UNIT 22 POLICY ANALYSIS: OPTIMISATION STUDIES

Structure

22.0 LEARNING OUTCOME

After studying this Unit, you should be able to:

- Understand the concept of optimisation;
- Differentiate between constrained optimisation techniques and unconstrained optimisation techniques; and
- Apply decision analysis to lake decision under risk.

22.1 INTRODUCTION

In economic decision analysis, optimisation is a way of determining the best option among alternative choices to achieve a goal or objective. This means finding a choice that optimises the value of an objective function. The optimisation could be maximisation of profit or minimisation of cost. For example, in the production process we may be interested in finding the output level that maximises the profit Also, as a policy analyst you may be interested in choosing a project that maximises the net present value. You have learned in Unit 20, how the decisions are taken in choosing alternative projects in capital budgeting problems.

In this Unit, you will learn some techniques for solving optimisation problems. We introduce you to the problem of linear programming, and how to solve the problem graphically. The optimisation techniques are highly mathematical oriented. They include probability, differential calculus, and lot of intricate computations.

Keeping the scope of this Unit, we [illegible] our discussion on understanding the philosophy and practical application of these [illegible]chniques. In this world of computers, a number of computer programmes are re[illegible] solve the problems by using these techniques. For example, computers using simplex methods can solve linear programming problems with hundreds and even thousands of variables.

22.2 CONSTRAINED AND UNCONSTRAINED OPTIMISATION PROBLEMS

The basic form of an optimisation psoblem is: identify the alternative means of achieving a given objective, and choose the alternative that accomplishes the objective in the best way. In an optimisation problem you have three elements: decision variables, objective function, and constraints. We will illustrate these elements with an example. Assume that you are a production manager in a manufacturing firm. The firm is producing two products P, and P_2 say tables and chairs respectively. Both the products require wood and labour resources. The following table 22.1 indicates resources required to manufacture aunit of table or chair and the total atnount of resources available.

Table 22.1: Resources Required/ Available for manufacturing

Resource	Product (requirement of resources per unit)		Total resources available
	Table (P,)	Chair (P,)	
Labour (hours)	5	10	100
Wood (sq.ft.)	40	20	400

The data highlight the resource requirement and available resources for the product. Now you may be interested to find the number of tables and chairs that can be manufactured given the limited availability of resources (labour and wood) in order to maximise the profit. Therefore, the unknown quantities are the number of chairs and tables that we may decide to manufacture, which we term them as decision variables. We may denote them symbolically as X, and X_2. Thus we can say that the decision variables refer to the candidates like products, services, projects, etc. that are competing with each other in sharing limited resources available on hand. In the above example, your objective is to maximise the production in terms of number of tables (X,) and number of chairs (X,).

Suppose you know the profit for each table is Rs. 10, and for each chair is Rs. 15. Since we do not know what will be the total profit, we indicate this as Z. Also, your objective is to maximise the profit (Z). We express maximisation of Z as a function of X, and X_2 and indicate symbolically as follows.

$$\text{Maximise } Z = 10X_1 + 15X_2$$

The above function is called objective function. Therefore, we can say the objective function is a mathematical relationship between decision variables whose values are to be kept in such a way as to maximise or minimise the function.

In our example we also know that the inputs labour and wood are limited resources. You have only 100 hours of labour and 400 square foot available of wood. That means you can use a maximum of 100 hours labour and 400 square foot of wood in the production process of tables and chairs. Also, you know that how much wood and labour is required to manufacture each table and chair. We can express these relationships in the following equations:

$$5X_1 + 10X_2 \leq 100$$

$$40X_1 + 20X_2 \leq 400$$

We also assume that the number of tables (X,)and number of chairs (X,) to be produced are positive. That is,

$X_1 \geq 0$ and $X_2 \geq 0$

The above inequalities are called constraints. Therefore, we can say that the limited resources expressed in terms of linear equalities or inequalities in terms of decision variables are called constraints.

In general, the above optimisation problem can be represented mathematically as:

$$\text{Optimise } Z = f(X_1, X_2, ..., X_n) \qquad (1)$$

$$\text{Subject to } g_j(X_1, X_2, ..., X_n) \begin{bmatrix} \leq \\ = \\ \geq \end{bmatrix} b_j \qquad j = 1,2,...m \qquad (2)$$

Where (1) is the objective function, (2) is the set of constraints, $X_1, X_2, ..., X_n$ are the decision variables, g_j are *the* input functions, b_j are the maximum amount of resources available. Depending on tlie nature of the problem, the optimisation may be maximisation or minimisation of the objective function.

The mathematical techniques used to solve an optilnisation problem shown in equations (1) and (2) above depend on the nature of the objective function (linearity or no linearity) and constraints. The above optimisation problem is also called constrained optimisation. The simplest situation is that of an unconstrained optilnisation problem. In such a problem you have only objective function and no constraints are imposed on the objective function. Here, our objective is either maximisation or minimisation of the objective function on the assumption that there is no scarcity of resources. We will discuss these two types of optimisation problems in the subsequent sections.

22.3 UNCONSTRAINED OPTIMISATION TECHNIQUE

To get optimum solution to the unconstrained problems, we use differential calculus. Some of the most commonly used rules of differentiation (for analysing unconstrained optimisation) are derivation of a constant; derivation of a power; derivation of function; and derivation of sum (difference) function.

Derivation of Constant Functions

A constant function can be expressed as:

$$y = a$$

Where y is a variable and a is a constant, The derivative of any constant function is zero.

$$\frac{dy}{dx} = 0$$

For example, consider the constant function $y = 5$, then $\frac{dy}{dx} = 0$

Derivative of Power Functions

A power function can be expressed as $y = ax^b$ where a and b are constants and y and x are variables. Then, $\frac{dy}{dx} = a\,b\,x^{b-1}$

Example 1: If $y = x^2$, then $\frac{dy}{dx} = 2x^{2-1} = 2x^1 = 2x$

Example 2: If $y = x^{10}$, then $\frac{dy}{dx} = 10x^{10-1} = 10x^9$

Example 3: If $y = 4x^6$, then $\frac{}{dx} = 4.6x^{6-1} = 24x^5$

Example 4: If $y = 4x$, then $\frac{dy}{dx} = 4x^{1-1} = 4x^0 = 4$

Sum (difference)Functions

Suppose a function $y = f(x)$ represents the sum (difference)of two separate functions $f_1(x)$ and $f_2(x)$, that is, the sum function $y = f_1(x) + f_2(x)$ or the difference function $y = f_1(x) - f_2(x)$.

Then, the derivation of the sum function is, $\frac{dy}{dx} = \frac{d}{dx}\frac{f_1(x)}{} + \frac{d}{dx}\frac{f_2(x)}{}$

The derivation of the difference function is, $\frac{dy}{dx} = \frac{d}{dx}\frac{f_1(x)}{} - \frac{d}{dx}\frac{f_2(x)}{}$

The above two results can be extended to finding the derivation of the sum (difference) of any number of functions.

Example 1: If $y = 4x + 5x^8$ then $\frac{dy}{dx} = \frac{d}{dx}\frac{(4x)}{} + \frac{d}{dx}\frac{(5x^8)}{} = 4 + 5.8x^7 = 4 + 40x^7$

Optimisation Conditions

The season for introducing differentiation rules for simple functions, above, is that these methods can be used to find optimal solutions to many kinds of maximisation and minimisation problems with unconstrained functions in managerial economics. There are two conditions for obtaining either maximisation or minimisation. These are called necessary and sufficient conditions. Sometimes, they are also referred as first order and second order conditions respectively.

The necessary (first order) condition for solving both maximisation or minimisation problems are $\frac{dy}{dx} = 0$.

The sufficient (second order) condition for solving maximisation problems is $\frac{d^2y}{dx^2} < 0$.

(Kindly note that $\frac{d^2y}{dx^2}$ indicates the derivative of the result of $\frac{dy}{dx}$.

The sufficient (second order) condition for solving minimisation problems is $\frac{d^2y}{dx^2} > 0$.

However, at certain points you may find $\frac{d^2y}{dx^2} = 0$. Such points are called point of inflexion. In such cases, the points are neither maximum nor minimum and no optimum result can be obtained.

By finding the $\frac{dy}{dx}$, equating it zero and then simplifying the objective function we can find the value of y. However, we go for second order condition, since the first order condition alone does not guarantee the optimum result. We summarise the optimisation conditions in the following table.

Table 22.2: Conditions for Optimisation

Nature of Condition	Maximisation Problems	Minimisation Problems
Necessary Condition	$\frac{dy}{dx}=0$	$\frac{dy}{dx}=0$
Sufficient Condition	$\frac{d^2y}{dx^2}<0$	$\frac{d^2y}{dx^2}>0$
Note that if the value of $\frac{d^2y}{dx^2}=0$ *then you cannot obtain optimisation* ***result for*** *the problem under study.*		

Examples for Profit or Revenue Maximisation

Example-1

For the following profit function, find the optimum output that maximises the profit.

$\Pi=-20+150Q-5Q^2$ where Π is total profit and Q is the output level.

Solution

The necessary condition for profit maximisation is the first order derivative set to zero. That is,

$$\frac{d\Pi}{dQ}=0$$

$$\frac{d\Pi}{dQ}=150-10Q$$

Setting the above equation to zero,

$$150-10Q=0$$

$$150=10Q$$

$$Q=\frac{150}{10}=15$$

The sufficient condition for profit maximisation is the second order derivative, which should be less than zero. That is, $\frac{d^2\Pi}{dQ^2}<0$

Since, $\frac{d\Pi}{dQ}=150-10Q$

$\frac{d^2\Pi}{dQ^2} = \frac{d(150-10Q)}{a} = -10$ which is negative as required for maximisation. Hence for the given function, the profit can be maximised for the output Q=15.

Example-2

For a firm (where Q is the level of output),the total cost function (TC) is given by $TC = 20 + 5Q + Q^2$. Also, the demand for the output of the firm is a function of price and the relationship is given as $Q = 25 - P$.

i) Given the total revenue TR equals price (P) per unit multiplied by the number of units sold, that is, TR = PQ. Also. the total profit (Π), is the diffrence between the total revenue and total cost (TC). That is, Π = TR-TC. Express the output as a profit function.

ii) Determine the output level where total revenue (TR) is maximised.

iii) Calculate the total output at which profit can be maximised.

iv) Determine the output that maximises the total profit.

V) Calculate the price at the profit maximisation output level.

Solution

i) Given the demand function $Q = 25 - P$, the price function is $P = 25 - Q$

And total revenue function (TR) is $TR = PQ = (25 - Q)Q = 25Q - Q^2$

The total profit (Π) is the difference between TR and TC

$\Pi = (TR - TC) = (25Q - Q^2) - (20 + 5Q + Q^2) = -20 + 20Q - 2Q^2$

ii) The total revenue function is $TR = 25Q - Q^2$

To maximise the TR, the necessary condition is $\frac{dTR}{dQ} = 0$

$$\frac{dTR}{dQ} = 25 - 2Q$$

$$25 - 2Q = 0$$

$$Q = \frac{25}{2} = 12.5$$

The sufficient condition to maximise TR, $\frac{d^2TR}{dQ^2} < 0$

$$\frac{d^2TR}{dQ^2} = \frac{d}{dQ}\frac{(25 - 2Q)}{} = -2$$

Since the second derivative is negative, the total revenue maximises at the output level Q=12.5.

iii) The total profit function is $\Pi = -20 + 20Q - 2Q^2$

To maximise the Π, the necessary condition is $\frac{d\Pi}{dQ} = 0$

$$\frac{d\Pi}{dQ} = 20 - 4Q$$

$$20 - 4Q = 0$$

$$Q = \frac{20}{4} = 5$$

The sufficient condition to meximise total profit, $\frac{d^2\Pi}{dQ^2} < 0$

$\frac{d^2\Pi}{dQ^2} = \frac{d}{dQ}\frac{(20-4Q)}{} = -4$ which is negative as desired. Since the second derivative is negative, the total profit maximises at output level Q=5.

iv) We obtained from (iii) the profit maximisation output level as Q=5.
Therefore, the total profit at Q=5 is,

$$\Pi = -20 + 20Q - 2Q^2$$

$$\Pi = -20 + 20(5) - 2(5^2) = -20 + 100 - 50 = 30$$

v) The price at Q=5 is,

$$P = 25 - Q \quad = 25 - 5 = 20$$

22.4 CONSTRAINED OPTIMISATION TECHNIQUES

In section 22.2 discussed above, you have learned the concepts of constrained and unconstrained optimisation problems. Also in section 22.3 you have learned how lo optimise (maximisation or minimisation) unconstrained optimisation problems using necessary and sufficient conditions of first order and second order derivatives respectively. In this section, you will learn how to solve constrained optimisation problems. There are many types of constrained optimisation problems like linear programming, non-linear programming, dynamic programming, etc. However, we will limit our discussion in this Unit to only linear programming technique.

22.4.1 Linear Programming

Linear programming is a technique for mathematical solution of a constrained optimisation problem. As such it has been used in policy analysis for formulating an objective function to be maximised or minimised subject to a set of resource constraints. For example, for apolitical candidate the objective function could be to maximise votes polled in his/her favour subject to resource constraints or minimise the total costs and still get winning votes. Originally, the linear programming problems are applied widely in business management. For example, to define a sales strategy that maximises the profits, or a production strategy that best uses the scarce resources,

For an effective application of linear programming, it requires a good understanding of the underlying modeling assumptions and pertinent interpretation of the analytical solutions obtained. Therefore, in this section, we discuss the details of the linear programming modelling, and its underlying assumption with examples.

Formulation of a Linear Programming Problem

Consider a company that produces two types of products P, and P,. The company has two factories, F, and F_2 both producing the products P, and P_2. The Factory F_1 produces 40 units per day of product P, or 60 units per day of product P_2. Similarly, factory F_2 produces 50 units per day of product P_1 or 50 units per day of product P,. The company's profit on product P_1 is Rs.200 per unit and on product P_2 is Rs.400 per unit. Suppose if the company sells all the produced units of products P_1 and P_2, how many units of each product should the company produce on a daily basis to maximise its profit?

The above problem is a constrained optimisation problem. Here our objective is to maximise the company's daily profit. Further, we need to maximise the profit by adjusting the levels of daily production of the two products P_1 and P_2. Therefore, the daily production levels of product P, and P, are the decision variables. These decision variables are:

X, is the number of units produced daily of product P,

X_2 is the number of units produced daily of product P_2

Let the daily profit on both products be Z.

The profit on one unit of P, is Rs.200, and on one unit of P_2 is Rs.400. The total profit function can be expressed as:

$$Z = 200X_1 + 400X_2 \rightarrow \quad (1)$$

In this problem, our objective is to maximise the profit and this can be expressed analytically as:

$$\text{Maximise } Z = 200X_1 + 400X_2 \rightarrow \quad (2)$$

The equation (2) is called objective function of the problem, and coefficients of the decision variables are called objective function coefficients,

Also, we have constraints of producing each product by two factories. Factory F_1 can produce 40 units of product P_1 or 60 units of product P_2 per day. Assuming that production of one unit of product P_1 (and P_2) requires a constant amount of processing time, it follows that for product P_1 it is 1/40 and for product P_2 it is 1/60. The total capacity in terms of time length required for producing X, units of P_1 and X_2 units of P_2 by F_1 is equal to $\frac{1}{40}X_1 + \frac{1}{60}X_2$. Hence, the constraints imposed on F, should not exceed its production capacity is expressed as:

$$\frac{1}{40}X_1 + \frac{1}{60}X_2 = 1.0 \rightarrow \quad (3)$$

By similar argument, the constraint imposed on F_2 is expressed as:

$$\frac{1}{50}X_1 + \frac{1}{50}X_2 = 1.0 \rightarrow \quad (4)$$

Finally, for the above constraints the production of X, and X_2 cannot be negative. That is,

$$\left.\begin{array}{l} X_1 \geq 0 \\ \\ X_2 \geq 0 \end{array}\right\} \quad (5)$$

Combining equations (2) to (5), the formulation of the linear programming problem is as follows:

$$\text{Maximise } Z = 200X_1 + 400X_2$$

$$\text{Subject to } \frac{1}{40}X_1 + \frac{1}{60}X_2 = 1.0$$

$$\frac{1}{50}X_1 + \frac{1}{50}X_2 = 1.0$$

$$X_1 \geq 0$$

$$X_2 \geq 0$$

22.4.2 Solution to the Linear Programming Problem through Graphical Method

Most of the linear programming problems require the use of computer to find optimum solutions. A graphical method of solving is useful to understand the concept of linear programming problem, its formulation and solution. However, graphical solution can be used only for two decision variables problem. A more general method known as Simplex Method is widely used for solving linear programming problems for larger number of variables. As this method requires a large number of computations, it is advised to use computers for solving such problems. We solve the maximisation problem formulated in section 2.4 graphically. Let the horizontal axis represent the number of units of product P, and the vertical axis represent the number of units of product P_2 produced. Since, output (X, and X,) of both the products are non-negative, the graphical method stat-ts with the first quadrant of the graph.

Now, we plot the constraints on a graph. Since both the constraints are inequalities, for the time being we consider them as equalities for the purpose of plotting them on the graph.

$$\frac{1}{40}X_1 + \frac{1}{60}X_2 = 1.0$$

$$\frac{1}{50}X_1 + \frac{1}{50}X_2 = 1.0$$

When plotted on a graph, the above equalities represent straight lines. A straight line can be completely specified by knowing any two points that fall on the line.

Let us consider the first equality constraint

$$\frac{1}{40}X_1 + \frac{1}{60}X_2 = 1.0$$

We can plot a straight line by substituting 0 for X, in the above equation:

$$\frac{1}{40}(0) + \frac{1}{60}X_2 = 1.0$$

$$\frac{1}{60}X_2 = 1.0$$

$$X_2 = 60$$

This represents the vertical interceptor on X_2 axis and is written as (0, 60). That is 0 units of product P_1 and 60 units of product P_2.

Similarly, when X_2=0

$$\frac{1}{40}X_1 + \frac{1}{60}(0) = 1.0$$

$$\frac{1}{40}X_1 = 1.0$$

$$X_1 = 40$$

It is also written as (40, 0) and represents 40 units of product P_1, and 0 units of product P_2 and corresponds to horizontal interceptor X1. Join these two interceptors to draw the constraint line as shown in the graph (figure 22.1). Since this constraint is less than or equal to constraint, the points below the line represent the region satisfied by $\frac{1}{40}X_1 + \frac{1}{60}X_2 = 1.0$ constraint. Any point that lies outside this line gives an infeasible solution.

Now let us consider the second equality constraint, that is,

$$\frac{1}{50}X_1 + \frac{1}{50}X_2 = 1.0$$

When X, = 0

$$\frac{1}{50}(0) + \frac{1}{50}X_2 = 1.0$$

$$X_2 = 50$$

This represents vertical interceptor on X_2 axis and is written as (0, 50).

When $X_2 = 0$

$$\frac{1}{50}X_1 + \frac{1}{50}(0) = 1.0$$

X, = *50*

This represents horizontal interceptor X_1 axis and is written as (50, 0). Using (0, 50) and (50, 0) points, we can draw a straight line on the graph. Since it is a less than or equal to constraint, the points'below this line represent the region satisfied by $\frac{1}{50}X_1 + \frac{1}{50}X_2 = 1.0$ constraint.

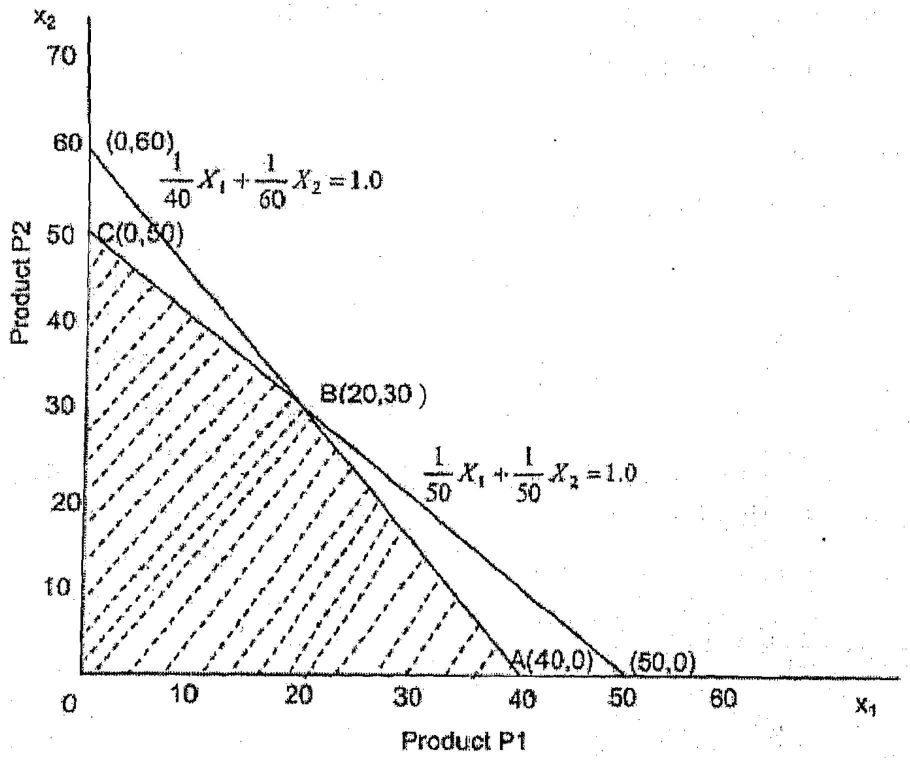

Fig. 22.1: Graph showing Feasible Region for Optimisation

Any combination of valu/es of X_1 and X_2, which satisfies the given constraints, is called a 'feasible solution' to the problem. The area OABC in the graph satisfied by the constraints is shown by shaded area is the feasible region or feasible set.

To find the best solution, we use our objective function $Z = 200X_1+400X_2$

For extreme point A (40, 0), X_1=40 and X_2=0, Z=200(40)+400(0) = 8000+0=8000

For extreme point C (0,50), X_1=0 and X_2=50, Z=200(0)+400(50)=0+20000=20000

Extreme point B is at the intersection of the two lines. We can solve these two equations to find B.

$$\frac{1}{40}X_1 + \frac{1}{60}X_2 = 1.0 \rightarrow \text{(a)}$$

$$\frac{1}{50}X_1 + \frac{1}{50}X_2 = 1.0 \rightarrow \text{(b)}$$

First, we can solve for X_1 or X_2 in one equation and then substitute that value in the second equation. Solving for X_1 in equation (a) we have:

$$\frac{1}{40}X_1 + \frac{1}{60}X_2 = 1.0$$

$$\frac{1}{40}X_1 = (1.0 - \frac{1}{60}X_2)$$

$$X_1 = 40(1.0-\frac{1}{60}X_2) = 40-\frac{40}{60}X_2$$

Substituting this value for X_1 in equation (b) **we** have

$$\frac{1}{50}\left(40-\frac{40}{60}X_2\right)+\frac{1}{50}X_2 = 1.0$$

$$40-\frac{40}{60}X_2+X_2 = 50$$

$$X_2-\frac{40}{60}X_2 = 50\text{-}40$$

$$\frac{60X_2-40X_2}{60} = 10$$

$$60X_2 - 40X_2 = 10\text{x}60$$

$$20X_2 = 600$$

$$X_2 = 30$$

Substituting this value for X_2 in either of our original equations (a) or (b) we can find X_1 value. Suppose we substitute $X_2 = 30$ in (b) then

$$\frac{1}{50}X_1 + \frac{1}{50}X_2 = 1.0$$

$$\frac{1}{50}X_1 + \frac{1}{50}(30) = 1.0$$

$$\frac{X_1 + 30}{50} = 1.0$$

$$X_1 + 30 = 50$$

$$X_1 = 50 - 30 = 20$$

Thus at corner point B, $X_1 = 20$ and $X_2 = 30$. From this we can compute Z:

$$Z = 200X_1 + 400X_2$$

$$Z = 200(20) + 400(30)$$

$$Z = 4000 + 12000 = 16000$$

Now we have to maximise objective function $Z = 200X_1 + 400X_2$.

This can be obtained from the table given below.

Table 22.3: Coordinates for Minimisation of objective functions

Extreme Point	**Coordinates**	**Profit Function $Z=200X_1+400X_2$**
A	$X_1=40, X_2=0$	8000
B	$X_1=20, X_2=30$	16000
C	$X_1=0, X_2=50$	20000

The maximum profit obtained at C(0, 50). That is $X_1 = 0$ and $X_2 = 50$. Hence, to maximise profit, the company should produce 50 units of product P_2. The maximum profit obtained from producing 50 units of product P_2 is 20,000.

Using Graphical Method for Minimisation

A firm is engaged in breeding pigs. The pigs are fed on various products grown on the farms. In order to ensure certain nutrient requirements, it was decided to buy one or two products (P_1, P_2). The nutrient constituents in each unit of the products P_1 and P_2 are given below:

Table 22.4: Requirements and constraints

Nutrient Constituents	**Nutrient Consisted in the Product**		**Minimum Requirement of Nutrient Constituents**
	P_1	**P_2**	
A	36	6	108
B	3	12	36
C	20	10	100

Product P_1 costs Rs.20 and Product P_2 costs Rs.40 per unit. Determine how much product P_1 and/or P_2 must be purchased so as to provide the pigs nutrients not less than the minimum required at the lowest possible cost.

Solution

First we formulate the above problem as a linear programming problem. Let X_1, and X_2 be the number of units (decision variables) of product P_1 and P_2 required. Here the objective is to minimise the cost. We are given the cost of product P_1 and P_2 as Rs.20 and Rs.40 respectively. Therefore, the objective function is:

$$\text{Minimise } Z = 20X_1 + 40X_2 \rightarrow (1)$$

Since the number of units of nutrient 'A' in product P_1 and P_2 are 36 and 6 respectively and the minimum requirement of 'A' is 108 units, we write this constraint as:

$$36X_1 + 6X_2 \geq 108 \rightarrow (2)$$

Similarly, the constraints for B and C nutrients are:

$$3X_1 + 12X_2 \geq 36 \rightarrow (3)$$

$$20X_1 + 10X_2 \geq 100 \rightarrow (4).$$

We also have the decision variables as non-negative, that is,

$$\left.\begin{array}{l} X_1 \geq 0 \\ X_2 \geq 0 \end{array}\right\} (5)$$

For plotting the constraints as straight lines in the graph, we rewrite the inequality constraints as equality constraints, that is,

$$36X_1 + 6X_2 = 108 \rightarrow (6)$$

$$3X_1 + 12X_2 = 36 \rightarrow (7)$$

$$20X_1 + 10X_2 = 100 \rightarrow (8)$$

Now for drawing straight line (6) we have

When $X_1 = 0$

$36(0) + 6X_2 = 108$

$6X_2 = 108$

$X_2 = 108/6 = 18$

We have the coordinates A(0, 18)

When $X_2 = 0$

$36X_1 + 6(0) = 108$

$X_1 = 108/36 = 3$

We have the coordinates B(3, 0)

Joining these two coordinate points we get straight line for equation (6) as shown in the graph. Similarly, for constraint (7) we have the coordinate points C(0, 3) and D(12, 0) and for constraint (8) we have the coordinate points E(0, 10), F(5, 0). Now we can draw the straight lines for constraint (7) and (8) as shown in the following graph.

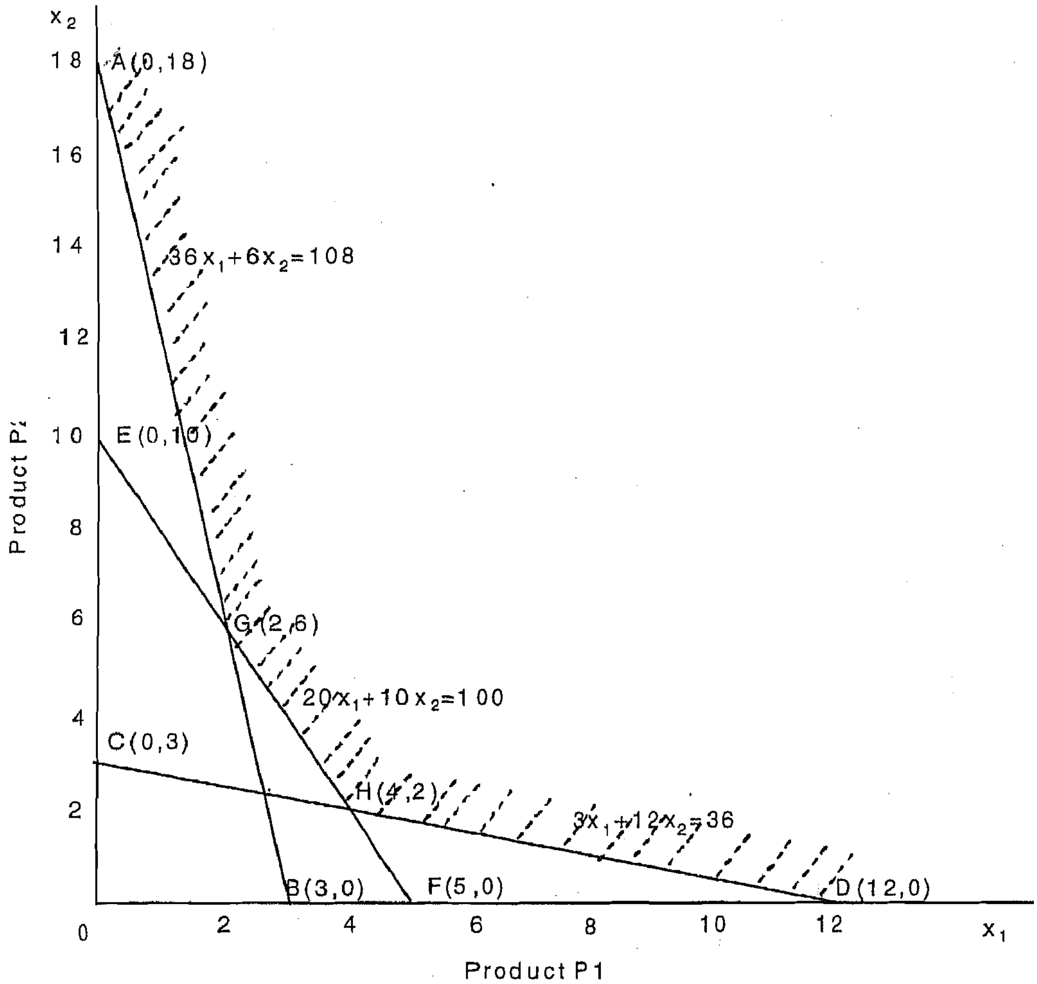

Fig. 22.2: Graph showing Feasible Region for Optimisation

The coordinate G is an intersection of constraints (6) and (8). Solving these two equations for X_1 and X_2 from (6), $36X_1 = 108-6X_2$; $X_1 = \frac{108-6X_2}{36}$; substituting this value in (8) we have

$$\frac{20(108-6X_2)}{36} + 10X_2 = 100$$

$$\frac{20(108-6X_2)+360X_2}{36} = 100$$

$$2160 - 120X_2 + 360X_2 = 3600$$

$$240X_2 = 3600 - 2160 = 1440$$

$$X_2 = 1440/240 = 6$$

Substituting $X_2 = 6$ in (8) we have

$$20X_1 + 10X_2 = 100$$

$$20X_1 + 10(6) = 100$$

$$20X_1 = 100 - 60 = 40$$

$$X_1 = 40/20 = 2$$

Therefore, the coordinate points at G are (2, 6).

Similarly, the equations (7) and (8) are intersecting at H.

From (7)

$$3X_1 + 12X_2 = 36$$

$$3X_1 = 36 - 12X_2$$

$$X_1 = \frac{36 - 12X_2}{3}$$

Substituting this value in (8), we have

$$\frac{20(36 - 12X_2)}{3} + 10X_2 = 100$$

$$720 - 240X_2 + 30X_2 = 300$$

$$210X_2 = 420$$

$$X_2 = 2$$

Substituting this value in (8) we have

$$20X_1 + 10(2) = 100$$

$$20X_1 = 100\text{-}20$$

$$X_1 = 80/20 = 4$$

Therefore, the coordinate points at H are (4, 2).

The feasible region is shown as AGHD as lowerpoints, since the inequality constraints a n greater than or equal to type ($\geq$). The Z values for these coordinates are shown in the following table.

Table 22.5: Coordinates for minisation of objective function

Corner Points	**Coordinates of Corner Points (X_1, X,)**	**Objective function 2 = $20X_1$+$40X_2$**	**Z value**
A	(0, 18)	20(0) + 40(18)	720
G	(2, 6)	20(2) + 40(6)	280
H	(4, 2)	20(4) + 40(2)	160
D	(12, 0)	20(12) + 40(0)	240

Since it is a cost minimisation problem, we have the value 160 at corner point H (4, 2) as minimum, that is, $X_1 = 4, X_2 = 2$. Therefore, the required products are P, = 4 units and $P_2 = 2$ units, and the minimum cost is Rs.160.

22.5 DECISION UNDER RISK

We take decisions every day. We know the outcome of some decisions at the time of taking decisions. These are called decisions with certainty, For example, suppose you want to invest Rs.10,00,000 as a fixed deposit in a Bank. Suppose the Bank is giving 10 per cent interest on fixed deposits for a five-year period. Also, you have an alternative option of investing this in a saving account where the Bank gives 6 per cent interest. You know that both investments are secure and guaranteed. Then there is a certainty in your decision of investing the money in a scheme that gives better returns. Cost-benefit analysis, constrained optimisation (linear programming), and unconstrained optimisation are some of the techniques used under certainty conditions that you have already learned.

Now we focus our discussion on decision making under risk. The objective here is to develop guidelines for making rational decisions, given the individual decision maker's attitudes towards risk. The decision makers according to their attitudes in decision-making under risk: may be of 3 types, as under:

- A risk seeker is one who prefers risk.
- A risk averter accepts only favourable conditions. He/she chooses less risky investments.
- Between these two extremes (risk seeker and risk averter) lies the risk neutral person. He/she considers the face value of money to be its true worth.

Decision Tables

Each alternative choice that involves risk (which we term as states of nature or outcomes) is associated with a probability of occurrence or non-occurrence of that choice. For example, the decision on motor vehicle insurance is associated with the probability of accident and/or theft. The decision here is to make insurance or not. This decision depends on the probability of occurrence of theft/accident. Suppose you are told that the probability of theft/accident is 0.05 (that is 5% chance), and no theft/accident is 0.95 (that is 5% chance). Also, given that the vehicle costs Rs.5,00,000 and the premium for insurance payment is Rs.5,000 per annum. Now you have four courses of action to expect. These are: a) make insurance and no accident/theft, b) make insurance and accident/theft, c) no insurance and no accident/theft, and d) no insurance and accident/theft. Each of these courses of action will cost you. If you go for insurance and accident/theft takes place still you loose Rs.5,000 towards premium. If you don't go for insurance and no accident/theft takes place you will not loose any money. Lastly, if you don't go for insurance and the accident1 theft takes place you will loose the cost of the vehicle. That is, Rs.5,00,000. We can summaries this problem using a decision table as shown below.

Table 22.6: Decision Table for Insurance Decision

Event	Courses of Action	
	Insurance	**No insurance**
Accident/theft	Loss (Rs.5,000)	Loss (Rs.5,00,000)
No accident/theft	Loss (Rs.5,000)	Loss (Rs.0)

The decision table indicates the relationship between pairs of decision elements (courses of action and'events). There is specific information for each course of action and event combination. This information is also expressed in terms of rupees. Therefore, every course of action-event combination has a payoff, which measures the net benefit that accrues from a given course of action and event. Sometimes, they are also known as conditional profit values.

The decisions on the courses of action under risk are based on the availability of prior information, knowledge, experience, or judgment, that we express in terms of event probability values.

Maximising the Expected Payoff

The expected payoff is a measure of expected returns associated with the probability of the events, that is,

Expected payoff = robability of first event X (payoff) of first event + probability of second event X payoff of second event + ... + probability of nth event X payoff of nth event.

Expected payoff = this is represented by $E(X) = P_1X_1 + P_2X_2 + \ldots + P_nX_n = \sum_{i=1}^{n} P_i X_i$

Where P_i is the probability of i^{th} event

X_i is the payoff of that event

E(X) is the expectedpayoff (of taking a course of action)

In our insurance example, the calculation of the expectedpayoff is shown in the following table:

Table 22.7: Calculation of Expected Payoff for Insurance Decision

Event	Probability	Courses of Action			
		Insurance		No Insurance	
		Payoff (Rs.)	Payoff x Probability (Rs.)	Payoff (Rs.)	Payoff x Probability
Accident/theft	0.05	-5,000	-250	-5,00,000	-25,000
No accident/theft	0.95	-5,000	-4,750	0	0
Expected Payoff			-5,000		-25,000

Kindly observe that the payoffs are shown as negative numbers since they represent loss in that event. Also, the total probability of all events is 1.00. We calculate the expected payoff for each course of action by multiplying each payoff by the respective event probability and then adding the products.

Then how does the decision maker choose a course of action among alternative courses of action available to him/her? The answer to this question is to choose that course of action for which the expected payoff is the highest. In our example of insurance, you can observe from the above table that the expected payoff is highest for insurance course of action. In other words, expected loss is Rs.5,000 when you go for insurance, and Rs.25,000 for not going for insurance.

Decision Tree Analysis

A decision problem can also be illustrated with a decision tree diagram. It is particularly convenient to display decision problems in the form of decision trees when the choices are to be made at different times or stages. As the name implies, a decision tree consists of network nodes, branches and probability estimates and pay off's. Nodes are of two types: (a) decision node indicated as a square (□), and (b) chance node indicated as circle (○) that illustrates the point of uncertainty. A basic guideline for constructing a decision tree diagram is that the flow should be chronological from left to right.

The alternative courses of action are shown as emanating from the decision point and then corresponding to each decision, the possible outcomes are shown emanating from the uncertainty point. The side of the outcome lists the probability and the consequence for each outcome, Our example of insurance decision can be illustrated using a decision tree as shown in the following figure.

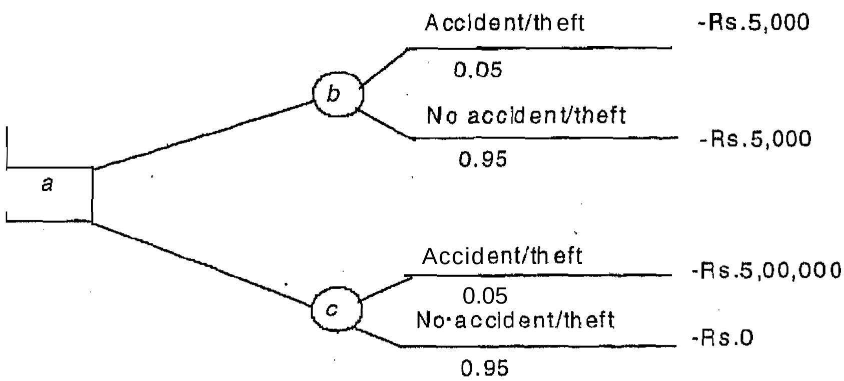

Fig. 22.3: Insurance Decision Tree

The initial choice at decision point a is 'insurance' or 'no insurance'. For each of these you have two events. In all you have four courses of action. The probabilities and payoffs associated with each event are also shown in the tree diagram: The main advantage of using a decision tree is that it helps in isolating each chain and follows it through the branches to the terminal event.

22.6 CONCLUSION

In this Unit, various optimisation techniques are introduced that may help you in your decision making process. The optimisation decisions may occur under certainty or risk conditions. The optimisation techniques under certainty conditions are further classified into constrained and unconstrained optimisation.

The necessary condition for optirnisation (for both maximisation and minimisation problems) is $\frac{dy}{dx} = 0$. The sufficient condition for maximisation problem is that $\frac{d^2y}{dx^2} < 0$. The sufficient condition for minimisation problem is that $\frac{d^2y}{dx^2} > 0$. To obtain optimisation results both necessary and sufficient conditions need to be met. In case the sufficient condition $\frac{d^2y}{dx^2} = 0$. Then for that problem no optirnisation result exists.

The linear programming helps to solve optimisation problems with constraints. To understand the basic philosophy and concepts, we have explained in this unit how to formulate and solve the linear programming problems graphically. The simplex method is widely used to solve linear programming problems. Many computer programmes are available as aids for carrying out the large number of computations required for solving linear programming problems.

The decisions taken under risk are explained with the help of expected payoffs and decision trees. Here, the decision is to choose an alternative where the expected payoff is maximised and risk is minimised.

22.7 KEY CONCEPTS

Constraints : These are the linear equalities or inequalities arising out of practical limitations.

Decision Trees : A useful way of analysing sequential decision problems.

Decision Variables : These are variables whose numerical values are of interest to the decision-maker to obtain optimal solution to problems.

Expected Payoff : A decision criterion that employs probability.

Feasible Set : The available set of alternatives to the decision-maker is called a feasible set.

Graphical Method : A method of finding solution to the linear programming problem through plotting equations (constraints) on a two dimensional graph.

Linear Programming : A linear deterministic model used to find optimum allocation of scarce resources.

Objective Function : In linear programming, objective function is a linear function of the decision variables expressing the objective of decision maker.

Optimisation : Optimisation is an act of selecting the best alternative out of available options to the decision-maker,

Payoff : The net benefit to the decision-maker, which accrues, from a given combination of decision alternatives or events.

Risk Averter : A risk averter is one who will select the less risky option in an investment situation.

Risk Neutral : A risk neutral person considers the face value of money to be its true worth.

Simplex Method : A mathematical procedure of finding optimum solution to the linear p ogramming problem.

22.8 REFERENCES AND FURTHER READING

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22.9 ACTIVITIES

1) Given the following total profit function

   $\Pi = -2500 + 200Q - 2Q^2$

   Where Q is the output level.

   Determine the output level that maximises profit.

2) Given the following total revenue function:

   $TR = 100Q - 2Q^2$

   Where Q is the output level

   Determine the output level that maximises the revenue.

3) Given the following average total cost function:

   $ATC = 5000 - 100Q + 2Q^2$

   Where Q is the output level.

   Determine the output level that minimises the average total cost.

4) These are two factories that manufacture three different grades of paper. There is demand for each grade. The company that controls two factories (**A** and B) has the contract to supply 16 tons of low grades, 5 tons of medium grade and 20 tons of high-grade paper, It casts Rs.10,000 per day to operate the first factory and Rs.20, 000 per day to operate the second factory. Factory **A** produces 8 tons of low grades, 1 ton of medium grade and 2 tons of high grade paper in one day's operation. Factory B produces 2 tons of low grade, 1 ton of medium grade and 7 tons of high grade paper per day. Formulate the above problem for a decision linear programming by applying technique.

5) Consider the problem given at (4) above. For how many days should each factory be operated in order to meet the orders most economically? Solve the problem graphically.

6) Describe the situations of decision-making under certainty and risk, and discuss the behaviour of risk averter, risk seeker, and risk neutral persons in investment decisions in the later situation.