximize productivity $Z = 8x_1 + 3x_2$, subject to the

i) constraint for manhours: $6x_1 + 4x_2 \le 25$

ii) constraint for charges $8 \times 20x_1 + 4 \times 8x_2 \le 440$
or, $160x_1 + 32x_2 \le 440$

iii) non-negative integer constraints $x_1, x_2 \ge 0$ are integers.

E2) Let x_1, x_2 be the number of dolls of X, Y types respectively to be manufactured. Then the L.P.P consists in maximizing profit $Z = x_1 + x_2$, subject to be constraints $2x_1 + 5x_2 \le 16$, $6x_1 + 5x_2 \le 30$, $(x_1, x_2) \ge 0$.
The non-integer optimal solution is obtained in the usual simplex method.

C_B	y_B	$c \rightarrow$ $x_B \downarrow$	1 y_1	1 y_2	0 y_3	0 y_4
1	y_2	$\frac{9}{5}$	0	1	$\frac{3}{10}$	$-\frac{1}{10}$
1	y_1	$\frac{7}{2}$	1	0	$-\frac{1}{4}$	$\frac{1}{4}$
		$\frac{53}{10}$	0	0	$\frac{1}{20}$	$\frac{3}{20}$

With a view to achieve optimal integer solution, Gomory's fractional cuts are added in successive iterations.

C_B	y_B	$c \rightarrow$ $x_B \downarrow$	1 y_1	1 y_2	0 y_3	0 y_4	0 s_1
1	y_2	$\frac{9}{5}$	0	1	$\frac{3}{10}$	$-\frac{1}{10}$	0
1	y_1	$\frac{7}{2}$	1	0	$-\frac{1}{4}$	$\frac{1}{4}$	0
0	S_1	$-\frac{4}{5}$	0	0	$-\frac{3}{10}$	$-\frac{9}{10}$ $* \uparrow$	$1 \rightarrow$
		$\frac{53}{10}$	0	0	$\frac{1}{20}$	$\frac{3}{20}$	0

C_B	y_B	$c \rightarrow$ $x_B \downarrow$	1 y_1	1 y_2	0 y_3	0 y_4	0 s_1
1	y_2	$\frac{17}{9}$	0	1	$\frac{1}{3}$	0	$-\frac{1}{9}$
1	y_1	$\frac{59}{18}$	1	0	$-\frac{1}{3}$	0	$\frac{5}{18}$
0	y_4	$\frac{8}{9}$	0	0	$\frac{1}{3}$	1	$-\frac{10}{9}$
		$\frac{31}{6}$	0	0	0	0	$\frac{1}{6}$

C_B	y_B	$c \rightarrow$ $x_B \downarrow$	1 y_1	1 y_2	0 y_3	0 y_4	0 s_1	0 s_2
1	y_2	$\frac{17}{9}$	0	1	$\frac{1}{3}$	0	$-\frac{1}{9}$	0
1	y_1	$\frac{59}{18}$	1	0	$-\frac{1}{3}$	0	$\frac{5}{18}$	0
0	y_4	$\frac{8}{9}$	0	0	$\frac{3}{1}$	1	$-\frac{10}{9}$	0
0	s_2	$-\frac{8}{9}$	0	0	$-\frac{1}{3}$ $* \uparrow$	0	$-\frac{8}{9}$	$1 \rightarrow$
		$\frac{31}{6}$	0	0	0	0	$\frac{1}{6}$	0

C_B	y_B	$c \rightarrow$ $x_b \downarrow$	1 y_1	1 y_2	0 y_3	0 y_4	0 s_1	0 s_1
1	y_2	1	0	1	0	0	-1	1
1	y_1	$\frac{25}{6}$	1	0	0	0	$\frac{7}{6}$	-1
0	y_4	0	0	0	0	1	-2	1
0	y_3	$\frac{8}{3}$	0	0	1	0	$\frac{8}{3}$	-3
		$\frac{31}{6}$	0	0	0	0	$\frac{1}{6}$	0

C_B	y_B	$c \rightarrow$ / $x_B \downarrow$	y_1 (1)	y_2 (1)	y_3 (0)	y_4 (0)	s_1 (0)	s_2 (0)	s_3 (0)
1	y_2	1	0	1	0	0	-1	1	0
1	y_1	$\frac{25}{6}$	1	0	0	0	$\frac{7}{6}$	-1	0
0	y_4	0	0	0	0	1	-2	1	0
0	y_3	$\frac{8}{3}$	0	0	1	0	$\frac{8}{3}$	-3	0
0	s_3	$-\frac{2}{3}$	0	0	0	0	$-\frac{2}{3}*\uparrow$	0	$1 \rightarrow$
		$\frac{31}{6}$	0	0	0	0	$\frac{1}{6}$	0	0

C_B	y_B	$c \rightarrow$ / $x_B \downarrow$	y_1 (1)	y_2 (1)	y_3 (0)	y_4 (0)	s_1 (0)	s_2 (0)	s_3 (0)
1	y_2	2	0	1	0	0	0	1	$-\frac{3}{2}$
1	y_1	3	1	0	0	0	0	-1	$\frac{7}{4}$
0	y_4	2	0	0	0	1	0	1	-3
0	y_3	0	0	0	1	0	0	-3	4
0	s_1	1	0	0	0	0	1	0	$-\frac{3}{2}$
		5	0	0	0	0	0	0	$\frac{1}{4}$

The optimal integer solution is $x_1 = 3, x_2 = 2$, max.$Z = 5$.

E3) The first non-integer solution can be found to be $\left(x_1 = \frac{8}{3}, x_2 = \frac{8}{3}, x_3 = 0, \text{max}.Z = 16\right)$. It generates two sub-problems I and II:

Sub-problem I

Maximize $Z = 3x_1 + 3x_2 + 13x_3$

Subject to $-3x_1 + 6x_2 + 7x_3 \le 8$

$6x_1 - 3x_2 + 7x_3 \le 8$

$x_1 \ge 3$

$x_1, x_2, x_3 \ge 0$ and are integers.

Sub-problem II

Maximize $Z = 3x_1 + 3x_2 + 13x_3$

Subject to $-3x_1 + 6x_2 + 7x_3 \le 8$

$6x_1 - 3x_2 + 7x_3 \le 8$

$x_1 \le 2$

$x_1, x_2, x_3 \ge 0$ and are integers.

Sub-problem II has the optimal non-integer solution $\left(x_1 = 2, x_2 = 2, x_3 = \frac{2}{7}, \text{max}.Z = \frac{110}{7}\right)$. This solution of Sub-problem II generates two Sub-problems III and IV represented by

Sub-problem III

Maximize $Z = 3x_1 + 3x_2 + 13x_3$

Subject to $-3x_1 + 6x_2 + 7x_3 \le 8$

$6x_1 - 3x_2 + 7x_3 \le 8$

$x_1 \ge 2, x_3 \ge 1$

Sub-problem IV

Maximize $Z = 3x_1 + 3x_2 + 13x_3$

Subject to $-3x_1 + 6x_2 + 7x_3 \le 8$

$6x_1 - 3x_2 + 7x_3 \le 8$

$x_1 \le 2, x_3 \le 0$

$x_1, x_2, x_3 \geq 0$ and are integers. $\quad$ $x_1, x_2, x_3 \geq 0$ and are integers.

- Sub-Problem III offers the solution $\left(x_1 = \frac{1}{3}, x_2 = \frac{1}{3}, x_3 = 1, \max.Z = 15\right)$.

There are two branches (i.e., Sub-problems V and VI) of this solution corresponding to arbitrarily chosen non-integer solution $x_2 = \frac{1}{3}$.

Sub-problem V

Maximize $Z = 3x_1 + 3x_2 + 13x_3$

Subject to $-3x_1 + 6x_2 + 7x_3 \leq 8$

$6x_1 - 3x_2 + 7x_3 \leq 8$

$x_1 \leq 2,\ x_2 \geq 1,\ x_3 \geq 1$

$x_1, x_2, x_3 \geq 0$ and are integers.

Sub-problem VI

Maximize $Z = 3x_1 + 3x_2 + 13x_3$

Subject to $-3x_1 + 6x_2 + 7x_3 \leq 8$

$6x_1 - 3x_2 + 7x_3 \leq 8$

$x_1 \leq 2,\ x_2 \leq 0,\ x_3 \geq 1$

$x_1, x_2, x_3 \geq 0$ and are integers.

Sub-Problem VI gives the solution $\left(x_1 = 0,\ x_2 = 0,\ x_3 = \frac{8}{7},\ \max.Z = \frac{104}{7}\right)$. The branches of this solution are Subproblems VII and VIII.

Sub-problem VII

Maximize $Z = 3x_1 + 3x_2 + 13x_3$

Subject to $-3x_1 + 6x_2 + 7x_3 \leq 8$

$6x_1 - 3x_2 + 7x_3 \leq 8$

$x_1 \leq 2,\ x_2 \geq 0,\ x_3 \geq 2$

$x_1, x_2, x_3 \geq 0$ and are integers.

Sub-problem VIII

Maximize $Z = 3x_1 + 3x_2 + 13x_3$

Subject to $-3x_1 + 6x_2 + 7x_3 \leq 8$

$6x_1 - 3x_2 + 7x_3 \leq 8$

$x_1 \leq 2,\ x_2 \leq 0,\ x_3 \leq 1$

$x_1, x_2, x_3 \geq 0$ and are integers.

The solution of Sub-problem VIII finally gives the optimal integer solution $(x_1 = 0, x_2 = 0, x_3 = 1, \max.Z = 13)$.

E4) Matrix minima method is used to get the initial basic feasible solution:

	I	II	III	IV	V
A	 4	50 1	50 2	 6	 9
B	10 6	 4	20 3	 5	90 7
C	30 5	 2	 6	90 4	 8

The cost is 1410, $Z_{11} = 1$ and all other $Z_{ij} = u_i + v_j - c_{ij} \leq 0$. Allocated $\theta > 0$ to the cell $(1, 1)$.

θ 4	50 1	$50-\theta$ 2	 6	 9	Let $\theta = 10$	10 4	50 1	40 2	 6	 9
$10-\theta$ 6	 4	$20+\theta$ 1	 5	90 7	T3 $\rightarrow$	 6	 4	30 3	 5	90 7
30 5	 2	 6	90 4	 8		30 5	 2	 6	90 4	 8

All $Z_{ij} \leq 0$. Therefore this is the optimal table. Optimal cost $= 1400$.

E5) The initial basic feasible solution is obtained by matrix minima method (T1).

Here Cost $= 4000$ and $Z_{12} = 4, Z_{22} = 1$.

Allocate $\theta > 0$ in (1, 2) of T2.

Choose $\theta = 100$ in T2. Get T3. Allocate θ in (2, 2) of T4.

From the loop and choose $\theta = 0$ and obtain the optimal table. Minimum cost $= 3600$.

T1 ↓

400 1	 2	100 6	300 2	 3	T2 →	400 1	θ 2	$100-\theta$ 2	300 2	 3
 3	 4	100 5	 8	500 1		 3	 4	100 5	 8	500 7
 3	100 1	100 1	 2	 6		 3	$100-\theta$ 1	$100+\theta$ 1	 2	 6
 4	 7	400 3	 5	 4		 4	 7	400 3	 5	 4

T3 ↓

400 1	100 2	 6	300 2	 3	T4 →	400 1	100 2	 6	300 2	 3
 3	 4	100 5	 8	500 1		 3	θ 4	$100-\theta$ 5	 8	500 1
 3	0 1	200 1	 2	 6		 3	$0-\theta$ 1	$200+\theta$ 1	 2	 6
 4	 7	400 3	 5	 4		 4	 7	400 3	 5	 4

Choose $\theta = 0$.

Test for optimality.

The optimal transportation cost $= 3600$.

E6) Initial basic feasible solution is obtained by North-West Corner Rule:

	M_1	M_2	M_3	M_4	Supply ↓
F_1	5 10	10 2	 20	 11	15
F_2	 12	5 7	15 9	5 20	25
F_3	 4	 14	 16	10 18	10
Demand →	5	15	15	15	

Cost $= 520$.

Test for optimality shows $Z_{14} = 4, Z_{31} \doteq 9$.

The solution is not optimal.

So θ is allocated at the cell $(3, 1)$ and the loop is formed:

{3, 1), (3, 4), (2, 4), (2, 2), (1, 2), (1, 1)}

	M_1	M_2	M_3	M_4	Supply↓
F_1	$5-\theta$......... ↓ 10	$10+\theta$ ↑ 2	20	11	15
F_2	↓ 12	$5-\theta$....... 7	15........... 9	$5+\theta$ ↑ 20	25
F_3	θ........... 4	 14	........→ 16	$10-\theta$ 18	10
Demand →	5	15	15	15	

Choose $\theta = 5$.

For the resulting table $T3$, cost $= 485$, $Z_{14} = 4$. The solution is not optimal.

T3↓			
10	15 2	20	11
12	0 7	15 9	10 20
5 4	14	16	5 18

For the table $T4$, choose $\theta = 10$.

For the next table $T5$, all $Z_{ij} \leq 0$. Also the cost $= 435$. This gives the optimal solution.

T4↓			
10	$15-\theta$← ↓ 2	 20	θ ↑ 11
12	$0+\theta$→ 7	15....... 9	$10-\theta$ 20
5 4	14	16	5 18

T5↓			
10	5 2	20	10 11
12	10 7	15 9	20
5 4	14	16	5 18

E7) Unbalanced problem.

Take credit amount for Private bank $= 750-(400+250)=100$.

Use matrix minima method to get initial b.f.s., at $T1$. All $Z_{ij} \leq 0$. The solution is optimal and minimum cost $=$ RS.116250.00 (since $20\% = 0.20$, etc.).

Alternative initial b.f.s. obtained by the same method is represented by $T2$.

This again is an optimal solution, for, it has the same minimum cost.

T1↓	U	V	W	X	Y	
P	20	18	100 18	17	17	100
N	150 16	150 16	100 16	15	16	400
C	50 15	15	15	125 13	75 14	250
	200	150	200	125	75	

T2↓ 20	100 18	18	17	17
200 16	50 16	150 16	15	16
15	15	50 15	125 13	75 14

E8) Here $C = 3, \lambda = \frac{48}{8} = 6/\text{hour}, \mu = \frac{1}{15} \times 60 = 4/\text{hour}, \frac{\lambda}{\mu} = \frac{3}{2}$, and hence $p_0 = 0.21$.

i) $L_s = 1.74$, ii) $L_q = 0.24$, iii) $W_s = 17.4$ minutes

iv) $W_q = 2.4$ minutes v) utilization factor $= \frac{\lambda}{C\mu} = \frac{1}{2}$; therefore the number of hours each day a tax adviser spends doing his job $= \frac{1}{2} \times 8 = 4$. On an average, a tax adviser is busy $4 \times 5 = 20$ hours base on 5-working day week.

vi) $p(n \geq C) = 0.236$.

vii) expected number of idle advisers at any specified time

$$= 3p_0 + 2p_1 + 1p_2 = 1.5 \text{ (approx.), since } p_n = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n \cdot p_0,\ n < C.$$

E9) i) 0.167 ii) 67%, iii) $\frac{1}{12}, \frac{3}{4}$.

E10) i) $p_n = \frac{1}{n!}(0.28)\,(1.2)^n,\ 0 \leq n < n;\ (0.28)\,(0.6)^n,\ n \leq n \leq 3$

ii) 0.28

iii) 3 (approx.)

E11) i) $L_q = 6.87$ ships

ii) $W_q = 7.21$ days, $W_s = 9.52$ days

E12) It is single channel problem. Here $\lambda = 12/\text{hr.}, \mu = 20/\text{hr}$.

$\therefore$ probability that the booth is busy $= \frac{\lambda}{\mu} = 0.6$ and $W_s = \frac{1}{\mu - \lambda} = \frac{1}{8}$ hr.

Let μ' be the new value of μ and $W_s' = \frac{1}{2} W_s$.

Hence, $W_s' = \frac{1}{\mu' - \lambda}$ gives $u' = 28/\text{hr}$.

$\therefore$ Number of booths $= \frac{\mu'}{\mu} = 2$.

E13) i) It is multi-channel problem. Here $c = 3, \lambda = \frac{5}{2}$ arrivals/hr.,

$\mu = \frac{3}{2}$ services/hr., and $\rho = \frac{\lambda}{c\mu} = \frac{5}{9}$. Also,

$$p_0 = \left[\sum_{n=0}^{n=2} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{3!}\left(\frac{\lambda}{\mu}\right)^3 \frac{1}{1-\frac{5}{9}} \right]^{-1} = \frac{24}{139},$$

$$p_1 = \frac{1}{1!}\left(\frac{\lambda}{\mu}\right) p_0 = \frac{40}{139}, \; p_2 = \frac{1}{2!}\left(\frac{\lambda}{\mu}\right)^2 p_0 = \frac{100}{417}.$$

$\therefore$ Expected number of idle adjusters at any specified time

$= 3p_0 + 2p_1 + p_2 = \frac{4}{3}$.

Probability that any adjuster is idle $= \frac{4/3}{3} = \frac{4}{9}$.

Probability that no adjuster is idle $= 1 - \frac{4}{9} = \frac{5}{9}$.

Expected weekly (5-day week) time an adjuster spends with claimants $= \frac{5}{9} \times 40\,\text{hrs} = 22.2\,\text{hrs}$.

iii) Average time an adjuster spends in the

$$\text{system} = \frac{\mu\left(\frac{\lambda}{\mu}\right)^C}{(c-1)!\,(c\mu-\lambda)^2} p_0 + \frac{1}{\mu} = 49.0 \text{ minutes.}$$

7.7 PRACTICAL EXERCISES

Session 3

1. Write a program in 'C' language to find the initial basic feasible solution of a transportation problem using
   i) North-west corner method, and
   ii) Matrix-Minimum method.

   Also, try this program on Example 8.

2. Write a program in 'C' language to find the operational measures L_s, L_q, W_s and W_q for M/M/C queueing model. Also, try this program on Example 14.

NOTES